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ABSTRACT

This paper reports further on the study described in SE 019 717. Correlational data relating success on proportionality tasks to understanding of related concepts and processes are presented. The five correlative tests involve (1) reducing fractions, (2) multiplying fractions, (3) facility with fractions, (4) inverse relations, and (5) more than/times as much. The results of the study suggest that a child's understanding of proportionality is dependent on the task being performed. It is suggested that in teaching proportionality concepts teachers use the simplest possible arithmetic. (SD)

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Adolescent Understanding of Proportionality: Skills Necessary for Its Understanding

by

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SE 019 719

Abstract

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Six proportionality problems involving prediction of the height of stick figures using ratios were administered to 32 seventh grade students. The effects of subject ability and task characteristics of equal or unequal, size of the unknown number and type of ratio were investigated. Ss performance was rated on a development scale. Results showed significant effects for type of ratio, size of unkpown, and ability. An exploratory study of correlative skills necessary for an understanding of proportionality was also undertaken. Implications of the findings for developmental theory, further research, and the teaching of proportions were discussed.

Adolescent Understanding of Proportionality

Introduction

TEACHER. "How do you solve the equation $2x = y$?

STUDENT. "Subtract 2 from both sides of the equation."

Why doesn't the child realize that subtracting is an inappropriate way to solve for x in the equation $2x = y$? Is it because he/she has not yet developed to a prerequisite cognitive stage? Is the problem one of application, i.e., does the child understand the necessary operation but not know when to use it? Could the child be taught the division process and application, as Bruner (1966) advocates?

An understanding of proportionality is necessary to the solution of the equation $2x = y$ and to similar equations. Thus all these questions are relevant to an understanding of how the concept of proportionality develops in students.

Cognitive psychologists believe that most children are not capable of handling metric proportion until the beginning of junior high school onwards (Lovell, 1971) The question of whether children can be taught to use the concept before that time has never been answered. A developmental acquisition of the concept of proportionality may be a necessary prerequisite to learning its effective use. If this is so, then the current practice of teaching fractions in the upper elementary grades may be wasteful, since this teaching is well in advance of the age at which development of the concept of proportionality theoretically occurs.

This problem has other implications for educational practice. The relationship between equivalent fractions is used continuously throughout high school in mathematics and science classes. Proportional relations may also be referred to in secondary courses in the social sciences where data and trends are analyzed. Yet many children are frustrated when certain concepts are developed in a way that tacitly assumes command of proportional reasoning. Instruction based on this assumption is most prevalent in mathematics and science courses. Pressure, density, intensity, flux, chemical composition, and other concepts involve ratios; proportional thinking is required for work with algebraic equations.

Information about how adolescents solve problems involving proportionality would have implications for how various concepts expressed mathematically could be taught. If the presence of a developmental sequence is confirmed, lessons involving proportionality could then be taught when adolescents are cognitively ready to learn them. Furthermore, there may be certain teaching conditions that would benefit students at one cognitive level and not another. Lastly if skills underlying the understanding of proportionality are isolated, teachers would be able to direct their attention to teaching these skills to remedy student difficulty with proportionality. The concept of proportionality is also important in areas apart from mathematics and science. The ability to manipulate proportions has been identified with Piaget's (1958) stage of formal operational reasoning. A study of the development of proportional reasoning therefore has theoretical implications. This means that an understanding of proportionality can be used to diagnose how well children reason abstractly. In one study (Kuhn et al., 1971), it was reported that 45 percent of those between the ages of 10 and 15 and 53 percent of those aged 16 to 20 are capable of reasoning on a formal operational level. Although all children probably become concrete operational reasoners in our schools, probably only half develop the ability to reason abstractly by the time they finish high school. If formal operational thinking develops during adolescence, it is necessary to investigate factors that inhibit or facilitate it acquisition.

An understanding of proportionality has been identified with Piaget's stage of formal operational reasoning, which theoretically emerges at the ages of 12 to 13 years (Inhelder & Piaget, 1958). Piaget investigated the child's acquisition of proportionality by examining children's reactions to situations such as equilibrium on a balance and shadow size. He found that younger children (seven to twelve years) dealt with these problems by using arithmetic solutions, whereas adolescents (13 to 17 years old) demonstrated understanding of proportional increase and decrease and reciprocity between various relations. Piaget also found that children demonstrated an intuitive understanding of proportionality before they could deal with it quantitatively.

Studies investigating the developmental acquisition of proportionality have compared children's performance across several tasks to determine under what circumstances and at what age level an understanding of proportionality becomes operational. Several investigators have found that children under the age of fourteen do not have a well-developed understanding of proportionality. There is general agreement however, that formal operational thought is prerequisite to the solution of problems involving proportional relationships (Lunzer & Pumphrey, 1966; Lovell & Butterfield, 1966). There is also considerable experimental evidence that children employ an additive strategy prior to the onset of proportional thought regardless of the materials used. (Lunzer & Pumphrey, 1966; Karplus & Peterson, 1970; Karplus & Karplus, 1972; Wollmen, et al., 1973)

It can be hypothesized that the concept of proportionality is a structured whole made up of independent, internalized actions which must be integrated in order for the concept to become fully operational. Using Piaget's results, possible candidates for these internalized actions are:

1. recognizing when an additive strategy is not suitable.
2. grasping an intuitive understand of proportionality.
3. expressing the proportional relations numerically.

