DOCUMENT RESUME

ED 108 893 SE 019 075

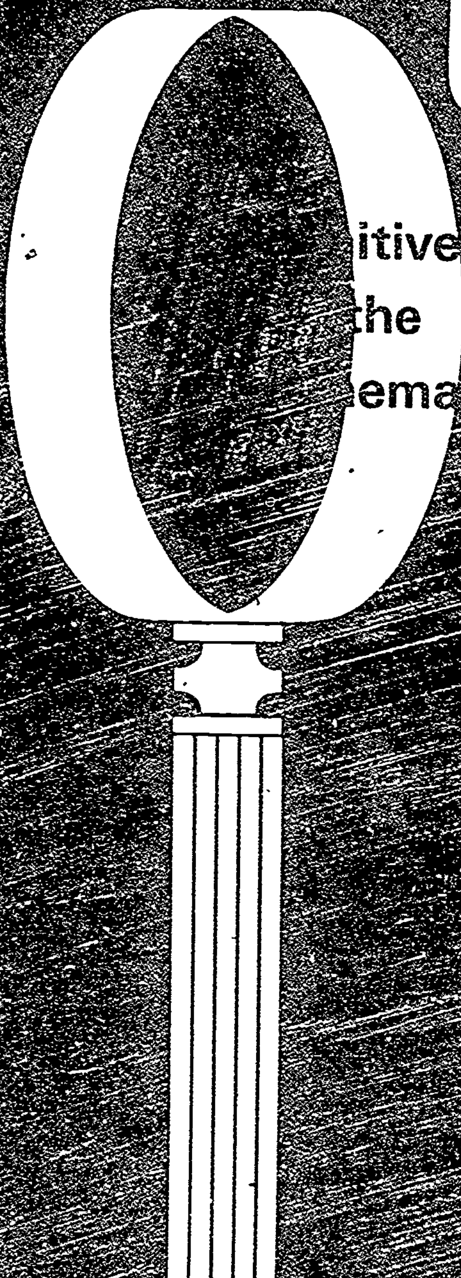
AUTHOR Lesh, Richard, Ed.
TITLE Cognitive Psychology and the Mathematics Laboratory. Papers from a Symposium.
INSTITUTION ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, Columbus, Ohio.
SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.
PUB DATE [74]
NOTE 149p.; occasional small print
AVAILABLE FROM Ohio State University, Center for Science and Mathematics Education, 244 Arps Hall, Columbus, Ohio 43210 ($2.75)

EDRS PRICE MF-$0.76 HC-$6.97 PLUS POSTAGE
DESCRIPTORS Activity Learning; *Cognitive Development; Conference Reports; Elementary School Mathematics; Elementary Secondary Education; Laboratories; Learning; *Mathematical Applications; Mathematical Concepts; *Mathematics Education; *Psychological Studies; *Research
IDENTIFIERS Northwestern University; Piaget (Jean)

ABSTRACT

This volume records the papers presented at a Northwestern University symposium concerning the articulation of cognitive psychology with mathematics education. Piagetian theories are described and extended to classroom applications in several of the papers; other psychological theories such as information processing are also discussed. All of the papers are concerned with students' learning mathematics in an active environment. Charles Smock's paper addresses the ways in which students organize mathematical ideas. Related to this question is Max Bell's paper on the role of applications in learning mathematics; Professor Bell explores the question of whether concepts must precede applications, or, conversely, application is a necessary part of the learning of concepts. Zoltan Dienes' discussion of finite geometries and Robert Davis' consideration of computer-assisted mathematics laboratories carry this theme further. The extension of Piaget's research to concepts ordinarily taught in the mathematics classroom, and to the development of children between the stages of concrete and formal operations is discussed by Leslie Steffe. The related issue of using Piagetian tasks in educational diagnosis is also discussed by Davis. An overview of psychological research as related to mathematics education, especially in the area of problem solving, is provided in Harry Beilin's paper. (SD)

ED108893



itive
the
ema

SE 019 075

itive

the

ema

Cognitive Psychology and the Mathematics Laboratory

Papers from A Symposium

sponsored by
The School of Education,
The Department of Mathematics,
and the
Center for the Teaching Professions
NORTHWESTERN UNIVERSITY

ERIC/SMEAC SCIENCE, MATHEMATICS, AND ENVIRONMENTAL EDUCATION INFORMATION ANALYSIS CENTER.
an information center to organize and disseminate information and materials on science, mathematics, and environmental education to teachers, administrators, supervisors, researchers, and the public
A joint project of The Ohio State University and the Educational Resources Information Center of NIE.

Mathematics Education Reports

Mathematics Education Reports are being developed to disseminate information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community. Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, the Conference Board of the Mathematical Sciences, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the editor.

Jon L. Higgins
Editor

This publication was prepared pursuant to a contract with the National Institute of Education, U.S. Department of Health, Education and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their judgment in professional and technical matters. Points of view or opinions do not, therefore, necessarily represent official National Institute of Education position or policy.

CONTENTS

Introduction v

An Overview of the Book
Richard Lesh ix

Discovering Psychological Principles for Mathematics Instruction
Charles D. Smock 1

Two Special Aspects of Math Labs and Individualization: Papert's Projects and Piagetian Interviews
Robert B. Davis 21

The Role of Applications in Early Mathematics Learning
Max S. Bell 43

Abstraction and Generalization: Examples Using Finite Geometries
Zoltan P. Dienes 59

An Application of Piaget - Cognitive Development Research in Mathematical Education Research
Leslie P. Steffe 85

Future Research in Mathematics Education: The View From Developmental Psychology
Harry Beilin 125

Introduction

The articles in this book are based on lectures which were given at a two-day symposium, "Cognitive Psychology and the Mathematics Laboratory," that was held as part of the 1973 dedication year ceremonies for Northwestern University's new School of Education building. The symposium was jointly sponsored by Northwestern's Mathematics Department, School of Education, and the Center for the Teaching Professions; and (in spite of the worst snow fall of the year) the symposium was attended by more than two hundred mathematics educators from throughout the country.

Rationale:

The rationale for the symposium grew out of discussions that were generated during a conference that was held at Columbia University in 1971 (ref., M.F. Rosskopf, L.P. Steffe, and S. Taback's Piagetian Cognitive-Development Research and Mathematical Education). Continued correspondences between various participants in the Columbia conference indicated a growing concern about the following issues.

1) While mathematics laboratories are gaining in fad appeal, precise meaning for such an instructional technique has remained only loosely and ambiguously defined. Elementary or junior high school teachers who have attempted to use such a teaching strategy have typically been forced to rely on a set of "rule of thumb" slogans, none of which are valid in all learning situations. "Concrete understanding before abstract"; "intuitive understanding before formalized"; "use activities, then symbols"; "use discovery rather than reception methods"; each of these slogans refer to distinct instructional variables which can occasionally specify contradictory approaches to teaching if their range of appropriateness is not qualified and coordinated by at least an embryonic theory of laboratory instruction.

2) Several cognitive theories (e.g., Piaget's) seem to offer at least a framework for a theory of instruction that could be used to give direction to the laboratory movement. However, the trend in education has typically been to use cognitive psychology to help justify preconceived instructional biases rather than to look at a theory in order to derive a consistent set of implications. Consequently, when a method of instruction is not effective in certain situations, the theory may be unjustifiably discredited (or rejected) rather than being modified or extended to cope with the new difficulties.

The above two problems are certainly not new to the history of curriculum change. In fact, if the name "Piaget" is replaced by "Dewey," most senior mathematics educators will be able to point out striking similarities between the "activity curriculum" movement of the 1920's and the "mathematics laboratory" movement of the 1970's. However, this cyclic history of curriculum change (i.e., enthusiastic adoption, followed by disillusionment, followed by rejection) indicates that theory building has not really been taken seriously by mathematics educators.

"Theory Building"-vs-"Theory Borrowing":

Perhaps it is unrealistic to continue to search for "outside" theories that can be "lifted" (without modification) and used in mathematics education. Perhaps the emphasis should shift from "theory borrowing" to "theory building." One of the main benefits to be derived from theory building is that the theory seldom has to be completely rejected when conflicts are detected or when difficulties occur.

In spite of the doctoral dissertation experiences that many mathematics educators have endured, theory building does not necessarily have to conjure up images of dull, "ivory tower" activities that make no real difference anyway. For a beginning, theory building can simply involve organizing a point of view that can form a basis for communication with other mathematics educators. In this way, individuals can profit by (and build on) the work of others. However, in order to avoid obvious errors and inconsistencies, theory building inevitably attempts to describe the range of applicability of its major principles, and to reconcile major conflicts within its point of view. Consequently, when difficulties arise, a theory should be more than a point of view that is simply accepted or rejected; it should be an explanatory "model" that can (and must) be gradually modified and reorganized to deal with progressively more complex situations.

While the history of science is filled with examples to illustrate the power of theory building, many mathematics educators would point out that mathematics education is more of a profession than a scientific discipline, and that "the best practice of the best practitioners is still better than the best theories of the best theorists." However, this observation does not mean that theory building could not be helpful, it simply reemphasizes the point that theory building in mathematics education is in a very primitive state. Certainly no currently available psychological theories (including Piaget's) is ready for wholesale adoption by mathematics educators. In fact, it seems unlikely that a lasting theory will ever become available which can be adopted (without modification) by mathematics education. Even if a particular theory seems to be especially relevant to the acquisition of mathematical concepts, the mark of a useful theory is measured as much by the questions it generates as by the questions it answers. For this reason, every theory carries with it the seeds of its own destruction which soon require it to be modified and incorporated into a more comprehensive theory. But, continuous modification in mathematics education cannot take place by continuously borrowing from outside mathematics education.

The recent boom in cognitive research has produced information about the development of mathematical concepts which was simply not available to curriculum designers even ten years ago. The question is whether the modern laboratory movement can organize this information into a theoretical point of view which will help it cope with some of the major problems that contributed to the downfall of its "activity curriculum" prototype of the 1920's.

The Issues:

The purpose of this book (and the symposium on which it was based), was to draw together a series of articles by some of the foremost authorities concerning the relationship between mathematics laboratories and cognitive psychology. An attempt was made to focus on issues which have been neglected in the laboratory movement. These issues and problem areas include: applications of mathematical ideas, concrete embodiments of mathematical ideas, computer assisted laboratory activities, clinical diagnosis of student errors, teacher training (using laboratory techniques), and directions for future research.

It is the hope of the authors and editor that the articles in this book will indicate some crucial problems and stimulate some useful ideas which may contribute to the success of the mathematics laboratory movement and to closer ties between cognitive psychology and mathematics education.

Richard Lesh
Editor

An Overview of the Book

Richard Lesh
Northwestern University

Chapter 1: Discovering Psychological Principles for Mathematics Instruction.

Author: Professor Charles Smock has been associated with the Piagetian school of cognitive psychology, and is currently the director of the "Mathemagenic Activities Program-Follow Through" at the University of Georgia. He has frequently worked with mathematics educators in the development of instructional materials, and has conducted research concerning the development of logical-mathematical concepts.

Topic: The question of how children learn cannot be neglected when considering the question of how they should be taught. The way that a child organizes a set of mathematical ideas may, or may not, correspond to the way a textbook or teacher organizes them. Beginning with the two major issues that constituted the rationale for Northwestern's symposium, Professor Smock isolates important cognitive variables having to do with the way children organize mathematical ideas, and he links these cognitive variables to instructional variables which are basic to a laboratory form of instruction.

Chapter 2: Two Special Aspects of Math Labs and Individualization - Papert's Projects and Piagetian Interviews.

Author: Professor Robert Davis is best known to mathematics educators as the founder and director of the Madison Project. The Madison Project has produced curriculum materials emphasizing the use of discovery exercises and concrete materials. Professor Davis is currently the director of the University of Illinois Curriculum Laboratories. In order to appreciate the perspective from which Davis' paper was written, it is important to mention that the curriculum laboratories are closely associated with the University of Illinois Plato Project which emphasizes flexible and creative uses of the computer in instruction.

Topic: Professor Davis' paper actually considers two separate but related issues. The first has to do with the diagnosis of student errors through "Piagetian" clinical interviews; and the second has to do with "computer assisted mathematics laboratories" which has been devised by Seymore Papert at M.I.T.

One of the great possibilities created by the University of Illinois' Plato Project was that detailed histories of formal mathematics instruction could be stored for large numbers of individual students. Such information could furnish exhaustive data about the ability of students to master a given concept depending on whether or not specific "prerequisite" concepts had already been introduced. Given this possibility, and Professor Davis' close affiliation with the Plato Project, it becomes even more impressive to notice that he has become an enthusiastic supporter of the "clinical interview" technique of diagnosing students' difficulties. Apparently Davis has concluded that such clinical techniques can furnish important information to supplement the kind of information which can be gathered by Plato.

To further reinforce the potential ties between computers and mathematics laboratories, Davis argues by analogy: "If a person really wants to learn French, going to France is better than taking a course; so, if a student wants to learn math, it may be best to go to 'mathematics land' (a place where mathematics is created and where communication takes place in 'mathematics')." As an example of one such mathematics land, Davis sites Seymore Papert's "Turtle Lab" (a computer assisted mathematics laboratory). The turtle lab attempts to demonstrate that a computer (or "turtle") is a perfect playmate to accompany the student through mathematics land since it is a creature that only communicates in "mathematics."

Chapter 3: The Role of Applications in Early Mathematical Learning

Author: Professor Max Bell is well known for his work with mathematical applications, mathematics laboratories, and innovative teacher training programs. He is currently the Chairman of the Elementary Education Department of the University of Chicago's School of Education.

Topic: Giving students real world experiences in order to learn useful mathematical concepts has usually been one aspect of laboratory approaches to teaching. A key problem with regard to the emphasis of mathematical applications in the curriculum can be stated as follows. If it is true that one must know a mathematical concept before one can apply it, then due to lack of time and training, teachers may often have to neglect applications in order to have time to simply "teach" the underlying concepts. On the other hand, learning to apply the concept may be a critical aspect in the initial acquisition of many mathematical concepts. Professor Bell distinguishes between "applications" and the kind of concrete "embodiments" that Dienes discusses, and he argues that even though educators are beginning to recognize the value of concrete embodiments, applications are still largely neglected. He goes on to clarify the role that mathematical applications might be able to play in motivating students and fostering the acquisition of process objectives (organizing data, formulating hypotheses, estimating answers, etc.).

Chapter 4: Abstraction and Generalization: Examples Using Finite Geometries.

Author: Professor Zoltan Dienes is director of the Psycho-Mathematics Research Center at the University of Sherbrooke, Canada. He has worked with Jerome Bruner at the Center for Cognitive Studies at Harvard University, and has directed major curriculum projects in Canada and Australia, and has been closely affiliated with the current mathematics education movement in England.

Topic: Professor Dienes has formulated a theoretical point of view which is a significant extension of Piaget's theory. Dienes' "learning cycles" model of mathematics instruction uses concrete "embodiments" and games to help children learn mathematical concepts. However, even though some of Dienes' principles of instruction (e.g., the multiple embodiment principle) have gained recognition and general acceptance, other aspects of his theory have been largely ignored by mathematics educators.

In this book, Professor Dienes uses examples about finite geometries to illustrate some of the processes which he believes are involved in the abstraction and generalization of mathematical ideas.

Chapter 5: An Application of Piaget-Cognitive Developmental Research in Mathematical Education Research.

Author: Professor Leslie Steffe has been one of the foremost mathematics educators who has attempted to interpret and investigate the meaning of Piaget's theory for mathematics instruction. Professor Steffe is associated with the University of Georgia.

Topic: In order to analyze the development of logical-mathematical thinking in children, Piaget has concentrated his efforts on children in the 5-7 and 10-12 year old age ranges. Consequently, Piaget's research has focused on the cognitive processes used by first graders (i.e., groupings) and sixth graders (i.e., INCR groups), while neglecting children at intermediate grade levels. For this reason, and since Piaget has avoided mathematical ideas that are typically taught in school, it is only possible to make relatively crude inferences about how children's mathematical thinking gradually changes from concrete operational mode of thinking to a formal operational mode.

Professor Steffe argues that certain mathematical structures may be able to describe the transitional phases through which elementary school children must pass, and that these mathematical structures may be even better models of children's logical-mathematical thinking than Piaget's groupings or INCR groups. If Steffe's hypotheses is correct, this fact could be tremendously useful to mathematics educators who would like to construct curriculum materials which are consistent with the "natural" development of logical thinking in children.

Chapter 6: Future Research in Mathematics Education: The View From Developmental Psychology.

Author: Professor Harry Beilin delivered one of the key addresses at the 1971 Columbia Conference on "Piagetian Cognitive-Developmental Research in Mathematics Education," and is one of the leading psychological authorities who has attempted to interpret the relevance of developmental psychology for mathematics education.

Presently, Professor Beilin is the editor of the Journal of Experimental Child Psychology, and is associated with the educational psychology and developmental psychology programs at the City University of New York Graduate Center.

Topic: Professor Beilin was assigned the awesome task of describing the directions that future research must take in order to further establish the relevance of cognitive psychology (and Piaget's theory in particular) toward the development of mathematics laboratory experiences for children.

Those mathematics educators who have maintained an interest in problem solving strategies will be especially interested in Professor Beilin's analysis of current trends in cognitive research.

Additional Presentation

The conference program contained an additional presentation by Professor John LeBlanc, director of the Mathematics Education Development Center. We regret that it was not possible to include Professor LeBlanc's paper in this book. For a description of his presentation, Training Teachers Using Model Techniques, interested readers should write to: Professor John LeBlanc, Mathematics Education Development Center, 329 South Highland Avenue, Bloomington, Indiana 47401.

Discovering Psychological Principles for Mathematics Instruction[1,2]

Charles D. Smock
University of Georgia

It is not that they can't see the solution -- they can't see the problem.

(Chesterton)

Science is not just a collection of laws, a catalogue of unrelated facts. It is a creation of the human mind, with its freely invented ideas and concepts . . . The only justification for our mental structures is whether and in what way our theories form . . . a link with the world of sense impressions.

(Einstein)

The drive to lead an intellectually satisfying life is . . . a product of long process of education . . . (and) . . . is an autocatalytic affair, growing with the practice of it.

(Bridgeman)

Currently, either the name Piaget or the term mathematics laboratory is sure to attract the interest of mathematics educators. While laboratories are gaining in fad appeal, precise meaning for such an instructional technique has remained only loosely and ambiguously defined. Teachers who have attempted to use such a teaching strategy have typically been forced to rely on a set of "rule of thumb" slogans, none of which has been shown to be valid in all learning situations.

"Concrete understanding before abstract;" "Intuitive understanding before formalization;" "Use activities, then pictures, then symbols;" "Use discovery rather than reception methods." All of these slogans refer to distinct instructional variables which occasionally specify contradictory approaches to teaching if their range of appropriateness is not qualified and coordinated by at least an embryonic theory of laboratory instruction.

---

[1]This paper is not for duplication or reproduction in any form without the permission of the author.

[2]This report is based on activities supported (in part) by the Mathemagenic Activities Program-Follow Through, C.D. Smock, Director, under Grant No. OEG-0-8-5224478-4617 (287) Department of HEW, U.S. Office of Education.

Mathematics educators appear to have little interest in theory construction and have a restricted view of the role of a mathematics laboratory. Theory is used to justify instructional biases and the laboratory is considered only an instructional device. Theory construction can help clarify the different types of roles a laboratory has in mathematics education research and practice. The mathematics laboratory should be a context for research on problems relevant to specific aspects of the instructional process (variation of teaching strategies and techniques) and on discovering those conditions critical for mathematical learning, and development of mathematical thinking in children, as well as fun and games for child and teacher. If educational researchers expect to accumulate knowledge relevant to theory based instructional practices, they must have "mathematics" laboratories.

However, in my opinion, Chesterton's remark is quite appropriate: "It is not that they can't see the solution -- they can't see the problem." We have yet to identify the fundamental dimensions of educational and instructional problems facing the mathematics educator. Key psychological principles for mathematics instruction and construction of a theory of instruction can be realized only after this first step has been achieved. A body of knowledge now exists in developmental-cognitive psychology that should have considerable utility for contributing to more refined theories of and strategies for instruction; i.e., for creating better school learning environments. One approach to the immediate task, then, is to search for suggestions from developmental psychology that pose relevant problems for mathematics educators for developing theories of mathematics instruction.

There is no paucity of choices of psychological models available from which to start the search. Berlyne, Bandura, Bower, Scandura, Skinner, Suppes -- and of course Piaget. Each has proposed a set of ideas (relevant to a theory of mathematics instruction) requiring theoretical and empirical study. The selection of any one model, however, brings with it many hidden presuppositions and is determined in no little part by one's own preconceived notion regarding human development, learning and education. As a group, these models represent a virtual wonderland of exciting ideas. Each of us can understand Alice's dilemma better as we explore their fantasies (Suppes, 1971).

A theory of instruction must begin with an adequate theory of learning and/or cognitive development. No longer can we accept that statement as "obvious" -- and go about the business of generating a multitude of methods based on unorganized intuitive rules constructed on the basis of inadequate knowledge of the process of cognitive development of children. Mathematics educators need to return to the beginning and ask not "how do we teach?" but rather "how do children learn?"

Modern developmental psychology provides a necessary, but not sufficient, body of knowledge for identifying some of the fundamental issues, constraints, and facts associated with the process of generating a theory of mathematics learning and instruction. But, to imply, and act as if, psychology has become relevant to mathematics learning only AP (i.e., after Piaget), misses a fundamental point about the relation of the science of psychology to the science of education, and distorts the history of both. On the other hand, I do not want Piaget's theory of cognitive development abandoned without clear understanding of why. The historical pattern in education and psychology seems to be one of enthusiastic adherence to a relatively novel theory -- with disappointment and rejection following close behind. The absence of serious controversial issues underlying much of the current research in cognitive development, and mathematics learning, increases my concern that much that is valuable in Piaget's theory may be lost. This symposium, perhaps, will provide a more relevant set of issues about which we can disagree than have surfaced to this point.

