DOCUMENT RESUME

ED 111 667 SE 019 642

AUTHOR Henle, James M.
TITLE Numerous Numerals.
INSTITUTION National Council of Teachers of Mathematics, Inc., Reston, Va.
PUB DATE 75
NOTE 49p.
AVAILABLE FROM National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia 22091 ($1.50, discounts on quantity orders)

EDRS PRICE MF-$0.76 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS Curriculum; *Learning Activities; Mathematical Concepts; *Mathematical Enrichment; Mathematics Education; *Number Concepts; *Number Systems; Problem Sets; Secondary Education; *Secondary School Mathematics

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This pamphlet consists of 17 brief chapters, each containing a discussion of a numeration system and a set of problems on the use of that system. The numeration systems used include Egyptian fractions, ordinary continued fractions and variants of that method, and systems using positive and negative bases. The book is informal and addressed to students. Answers to problems are included. (SD)

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NUMEROUS NUMERALS

JAMES M. HENLE

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

NCTM

Library of Congress Cataloging in Publication Data:

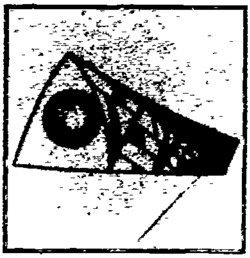
Henle, James M

Numerous numerals.

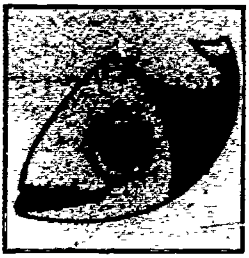
SUMMARY: Seventeen essays, with exercises, explaining both "new" numeration systems, such as fracimals, frictions, zerones, and negaheximals, and such familiar material as continued fractions and bases.

1. Numeration—Juvenile literature. [1. Number systems] I. National Council of Teachers of Mathematics. II. Title.
QA141.3.H46 513'.5 75-19347

Printed in the United States of America



CONTENTS



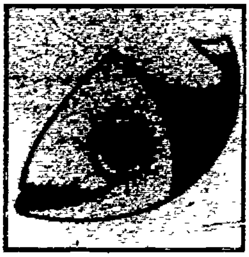
TO THE TEACHER

The primary reason for writing this booklet was to test a theory of education that I believe holds great promise. The theory calls for a mode of teaching that is unquestionably as exciting, challenging, unpredictable, and hazardous as life itself. When it is successful, both teacher and student may find that it has given them the most rewarding kind of mathematical experience they have ever had.

The technique is very simple: Teach something you do not know. Choose a topic foreign both to you and to your students. You will not "teach" the subject in the ordinary sense of the word; instead, you and your students will explore the subject together. You will lead when necessary, but frequently you will find your students leading—a situation they will enjoy immensely. The best topics to choose are those where the answers either are not known or cannot easily be found. When you do this, your teaching becomes nothing less than true mathematical research, and the questions you pose and the answers you discover are real achievements in the academic world.

The natural difficulty with this approach to teaching is finding suitable subjects. After all, the subject must be relatively unexplored (at least by you), it must be easily comprehensible to your students, and it would be nice if it had something to do with the rest of what you teach. A temporary solution to this problem (any solution to this problem must, by definition, be temporary) is this booklet. Each chapter is devoted to a different system of numeration. Nearly everyone will be familiar—perhaps too familiar—with the chapter on bases. Some will be familiar with a few others, but more than half these systems were written only for this booklet and have never, to my knowledge, been explored.

A system is explained in each chapter, with examples, exercises, and answers provided. From there, you are on your own. Occasionally I shall suggest a possible line of inquiry, but in general, you and your students should systematically explore the possibilities and limitations of the system, such as what numbers can be written in the system, whether one number can be written



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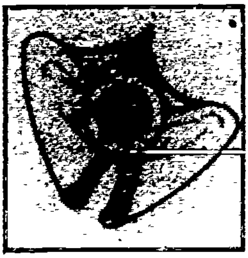
The material is aimed at the seventh-grade level and above, although bright sixth graders should have no difficulty with it, and certain chapters (5 through 9, 12, and 13) could even be given to fifth graders. The "Harder Stuff" at the end is for students who have completed half a year of algebra. You will find that the systems given here *are* related to the standard curriculum, working with these systems, a student cannot help but improve his facility and familiarity with numbers—integers and fractions, positive and negative.

The very nature of the teaching technique forbids me to give you the answers, or even the questions. (I do not know all of them myself.) Frequently, the greatest steps forward in mathematics are taken when someone asks the right questions.

Good luck!



1 EGYPTIAN FRACTIONS

The ancient Egyptians did not write fractions the way we do. They had no symbol for $\frac{2}{3}$ or $\frac{4}{5}$, but they did have symbols for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and so on. These fractions are called *unit* fractions because they have a 1 on top. The Egyptians used these fractions to make other fractions, like $\frac{2}{3}$ and $\frac{4}{5}$, for example, they might write $\frac{2}{3}$ as $\frac{1}{2} + \frac{1}{6}$, or they could write $\frac{4}{5}$ as $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$. This was the rule: Write the fraction as the sum of two or more *different* unit fractions. For example, they couldn't write $\frac{2}{3}$ as $\frac{1}{3} + \frac{1}{3}$; they had to use different fractions, like $\frac{1}{2} + \frac{1}{6}$. They couldn't write the fraction $\frac{2}{5}$ as $\frac{1}{5} + \frac{1}{5}$; they had to write it as $\frac{1}{3} + \frac{1}{15}$, or $\frac{1}{4} + \frac{1}{10} + \frac{1}{20}$, or $\frac{1}{5} + \frac{1}{6} + \frac{1}{30}$.

Write these as the Egyptians might have written them:

1. $\frac{3}{4}$	**3.** $\frac{3}{5}$	**5.** $\frac{5}{8}$
2. $\frac{5}{6}$	**4.** $\frac{3}{8}$	**6.** $\frac{7}{8}$

Although the Egyptians had a symbol for $\frac{1}{2}$, we can carry the rule one step further and say that *any* fraction must be expressed as at least two different unit fractions. For example, to express $\frac{1}{2}$, you have to write $\frac{1}{3} + \frac{1}{6}$; and you can't just write $\frac{1}{7}$—you have to write $\frac{1}{8} + \frac{1}{56}$.

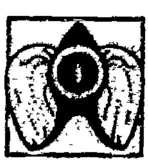
Write these as the Egyptians might have written them:

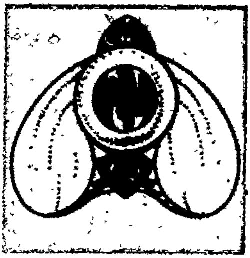
7. $\frac{1}{3}$ 10. $\frac{1}{6}$ 13. $\frac{4}{9}$ 16. $\frac{8}{9}$

8. $\frac{1}{4}$ 11. $\frac{1}{9}$ 14. $\frac{5}{9}$

9. $\frac{1}{5}$ 12. $\frac{2}{9}$ 15. $\frac{7}{9}$

Maybe you can find a system for writing these things.





2 FRACIMALS

Here are some examples of fracimals:

.3.2. .6.4.3. .2.2.3.7.