It appears that adolescents progress from the use of an additive strategy to one which combines components of addition with components of proportion. At some point, the use of this combined strategy gives way to an understanding of proportionality with numerical facility. As of yet an intuitive and/or logical understanding of the concept without numerical facility has not been identified.

Investigations to date have not explored the child's ability to handle very complex proportions. It is possible that with more complex proportions an intuitive understanding of the problem without mathematical facility would appear.

According to Piagetian theory, the structured whole corresponding to the concept of proportionality consists of separate schema that have been successfully integrated. The ability to recognize situations in which a proportional strategy rather than an additive strategy is required may be one of these separate schema. Other skills might also be necessary for the development of an operational understanding of proportionality.

If such other skills or schema existed, then their integration into the structured whole would allow the subject to solve proportion tasks successfully. Subjects who could solve proportion problems successfully would be expected to possess these skills. Subjects who were missing any one of these skills would be expected to have some difficulty with proportion tasks.

Informal observations and discussions with children as they tried to solve proportion problems indicate that those with no understanding of proportionality seem to lack several basic understandings. One such understanding is the ability to make a distinction between "bigger than" and "times as much." Subjects who use an additive strategy appear to be usually unsuccessful in making such a distinction. They may know that 6 is 2 more than 4, but they do not realize that 6 is also 1 $^{1}/2$ times as big.

A second understanding that may be necessary for the successful solution of proportion problems is the ability to understand inverse relations between unit size and the number of units used in a measuring task. Children who cannot solve proportion problems may not realize that it will take fewer large units than small units to measure a distance. If these skills represent independent structures that must be integrated prior to the attainment of the proportionality concept, it could be predicted that there might be a relation between the strategies that children use in solving proportion problems and performance on tests of these skills. At the very least, all proportional thinkers could be expected to perform successfully on such tasks.

Objectives of the Study

Questions raised by a review of the literature provided a focus for the present investigation. The first was: how generalizable is performance on one proportion task to performance on another? If the attributes of a task are systematically altered, subject performance across the variations would indicate how consistent subject performance is as well as what task variations cause difficulty. If subject performance on variations of a common task were consistent, the hypothesis that performance on one task is generalizable to performance on a slightly different task would be supported. On the other hand, if the strategies subjects use were inconsistent and apparently dependent on the material's content, then one would begin to have an understanding of limitations of children's understanding of proportionality.

The second question concerns the identification of correlates of maturity and subject performance on proportion tasks. Specific and independent skills thought necessary to an understanding of proportionality have been described above. Tests of these skills were constructed. The assumption was made that an understanding or skill may be necessary for proportional reasoning if it is related to performance on the criterion task of proportionality.

Method

Subjects

The subjects were 32 seventh-grade students from a San Francisco Bay area school. All were twelve to thirteen year olds from white, middle class backgrounds. Teacher assessments were used to classify subjects as high or low ability students.

Criterion Task of Proportionality

Robert Karplus has devised a test to determine the level of abstract reasoning children use in a ratio and proportion task (Karplus & Peterson, 1970). In the Karplus task children were presented with a drawing of a large stick figure (Mr. Tall), the height of which was measured with large paper clips (biggies). A drawing of a smaller figure (Mr. Short), was then presented and measured. Subjects were asked to measure Mr. Short with small paper clips (smallies), to predict the height of Mr. Tall in smallies, and to write an explanation of how they arrive at their prediction.

Subsequently Karplus modified his task to prevent children from relying on perceptual cues. In the altered task, the Ss were asked to predict the height of Mr. Tall, without seeing the figure. The Ss were supplied with the same information as in the initial task (i.e., Mr. Tall's height in biggies). The important difference was that they were unable to rely on any perceptual comparisons between the two figures to help them solve the problem because they saw only one figure (Mr. Short).

In this study, proportionality problems were organized into test booklets. Each subject received a test booklet containing six tasks. Each task was presented on two pages. The first page showed a stick figure (Mr. X) measured by two sets of different colored loops. The subject was asked how many loops of each color it took to measure Mr. X. The second page showed a different sized stick figure (Mr. Y), measured by only one set of colored loops. The subject was asked how many loops of this color it took to measure Mr. Y, to guess how many loops of the other color it would take to measure Mr. Y, and to explain how he/shearrived at that answer.

Stimulus characteristics of the tasks included Difference (Equal/ Unequal), Size (Larger/Smaller), and Type (Simple/Complex/Complex Multiple).

The presence or absence of a repeated difference between the measurements was designated as Difference (Equal/Unequal). Values for the problems were chosen so that there was sometimes a repeated difference ($^{4}/6 = {}^{6}/x$), and sometimes not ($^{4}/6 = {}^{10}/x$). The extent to which the numbers used influenced subjects to use a differencing strategy (subtracting the numerator from the denominator or one numerator from the other) could then be assessed.

The second stimulus characteristic investigated was Size (Larger/ Smaller). In some of the problems the unknown number was larger than the numbers already known, in other it was smaller.

The third stimulus characteristics was Type (Simple/Complex/ Multiple Complex). Three possible relationships were used: a) small whole numbers involving factors of 2, 3, etc with the unknown always an integer; b) complex multiples involving factors of 1 $^1/2$, 1 $^1/4$, $^2/3$, etc. with the unknown always an integer; and c) complex ratios involving more complex factors with the known always a mixed number. It was expected that most Ss would be able to solve the small whole numbers successfully. The complex multiples were expected to present greater difficulty, and the complex ratios the most.