Many psychologists, including myself, consider Piaget's clarification of the necessary bases of theory construction as important as his cognitive developmental theory per se. Implicitly, and explicitly, Piaget was greatly influenced by advances in theoretical physics (Bridgeman, 1929) during the 1920's and 30's. The fundamental aspect of relativity theory which cannot be ignored in psychological theorizing is that conceptual judgments are always relative to the position of the observer. Analysis of knowledge acquisition requires a description of the operational basis of these acquisitions; i.e., the mental operations of the individual that are associated with the construction and maintenance of consistent patterns (structure) of his continually transforming relations with the physical and social environments. Thus, Piaget's emphasis on a constructivistic theory of knowledge (Piaget, 1968, 1971) is indissoluble from his interpretation of operationism -- that is, the need for operational analysis of the process of knowledge acquisition. "Reality" is constructed, not eminent in mind, man or stimulus, and applies to the child and, contrary to some interpretations (e.g., Stevens, 1935), theorist alike.

The form of epistemology typical of American psychologists (cf. Mischel, 1971) has been naive realism, and that orientation has been quite useful. We need not abandon it completely, but, it should now be obvious that our epistemological preconceptions, whatever they may be, are part of our view of the child. Kessen (1966) states the issue clearly: "The child who is confronted by a stable reality that can be described adequately in the language of contemporary physics, is a child very different from the one who is seen facing phenomenal disorder from which he must construct a coherent view of reality" (pp. 58-59).

Analysis of cognitive learning and development, then, is always "biased" by the fact of a context of preconceived ideas of reality (i.e., western culture) and a particular set of concepts or theory and selected data. Piaget's approach to the analysis of the development of children's conception of space provides us an excellent example (Piaget, J. and Inhelder, B., 1956). The designation of a conception of space toward which the child will develop, i.e., that conception held by the educated adult, is the critical first step. Once the endpoint for this development has been stated, observations and interpretation of the child's behavior are organized around these specifications. Observations not congruent with, and not structurable in terms of the specified endpoints, no matter how reliable, cannot be considered a relevant part of development. This is not an example of bad science or inappropriate procedures but rather illustrates that conclusions about the child's cognitive apparatus may be based as much on the construction of reality imposed on him in the first place as on the reliability and generality of the observations we make. Whether we are engaged in instructional practices or instructional research, or theory building, there is, for each of us, a set of guiding propositions that constitutes a theory of learning and development. These "fantasies" or "freely invented ideas and concepts" provide a particular coherent view of the developing child and of the critical determinants of the learning process.

Piaget's epistemology, and his biological background, thus predispose him toward an operational and structural analysis of the knowledge acquisition process (Piaget, 1967; 1968; 1970; 1971). The essentials of his position require only brief review there. Knowledge is defined as invariance under transformation (a most familiar concept to mathematicians). The construction of invariances in organism-environment relations takes place through the operation of two complimentary biological adaptation processes, both of which are under the control of the internal self-regulating mechanism of equilibration.

One of the two processes (assimilation) concerns the application of existing cognitive operational systems (structures) to the processing of environmental (sensory) data. External data or events are incorporated into existing structures through both on-going physical and mental activity. Such events and the products of new experience can be incorporated into the cognitive structure only to the extent it is consistent with existing functional structures.

Accommodation is the complementary process whereby adaptation occurs by integration of existing structures with other functional structures and/or by differentiation of new structures under confrontation with new experience.

Activities such as play, practical or symbolic, represent assimilative activity: whereas memory, in the sense of invoking past events, and imitation are accommodative since only existing structures are brought to bear on particular events or set of events. Assimilation is an active constructive process by which the data from experience are transformed and integrated with an already generalized cognitive structure. Accommodative activity, on the other hand, is associated with the process whereby application of existing structures are brought to bear on particular new events or sets of events, i.e., events or sets of events to which these structures have not been applied previously.

Too often instructional theory and practice have emphasized assimilation (i.e., "play") or accommodation (i.e., imitation) activity and neglected the role of equilibration of these complementary processes for cognitive learning. Appropriate generalization of Piaget's ideas to instructional theory and practice requires consideration of three additional factors associated with Piagetian theory: "logic," operative vs. figurative thought, and equilibration.

Operational Structures and "Logic".

Piaget (1970) recently elaborated his position that human beings possess the same biological structures and functions that, in "exchange" with the common features of the natural world, generate mental (operational) structures and functions characteristic of each stage of development. Logical thought, in the Piagetian sense, then is universal and of fundamental importance to an understanding of development and learning. But, whereas Chomsky maintains the human mind is "programmed" at birth with cognitive structures (i.e., mental representation of a universal grammar), Piaget accounts for the universality and stability of structures across cultures (Piaget, 1967; Goodnow, J., 1962; Goodnow, J., and Bethon, G., 1966; Greenfield, 1966; Maccoby, M. and Modiano, N., 1966) in terms of the self-regulation mechanism of equilibration. Thus, Piaget proposes that the mind at any point in development (i.e., life) is the unfinished product of continual self-construction (1971); i.e., "logical" processes are generative and not fixed. Structures are not performed, but are self-regulatory, transformational systems with the functional factors in that construction being the processes of assimilation and accommodation.

Intelligence, the basis for knowledge acquisition, consists, then, of two aspects: adaptation (with the complementary processes of assimilation-accommodation and the self-regulatory mechanism of equilibration) and organization consisting of sets of mental operations that form the basis for maintaining invariance under transformation (i.e., knowledge). It follows from these considerations that there is an inherent logic to development; i.e., operational systems consist of elements and laws of combination of those elements that form a "logical" closed system. These mental structures are observable in the actions of the organism in its environment and further, are describable in terms of formal or logico-mathematical structures. Genetic psychological analysis of these structures is a necessary prerequisite to an understanding of thought processes. "No structure without genesis, no genesis without structure" (Piaget, 1968). During the sensorimotor period of development, action structures of the individual are revealed in practical groups, i.e., the observable coordinated actions of the individual (Forman, 1973). During the pre-operational period, the child constructs representations (figurative structures) which do not have the operational property of reversibility. Piaget was able to identify operational structures with clear mathematical system properties in children between ages five and seven. The discovery of a resemblance between the structure of the mental action system (reasoning or thought) and mathematical structures (i.e., mathematical groups and lattices) had a profound effect on Piaget's thinking. Thought, it would appear, has the same, or similar properties, as mathematical group structures, both of which are governed by the same internal logic.[3]

---

[3]Piaget never has tried to find a mathematical "model" to "fit" the observed facts of behavior; rather the mathematical aspect of Piaget's theory is unique in that he assumes, somewhat a reminescent of Boole's "laws of thought," an identity between the inherent logic of thought processes and certain formal mathematical systems that have become "externalized," through inductive reasoning, and guide the action-patterns of the individual.

The basic structuralistic approach of Piaget involves finding logical mathematical systems that describe the thought processes of an individual. Mathematical group and lattice theory offers algebraic systems (Piaget, 1957; Flavell, 1963) for describing operational thought and Piaget has tried to take maximum advantage of that approach. To review the fundamentals: a mathematical group is a system consisting of a set of elements, together with an operation (law of combination), which yields the following system properties: (1) when applied to the elements of the set, the combinatorial operation will yield only elements of the set; no elements external to the set can be produced; (2) each set contains a neutral (or identity) element that when combined with any other element of the set yields no change; (3) each element of the set has an inverse which in combination with any element yields the neutral or identity element; (4) the combinatory operation (and its inverse) is associative; i.e., $[(n + m) + p = n + (m + p)]$. Piaget found it necessary to generate a "grouping" model with additional properties (i.e., both group and lattice properties) to describe the concrete operational structures. The properties of these psychological groupings are not derived from the properties of things, but from modes of acting upon things. Thus, the elements of psychological groups are, themselves, transformations that characterize the individual's operations as he acts upon incoming sense data.

The revelations emerging from relativity theory require a constructionistic position with respect to the nature of knowledge; i.e., understanding of knowledge acquisition requires a description and characterization of the mental operations and operational systems applied to the data of experience. Piaget's emphasis on structural analysis thus makes contact with the epistemological implications emerging from relativity theory; the biologists' emphasis on development as the formation, differentiation and hierarchical integration of functional (action) structures; and the mathematicians' emphasis on formalized systems that permit description of these structures. The task of the developmental psychologist is to describe the nature of action structures of the child at each point in development and, as much as possible, to formalize those descriptions in terms of logico-mathematical terms.

The classical "conservation" tasks, if administered appropriately, form one basis for generating observation of coordinated actions that appear to reflect such mental operational structures. The available evidence appears to support the possibility that such operational (mental) structures "exist" both in terms of replicability of developmental trends in task solution and training studies (cf: Beilin, 1971a). At the same time, neither psychological nor educational researchers have yet devoted sufficient attention to the problems of the validity (i.e., internal consistency) of the grouping structures (Clary, 1970; Green, D., 1971), nor to the role of such structures in learning (Berlyne, 1961; Bruce, 1971; Inhelder, B. and Sinclair, H., 1969), beyond these classical situations.

Only recently have mathematics educators become interested in Piaget's views of fundamental logico-mathematical relations, such as his ideas about the logical properties of number and space. Beilin (1971b) points out that philosophers of science generally have emphasized the desirability of isolating philosophical and logical systems from psycho-logical matters. Psychologists, mathematicians and logicians generally have maintained this position with respect to Piaget. However, a significant part of his psychological theory has mathematical and logical content which cannot be ignored (Leskow, S. and Smock, C.D., 1970; Alonzo, M., 1967) by either psychological or mathematics learning researchers. Mathematics educators rightly should be directed in part, to the analysis of the logical and mathematical veracity of Piaget's system and to the correspondence between the characteristics of the psycho-logic systems to those logic structures derived from purely mathematical analysis. Recent work from Steffe's laboratory (e.g., Kidder, 1973; Johnson, D., 1971; Lesh, R., 1971; Johnson, M., 1971) represents an excellent beginning in this direction.

Role of Experience and Equilibration

Experience is not a unique factor in development according to Piaget. Merely being exposed to particular experiences is conducive neither to cognitive activity nor to developmental change. Children may or may not make discoveries in the course of play, and watching a laboratory experiment (or conducting one) may or may not help a child acquire a particular concept. Equilibration (Langer, J., 1973) is the central factor in structural changes whether the reference is to stage or concept learning. Equilibration is the process of intrinsic (self) regulation that balances assimilatory and accommodative processes, compensates for external and internal disturbances and makes possible the development of more complex, hierarchically integrated operational structures. Disequilibrium occurs as the child assimilates data from exchanges with the environment into existing mental structures. As cognitive structures change to accommodate to the new information, equilibrium is restored. The equilibration process is one of auto-regulation -- both of the transformations of information based on existing cognitive structures and of changes through accommodation. In any case, the child must be exposed to environmental input that "engages" the functional structures; i.e., he must be involved in a personal striving to understand or "accept" the task as a "problem."

A basic question for instructional theory and practice is: What are the processes and conditions that motivate and insure engagement or acceptance of the problem task by the child? The source of "interest" that promotes striving for problem solution is contingent on assimilative-accommodative activity but the specifics remain unclarified in Piaget's theory (cf: Mischel, 1972). What Piaget means by the "need" of structures to function is a "primitive" factor not unlike the notion of competence "drive" suggested by White (1969). Within a structuralist framework -- if a structure exists, it must function -- cognitive structures appear to have a dimension of openness that make probable continual sources of disequilibrium from interaction of the internal operational and/or figurative structures activated as well as by exchanges involving novel environmental input. In any case, natural or lifelike contexts seem to provide excellent situations in terms of promoting cognitive change. Despite lack of specifications, Piaget is quite explicit on his position: "It is not necessary for us to have recourse to separate factors of motivation in order to explain learning, not because they don't intervene... but because they are included from the start in the concept of assimilation... to say that the subject is interested in a certain result or object thus means that he assimilates it or anticipates an assimilation and to say that he needs it means that he possesses schemas requiring its utilization" (Piaget, 1959).

Cognitive conflict, or the awareness of a momentary disequilibrium, represents a need to establish consistency (equilibrium) between the existing schemas and/or novel information and is motivation for cognitive activities. Both applying an existing schema and elaborating new ones in the course of development stem from simply the overriding need to make "sense" of present problems by fitting them coherently into schemas "learned" in the course of solving prior problems.

The notion that disturbances introduced into the child's systems of prior schemas lead to the adoption of a strategy for information processing is the fundamental difference between the equilibration and associationistic theories of learning (Piaget, 1957b). For associationistic theories of learning, "what is learned" depends on what is given from the outside (copy theory) and the motive that facilitates learning is an inner-state of some sort of other. Equilibration theory holds, however, that learning is subservient to development; i.e., what is learned depends on what the learner can take from the given by means of the available cognitive structures. Further, cognitive disequilibrium (functional need) is what motivates learning (i.e., questions of felt lacunae arising from attempts to apply schemas to a "given" situation).

The child then will take interest in what generates cognitive conflict; i.e., in what is conceived as an anomaly. If the task demands are too novel as to be unassimilable or so obvious as to require no mental work, the child will not be motivated.

After the period of sensori-motor development, equilibration becomes a process of compensating for "virtual" rather than actual disturbances. At the operational level, intrusions "can be imagined and anticipated by the subject in the form of the direct operations of the system -- the compensatory activities will also consist of imagining and anticipating the transformations to an inverse sense" (Piaget, 1967). Further, there need be no external intrusions in order for the equilibration process to be activated. For example, the acquisition of conservation concepts is, in Piaget's view, "not supported by anything from the point of view of possible measurement or perception -- it is enforced by logical structuring much more than by experience" (Piaget, 1967). It is the "internal factors of coherence -- the deductive activity of the subject himself" that is primary. Equilibration, we noted earlier, is a response to internal conflict between conceptual schemas rather than a direct response to the character of outside structure factors. Equilibration is a matter of achieving "accord of thought with itself" in the service of establishing accord of thought with things.

Unfortunately, little empirical investigation has been oriented to questions of situational determinants of curiosity (Smock, C.D. and Holt, B.G., 1962) of children at various stages of development and with different experiential backgrounds; i.e., what children recognize as problematic, and what kinds of incongruities are sufficient to motivate change in concepts and/or beliefs.

Operative and Figurative Thought

A considerable amount of confusion concerning Piagetian theory and its implication for both research and instructional practice derives from a failure tc consider the figurative and operative aspect of intellectual functioning. In general terms, the distinction is between the selection, storage and retrieval, and the coordination and transformation of information (Inhelder, et. al., 1966). More specifically, the development of any sequence of psychological stages, a la Piaget, consists of an interactive process of equilibrating functional structures of the organism with the event-structures of the environment. Figurative and operational processes represent two types of functional structures necessary to account for knowledge acquisition, development and learning. Figurations are defined as those action schemata that apprehend, extract and/or reproduce aspects of the physical and social environment. Such action schemata include components of perception, speech, imagery, and memory. Figurations and associated acts are based on physical, as contrasted to logico-mathematical, experience and constitute the "empirical" world; i.e., empirical truth is no more than the "representation of past events in memory."

Operations, on the other hand, do not derive from abstractions from objects and specific events; rather, operational knowledge is derived by abstractions from coordinated actions on those events. Thus, operations are those action schemata that construct "logical" transformations of "states." Such logical systems of transformations operate either upon representations of events, or on the cognitive system's own logical operations, i.e., reflexive operations.

Figurative and operative structures are two parallel streams with their genetic or developmental origins in the same source (Piaget, 1967, 1968; Piaget, 1970, 1971) -- the sensori-motor structures. Logical (operational) structures are not generated from the figurative schemata, i.e., not from perception, memory, etc. Reciprocally, figurative structures do not derive from operative schemas but from the representations of past states of events derived from physical experience. And most importantly, figurative structures do not derive from each other, but have unique bases in sensori-motor schema. Imagery, for example, is a derivative of deferred sensori-motor or imitation (Piaget, 1951; 1952; 1971) and not perception.

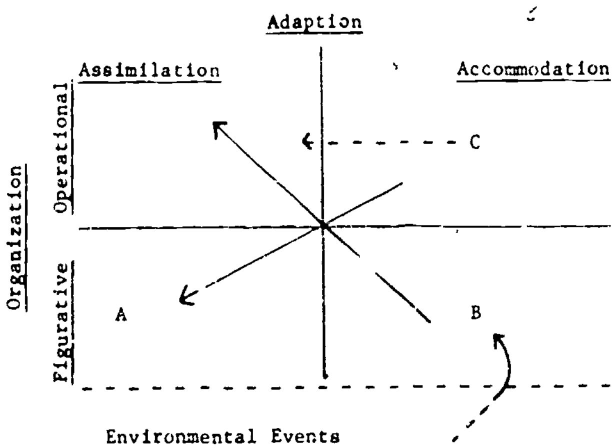
The postulation of these quite distinct functional structures is one of the cornerstones of Piaget's theory of knowledge acquisition and cognitive development (cf: Furth, 1969). Both the source and function of the structures are theoretically distinct. Operative structures derive from abstraction from coordinated actions, figurative structures derive from sensori-motor and perceptual activity. Operative structures produce "logical" transformations (conservation of invariants) whereas figurative structures reproduce sensory-perceptual consequences of environmental configurations. The variant operative structures of the intuitive, the concrete, and the formal levels form the discontinuous sequence of stages of cognitive development. On the other hand, figurative structures are static and dependent directly upon the data of experience (sensory-perceptual consequences of stimulation). Thus, Piaget makes the fundamental assumption that all knowledge acquisition activity is constructive, but the construction of figural representations is quite a distinct process from that constructive activity at the operative level.

Logically, there are three possible relations between the figurative and operational structures. First, they may be unrelated and, if so, as mentally segregated functional structures, do not set limits on the functioning and development of each other. Second, psychological phenomena might be reduced to one of the types of structures. Langer (1969) suggests that subjective idealists, perhaps, try to reduce psychological phenomena to assimilatory operations; and there are many theorists who try to reduce all mental phenomena to accommodatory figurations while naive realists propose that such processes are figurative; i.e., perception is knowledge (Michotte, 1963; Garner, 1962). Third, and the one proposed by Piaget, is that of partial communication between figurative and operative structure within the constraints of assimilation and accommodation processes.

The relations, and the potential form of interaction are schematically presented in Figure 1 below.

Figure 1

Relations of Two Invarient Processes
of Adaptation and Two Types of Cognitive Structures



Langer (1969) has examined the organizational and developmental (i.e., transitional) impact of accommodatory figurations on assimilatory operations (See B, Figure 1); i.e., how does the child mentally extract and/or represent empirical information about physical and social objects and the consequences of that empirical activity for the construction of logical concepts. Imitation of an observed event, comparison of one's predictions with the outcome of a physical deformation, comparison of observation or appearance with the way things really are, represent different modes of introducing conflict and cognitive-structural change. Generally, his findings are confirmatory, but not definitive with respect to the Piagetian hypotheses. In any case, if the development of each type of functional structure has implications for, but not direct causal effects upon, the functional structure and development of the other, current paradigms for the study of learning mathematical concepts will require considerable modification. The work of the Geneva group mentioned earlier concerning, for example, memory (see A, Figure 1) and Langer's (1969b) analysis of the impact of accommodatory figurations (i.e., imitation, etc.) on assimilatory operations represents beginnings in this direction.

Analysis of learning, in the context of Piagetian theory, poses requirements for much more detailed empirical analysis than has been generally recognized. On the one hand, researchers attempting to assimilate Piaget to their own conceptual structures concentrate on experimental procedures whereby the subject is required only to remember event contingencies (e.g., response-reward associations or "a" follows "b" follows "c"). Such procedures certainly produce change in "behavior" (e.g., Gellman, 1969; Mehler and Bever, 1967; Bever and Mehler, 1968); however, failure on transfer tasks, and a lack of persistence of task solution over time indicates that a figurative process underlies the change in performance. On the other hand, the accommodators (i.e., those more favorable toward Piaget's theory) often fail to generate experimental paradigms that adequately differentiate between the figurative and operational knowledge (Wallach, L. and Sprott, 1964) or assume that "external disparity" (appearance vs. "reality") is sufficient to establish disequilibrium (conflict) between logical necessity (derived from the operational structures) and perceptual pregnance (cf: Bruner, 1966). For example, situations designed to establish disparity between the child's predictive judgment of the outcome of a transformation and his observation of the actual outcome may, in fact, generate little or no cognitive conflict. Certainly, a most parsimonious explanation of many negative findings in training studies is that such disparity is external to the child's logical operational system.

Implications for Learning and Instruction

In some form or other, the goals of American educators have always been stated in terms of "optimizing" the intellectual, social, etc., development of individual children; a vague statement, obviously, and subject to a variety of interpretations. Whatever imperatives that goal implies, the educational and instructional processes must be based upon an understanding of the nature of psychological development of children. Whether we want to produce individuals who will strive to maintain the status quo; individuals who desire and accept change; people content to be technologists (i.e., skilled labor); or problem solvers; it is necessary to understand the basic processes of child development and the conditions that permit "quality control of the product" (if I may use current jargon).

The issue is important because science (i.e., theory and research) can only yield "what is" and not what "ought to be." We are fortunate, in one sense, that the science of psychology (and of pedagogy) is young and imperfect; the proposed models and methods for educating young children are no less imperfect and are influenced as strongly by current social thought and individual philosophical biases, as by an understanding of the laws of psychological development. Such a state of affairs, while producing wasted efforts, spurious claims, more rediscoveries than discoveries, etc., can at least provide time for the development of articulated sets of societal goals for education.

The best that can be hoped for, under the current conditions of our knowledge, is development of preliminary "models" for instruction. Such models can provide, at least, a schematic set of principles and guidelines for constructing a learning environment consistent with the admittedly inadequate theories and knowledge of psychological growth. However, we should try not to violate recent advances in theory and known laws of child development.

