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ABSTRACT

A hierarchy for learning to solve different types of addition with fractions problems was hypothesized on the basis of both content analysis and psychological considerations. Problem types were defined according to the relationship of the two denominators to each other (e.g., equal, prime, etc.) Students in grades 4 through 8 were each given 45 additional problems to perform. Papers which were totally correct or totally incorrect were deleted leaving a sample of 200. These papers were analyzed using both the Walbesser Technique and Pattern Analysis. No ordering of the tasks was found to yield acceptable levels for all of the Walbesser ratios (consistency, adequacy, completeness). However, with few exceptions, task comparisons yielded acceptable values on two of the three ratios. The empirically determined sequence was analyzed and seven implications for teaching addition with fractions were determined. (SD)

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AN INTRACONCEPT ANALYSIS OF RATIONAL NUMBER ADDITION: A VALIDATION STUDY

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In previous studies, researchers have attempted to generate learning hierarchies using task analysis based primarily on epistemological considerations (Gagne & Paradise, 1961; Gagne, 1962; Cox & Graham, 1966; Uprichard, 1970; Okey & Gagne, 1970; Harke, 1971; Riban, 1971; Phillips & Kane, 1973; Miller & Phillips, 1975). Studies of this type conducted in the early sixties provide substantial evidence to support the hierarchical structure of knowledge (Gagne & Paradise, 1961; Gagne & Brown, 1961; Gagne, 1962, 1963' Gagne, Mayor, Garstens & Paradise, 1962; Gagne & Staff, 1965). An examination of results from recent studies (Neidermeyer, Brown & Sulzen, 1969; Brown, 1970; Phillips & Kane, 1973; Callahan & Robinson, 1973) suggests that optimal learning sequences can be developed by sequencing instructional materials according to validated learning hierarchies. However, both Gagne (1968) and Pyatte (1969) have pointed out that the determination of an optimal or hierarchical sequence of subtasks from simplest to most complex is not easily achieved.

Numerous hierarchy validation techniques have appeared in the literature. Critical analyses of the efficacy of these techniques are also reported (Resnick & Wang, 1969; Eisenberg & Walbesser, 1971; White, 1973; White, 1974a, 1974b: Phillips, 1971, 1974). Many of these techniques are concerned with the analysis of data collected on hypothesized hierarchies. While there is considerable room

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for improvement, techniques such as the Walbesser method (Walbesser, 1968) and Pattern analysis (Rimoldi & Grib, 1960) have been used effectively in validating hypothesized hierarchies (Harke, 1971; Phillips & Kane, 1973; Miller & Phillips, 1975).

The investigators in the present study were concerned with the generation and validation of an hypothesized learning hierarchy. More specifically, the purposes of the study were (1) to develop a learning hierarchy for rational number addition using the intraconcept analysis technique, and (2) to test the validity of the hypothesized hierarchy using the Walbesser Technique and Pattern Analysis. The objective in doing an intraconcept analysis is to generate a series of tasks, each of which represent an operationalized level of a given concept. This technique differs from the more widely used task analysis model in that consideration is given to both psychological and content (discipline) factors in developing and hierarchically ordering tasks. Because of this, the authenticity of many tasks generated using the intraconcept analysis technique may be questioned by content area specialists. However, many educators (including the authors), who have worked in clinical situations with young children experiencing difficulty in learning mathematics question whether one can simply impose mathematical or adult logic on children. The present study was conducted in hopes of shedding some light on the epistemological-psychological balance needed in developing efficient or optimal instructional sequences in mathematics.

