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ABSTRACT

A Markov model for predicting performance on criterion-referenced tests is presented. The model is expressed mathematically as a function of transition matrix, a current state vector, and a future state vector. The matrix is defined in terms of conditional probabilities, i.e., the probability of making a transition to a specific future performance state given data pertaining to the student's current performance state. Performance is expressed in terms of mastery, a theoretical construct that is defined in the paper. State vectors indicate either the probability of mastery or the degree of mastery. The current state vector can be computed from available observed criterion test scores. Three examples are included which indicate how transition matrices may be computed. An example is also provided which shows how the model can be used to predict future performance. Finally, a research application and a management application of the Markov model are mentioned. (Author)

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A Markov model for predicting performance on criterion-referenced tests is presented. The model is expressed mathematically as a function of a transition matrix (T), a current state vector (V_c), and a future state vector (V_f). The matrix is defined in terms of conditional probabilities; i.e., the probability of making a transition to a specific future performance state given data pertaining to the student's current performance state. Performance is expressed in terms of mastery, a theoretical construct that is defined in the paper. State vectors indicate either the probability of mastery or the degree of mastery. The current state vector can be computed from available observed criterion test scores.

Three examples are included which indicate how transition matrices may be computed. An example is also provided which shows how the model can be used to predict future performance. Finally, a research application and a management application of the Markov model are mentioned.

A MARKOV MODEL FOR PREDICTING PERFORMANCE ON CRITERION-REFERENCED TESTS

Introduction

Criterion-referenced tests are becoming a common method of assessing pupil performance with respect to a specified content objective. Such tests are intended to provide absolute measures of proficiency. According to Cronbach (1969), too much attention has been given to comparative interpretations and too little to absolute content-referenced measurement. Kriewall (1969) points out:

> . . . what is required to guide instructional decisions is curriculum-specific data that can be meaningfully interpreted in the absence of pupil-group data. Hence one needs absolute measures of an individual's proficiency with respect to a well-defined body of content or set of skills.

Absolute measures of an individual's proficiency can aid the teacher in making decisions that are intended to improve subsequent pupil performance. If criterion test scores are used for instructional decision-making, a method of predicting performance on subsequent tests may be of considerable value; especially when (a) the instructional program consists of an ordered set of units and (b) each unit has a set of subobjectives that are related to the program objective. The prediction model presented in this paper was developed specifically for instructional programs which include criterion-referenced tests.

Using a prediction model, a teacher may select instructional activities that appear to have a high pay-off for improving predicted performance on subsequent units of instruction. Without a prediction model, the teacher desiring to improve subsequent performance must use performance on units of instruction for which observed scores are available. The strategy aimed at improving predicted performance on subsequent units of instruction is seldom used, not because it has been ineffective, but because it requires a reliable prediction model that is convenient to use.

Definition of Terms

Before discussing the prediction model, the following terms need to be defined: mastery, probability of mastery, degree of mastery, state vectors, and transition matrix.

Mastery is a theoretical construct used to represent the maximum performance level for a specified content objective when performance is measured with a criterion test which makes no assessment errors. An N item test which makes no assessment errors would result in an observed score of N for all students who are in the total mastery state; likewise, all students with an observed score of zero would be classified in the total non-mastery state.

Two types of assessment errors may occur. Type I errors occur when a student who has mastered an outcome answers some of the items incorrectly. Type II errors occur when a student who has not mastered the objective answers some of the items correctly. Carelessness is likely to result in Type I errors; guessing is likely to result in Type II errors. Both types of errors may also result when criterion test performance is not dependent solely on the specified content objectives for which the test was intended.

Since criterion tests which make no assessment errors cannot be constructed, observed scores are not completely reliable indicators of the student's true performance state. Observed scores can be converted, however, to values which are estimates of performance relative to mastery or non-mastery; two such values are probability of mastery, P(M), and degree of mastery, D(M). When probability of mastery is used to represent criterion performance, it is assumed that each student is in either the total mastery state or the total non-mastery state. Probability of mastery, therefore, is a measure of confidence that a student is in the total mastery state. On the other hand, when degree of mastery is used to represent criterion performance, it is assumed that students can be in any state along a continuum from total mastery to total non-mastery. Degree of mastery, therefore, is an estimate of how close a student is to the total mastery state. In this paper both probability of mastery and degree of mastery will be represented by a number between zero and one. Methods of computing degree of mastery and probability of mastery measures have been presented by Kriewall (1969) and Emrick and Adams (1970), respectively. Besel (1971) has compared the two methods.