Piaget (1961b) has rightfully declined to generalize his theory to specifics for educational practice. He has, however, suggested a theory of knowledge acquisition which has contributed to clarification and integration of a particular set of propositions concerning psychological development (many of which have a long history in child psychology and education). If we accept the fact that his theory of cognitive development is not yet a conceptually or empirically "closed system," several deductions concerning the construction of "optimum" environments can be generated. A modest attempt in this direction has been made at the University of Georgia-Follow Through Program (Smock, 1969). The initial stage of that model is based on an attempt to generate a set of instructional principles based on our understanding of Piaget's theory of cognitive development. Though many of the basic propositions upon which the model is based are not inconsistent with Piaget's thinking about knowledge acquisition, the interpretation is that of the modeler. It is influenced, therefore, by numerous sources of bias, misunderstanding, distortions, etc., that are inevitable under conditions where abstract theoretical concepts are not represented in unequivocal abstract or logico-mathematical terms.

It now is clear the child can no longer be considered a passive recipient of stimulation, nor can external reinforcement be considered a primary factor in learning and behavioral change. The introduction of "mediation responses" (verbal or otherwise) is not able to account for the complexities of observed changes in behavioral organization during the course of psychological growth during childhood. Many psychological theorists have adopted, in one form or another, the idea that human organisms actively respond to their environment and that the patterning of these responses reflects a "plan" or "set of cognitive operations." In other words, the child interprets environmental event input, but these interpretations are controlled by his capabilities for generating rule systems for coordinating and transforming the input to "match" a scheme, plan, or a mental operational structure. Analysis of the "rule systems" characterizing cognitive development, thinking, and learning requires specifications of the properties of, and antecedent conditions for, acquisition and structuralization of environmental events (mental representation/figurative knowledge) and of mental actions (operative knowledge) involved in coordination and transformation of those representations. The study of the development of rule systems defined as such is coincident, then, with the systematic investigation of the "inherent logic" of development of operational and figurative thought processes.

Intelligence, first of all, is considered no more, and no less, than biological adaptation; i.e., adaptation at any level of complexity reflects "intelligent" activity. "Knowledge" consists of two types of functional structures (figurative and operational) that construct invariants in organism-environment relations. These invariants are derived from abstractions from objects (physical experience) in the first case, and from coordinated actions (logico-mathematical experiences) in the second. Intelligence, then, refers to both types of cognitive learning and development and is defined in terms of functions (thus, thinking, reasoning) rather than content (i.e., words, verbal responses, associations, etc.). Analysis of conditions for congitive learning and development must begin with the identification of components of behavioral organization (structure) that reflect particular coordinated action-modes of the child as he is confronted with changing intrinsic (maturation and prior cognitive acquisitions) and extrinsic (physical and sociolinguistic) factors.

Cognitive structures of systems of coordinated (mental) actions proceed through invariant stages of structural change with autogenetic development. The successive differentiation and hierarchical integration of these cognitive structures permit the individual to cope with increasingly complex social and physical "realities." The process of cognitive development involves the changing characteristics of transformational rule systems (virtual and/or cognitive operations) characterizing the child's mode of adaptation. Neither the maturational structure of the organism nor the "teaching" structure of the environment is the sole source of reorganization; rather, it is the structure of the interaction between the child and the environment that provides the basic intellectual development, i.e., construction by the child.

Optimal conditions for structural organization and reorganization require: a) an optimal degree of discrepancy between environmental-demand structures and the functional-psychological structures -- both figurative (i.e., perceptual activity, images, memories) and operative; and b) social-learning conditions that demand "spontaneous" or "constructive" activity by the child.

Several implications for the construction of theoretically appropriate learning environments are implied in these general principles. First, structural change, for example, depends upon experience but not in a way that traditional learning theorists conceive experience; i.e., learning interpreted as pairing of specific objects and responses, direct instructions, modeling, etc. Rather, the functional genetic view holds that the cognitive capacities determine the effectiveness of training. For example, ability to solve class inclusion problems implies that the child already has the requisite single and multiple classification operational system for classes (i.e., combination, reversibility, etc.) in addition to appropriate information selection, storage, and retrieval abilities. At the same time, while experience is necessary for developmental progress, and appropriate enrichment of the environment can accelerate such development, experience cannot change the sequence, structuring, or emergence of action modes in the process of developmental change. In other words, organization of experience is not provided solely by the environment nor by the internal structures of the child.

Second, the structure of learning environment also must be considered relative to two frames of reference The cognitive development of the child first must be analyzed in terms of the operational systems controlling his interpretation of environmental events. Such operations, or "transformation responses," are expressed behaviorally in the coordinated actions of the child as he is confronted with changes in the physical and social world. For example, the mental operations of associativity, reversibility, etc., can be inferred from the manner in which the child attempts to solve problems involving regular environmental contingencies (causality), understanding of spatial relations, etc. The content of substantive areas (e.g., science, mathematics) then must be analyzed and structured in terms of their own logical sequence and interlockingness with other contents. Certain concepts in the physical sciences, language, and mathematics, for example, each have their own inherent sequence and structure. Thus, certain concepts and information are necessary precursors to subsequent understanding of higher order concepts. Further, the interlocking nature of these "contents," in some cases, may be independent of the psychological state of the child. Optimal educational conditions require, then, thorough understanding of the psychological-cognitive capacities of the child as well as the sequential structuring of concepts within a particular curriculum area.

Third, the striving for equilibration between assimilatory and accommodative processes under both intrinsic or extrinsic pressures underlies the adaptive process. Optimum conditions for structural reorganization (learning in the broad sense) require disequilibration. This condition is met when there is an appropriate "mismatch" between the cognitive capacities of the child and the conceptual demand level of the learning task. Too little or too much "pressure" results in over-assimilation or over-accommodation tendencies respectively, but does not promote developmental changes in cognitive structures.

Fourth, facilitation of learning requires analysis of two levels of cognitive functioning -- figurative and operative processes. The first (i.e., figurative thought) is most emphasized by those theorists (particularly behaviorists) recommending a direct tuition approach to instruction. The operational theory of intellectual development does not deny the value of "provoked" learning (i.e., through imitation, algorithms, etc.). Rather, it must be recognized that such learnings are considered limited because of lack of generalization or transfer to new situations and because the basic (i.e., operational) intellectual processes concerned with problem solving and reasoning are not much affected.

While there is some doubt that much acceleration of structural reorganization is possible through environmental enrichment, early childhood education should provide opportunities for utilization of relevant cognitive operational structures. Generalization of conceptual learning across content areas rather than the building of specific knowledge and skills (e.g., a large vocabulary) should be emphasized since the latter cannot directly accelerate operational system change and may, in fact, retard development of these "deeper" competence structures.

In any case, the nature and variety of the child's "exchanges" with the environment need to be considered in educational planning. The nature of the interaction refers to the relative emphasis on autogenesis (self-directed) as contrasted to exogenesis (environmental or teacher-directed) structure of the learning environment. The position of the functional genetic position can best be summarized in the old adage -- "You can lead a horse to water, but you can't make him drink - unless you feed him salt." Thus, the task of the teacher is to engineer an educational environment consisting of curriculum materials, social interactions, and directed activities that provide appropriate "salt" for each child. Sequentially structured curricula could be designated to provide an optimum degree of environmental structure and level of conceptual material to permit appropriate balance of assimilatory and accommodatory activity.

The amount of interaction (i.e., enrichment) refers to the variety of structured curriculum contents which are relevant to the child's physical, social, and symbolic experiences. The interlocking nature of substantive curriculum areas makes it possible to provide a variety of experiences relevant to acquisition of the cognitive "products" that provide representation of the environment (memories, vocabulary, etc.) and, at the same time, to facilitate the development of coordinated rule systems associated with cognitive operational development. For example, analysis of the visual environment (attention or observational skills) as well as cognitive operational structures (e.g., conservation of area) can be emphasized in science, social studies, mathematics, art, etc.

The engineering of an educational or "learning environment" based on the preceding consideration necessarily involves some specification of: 1) the child's cognitive developmental level; 2) the physical structures, including curriculum materials; and 3) the social or inter-personal structures. The organization of these "elements" should be such that the equilibration, between different cognitive systems and/or between intrinsic functional structures and environmental structures, is achieved. Thus, sequentially structured sets of curriculum materials and of social interaction situations are necessary to provide the "pressure" necessary for learning (adaptation). A variety of specific learning environments needs to be available to maximize the probability of each child finding activities that attract or "trap" him into interacting with the physical (e.g., science) and social (e.g., social studies) environment at both the behavioral and symbolic levels (e.g., art, role playing, music). Finally, the physical and social environments should be arranged so that considerable freedom of movement, within the structure of a variety of contents, is possible, i.e., "a modified open-structure classroom." A careful balance between relatively high and low structured learning situations and between group and individual learning activities should be maintained.

The Mathemagenic Activities Program, a model developed in the context of enriching the educational environments of economically deprived children[4], is based on three explicit principles derived from the considerations discussed above. Specifically, the MAP principles of change -- whether the target for change is the individual organism (child) or a complex social system (e.g., Local Education Authority) are based on the above assumptions concerning the role of experience in learning and development. First, the source of motivation to change is provided by a discrepancy (disequilibration) between different conceptual systems (ideas) and/or between previously acquired conceptual systems and environmental task demands. Thus, an appropriate mismatch (M) is necessary to generate exploratory activities and insure the individual has the prerequisite conceptual basis for learning higher order concepts.

Second, since coordinated actions (practical and mental) are the bases for the knowledge acquisition, the learning environment must be structured so that specific task demands include appropriate practical, perceptual, and mental activity (A).

Third, the learning environment must include provisions for personal, self-regulatory (P) constructions. Knowledge acquisition involves construction of invariants from properties of objects (physical experience) and from the child's actions on objects (logico-mathematical experiences). Optimal conditions for facilitating new "constructions" (concepts learning) involve a balance between tasks that are highly structured (in which the child merely "copies" or imitates the correct solution) and tasks that permit the child to generalize and discover new applications of his concepts. Practically, self-regulation implies a variety of task options available to the child, the number of options may well vary with the nature of the task, level of development, and many other factors. However, MAP proposes that options -- in terms of level of task difficulty, mode of learning, and choice of activity itself -- are necessary ingredients of developmental change, whether the target be a child, a teacher, or an educational system.

The implied educational model requires significant changes in the teachers' role definition and teaching strategies (as well as tactics). The need for sensitivity to the child's capabilities, and the structuring of learning situations that promote self-regulated, "constructive" knowledge acquisition, together with thorough acquaintance with available technological aids, require an "educational engineer" in the best sense of that term.

---

[4]Mathemagenic Activities Program: A Model for Early Childhood Education, prepared by C.D. Smock (A preliminary statement of the conceptual basis for the Mathemagenic Activities Program for the Follow Through Program appeared in Terminal Report, Research and Development Center for Early Educational Stimulation, University of Georgia, August, 1970).

References

Alonso, M.R. Spatial analogies problems test: A preliminary validity study. Unpublished master's thesis, University of Georgia, 1970.

Beilin, H. Cognitive capacities of young children: A replication. Science, 1968, 162, 920-921.

Beilin, H. The development of physical concepts. In T. Mischel (Ed.), Cognitive development and epistemology. Academic Press, Inc., 1971.

Beilin, H. Stimulus and cognitive transformation. In D. Elking and J.H. Falvell (Eds.), Studies in cognitive development: Essays in honor of Jean Piaget. New York: University Press, 1969.

Beilin, H. The training of acquisition of logical operations. Piagetian cognitive-development research and mathematical education, National Council of Teachers of Mathematics, Inc., 1971.

Berlyne, D.E. Structure and direction in thinking. New York: John Wiley and Sons, 1965.

Bever, T.G.; Mehler, J. and Epstein, J. What children do in spite of what they know. Science, 1968, 162, 921-924.

Bridgeman, P.W. The logic of modern physics. New York: Macmillan Co., 1927.

Bruce, A.J. The effects of age and stimulus structure on serial learning. Unpublished doctoral dissertation, University of Georgia, 1968.

Bruner, J.S. The process of education. Harvard University Press, 1961.

Bruner, J.S , et.al. Studies in cognitive growth. New York: John Wiley and Sons, 1966.

Clarey, S.A. The presence of grouping properties in Jean Piaget's logical grouping I. Unpublished master's thesis, University of Georgia, 1971.

Flavell, J.H. The developmental psychology of Jean Piaget. New York: D. Van Nostrand Co., 1963.

Forman, G.E. The early growth of logic in children: Influences from the bilateral symmetry of human anatomy. Paper presented at Biennial Meeting of the Society for Research in Child Development, Philadelphia, 1973.

Garner. W.R. Uncertainty and structure as psychological concepts New York: John Wiley and Sons, 1962.

Gilman, R. Conservation acquisition: A problem of learning to attend to relevant attributes. Journal of Experimental Child Psychology, 1969, 7, 167-187.

Goodnow, J. A test of milieu effects with some of Piaget's tasks. Psychological Monographs, 1962, 76 (36), whole no. 555.

Goodnow, J. and Bethon, G. Piaget's tasks: The effects of schooling and intelligence. Child Development, 1966, 37, 573-582.

Green, D.R., Ford, M.P. and Flamer, G.B. (Eds.) Measurement and Piaget. McGraw-Hill Book Company, 1971.

Greenfield, P. On culture and conservation, and on culture and equivalence II. In J.S. Bruner (Ed.), Studies in cognitive growth. New York: John Wiley and Sons, 1966.

Gruen, G.E. Note on conservation: Methodological and definitional considerations. Child Development, 1966, 37, 977-983.

Hall, V.C. and Kingsley, R. Conservation and equilibration theory. Journal of Genetic Psychology, 1968, 113, 195-213.

Hatano, G. and Suga, Y. Equilibration and external reinforcement in the acquisition of number conservation. Japanese Psychological Research, 1969, 11, 17-31.

Hunt, J.M. Intrinsic motivation and its role in psychological development. In D. Levine (Ed.), Nebraska symposium on motivation. Vol. 13, Lincoln, Nebraska: University of Nebraska Press, 1965.

Inhelder, B. Comments on "The course of cognitive growth," American Psychology, 1966.

Inhelder, B., Bovet, M. and Sinclair, H. Developpement et appretissage. Schweizerische Zeitschrift fur psychologie und ihre Anwendungen, 1967, 26, 1-23.

Inhelder, B. and Sinclair, H. Learning cognitive structures. In P.J. Mussen, J. Langer, and M. Covington (Eds.), Trends and issues in developmental psychology. New York: Holt, Rinehart, and Winston, 1969.

Johnson, D.C. Learning of selected parts of a boolean algebra by young children. Presented at the meeting of the American Educational Research Association, Chicago, Illinois, 1972.

Johnson, M.L. Learning of classification and seriation by young children. Unpublished doctoral dissertation, University of Georgia, 1971.

Kidder, F.R. An investigation of nine, eleven, and thirteen year-old children's comprehension of Euclidean transformation. Unpublished doctoral dissertation, University of Georgia, 1973.

Lesh, R.A. The generalization of Piagetian operations as it relates to the hypothesized functional interdependence between classification, seriation, and number concepts. Unpublished doctoral dissertation, Indiana University, 1971.

Leskow, S. and Smock, C.D. Developmental changes in problem-solving strategies: Permutation. Developmental Psychology, 1970, 2, 412-422.

Lovell, K. Concepts in mathematics. In J. Klausmeier and W. Harris, Analyses of concept learning. New York: Academic Press, 1966.

Lovell, K. The growth of basic mathematical and scientific concepts in children. London: University of London Press, 1961.

Maccoby, M. and Modiano, N. On culture and equivalence. In J.S. Bruner (Ed.), Studies in cognitive growth. New York: John Wiley and Sons, 1966.

Mehler, J. and Bever, T.G. Cognitive capacity of very young children. Science, 1967, 158, 141-142.

Michotte, A. The perception of causality. Methuen & Co., Ltd., 1963.

Mischel, T. (Ed.) Cognitive development and epistemology. New York: Academic Press, 1971.

Murray, B. Cognitive conflict and reversibility training in the acquisition of length conservation. Journal of Educational Psychology, 1968, 59, 82-87.

Pascual-Leone, J., and Bovet, M.L apprentissage de la quantification de l'inclusion et la theórie opératoire. Acta Psychologica, 1966, 25, 334-356.

Piaget, J. Apprentissage et connaissance. In La logique des apprentissages, by M. Goustard; P. Greco; B. Matalon, and J. Piaget, pp. 159-188. Études d'épistémologie génétique, vol 10. Paris: Presses Universitaires de France, 1959.

Piaget, J. The child's conception of number. London: Routledge and Kegan Paul, 1952.

Piaget, J. Genetic epistemology. New York: Columbia University Press, 1966.

Piaget, J. Logic and psychology. New York: Basic Books, Inc., 1957a.

Piaget, J. Logique et equilibre dans les comportements du sujet. In L. Apostel, B. Mandelbrot, and J. Piaget (Eds.), Logique et equilibre. Études d'épistémologie génétique, 2. Paris: Presses Univ. de France, 1957b.

Piaget, J. The origins of intelligence in children. New York: International Univ. Press, 1952.

Piaget, J. Play, dreams, and imitation in childhood. New York: Norton, 1951.

Piaget, J. Psychology and epistemology. New York: Grossman Publishers, 1971a.

Piaget, J. Science of education and the psychology of the child. New York: The Viking Press, Inc., 1971b.

Piaget, J. Six psychological studies. New York: Random House, 1967.

Piaget, J. Structuralism. New York: Basic Books, Inc., 1970.

Piaget, J. Quantification, conservation, and nativism. Science, 1968, 162, 976-981.

Piaget, J. and Inhelder, B. The child's conception of space. London: Routledge and Kegan Paul, 1956.

Piaget, J. and Inhelder, B. Mental imagery in the child. New York: Basic Books, Inc., 1971.

Price-Williams, D., Gordon, W. and Ramirez III, M. Skill and conservation: A study of pottery-making children. Developmental Psychology, 1969, 1, 769.

Shantz, C. A developmental study of Piaget's theory of logical multiplication. Unpublished doctoral dissertation, Purdue, University, 1966.

Sigel, I.E. Reflections. In I.E. Sigel and F.H. Hooper (Eds.), Logical thinking in children. New York: Holt, Rinehart and Winston, 1968.

Sigel, I.E. and Hooper, F.H. (Eds.) Logical thinking in children. New York: Holt, Rinehart, and Winston, Inc., 1968.

Smedslund, J. Patterns of experience and the acquisition of conservation of weight. Child Development, 1968, 39, 515-526.

Smock, C.D. Mathemagenic activities program: A model for early childhood education. (Working Paper #1) Athens, Georgia, Mathemagenic Activities Program, University of Georgia, 1969.

Smock, C.D. and Holt, B.G. Children's reactions to novelty: An Experimental study of "curiosity motivation." Child Development, 1962, 33, 631-642.

Stevens, S.S. The operational basis of psychology. American Journal of Psychology, 1935, 47, 323-330.

Suppes, P. Facts and fantasies of education. (Psychology and Education Series, Tech. Rep. No. 193) Stanford, California, Institute for Mathematical Studies in the Social Sciences, 1972.

Wallach, L., and Sprott, R.L. Inducing number conservation in children. Child Development, 1964, 35, 1057-1071.

White, R.W. Motivation reconsidered: The concept of competence. Psychological Review, 1959, 66, 297-333.

Wholwill, J.F. The place of structured experience in early cognitive development. Interchange, 1970, 1, 13-27.

Two Special Aspects of Math Labs and Individualization: Papert's Projects and Piagetian Interviews

Robert B. Davis
Director, The Madison Project
University of Illinois (Urbana)

Math labs and individualization have moved into a new era. For the next few years, the important new emphasis will be on "Turtle Labs", PLATO terminals, and new diagnostic procedures.

What Is Old

During most of the decade 1963-1972 the effort toward "math labs" and manipulatable materials in the United States was in the direction of learning how to use a "lab" and "exploratory" approach in school mathematics. This involved learning about appropriate physical materials, from Cuisenaire rods and geoboards to pebbles, bottle-caps, and string; it involved identifying worthwhile explorations that students could undertake (as in the formula for the number of moves in the Tower of Hanoi puzzle, or in Pick's Theorem for geoboards, etc.); it involved developing the strands of mathematics (such as graphs and functions) that could make the lab explorations fruitful; it involved re-consideration of classroom layout (e.g., creation of a "math lab corner") and a time schedule that would facilitate a hands-on approach; and it involved developing an explicit rationale to explain to outsiders what this was all about. The recent NCTM Annual Meeting in Houston, Texas, was dramatic testimony to the completion of one stage in this historical process: among the commercial exhibits, more space was devoted to manipulatable materials than to traditional textbooks, and the major excitement was around manipulatable physical materials, computers, self-study machines, and so on.

The "how-to-do-it" decade drew heavily upon the work already done in Great Britain; the general impact of British programs is well-summarized in two books by Edith Biggs, one of the leaders of the British movement:

> The Schools Council, Mathematics in Primary Schools, Curriculum Bulletin No. 1, London, England, Her Majesty's Stationery Office, 1966.
>
> Edith Biggs and James MacLean, Freedom to Learn. An Active Learning Approach to Mathematics. Addison-Wesley (Canada), 1969.

Another excellent book, covering virtually every aspect of the "experience" approach to learning mathematics is:

> E.M. Williams and Hilary Shuard, Primary Mathematics Today, Longman, 1970.

Among observers of the American scene, Silberman has viewed this movement with approval (Charles E. Silberman, Crisis in the Classroom. The Remaking of American Education. Random House, 1970. Cf. especially pages 296-297.), and Morris Klein has almost completely ignored it

(Why Johnny Can't Add. The Failure of the New Math. St. Martin's Press, New York City, 1973).

As the use and rationale for math labs developed further, many educators were led to "open education." But -- while most of this work has not yet been effectively implemented in most schools -- on the research level this is now all "old hat", at least in the sense that excellent programs do exist in some places (e.g., in the Bank Street School in New York City), and excellent written discussions already exist (cf. Sections C and D in the bibliography).