PROCEDURE

Development of Hierarchy

The addition of rationals in fraction form was analyzed using an intraconcept analysis technique. The specific strategy used in the analysis was as follows:

a) Problems were divided into two levels: Like Denominators, and Unlike Denominators. Classes within each level were identified by the nature of (prime and composite) and relationship between (multiple or not) denominators of the two rationals added. The classes, in their hypothesized order of difficulty, were:

Like Denominators

I. Composites. (1/4 + 2/4 = ___)

II. Primes. (1/5 + 1/5 = ___)

Unlike Denominators

III. Composites- one composite multiple of other. (1/4 + 1/8 = ___)

IV. Prime, composite- composite a multiple of prime. (1/5 + 1/10 = ___)

V. Primes. (1/5 + 1/7 = ___)

VI. Prime, composite- composite not multiple of prime. (1/5 + 1/9 = ___)

VII. Composites- not multiples. (1/6 + 1/8 = ___)

b) Within each denominator class an attempt was made to generate five sum categories.[1] The sum categories were operationally defined in terms of renaming. The five categories in their hypothesized order of difficulty were:

I. Sum- proper fraction in simplest form without renaming. (1/4 + 2/4 = 3/4)

II. Sum- improper fraction renamed to one (1). (1/4 + 3/4 = 4/4 = 1)

III. Sum- proper fraction renamed to simplest form. (1/4 + 1/4 = 2/4 = 1/2)

IV. Sum- improper fraction renamed to mixed numeral in simplest form,one renaming. (3/4 + 2/4 = 5/4 = 1 1/4)

V. Sum- improper fraction renamed to mixed numeral in simplest form, two renamings. (3/4 + 3/4 = 6/4 = 1 2/4 = 1 1/2)

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[1]Five sum categories do not exist in some classes.

c) Two tasks involving mixed numerals (whole number sums less than or equal to nine) were associated with each sum category within a class. A sum category and its two related tasks is presented below.

Sum Category I	Task #1	Task #2
1/4 + 2/4 = ___	1 1/4 + 2/4 = ___	1 1/4 + 1 2/4 = ___

Analysis of Hierarchy

The procedure as described yielded eighty-nine specific tasks. In order to hold testing to a manageable size, some tasks within classes were collapsed. The final hierarchy consisted of 45 tasks including (a) all existing sum categories within identified classes and (b) a sampling of tasks involving mixed numerals within classes. Based upon the hypothesized ordering of the subordinate tasks, a test was constructed to assess mastery at each level in the hierarchy. "Pass" was defined as correct sum expressed in simplest form. For ease in computation the tasks appeared in vertical form.

The test was administered to 251 students in grades 4 through 8 in order to obtain a wide range of ability levels. The majority of the Ss were in grades 5 and 6. Subjects who passed or failed all 45 items were excluded from the study since inferences about the order of items cannot be based on responses from these students (Phillips & Kane, 1975). The resulting sample contained 200 Ss.

The patterns of responses for each transfer in the hierarchy were analyzed using both the Walbesser Technique (Walbesser, 1968) and Pattern Analysis (Rimoldi & Grib, 1960). The Walbesser Technique is based on the 2 X 2 contingency table of pass-fail responses (Figure 1).

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Item 2 \ Item 1	fail (-)	pass (+)
pass (+)	-+	++
fail (-)	--	+-

Figure 1. Contingency Table

Using the frequency of students falling in each cell above, the following three ratios were computed for every possible relationship in the hierarchy (i.e. level 1 with 2, 1 with 3, ... 1 with 45; 2 with 3, 2 with 4, ... 2 with 45, etc.).

(1) Consistency Ratio = $\frac{(++)}{(++) + (+-)}$

The value of this ratio is a measure of how consistent the data are with the hypothesized dependency.

(2) Adequacy Ratio = $\frac{(++)}{(++) + (-+)}$

The value of this ratio is a measure of the adequacy of the identified levels.

(3) Completeness Ratio = $\frac{(++)}{(++) + (--)}$

The value of this ratio is a measure of the effectiveness of instruction.

The level of acceptability used for each of these ratios was that determined by Phillips (1971) instead of those proposed by Walbesser since no instructional sequences were involved. These levels are: (1) Consistency Ratio .85, (2) Adequacy Ratio .70, (3) Completeness Ratio .50.