It may be possible to assume a probability of mastery equal to one for all degree of mastery scores greater than or equal to a specified criterion value and a probability of mastery equal to zero for all degree of mastery scores less than the specified criterion value. Such an assumption is not unreasonable if (a) the criterion test does not make too many assessment errors; and (b) an appropriate "cutting score" can be determined. The prediction model presented in this paper does not require cutting scores to be selected.

State vectors represent criterion performance at a specified point in time for a student or group of students. A state vector (V) is a function of probability of mastery or degree of mastery. Mathematically, the state vector for unit i is defined as follows:

$$V_i = \begin{bmatrix} \overline{M}_i \\ M_i \end{bmatrix}$$

where M_i and $\overline{M}_i$ represent values indicative of performance relative to mastery and non-mastery respectively; e.g., P(M) and 1-P(M) for unit i. Current state vectors (V_c) exist for all units of instruction that have been completed. Future state vectors (V_f) can be calculated by using one current state vector and the appropriate transition matrix (T).

Transition matrices indicate the probability of a specific performance state after completing unit j given knowledge about performance on a criterion test for unit i. Using the notation P (Y_j/X_i) to represent the conditional probability of state Y after completing unit j given that the student was in state X after completing unit i, the mathematical definition of the transition matrix T_{ij} is as follows:

$$T_{ij} = \begin{bmatrix} P(\overline{M}_j/\overline{M}_i) & P(\overline{M}_j/M_i) \\ P(M_j/\overline{M}_i) & P(M_j/M_i) \end{bmatrix}$$

where M represents mastery; $\overline{M}$, non-mastery; $P(M_j/\overline{M}_i)$ equals $1-P(\overline{M}_j/\overline{M}_i)$; and $P(\overline{M}_j/M_i)$ equals $1-P(M_j/M_i)$.

Methods of Computing Transition Probabilities

A variety of methods can be used to compute the conditional probabilities, depending on the data available and the assumptions made. For example, if N students are classified in either the mastery or non-mastery state for units i and j, then a counting procedure can be used to determine the frequencies associated with each transition path; i.e., $F(\overline{M}_i, \overline{M}_j)$, $F(\overline{M}_i, M_j)$, $F(M_i, \overline{M}_j)$ and $F(M_i, M_j)$. These frequencies can then be used to compute the conditional probabilities. The mathematical equations are as follows:

$$P(\overline{M}_j/\overline{M}_i) = F(\overline{M}_i, \overline{M}_j)/F(\overline{M}_i)$$

$$P(M_j/\overline{M}_i) = F(\overline{M}_i, M_j)/F(\overline{M}_i)$$

$$P(\overline{M}_j/M_i) = F(M_i, \overline{M}_j)/F(M_i)$$

$$P(M_j/M_i) = F(M_i, M_j)/F(M_i)$$

where $F(\overline{M}_i)$ equals $F(\overline{M}_i, \overline{M}_j) + F(\overline{M}_i, M_j)$ and $F(M_i)$ equals $F(M_i, \overline{M}_j) + F(M_i, M_j)$.

If probability of mastery, $P(M)$, and probability of non-mastery, $P(\overline{M})$, are assumed to be either zero or one, an equivalent set of equations can be expressed for a group of N students:

$$P(\overline{M}_j/\overline{M}_i) = \frac{\sum^{N} P(\overline{M}_j)\, P(\overline{M}_i)}{\sum^{N} P(\overline{M}_i)}$$

$$P(M_j/\overline{M}_i) = \frac{\sum^{N} P(M_j)\, P(\overline{M}_i)}{\sum^{N} P(M_j)\, P(\overline{M}_i)}$$

$$P(\overline{M}_j/M_i) = \frac{\frac{\sum^N P(\overline{M}_j)\, P(M_i)}{\sum^N P(M_i)}}{\frac{\sum^N P(M_j)\, P(M_i)}{\sum^N P(M_i)}}$$

If it is not assumed that $P(M)$ and $P(\overline{M})$ are either zero or one, the same equations can be used to compute transition matrices that may be more reliable for predicting future state vectors.