Here is what they mean:

$$.3.2. = \frac{1}{3} + \frac{1}{3 \times 2} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$.6.4.3. = \frac{1}{6} + \frac{1}{6 \times 4} + \frac{1}{6 \times 4 \times 3} = \frac{1}{6} + \frac{1}{24} + \frac{1}{72} = \frac{16}{72} = \frac{2}{9}$$

$$.2.2.3.7. = \frac{1}{2} + \frac{1}{2 \times 2} + \frac{1}{2 \times 2 \times 3} + \frac{1}{2 \times 2 \times 3 \times 7} = \frac{1}{2} + \frac{1}{4} + \frac{1}{12} + \frac{1}{84} = \frac{70}{84} = \frac{5}{6}$$

Figure out these fracimals:

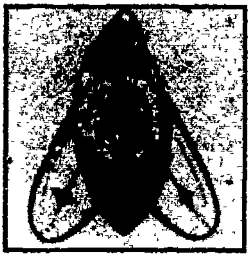
1. .2.
2. .5.2.
3. .2.5.
4. .3.3.2.
5. .3.3.3.2.
6. .4.3.
7. .4.4.3.
8. .2.6.3.
9. .5.2.3.
10. .5.2.4.3.
11. .3.5.2.

You may not use 1's in fracimals. If you could, they would be too easy to write. For example, if you wanted to write $\frac{7}{13}$, you could just write .13.1.1.1.1.1.1. . But this is illegal; the right way to write it is .2.13., or even .3.2.5.7.13. .

Write these as fracimals:

12. $\frac{1}{2}$
13. $\frac{1}{3}$
14. $\frac{1}{4}$
15. $\frac{2}{3}$
16. $\frac{2}{5}$
17. $\frac{3}{5}$
18. $\frac{3}{4}$
19. $\frac{3}{8}$
20. $\frac{5}{8}$
21. $\frac{7}{8}$
22. $\frac{5}{6}$
23. $\frac{4}{9}$
24. $\frac{13}{18}$

I think there are some patterns, but I'm not sure. See if you can find any.



3 CONTINUED FRACTIONS

In chapter 1, we saw a way to write fractions using only unit fractions, that is, fractions with a 1 on top. Another way is to use continued fractions. Before I explain it, remember one important fact.

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}$$

For example,

$$\frac{1}{\frac{2}{3}} = \frac{3}{2}, \quad \text{and} \quad \frac{1}{\frac{13}{7}} = \frac{7}{13}.$$

Now, here are some examples of continued fractions:

$$1 + \frac{1}{3},\ 2 + \cfrac{1}{1 + \cfrac{1}{6}},\ 2 + \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{3}}},\ 1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{3 + \cfrac{1}{2}}}}},$$

What do they equal?

$$1 + \frac{1}{3} = 1\frac{1}{3}, \quad \text{or} \quad \frac{4}{3}.$$

$$2 + \cfrac{1}{1 + \cfrac{1}{6}} = 2 + \cfrac{1}{1\frac{1}{6}} = 2 + \cfrac{1}{\frac{7}{6}} = 2 + \frac{6}{7} = \frac{20}{7}.$$

$$2 + \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{3}}} = 2 + \cfrac{1}{3 + \cfrac{1}{7\frac{1}{3}}} = 2 + \cfrac{1}{3 + \cfrac{1}{\frac{22}{3}}}$$

$$= 2 + \cfrac{1}{3 + \frac{3}{22}} = 2 + \cfrac{1}{\frac{69}{22}} = 2\frac{22}{69} = \frac{160}{69}.$$

$$1+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{3+\frac{1}{2}}}}}=1+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{\frac{7}{2}}}}}$$

$$=1+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{5+\frac{2}{7}}}}=1+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{\frac{37}{7}}}}$$

$$=1+\cfrac{1}{4+\cfrac{1}{2+\frac{7}{37}}}=1+\cfrac{1}{4+\cfrac{1}{\frac{81}{37}}}$$

$$=1+\cfrac{1}{4+\frac{37}{81}}=1+\cfrac{1}{\frac{361}{81}}$$

$$=1+\frac{81}{361}=\frac{442}{361}.$$

Simplify:

1. $1+\frac{1}{5}$

2. $2+\cfrac{1}{2+\frac{1}{2}}$

3. $3+\frac{1}{7}$

4. $4+\cfrac{1}{1+\frac{1}{2}}$

5. $2+\cfrac{1}{7+\frac{1}{3}}$

6. $1+\cfrac{1}{6+\cfrac{1}{2+\frac{1}{3}}}$

7. $2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}$

8. $5+\cfrac{1}{4+\cfrac{1}{3+\frac{1}{2}}}$

9. $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}}$

10. $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}}}}$

Write as continued fractions:

11. $\frac{5}{4}$
12. $\frac{16}{3}$
13. $\frac{5}{3}$
14. $\frac{7}{5}$
15. $\frac{8}{3}$
16. $\frac{10}{7}$
17. $\frac{43}{30}$
18. $\frac{37}{32}$
19. $\frac{3}{2}$
20. $\frac{5}{3}$
21. $\frac{8}{5}$
22. $\frac{13}{8}$
23. $\frac{21}{13}$
24. $\frac{34}{21}$
25. $\frac{79}{47}$

There must be an easier way to do these! Have you found one?

4 DISCONTINUED FRACTIONS

Discontinued fractions are just like continued fractions except that all the + signs are changed to − signs. For example,

$$2-\cfrac{1}{5-\cfrac{1}{3-\cfrac{1}{2}}}$$

is calculated in the same way as the continued fractions:

$$2-\cfrac{1}{5-\cfrac{1}{2\frac{1}{2}}}=2-\cfrac{1}{5-\cfrac{1}{\frac{5}{2}}}=2-\cfrac{1}{5-\frac{2}{5}}=2-\cfrac{1}{\frac{23}{5}}=2-\frac{5}{23}=\frac{41}{23}$$

Simplify:

1. $3-\frac{1}{4}$

2. $2-\cfrac{1}{3-\cfrac{1}{3}}$

3. $2-\cfrac{1}{1-\cfrac{1}{4}}$

4. $3-\cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{4}}}$

5. $4-\cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{4}}}}$

6. $1-\cfrac{1}{2-\cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}}}$

Write as discontinued fractions:

7. $\frac{1}{2}$

8. $\frac{2}{3}$

9. $\frac{8}{3}$

10. $\frac{20}{7}$

11. $\frac{1}{3}$

12. $\frac{1}{4}$

13. $\frac{4}{3}$

14. $\frac{3}{8}$

15. $\frac{117}{41}$

5 FRICTIONS

What does the fraction $\frac{4}{5}$ mean? It means $4 \div 5$, and if you divide 4 by 5, you get .8, which is another name for $\frac{4}{5}$. In other words, a fraction is just another name for a division problem—that is, every fraction $\frac{a}{b}$ is equal to $a \div b$.

A *friction*, however, is another name for a *subtraction* problem; thus $-\frac{a}{b}-$ is equal to $a - b$. For example:

$$-\frac{3}{2}- = 3 - 2 = 1$$

$$-\frac{17}{4}- = 13$$

$$-\frac{8}{11}- = -3$$

What numbers do these equal?

1. $-\frac{6}{4}-$
2. $-\frac{3}{8}-$
3. $-\frac{271}{365}-$

Write as frictions:

4. 7
5. −9
6. 37

You know some special rules for adding, subtracting, multiplying, and dividing *fractions*. Now try to find some rules for working with *frictions*. My students found ways to add and subtract them, but we couldn't figure out how to multiply them.

6 ZERONES

A zerone (rhymes with *tone*) is a sequence of ones with one zero in it, like this:

$$1101_z, \quad 10111_z, \quad 011_z, \quad 110_z$$

I put the *z* there so they won't be confused with base 2 (see chapter 12); *z* stands for *zerone*. This is how you figure them out (start at the right):

In 1101_z,

$$\begin{array}{lr|r|}
\cline{3-3}
\text{the 1 means} & 1 & 1 \\
\text{the 0 means} & 0+1 & 1 \\
\text{the 1 means} & 1+0+1 & 2 \\
\text{the 1 means} & 1+1+0+1 & +\ 3 \\
\cline{3-3}
 & & 7 \\
\cline{3-3}
\end{array}$$

And so 1101_z is equal to 7.

