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ABSTRACT

This study is one of several conducted by the authors in their investigation of the use of "higher order rules" in the solution of problems. The focus of the current experiment was determination of the compatibility of identified rules with the knowledge of average teenagers, and of the extent to which instruction in higher order rules facilitate performance on geometric construction problems. Higher order rules were presented as flow charts defining four paths. Thirteen construction problems, each solvable by using these rules, were defined. Coincidentally, these problems involved 13 lower order rules. Thirty college students, enrolled in a geometry course, served as subjects. Two pretests, three posttests, and seven instructional sessions were used in a repeated measures design; the sequences in which higher order rules were studied defined the treatments. Overall, findings indicated that instruction on higher order rules was effective and was relatively efficient. Sequential effects were also discovered. (SD)

# DIAGNOSIS AND INSTRUCTION OF HIGHER ORDER RULES FOR SOLVING GEOMETRY CONSTRUCTION PROBLEMS

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The viability of systematic analyses of real problem domains in terms of rules and higher order rules has been demonstrated by Scandura, Durnin, and Wulfeck (1974) and Durnin and Scandura (1973). The practical importance of such analyses in such areas as artificial intelligence and education, however, is still an open question. Although an attempt was made to insure in those analyses that the rules identified reflect human knowledge, it was not demonstrated there that they do. Rigorous tests of this thesis require experimental data. Furthermore, even if the rules do turn out to be compatible with what human subjects are likely to know, it is not clear whether, and to what extent, instruction in the higher order rules will result in improved problem solving performance.

The research reported in this paper deals with these questions in education with respect to the geometry construction analyses in Scandura et al. (1974). Specifically, the purpose of this research was to determine the extent to which: (1) the basic higher order rules identified in the analyses are compatible with the knowledge had by a group of average ability teenagers, (2) instruction in the higher order rules facilities performance on geometry construction problems and (3) instruction in some higher order rules influences (i.e., facilitates or hinders the learning and/or use of) subsequent ones.

A total of four paths of the two loci, similar figures, and auxiliary figures higher order rules were considered in the study. According to the analysis given in Durnin and Scandura (1973), the central question in determining behavioral compatibility is whether the paths of the higher order rules act in (near) atomic fashion. The major task, here, is to determine whether the ability, or lack thereof, to appropriately combine available lower order rules in one problem situation is reflected in other problem situations of the same type. This ability can be determined either directly in higher order task situations, where the subject is required to derive solution procedures for given problems, or indirectly, as we have done below, by asking the subject to actually solve problems (i.e., derive solution procedures and then use them).

In theory, when paths of a rule act in atomic fashion with a given population of subjects, inadequacies determined through testing can be overcome through direct instruction on the paths involved. If a person can solve problems whose solution rules require one path of the similar figures higher order rule, but not problems involving the other path, for example, then instruction presumably would be required only on the latter path.

Previous research provides more or less definitive guide lines on how to proceed and what to expect with regard to the first two questions. Thus, the research reported in Durnin and Scandura (1973) suggests that introspection as to how one actually solves a class of problems often results in the identification of procedures which appropriately partition the class (into equivalence classes). Although we know of no empirical research in the literature which bears on the second goal, members of the MERG group have developed materials for diagnostic testing and remediation in the arithmetical skills which are based directly on these ideas (for details, see Scandura, 1972). Formal data concerned with instruction have not been obtained but informal tryouts attest to the effectiveness of the remedial materials.

With regard to the third goal, very little can be said on the basis of available evidence. The fact that various paths of the higher order rules share many steps in common suggests that there might be positive transfer from one path to another. Thus, having learned one path, there is apt to be less to learn on subsequent ones so that learning them will require less time. On the other hand, one could argue that similarities among the higher order paths could result in interference. In attempting to generate solution procedures to given problems, the subjects might use the wrong paths.