The following three examples illustrate how conditional probabilities might be computed for three students whose observed scores on units one and two were (4, 5), (2, 3), and (5, 3), respectively:

Example 1

Assume that an observed score of 4 or 5 indicates mastery and that an observed score less than 4 indicates non-mastery. Therefore, the required frequency counts are as follows:

$F(M_1) = 2$

$F(\overline{M}_1) = 1$

$F(\overline{M}_1, \overline{M}_2) = 1$

$F(\overline{M}_1, M_2) = 0$

$F(M_1, \overline{M}_2) = 1$

$F(M_1, M_2) = 1$

The conditional probabilities are indicated in the following transition matrix:

$$T_{12} = \begin{bmatrix} 1 & .5 \\ 0 & .5 \end{bmatrix} = \begin{bmatrix} P(\overline{M}_2/\overline{M}_1) & P(\overline{M}_2/M_1) \\ P(M_2/\overline{M}_1) & P(M_2/M_1) \end{bmatrix}$$

Example 2

Assume that the probability of mastery is one for observed scores of 4 or 5 and that the probability of mastery is zero for observed scores less than 4. The probabilities of mastery and non-mastery are, therefore, as follows for the three students:

Student #	$P(M_1)$	$P(\overline{M}_1)$	$P(M_2)$	$P(\overline{M}_2)$
1	1	0	1	0
2	0	1	0	1
3	1	0	0	1

The following sums are needed to compute conditional probabilities:

$$\sum^{3} P(\overline{M}_2)\ P(\overline{M}_1) = 1$$

$$\sum^{3} P(M_2)\ P(\overline{M}_1) = 0$$

$$\sum^{3} P(\overline{M}_2)\ P(M_1) = 1$$

$$\sum^{3} P(M_2)\ P(M_2) = 1$$

$$\sum^{3} P(M_1) = 2$$

$$\sum^{3} P(\overline{M}_1) = 1$$

Applying the appropriate equations results in the following conditional probabilities:

$P(\overline{M}_2/\overline{M}_1) = 1/1 = 1$

$P(M_2/\overline{M}_1) = 0/1 = 0$

$P(\overline{M}_2/M_1) = 1/2 = .5$

$P(M_2/M_1) = 1/2 = .5$

These values are the same as those computed previously by the frequency method.

Example 3

Assume that values for the probability of mastery can be computed for any observed score from 0 to 5. (Appendix A describes a procedure for computing P(M/X), where X is an observed score from 0 to 5.) For this example, the following values will be used:

Student #	$P(M_1)$	$P(\overline{M}_1)$	$P(M_2)$	$P(\overline{M}_2)$
1	0.80	0.20	1.00	0.00
2	0.10	0.90	0.30	0.70
3	0.90	0.10	0.20	0.80

The following sums are obtained for the probabilities shown:

$$\sum^{3} P(\overline{M}_2)\ P(\overline{M}_1) = 0.71$$

$$\sum^{3} P(M_2)\ P(\overline{M}_1) = 0.49$$

$$\sum^{3} P(\overline{M}_2)\ P(M_1) = 0.79$$

$$\sum^{3} P(M_2)\ P(M_1) = 1.01$$

$$\sum^{3} P(M_1) = 1.80$$

$$\sum^{3} P(\overline{M}_1) = 1.20$$

These sums result in the following conditional probabilities:

$$P(\overline{M}_2/\overline{M}_1) = 0.71/1.20 = 0.59$$

$$P(M_2/\overline{M}_1) = 0.49/1.20 = 0.41$$

$$P(\overline{M}_2/M_1) = 0.79/1.80 = 0.44$$

$$P(M_2/M_1) = 1.01/1.80 = 0.56$$

It should be noted that the probabilities computed in this example differ from the values obtained in the pfevious two examples.