A fourth stimulus characteristic was labeled materials. Two sets of proportion problems were constructed. Each set included the twelve possibilities in crossing three levels of Type with two levels of Size and two levels of Difference. The sets were designed to provide information about generalizability across particular numbers and were designated as Form (A/B). Task variations and examples are summarized in Table 1.

Insert Table 1 about here

Fractional factorial designs were used to designate what stimulus characteristics and their order were to each subject.

The first two proportionality problems in each test booklet involved variation in only three stimulus characteristics, since just one type of ratio was involved. Since there were three factors of two levels each, it was possible to generate eight basic tasks.

Each of the remaining four proportionality tasks in the booklet was described by four factor values, one from each of the following pairs of task characteristics: Size (Smaller/Larger), Difference (Equal/Unequal), Type (Complex/Multiple Complex), Form (A/B). Since

TABLE 1

EXAMPLES OF TASK VARIATIONS

	Form A				Form B			
	Small Number		Large Number		Small Number		Large Number	
	Equal Difference	Unequal Difference	Equal Difference	Unequal Difference	Equal Difference	Unequal Difference	Equal Difference	Unequal Difference
Simple Whole Multiple	$1/2 = 2/x$	$1/2 = 3/x$	$4/2 = 2/x$	$6/3 = 2/x$	$1/3 = 3/x$	$1/3 = 4/x$	$9/3 = 3/x$	$6/2 = 3/x$
Complex Multiple	$4/6 = 6/x$	$4/6 = 14/x$	$9/6 = 6/x$	$12/9 = 8/x$	$4/10 = 10/x$	$4/10 = 14/x$	$25/10 = 10/x$	$25/10 = 15/x$
Complex Ratio	$5/7 = 7/x$	$5/7 = 8/x$	$7/5 = 5/x$	$7/5 = 4/x$	$3/5 = 5/x$	$3/5 = 7/x$	$9/5 = 5/x$	$9/5 = 4/x$

there were four factors of two levels each, 16 basic tasks were generated. The proportions of the small whole multiple type were omitted from these problems because they were considered relatively easy for the subjects to solve and therefore relatively insensitive to experimental manipulation.

Correlative Skills

Subjects can have difficulty with proportionality problems for at least two reasons. They may lack manipulative facility with fractions and/or conceptual understanding of proportionality. To assess how well the subjects in the study handle fractions and concepts related to proportionality, correlative tests investigating the following were administered: reducing fractions, multiplying fractions, overall facility with fractions, concept of more than/times as much", and concept of "inverse relations."

Procedure

All subjects were presented with the test booklets in one group session. Each subject was asked to read the directions describing the proportionality task in the test booklet while the test administrator read them aloud. Questions were solicited. Before they began, Ss were reminded that the object of the problem set was to determine how they went about solving problems of this nature rather than whether they got a right or wrong answer. The Ss were also reminded not to look back at any figure once they answered questions about it, unless they were directed to do so. They were also told to answer all questions to the best of their ability. Tests of correlative skills were administered in a second testing sessions on the day following the proportionality test session. Each of the test sessions was approximately 45 minutes long.

Results

Subject responses to each of six proportionality problems were scored on a ten-point scale. Each point on the scale was designed to reflect a different strategy that could be used to solve the proportion problems. The scale consisted of the following categories:

1. N - No explanation

2. I - An explanation referring to estimates without reference to the data.

3. IC - An explanation using the data haphazardly.

4. S (scaling) - An explanation based on a change of scale that the subject does not justify in terms of the data.

5. A (addition) - An explanation focussing on a single difference, and solving the problem by addition.

6. AS (addition and scaling) - An explanation in which the difference between measurements is first isolated and then related by multiplication to one of the measurements.

7. IP (incomplete proportion) - An explanation making use of one ratio involved in the proportion, but not applying the ratio correctly.

8. PC (proportion concrete) - An explanation using the correct ratio of measurements but applying it by actually measuring off the ratio on the figure given.

9. AP (addition and proportion) - An explanation using the correct ratio but applying it by addition.

10. R (ratio) - An explanation using a proportion or deriving the scale ratio from the data, and applying the ratio in a proportion.

This scale is similar to the one used by Karplus. He has reported his results both in terms of a ten point category scale and in terms of a three point collapsed scale. He stated that the levels of the latter are indicative of preconcrete, concrete, and formal operational thought. This study followed a similar procedure; a nine-point category scale was used as well as a three point category scale. However, the levels of the latter are given a somewhat different meaning than that employed by Karplus. Scores of 0 to 2 were considered indicative that the subjects had no idea how to solve the proportion problems. Scores of 3 to 5 were taken as indicative that the subjects focused on a pattern independent of of the ratio of the numbers. Scores of 6 through 9 were thought reflective of subjects who used a ratio to solve the proportion problems. These three categories were designated as non-patterned, patterned (inappropriate) and proportional, respectively.

Two scorers used the scale to score responses independently. Also all responses were coded as correct (3), almost correct (2), or incorrect (1). The correlation of the two sets of strategy scale scores was .80, indicating good inter-rater reliability.