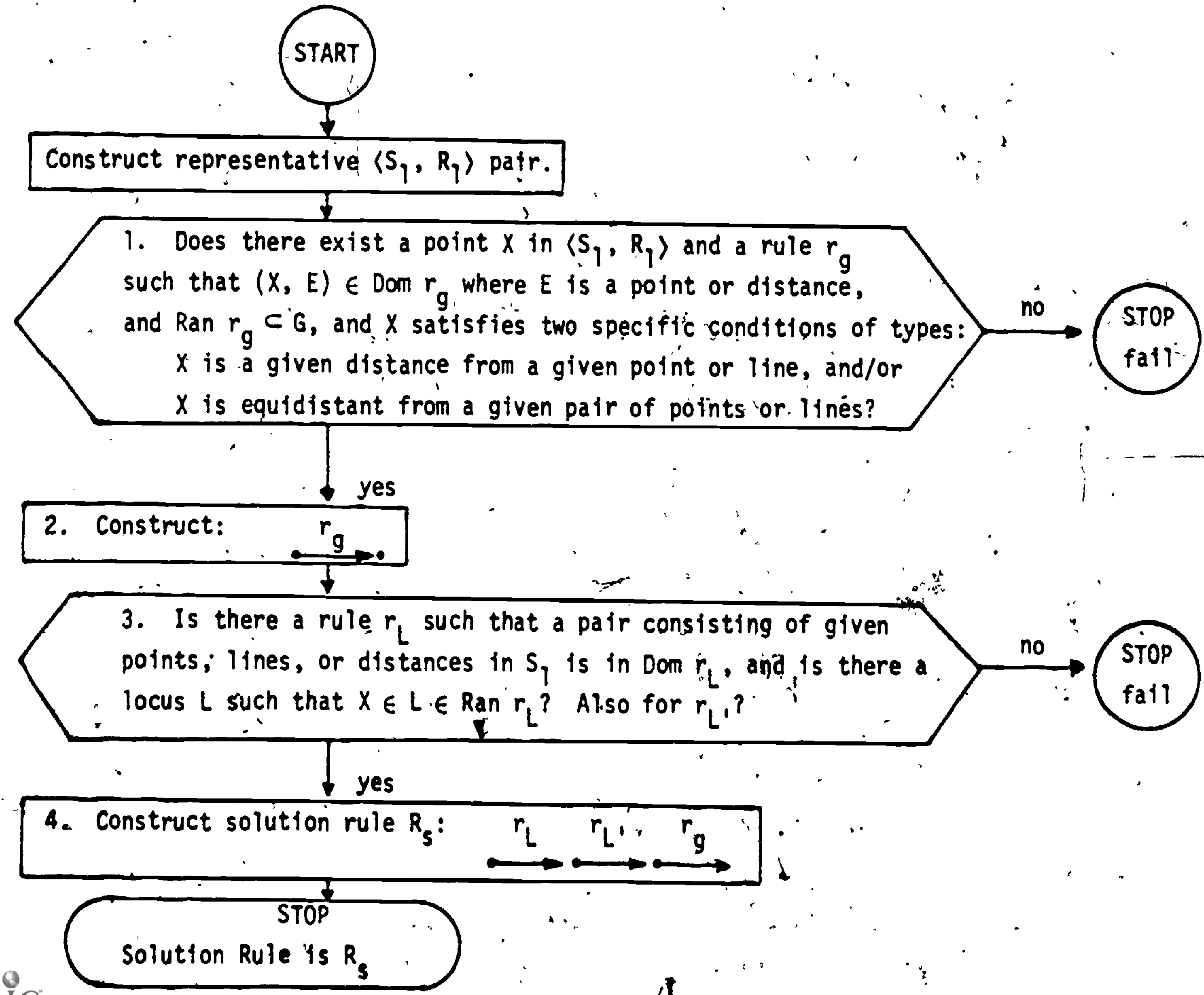
The present study was designed both to provide answers to the first two questions and to determine transfer and relative learning efficiency resulting from prior training.