What Is New

But the point is that we are entering a new era, and the task before us is no longer the light-hearted discovery of delightful practices in England, but rather the job of looking beneath the surface to glimpse the foundation we are building on. Consequently, the present note deals with three specialized topics that are suggestive of the tasks that face us nowadays. The first of these is the PLATO project the second is Seymour Papert's use of "Turtle Labs" in elementary school mathematics, which may point to a wholly new approach both to curriculum and to learning experiences and the third is an adaptation of Piagetian interview procedures by Jack Easley, Stanley Erlwanger, and their colleagues, which has demonstrated a quite unprecedented capability of revealing a child's ideas about mathematics The children's ideas, as thus identified, are far different from what anyone suspected. In fact, this discrepancy is potentially the most revolutionary thing in mathematics education today, and surely one of the most interesting.

PLATO

The PLATO Project at the University of Illinois in Urbana-Champaign is the largest educational computer system in the world. Its Director is Professor Donald Bitzer, a twentieth-century Thomas Edison who has personally invented many new hardware and software devices that make the system simultaneously one of the most flexible, yet also the cheapest, of the available computer systems. Earlier versions of the system -- called PLATO I, PLATO II, and PLATO III -- have provided computer-based instruction at the University of Illinois for the past twelve years. The greatly improved PLATO IV system is now being phased in and -- besides continuing to provide university-level instruction -- will soon begin to offer instruction in several community colleges, and in reading and mathematics at the elementary school level.

This is perhaps not the place for a detailed description of the uses and potentialities of PLATO, but at a more modest level it is surely true that a PLATO terminal in the corner of the room can be the most important item in a math lab, and a cluster of PLATO terminals can provide the best delivery system for individualized programs that presently exist. For information write to: Computer-based Education Research Laboratory, University of Illinois, Urbana, Illinois 61801.

Papert's "Turtle Labs"

An entirely different use of computers (and other technology) is being developed by Seymour Papert and his colleagues, at the Artificial Intelligence Laboratory at M.I.T. Papert does not use his computers to "teach" children anything; rather, he uses the computers to provide a new kind of environment that a child can live in. Papert's rationale is subtle and profound, and cannot be reduced to a few brief propositions -- but perhaps some brief remarks can serve to suggest Papert's analysis. "Suppose", Papert says, "your daughter takes a course or two in French. Will she learn to speak French?" "No", he answers, "probably not -- not really. But does this mean that she is somehow unable to learn to speak French? Clearly not, for if she grew up in France she would speak French, quite fluently." This, then, gives us an important clue -- perhaps, instead of taking a course in French, we should live, for a time, in France. Perhaps, instead of taking a course in mathematics, a child should live for a while in Mathland. But where is Mathland? It is where children can talk with Mathematical Beings about matters that interest children -- using, of course, the language Mathematics.

Well, where on earth is that?

Nowhere, Papert replies, unless we build it somewhere. He has built it somewhere -- specifically, in schools in Lexington and Concord, Massachusetts, and in Exeter, England. The Mathland that Papert has built he calls a "Turtle Lab." In it there are music boxes that produce music, under computer control. There are electric typewriters that type whatever the computer tells them to type. There are some electrically-operated wheeled vehicles that role around the floor, following orders that they receive from a computer. There is a television tube ("CRT") that displays whatever the computer orders it to display. There is a marionette show where the choreography is determined by the computer.

And who tells the computer what to do? The child, of course! But since the computer speaks Mathematics, the child must converse with the computer in Mathematics.

Notice that the child does NOT first "study" mathematics, then "apply" it to the computer. No more does he first study English, then apply it to telling his parents that he wants a bottle, or the blanket to cover him up. He begins using English -- and, in the Turtle Lab, mathematics -- to express himself. A child's initial learning of English comes from using it to pursue worthwhile goals. Similarly, in the Turtle Lab, a child acquires a mastery of mathematics by using it to pursue worthwhile goals.

There is, in fact, a special version of the language Mathematics -- in reality it is called LOGO -- that Papert developed that allows any child who can read, write, and count, to tell computers what he wants them to do. Papert has arranged for his computers to be programmed so that they understand the LOGO language, and hence carry out the things the children tell them to do In a sense, Papert's LOGO-speaking computer is an infinitely docile, perfectly obedient pet -- a kind of super-dog -- that never tires of carrying out the child's commands. Of course, computers are so obedient -- or, if you prefer, so stupid -- that you must always be careful to tell them exactly what you mean, because they will follow your instructions to the word, to the letter, to the decimal point. You must make sure you say what you mean ..., and you must say it in the language Mathematics.

In order to permit comparisons between Papert's Turtle Lab and various other learning settings, we need to mention briefly one or two other aspects of Papert's approach. "You learn", Papert argues, "by doing, by thinking about what you are doing, and by talking about what you are doing." Papert's program is individualized in the same sense that wood-working shop is: one child is making a bookcase, which he will take home, another child is making a toy sail boat, which he will take home. In the Turtle Lab an analogous thing happens: one child is working out the choreography for a puppet show, another child is programming the computer so that one of the "turtles" will become skillful at escaping from mazes. In this sense, the Turtle Lab is individualized. But in a more important sense it is not individualized at all. Just as the child making the sail-boat may ask advice from the child making the bookcase, so the children in the Turtle Lab often talk with one another to get suggestions on how to overcome various technical problems.

To make such discussions even more fruitful, Papert has developed a heuristic meta-language, reminiscent of Polya, to aid children in talking about their work. They divide large problems into smaller ones, they look for "bugs" (computer talk for inadequacies in algorithms), they try simpler analogous problems, etc. This emphasis on analyzing what you are doing is a central, and major, part of Papert's learning environment.

It is also worth pointing out that in the Turtle Lab, as in woodworking shop, children work on projects, not merely assigned problems. Projects are (mainly) chosen by the children themselves, often designed by the children themselves, and are permanently stored in the computer for use whenever desired. This makes possible a great cumulative power. At the start of the school year, in September, children program the computer to execute relatively simple things. These computer programs are called procedures. A September procedure might be as simple as having the computer draw a triangle on the CRT. But after it is written, each procedure is named, and the computer will execute the procedure whenever the name is used in later procedures. One fifth-grade boy extended an early triangle-drawing procedure to a procedure that would draw an arbitrary broken line -- which could close on itself to form a triangle, or a square, or an octagon, etc., but need not do so. Using this procedure, the boy wrote other procedures to draw fish, aquatic plants, and finally a whole aquarium scene.

Looking at the aquarium scene one is overwhelmed by its complexity -- to think that a 10-year-old boy told a computer how to draw it! -- but the important point is that it is built out of less complex pieces that the boy assembled all year long. Nor did he, in September, have any idea that he would make the aquarium scene months later. Rather, he began by making something that looked as if it might be useful. Then he made more things that looked as if they might be useful -- or might be fun to make. As he made new things, he saw opportunities to incorporate as components some of the things he had done earlier. Thus he came to build very complicated things that were, in fact, made up of simpler things used as building blocks.

This cumulative use of past work has great power, and, emotionally, it shows a child that the things he has been making are worthwhile. It is the opposite of making something only to throw it away. In the Turtle Lab, you make things in order to keep them, enjoy them, and use them -- perhaps as pieces in something even more complicated.

Information on Papert's Turtle Labs can be obtained by writing to him at this address: Massachusetts Institute of Technology, 545 Technology Square, Cambridge, Massachusetts 02139.

A Child's View of Mathematics

Until recently it had been fashionable to say that "ideas" are inaccessible, we cannot know them, and therefore we must content ourselves with considering "behavior." A witticism summarized 5 decades or so of American psychology by saying: "Man, having lost his soul and his freedom, has now lost his mind." This was an unfortunate point of view for mathematics, because mathematical entities are ideas, they exist nowhere except in the human mind. The study of mathematics is necessarily the study of concepts, of cognitive constructs. Deprived of a concern for ideas, one is automatically *a fortiori* deprived of mathematics.

For some time there has been intellectual warfare between two factions of educators who espouse opposite sides of this question: the one side emphasizing a focus on the child's ideas, while the other side advocates the avoidance of "ideas", and a focus on externally-observable "behavior". (Cf., e.g. Rising, et al., 1973.) To this abstract intellectual dispute there has recently been added a lively empirical confrontation. The "behavior" approach, in this fight, is presently represented by a paper-and-pencil "individualized" mathematics curriculum created according to the behaviors to be developed, which was authored by some of the leading proponents of the method of focussing on observable behaviors. The "idea" side is represented by Professor Jack Easley, of the University of Illinois, and a group co-workers, notably Stanley Erlwanger, who have adapted Piaget's clinical interview technique in order to investigate children's ideas about mathematics. The Easley-Erlwanger group did not in fact plan a confrontation. What they had in mind was an exploration of children's ideas about mathematics. Fate provided a direct confrontation of the two philosophies. By accident, many of the children studied by the Easley-Erlwanger group have, for several years, been students in the "individualized" curriculum created according to "behavior" criteria. Hence -- by accident -- the two points of view are now engaged in direct battle on an empirical battle ground. What ideas do children have about mathematics, and where do these ideas come from? But the children being studied are in a school program that de-emphasizes "ideas" -- hence the dramatically focused confrontation.

What Easley and Erlwanger have been finding, in pursuit of children's mathematical ideas, is perhaps the most exciting thing now going on in mathematics education.

The case of Benny is typical. Benny is a sixth grader, has been in the individualized math program since the beginning of grade two, and was identified by his teacher as doing well in mathematics, being in fact one of the most successful students in the class.

Erlwanger has audiotaped many hours of interviews with Benny; these tapes are now being carefully transcribed. I make use, here, of a preliminary transcription that may have involved some minor paraphrasing of language, but the actual mathematics has been carefully checked for accuracy. It is just as Benny did it.

Converting fractions to decimals:

> E: How would you write 2/10 as a decimal or decimal fraction?
>
> B: One point two (writes: 1.2)
>
> E: And 5/10?
>
> B: (writes: 1.5)

Asked to explain his procedure, Benny said (for the 5/10 example):

> B: The one stands for ten then there's the decimal point; then there's 5 -- shows how many ones.

Other conversions made by Benny:

> 400/400 = 8.00
>
> 9/10 = 1.9 ("The decimal means it's dividing till you can get from one nine that will be 19, and in that 1.9, the decimal shows how many tens.")
>
> 429/100 = 5.29
>
> 3/1000 = 1.003
>
> 1/8 = .9
>
> 1/9 = 1.0
>
> 4/6 = 1.0

It is probably already clear that Benny has his own methods for doing things. They <u>are</u> methods, and he uses them very methodically -- even thoughtfully. But, unfortunately, Benny has not learned good ways to think about arithmetic.

Here is some more of Benny's work:

E: And 4/11 ?

B. 1.5

E: Now does it matter if we change this (pointing to the 4/11) and write instead 11/4 ?

B: It won't change at all; it will be the same thing ... (writes: 11/4 = 1.5)

E: How does this work? 4/11 is the same as 11/4 ?

B: Yah ... because there's a ten at the top. So you have to drop that 10 ... take away the 10, put it down at the bottom ...

(writes: 11/4

then ~~1~~1/4

then ~~1~~1/14

then 1/14.)

So, really, it will be 1/14. So you have to add these numbers up, which will be 5 ... then 10 ..., so 1.5.

For the inverse process, converting a decimal to a fraction, one gets a choice of many "correct" answers: thus .5 can be 3/2, or 2/3, or 1/4, or 4/1.

<u>Addition of decimals</u>:

E: Like, what would you get if you add .3 + .4?

B: That would be ... (writes: .3 + .4 = .07).

E: How do you decide where to put the point?

B: Because there's two points; at the front of the four and the front of the three. So you have to have two numbers after the decimal, because ..., you know ... two decimals. Now like if I had .44 + .44, I have to have four numbers after the decimal.

E: How about 3/10 + 4/10

B: (writes: 3/10 + 4/10 = 7/10)

Further answers by Benny:

4 + 1.6 = 2.0

7.48 - 7 = 7.41

Addition of fractions:

Answers by Benny:

2/1 + 1/2 = 3/3 = 1

(" 2/1 + 1/2 is just like saying 1/2 + 1/2, because 2/1, reverse that,(writes 1/2), so it will come out one whole no matter which way. One is one.")

Recall that Benny had been identified by his teacher as a very successful student in mathematics, perhaps the best in his class. Nearly every 5th or 6th grader studied by Erlwanger has had mainly wrong ideas about arithmetic -- although many of these children were regarded as successful by their teachers. Apparently the usual diagnostic procedures used in school are not adequate to pick up the erroneous ideas many of these children have. Moreover, the children range in I.Q. from 100 to well over 120, one would expect that they should be able to understand mathematics.

Returning to Benny's work, shown above, it is important to note that there is a pattern to all of this. Not only is Benny consistent within his own scheme, but as one looks at a large number of children and compares their methods with the instructional program, one sees that where the students have gone astray, their errors match up with identifiable features of the program of instruction that they are experiencing in school. Obviously, one cannot prove that the instructional method is responsible for the particular kinds of errors that have been built into the children's cognitive schemata, without undertaking a sizeable study of children in different school curricula. This has not yet been done -- but Erlwanger and Easley do have it on their agenda for the future. For the moment one can merely look at the children's ideas, look at the course of instruction, and ask: is it reasonable or credible that this course of study should lead children to form these ideas?

Features of the Program of Instruction

The program of instruction that Benny and the other children were pursuing in school had, among others, the following features:

1. Working alone. It was "individualized" in the sense that each student worked alone, by himself. This had several consequences, noted below.

2. Paper-and-pencil. The program existed on pieces of paper -- printed booklets, written answers, written judgements (by the teacher) on some of those answers (but very sketchy judgements, mainly limited to "right" and "wrong").

3. Absence of physical materials. The program did not use any "math lab" materials or experiences: no measuring cups, no actual cooking, no meter sticks, no making maps of the school yard, no Cuisenaire rods, no Dienes MAB blocks etc.

4. No discussions. Because each child worked alone, he had no opportunity to hear (or participate in) classroom discussions, peer-group teaching, explanations of what he thought he was doing, etc.

5. No heuristic meta-language. Obviously, then, no special way of talking about what you are doing was developed (quite in contrast to Papert's very careful development of a heuristic meta-language).

6. No diagnosis. No serious diagnosis of a child's ideas was carried out. (The teacher was unaware, for example, that many of the children considered that, in

$$8.3$$

the 8 indicated "the wholes", and the 3 indicated "the parts", and that they therefore believed that

$$8.3 = 3/8.)$$

7. Section tests. The material was divided into sections (or "levels"), and, in order to complete a section and move on to the next section, a child had to take a section test and get at least 80% of the answers right. This arrangement was intended to avoid cases like Benny's, but it clearly failed to do so (Why it failed is an interesting question!).

8. Checking answers. As a child worked on problems, either he would check them himself against an answer key, or a para-professional would check them.

9. Absence of counter-examples. Given good diagnostic work and a high level of teaching, one of the most important activities would undoubtedly have been confronting students with contradictions and counter examples to their wrong generalizations. Given the conditions of this particular curriculum, this sort of activity was impossible, and did not occur.

10. Competition and haste. The program was highly competitive, and the competition was based on speed. A race-track drawing in the classroom showed who was the leader (in the sense of having completed the most "levels" or sections), who was next, etc. Every child's standing in class was displayed for everyone to see. (Comparisons were only on speed; no consideration was given to creativity, depth of understanding, etc.)

11. Homogeneous problem sets. Any single problem set dealt with one subject. Consequently, careful analysis of how to attack a problem was never involved; looking at a problem for clues as to how to proceed didn't occur. You did what was being covered in that particular lesson.

12. Task and feedback. A typical task -- say, adding decimals -- was introduced by a sketched-out instance, such as:

$$.2 + .3 = .5$$

A student was supposed to stare at this example, then go and do likewise. What this meant was that he guessed at a pattern, then tried to use his pattern to answer similar questions, after which he compared his answers with those in the key. If they disagreed, he used this feedback -- very different in essence from a counter-example or a sharply pinpointed contradiction -- to re-adjust his pattern, and to build up his own rationalization as to what was going on.

This question is so important that it deserves further discussion. Suppose you have "solved" a problem -- the subsequent discussion depends on the kind of problem -- b[illegible] say it was

$$1/2 + 1/3 = 2/5.$$

Notice that there are many different kinds of feedback that you might receive, and notice how important the [illegible]nces are. Here are a few possibilities:

i) You are told that your answer is wrong (this is the only feedback which this program usually provided);

ii) You are asked to make a picture of a pie (or a rectangle), color 1/2 of it blue, 1/3 of it red, and see how much the sum seems to be;

iii) Someone goes over your procedure with you, one step at a time, and discusses each step.

Thinking of Mathematics as Chinese Orthography

For these children, mathematics consists of symbols on papers. The object of the game is to get the right symbols in the right places. (I might take this attitude if you asked me to copy some Chinese characters.)

This was revealed [illegible] Berry's examples, considered above.

It is also shown by the following work of a 7th grade student (not in Erlwanger's sample):

Q: How else might you write

$$\frac{6^5}{6^2}?$$

A: (by 7th grade boy):

$\frac{6^5}{6^2} = \frac{6^3}{6}$ subtracting exponents

$= \frac{5^{13}}{6}$ he tried to take 6 from 3, couldn't, and "regrouped" the numerator.

$= 5^7$ subtracting 6 from 13.

You couldn't make this kind of error if exponents meant something to you: if you had a clear notion of the size of various numbers, etc. You can only make this kind of error when you are applying rote memorized procedures to relatively meaningless symbols.

The Small Procedures Are Correct

In the examples above, and in nearly all we have studied, the small "pieces" of procedures are correct. They are by no means random. Indeed, in most cases they could be classified as regression to earlier ideas. Cf. the example with exponents. When the boy found himself confronted with

$$\frac{6^3}{6},$$

since it did not have for him a clear meaning he had to do something. What he did was to use the correct algorithm for this problem:

$$\begin{array}{r} 63 \\ -\ 6 \\ \hline \end{array}$$

namely, "re-group" as follows

$$\begin{array}{r} {}^{5}\,{}^{1} \\ \cancel{6}\,3 \\ -\ \ 6 \\ \hline \end{array}$$

and subtract

$$\begin{array}{r} {}^{5}\,{}^{1} \\ \cancel{6}\,3 \\ -\ \ 6 \\ \hline 5\,7 \end{array}.$$

Similarly, Benny's rule for placing the decimal point in the problem

$$.3 + .4 = .07$$

<u>was</u> <u>correct</u>, but belonged with the problem

$$.3 \times .4 ,$$

and not with the problem in addition.

Throughout hundreds of examples the pattern remains clear: children use correct mini-procedures, but don't always pick the correct one for the actual problem at hand. The little pieces are correct, but belong somewhere else.

How much practice did the school program provide in the problem of <u>selecting</u> procedures? Very little, since most problem sets were homogeneous. Once you found, perhaps by trial and error, some procedure that yielded an answer the teacher would accept, you merely used this same procedure on all the other problems on that page.

<u>Confusion of Similar Stimuli</u>

It is especially worthwhile to compare, side-by-side, the actual problems children were supposed to be working on, with the problems from which their (inappropriately chosen) mini-procedures came. Here are a few:

i) Benny (grade 6; "at the top of his class" according to teacher's rating):

$$.3 + .4 = .07$$

(That is to say, while Benny's addition mini-procedure to add 3 and 4, getting 7, was correct and appropriate, his mini-procedure for placing the decimal point was not appropriate for this problem -- but it would have been appropriate for

$$.3 \times .4$$

Hence, in selecting a mini-procedure for locating the decimal point, Benny confused these two stimuli"

$$.3 + .4 \quad \text{vs.} \quad .3 \times .4$$

ii) Benny, again.

$$\frac{4}{11} = 1.5$$

The partial procedure concerned with the digits ("use 4 and 11 to get 15") is a correct one, but Benny has confused the stimuli

$$\frac{4}{11} \quad \text{vs.} \quad \begin{array}{r} 4 \\ \underline{11} \end{array}$$

In the vast majority of these cases, the two stimuli which were confused were, in fact, actually quite similar. Apparently the children haven't learned to make finer discriminations between rather similar stimuli.

But, before one adds to the curriculum some units on discriminating rather similar stimuli, it is worth noting that, in curricula based on the meanings of the symbols, and giving priority to thinking about these meanings, it appears to be the case that this problem of stimulus confusion is far less severe. Perhaps it is not advantageous to conceptualize these symbols as stimuli -- maybe it is better to consider them as marks that are supposed to refer to some important idea, and to focus attention on these ideas.

Children Make Up Their Own Rules

In this school's course of instruction, relatively little attention is paid to why something is, or is not, true. There is also no chance at all for a student to explain his line of thought to a teacher. Consequently, whatever ideas a child gets -- if they produce, for the time being, acceptable answers -- will go unchallenged and unanalyzed.

Consider the case of Natalie, a 10-year old fifth grader, whose recorded I.Q. is listed as 99, but whose actual I.Q. is almost certainly higher than that.

Asked to add

$$.4 + .3$$

Natalie wrote

$$.4 + .3 = .7$$

This was like one of her usual problem sets. She wrote the answer the teacher would expect -- hence no one discussed with Natalie how she was thinking about this problem.

Asked to add

$$4. + 3. =$$

Natalie wrote

$$4. + 3. = 7.$$

Again, what one would expect. Again, no one discussed with Natalie how she was thinking about this problem. After all, she got the right answer, didn't she?

Here is part of Erlwanger's interview:

E: $.4 + 3. =$

N: $.4 + 3. = .7.$

Notice the creation of a new symbol

$$.7.$$

which presumably can have no meaning to Natalie as a number. How large is .7.?

But Natalie has finally revealed to us something of the way she is thinking about those little periods that they use in mathematics. "If they're on the left in the question", Natalie seems to think, "put them on the left in the answer." Consequently, when -- for the first time -- Natalie encountered them on both left and right in the question, she decided to put them on both the left and the right in the answer.

Children make up their own rules. It often pays to find out what those rules are. It is not enough to know that, for the moment, those rules seem to be producing right answers. They may still be wrong rules.