Task Characteristics

The first two booklet problems contained task characteristics of Size (Large/Small), Difference (Unequal/Equal), and Form (A/B). Only ratios involving simple whole number were used in these proportionality problems. The one between subjects' variable investigated was Ability (High/Low). Contrast scores were derived by subtracting performance on one level of a factor from performance on the second level of the factor. A univariate analyses of variance (ANOVA) on contrast scores was used to analyze the data. None of the characteristics varied in these problems appeared to affect subject performance significantly (Table 4). The means of subject scores on the two tasks (Table 5) indicate, however, that subjects did solve problems involving simple fractions with nearly proportional strategies.

Insert Tables 4 and 5 about here

TABLE 4

SUMMARY OF AN ANALYSIS OF VARIANCE FOR THE TASK FACTORS GENERATED FROM SUBJECTS SCORES ON TWO SESSION IA PROBLEMS

Source	d.f.	Mean Square	F
Size	1	0.0	0.0
Difference	1	12.50	3.39
Form	1	1.13	.30
Size x Difference	1	15.12	4.10
Error Term	16	3.69	
Ability	1	128.00	3.61
Error Term	16	35.40	

TABLE 5

MEANS AND STANDARD DEVIATIONS FOR PROBLEMS IN SESSION IA

Factor	Level	Mean Correct	S.D.
Ability	High	7.69	2.78
	Low	5.44	3.64
Size	Large	6.56	3.38
	Small	6.56	3.48
Difference	Unequal	6.69	3.28
	Equal	6.44	3.57
Form	B	6.09	3.65
	A	7.03	3.13

The other four booklet problems varied task characteristics of Type (Complex/Multiple Complex), Difference (Unequal/Equal), Size (Large/Small), and Form (A/B). A univariate ANOVA on contrast scores was again used to analyze the data.

The mean values in Table 6 indicate that for this group of seventh grade subjects responses on four tasks were primarily patterned (inappropriate). The average range of responses was at the upper end of the patterned (inappropriate) category for the easy level of a factor and at the lower end of the patterned (inappropriate) category for the harder level of a factor.

Insert Table 6 about here

This use of patterned (inappropriate) responses was quite logical from the subject's point of view. Subjects perceived a pattern which exists among the number involved and applied it. Examples of such patterned responses are:

1. Given: Mr. Al is 1 red and 3 blues; Mr. Bob is 4 reds. How many blues does it take to measure Mr. Bob?

   Answer: Seven

   Subject Response: On the first page it took 1 red to measure Mr. Al and 3 blues. Then the second time it took 4 reds, so I figured they just added the reds to tiny chains, so they added 1 + 3 which gives 4 red chains. Then I thought since it takes 3 blues and 4 reds, why not add them and the measure for blues.

2. Given: Mr. Ron is 9 reds and 5 greens; Mr. Sam is 4 reds. How many greens does it take to measure Mr. Sam?

   Answer: 0

TABLE 6

MEANS AND STANDARD DEVIATIONS FOR PROBLEMS IN SESSION IB

Factor	Level	Mean Correct	S.D.
Ability	High	4.79	2.54
	Low	3.35	2.04
Ratio	Complex ratio	3.72	1.99
	Complex multiple	4.72	2.63
Size	Small	3.72	2.54
	Large	4.44	2.24
Difference	Unequal	3.93	2.18
	Equal	4.21	2.63
Form	B	3.83	2.36
	A	4.33	2.46

Subject Response: It took 9 reds to measure Mr. Ron. It also took 5 greens which is 4 less. And it took 4 reds to measure Mr. Sam, so I thought you'd subtract 4 from 4 which would give zero.

3. Given: Mr. Lou is 3 blacks and 5 blues; Mr. Moe is 5 blacks. How many blues does it take to measure Mr. Moe?

Answer: 3

Subject Response: Just the opposite.

Contrary to expectation, Difference was not a significant effect. The Form factor also failed to affect subject performance differentially. The only significant task characteristic effects were Type and Size. (Table 7)

Insert Table 7 about here

The Ss were tested on their ability to solve three types of proportion problems: simple multiple ($^1/2 = ^2/x$), complex multiple ($^4/6 = ^6/x$), and complex ratio ($^5/7 = ^7/x$). Ss used more sophisticated strategies to solve the simple multiples than they used to solve problems of the other two ratio types. Also a significant difference in performance between the other two types appeared favoring the complex multiples. Table 8 shows the distribution of responses on the category scale for the type factor. Although subject responses were almost equally distributed between the patterned and proportional categories for problems of the complex multiple type, this distribution was skewed to the left for problems of the complex ratio type. Only four subjects used a proportional strategy on both levels of the ratio factor. The other eight subjects who

TABLE 7

SUMMARY OF AN ANALYSIS OF VARIANCE FOR THE TASK FACTORS GENERATED FROM SUBJECT SCORES ON FOUR PROPORTIONALITY PROBLEMS

Source	df	MS	F
Size	1	72.00	8.00*
Ratio	1	72.00	22.22**
Order	1	8.00	.88
Error	16	9.00	
Difference	1	8.00	.55
Error	16	14.63	
Size by Difference	1	4.50	.40
Ratio by Difference	1	4.50	.40
Error	16	11.38	
Ability	1	210.00	4.80*
Ability x Ratio	1	28.12	3.12
Ability x Size	1	45.12	5.01*
Error	16	9.00	
Ability x Difference	1	.12	.06
Error	16	14.63	

*$p < .05$

**$p < .01$

used a proportional strategy to solve complex multiples used either a patterned (inappropriate) or a non-patterned strategy to solve complex ratios.