## METHOD

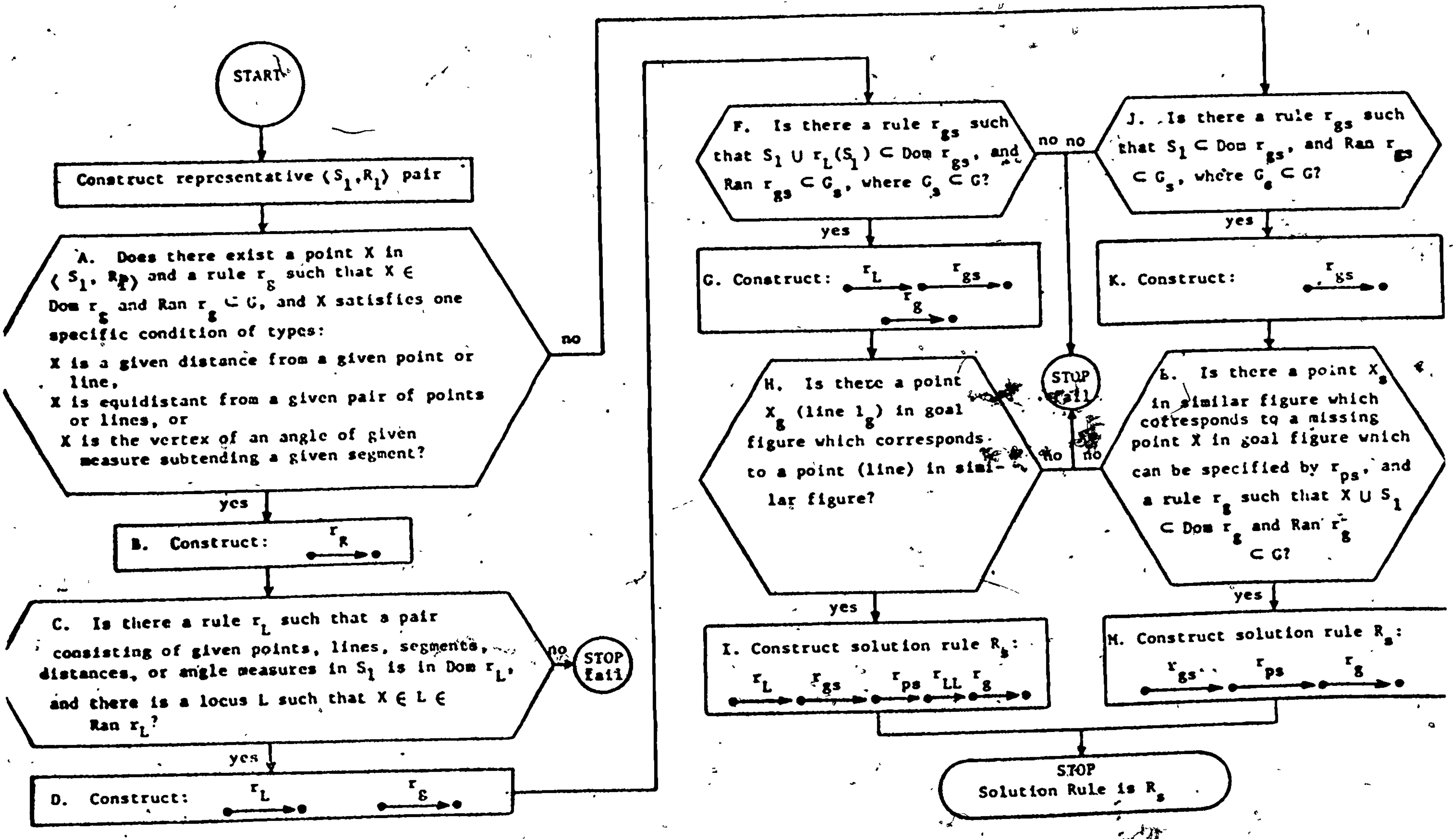
### Tasks and Materials

Four paths of three higher order rules were considered in the study. Figure 1 depicts the two loci rule, and Figure 2 depicts the two paths of the similar figures rule in Scandura et al. (1972). Path 1 (restricted similar figures) involves steps J, K, L, and M, and path 2 involves steps A, B, C, D, F, G, H, and I. The fourth path involves the auxiliary figures rule (Scandura et al., 1974). For instructional purposes, the decisions and operations of each path were written, respectively, as simple lists of questions and imperative statements.

FIGURE 1



## FIGURE 2


START
Construct representative $(S_1, R_1)$ pair
A. Does there exist a point X in $\langle S_1, R_1\rangle$ and a rule $r_g$ such that $X \in$ Dom $r_g$ and Ran $r_g \subset G$, and X satisfies one specific condition of types:
X is a given distance from a given point or line,
X is equidistant from a given pair of points or lines, or
X is the vertex of an angle of given measure subtending a given segment?
no
yes
B. Construct: $r_R$
C. Is there a rule $r_L$ such that a pair consisting of given points, lines, segments, distances, or angle measures in $S_1$ is in Dom $r_L$, and there is a locus L such that $X \in L \in$ Ran $r_L$?
no
STOP fail
yes
D. Construct: $r_L$ $r_g$
F. Is there a rule $r_{gs}$ such that $S_1 \cup r_L(S_1) \subset$ Dom $r_{gs}$, and Ran $r_{gs} \subset G_s$, where $G_s \subset G$?
no no
J. Is there a rule $r_{gs}$ such that $S_1 \subset$ Dom $r_{gs}$, and Ran $r_{gs} \subset G_s$, where $G_s \subset G$?
yes
G. Construct: $r_L$ $r_{gs}$ $r_g$
yes
K. Construct: $r_{gs}$
H. Is there a point $X_g$ (line $l_g$) in goal figure which corresponds to a point (line) in similar figure?
STOP fail
no
no
L. Is there a point $X_s$ in similar figure which corresponds to a missing point X in goal figure which can be specified by $r_{ps}$, and a rule $r_g$ such that $X \cup S_1 \subset$ Dom $r_g$ and Ran $r_g \subset G$?
yes
I. Construct solution rule $R_s$: $r_L$ $r_{gs}$ $r_{ps}$ $r_{LL}$ $r_g$
yes
M. Construct solution rule $R_s$: $r_{gs}$ $r_{ps}$ $r_g$
STOP
Solution Rule is $R_s$

The experimental tasks were 13 geometry construction problems (see Table 1) taken from Scandura et al. (1974). These problems may be categorized according to which of the four paths of the higher order rules may be used to generate an appropriate solution rule. The two loci higher order rule constituted one path; the similar figures rule yielded two paths (restricted similar figures and similar figures); the auxiliary figures rule yielded one.