The Absence of Meaning

Perhaps this is really a restatement of the Chinese orthography theme. Benny again provides an example:

Asked to add

$$.3 + .4,$$

Benny wrote

$$.3 + .4 = .07$$

Asked, immediately thereafter, to add

$$\frac{3}{10} + \frac{4}{10},$$

Benny wrote

$$\frac{3}{10} + \frac{4}{10} = \frac{7}{10}.$$

Benny was not disturbed by any incongruity here; yet obviously for anyone to whom the meanings of these symbols are important, the gross inequality in the size of 7/10 and .07 is conspicuous, salient, and arresting -- not a matter that one could easily overlook, by any means. Yet Benny did not see it at all (or else he didn't see the relation between

$$\frac{3}{10} + \frac{4}{10}$$

and

$$.3 + .4 \ ).$$

The Child's Use of Feedback

We have remarked earlier that the school program provided the children with no feedback that related to meaning -- only feedback about right or wrong answers. Hence, for the children, math became a kind of guessing game. You tried something, and waited to see what happened. If necessary, you adjusted things, and tried again. This use of feedback is appropriate and inevitable -- in math, or in learning to bowl, or learning to ride a bicycle -- but the feedback must relate to central meanings, not to peripheral details. Math as a meaningless guessing game was revealed in the language used by many children. For example, Lori, a 6th grade girl with an I.Q. of 123, asked to write

$$\frac{21}{1000}$$

as a decimal, said:

"I'd probably write 1000.21."

The use of the phrase "I'd probably write ..." is as significant and as characteristic as the wrong answer she did write.

Benny stated this even more emphatically:

E: It (the process of finding answers to problems) seems to be like a game.

B: (Emotionally) Yes! It's like a wild goose chase!

E: So you're chasing answers the teacher wants?

B: Yah! Yah!

E: Which answer would you like to put down?

B: (Shouting) ANY! As long as I know it could be the ri t answer.

Speaking of his teacher, and her paraprofessional aide, Benny says:

B: They mark it wrong because they just go by the key. They don't go by if the answer is true or not. They go by the

> key. It's like if I had 2/4, they wanted to know what it was, and I wrote down one whole number, and the key said a whole number, it would be right ... no matter if it was really wrong.

What Is Mathematics? What Do the Children Think Mathematics Is?

I have argued elsewhere that one of the main goals of mathematics instruction is to end up with children who believe that mathematics is a reasonable and sensible response to a reasonable and sensible problem. One of Papert's 5th grade students defined mathematics by saying: "For any challenge, there's a smart way to approach it, and a dumb way to approach it. If you approach it the smart way, you're doing mathematics."

The children in Erlwanger's sample see mathematics very differently from this.

We've already seen the view that mathematics is "a goose chase", a kind of blind man's bluff in search of answers that the teacher will accept.

But Benny also had a different view of the nature of mathematics -- a far subtler view, that is intriguingly dualistic. Here is Benny's description, as paraphrased by Erlwanger:

> Benny's view about answers is associated with his understanding of operations in mathematics. He regards operations as merely rules; for example, to add 2 + .8, he says: 'I look at it like this: 2 + 8 = 10; put my 10 down; put my decimal in front of the zero.' However, rules are necessary in mathematics 'because if all we did was to put any answer down, (we would get) 100 every time. We must have rules to get the answer right.' He believes that there are rules for every type of problem; for example, he says: 'In fractions we have 100 different kinds of rules.' These rules were 'invented by a man or someone who was very smart.' This was an enormous task because 'It must have took this guy a long time ... about 50 years ... because to get the rules he had to work all of the problems out like that ...'

Hence we see Benny's view of math as arbitrary, expressed (for example) in the 100 rules for fractions that were worked out by "someone who was very smart." The rules serve the game-like purpose of guaranteeing that there will be some winners and some losers: "because if all we did was to put an answer down, (we would get) 100 every time," which Benny obviously considers unacceptable. In all of this we see the arbitrary, game-like, senseless face of mathematics -- as experienced by Benny.

But Benny also sees an objective reality lurking there somewhere, for he says

> It must have took this guy a long time ... about 50 years ... because to get the rules he had to work all of the problems out like that ...

So, apparently, "this guy" was working against some objective criteria, criteria obviously unknown to Benny But Benny doesn't seem to want to know these objective criteria -- Benny wants to know the 100 rules that "this guy" finally ended up with.

Both sides of this duality are expressed by Benny elsewhere in the interviews, as well. For example, as Erlwanger writes:

> Benny also believes that the rules are universal and cannot be changed. The following excerpt illustrates this view:
>
> E: What about the rules. Do they change or remain the same?
>
> B: Remain the same.
>
> E: Do you think a rule can change as you go from one level to another?
>
> B: Could, but it doesn't. Really, if you change the rule in fractions it would come out different.
>
> E: Would that be wrong?
>
> B: Yes. It would be wrong to make our own rules; but it would be right. It would not be right to others because if they are not used to it and try to figure out what we meant by the rule it wouldn't work out.

Benny's objectivity here seems, however, considerably less objective -- not based on an underlying reality, but rather on a desire to be fair to other people by not changing the rules in the middle of the game.

On Eliminating Cognitive Dissonance

In Erlwanger's interviews, Benny got wrong answers nearly all the time. (That his teacher nonetheless considered him very successful is something of a mystery, but perhaps her own view of mathematics was that it was a kind of Easter egg hunt, and she found Benny an avid and energetic hunter for hidden treasures. This would contrast with my own view, which presumes the existence of a unique reality for which we are attempting to construct various descriptions, and a child's job in learning mathematics is to build up inside his own head cognitive structures that reflect this reality with the most profound accuracy that can be attained. In this latter view, Benny was doing very badly, since he nearly always missed the central point mathematically, but in terms of the Easter egg view -- there being no profound grand strategy determining the placing of the Easter eggs, and hence no external structure to be faithfully modelled cognitively -- Benny's energetic pursuit could be considered the ideal pattern of learning. You find Easter eggs by scurrying around, all over the place.)

In any event, Benny's answers in the interviews were mostly wrong. From this, and from remarks he made during the interviews, it seems safe to infer that Benny's answers in class were also mainly wrong.

Now, in this particular school program, a child checks his answers against a key, to see whether he is right or wrong. This must have been a sore trial for Benny. But a very interesting -- and in some senses tragic -- thing occurred: Benny thought about this situation at length, and finally succeeded in developing a personal philosophy that brought him peace of mind. So much peace of mind, unfortunately, that he lost all motivation to make his answers match other people's. In this he resembled some psychotics: he was so contented with his world view that he didn't want to change it.

What Benny did was this: he noticed that you might have an answer like,

$$\frac{2}{4},$$

whereas the key might have an answer like

$$\frac{1}{2}.$$

From "just looking at them" you might conclude that these answers were different -- but there is a way out! There exists a rule that lets you do something to one of these

$$\frac{2}{4} = \frac{2 \times 1}{2 \times 2} = \frac{\cancel{2} \times 1}{\cancel{2} \times 2} = \frac{1}{2}$$

so that you obtain the other. They looked different, but once you used the appropriate transformation rule on them, they were really the same!

Benny generalized this: if only you knew enough of these transformation rules, you could always show that your answer really matched the answer in the key! Benny had invented a large repertoire of such rules. Here is one out of many:

E: How would you write

$$\frac{11}{4}$$

as a decimal?

B: (writes) $\frac{11}{4} = 1.5$

E: Now does it matter if we change this (pointing to the $\frac{4}{11}$)

and write instead $\frac{11}{4}$?

B: It won't change at all; it will be the same thing ... (writes: $\frac{11}{4} = 1.5$)

E: How does this work? $\frac{4}{11}$ is the same as $\frac{11}{4}$?

B: Yah ... because there's a ten at the top. So you have to drop that 10 ... take away the 10, put it down at the bottom ... (writes):

$$\frac{11}{4}$$

then $\frac{\cancel{1}1}{4}$

then $\frac{\cancel{1}1}{14}$

then $\frac{1}{14}$.)

So, really, it will be $\frac{1}{14}$. So you have to add these numbers up, which will be 5 ... then 10 ... so 1.5.

These rules had one good effect on Benny -- they let him be happy about his "success" in mathematics -- but they also had a very unfortunate effect. It was next to impossible to convince Benny that any of his answers were wrong. What tools could you use to convince him? Show him the correct answer? No, he responded by showing you how his answer was really the same as yours. Show him contradictions? No, his transformation rules enabled him to reconcile any apparent contradictions. Show him that his answer was nonsense -- for example, much too large, or much too small? No, all the symbols were meaningless to Benny. Considerations involving meaning did not reach him. To Benny, all answers were nonsense, so his were exactly as good as yours.

Implications

Adaptations of Piaget's clinical interview procedures seem to be capable of revealing much more of what is in a child's mind than other common methods can. If this potential can be developed by teachers -- and by parents -- we can get a far deeper assessment of the successes and failures of our school programs. And if, as has been the case in the Erlwanger study, many of these programs are revealed as catastrophic failures, then various vigorous actions may develop in consequence.

Meanwhile, for those of us who (as I do) work on the creation of school programs, here is a specially significant kind of handwriting on the wall.

For details, write to: Stanley Erlwanger, 1210 West Springfield Avenue, Urbana, Illinois 61801.

PIBLIOGRAPHY

A. Papert's Turtle Labs

Wallace Feurzeig and George Lukas, "LOGO - A Programming Language for Teaching Mathematics", Educational Technology, vol. XII, no.3 (March, 1972), pp. 39-46.

Seymour Papert and Cynthia Solomon, "Twenty Things to Do With a Computer", Educational Technology, vol. XII, no. 4 (April, 1972), pp. 9-18.

See also an untitled 16 mm. film, showing 5th graders working in a Turtle Lab, that is available from Professor Papert.

B. Erlwanger's Diagnostic Interviews (and other closely related matters)

Robert B. Ashlock, Error Patterns in Computation, Charles Merrill (paper back), 1972.

Robert B. Davis, "The Structure of Mathematics and the Structure of Cognitive Development", Journal of Children's Mathematical Behavior, vol. 1, no. 1 (Winter, 1971-72), pp. 71-99.

Robert B. Davis and Rhonda Greenstein, "Jennifer", New York State Mathematics Teachers Journal, vol XIX, no. 3 (June, 1969), pp. 94-103.

J.A. Easley, Jr. and Klaus G. Witz, "Individualized Instruction - Some Observations from the Ivory Tower", Educational Technology, vol. XII, no. 3 (March, 1972), pp. 50-52.

Stanley Erlwanger, "The Case of Benny", (mimeographed). University of Illinois, 1973.

__________, The Observation-Interview Method as a Technique for Studying Children's Mathematical Thinking. Proceedings of Math Ed Conference at Florida State University. (to appear)

Herbert Ginsburg, "The Case of Peter: Introduction and Part I", Journal of Children's Mathematical Behavior, vol. 1, no. 1 (Winter, 1971-72), pp. 60-70.

__________, "Children's Knowledge and Individualized Instruction", Educational Technology, vol. XII, no. 3 (March, 1972), pp. 8-12.

Gerald Rising and Burt Kaufman, Editors, special issue of Educational Technology devoted to the case against "behavioral objectives" - fall, 1973 (to appear).

C. Math Labs

In addition to the references given in the body of the paper, see also:

Rose Grossman, "Individualizing Occurs When Thinking Becomes Viable", Educational Technology, vol. XII, no. 3 (March, 1972), pp. 36-39.

Lyn McLane, "Individualized Mathematics Programs in Elementary Schools", Educational Technology, vol. XII, no. 3 (March, 1972), pp. 15-20.

Charles Wilbur, "Record Keeping in a Math Lab", Educational Technology, vol. XII, no. 3 (March, 1972), pp. 34-36.

Especially important are the famous Nu ld Guides, available in the USA from John Wiley and Sons, Inc., 605 Third Avenue, New York, NY 10015.

See also the film I Do ... and I Understand, available from: Radim Films, 220 West 42nd Street, New York, NY 10036.

D. Open Education

The pursuit of individualization and math labs has led many U.S. educators to an interest in so-called "Open Education." An excellent reference on this is:

Kathleen Devaney, Developing Open Education in America: A Review of Theory and Practice in the Public Schools. Far West Laboratory for Educational Research and Development, 1855 Folsom Street, San Francisco, CA 94103 (1973).

See also:

Molly Brearley, et al., Fundamentals in the First School, Basil Blackwell, Oxford, England (1969).

Casey and Liza Murrow, Children Come First, American Heritage Press, New York, 1971.

Lady Bridget Plowden, et al., Children and their Primary School, (2 vols) London, England: Her Majesty's Stationery Office, 1967.

Joseph Turner, Making New Schools. The Liberation of Learning, David McKay Co., 1971.

See especially the film strip:

Sue Monell, A Teacher Looks at her Classroom, available from: Follow Through Program, Bank Street College of Education, 216 West 14th Street, New York, NY 10011.

The Role of Applications in Early Mathematics Learning

Max S. Bell
University of Chicago

I hope during this hour to open up for examination the role of applications of mathematics in early mathematical learning. It seems to me that this unsolved and neglected problem warrants considerably more attention than it has received up to now, if only because the ability to use mathematics lies at the heart of our objectives for mathematics education. Indeed I would say that mathematics education has been a failure for any person who is unable to comfortably and naturally make use of mathematics in a wide variety of situations. (I believe it follows from this that mathematics education is a failure for very large numbers of people -- perhaps the majority -- but that is a matter to be argued elsewhere.) It is possible, of course that early attention to application has nothing much to do with eventual ability to apply mathematics; still, I believe we should open up the question for investigation and reflection.

By "applications" I mean exercises that link things in mathematics to aspects of actual real world happenings. "Applications" will be considered to be different than "models" in the sense of paper and plastic and string representations, and different than invented games with mathematical content or other "embodiments" of mathematical things in the Dienes sense; though these are all unquestionably useful in learning mathematics. By "early" learning I have in mind about kindergarten through third grades or, in other terms, the late preoperational and early concrete operations parts of a child's development. By a "role" for applications, I do not have in mind any exclusive dependence on applications but merely an end to the nearly total neglect that prevails now. I am tentatively assuming that such neglect is hazardous because in the same way that important aspects of a child's general cognitive ability are probably built (or mature naturally) during the early school years, important aspects of his ability to use mathematics are probably also developing. Therefore, it may be that a poverty of early experience with applications represents a developmental loss that is difficult or impossible to make up later.

In my remarks I first want to sort out various categories of mathematics and real world links, and draw a distinction between "embodiments" as a link and "applications/mathematical" models as a link. Next, and briefly, I will try to establish that a problem exists; that applications in early learning are neglected. Then I will briefly explore several possibly good reasons for this neglect. Then, by a series of simple-minded examples, I will argue for the proposition that early attention to applications could open up many of the things we really should accomplish in mathematics education. Along the way I will make some random comments on possible connections between cognitive psychology and an early concern for applications, with no claim, however, to consistency, comprehensiveness, or expertness with respect to models of cognitive development.

Mathematics and Real World Links

I believe that we are often unclear about what we mean by "applications." There are many possible cross-connections within the world of reality, within the world of mathematics, and between these two worlds. To call all such connections "applications" makes the word mean too many things. It may clarify the matter at hand to sort out the possibilities; Figure 1 is a rough attempt to do this.

Figure 1: A Diagram of Links Between and Within Mathematical and Real Worlds

	The World of Reality		The World of Mathematics
1.	Real world situation ↔ Another (possibly simpler) real world situation		
2.			Mathematical item[1] Another (possibly more complex) mathematical item
3.	"Embodiments" in real world materials or operations on such materials	←	Mathematical item
4.	Problem situation	→	Mathematical Model

[1] The "Mathematical item" might be a single thing such as a number or equation, or a mathematical structure, or what have you.

The first part of the diagram concerns moving between various levels of abstraction while still within the real world, with no explicit resort to the world of mathematics. This would include various sorts of simulation and simplification within the real world. While there is no direct resort to mathematics, the thinking processes often resemble those useful in mathematics, and it is my belief that activities here may build important and possibly crucial background for mathematical learning.

The second situation diagrammed in Figure 1 is the well known one of moving among various levels of abstraction strictly within the world of mathematics. For example, one develops a certain piece of mathematics, then uses it as a starting point for another mathematical concept or theory. As a matter of fact, and unfortunately in many cases, the great bulk of mathematics instruction falls here. But I believe this should not be the case with respect to early mathematical learning. At some point beyond early learning, this may be the most efficient way to learn mathematics, but I can see no stage where it should be the exclusive means to mathematical learning.[2]

The possibilities indicated by the third situation diagrammed in Figure 1 have been extensively developed by Dienes, Davis, the Papys, and many others. Here one starts with mathematical ideas or concepts to be taught and looks for links to the world of reality via "embodiments" of those concepts in operations on real world objects. It may well be that teaching strategies exploiting possibilities in this area are the ones that fit child development patterns best, certainly their use in much of the Piaget oriented research is striking. I do not believe in panaceas or single dimension solutions, but I believe there is no question about the fruitfulness of the embodiments approach to early mathematical learning, and much of the discussion in this symposium will predictably be centered on it.

But at least for this one hour let us direct our attention to the fourth area indicated by Figure 1. Here one does not start with mathematical ideas and then seek embodiments of these in the real world, but rather starts with problematical situations in the actual real world, and seeks solutions by way of the intervention of mathematical models. This is very common now in scholarship, commerce, and industry, and has proved to be very fruitful.[3]

---

[2] Perhaps an emphasis exclusively on this area is misplaced even in graduate level mathematics courses. In an essay in the American Mathematical Monthly, E.T. Parker credits his success in a breakthrough on a previously unsolved conjecture of Euler to a graduate level group theory course in which his professor began every class session by referring to some aspect of the rigid motions of a square as an embodiment of the mathematics to be discussed (Parker).

[3] What is meant by the phrase "mathematical models" has been dealt with in many places (e.g., several articles in Bell, 1967). Here is a neat capsule summary: "The use of applied mathematics in its relation to a physical problem involves three stages: (1) a dive from the world of reality into the world of mathematics; (2) a swim in the world of mathematics; (3) a climb from the world of mathematics back into the world of reality carrying a prediction in our teeth." (John Synge, quoted in the American Mathematical Monthly, 68 (October, 1961), p. 799). The "dive" is the province of problem solving strategies and its result is a mathematical model. The "swim" depends on ability to manipulate, transform, and extend the mathematics itself.

For the remainder of this discussion I will use "applications" as a single word to indicate activity on behalf of mathematical modelling of real world sit ations - the fourth area diagrammed in Figure 1. There seems to be no disagreement to the proposition that such applications can enrich mathematical learning from at least the middle school years on. (It remains true, however, that in spite of endless and repeated recommendation, such activity is not yet a viable part of mathematics instruction at any level.) Many suggestions for the middle and later years are beginning to appear (Bell 1972, Tanur, Mosteller, ECCP).

But in this hour we will leave that aside in favor of a preliminary attempt to sort out what contribution attention to this fourth sort of link -- applications -- might make to facilitating the early learning of mathematics.

Are Applications Neglected in Early School Mathematics?

Having stated a problem, I should show that it exists. Is this fourth sort of link, act.al uses of mathematics from the actual world, in fact neglected in grades K-3? The answer is certainly yes with respect to the textbooks that, rightly or wrongly, dominate the child's early learning of mathematics. Examination of most such books reveals at most a few thinly scattered pages of verbal problems that very seldom make interesting connections with genuine real world issues. The answer is yes with respect to journals such at The Arithmetic Teacher -- looking over a recent year of that journal I found only three articles that gave attention to some application of mathematics and none of these were directed at the early elementary years. The answer is probably yes with respect to research in mathematics education. I have not carried out a thorough survey of all the research literature but an informal survey plus examination of several reviews of research in mathematical education turned up very little of interest on the topic under consideration here. The charge of neglect is most certainly valid with respect to the training of elementary school teachers. Again, I have not made an exhaustive survey but I know of just a single place giving courses for teachers that are explicitly directed at what might be done with applications in the school mathematical experience of youngsters. (The University of Chicago offers such courses in summer sessions.) Also, if one can judge by published textbooks for mathematics and methods courses for elementary school teachers, a concern for applications is certainly not a major emphasis in such courses.

On the other hand the neglect may not be total. There has been considerable activity on behalf of early school science by a half dozen or more curriculum writing groups. The AAAS, Minnemast, FSS, SCIS, and other projects have proposed science oriented applications for young children that would most certainly be helpful in the learning of mathematics (Hurd). However, that doesn't let us off the hook; and in any cas a science educator colleague tells me that such materials are by no means widely used in primary schools. To the extent that mathematics education curriculum projects elsewhere - for example, the Nuffield Project materi. - have inf enced United States practice, this would also lessen neglect of applications for they tend to give considerable attention to the interplay between a child and his real world. If any schools have survived in the United States that use soundly implemented John Deweyan principles, then applications of mathematics in the sense of the fourth section of Figure 1 cannot have been neglected. (I mention this last unlikely possibility only to remind us of one of the roots of much of the discussion that will take place during this symposium.)

Some Possible Reasons for Neglecting Applications in Early Mathematical Learning

Mathematics and real world links in the embodiments sense may be catching on as a significant factor in early mathematics teaching, but I believe that we must conclude that such links in the applications (mathematical models) sense play virtually no role at present in early mathematics teaching in the United States. Such widespread neglect of what on its face would seem to be important cannot have happened without reasons, and it may be that some of these reasons would compel us to the conclusion that neglect is warranted and ought to continue. I wish to examine that possibility now.

Leaving aside bad reasons for neglect, here are some possibly good reasons: first, it may be that "Young students, by and large, are not interested in applications .... They are the purest of pure mathematicians, and somewhere during the maturation process, they become sullied and begin demanding applications of the subject" (Beberman, p.11). Second, there may be dissonance and conflicts between cognitive development patterns and the requirements of applications. Third, the disparity between the requirements of applications and the meager computational skills available at early school levels may be impossible to bridge. Fourth, even if the above problems could be resolved, working on the problem of applications for the young might still justifiably be assigned low priority given the many other difficult problems we face. Finally, there may be an overall and pervasive dissonance between attention to applications and our proper objectives and curriculum - that is, with what we think children must accomplish in school mathematics. I will consider each of these briefly in the next few paragraphs, and several of them at greater length in the section that follows.