Insert Table 8 about here

Subjects performed better with proportions in which the unknown was larger than the known numbers than they did with proportions in which the unknown was smaller than the known numbers. Table 9 indicates that half the responses were of a patterned type on proportions involving larger answers, with approximately a quarter of the responses of the non-patterned type and the remainder proportional. This distribution changed when the unknown was smaller than the known numbers. Although the decrease in the proportional category was not great, a considerable number of the subjects who used a patterned strategy reverted to a non-patterned one and were totally unable to solve the problem.

Insert Table 9 about here

The reversion was most noticeable for the low ability subjects as evidenced by a significant ability by size interaction (Figure 1). The performance of the high ability subjects reverted an average of only one category response when they were faced with proportions whose unknown was smaller. Performance of the low ability subjects dropped substantially in this condition; they gave primarily unpatterned responses.

Insert Figure 1 about here

TABLE 8

DISTRIBUTION OF RESPONSES ON THE THREE CATEGORY SCALE FOR THE RATIO FACTOR

Item Type	Categories		
	Non-patterned	Patterned	Proportional
complex multiple	13	27	24
complex ratio	23	33	8

TABLE 9

DISTRIBUTION OF RESPONSES ON THE THREE CATEGORY SCALE FOR THE SIZE FACTOR

Item Type	Categories		
	Non-patterned	Patterned	Proportional
Answer Larger	12	34	18
Answer Smaller	24	26	14

The only between subjects variable which was significant was Ability. Subjects designated as high ability students by their teacher performed at the upper end of the patterned (inappropriate) category, whereas lower ability students performed at the lower end of the same category (Table 10). Although there were an almost equal number of responses in the patterned (inappropriate) category for high and low ability subjects, more proportional strategies were used by high ability subjects and more non-patterned strategies were used by low ability subjects.

Insert Table 10 about here

Correlative Skills

Subjects were given the following five skills tests: reducing fractions; multiplying by fractions; an overall test of facility with fractions; a test investigating the concept of the inverse relation between a measuring unit and the number of units needed to measure; and a test investigating the concept of more than and times as much. The means for the five tests are listed in Table 11. Each test contained nine problems. Each problem was scored as correct (3), almost correct (2), incorrect (1), or nor answer (0). Subject performance was significantly better on the test of reducing fractions as compared to the other four tests ($F = 45.88$, p .01) and on More than/Times as much compared to the test of inverse relations ($F = 7.19$, p .05).

TABLE 10

MEANS AND DISTRIBUTION OF HIGH AND LOW ABILITY SUBJECTS' RESPONSES ON THE THREE CATEGORY SCALE

	Mean	Categories		
Ability		Non-patterned	Patterned-Inappropriate	Proportional
High	4.80	12	32	20
Low	3.36	24	28	12

One of the aims of this investigation was to determine whether any of the skills implicit in each test had a bearing on the subject's ability to solve proportion problems. This part of the investigation was primarily exploratory and any connections between performance on the proportionality problems and underlying skills would be tentative and tenuous for several reasons. Although on the face of it, each test may seem to be asking different questions, in actuality there is a high degree of interrelation between performance on any two tests. Reducing Fractions and Multiplying Fraction showed a correlation of .54. The Multiplying Fractions and Facility with Fractions correlated .60. Facility with Fractions and More than/Times as Much tests correlated .35.

When partial correlations were calculated, these zero order correlations were subject to change. For example, the correlation of Facility with Fractions with More than/Times as Much rose to .45 when performance on the Inverse Relations test was controlled for. This gain was due to the fact that the Inverse Relations test correlated negatively with Facility with Fractions and very slightly with performance on the test of More than/Times as Much. In essence, this means that there was a stronger relationship between the concept More than/Times as Much and overall Mathematical Facility when the concept of the Inverse Relation of the measuring units was controlled.

The problem of multicolinearity of these measures was compounded when their relationship with proportionality as measured with the three category scale (PAVER), the size contrast (LS) and the Ratio contrast (CRCM) was investigated. The average performance of subjects across the four porportionality tasks was calculated. This average score was then classified as being either non-patterned (0-2.5), patterned (inappropriate) (2.6-5.5) or proportional (5.6-9.0) depending on its value. The size contrast was calculated by subtracting each subjects performance on larger answers from their performance on smaller answers. The same procedure was followed to calculate the value of the ratio contrast. Each subject's performance on complex multiples was subtracted from their performance on complex ratios.

In order to make sense of the full set of data a factor analysis using both a varimax and oblique rotation was carried out. The factors obtained from the varimax rotation were used because this analysis also yielded a graphical presentation of the variables. Both analyses, however, yielded the same results. The correlation matrix for the variables included in the factor analysis is found in Appendix E.

Table 22 shows the variables, their communality, and the values of the correlations of each variable with the rotated factors. The communality indicates the amount of variance that one variable shares with at least one other variable in the set. It is apparent that the amount of overlap is large especially for Multiplying Fractions, Facility with Fractions, the average Proportionality Score, and the size contrast.

Although there was overlap between the variables as far as variance accounted for is concerned, there was a nice separation of variables by factor. The tests that have to do with the handling of fractions all load on factor 1 which accounts for 53.7 percent of the variance; the average proportionality score and ability load on factor 2 which accounts for 26.6 percent of the variance; the size contrast, the ratio contrast and correlative test of Inverse Relations, load on factor 3 which accounts for 19.7 percent of the variance.