The pattern analysis technique was used to analyze the responses for the complete hierarchy on a subject by subject basis. The index of agreement given by the pattern analysis indicates the amount of agreement or correlation between two patterns. In this case, the index of agreement indicates the agreement between the observed and expected patterns. If the tasks were truly hierarchical, where each subtask was a necessary prerequisite to the next, once a learner failed a given level he would be expected to fail all subsequent levels. Thus, the expected pattern was defined as one where no correct responses followed an incorrect response.

RESULTS

The initial hypothesized hierarchy developed using an intraconcept analysis is given in Table 1. A computer program based on the Walbesser Technique

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Insert Table 1 About Here

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was used to give the pass-fail response patterns between all relationships. That is, item 1 was paired with all 45 items; item 2 with all items; etc.; until all possible pairs of items were considered.

In order to analyze the hypothesized hierarchical sequence, the consistency, adequacy, and completeness ratios for each relationship within the hierarchy were examined. No ordering of the 45 tasks yielded acceptable levels in all three ratios. The ordering which yielded the best fit to the data is shown in Table 2.

Insert Table 2 About Here

The pattern of responses obtained from the hypothesized ordering of subordinate tasks yielded an index of agreement of .68. After final revision, the empirical sequence yielded an index of agreement of .70 which indicates a slightly higher agreement between the expected and observed response patterns. No statistical test of significance for the index of agreement has been developed.

The internal consistency of the test based on the initial hierarchy was determined using the Kuder-Richardson Formula 20 (Nunnally, 1967). The value of this coefficient was .98. The pattern of responses for the empirical sequence was analyzed to determine if the ordering exhibited a hierarchical structure based on item difficulty. Item difficulty of each task in the ordered sequence is given in Table 3. Although there would be some discrepancies

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Insert Table 3 About Here

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between a hierarchical sequence based on item difficulty and the ordered sequence determined, the general pattern of item difficulty of the latter is acceptable. Further, item difficulty alone is not considered an adequate technique for determining hierarchical relationships (Phillips, 1971).

Since many task comparisons involving mixed numerals resulted in low Walbesser ratios (see Table 2), an empirical sequence without mixed numerals was determined. The index of agreement for this sequence which is presented in Table 4 was .75.

Insert Table 4 About Here.

DISCUSSION AND IMPLICATIONS

An examination of data in Table 2 (Walbesser ratios) indicates that with few exceptions task comparisons meet specified criteria on two of the three

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ratios. The consistency ratio was met in 35 of 44 comparisons, the adequacy ratio in 43 of 44, and the completeness ratio in only 27 of 44. It should be noted, however, that the first two ratios reported are more critical when determining a hierarchy from test data. A considerable improvement in the number of comparisons meeting criteria on ratios was observed for the hierarchical sequence determined when tasks involving mixed numerals were omitted (Consistency, 26/28; Adequacy, 28/28; and Completeness, 20/28; see Table 4). The index of agreement for the twenty-nine task hierarchy (no mixed numerals) was greater than that of the forty-five task hierarchy .75 vs .70. This difference may be directly related to the number of tasks in each hierarchy.

The empirically determined sequence is presented in Table 5 in terms of

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Insert Table 5 About Here

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the defining characteristics of each task. An examination of the sequence tends to support the following implications for teaching youngsters the addition of rationals in fraction form.

1. Tasks involving like denominators should be taught before those involving unlike denominators, with one exception-sum category V.
2. In working with like denominators one should teach tasks within a given sum category before preceding to the next one. The sum categories in hierarchical order are I, II, IV, III, V.
3. Within a sum category tasks involving like-composite denominators should be taught before those with like-prime denominators.
4. In working with unlike denominators, one should teach tasks within a given sum category before preceding to the next one. The sum categories in hierarchical order are I, II, III, IV, V.