First, whether or not lack of interest of children is a barrier to using applications seems to me to be very much an open question that cannot be answered until we put our best inventive efforts into a fair trial. That is, if we make it an objective to use applications in early mathematical teaching, then we are immediately thrown on the question of appropriate learning experiences. We must, in short, carry out feasibility studies, using with children the best materials we can devise.

Second, we must consider the possibility of dissonance between developmental patterns and early attention to applications. If one takes a behaviorist view towards early mathematical learning, there should, in principle, be little dissonance for I take the behaviorist point of view to be "if you want it, do it." That is, it is a matter of outlining learning hierarchies where, for whatever applications exercise you wish to use, the appropriate prerequisites have been attended to earlier, and the reinforcement structure is such that the present piece of material is likely to be learned. Indeed, Robert Gagne, who is far more in the behaviorist camp than in the Piagetian camp, was one of the principal architects of the apparently successful AAAS science program for early science learning, and it has many mathematical elements in it. If, on the other hand, one takes a Piagetian point of view, there are probably some restraints on considering applications. For example, some applications would have to wait for the formal operations stage. But I don't believe the restraints are very severe. Attention to many applications that are tied to a child's direct experience should be possible even in the late pre-operational and in the concrete operations stages.

As to the third possible reason for neglect, disparity between the rather slow development of computational and algorithmic skills in children and the requirements of significant applications, I believe there are at least three possible ways to bridge the gap. In the first palce, the requirements of calculation can be embodied in very concrete calculating devices, e.g., Napier bones, Papy minicomputer, Hassler Whitney's minicomputer (Van Arsdel, Whitney). A second way to finesse this problem would be for teachers to act as "consultants" or "answer machines" so that when youngsters have formulated a problem to the point where calculation is called for they can call for an answer to the particular computation required. A third answer, and one that I t k may very well be forced upon us whether we like it or not, lies in : widespread use in early mathematical learning of the small electronic calculating machines that are widely advertised these days. These pocket-size calculators use computer chip technology and can do any operations in any order with eight or more digits available for input or output data. Furthermore, they give instantaneous results with the decimal point automatically placed. Some come with memories and with built-in standard programs. (One I have seen has a special percent key -- surely a sad comment on how little people have learned from the mathematics education.) Hence I can imagine that a first grader might understand perfectly well that the proper maneuver to get the total number of first graders in his school is addition of 31, 29, 33, 27, 21, and 30, but would not have the technical skill or the patience for drudgery that would let him get an answer. If so, punching the numbers into such a calculator would solve the problem. Another example: except for drudgery and lack of skill, a third grader might very well be able to balance the family check book for a given month with possibly considerable insight into family finance resulting from the exercise. With such a machine as this, the drudgery and the demands on technical skill are removed. These machines are already very cheap and very available, and I think that they may well cause a revolution in what is appropriate to do in the school teaching of mathematics - a revolution we should even now anticipate and plan for.[4]

---

[4]Recently one indicator of American taste, The New Yorker, had a cartoon with four ladies finishing up a luncheon and one saying, "But according to my pocket calculator, my share of the check including tax and tip is five dollars and forty-two cents." These machines are also the subject of the cover article in the March 1973 Popular Science.

Now a comment on possible neglect because of justifiably low priority among many problems. I would have some sympathy for this point of view if achieving more emphasis on applications required considerable expenditure of energy and resources in isolation from our other problems. But, on the contrary, attention to applications in early mathematical learning might have considerable payoff in connection with, or as a means to, solution of many of our other problems. For example, "disinterest," "lack of motivation," and "poor attitudes," are perennial problems in mathematical learning, and really good applications material might help overcome them. For another example, better evaluation of what a child really knows might be promoted by setting up an interesting application, then observing how the child attacks the problem. Similarly, observation of children engaged in applications might reveal interesting things about cognitive development patterns, especially if the applications were highly involving of the child's energy and attention.

The final reason for neglect that I listed was that attention to applications may simply be out of step in fairly pervasive ways with the school curriculum as it "ought" to be or with what we really want as end result of schooling in mathematics.* I cannot deal with this objection briefly since it requires a definition of what we want from the school mathematics experience. The next section attempts such a definition, and also implicitly attempts a rebuttal to this and several of the other reasons given for neglecting applications in early learning.

<u>What Role Can Applications Play in Early Learning of the Mathematics Children Should Eventually Have in Hand?</u>

It is my firm belief that anyone who proposes any objective for mathematical education is obliged to also consider what learning experiences might be attempted to test the feasibility of that proposed objective. In my mind the great strength of the work of such people as Dienes, Davis, the Papys, and others has been their commitment to proposing learning experiences that support their announced convictions. Hence I feel obliged to give some attention here to an indication that applications <u>can</u> play a role in the early experience of children in opening up and developing important mathematical outcomes. For a variety of reasons I cannot think it appropriate to tie such a discussion to a standard "scope and sequence" diagram of present school content. Instead, I will use an outline (shown here as Figure 2) that has been more extensively developed in another place (Bell 1973). This outline summarizes my own view of what we <u>really</u> want as minimum net residue in the minds and guts of people after they have served their required 8, 9, or 10 year sentence in school mathematics classrooms. I believe that there has been insufficient consideration of such endpoint objectives, but that argument belongs elsewhere. For the present, let me indicate the spirit in which this "tentative list" should be considered, then move on to the attempt to show how applications could contribute in the early stages of development of some of the items in the list.

---

*For example, "Applications tend to give students ... the idea that mathematics has no right to its own existence, and so you are shutting off potential mathematicians." (Beberman 1963, p.12)

1. The main uses of numbers (without calculation)	1.1 Counting 1.2 Measuring 1.3 Coordinate systems 1.4 Ordering 1.5 Indexing 1.6 Identification numbers, codes 1.7 Ratios
2. Efficient and informed use of computation algorithms	2.1 Intelligent use of mechanical aids to calculation
3. Relations such as equal, equivalent, less or greater, congruent, similar, parallel, perpendicular, subset, etc.	3.1 Existence of many equivalence classes 3.2 Flexible selection and use of appropriate elements from equivalence classes (e.g., for fractions, equations, etc.)
4. Fundamental measure concepts	4.1 "Measure functions" as a unifying concept 4.2 Practical problems role of "unit", instrumentation, closeness of approximation 4.3 Pervasive role of measures in applications 4.4 Derived measures via formulas and other mathematical models
5. Confident, ready, and informed use of estimates and approximations	5.1 "Number sense" 5.2 Rapid and accurate calculation with one and two digit numbers 5.3 Appropriate calculation via positive and negative powers of ten 5.4 Order of magnitude 5.5 Guess and verify procedures recursive processes 5.6 "Measure sense" 5.7 Use of appropriate ratios 5.8 Rules of Thumb rough conversions (e.g., "a pint is a pound"); standard modules 5.9 Awareness of reasonable cost or amount in a variety of situations
6. Links between "the world of mathematics" and "the world of reality"	6.1 Via building and using "mathematical models" 6.2 Via concrete "embodiments" of mathematical ideas
7. Uses of variables	7.1 In formulas 7.2 In equations 7.3 In functions 7.4 For stating axioms and properties 7.5 As parameters
8. Correspondences, mappings, functions, transformations	8.1 Inputs, outputs, appropriateness of these for a given situation 8.2 Composition ("If this happens, and then that, what is combined result?") 8.3 Use of representational and coordinate graphs
9. Basic logic	9.1 "Starting points": agreements (axioms) and primitive (undefined) words 9.2 Consequences of altering axioms (rules) 9.3 Arbitrariness of definitions, need for precise definition 9.4 Quantifiers (all, some, there exists, etc.) 9.5 Putting together a logical argument
10. "Chance," fundamental probability ideas, descriptive statistics	10.1 Prediction of mass behavior vs. unpredictability of single events 10.2 Representative sampling from populations 10.3 Description via arithmetic average median, standard deviation
11. Geometric relations in plane and space	11.1 Visual sensitivity 11.2 Standard geometry properties and their application 11.3 Projections from three to two dimensions
12. Interpretation of informational graphs	12.1 Appropriate scales, labels, etc. 12.2 Alertness to misleading messages
13. Computer uses	13.1 Capabilities and limitations 13.2 "Flow chart" organization of problems for communication with computer

Figure 2: A Short and Tentative List of What is "Really" Wanted as a Minimum Residue for Everyman from the School Mathematics Experience

For example, consider equivalence classes (3.1 and 3.2 in the list). Through a long series of exposures and experiences starting in kindergarten (or before) a youngster should come to know that most mathematical things come in many equivalent forms and that much of school mathematics deals with conversion from one form to another. He should also realize that problem solving both of textbook exercises and real life problems frequently involves recognition of equivalence plus good judgment about which of a number of possibilities are appropriate for use in a particular situation. (For example, "3 + 7" and "10" are each sometimes useful; and in calculating "1/2 + 1/3," "3/6" is a more helpful form than "1/2".) Again, "measure functions" (see 4.1) may seem to be fancy vocabulary to confuse the uninitiated but what is in mind is merely that in virtually every measurement situation, one has a set of real world objects (or happenings) on the one hand, and a set of numbers on the other hand, with the task of assigning numbers uniquely to objects (or happenings). Thus, measures of public opinion and measures of length are not altogether different, and the processes involved in volume measure and length measure are very similar. Through an extended and varied development over a number of years, everyman should understand this commonality of approach. Similar comments apply to the other items on the list. What are wanted are durable and correct intuitions and gut feelings, friendly familiarity, and genuine competence with respect to those mathematical things that can help individuals sort out their increasingly complicated worlds. In no case will a single experience or unit of work accomplish what is needed, and some part of nearly everything can be worked on nearly anywhere in the school experience of youngsters.

The question for this hour is the contribution that applications could make in the early learning of things such as those in my list. As I make a number of random suggestions I don't mean to suggest that all will actually work nor that they exhaust the possibilities. I merely wish to suggest that feasibility investigations are possible. If we have learned anything over the past fifteen years or so, it should be that the best way to find out what kids can do is to try something.

For starters, numbers pervade the actual world and it should be possible early on to call attention to some of their many uses. Counting and measuring are already widely attended to in schooling so let us see what might be done with some other sorts of uses. Might a youngster be led to notice that a room number such as 213 really conceals a pair of numbers - one giving the floor and the other the room number on that floor, with the 13 also indicating where on the second floor and perhaps even on which side of the hall? To contribute both to coordinate systems and ordering (1.3 and 1.4 in the list) I wonder if a youngster might be given the assignment of walking up and down the block he lives on a few times and trying to make sense of the house numbers? I don't know at what age a youngster would be able to sort out all the details of the coordinate and ordering system thus embodied but he might very well notice that all the numbers are of the sort 5403, 5411, 5413, 5450, 5478, etc., and the "54" might take on some significance. Further, the child may notice that smaller numbers come before larger numbers on the block; that houses an equal distance apart have addresses that jump by about the same amount; possibly that even numbers are on one side of the street and odd numbers are on the other; and that if he goes into the next block, all the numbers begin with 55.

As other examples, might we not alert the youngster to the pervasive use of numbers as identification codes (1.6); for example, license plates, telephone numbers, zip codes, highway numbers, etc.? Might we make something of "take a number" in bakeries and other places to impose a fair order of service on a crowd of customers? Can we also ask where it might not be appropriate to impose an order of service based on arrival time -- in a hospital emergency room, for example?

Moving on, there must be many ways of having youngsters sort themselves into equivalence classes (3.1). For example, note how far each lives from school and put all those the same distance away in the same group for some activity. (Some of the Nuffield and Papy arrow graph exercises are based on such exercises.)

With respect to measure function (4.1), we perhaps only need to make processes more explicit in many actual measure situations; e.g., the school nurse lining up a class and weighing each pupil, or a teacher returning youngster's work marked with "good," "you can do better," etc., or even ☺ versus ☹. With respect to the pervasiveness of measure (4.3) it would be interesting to find out when children can be led to notice how many measures surround them by simply asking them from time to time for lists of as many situations as they can think of in their everyday life where measures are assigned.

With respect to indirect measure via formulas (4.4), the Piagetian research suggests that anything very complicated in this line may need to wait on the formal operations stage. But some exploration of not direct measures may be possible by way of, for example, rubbing young children's noses in the fact that when reading temperature on a thermometer, they are reading a length with respect to the red line on the thermometer; that time is judged by amount of rotation of clock hands, and so on.

Let us move down the list to "order of magnitude" (5.4 on the list). It has been pointed out that an order of magnitude change in technology frequently changes things qualitatively and not merely quantitatively (Hamming 1963). For example, moving from horse and buggy travel at five miles an hour to automobile travel at 50 mph changes the entire sociology and habits of a country in very fundamental ways. Again, I'm not sure where in children's development such ideas are accessible, but perhaps we could draw from their experience with the world with such questions as these: "Suppose you had 2¢, what could you do with it? Suppose you had 20¢, then what? $2? $20? $200? $2,000?" and so on. Or: "Where could you get to by walking for the next couple of hours? Where to by automobile? Where to in an airplane?"

To consider guess and verify procedures (5.5) I believe quite simply that we must work them in whenever possible. That is, we should watch for opportunities based on children's experience where it is appropriate to ask "About how much...?" and then "Why do you think so?"

The use of variables (7.1 to 7.5 in the list) has been explored in a variety of existing materials by the use of "frames." To the extent that functions and equations expressed using frames can be tied to children's everyday experience with the real world, this would qualify for what I want done with applications in early learning. Similarly, consideration of inputs and outputs (8.1) can be supported by a variety of classroom and everyday activities.

The Nuffield books have a good deal of activity centered around representational (arrow) graphs (8.3) that could well be adapted for American classrooms. (Coordinate graphs related to actual applications perhaps must be deferred beyond the early stages I am discussing here.)

With respect to logic (9 in the list), we all know about the Piagetian findings with respect to reasoning from arbitrary hypotheses but I believe that it is widely agreed that much less formal reasoning based directly on a child's experience is accessible in the concrete operation stage. For example, youngsters at play <u>do</u> make up arbitrary rules for games and as disputes arise they argue through the consequences of their rules. They also change rules and argue from the new "axioms."

With respect to my category 12, informational graphs, we have many suggestions in the early Nuffield mathematics materials and it seems clear that the child's actual experience provides ample material to be exploited here.

Finally in this brief sampling, it may be that flow charting exercises belong beyond the early learning that is my focus here, but perhaps not. Piagetian findings indicate that one-way functions are available fairly early and perhaps it isn't too much to ask youngsters to sequence events that they are directly involved in by what would amount to an informal flow chart.

The examples I have given here are (deliberately) quite simple-minded. My intention has been to indicate that applications from a child's real world can play a role in the opening up of many important mathematical concepts as well as in their further development. Two things would be required, it seems to me, and neither of them is very common in early schooling. First, we would need to have teachers themselves enough at home with such mathematical ideas as are suggested by the list displayed as Figure 2 that their own knowledge and intuitions would alert them to possibilities for using the children's experience in opening up and developing these ideas. Second, even with such awareness of where mathematics education is headed and what it would mean to arrive there, teachers would need many suggestions with respect to things they might actually <u>do</u> with youngsters. For this, we would need to go to work and produce a large number of sample experiences and dialogues to illustrate ways in which each of the things listed above could be opened up. The first task would be to sensitize kids to notice what goes on in their world. Next, there should be more explicit development of concepts via applications based on children's experiences (as well as through embodiments) through the concrete operations stage. Finally, in the formal operations stage, there should be a variety of problem situations firmly rooted in the real world but not now necessarily in a child's direct experience. These would require for their solution fairly sophisticated formulation and exploitation of mathematical models. For this last many examples are already available, if not yet very much used. But for early mathematical learning, accumulation of the required large number of suggestions has barely begun, except in the science units noted earlier.

Some Further Remarks About Applications Related To Early Mathematical Learning

I wish now to continue with some more or less random comments and speculations on the subject at hand.

So far I have referred to the findings of developmental psychology research mainly to speculate on limitations that developmental patterns might put on the use of applications in early learning. (For example, metric proportion may not be available as a mathematical model until the formal operations stage.) But one wonders if observing children as they deal with actual applications from their own direct experience might not throw additional light on developmental patterns. For example, Sinclair notes that "Until nine years of age, there is confusion in comparing two moving objects or persons ... going further usually implies taking more time" (Sinclair). But suppose a teacher were to get the cooperation of some parents and set up or find situations where John, Mary, and Jack have actually travelled quite different distances in about the same time (a near place by automobile and a far place by airplane; or to school on a given morning by bicycle versus walking versus automobile, for example). In the context of direct personal experience, how will John, Mary, and Jack themselves respond at different ages to the question "How can that happen?" and to what extent will their classmates also engage the problem? That is, I wonder if the medium of applications from the real world might not be fruitfully used in developmental studies -- at some loss in precision and replicability but perhaps with some gain in motivation and concreteness. To be sure, standardized concrete situations with action called for are already common in such research but I have in mind longitudinal studies of what a given child's development lets him make of his own direct world experience when explicitly alerted to some aspect of it, especially with reference to appropriate mathematical models and concepts.

One way to attend early to applications of mathematics would be to call attention to aspects of the child's experience that he may not be able to "explain" at a given developmental level but which would be interesting in their own right and which might set the stage for later development. For example, there is a neat and rather profound link between the world of reality and the world of mathematics by way of biological consequences of scaling laws that indicate how changes in linear dimensions relate to changes in area and volume (Haldane). Consideration of the complicated proportions among linear, squared and cubed measures must wait for the middle and upper school years but direct experience can show much earlier, for example, that mice eat much more in relation to their size than do children, horses, or elephants. (This is, of course, because they have more skin surface area relative to their volume and weight so relatively more body heat is radiated.) Children can observe that a young baby is about half as long as a first grader, but may only weigh about an eighth as much; and that babies (with relatively more surface area) are much more easily chilled or dehydrated than older children. Hippopotamuses have quite different structures than gazelles, partly because of scale effects. Such things may be observable, if not explainable, at Piaget's concrete operations stage and would support more sophisticated later work. To explore such possibilities we again need to describe a great many spiral developments where the observations and experiences at the beginning of the spiral are clearly spelled out in terms of things that can be drawn from a child's actual experience. Observing children's reactions as we follow through such spirals might also considerably extend our knowledge of developmental patterns in children.

The original suggestion of the organizer of this symposium was that I try to relate Ausubel's advanced organizer notion to applications. I decided instead on a broader theme but do wish to comment briefly in this area. You are no doubt familiar with Ausubel's definition of an advance organizer as "introductory background material presented at a higher level of abstraction, generality and inclusiveness than the material itself and designed to serve as organizing or anchoring framework as the material is learned" (Ausubel). I can imagine situations (probably in the formal operations stage) where a given application is indeed inclusive enough to provide an anchoring framework for the learning of some mathematics. For example, an applications situation with linear programming as the appropriate mathematical model would serve very nicely as an advance organizer for systems of equations and inequalities. An application - say predicting an election - calling for collection of data, then manipulation of the data using statistical models might serve very well as a framework in which to integrate certain statistical concepts. But in general, and especially in early mathematical learning, I believe most useful and accessible applications make too few demands on the appropriate mathematics to serve as advance organizers of that mathematics in the Ausubel sense - that is, the applications are not general or inclusive enough. There is another sense, however, in which explicit attention to applications in the early school experience might serve as advance organizers; not to give children ways of assimilating mathematical theory but rather ways to assimilate their everyday experience. That is, if Ausubel is correct in asking for integrating structures into which new data are incorporated, it may be that explicit attention to applications of mathematics will give a youngster structures into which he will absorb aspects of his everyday experience that otherwise might go unnoticed or unacknowledged. He might be sensitized to the very pervasive use of numbers in his surroundings, for example, in ways that his school instruction in arithmetic would not cause, even if enriched with a laboratory approach via embodiments. Similarly, he might be sensitized to the pervasiveness of measures in his world.

Finally, let me try to tie my topic more explicitly into the general symposium themes. First, it should be clear that what I am calling applications here are in the general spirit of a "laboratory" approach, especially in the sense advocated by Moore and Perry around 1900 and later by Dewey and those of like mind. It must be recognized, however, that mathematics laboratories as they exist today as physical locations in schools contain mainly materials in the "embodiments" mode, plus a variety of drill and practice materials. They are not generally set up to emphasize applications in the sense of my discussion today. To make mathematics laboratories useful for teaching applications, the usual "laboratory" collection would need to be augmented with a number of workcards, etc. that sensitize children to their own real world experience, and that also direct their attention to real world problems in ways that invite them to engage in the mathematics modelling process.[6] Tools and instruments to help them in these encounters should also be present in the laboratory.

To turn to the second conference theme, my remarks so far have made several connections with cognitive psychology. There was some speculation on limitations that might be put on an applications approach if Piaget-postulated development patterns prove out. I made several suggestions to the effect that applications might be used in exploration of developmental patterns, on the assumption that the Piagetian research does not yet tell the whole story. I noted in passing that at least one excellent elementary science program, which includes some applications in the spirit of my remarks today, has been worked out on behaviorist rather than Piagetian assumptions -- the Robert Gagne influenced AAAS science program. I have also remarked that much of what I am talking about, and much else consistent with the symposium discussion, goes back to Dewey and those of like mind. That is, at the moment I find the Piagetian assumptions most helpful but I believe that a more eclectic selection from various theories or taking an avowedly behavioristic view would lead to the same positive conclusions about the likely effectiveness of applications as an aid to early mathematical learning.