### TABLE 1

### Geometry Construction Tasks

| Test | Problem Number | Solution Type | Problem Statement |
|---|---|---|---|
| Pretest I | 1 | Two loci | Given a line and a point not on the line, and a radius R, find a circle having the given radius R, which is tangent to the line, and passes through the given point. |
| Pretest I | 2 | Path one similar figures | Given angles B and C and the altitude $H_a$, construct the triangle. |
| Pretest I | 3 | Two loci | Given side a, and the median $M_a$, and the height $H_a$, construct the triangle. |
| Pretest I | 4 | Path one similar figures | Given angles B and C and the angle bisector $D_a$, construct the triangle. |
| Posttest I | 5 | Two loci | Given sides a and b and the median $M_a$, construct the triangle. |
| Posttest I | 6 | Path one similar figures | Given angles B and C and side b opposite angle B, construct the triangle. |
| Posttest II | 7 | Two loci | Given two intersecting lines and a radius R, construct a circle with radius R tangent to the two given lines. |
| Posttest II | 8 | Path one similar figures | Given angles B and C and the median $M_a$, construct the triangle. |
| Choice Test | 9 | Two loci and Path one similar figures | Given right triangle ABC with right angle at B, inscribe a square in it such that two sides of the square lie on the legs (AB and BC) of the triangle and the fourth vertex of the square (the intersection of the other two sides) is on AC. |
| Pretest II | 10 | Path two similar figures | Given two intersecting lines m and n and a point A not on either line, construct a circle tangent to lines m and n which passes through point A. |
| Pretest II | 11 | Path two similar figures | Given two intersecting lines m and n and a point P on line m, construct a circle whose center is on line m, which passes through point P and is tangent to line n. |
| Posttest III | 12 | Path two similar figures | Given line m and points A and B on the same side of line m, construct a circle tangent to line m which passes through points A and B. |
| Posttest III | 13 | Auxiliary figures | Given sides a and c and the altitude $H_b$, construct the triangle. |

The lower order rules needed (in addition to the higher order rules) for solving the experimental problems are shown in Table 2. For purposes of instruction, these rules were refined to the level of actual compass settings and placements. Each consisted of a sequence of operations to be performed. Accompanying sketches illustrated the result of each operation.

TABLE 2

Lower Order Rules

| | | |
|---|---|---|
| 1. | Circle rule | Construct the locus of points at a given distance from a given point. |
| 2. | Median locus circle rule | Construct the locus of points at a given distance from the midpoint of a given segment. |
| 3. | Point-line circle rule | Determine the distance between a given point and a given line and then construct the locus of points at the obtained distance from the given point. |
| 4. | Parallel line rule | Construct the locus of points at a given distance from a given line. |
| 5. | Angle bisector rule | Construct the locus of points equidistant from two given intersecting lines. |
| 6. | Triangle rule | From a point not on a given line segment, draw segments to the endpoints of the given segment (i.e., construct a triangle given a side and an opposite vertex). |
| 7. | Perpendicular bisector rule | Construct the locus of points equidistant from two given points. |
| 8. | Similar triangle rule | Construct an arbitrary triangle from a pair of given angles, and construct on it parts corresponding to other given segments. |
| 9. | Goal triangle rule | Construct a triangle having some part a given length similar to a given triangle with a corresponding part. |
| 10. | Point of similarity rule | Select a point of intersection of two lines through corresponding points of goal and similar figures as the point of similarity, then construct a line through the point of similarity and a point on the similar figure, to intersect the goal figure at a corresponding point, from which the goal figure may be constructed. |
| 11. | Similar square rule | Construct an arbitrary square in a right triangle with two of its sides contained in the legs of the triangle. |
| 12. | Goal square rule | Given a right triangle and a point on its hypotenuse, construct a square with that point as one vertex, such that its two opposite sides are contained in the legs of the triangle. |
| 13. | Similar circle rule | Construct an arbitrary circle with its center on one line and tangent to another line. |

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*The fact that there were 13 tasks and 13 lower order rules is strictly happenstance. The only connection is that the senior author was married on August 13 and is in the 13th year of marriage.

Descriptions of all problems and rules were reproduced on 21.59 cm. x 27.94 cm. (8 1/2" x 11") paper. Each problem appeared on a separate page so that constructions could be done on that page. The 13 problems were arranged into six separate tests as shown in Table 1.

The instructional materials were arranged into seven training booklets. Booklet 1 contained lower order rules 1-10 and a sample task for each. Booklet 2 contained 10 review tasks for the rules of Booklet 1. Booklet 3 contained path 1 of the two loci higher order rule, with the two Pretest I, two loci problems as practice. In parallel fashion, Booklet 4 contained path 1 of the similar figures higher order rule with the two Pretest I, similar figures problems as practice. Booklet 5 contained lower order rules 11 and 12 of Table 2 with corresponding practice tasks and Booklet 6 contained Rule 13 from Table 2 with a practice task. Booklet 7 contained the second path of the similar figures higher order rule along with the two problems from Pretest II as practice.

Pencils, compasses, and straightedges were available where subjects did not provide their own.