Likewise, 10111_z means

$$\begin{array}{r}
1 = 1 \\
1+1 = 2 \\
1+1+1 = 3 \\
0+1+1+1 = 3 \\
1+0+1+1+1 = \underline{4} \\
13
\end{array}$$

011_z means

$$\begin{array}{r}
1 = 1 \\
1+1 = 2 \\
0+1+1 = \underline{2} \\
5
\end{array}$$

and 110_Z means

$$\begin{array}{r} 0 = 0 \\ 1 + 0 = 1 \\ 1 + 1 + 0 = \underline{2} \\ 3 \end{array}$$

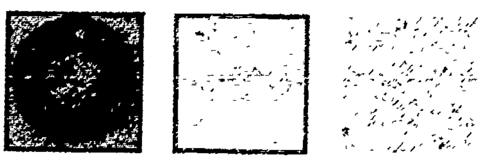
Figure out what these equal:

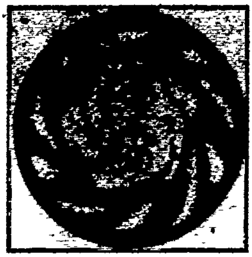
1. 101_Z
2. 11011_Z
3. 1101111_Z
4. 111111110_Z

Write these as zerones (remember to use only one zero):

5. 1	9. 5	13. 9	17. 13
6. 2	10. 6	14. 10	18. 14
7. 3	11. 7	15. 11	19. 15
8. 4	12. 8	16. 12	20. 47

Most of my students liked zerones.





7 SQUARIMALS

Squarimals look just like zerones, with this exception. you are allowed to have as many zeros as you like. For example: 1101q, 10101_q, 01001_q. (The *q* stands for *squarimal*.) This is how you figure them out (again starting at the right):

In 1101q,

the 1 means	$(1)^2$	1
the 0 means	$(0+1)^2$	1
the 1 means	$(1+0+1)^2$	4
the 1 means	$(1+1+0+1)^2$	+ 9
		15

And so 1101_q is equal to 15.

Similarly, 10101_q means

$$\begin{aligned} (1)^2 &= 1 \\ (0+1)^2 &= 1 \\ (1+0+1)^2 &= 4 \\ (0+1+0+1)^2 &= 4 \\ (1+0+1+0+1)^2 &= \underline{9} \\ & 19 \end{aligned}$$

and 01001_q means

$$\begin{aligned} (1)^2 &= 1 \\ (0+1)^2 &= 1 \\ (0+0+1)^2 &= 1 \\ (1+0+0+1)^2 &= 4 \\ (0+1+0+0+1)^2 &= \underline{4} \\ & 11 \end{aligned}$$

Find out what these squarimals mean:

1. 001q
2. 1111q
3. 0101q
4. 100101q
5. 1001001001q
6. 0111q

Write these numbers as squarimals (try to use as few zeros as possible):

7. 2
8. 5
9. 10
10. 15
11. 16
12. 17
13. 21
14. 29
15. 37
16. 53
17. 129

I would like to know a lot more about these numerals!

8 PHEROCIMALS

A pherocimal is a sequence of ones and twos, like this: 2122p, 22121p, 121p, and 212122p. (The *p* stands for *pherocimal.*) You are not allowed to have two ones together, but it doesn't matter how many twos are together. This is how you calculate them (start on the right):

In 2122p,

the 2 means	2	2
the 2 means	2×2	4
the 1 means	$1 \times 2 \times 2$	4
the 2 means	$2 \times 1 \times 2 \times 2$	+ 8
		18

And so 2122p is equal to 18.

In 22121p,

$$
\begin{aligned}
1 &= 1\\
2 \times 1 &= 2\\
1 \times 2 \times 1 &= 2\\
2 \times 1 \times 2 \times 1 &= 4\\
2 \times 2 \times 1 \times 2 \times 1 &= \underline{8}\\
& 17
\end{aligned}
$$

In 121p,

$$
\begin{aligned}
1 &= 1\\
2 \times 1 &= 2\\
1 \times 2 \times 1 &= \underline{2}\\
& 5
\end{aligned}
$$

And in 212122p,

$$\begin{array}{r}
2 = 2 \\
2 \times 2 = 4 \\
1 \times 2 \times 2 = 4 \\
2 \times 1 \times 2 \times 2 = 8 \\
1 \times 2 \times 1 \times 2 \times 2 = 8 \\
2 \times 1 \times 2 \times 1 \times 2 \times 2 = \underline{16} \\
42
\end{array}$$

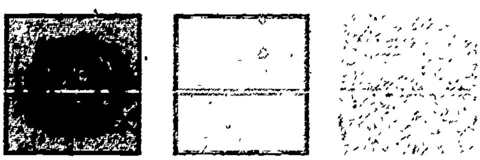
What do these equal?

1. 221p
2. 12122p
3. 21212p
4. 22221p
5. 221221221p

Change these to pherocimals (remember you can't have two ones together):

6. 1
7. 2
8. 3
9. 4
10. 5
11. 6
12. 7
13. 8
14. 9
15. 10
16. 19
17. 23
18. 45

Keep exploring!



9 SUPER PHEROCIMALS

Super pherocimals look exactly like pherocimals, but they are slightly different. Here are some examples:

22122sp, 222121sp, 221sp

The *sp* stands for *super pherocimal*, and here is what you do with them:

In 22122sp,

$$\begin{array}{rrr}
2 = & 2 & \\
2 \times 2 = & & 4 \\
1 \times 2 \times 2 = & 4 & \\
2 \times 1 \times 2 \times 2 = & & 8 \\
2 \times 2 \times 1 \times 2 \times 2 = & +16 & + \\
\hline
 & 22 \;\; - & 12
\end{array}$$

and $22 - 12 = 10$.

And so $22122sp = 10$.

For 222121sp,

$$\begin{array}{rrr}
1 = & 1 & \\
2 \times 1 = & & 2 \\
1 \times 2 \times 1 = & 2 & \\
2 \times 1 \times 2 \times 1 = & & 4 \\
2 \times 2 \times 1 \times 2 \times 1 = & 8 & \\
2 \times 2 \times 2 \times 1 \times 2 \times 1 = & & 16 \\
\hline
 & 11 \;\; - & 22
\end{array}$$

and $11 - 22 = -11$. So $222121sp = -11$.

In 221sp,

$$\begin{array}{rrr}
1 = & 1 & \\
2 \times 1 = & & 2 \\
2 \times 2 \times 1 = & 4 & \\
\hline
 & 5 \;\; - & 2
\end{array}$$

and 5 − 2 = 3. So 221sp = 3.

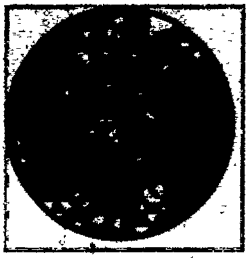
Figure these out:

1. 212sp
2. 2212sp
3. 22121sp
4. 122sp
5. 12122sp
6. 1212122sp
7. 2222221sp
8. 212121212121sp

Change to super pherocimals (as before, you can't have two ones together):

9. 1
10. 2
11. 3
12. 4
13. 5
14. 6
15. 0
16. −1
17. −2
18. −3
19. −4
20. −5
21. −6
22. 17
23. −17

These numerals are pretty crazy, but I sort of like them. I wonder if there is an easy way to change a pherocimal into a super pherocimal?