Summary

I have attempted during this hour to open up for further discussion and investigation some possibilities for using applications of mathematics from the every day experience of young children to enrich and support early mathematical learning. I tried to sort out some of the main links within and between the world of reality and the world of mathematics (Figure 1). I claimed that work strictly within the world of mathematics dominates the school mathematics experience although the exploitation of "embodiments" of mathematics concepts is making some inroads on this dominance. I noted that the applications business of starting with real world situations and finding ways to fit mathematical models to them is a largely undeveloped possibility in early learning and examined some reasons for this neglect.

---

[6]At least one such workcard collection suitable for middle and later school grades has been published (Friebel).

I offered a possibly peculiar listing of the ultimate outcomes we might aim for in school mathematics (Figure 2) and tried to show that there might well be in the use of applications from the child's real world a number of opportunities to begin spirals of experience leading to gut level mastery of certain very important concepts. I concluded with a series of more or less random remarks about possibilities for exploiting applications from the real world in exploring cognitive developmental patterns in children and also to provide "organizers" to help a child integrate his ever [illegible] encounters with various uses of mathematics into his developing co[illegible]ve patterns. (Our colleagues in science are probably well ahead of us i[illegible]is area and we should look carefully at what they are doing and the results they have achieved.)

If we are to become serious about exploiting applications in the child's world in early mathematics learning a good deal of work would have to go into providing teachers with suggestions, sample dialogues, outlines of spiral developments, and so on. I am not optimistic that such efforts will take place very soon, if only because mathematics education is faced with plenty of problems already. But the fact is that a child does live in the real world and is surrounded by countless applications of mathematics; surely this could be a potent learning resource. The ability to use mathematics lies at the heart of our objectives for mathematics education. I do not believe we can afford to neglect much longer the potential of the use of applications to improve the early mathematical learning experience.

References

Ausubel, David P. "The Use of Advance Organizers in the Learning and Retention of Meaningful Verbal Learning," Journal of Educational Psychology, 51:267-72 (1960).

Beberman, M. "Statement of the Problem," in Friedman, D. (ed.), Proceedings of a UICSM Conference on the Role of Applications in a Secondary School Mathematics Curriculum. Urbana, Illinois: University of Illinois Committee on School Mathematics, 1964.

Bell, M.S. Mathematical Uses and Models in Our Everyday World. Studies in Mathematics, Volume 20. Stanford, California: School Mathematics Study Group, 1972.

Bell, M.S. (ed.) Some Uses of Mathematics: A Source Book for Teachers and Students of School Mathematics. Studies in Mathematics, Volume 16. Stanford, California: School Mathematics Study Group, 1967.

Bell, M.S. "What Does Everyman Really Need from School Mathematics," University of Chicago, mimeo, 1973.

Engineering Concepts Curriculum Project. The Man Made World. New York: McGraw Hill, 1971.

Friebel, A.C. and Gingrich, C.K. Math Applications Kit. Chicago: Science Research Associates, 1971.

Haldane, J.B.S. "On Being the Right Size," in Newman, J.R. (ed.), The World of Mathematics, Volume 2. New York: Simon and Schuster, 1956, pp. 952-957.

Hurd, P.D. and Gallagher, J.J. New Directions in Elementary Science Teaching. Belmont, California: Wadsworth Publishing Company, Inc., 1968.

Mosteller, et al. (eds.) Statistics by Example (Four Volumes). Reading, Mass.: Addison-Wesley Publishing Company, 1973.

Parker, E.T. "A Memorable Teacher," The American Mathematical Monthly, 72 (Dec. 1965) 1127-28.

Sinclair, H. "Different Types of Operatory Structures," in Rosskopf, et al. (eds.), Piagetian Cognitive Development Research and Mathematical Education. Washington, D.C.: National Council of Teachers of Mathematics, 1971.

Tanur, R., et al. (eds.) Statistics: Guide to the Unknown. San Francisco: Holden-Day, 1972.

Van Arsdel, J. and Lasky, J. "A Two Dimensional Abacus -- The Papy Minicomputer," The Arithmetic Teacher, 19 (October 1972) 445-51.

Whitney, H. "A Mini-Computer for Primary Schools," Mathematics Teaching, 52:4-7.

ABSTRACTION AND GENERALIZATION EXAMPLES USING FINITE GEOMETRIES

Zoltan P. Dienes
Director of Psychomathematics
Research Centre
Sherbrooke, Canada

It is beginning to filter down into the inner circles of mathematics education that what mathematics a child learns is of little importance because the specific content will for the most part be forgotten anyhow. What matters is the kind of mental discipline that he acquires and the kind of mental habits and techniques that he learns. Such interests and skills would include the following: an interest in generalization; an appreciation of an abstract structure; the ability to decode a coding system or to transcode from one system to another system; and the ability to look for necessary or sufficient conditions for certain properties to hold. Provisions for acquiring these kinds of competences in children should be the aim of mathematics education, instead of saturating young heads with useless definitions and terms, however exact these may be from the point of view of the abstract mathematician.

In this paper, I would like to show how some of these ideas can be encouraged to grow. The field I have chosen is finite geometries. Naturally, practically any other part of mathematics could have been taken and similar kinds of arguments used.

The Seven-Point Geometry. Points and Lines.

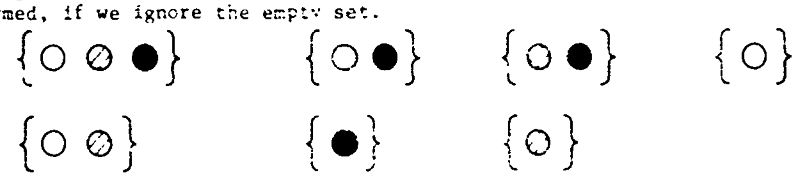
We could start with the following game. Let us say that seven children are together and they want to play a game. This game can only be played by three children, and the other four have to watch. How could we arrange the children in groups of three so that every child plays three times, and so that each child plays with every other child one time, and only one time? There will be seven such teams of three.

If the seven children are:	The teams could be:
John, Jack, Jess	(John, Jack, Joan)
Joan, Jasper, Joey,	(John, Johanne, Jasper)
and Johanne	(John, Jess, Joey)
	(Jack, Jess, Jasper)
	(Jack, Johanne, Joey)
	(Jess, Johanne, Joan)
	(Joan, Joey, Jasper)

This problem will lead to the consideration of a seven-point geometry in which the children are the points, and the groups of children are the lines.

The game can be made more concrete by using seven objects. The objects can have two colors (red and green), two shapes (square and circle), and two sizes (large and small). We will then have seven objects if we get rid of the small green circle. And, to form sets of three objects each, we pick any two of the remaining objects, and look at the small green circle. If we put the small green circle next to one of the objects, we will see that they are different from each other in certain ways. The remaining object has to differ from some other object in the same ways. This determines the third object in the set of three objects. For example, if we take a small red circle and a large green circle as our first two objects, which is the third object? Place the small green circle next to the small red circle. We see that there is a difference in color. Therefore, we must put the large red circle next to the large green circle, so that there is also a difference in color (and only in color) between the latter two objects. So the small red circle, the large green circle, and the large red circle belong to the same set of objects. Of course, we could have put the small green circle next to the large green circle; then, we would have noticed a difference in size; so then the large red circle would still have to have been put next to the small red circle because the small red circle and the large red circle are different in size, and in size only. In this way, we establish a binary function. Given any two objects, a third one will be determined. This corresponds to the fact that, given any two points, a straight line is determined.

Instead of using eight objects, we could have used eight sets. To make the sets, we could use four red objects, four blue objects, and four yellow objects. The first problem is to construct sets of objects out of these. The sets will be constructed so that there is never more than one of any given color in each set. Using this rule, seven different sets can be formed, if we ignore the empty set.



We can formulate the binary function in the following way: given two sets, the third set that is associated with the first two is determined by saying that if there are already two of a color in the first two sets, we must not put this color in the third set; if a color occurs only once in the first two sets, then this color must occur in the third of the three sets.

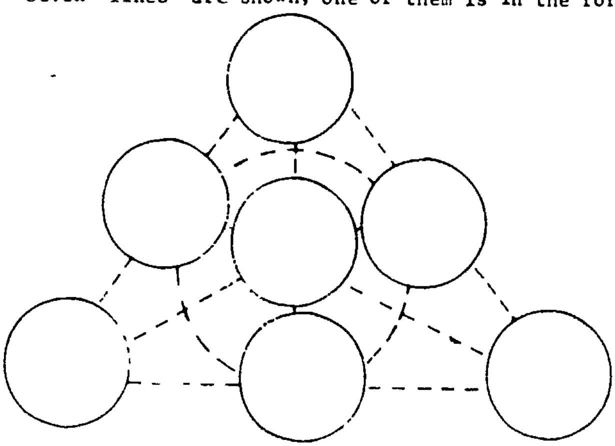
This problem is completely identical in structure to the first problem. Of course, this identity of structure is evident to a mathematician but not so evident to a child. This is just what makes a mathematician what he is; namely, the fact that he is able to recognize identity of structure. In other words, he can think in terms of abstraction. To learn to abstract, we need to practice the process of abstraction. We might proceed as follows: one group of children can play the seven-object game, and another group will play the seven-sets game. These two games can then be compared. Which set corresponds to which object? Let us say that object A and object B determine object C in the object game, and that to object A corresponds set X, to object B corresponds set Y, and to object C corresponds set Z. Set X and set Y must determine set Z. If that is not so, then either we have not found a way to compare the object-game and the set-game, or else this is impossible. So, in the case of comparable or isomorphic games, children should learn how to construct the isomorphism in question.

child game		object game		set game
John	⟷	$\boxed{g}$	⟷	(red)
Jack	⟷	g	⟷	(yellow)
Jess	⟷	r	⟷	(blue)
Joan	⟷	$\boxed{g}$	⟷	(red, yellow)
Jasper	⟷	r	⟷	(yellow, blue)
Joey	⟷	$\boxed{r}$	⟷	(red, blue)
Johanne	⟷	$\boxed{r}$	⟷	(red, blue, yellow)

Isomorphisms for the three games that have been considered so far are shown above. To that these isomorphisms work, look at the following diagram. The "triads" in one game whould correspond to the "triads" in each of the other games.

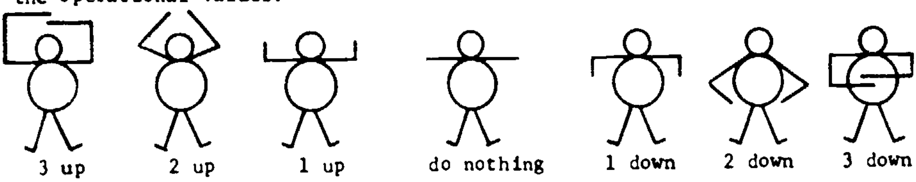
Joan	and	Jasper	"only play with"	Joey
↕		↕		↕
$\boxed{g}$	and	r	"determine the object"	$\boxed{r}$
↕		↕		↕
(red, yellow)	and	(yellow, blue)	"determine the set"	(red, blue)

Here is a useful representation of the abstraction that was just described. Seven "lines" are shown; one of them is in the form of a circle.

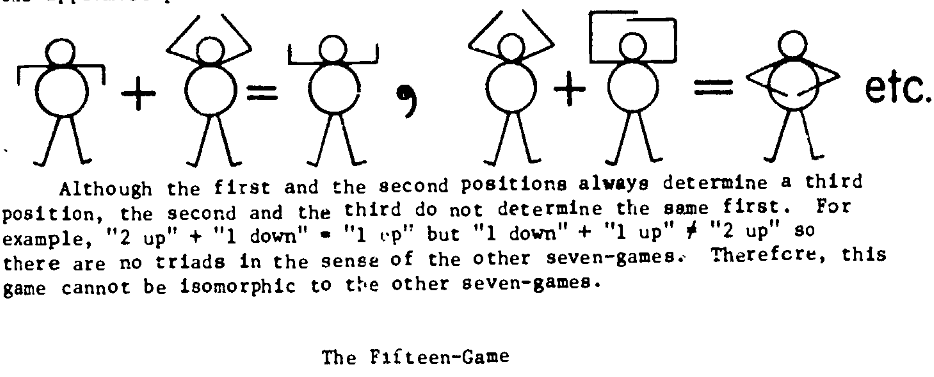


In the case of non-isomorphic games, children should be able to discover why the two games are not isomorphic. This will lead to arguments leading to different conjectures. If they were the "same" game, then certain things would happen; but you know that this is not so, and so the two games are not the same. So the idea of sameness will acquire a kind of "isomorphism color" instead of an "identity color".

Take the following "seven-element-game" with these arm positions as the operational values:



Three children play; the first assumes a position of his choice, the second is the operator and the third has to show the result of the operation. We should specify that to go "up" from the uppermost position means to assume the lowermost one, to go "down" from the lowermost position means to assume the uppermost position. Here are some "additions":



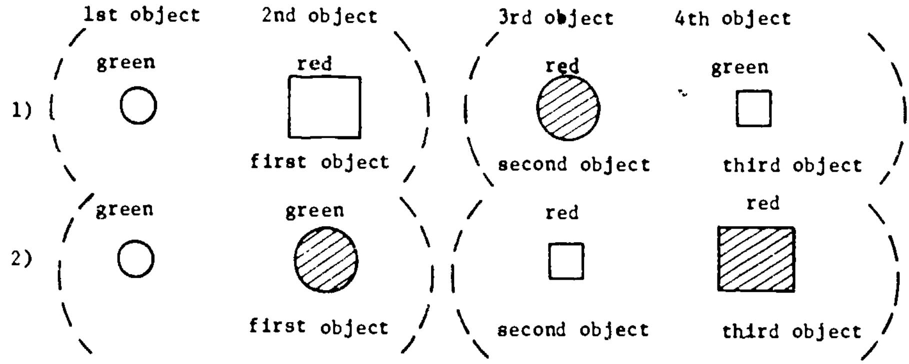
Although the first and the second positions always determine a third position, the second and the third do not determine the same first. For example, "2 up" + "1 down" = "1 up" but "1 down" + "1 up" ≠ "2 up" so there are no triads in the sense of the other seven-games. Therefore, this game cannot be isomorphic to the other seven-games.

The Fifteen-Game

Mathematicians are inveterate generalizers. At any meeting of mathematicians when one talks to another about a theorem he has just discovered, the first thing the other one will say: "Oh yes! That's fine, but is it not applicable to a more general situation? And if not, why not"? How can we encourage this type of inquiry? Clearly we can do so by putting children in situations where generalizations are possible, and indeed even quite obvious. In order to encourage them to do generalizations, we should first show them how one game can relate to another game by one being more general than the other.

Points, Lines and Planes

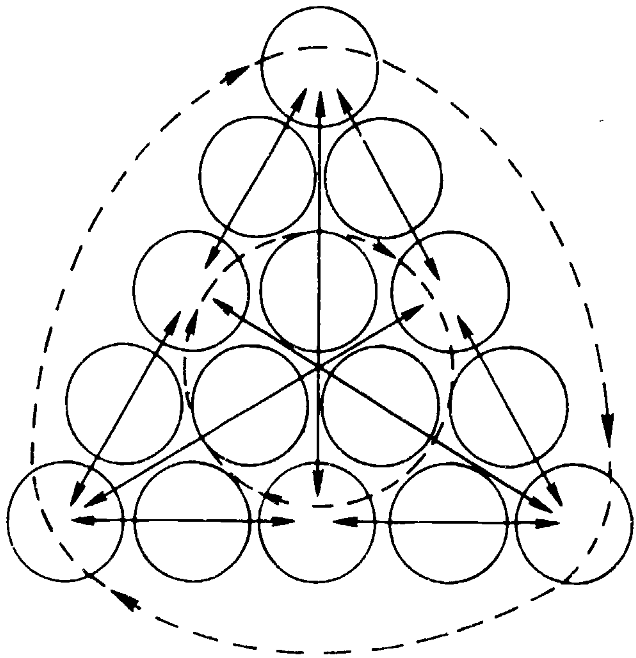
Instead of taking seven objects, we can take fifteen. In other words, instead of taking color, shape, and size, we can take color, shape, size, and thickness. There will be thin objects and thick objects as well as large ones and small ones, as well as circles and squares, as well as green ones and red ones. So we single out the small thin green circle in this case as the comparison object which is put against another object to make a pair. This determines another similar pair. The differences between the testing object and the first object must determine the differences between the second and the third object. Here are two examples:



A similar game can be played using sets of objects if we take eight red objects, eight blue objects, eight yellow objects, and eight green objects. Disregarding the empty set, we should be able to make fifteen sets such that in each one there is never more than one of each color. Again we can make the same requirement about the third set as we made in the 7-sets game. A third set is determined, given any first and second sets by completing the colors of which there is only one in the first and the second sets. We must not put a color in the third set if it already occurs twice in the first two sets, and we must put in a color which occurs only once in the first and the second sets. In this way, we have the same binary function as before, only we have more sets. In this game there are 35 "triads" that can be made.

There are certain novel features in the 15-game which are not present in the 7-game. In the seven-set game, if we are given three sets which do not belong to the same triad and try to extend by finding triads that can be formed from these three sets, then we eventually obtain all of the seven sets. This is not so in the 15-game. In the 15 game, given any three sets which do not belong to the same triad, we can never reach more than 7 sets. Here we have a substructure of the 15 structure which is isomorphic to the seven structure previously studied. But this time we can go beyond it. Such a 7 structure can be called an extension determined by our first three sets. It is not desirable to call them planes because the whole game occurs in one plane, and so this might confuse children. But if we take any four seta that are not in the same extension, then by constructing third seta in triads determined by pairs of sets, we can eventually reach all the 15 sets. It will be interesting to find that there are the same number of extensions as there are sets (i.e. there are exactly 15 possible extensions).

Naturally, the same game can be played with objects instead of sets, and an isomorphism can be constructed between the set of objects and the set of sets. Eventually a system can be developed in which we can represent either the objects or the sets or any other concrete version of this 15 point geometry. For example, here is an interesting representation which takes the form of a triangle.



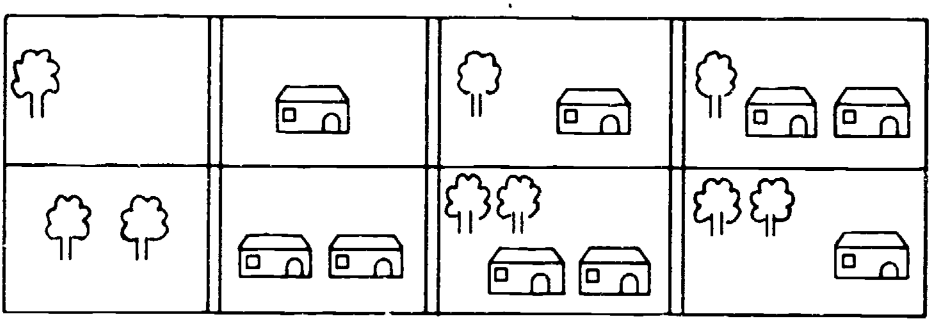
The way that the 15-game is more general than the 7-game is that in the 15-game there are elements, triads, and extensions, whereas in the 7-game there are only elements and triads. We have not yet extended the number of elements in a triad. But this would be another way of making up a more general game than before.

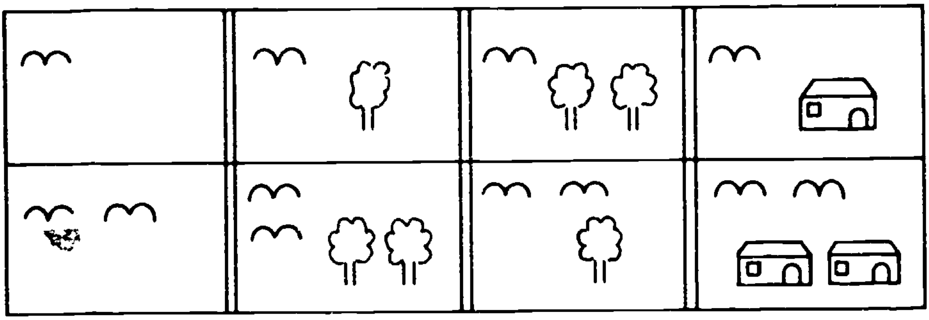
Clearly our original sets (or our objects) correspond to points in geometry, our triads to lines, and our extensions to planes.

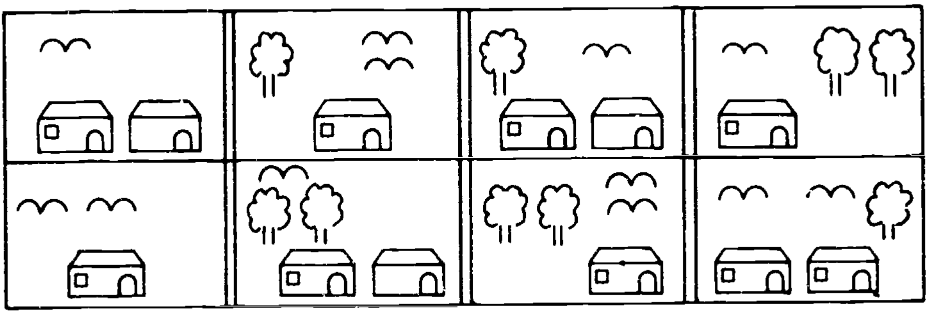
The Thirteen-Game. Four Points in a Line.

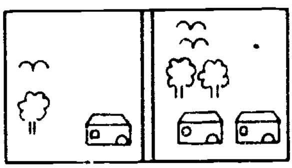
Let us think of another way that we can play a more general game. Let there be four points in a line and four lines through a point. How do we play such a game? We can tell the following story: there are thirteen children who want to play a game in which four children at a time can participate. The way to be fair in this game is to get everybody to play the game four times, and for everybody to play once and only once with every other child. Furthermore, we can require that out of any two groups of children playing, there should be one and only one in common. This is already included of course in the requirements that no child should play with the same child twice.