### Subjects, Design, and Procedure

The subjects were 30 Trenton State College students enrolled in an undergraduate college geometry class.

A repeated measures design was used. The first phase involved lower order rule training (Booklets 1 and 2) and Pretest I. Its main purpose was to obtain information regarding the adequacy of the two loci higher order rule and path one of the similar figures rule as a basis for assessing the (higher order) behavior potential of subjects. A secondary purpose was to obtain success or failure profiles, so that the subjects could be stratified before assignment to experimental groups.

The first meeting with the subjects occurred during a regularly scheduled 75-minute class period. One instructor and two experimental assistants were available to help the subjects and to evaluate their work. They were given Booklet 1 and instruction on lower order rules 1-10 contained in it. The steps of each rule were read aloud and the corresponding constructions were performed on the blackboard. Each subject then completed the corresponding practice problem.

During a second regular class meeting, the subjects were given the practice problems in Booklet 2 and were required to perform at least one correct construction for each of rules 1-10. Achievement of this criterion level was verified by one of the experimenters. As soon as they reached criterion, individual subjects were given Pretest I. All subjects were instructed to attempt all problems in Pretest I before the period ended; no subject "ran out of time." Pretest problems were scored "passed" if a correct solution figure was constructed. Minor deviations ("compass errors") were allowed. Each pretest was scored individually by three experimenters; there was no disagreement. The pretest results were used to stratify the subjects as shown in Table 3.

TABLE 3

Pretest Results

| Similar Figures Problems | Two Loci Problems | | | |
|---|---|---|---|---|
| | Passed Both | Passed One | Passed None | Σ |
| Passed Both | 6 | 1 | 0 | 7 |
| Passed One | 2 | 1 | 2 | 5 |
| Passed None | 2 | 6 | 10 | 18 |
| Σ | 10 | 8 | 12 | 30 |

On the basis of the pretest results, the subjects were randomly divided into two groups of 15 each, the two-loci-then-similar-figures (TS) group and the similar-figures-then-two-loci (ST) group, with the constraint that each of the cells in Table 3 was split evenly. (The two "singleton" subjects were placed in different groups.) Individual or small group sessions were arranged with each subject for all subsequent training and testing. Throughout the experiment, each subject retained all instructional materials, but not completed tests.

During the remainder of the study, instruction was provided on three of the four higher order paths and performance was measured on both within and extrascope problems. At the third meeting, the TS subjects were given Booklet 3, path one of the two loci higher order rule, and instruction in how to apply the rule. Specifically, they were shown how to determine whether particular lower order rules from Booklet 1 (which was available) were relevant to solution, and if they were, how to combine them so as to generate solution rules for the Pretest I, two loci problems (1 and 3). No actual constructions were performed. (One TS subject failed to attend this or any other instructional session and was dropped from the study.) The subjects in group ST received Booklet 4, path one of the similar figures higher order rule, and instruction on the application of that rule using problems 2 and 4 from Pretest I. After three subjects had been trained, the instruction was modified slightly so that additional emphasis was given to the stopping decisions (i.e., to conditions where the rule did not apply).

Immediately following instruction each subject was given Posttest I. The subjects in both treatment groups received exactly the same problems. Booklet I containing statements of rules 1-10 was available throughout. Also Booklets 3 and 4 containing the higher order rules were available to the subjects in groups TS and ST, respectively. Following Posttest I, one subject in the ST group became ill (no causal relationship implied) and had to be dropped from the experiment. (The two subjects dropped from the study had both failed all pretest problems, and had been assigned to different training groups.)

At the fourth meeting, those subjects who had received the two loci training received path one, similar figures training and vice versa. Instruction was given exactly as before. Posttest II paralleled Posttest I and followed immediately after training. Booklets 1, 3 and 4 were available to the subjects throughout the testing.

At each subject's fifth meeting, he was given Booklet 5 containing two new lower order rules (11 and 12), and training proceeded as with Booklet 1. With all previous training booklets available, the subjects then took the Choice test problem. (This problem could be solved by either of the two higher order rules on which the subjects had been trained.)

Next, at the sixth meeting, subjects were trained on the lower order rule (13) in Booklet 6. With this rule and all previously learned rules also available, the subjects then took Pretest II. The purposes of Pretest II were similar to those of the first Pretest, but dealt with the second path of the similar figures higher order rule. (At this point, two additional subjects who were failing the course, dropped out of the study. The remaining 26 subjects completed the experiments, 13 in each group.)

Finally, at the seventh meeting, each subject was trained as before on Booklet 7, the second path of the similar figures higher order rule, using the problems in Pretest II. After training, the subjects were given Posttest III. One problem of Posttest III was within the scope of the second similar figures path; the other was an auxiliary figures problem not solvable by using any of the three higher order paths on which instruction was provided.

Approximate times required by each subject were recorded for each session of the experiment.