10 CONTINUED FRICTIONS

Continued frictions are like continued fractions (remember those?), but with three differences:

1. Use frictions instead of fractions.
2. Use $\times$ instead of $+$.
3. Use only 1's and 2's.

Here are some examples:

$$1 \times \frac{1}{2},\quad 1 \times \frac{1}{2 \times \frac{1}{2}},\quad 2 \times \frac{1}{1 \times \frac{1}{2 \times \frac{1}{2 \times \frac{1}{2}}}}.$$

These are easy to solve:

$$1 \times \frac{1}{2} = 1 \times (1 - 2) = 1 \times -1 = -1$$

$$1 \times \frac{1}{2 \times \frac{1}{2}} = 1 \times \frac{1}{2 \times -1} = 1 \times \frac{1}{-2} = 1 \times 3 = 3$$

$$2 \times \frac{1}{1 \times \frac{1}{2 \times \frac{1}{2 \times \frac{1}{2}}}} = 2 \times \frac{1}{1 \times \frac{1}{2 \times \frac{1}{2 \times -1}}} = 2 \times \frac{1}{1 \times \frac{1}{2 \times \frac{1}{-2}}}$$

$$= 2 \times \frac{1}{1 \times \frac{1}{2 \times 3}} = 2 \times \frac{1}{1 \times \frac{1}{6}}$$

$$= 2 \times \frac{1}{1 \times -5} = 2 \times \frac{1}{-5} = 2 \times 6 = 12$$

Since the numbers on top aren't important, we can write each continued friction as a sequence—for example, the three continued frictions above would be 12c, 122c, and 21222c (the c stands for *continued friction*). Similarly, 2212c would be expressed like this:

$$2 \times \cfrac{1}{2 \times \cfrac{1}{1 \times \cfrac{1}{2}}}$$

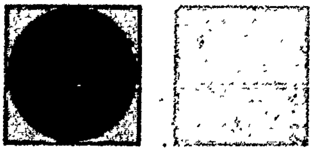
Compute these:

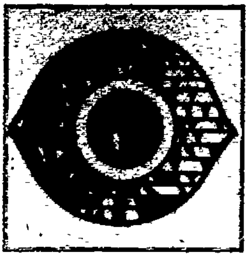
1. $2 \times \frac{1}{2}$
2. $2 \times \cfrac{1}{1 \times \cfrac{1}{2}}$
3. $1 \times \cfrac{1}{2 \times \cfrac{1}{1 \times \cfrac{1}{2}}}$
4. 22c
5. 212c
6. 1212c
7. 2122c
8. 12122c
9. 221c
10. 22121c
11. 2212121c
12. 1222222c
13. 2212212c

Write these as continued frictions:

14. 1
15. 2
16. 3
17. 4
18. 5
19. 0
20. −1
21. −2
22. −3
23. −4
24. −5
25. 13
26. −13

I have to admit this is a pretty strange number system, but I think there is a pattern somewhere.





11 DISCONTINUED FRICTIONS

About two hours after I finished that last chapter, I suddenly thought of this: Suppose we take the continued frictions and change all the $\times$ signs to $\div$ signs. I tried it, and I really got into a mess. For example:

$$1 \div \cfrac{1}{2 \div \cfrac{1}{2}} = 1 \div \frac{1}{2 \div -1} = 1 \div \frac{1}{-2} = 1 \div 3 = \frac{1}{3}$$

and

$$2 \div \cfrac{1}{1 \div \cfrac{1}{1 \div \cfrac{1}{2}}} = 2 \div \cfrac{1}{1 \div \cfrac{1}{1 \div -1}} = 2 \div \cfrac{1}{1 \div \cfrac{1}{-1}} = 2 \div \frac{1}{1 \div 2}$$

$$= 2 \div \cfrac{1}{\cfrac{1}{2}} = 2 \div \frac{1}{2} = 4$$

Again, only the bottom numbers are important, so we can write them in a line. In other words, the two examples above would be written as 122_d and 2112_d (the *d* stands for *discontinued friction*).

As soon as I started to explore this, I ran into trouble. I won't tell you all the problems I had, so you can have them too, but I will tell you about one of them. I wondered if I could write any number—positive or negative, integer or fraction—and so I picked an innocent-looking fraction, $\frac{4}{7}$, and tried to write it as a discontinued friction. Two hours later, I finally had it: $\frac{4}{7} = 21212221122_d$!

Find out what these are:

1. $1 \div \frac{1}{2}$
2. $2 \div \cfrac{1}{1 \div \cfrac{1}{2}}$
3. $1 \div \cfrac{1}{1 \div \cfrac{1}{2 \div \cfrac{1}{2}}}$
4. 1112_d
5. 21212_d
6. 11222_d
7. 2222_d
8. 11112_d
9. 211221122122_d

Try writing these as discontinued frictions:

10. -2 11. $\frac{2}{3}$ 12. $\frac{1}{2}$ (*Hint:* They are all short ones.)

If you can make some sense out of this system, please let me know!

12 NUMBER BASES

You probably know already that we generally write numbers in base 10. This means that in a number like 3,479 each digit means something:

the 3 means three 1,000's	3,000
the 4 means four 100's	400
the 7 means seven 10's	70
the 9 means nine 1's	+ 9
	3,479

Every number in base 10 fits the following pattern:

$$\cdots \boxed{100{,}000}\ \boxed{10{,}000}\ \boxed{1{,}000}\ \boxed{100}\ \boxed{10}\ \boxed{1}$$

But this pattern can also be written as

$$\cdots \boxed{10 \times 10 \times 10 \times 10 \times 10}\ \boxed{10 \times 10 \times 10 \times 10}\ \boxed{10 \times 10 \times 10}\ \boxed{10 \times 10}\ \boxed{10}\ \boxed{1};$$

which shows you why we call it base 10. It also shows you how we could set up a different base—for example, base 6:

$$\cdots \boxed{6 \times 6 \times 6 \times 6 \times 6}\ \boxed{6 \times 6 \times 6 \times 6}\ \boxed{6 \times 6 \times 6}\ \boxed{6 \times 6}\ \boxed{6}\ \boxed{1},$$

which is the same as

$$\cdots \boxed{7{,}776}\ \boxed{1{,}296}\ \boxed{216}\ \boxed{36}\ \boxed{6}\ \boxed{1}.$$

For example, this is what 3452 means in base 6:

the 3 means three	216's	648
the 4 means four	36's	144
the 5 means five	6's	30
the 2 means two	1's	+ 2
		824

So 3452 in base 6 is 824 in base 10. Since 3452 can mean different things in different bases, we write it as 3452_6 when we mean base 6 and simply as 3452 (or 3,452) when we mean base 10.

Write in base 10:

1. 35_6
2. 41_6
3. 133_6
4. 352_6
5. 23451_6

Notice that we don't need to use the digits 6, 7, 8, or 9 when writing in base 6. If we want to write 7 in base 6, we don't write 7_6; we write 11_6. When we want to write 48 in base 6, we don't write 80_6; we write 120_6.

Write in base 6:

6. 9
7. 19
8. 37
9. 35
10. 79
11. 333
12. 215

We can write in other bases, too—for example, in base 2:

... $\boxed{2 \times 2 \times 2 \times 2 \times 2 \times 2}$ $\boxed{2 \times 2 \times 2 \times 2 \times 2}$ $\boxed{2 \times 2 \times 2 \times 2}$ $\boxed{2 \times 2 \times 2}$ $\boxed{2 \times 2}$ $\boxed{2}$ $\boxed{1}$,

or

... $\boxed{64}$ $\boxed{32}$ $\boxed{16}$ $\boxed{8}$ $\boxed{4}$ $\boxed{2}$ $\boxed{1}$

Write in base 10:

13. 10_2
14. 11_2
15. 1011_2
16. 110101_2
17. 100010_2
18. 111111_2

In base 2, we use only the digits 0 and 1.

In base 3, we use only the digits 0, 1, and 2.