Insert Tables 11 and 12 about here

Each test had an equal number of questions designed to be easy, medium, and difficult. There was some internal consistency within each test as indicated by the correlations between subtests (Table 12).

These correlations indicate that subjects who could do the medium questions on each test had a higher probability of completing the difficult questions successfully.

Insert Table 13 about here

Table 11

MEANS AND STANDARD DEVIATIONS OF SUBJECT PERFORMANCE ON FIVE CORRELATIVE TESTS

Test	Mean Correct	S.D.
Reducing Fractions	24.88	3.66
Multiplying Fractions	21.62	5.77
Facility with Fractions	20.78	4.15
Inverse Relation	17.88	3.96
More than/Times as Much	21.97	3.30

Table 12

CORRELATIONS FOR SUBTESTS OF THE CORRELATIVE TESTS OF REDUCING FRACTIONS AND MULTIPLYING FRACTIONS

	Tests					
	Reducing Fractions			Multiplying Fractions		
subtest	easy	medium	difficult	easy	medium	difficult
easy		.19	.37		.65	.60
medium			.63			.81

Table 13

FACTOR ANALYSIS RESULTS FOR THE CORRELATIVE TESTS, AVERAGE PROPORTIONALITY SCORE, SIZE CONTRAST, RATIO CONTRAST, AND ABILITY

Variable	Communality	Factor 1	Factor 2	Factor 3
Reducing fraction	.48	.68	-.00	.15
Multiplying fractions	.74	.44	.09	-.17
Facility with fractions	.60	.71	.30	-.06
Inverse relations	.24	-.01	.00	.48
More than/Times as much	.36	.41	.36	.24
Average proportionality score	.84	.07	.90	.09
Ability	.31	.37	.42	.05
Ratio contrast	.41	-.05	-.22	.60
Size contrast	.60	.12	.28	.71

<u>Discussion</u>

According to Piaget, the ability to understand the concept of proportionality develops between the ages of twelve and thirteen. The results of subsequent research suggest that such concepts develop even later. In the sample of seventh-graders investigated here, only one-fourth of the problems were solved using a strategy that illustrates a well-developed understanding of the concept. And of that 25 percent, only two-thirds of the problems were solved correctly. These results are consistent with other work describing the concept of proportionality beginning to develop around the ages of thirteen and fourteen.

The seventh-grade subjects demonstrated little flexibility in their understanding of proportionality, as illustrated by the large effect the task characteristics of <u>Size</u> and <u>Type</u> had on performance. Eighteen percent of the items involving complex multiples were solved using proportional strategies, but only 6 percent of the complex ratios were solved using proportional strategies. Likewise, 14 percent of the proportions involving large answers were solved using proportional strategies, while only 9 percent of the proportions involving small answers were similarly solved.

One question raised by these results is whether the concept of "proportional" reasoning is indicative of abstract thought or merely a component of general ability. Although ability significantly affects subject performance, the effect of task characteristics occurred independent of ability for the <u>Type</u> characteristic. Thus even children judged as superior-performing students by their teacher had difficulty solving the more complex proportions. This was not the case, however, for the factor Size. Low-ability subjects who tended to be concrete operational thinkers responded to this factor by using preoperational strategies. Future research is needed to separate out the effect of general ability from the acquisition of developmental concepts.

It seems possible for Ss to have an intuitive understanding of proportionality without concurrently having the mathematical facility to solve proportion problems. The fact that Ss could solve the easier of the two levels of Size and Type problems indicates that they had some intuitive understanding of proportionality and some mathematical facility with problems of this sort. But there is also a limit either to this intuitive understanding or to their mathematical tools. Which limits which is not clear.

These results suggest that those investigating the developmental acquisition of proportionality must be careful not to generalize too quickly from performance on any one proportion task to the concept of proportionality in general. Subjects, especially those transitional between concrete and formal operational thinking (patterned inappropriate versus proportional strategies), may be quite capable of reasoning through proportions of moderate difficulty. However, when faced with a more demanding task, these same subjects might revert to the use of patterned concrete strategies.

A more valid use of these tasks for assessing the level of competence with proportionality would be to administer at least two proportions -- for example, a complex multiple and complex ratio. Subjects who solved both could be designated as having the concept in hand. Those who solved only the easier of the two could be considered as transitional with respect to an understanding of the concept. The use of such tasks as a diagnostic tool might be especially helpful to teacher of subject matter which requires an understanding of proportionality.

A third question involves components of proportional thought. The only variable to load on the same factor as the contrast values of the two significant main effects of size and ratio was the test involving the inverse relation of a measuring device to the units needed for measuring.

Figure 1 and 2 illustrate the frequency distribution of subject responses to the correlative test Inverse Relations and the average contrast scores due to the size and ratio factor. Subjects who received high scores on the Inverse Relations Test altered their strategy on the two different types of ratio less than subjects who earned a low score. Although this same pattern was repeated for subject performance on the size factor, subjects who earned high scores on the Inverse Relations Test also tended to use more sophisticated strategies when the unknown was smaller, rather than larger, than the known numbers.