## RESULTS AND DISCUSSION

### Assessment Results

Pretests I and II contained a total of six problems grouped on an a priori basis according to their solvability via the three higher order paths on which training was provided. To test the behavioral atomicity of the identified higher order rules, contingencies among within-class (path) problems were examined. Table 4 presents the Pretest I and II results on the three classes of problems. On Pretest I, the subjects' performance on the first two loci problem (problem 1 from Table 1) was significantly correlated with performance on the second two loci problem (problem 3) (Fisher's exact probability = .00485; one tailed), and similarly for the path 1, similar figures problems (2 and 4) (exact probability = .00165; one tailed) and, on Pretest II, for the path 2 problems (10 and 11) (exact probability = .00794; one tailed).

TABLE 4

Results of Pretests I and II

| | Pretest I | | | | Pretest II | |
|---|---|---|---|---|---|---|
| | Two loci<br>Problem 2 | | Path 1 of similar figures<br>Problem 2 | | Path 2 of similar figures<br>Problem 2 | |
| Problem 1 | Pass | Fail | Pass | Fail | Pass | Fail |
| Pass | 10 | 1 | 7 | 3 | 2 | 1 |
| Fail | 7 | 12 | 2 | 18 | 0 | 25 |

These results strongly suggest that the identified paths, both collectively and individually, acted in atomic fashion for the experimental subjects. Some of the deviant cases, furthermore, are due to two particular subjects who initially were obviously uncooperative but later applied themselves. Nonetheless, the relatively large number (5) of remaining "fail-pass" cases on the two-loci problems requires some discussion. In particular, this result suggests the possible desirability of further refinement of the two-loci higher order rule into a larger number of distinct paths. This would require analysis of the atomic operators and paths in terms of sub-operators and sub-decisions and, thereby, substitution of a number of paths with more limited domains for the original path. Because the various decisions of this rule involve disjunctions of properties, a basis for such refinement follows directly. The second decision making capability of the two loci higher order rule, for example, refers to a disjunction (A or B or C) of properties, any one of which, if satisfied, is sufficient to direct a computation to a particular sub-operator. With some subjects, at least, it is certainly possible that the ability to decide, say, on whether there is a rule containing a point and a line in its domain is independent of the ability to decide on whether a domain contains segments or angle measures. In this case, the ability to solve the problem involving property A would say nothing as regards the ability to solve a problem involving property B, as was the case with problems 1 and 3. In effect, such refinement would not only be consistent with our results but would follow directly from our analysis.

## Instructional Effectiveness

Over the entire experiment there were 51 cases where subjects failed all pretest problems from a given class prior to training on the path corresponding to that class. On posttests immediately following such training, new problems from the same classes were solved in 45 (88.2%) of the 51 cases. (The 95% confidence interval for this percentage is 79.3% to 97.1%.) In addition, there were 14 cases where subjects had solved only one pretest problem in a given class. The (new) posttest problems were solved in 13 of those cases.

Table 5 summarizes the results from Posttests I, II, and III on problems for which training immediately preceeded testing, arranged according to the number of pretest problems passed.

TABLE 5

Results on Posttest Problems Within Scope of Immediately Preceeding Training

| Number of pretest problems within scope of training passed prior to training | N | Number of Ss passing within scope problem after training |
|---|---|---|
| **Posttest I** | | |
| 0 | 14 | 11 |
| 1 | 6 | 6 |
| 2 | 9 | 9 |
| **Posttest II** | | |
| 0 | 13 | 12 |
| 1 | 7 | 6 |
| 2 | 8 | 8 |
| **Posttest III** | | |
| 0 | 24 | 22 |
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| **Total Cases** | | |
| 0 | 51 | 45 |
| 1 | 14 | 13 |
| 2 | 18 | 18 |

After training on either the two loci higher order rule or path one of the similar figures rule, 11 of the 14 subjects who failed both Pretest I problems in a given class succeeded on the corresponding Posttest I problem (binomial, $p < .05$). Not surprisingly, all six subjects who solved one Pretest 1 problem and all nine subjects who had solved both Pretest I problems also solved the Posttest I problem.

On Posttest II, 12 of 13 subjects who failed both pretest problems in a given class succeeded after training on the higher order rule for that class. Of seven subjects who solved one pretest problem, six solved the corresponding Posttest II problem after training. All eight subjects who solved both pretest problems succeeded following training. On Posttest III, 22 of 24 subjects who failed both Pretest II problems succeeded following training. Two other subjects, who had solved one or both pretest problems, also succeeded on the posttest problem.

Overall, there were seven cases of posttest failure out of 83 cases where success was expected. Five of the seven discrepancies occurred on Posttests I and II. After failure, these five subjects were retrained on the respective higher order paths, and retested. All five succeeded on the second trial.