In base 6, we use only the digits 0, 1, 2, 3, 4, and 5.

In base 10, we use only the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Write in base 2:

19. 3
20. 6
21. 9
22. 18
23. 36
24. 72
25. 30
26. 79
27. 127

Write in base 3:

28. 4
29. 13
30. 23
31. 5
32. 15
33. 45
34. 135
35. 101
36. 242

Write in base 5:

37. 7
38. 13
39. 28
40. 8
41. 40
42. 200
43. 199
44. 624

Numbers can be written in any whole-number base. Each base is a numeration system, or a system for writing numbers. Computers use base 2, but I'm not sure what the other bases are good for.

13 NEGATIVE NUMBER BASES

There is no reason why a negative number can't be a base; for instance, here is base -10:

$$\cdots \boxed{-10\times-10\times-10\times-10\times-10}\ \boxed{-10\times-10\times-10\times-10}\ \boxed{-10\times-10\times-10}\ \boxed{-10\times-10}\ \boxed{-10}\ \boxed{1},$$

which is the same as

$$\cdots \boxed{-100{,}000}\ \boxed{10{,}000}\ \boxed{-1{,}000}\ \boxed{100}\ \boxed{-10}\ \boxed{1}\ .$$

For example, in 835_{-10},

the 8 means eight 100's
the 3 means three -10's
the 5 means five 1's

So

$$835_{-10} = 800 - 30 + 5 = 775.$$

Write in base 10:

1. 4_{-10}
2. 14_{-10}
3. 102_{-10}
4. 348_{-10}
5. 3671_{-10}
6. 83649_{-10}

Write in base -10:

7. 7
8. 100
9. 105
10. 90
11. 91
12. 82
13. 43
14. 190
15. 177
16. 488
17. 12,224
18. 4,675
19. 2,788,265
20. 3,456,738,214,337
21. 99
22. 999
23. 9,999
24. 99,999
25. -10
26. -36
27. $-1{,}265$

Base −2 is also fun:

…	−128	64	−32	16	−8	4	−2	1

Write in base 10:

28. 101_{-2}
29. 1101_{-2}
30. 10111_{-2}
31. 10110010_{-2}

Write in base −2:

32. 5
33. 2
34. 3
35. 12
36. 13
37. 22
38. 23
39. 56
40. 57
41. 143
42. −1
43. −3
44. −13
45. −37

Write in base −6:

46. 13
47. 27
48. 53
49. 148
50. −23

Write in base −3:

51. 13
52. 27
53. 53
54. 148
55. −23

14 BALANCED TERNARY

This system looks a lot like base 3 and base −3. You use the same pattern:

… $\boxed{3 \times 3 \times 3 \times 3 \times 3}$ $\boxed{3 \times 3 \times 3 \times 3}$ $\boxed{3 \times 3 \times 3}$ $\boxed{3 \times 3}$ $\boxed{3}$ $\boxed{1}$;

that is,

$\boxed{243}$ $\boxed{81}$ $\boxed{27}$ $\boxed{9}$ $\boxed{3}$ $\boxed{1}$ …

But instead of using the digits 0, 1, and 2, you use 0, +, and −. The + means +1, the − means −1, and the 0 means 0. Here are some examples:

For +−0+ in balanced ternary,

the + means +1(27)	+27
the − means −1(9)	−9
the 0 means 0(3)	0
the + means +1(1)	+1
	+19

For −++0−+,

the − means −1(243)	−243
the + means +1(81)	+81
the 0 means 0(9)	0
the − means −1(3)	−3
the + means +1(1)	+1
	−137

Write in base 10:

1. +−
2. −+0
3. +00−+
4. −−0+−
5. +−0+−0
6. +−000−+

Write in balanced ternary:

7. 1
8. 2
9. 3
10. 12
11. 17
12. 25
13. 43
14. 164
15. −1
16. −2
17. −3
18. −12
19. −17
20. −25
21. −43
22. −164

15 HEXIMALS

The base-10 system is sometimes called the *decimal* system. With the decimal point, the complete base-10 system looks like this:

$$\cdots \boxed{10{,}000}\boxed{1{,}000}\boxed{100}\boxed{10}\boxed{1}\ .\ \boxed{\frac{1}{10}}\boxed{\frac{1}{100}}\boxed{\frac{1}{1{,}000}}\boxed{\frac{1}{10{,}000}} \cdots$$

For example, when we write .234,

the 2 means two	$\frac{1}{10}$s	$\frac{2}{10}$
the 3 means three	$\frac{1}{100}$s	$\frac{3}{100}$
the 4 means four	$\frac{1}{1{,}000}$s	$\frac{4}{1{,}000}$

So

$$.234 = \frac{2}{10} + \frac{3}{100} + \frac{4}{1{,}000} = \frac{234}{1{,}000} = \frac{117}{500}.$$

When we do this with base 6, we get *heximals*. The complete heximal system looks like this:

$$\cdots \boxed{1{,}296}\boxed{216}\boxed{36}\boxed{6}\boxed{1}\ .\ \boxed{\frac{1}{6}}\boxed{\frac{1}{36}}\boxed{\frac{1}{216}}\boxed{\frac{1}{1{,}296}} \cdots$$

The point, of course, is called a heximal point. Here are some examples:

$$.2_6 = \frac{2}{6} = \frac{1}{3}$$

$$.15_6 = \frac{1}{6} + \frac{5}{36} = \frac{11}{36}$$

$$.234_6 = \frac{2}{6} + \frac{3}{36} + \frac{4}{216} = \frac{94}{216} = \frac{47}{108}$$

Write these as fractions:

1. $.5_6$
2. $.23_6$
3. $.105_6$
4. $.333_6$
5. $.555_6$

Change to heximals:

6. $\frac{1}{6}$
7. $\frac{2}{6}$
8. $\frac{1}{3}$
9. $\frac{1}{2}$
10. $\frac{2}{3}$
11. $\frac{1}{36}$
12. $\frac{1}{18}$
13. $\frac{1}{12}$

Remember, in base 6 use only the digits 0, 1, 2, 3, 4, and 5.

Keep going:

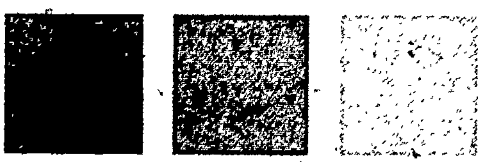
14. $\frac{5}{36}$
15. $\frac{7}{36}$
16. $\frac{1}{4}$
17. $\frac{4}{9}$
18. $\frac{7}{6}$
19. $\frac{3}{2}$
20. $\frac{13}{3}$
21. $\frac{43}{216}$
22. $\frac{1}{8}$

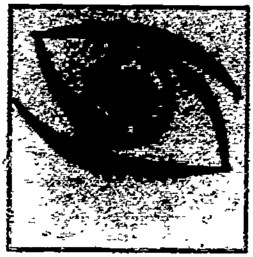
You can do the same thing in other bases.

Write $\frac{1}{2}$ in:

23. base 2
24. base 4
25. base 8
26. base 10

Don't stop here. Keep exploring with more bases.