5. Tasks involving adding with mixed numerals should not be introduced early in the development of rational number addition skills.
6. Tasks involving adding with two mixed numerals within a denominator class and sum category should be taught before those tasks involving one mixed numeral.
7. Sequencing of unlike denominator classes is difficult and not completely determined by this study. However, a possible teaching sequence might be III, IV, VI, V, VII.

In general, the results of this study, though limited to the addition of rational numbers in fraction form, support the notion that both epistemological and psychological factors be considered when developing teaching sequences in mathematics. Some of the implications above would not necessarily be derived from logical analysis alone. Also, in interpreting the results of this study one must be conscientious of the limitations of indirect validation procedures. For example, confounding variables such as prior educational experience of subjects and errors of measurement must be considered.

TABLE 1

Initial Hypothesized Hierarchy

1. 1/4 + 2/4 = ___
2. 1/5 + 3/5 = ___
3. 2 1/6 + 4/6 = ___
4. 3 1/8 + 4 2/8 = ___
5. 1/4 + 3/4 = ___
6. 3/7 + 4/7 = ___
7. 1/6 + 1/6 = ___
8. 5/8 + 4/8 = ___
9. 2/3 + 2/3 = ___
10. 4 3/5 + 4/5 = ___
11. 5 3/7 + 1 5/7 = ___
12. 3/4 + 3/4 = ___
13. 2 3/6 + 5/6 = ___
14. 6 5/8 + 2 5/8 = ___
15. 1/3 + 1/9 = ___
16. 1/4 + 1/8 = ___
17. 2 1/12 + 2/4 = ___
18. 3 1/8 + 5 1/16 = ___
19. 1/3 + 1/5 = ___
20. 1/5 + 8/10 = ___
21. 3/4 + 3/12 = ___
22. 1/3 + 2/6 = ___
23. 1/4 + 2/3 = ___
24. 3 2/5 + 2/10 = ___
25. 4 1/3 + 3 3/9 = ___
26. 2/3 + 3/6 = ___
27. 5/8 + 7/10 = ___
28. 2/3 + 3/5 = ___
29. 5 3/5 + 3/7 = ___
30. 3 1/2 + 4 2/3 = ___
31. 2/3 + 6/9 = ___
32. 3/4 + 4/8 = ___
33. 1/3 + 1/10 = ___
34. 2/6 + 3/7 = ___
35. 1/4 + 1/6 = ___
36. 3/6 + 4/3 = ___
37. 6 4/8 + 5/10 = ___
38. 5 2/4 + 2 3/6 = ___
39. 1/6 + 3/10 = ___
40. 1/3 + 7/10 = ___
41. 3/8 + 11/12 = ___
42. 4/6 + 4/7 = ___
43. 3 2/3 + 5/10 = ___
44. 4 3/5 + 3 4/8 = ___
45. 3/4 + 3/6 = ___

TABLE 2

Walbesser's Ratios for the Empirical Ordering (N=200)

Level	Consistency	Adequacy	Completeness
1-2	.98	.96	.97
*2-4	.95	.76	.97
*4-5	.85	.84	.83
5-6	.99	.99	.76
6-8	.98	.84	.73
8-9	.97	.98	.65
9-7	.86	.85	.69
*7-11	.87	.79	.66
*11-3	.70	.70	.67
*3-10	.72	.60	.60
*10-16	.78	.80	.56
16-15	.95	.93	.59
15-33	.94	.88	.58
33-19	.93	.94	.55
19-35	.93	.91	.54
35-20	.94	.88	.55
20-21	.96	.94	.56
21-36	.97	.90	.53
36-12	.83	.83	.51
*12-14	.90	.90	.51
*14-13	.94	.96	.51
*13-23	.81	.88	.52
23-22	.89	.92	.53
22-39	.94	.80	.48
39-34	.81	.88	.41
34-27	.98	.72	.47
27-26	.95	.97	.54
26-40	.95	.90	.52
40-28	.85	.89	.49
28-41	.89	.80	.46
41-31	.92	.82	.46
31-32	.93	.95	.50
32-42	.93	.79	.46
*42-18	.90	.73	.48
*18-25	.93	.76	.48
*25-17	.60	.78	.35
*17-24	.61	.73	.32
*24-30	.77	.94	.44
*30-38	.94	.92	.48
*38-44	.88	.81	.41
*44-29	.88	.79	.42
*29-37	.87	.81	.42
*43-45	.74	.85	.36