Inspection of the seven individual problem attempts which resulted in failure showed that, in six of those cases, mistakes occurred at points corresponding to disjunctive decision points in the two loci rule or in path 2 of the similar figures rule. As noted earlier, disjunctive decisions may be broken up (refined) to form separate paths. This suggests that more explicit attention to the molar nature of such (disjunctive) decisions might possibly have reduced even the small number of inconsistencies noted.

Instruction in the higher order rules was not only effective but also was relatively efficient. Subjects were able to solve relatively complex construction problems, once they knew the component rules involved, after only about 75 minutes of higher order rule training. Instruction on the lower order rules took an average of about 100 minutes. In all, less than three hours of actual instruction was required.

## Sequence Effects

In addition to the positive assessment and instruction results, a number of interesting sequence effects were found. On Posttest I, positive transfer to problems, for which training had not been provided, occurred in approximately 40 per cent of the cases. Of 13 subjects who had failed both Pretest I problems in the class for which no training was given, five solved the corresponding Posttest I problem. (Two subjects solved the two loci problem; three solved the path one, similar figures problem.) Three of seven subjects who had solved one of the untrained problems on Pretest I solved the Posttest I problem. (All three solved the two loci problem after path one, similar figures training.) On the other hand, one subject who had previously solved both Pretest I two loci problems failed the Posttest I two loci problem after being trained on the path one, similar figures rule. Apparently, this was due to his misunderstanding of the posttest instructions; the subject thought he was required to use the trained higher order rule (which was inadequate). After this misapprehension was corrected, the subject was retested and passed the problem. (It was at this point that more stress in the instruction was placed on when a higher order rule would not work -- i.e., when to stop.).

The results on Posttest II, restricted to the problem for which training had been given prior to Posttest I, suggests that training on the second higher order rule did not interfere with earlier training. Twenty-seven of the 28 subjects who completed Posttest II solved the problem corresponding to the first-trained rule, after second training. The remaining subject passed the problem on a second trial.

This lack of interference was also reflected in performance on the Choice problem, which could be solved using either the two loci or path one similar figures rules. All 28 subjects solved the problem. There were 17 two loci solutions and 11 path one similar figures solutions; this difference was not significant. Furthermore, there were no significant differences as to solution preference due to order of training. Group ST had eight and six, respectively. This suggests that when two (or more) rules are available at the time of problem solving (as they were), selections do not depend on when the rules were originally learned.

It is also of interest to note that the higher order rule selections after learning were in approximately the same 3:2 ratio as observed success on Pretest I, where 18 and 12 subjects, respectively, solved one or both of the two loci and similar figures problems. This observation suggests that subject preference after learning is somehow related to subject likelihood of having learned two loci-like higher order rules prior to training. Our data are inadequate to determine why this is so but it could have something to do with the involvement of similar, previously learned selection rules (cf. Scandura, 1974).

Positive transfer was also found from training on one higher order rule to the next. When given first, the two loci training required an average of about 32 minutes (25-60 min. range) and the similar figures training, 31 minutes (20-65 min. range). When the training came second, the corresponding times were 21 minutes (15-45 min. range) and 28 minutes (18-60 min. range). The third training session on path two of similar figures higher order rule required only about 18 minutes (10-30 min. range), even though the relatively large number of failures on the corresponding Pretest II problem suggests that this path was more difficult than the others. Overall, then, ignoring the particular training involved, first training took an average of 32 minutes; second training, 25 minutes; and third training, 18 minutes. Differences among these means were highly reliable ($F_{2, 47} = 34.70$, $p < .005$). Individual comparisons of first and second training and second and third were also reliable ($t_{25} = 4.85$, $P < .001$ and $t_{25} = 4.14$, $P < .001$, respectively).

Rather surprisingly, performance on the auxiliary figures problem, which could not be solved using any of the three trained higher order rules, depended on the sequence in which the higher order rules were learned ($\chi^2 = 7.58$, df = 1, $p < .01$). Eleven of the 13 TS subjects solved it while only three of the 13 ST subjects succeeded. No other relation was observed between the results on this problem and on any of the previous ones.

It is impossible to say with any certainty the source of this rather striking sequence effect. One possibility is that, in attacking the problem, subjects may have tended to select the first higher order rule on which they were trained. In this case, the TS subjects could have had an advantage because the auxiliary figures problem may be solved by repeated application of a variant of the two loci rule. Equivalently, it is possible that the subjects combined the higher order rules into more encompassing rules (see the combined two loci-similar figures rule in Scandura et al., 1974) as they were learned. If so, those subjects who tried the two loci path of the combined rule first, most likely the TS subjects, would again have an advantage, especially if the effects of limited memory are taken into account (cf. Voorhies & Scandura, 1973).

Both of these explanations, unfortunately, imply differential solution type preferences on the choice test problem. Since no such effect occurred, some alternative accounting seems necessary. One plausible explanation stems from the fact that the similar figures and auxiliary figures higher order rules may be regarded as progressive generalizations of the two loci rule. That is, all of the higher order rules begin by identifying constructable elementary figures upon which further operations may act to generate a goal figure. In the two loci rule, the elementary figure is "the missing point X." In the similar figures rule, the elementary figure is more general; it is no longer a "degenerate" point, but is still constrained by similarity. Finally, in the auxiliary figures rule, the elementary figure is arbitrary. (For details, see Scandura et al., 1974.)