These results suggest two possible interpretations. First, this factor may be solely indicative of strategy stability. Those subjects who understand inverse relationships seem to use a smaller range of strategies on the two levels of the factor.

Alternatively, this result suggests a possible conceptual understanding that may be required for solution of proportion problems.

The inverse relations skill test asked Ss to estimate how many units of either a larger or smaller measuring unit were needed to measure a certain length. Ss able to make such estimates also did equally as well or better with proportions in which the unknown was smaller than with proportions in which the unknown was larger. This implies that they knew the range and direction of an answer to a proportion problem.

A surprising result from the factor analysis is that skill tests of facility with fractions load on a different factor than tasks involving proportionality. These tests measure the amount of mechanical facility Ss have with fractions, i.e., reducing fractions, multiplying and dividing fractions. Proportionality tasks, however, demand a knowledge of this facility but also an understanding of how and when to use it in an appropriate situation. Such a result has educational implications. It suggests that drill alone may be insufficient in teaching proportionality. The teaching of fractions must be supplemented with tasks which help students conceptualize what they are doing with these numbers. This suggestion must be underlined. The Ss involved in the study had just finished a unit on fractions in which proportionality, equivalent fractions, and problem solving with fractions had been taught. Yet, by and large, their approach to solving these problems seemed particularly uninfluenced by the effect of their lessons -- much to the dismay of their teacher.

The results of this study indicate that a child's understanding of proportionality is dependent on the content of a proportionality task, especially for children transitional between concrete and abstract thought. Before the concept of proportionality can be used as a unitary indicator of formal operational thought, more research must be undertaken with children judged as formal operational reasoners to determine how variable their performance is both within and across different tasks. It may be that adolescents' understanding of proportionality depends to a great extent on the complexity of the physical relations inherent in a science or math task, much as performance on conservation tasks depends on whether the task involves the substance, weight, or volume of materials (Uzgiris, 1964). The results of such research may indicate that the use of proportionality occurs in a developmental sequence across a certain set of tasks. Such research has implications for when it would be best to teach various concepts requiring an understanding of proportionality.

Furthermore, these results suggest that when teachers are teaching concepts which require an understanding of proportionality, they have a dual task -- getting the concept across as well as teaching how proportionality is related to that concept. When introducing such a concept, teachers might facilitate their student's understanding by using number relationships the students can handle. An in depth treatment of such concepts might be more profitable when chidlren are older and have more facility with formal operational thought.

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List of Figure Captions

1. Frequency distribution of subject performance on the inverse relations test by average contrast scores on the ratio factor (complex ratio/complex multiple).
2. Frequency distribution of subject performance on the inverse relations test by average contrast scores on the size factor (small/large).