*indicates comparison involving mixed numerals

TABLE 3

Item Difficulty-Empirical Hierarchy (45 Tasks)

Task	Difficulty	Task	Difficulty
1	.94	.31	.54
2	.91	.32	.53
*4	.74	.42	.43
5	.76	*.18	.55
6	.73	*.25	.45
8	.64	*.17	.35
9	.66	*.24	.42
7	.64	*.30	.51
*11	.59	*.38	.50
*3	.61	*.44	.42
*10	.48	*.29	.47
16	.59	*.37	.46
15	.59	*.43	.38
33	.57	.45	.41
19	.57		
35	.54		
20	.58		
21	.55		
36	.53		
12	.51		
*14	.51		
*13	.50		
23	.55		
22	.53		
39	.46		
34	.42		
27	.56		
26	.53		
40	.51		
28	.50		
41	.47		

*indicates problems involving mixed numerals

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TABLE 4

Walbesser's Ratios for the Empirical Ordering- No Mixed Numerals (N=200)

Level	Consistency	Adequacy	Completeness
1-2	.98	.96	.97
2-5	.95	.76	.96
5-6	.99	.99	.76
6-8	.98	.84	.73
8-9	.97	.98	.65
9-7	.86	.85	.69
7-12	.95	.74	.60
12-16	.76	.86	.56
16-15	.95	.93	.59
15-33	.94	.88	.58
33-19	.93	.94	.55
19-35	.93	.91	.54
35-20	.94	.88	.55
20-21	.96	.94	.56
21-36	.97	.90	.53
36-23	.92	.87	.53
23-22	.89	.92	.53
22-39	.94	.80	.48
39-34	.81	.88	.41
34-27	.98	.72	.47
27-26	.95	.97	.54
26-40	.95	.90	.52
40-28	.85	.89	.49
28-41	.89	.80	.46
41-31	.92	.82	.46
31-32	.93	.95	.50
32-45	.93	.79	.46
42-45	.86	.81	.40

TABLE 5

Empirical Order of Tasks by Sum & Denominator's

Denominator Classes		Sum Categories: I. Proper fraction simplest form, without measuring	II. Improper fraction, renamed to one	III. Proper fraction, renamed to simplest form	IV. Improper fraction, renamed to mixed numeral simplest form, one renaming	V. Improper fraction, renamed to mixed numeral simplest form, two renamings
LIKE	I. Composites	1	4	8	6	20
	a. One mixed numeral	10				22
	b. Two mixed numerals	3				21
	II. Primes	2	5		7	
	a. One mixed numeral				11	
	b. Two mixed numerals				9	
UNLIKE	III. Composites - one composite multiple of other.	12	18	23	27	33
	a. One mixed numeral	37				
	b. Two mixed numerals	35				
	IV. Prime, Composite - composite multiple of prime.	13	17	24	28	32
	a. One mixed numeral			38		
	b. Two mixed numerals			36		
	V. Primes	15			30	
	a. One mixed numeral				42	
	b. Two mixed numerals				39	
	VI. Prime, Composite - composite not multiple of prime.	14		26	29	34
	a. One mixed numeral					43
	b. Two mixed numerals					41
	VII. Composites - not multiples	16	19	25	31	45
	a. One mixed numeral		37			
	b. Two mixed numberals		40			

[1]Numerals in table do not correspond to test items.

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