16 NEGAHEXIMALS

Of course, we can do the same thing in base -6. The *negaheximal* system would look like this:

$$\cdots\ \boxed{1{,}296}\ \boxed{-216}\ \boxed{36}\ \boxed{-6}\ \boxed{1}\ .\ \boxed{-\tfrac{1}{6}}\ \boxed{\tfrac{1}{36}}\ \boxed{-\tfrac{1}{216}}\ \boxed{\tfrac{1}{1{,}296}}\ \cdots$$

The point is called the negaheximal point. Some examples:

$$.2_{-6} = -\frac{2}{6} = -\frac{1}{3}$$

$$1.1_{-6} = 1 - \frac{1}{6} = \frac{5}{6}$$

$$2.03_{-6} = 2 + \frac{3}{36} = 2\,\frac{1}{12}$$

Write as fractions:

1. $.1_{-6}$
2. 1.4_{-6}
3. $.14_{-6}$
4. 2.13_{-6}

Write in base -6:

5. $\frac{1}{6}$
6. $\frac{1}{3}$
7. $\frac{1}{2}$
8. $\frac{1}{36}$
9. $\frac{1}{18}$
10. $\frac{1}{9}$
11. $\frac{1}{4}$
12. $-\frac{1}{6}$
13. $-\frac{1}{2}$
14. $-\frac{1}{36}$
15. $-\frac{1}{9}$
16. $-\frac{1}{4}$

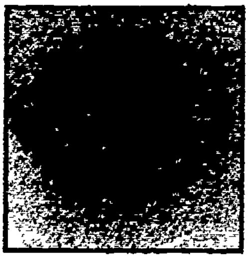
Write in base -10:

17. $\frac{1}{2}$
18. $\frac{3}{10}$
19. $\frac{7}{5}$
20. $\frac{17}{25}$
21. $-\frac{17}{25}$

Write $\frac{1}{2}$ in these bases:

22. -2
23. -4
24. -8

See if you can find out more about these bases than I have.



17 BALANCED TERNARY AGAIN

We can do the same thing with balanced ternary:

$$\cdots \boxed{729}\boxed{243}\boxed{81}\boxed{27}\boxed{9}\boxed{3}\boxed{1}\ .\ \boxed{\tfrac{1}{3}}\boxed{\tfrac{1}{9}}\boxed{\tfrac{1}{27}}\boxed{\tfrac{1}{81}}\boxed{\tfrac{1}{243}}\boxed{\tfrac{1}{729}}. \cdots$$

The point is called the *ternary* point. Here are some examples:

$$+.- = 1 - \frac{1}{3} = \frac{2}{3}$$

$$.0++ = \frac{1}{9} + \frac{1}{27} = \frac{4}{27}$$

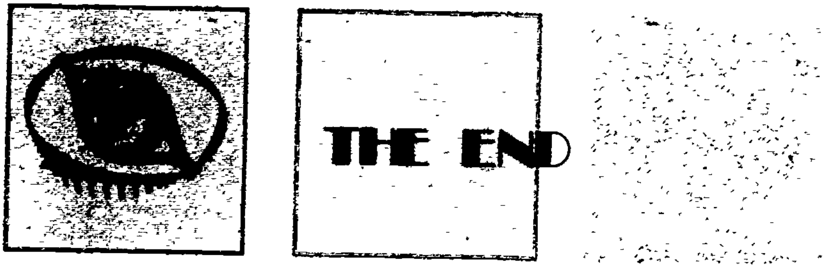
Compute these:

1. .++
2. .+−
3. −.+
4. +0.−−
5. +0−.00+−+

Change to balanced ternary:

6. $\frac{1}{3}$
7. $\frac{1}{9}$
8. $\frac{8}{9}$
9. $\frac{7}{9}$
10. $\frac{16}{27}$
11. $\frac{43}{81}$
12. $-\frac{1}{3}$
13. $-\frac{1}{9}$
14. $-\frac{8}{9}$
15. $-\frac{7}{9}$
16. $-\frac{16}{27}$
17. $-\frac{43}{81}$

Do you think this is a good system? Good for what?



From now on, you're on your own. I hope you had fun, and I hope you found out a lot about these numerals. If you did, please write me at the address below and let me know—there are a lot of mysteries I'm counting on you to solve.

There is no reason why you have to stop now; there are several things you can do. Here are a few suggestions:

1. Make a chart comparing base 3, base −3, and balanced ternary. Since they all are based on the same chart, there should be a connection between them.

2. Make a chart comparing base 2, base −2, zerones, squarimals, pherocimals, super pherocimals, continued frictions, and discontinued frictions. They can all be written as a sequence of two things—some use 0 and 1, and some use 1 and 2. There *should* be a connection!

3. Invent your own numeral system! It can be completely different, or it can be based on one of the systems here. For example, I think somebody could invent super fracimals, or maybe messimals. Somewhere out there, fructions are just waiting to be invented, and once you have those, continued and discontinued fructions will be a cinch!

Let me know if you come up with something interesting.

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HARDER STUFF

For most of this stuff we need a formula:

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + a^5 + \cdots$$

For example:

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots \\ &= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \end{aligned}$$

Harder stuff from chapter 2:

Do you remember what $.33333\overline{3}\ldots$ means? This is ordinary base 10, and it means this:

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1{,}000} + \frac{3}{10{,}000} + \cdots$$

By our formula, this equals

$$\begin{aligned} & \frac{3}{10}\left[1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1{,}000} + \cdots\right] \\ &= \frac{3}{10}\left[1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right] \\ &= \frac{3}{10}\left[\frac{1}{1-\frac{1}{10}}\right] = \frac{3}{10}\left[\frac{1}{\frac{9}{10}}\right] = \frac{3}{10} \times \frac{10}{9} = \frac{30}{90} = \frac{1}{3}. \end{aligned}$$

This is called a repeating decimal.

We can do the same thing for fracimals. For example:

$$\begin{aligned}
.3.3.3.3.\overline{3}.\ \ldots &= \frac{1}{3}+\frac{1}{3\times 3}+\frac{1}{3\times 3\times 3}+\frac{1}{3\times 3\times 3\times 3}+\cdots \\
&= \frac{1}{3}\left[1+\frac{1}{3}+\frac{1}{3\times 3}+\frac{1}{3\times 3\times 3}+\cdots\right] \\
&= \frac{1}{3}\left[1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots\right] \\
&= \frac{1}{3}\left[\frac{1}{1-\frac{1}{3}}\right]=\frac{1}{3}\left[\frac{1}{\frac{2}{3}}\right]=\frac{1}{3}\times\frac{3}{2}=\frac{3}{6}=\frac{1}{2}
\end{aligned}$$

These are called *repeating fracimals.*

Evaluate these:

A. $.2.2.2.2.\overline{2}.\ \ldots$

B. $.4.4.\overline{4}.\ \ldots$

C. $.5.5.\overline{5}.\ \ldots$

D. $.2.3.2.3.\overline{2.3}.\ \ldots$

Harder stuff from chapter 15:

We can also have repeating heximals. For example:

$$\begin{aligned}
.3333\overline{3}\ldots_6 &= \frac{3}{6}+\frac{3}{36}+\frac{3}{216}+\frac{3}{1{,}296}+\cdots \\
&= \frac{3}{6}\left[1+\frac{1}{6}+\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^3+\cdots\right] \\
&= \frac{3}{6}\left[\frac{1}{1-\frac{1}{6}}\right]=\frac{3}{6}\left[\frac{1}{\frac{5}{6}}\right]=\frac{3}{6}\times\frac{6}{5}=\frac{18}{30}=\frac{3}{5}
\end{aligned}$$

Try these:

E. $.111\overline{1}\ldots_6$

F. $.222\overline{2}\ldots_6$

G. $.444\overline{4}\ldots_6$

H. $.2323\overline{23}\ldots_6$

Harder stuff from chapter 16:

We can also have repeating negaheximals. For example:

$$\begin{aligned}
.3333\overline{3}\ldots_{-6} &= -\frac{3}{6}+\frac{3}{36}-\frac{3}{216}+\frac{3}{1{,}296}-+\cdots \\
&= -\frac{3}{6}\left[1+\left(-\frac{1}{6}\right)+\left(-\frac{1}{6}\right)^2+\left(-\frac{1}{6}\right)^3+\cdots\right] \\
&= -\frac{3}{6}\left[\frac{1}{1-\left(-\frac{1}{6}\right)}\right]=-\frac{3}{6}\left[\frac{1}{\frac{7}{6}}\right]=-\frac{3}{6}\times\frac{6}{7}=-\frac{18}{42}=-\frac{3}{7}
\end{aligned}$$