Because the TS subjects were taught the procedures in a "natural" order of generalization, while the ST subjects were not, the former may have been more likely to have "induced" a generalization procedure. More specifically, the TS subjects may have learned a "higher, higher order rule" for making generalizations. Such a rule could have been used to derive some form of auxiliary figures higher order rule, which in turn would have allowed derivation of an adequate solution rule.

## CONCLUSIONS AND IMPLICATIONS FOR MATHEMATICS EDUCATION

In view of the clarity of these results, it would appear that the identified higher order rules can be used effectively and efficiently both to diagnose difficulties subjects are having with geometry construction problems and to provide instruction in how to solve such problems. Furthermore, training on prior higher order rules seemed to facilitate the learning of later ones.

This is not the first time that beneficial effects have been found for instruction in heuristics. Ennis et al (1969), for example, found that training subjects in general heuristics such as means-ends analysis and planning improved mathematical problem solving. At the present time, Hatfield* also has a study under way in which he is trying to facilitate performance in mathematical problem solving by the informal (clinical) introduction of heuristics.

Although the present study is based on more rigorous and exacting analyses of heuristics in terms of higher order rules (as well as lower order ones), it must not be thought that these varying kinds of studies are incompatible. The present research is completely neutral as regards how information is to be imparted. Motivating the child to learn and the actual mode of presentation in the classroom is up to the teacher's judgement. However, because our higher order rules have been so explicitly characterized, instruction in whatever form (including diagnosis of individual sources of difficulty) is potentially more efficient and feasible than with loosely formulated heuristics.

Although a considerable degree of transfer was evident from training on one higher order rule to another, it is still an open question as to whether explicit instruction in higher order rules (by whatever means) also helps the learner develop new "heuristics" on his own. Earlier, of course, Roughead and Scandura (1968) found that "what is learned" in making simple discoveries can be presented in expository form with equivalent results. In the present case, however, the task of identifying "what is (to be) learned" is far from trivial and, initially, may require more informal, inductive methods (cf. Lowerre and Scandura, 1974). Whatever the answer, there is certainly no reason why the teacher might not encourage discovery (of higher order rules) in addition to whatever explicit training is provided. Indeed, one good teaching strategy would appear to be to present a variety of situations where learners are required to discover higher order rules. Even in the present study, the higher order rules were not taught explicitly as formal (some would say "rote") procedures. Representation of the rules as flow diagrams simply made the experimenters more aware of exactly what it was that was to be taught.

In spite of the positive nature of these results, it should not be forgotten that they deal primarily with the question of how subjects perform in particular problem solving situations given what (rules) they know on entering into the situation. Any complete prescription for problem solving instruction must deal in detail with the course of solving whole classes of problems. Our findings concerning the sequential effects of instruction on higher order rules has demonstrated the importance of such study, and is one step in this direction, but it is a small one indeed. Consider the complications introduced in considering a continually changing set of lower order rules (as learning progresses), not to mention the difficulties in attempting to explicate precisely the source of the sequential effects we observed with the higher order rules. Nonetheless, we are optimistic concerning the progress that might be made in this direction, and considering the obvious implications for mathematics education, to use a time-worn phrase, "we had better begin."

*Personal communication.

# References

Durnin, J. H. and Scandura, J. M. An algorithmic approach to assessing behavioral potential: Comparison with item forms and hierarchical technologies. Journal of Educational Psychology, 1973, 65, No. 2, 262-272.

Ennis, R. H., Finkelstein, M. R., Smith, E. R., and Wilson, N. H. Conditional logic and children. Ithaca, N.Y.: Cornell Critical Thinking Project, 1969.

Lowerre, G. and Scandura, J. M. Conceptually based development of individualized materials for critical thinking based on logical inference. Reading Research Quarterly, 1973-74, IX, 186-205.

Roughead, W. G. and Scandura, J. M. "What is learned" in mathematical discovery. Journal of Educational Psychology, 1968, 59, 283-289.

Scandura, J. M. Rule selection, problem definition (subgoal formation), and retrieval in problem solving. University of Pennsylvania Structural Learning Series Report 79, 1974.

Scandura, J. M. Structural learning I: The theory and empirical research. New York: Gordon and Breach Science Publishers, 1973.

Scandura, J. M. A plan for the development of a conceptually based mathematics curriculum. Instructional Science, 1972, 247-263.

Scandura, J. M., Durnin, J. H., Wulfeck, W. H. II. Higher order rule characterization of heuristics for compass and straightedge constructions in geometry. Artificial Intelligence, 1974, in press.

Voorhies, D. and Scandura, J. M. Determination of memory load in information processing, 1973.