Figure 1

Average Contrast Scores

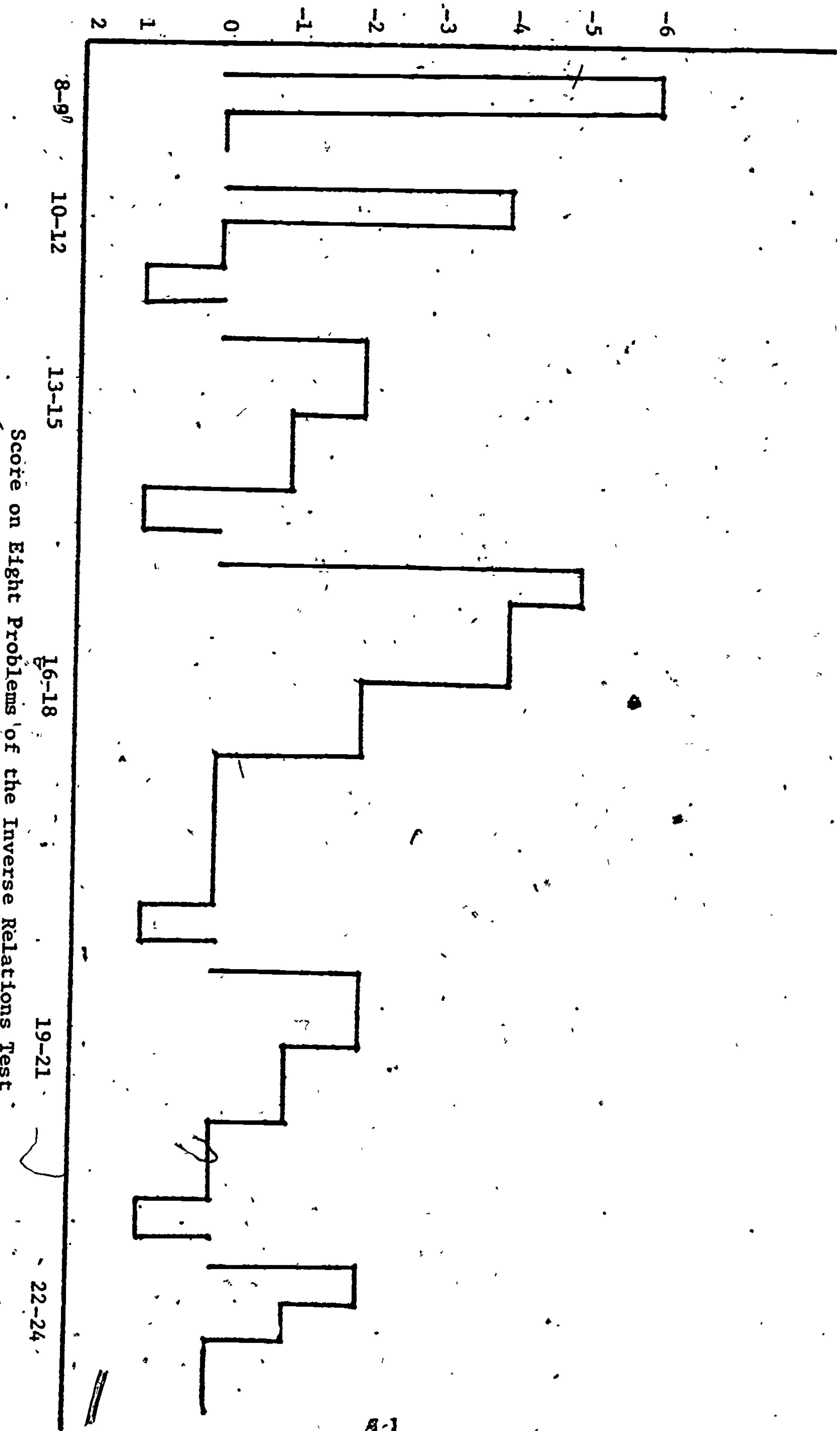
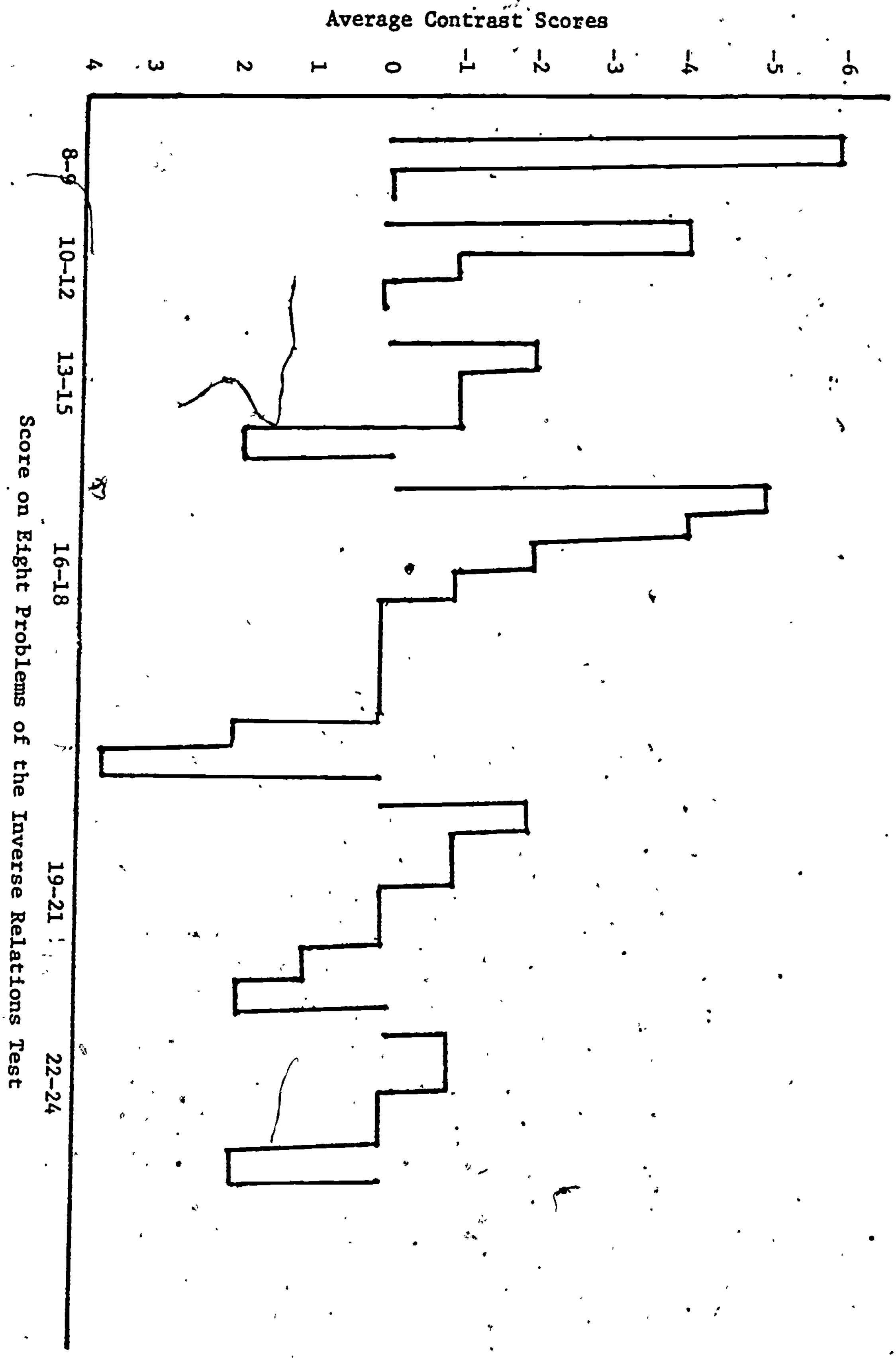


Figure 2


Average Contrast Scores
4
3
2
1
0
-1
-2
-3
-4
-5
-6
8-9
10-12
13-15
16-18
19-21
22-24
Score on Eight Problems of the Inverse Relations Test