See what you can do with these:

I. $.111\overline{1}\ldots_6$

J. $.222\overline{2}\ldots_6$

K. $1.555\overline{5}\ldots_6$

L. $.23232\overline{3}\ldots_6$

Harder stuff from chapter 17:

We can also have repeating balanced ternary. Here's one example:

$$.+++++\overline{+}\ldots = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}$$

Another example:

$$\begin{aligned}
.-0-0\overline{-0}\ldots &= -\frac{1}{3}-\frac{1}{27}-\frac{1}{243}-\frac{1}{2{,}187}-\cdots\\
&= -\frac{1}{3}\left[1+\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+\cdots\right]\\
&= -\frac{1}{3}\left[1+\frac{1}{9}+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^3+\cdots\right]\\
&= -\frac{1}{3}\left[\frac{1}{1-\frac{1}{9}}\right] = -\frac{1}{3}\left[\frac{1}{\frac{8}{9}}\right] = -\frac{1}{3}\times\frac{9}{8} = -\frac{9}{24} = -\frac{3}{8}
\end{aligned}$$

Figure these out:

M. $.-----\overline{-}\ldots$

N. $.+-+-\overline{+-}\ldots$

O. $.0+0+\overline{0+}\ldots$

P. $.-+\overline{-+}\ldots$

Harder stuff from chapter 3:

We can have repeating continued fractions too, but they are a little harder to figure out. For example:

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdot}}}$$

To find out what this is, call it X. Then we know

$$\frac{1}{X}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdot}}}},$$

which is also equal to $X-2$.

So we have $\frac{1}{X} = X - 2$, since $1 = X^2 - 2X$, $X^2 - 2X - 1 = 0$. Using the quadratic formula, we find that

$$X = \frac{2 \pm \sqrt{4 - -4}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2},$$

and since X must be positive,

$$X = 1 + \sqrt{2}.$$

Try these:

Q. $3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdot}}$ S. $4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdot}}$

R. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdot}}$ T. $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdot}}}}}$

Harder stuff from chapter 4:

I'd like to do the same thing with discontinued fractions, but I can't always do it. For example, if

$$X = 1 - \cfrac{1}{1 - \cfrac{1}{1 - \cdot}},$$

then we get

$$X - 1 = -\frac{1}{X}.$$

Therefore, $X^2 - X = -1$, and so $X^2 - X + 1 = 0$. Thus

$$X = 1 \pm \frac{\sqrt{1 - 4}}{2},$$

which is impossible.

Some of these repeating discontinued fractions do work if you're lucky. If you like, you can try these:

U. $$2-\cfrac{1}{2-\cfrac{1}{2-\cdot}}$$

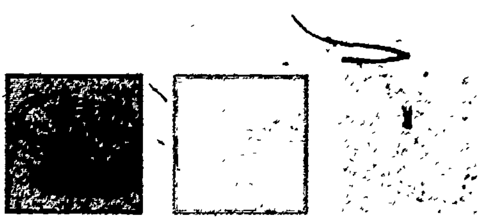
V. $$3-\cfrac{1}{3-\cfrac{1}{3-\cdot}}$$

W. $$2-\cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{1-\cdot}}}}}$$

X. $$1-\cfrac{1}{2-\cfrac{1}{1-\cfrac{1}{2-\cdot}}}$$

Harder stuff from chapter 11:

Egad! Repeating discontinued frictions? Well, I don't know. Try them! I wish you lots of luck!



THERE IS NO BIBLIOGRAPHY

A bibliography is a list at the end of a book. The list tells you where you can find out more about the stuff in the book. This book doesn't have a bibliography because most of the stuff in it is new, and nobody else has ever written about it. So if you want to read another book about these systems, you'll have to wait until you or somebody else writes one!

THE ANSWERS

1. Egyptian Fractions

1. $\frac{1}{2}+\frac{1}{4}$ 2. $\frac{1}{2}+\frac{1}{3}$ 3. $\frac{1}{2}+\frac{1}{10}$ 4. $\frac{1}{4}+\frac{1}{8}$ 5. $\frac{1}{2}+\frac{1}{8}$ 6. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
7. $\frac{1}{4}+\frac{1}{12}$ 8. $\frac{1}{5}+\frac{1}{20}$ 9. $\frac{1}{6}+\frac{1}{30}$ 10. $\frac{1}{7}+\frac{1}{42}$ 11. $\frac{1}{10}+\frac{1}{90}$
12. $\frac{1}{9}+\frac{1}{10}+\frac{1}{90}$ 13. $\frac{1}{3}+\frac{1}{9}$ 14. $\frac{1}{2}+\frac{1}{18}$ 15. $\frac{1}{2}+\frac{1}{18}+\frac{1}{9}+\frac{1}{10}+\frac{1}{90}$
16. $\frac{1}{2}+\frac{1}{3}+\frac{1}{18}$

2. Fracimals

1. $\frac{1}{2}$ 2. $\frac{3}{10}$ 3. $\frac{3}{5}$ 4. $\frac{1}{2}$ 5. $\frac{1}{2}$ 6. $\frac{1}{3}$ 7. $\frac{1}{3}$ 8. $\frac{11}{18}$ 9. $\frac{1}{3}$ 10. $\frac{1}{3}$ 11. $\frac{13}{30}$
12. .2. 13. .3. 14. .4. 15. .2.3. 16. .3.5. 17. .2.5. 18. .2.2.
19. .4.2. 20. .2.4. 21. .2.2.2. 22. .2.2.3. 23. .3.3. 24. .2.3.3.

3. Continued Fractions

1. $\frac{6}{5}$ 2. $\frac{12}{5}$ 3. $\frac{22}{7}$ 4. $\frac{14}{3}$ 5. $\frac{47}{22}$ 6. $\frac{52}{45}$ 7. $\frac{29}{12}$ 8. $\frac{157}{30}$ 9. $\frac{169}{70}$
10. $\frac{985}{408}$ 11. $1+\frac{1}{4}$ 12. $5+\frac{1}{3}$ 13. $1+\cfrac{1}{1+\frac{1}{2}}$ 14. $1+\cfrac{1}{2+\frac{1}{2}}$

15. $2+\cfrac{1}{1+\frac{1}{2}}$ 16. $1+\cfrac{1}{2+\frac{1}{3}}$ 17. $1+\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}$

18. $1+\cfrac{1}{6+\cfrac{1}{2+\frac{1}{2}}}$ 19. $1+\frac{1}{2}$ 20. $1+\cfrac{1}{1+\frac{1}{2}}$

21. $1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}$ 22. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}$

23. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}$

24. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}$

25. $1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{7+\frac{1}{2}}}}$

4. Discontinued Fractions

1. $\frac{11}{4}$ 2. $\frac{13}{8}$ 3. $\frac{2}{3}$ 4. $\frac{2}{3}$ 5. $\frac{2}{3}$ 6. $\frac{3}{2}$ 7. $1-\frac{1}{2}$ 8. $1-\frac{1}{3}$ 9. $3-\frac{1}{3}$

10. $3-\frac{1}{7}$ 11. $1-\cfrac{1}{2-\frac{1}{2}}$ 12. $1-\cfrac{1}{2-\cfrac{1}{2-\frac{1}{2}}}$

13. $2-\cfrac{1}{2-\frac{1}{2}}$ 14. $1-\cfrac{1}{2-\cfrac{1}{2-\frac{1}{3}}}$ 15. $3-\cfrac{1}{7-\frac{1}{6}}$