The Markov Prediction Model

The Markov prediction model enables future state vectors for a student to be predicted using a single current state vector and the appropriate transition matrix. Transition probabilities are based on test scores obtained for a sample of students who have previously completed the necessary units of instruction. Mathematically, the prediction model is represented by the following matrix equation:

$$V_{fj} = T_{ij} \cdot V_{ci}$$

where V_{fj} is the future state vector for unit j, V_{ci} is the current state vector for unit i, and T_{ij} is the transition matrix from unit i to unit j.

Normally, transition matrices will exist only for those cases where j equals i + 1, i.e., T_{12}, T_{23}, T_{34}, and so on. To obtain the matrix T_{ik}, where k is greater than one, requires a series of matrix multiplication operations. For example, T_{14} is equal to $(T_{12} \cdot T_{23}) \cdot T_{34}$. To illustrate, suppose T_{12}, T_{23}, and T_{34} are represented by the following matrices:

$$T_{12} = \begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} .5 & .2 \\ .5 & .8 \end{bmatrix}$$

$$T_{34} = \begin{bmatrix} .3 & .4 \\ .7 & .6 \end{bmatrix}$$

The matrix T_{13} is obtained by multiplying T_{12} and T_{23} as follows:

$$T_{13} = \begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix} \begin{bmatrix} .5 & .2 \\ .5 & .8 \end{bmatrix} = \begin{bmatrix} .35 & .32 \\ .65 & .68 \end{bmatrix}$$

Now, multiplying T_{13} and T_{34} results in the matrix T_{14}:

$$T_{14} = \begin{bmatrix} .35 & .32 \\ .65 & .68 \end{bmatrix} \begin{bmatrix} .3 & .4 \\ .7 & .6 \end{bmatrix} = \begin{bmatrix} .329 & .332 \\ .671 & .668 \end{bmatrix}$$

In order to illustrate how the Markov model can be used to predict future performance on a criterion test, the future state vectors will be computed using the matrices, T_{12}, T_{13}, and T_{14} for a student whose probability of mastery on the first unit is 0.8. The calculations are as follows:

$$V_{f2} = \begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix} \cdot \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .32 \\ .68 \end{bmatrix}$$

$$V_{f3} = \begin{bmatrix} .35 & .32 \\ .65 & .68 \end{bmatrix} \cdot \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .326 \\ .674 \end{bmatrix}$$

$$V_{f4} = \begin{bmatrix} .329 & .332 \\ .671 & .668 \end{bmatrix} \cdot \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .3314 \\ .6686 \end{bmatrix}$$

The future state vectors indicate that the probability of mastery is 0.68 for unit 2, 0.674 for unit 3, and 0.6686 for unit 4. It should be noted that the probability of mastery for unit 2 will vary between 0.6 and 0.7 no matter what the student's probability of mastery was for unit 1.

Applications of the Markov Model

One of the most useful applications of the Markov model is in predicting the effects of various instructional sequences on subsequent performance. Suppose, for example, that transition matrices have been computed which are based on two different methods of prescribing second instruction, P_1 and P_2. Since one prescription method may not be the most effective method for all units of instruction, the problem is to find a sequence that results in the highest probability of mastery based on a posttest for the instructional program. There are 2^{10} (or 1024) possible prescription sequences that would have to be tried in order to determine the optimum sequence. One approach to the problem would be to use the Markov model to predict posttest performance for all possible prescription sequences using a computer and then select those sequences that result in the highest probability of mastery; the selected sequences can then be evaluated with an experimental design.

A second application pertains to the selection of an optimal decision strategy. In business, the following question arises: Should decisions be made to maximize current profits or to maximize future profits? In education, a similar question arises: Should instructional prescriptions be selected to maximize current performance relative to unit objectives or to maximize future performance relative to end-of-program objectives?

A markov model enables prescriptions to be based on a strategy which maximizes future predicted performance. The method may be evaluated by comparing it with a strategy that maximizes current performance. Evaluation of the two prescription methods would likely involve a direct comparison of actual posttest performance.

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