5. Frictions

1. 2 2. -5 3. -94 4. $-\frac{8}{1}$ 5. $\frac{1}{10}$ 6. $\frac{39}{2}$

6. Zerones

1. 4 2. 12 3. 25 4. 36 5. 10_z 6. 01_z 7. 110_z 8. 101_z 9. 011_z
10. 1110_z 11. 1101_z 12. 1011_z 13. 0111_z 14. 11110_z 15. 11101_z
16. 11011_z 17. 10111_z 18. 01111_z 19. 111110_z 20. 1111111011_z

7. Squarimals

1. 3 2. 30 3. 10 4. 23 5. 58 6. 23 7. 01_q 8. 11_q 9. 0101_q
10. 1101_q 11. 11001_q 12. 110001_q 13. 1010001_q 14. 0101001_q
15. 11010001_q 16. 10010101_q 17. 011111001_q

8. Pherocimals

1. 7 2. 26 3. 20 4. 31 5. 147 6. 1p 7. 2p 8. 21p 9. 12p
10. 121p 11. 22p 12. 221p 13. 212p 14. 2121p 15. 122p
16. 21221p 17. 12221p 18. 21212121p

9. Super Pherocimals

1. 4 2. −4 3. 5 4. 2 5. 2 6. 2 7. 43 8. −63 9. 1sp
10. 2sp 11. 221sp 12. 212sp 13. 22121sp 14. 222sp 15. 12sp
16. 21sp 17. 22sp 18. 2121sp 19. 2212sp 20. 2221sp 21. 2122sp
22. 221212121sp 23. 22121221sp

10. Continued Frictions

1. −2 2. 4 3. −3 4. −2 5. 4 6. −3 7. −4 8. 5 9. 2
10. 2 11. 2 12. 43 13. 26 14. 1c 15. 2c 16. 122c 17. 212c
18. 12122c 19. 21c 20. 12c 21. 22c 22. 1212c 23. 2122c
24. 1222c 25. 1212212c 26. 122212c

11. Discontinued Frictions

1. −1 2. 1 3. $\frac{3}{2}$ 4. 2 5. impossible 6. $-\frac{1}{2}$ 7. 6 8. −1 9. $\frac{4}{3}$
10. 22d 11. 222d 12. 112d

12. Number Bases

1. 23 2. 25 3. 57 4. 140 5. 3,415 6. 13_6 7. 31_6 8. 101_4 9. 55_6
10. 211_6 11. 1313_6 12. 555_6 13. 2 14. 3 15. 11 16. 53 17. 34
18. 63 19. 11_2 20. 110_2 21. 1001_2 22. 10010_2 23. 100100_2
24. 1001000_2 25. 11110_2 26. 1001111_2 27. 1111111_2 28. 11_3
29. 111_3 30. 212_3 31. 12_3 32. 120_3 33. 1200_3 34. 12000_3
35. 10202_3 36. 22222_3 37. 12_5 38. 23_5 39. 103_5 40. 13_5
41. 130_5 42. 1300_5 43. 1244_5 44. 4444_5

13. Negative Number Bases

1. 4 2. −6 3. 102 4. 268 5. −2,469 6. 77,569 7. 7_{-10}
8. 100_{-10} 9. 105_{-10} 10. 110_{-10} 11. 111_{-10} 12. 122_{-10}
13. 163_{-10} 14. 210_{-10} 15. 237_{-10} 16. 528_{-10} 17. 28384_{-10}
18. 16735_{-10} 19. 3392345_{-10} 20. 4664879826477_{-10} 21. 119_{-10}
22. 19019_{-10} 23. 10019_{-10} 24. 1900019_{-10} 25. 10_{-10} 26. 44_{-10}
27. 2875_{-10} 28. 5 29. −3 30. 19 31. −146 32. 101_{-2} 33. 110_{-2}
34. 111_{-2} 35. 11100_{-2} 36. 11101_{-2} 37. 1101010_{-2} 38. 1101011_{-2}
39. 1001000_{-2} 40. 1001001_{-2} 41. 110010011_{-2} 42. 11_{-2} 43. 1101_{-2}
44. 110111_{-2} 45. 101111_{-2} 46. 141_{-6} 47. 123_{-6} 48. 245_{-6}
49. 404_{-6} 50. 41_{-6} 51. 221_{-3} 52. 12000_{-3} 53. 11012_{-3}
54. 21221_{-3} 55. 1121_{-3}

14. Balanced Ternary

1. 2 2. −6 3. 79 4. −106 5. 168 6. 484 7. + 8. + − 9. + 0
10. + + 0 11. + − 0 − 12. + 0 − + 13. + − − − − +
14. + − 0 0 + − 15. − 16. − + 17. − 0 18. − − 0
19. − + 0 + 20. − 0 + − 21. − + + + − 22. − + 0 0 − +

15. Heximals

1. $\frac{5}{6}$ 2. $\frac{5}{12}$ 3. $\frac{41}{216}$ 4. $\frac{43}{72}$ 5. $\frac{215}{216}$ 6. $.1_6$ 7. $.2_6$ 8. $.2_6$ 9. $.3_6$
10. $.4_6$ 11. $.01_6$ 12. $.02_6$ 13. $.03_6$ 14. $.05_6$ 15. $.11_6$ 16. $.13_6$
17. $.24_6$ 18. 1.1_6 19. 1.3_6 20. 4.2_6 21. $.111_6$ 22. $.043_6$
23. $.1_2$ 24. $.2_4$ 25. $.4_8$ 26. .5

16. Negaheximals

1. $-\frac{1}{6}$ 2. $\frac{1}{3}$ 3. $-\frac{1}{18}$ 4. $\frac{23}{12}$ 5. 1.5_{-6} 6. 1.4_{-6} 7. 1.3_{-6} 8. $.01_{-6}$
9. $.02_{-6}$ 10. $.04_{-6}$ 11. 1.53_{-6} 12. $.1_{-6}$ 13. $.3_{-6}$ 14. $.15_{-6}$
15. $.12_{-6}$ 16. $.23_{-6}$ 17. 1.5_{-10} 18. 1.7_{-10} 19. 2.6_{-10} 20. 1.48_{-10}
21. $.72_{-10}$ 22. 1.1_{-2} 23. 1.2_{-4} 24. 1.4_{-8}

17. Balanced Ternary Again

1. $\frac{4}{9}$ 2. $\frac{2}{9}$ 3. $-\frac{2}{3}$ 4. $\frac{23}{9}$ 5. $8\frac{7}{243}$ 6. . + 7. . 0 + 8. + . 0 −
9. + . − + 10. + . − − + 11. + . − − − + 12. . −
13. . 0 − 14. − . 0 + 15. − . + − 16. − . + + −
17. − . + + + −

Harder Stuff

A. 1 B. $\frac{1}{3}$ C. $\frac{1}{4}$ D. $\frac{4}{5}$ E. $\frac{1}{5}$ F. $\frac{2}{5}$ G. $\frac{4}{5}$ H. $\frac{3}{7}$ I. $-\frac{1}{7}$ J. $-\frac{2}{7}$
K. $\frac{2}{7}$ L. $-\frac{9}{35}$ M. $-\frac{1}{2}$ N. $\frac{1}{4}$ O. $\frac{1}{8}$ P. $-\frac{1}{4}$ Q. $\frac{3+\sqrt{13}}{2}$
R. $\frac{1+\sqrt{5}}{2}$ S. $2+\sqrt{5}$ T. $\frac{1+\sqrt{3}}{2}$ U. 1 V. $\frac{1+\sqrt{3}}{2}$
W. impossible X. impossible