Children find this game already quite difficult, and we might help by giving them a different kind of embodiment. For example, we could take the 27 elements vector-space, with which children will play quite readily. They can be asked to draw 27 pictures so that in each picture there are either zero, one or two trees, either zero, one or two birds, and either zero, one or two houses. You throw away the picture on which there is nothing (which annoys children in any case) and so we are left with 26 pictures. An "addition" game can easily be worked out by saying that if you put two pictures together, there is a third picture always associated to the first two by adding the number of houses, the number of birds, and the number of trees with the proviso that if you get three trees you cut them down, if you get three houses, you sell them, if you get three birds you let them fly away. Of course, if you get four birds, you let three of them fly away, and keep one, and so on. In other words, the rules are for playing a modulo three game. There are three scalars in the vector space: zero, one, and two. The zero is somewhat trivial; multiplying a picture by zero turns it into e empty set. The scalar one is also somewhat trivial; multiplying a picture by one naturally yields the same picture. The scalar two, when used as a multiplier, turns a two into a one, a one into a two, and leaves a zero unaltere.. We can ask children to find a partner for each picture so that for each picture there is always a definite rule that gives another picture to go with it. They will very soon get to the point of saying: "Yes, every time there are two trees in a picture, its partner will have one, or if it has got two houses, its partner will have one, and if it has got one bird, its partner will have two. If it has got none of something, its partner will not have any of that either, and so on."



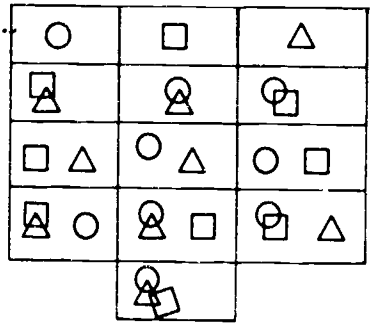






In this case, the partner-picture and the picture can be stuck back to back so that instead of 26 pictures we now only have thirteen. If we play the addition games with these cards, we obtain the thirteen teams of our initial problem in the following way. We draw the 26 pictures in pairs. Then, if we are given any two double-pictures, only two further double-pictures can be obtained by modulo three addition. By this means, we obtain a second pair of double-pictures. In other words, we generate sets of four pictures in which the initial pair is included. Therefore, we have constricted a model of a geometry in which a "line" will contain four "points" and every "point" will have four "lines" going through it. Very soon the children will associate the names of the thirteen children to one of the double-faced cards, and they will have an idea of how to play the game.

In this way we have generalized from three points in a line to four points in a line, but of course we have neglected to generalize from points and lines to points and lines and planes. We have also neglected to abstract, because we have not provided several embodiments. To do so, we would have to find some kinds of interesting isomorphisms that can be established between a new embodiment and the one we have already established. The search for such isomorphic embodiments should form an important part of the stock in trade of the imaginative mathematics teacher of the future.



Here is such a new embodiment. It is possible to take figures of people instead of circles, squares, and triangles. When the geometrical figures are overlapping, the people could hold hands. We would need three kinds of people. If we want to use the children in the class, we can make two kinds of hats. One kind of hat is put on some of the children, another kind on the other children, and the remaining children would wear no hat. For example, children that correspond to overlapping figures of the ○ , □ , △ game will hold hands, and the children that correspond to non-overlapping figures will stand apart to make the distinction. In this way, we can generate the thirteen elements of this geometry out of real children or out of plastic figures.

It will be interesting to pose the problem of which of these new elements correspond to which of the pictures in the thirteen double-faced picture game. The correspondence is very simple. We have to associate squares, circles, and triangles to the three different kinds of objects in the pictures. For example: a tree could be a triangle, a bird could be a circle and a house could be a square. Now if there are the same number of trees as birds on a card, then on the back of the card there will also be the same number of trees as birds because this property is not altered by taking the additive inverse. In this case, the circle and the triangle will be overlapping. If there is a different number of trees from birds, the circle and the triangle will be separate. If there is no bird, then there will be no circle; if there is no house, there will be no square, and so on. Now if the children have discovered how to find the missing two double-faced pictures, given any two double-faced pictures, then they should be given the task of finding the rule of how to determine the other two of the geometrical figures when any two of them are given. In other words, what is the necessary and sufficient condition that four of these figures should be "aligned." They will very soon find a necessary condition. A necessary condition is that in the set of four there should be three of each figure of which there are any. For example, if there are any circles at all in the set of four figures, there must be three circles. The same must be true for the triangles; if there are any, there must be three. Unfortunately, this is not a sufficient condition.

The necessary and sufficient condition is rather complex and can be expressed disjunctively as follows: given four figures, they are aligned if and only if for each one of the three possible pairs circle-triangle, circle-square, triangle-square one or the other of the following is true.

a. The pair occurs in the same way three times, and is missing altogether from the fourth picture, or

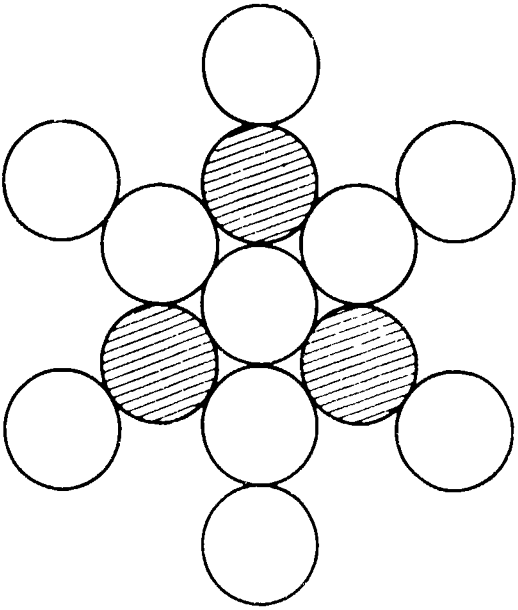
b. the pair occurs in all four possible ways. That is, overlapping, separate from each other, the first one without the second, and the second one without the first.

For example, in a set of four, we may have the square and the triangle occuring three times overlapping and the fourth time no square and no triangle, or we may have the square and the triangle occuring once overlapping, once separate, once the square without the triangle and once the triangle without the square. This is quite a sophisticated, disjunctive necessary and sufficient condition. But, children of nine or ten are able to handle this degree of complexity, particularly if they have been brought up to consider disjunctions and conjunctions as part and parcel of their logic course.

Representations

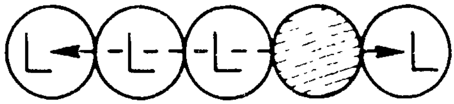
We have sketched a generalization exercise from the seven-game to the thirteen-game, meaning that from three elements aligned, we now have four elements aligned. We have also sketched an abstraction exercise of the thirteen game by giving two different concrete models of the same abstract structure. It is time to find a representation.

<u>13-game</u>

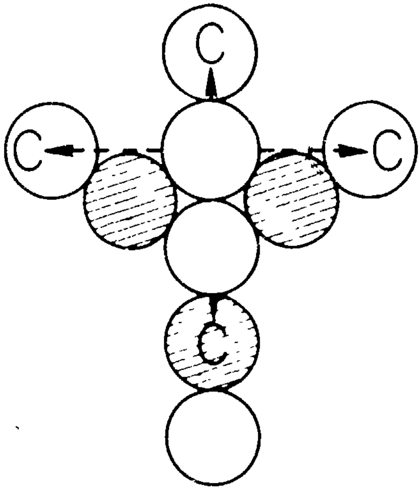


Here is a representation of thirteen elements, twelve elements placed around a center. The thirteen sets of four are arranged in four shapes. There are three lines, three crosses, three triangles, and four Y's. One of the Y's is in the middle and the other three Y's are around the sides with the filled-in circle in the middle. The crosses have the filled-in circle at the base and the triangles have it in the middle of the mid-point of the base. The triangles are isosceles. In this way, the arrangement is completely symmetrical about the three axes of symmetry of the figure.

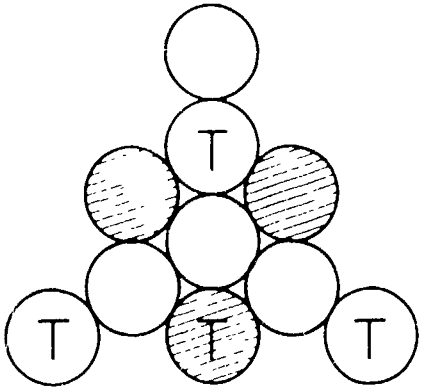
Lines



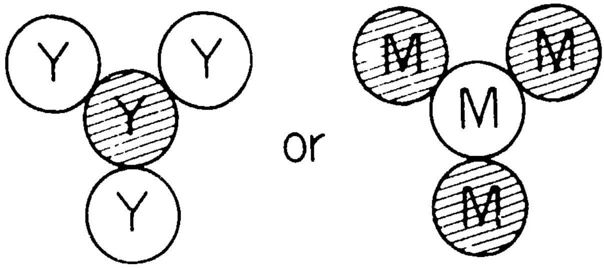
Crosses



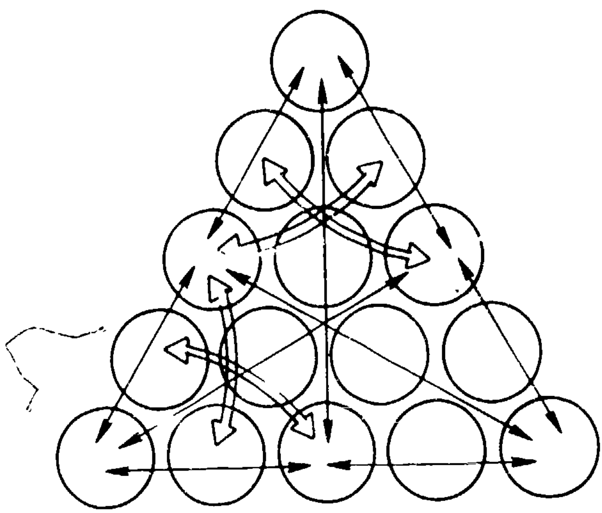
Triangles



Y-shapes

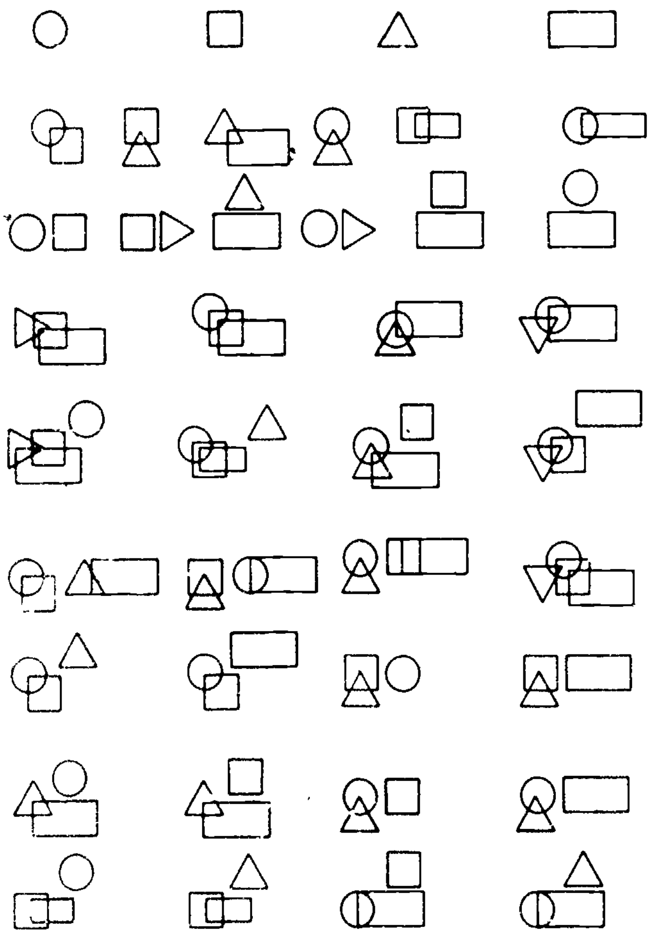


Let us now go back to consider the fifteen-game. This can also be symmetrically represented about three axes: whatever happens on one side of an axis, happens on the other side of the same axis also. It is interesting to give a certain number of requirements in this game as to where the triads should be situated and see what kind of solution the children come up with, while satisfying the requirements. It will also be interesting to give some requirements which fully determine the solution and others which do not. For example, if we require that the triads should be spaced starting from the vertices of the triangle as well as along the inner, the middle, and the outer triangles, then the positioning of the rest of the triads is determined. But if we require the "knight's" moves and the straight line segments from the vertices but only those along the sides, then the solution is not determined. In the thirteen game also, the solution given is not unique but there are not too many variations possible.



We have not generalized the 13-game to ir^l"de an extra dimension. To do this, we can generalize the four elemen a line problem to a problem of points, lines, and planes. This ca. be done by having four shapes instead of three. T e same rules apply for generating "four-somes" as in the 13-game. There will be forty possible points, a line will have four points, and a plane will have thirteen. This is an easy generalization to make, once children have played with the three shapes, they will generalize to four shapes. This is an example to show the kind of situation which we can contrive so as to train a child's direction of thinking towards generalization rather than towards learning by rote what the teacher presents.

The 40-game



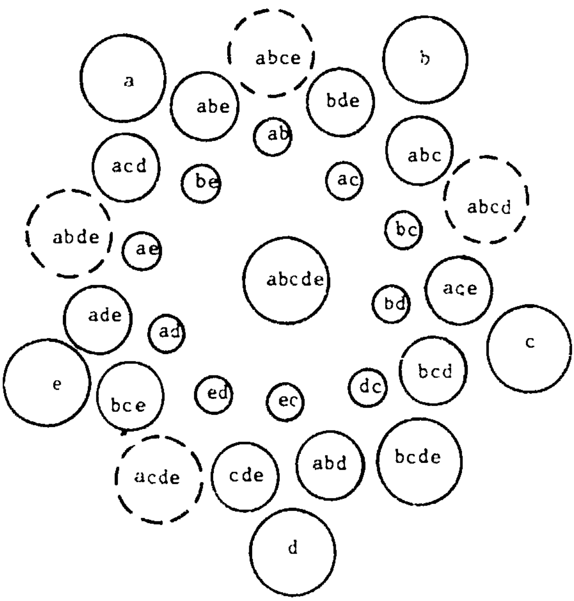
Two Different 31-Games

There is another interesting 31-element extension that could admit two possible solutions. One of these extensions can be derived from the one-hundred and twenty-five element vector space, itself based on the five element field. We can also derive one from the thirty-two element vector space that is based on the two-element field. In this case, we remove the empty set and we have a thirty-one element projective geometry which consists of either thirty-one objects with five different attributes (each attribute having two different values) or of sets of objects made up out of sixteen red ones, sixteen blue ones, sixteen yellow ones, sixteen green ones, and sixteen black ones. Out of these we could make thirty-one sets in such a way that no more than one of each color is put in each set, not counting the empty set. The same rules as before can apply for making the triads. The following diagram will act as a representation which takes account of the fact that we are dealing with the fifth power of two. Children will readily place the single-element sets round the first "circle", the two-element sets round the second "circle", and so on. (see diagram below)

The First 31-Game

Many triads will readily be found, such as (ab, ac, bc) or (a, abcde, bcde). There are extensions (i.e., planes) of seven, and "super-extensions" (i.e., spaces) of fifteen. It will be interesting to find at least some of these.

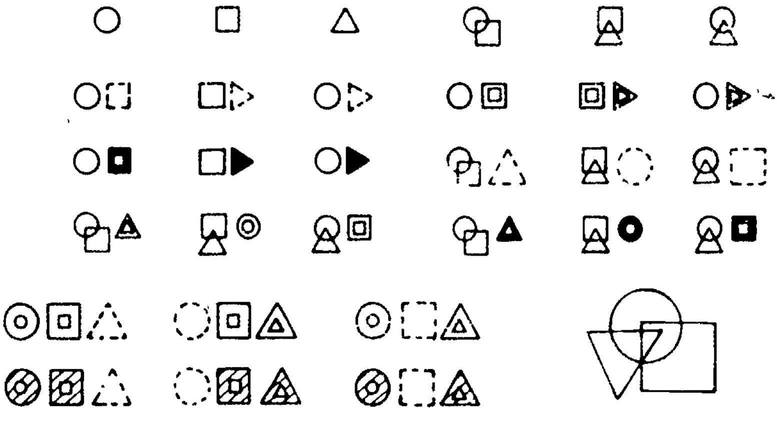
Suggestions such as "Try to make your arrangement look pretty!" will encourage children to look for symmetrical solutions.



If each letter is drawn as a flower of a particular kind, the above distribution provides an interesting bouquet.

Interestingly enough we can build a thirty-one element geometry out of the 125-element three dimensional vector space. In this case, children need to draw 124 pictures with trees, birds, and houses. They are allowed to draw either zero, or one, or two, or three, or four of each kind of object. So the largest number of objects on a picture will be four trees, four birds, and four houses. Before they can build a geometry out of this, they will have to know about multiplication modulo five. For example, if we take a picture with one tree, two birds, and three houses, and "multiply by two" we shall have two trees, four birds and one house. If we multiply again by two, we shall have four trees, three birds, and two houses; if we do so again, we should have three trees, one bird, and four houses. Of course, if we multiply again by two we get back to one tree, two birds, and three houses. In other words, instead of putting the pictures in pairs as we did when we had twenty-six pictures, we will now have to put them in sets of four. So the pictures just described would have to be stuck on a large piece of cardboard and be considered as one element. In this way, instead of 124 elements, we will have again the magic number of thirty-one, and we can play the "adding game". We will find that to each pair of four-somes of pictures, will correspond four other four-somes and no more. This is natural because to any two vectors represented on the two four-somes of pictures, by adding we can obtain any member of the two dimensional subspace in the three dimensional vector space. This subspace will contain twenty-five elements if we include the empty set, which represents the zero vector. So it is not surprising that we only obtain twenty-four pictures and the empty set, representing the twenty-five elements of the two-dimensional subspace. At this point, the mathematician can perform a slight pedagogical cheat, because he knows in advance how the game is going to work out. The mathematician can use his mathematical knowledge to help children have fun in discovering mathematics that we ourselves were probably shown ready-made by some teacher or professor.

To map the path for our young clients, we must provide another embodiment of a situation in which a one, two, three, a two, four, one, a four, three, two and the three, one, four are represented by one and the same diagram. Here is a suggested solution to this psychological problem:



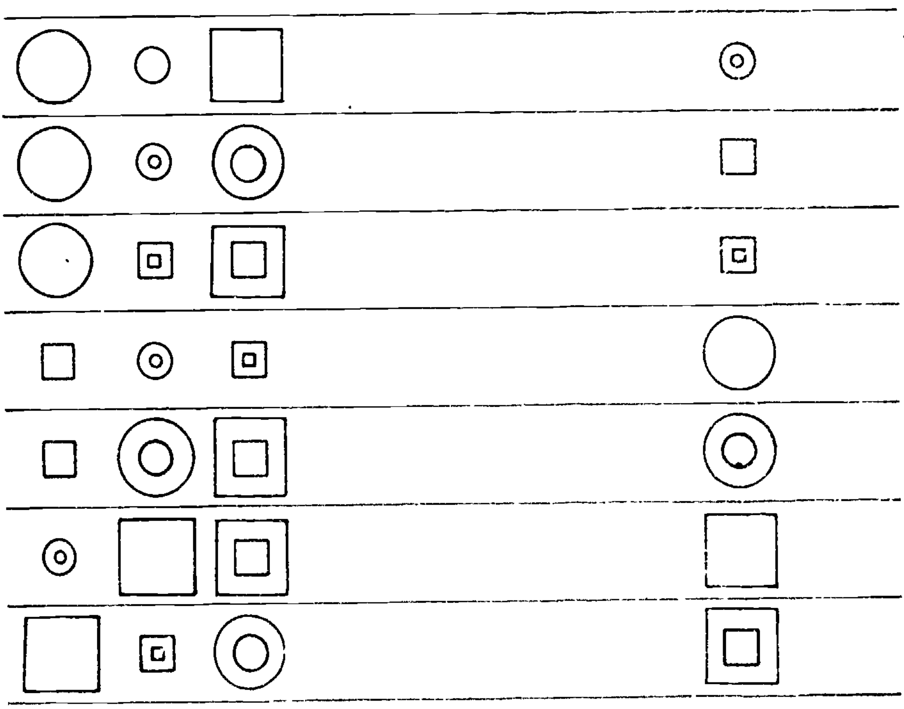
The problem can be put in the following way: the diagrams are drawn on one side of a card and the four corresponding vectors (i.e., those that belong to the same one dimensional vector space) are written on the other side. Children have to practice telling what vectors are on the other side of a diagram and what diagram is on the other side of a set of four vectors. To solve this transcoding problem, they have to find out what is constant as they go from one vector to the next on the same card. How is one, two, three, like a two, four, one? It is not difficult to see. Two is the double of one, and four is the double of two, and three is multiple of four of two, and one is multiple of four of four, and so on. In other words, the digits are in the same proportion to each other in 123 as in 241. This is something children will discover with great pleasure. They will probably write: 1 x 2 2 x 4 3 and 2 x 2 4 x 4 1.

Duality

Now let us come to another interesting point. There is a principle of duality at work in these games. Take for instance the seven game or the thirteen game or any other similar game. The problem is to find corresponding names for the dual objects. For instance, let us draw the diagrams for the "points" in the thirteen-game in black, and then draw the corresponding diagrams for the "lines" in red. There will be thirteen red diagrams, each diagram representing a line and thirteen black diagrams, each diagram representing a point. So each red picture corresponds to just four black pictures. But each black picture corresponds to exactly four red pictures. For example, if we take the picture of the red circle by itself, it will be associated with the black triangle, the black square, the black triangle-square overlapping and the black triangle and square separate. In that case, if we take the black circle, this black circle should be associated to the red triangle, the red square, the red square and triangle separate, and the red square and triangle overlapping. The problem of naming the lines by means of red diagrams amounts to asking which set of four black ones should be associated with which red diagram and visa-versa. The solution to this problem in the case of the seven-game and in the case of the thirteen will be found in the following diagrams.

Duality in the 7-game

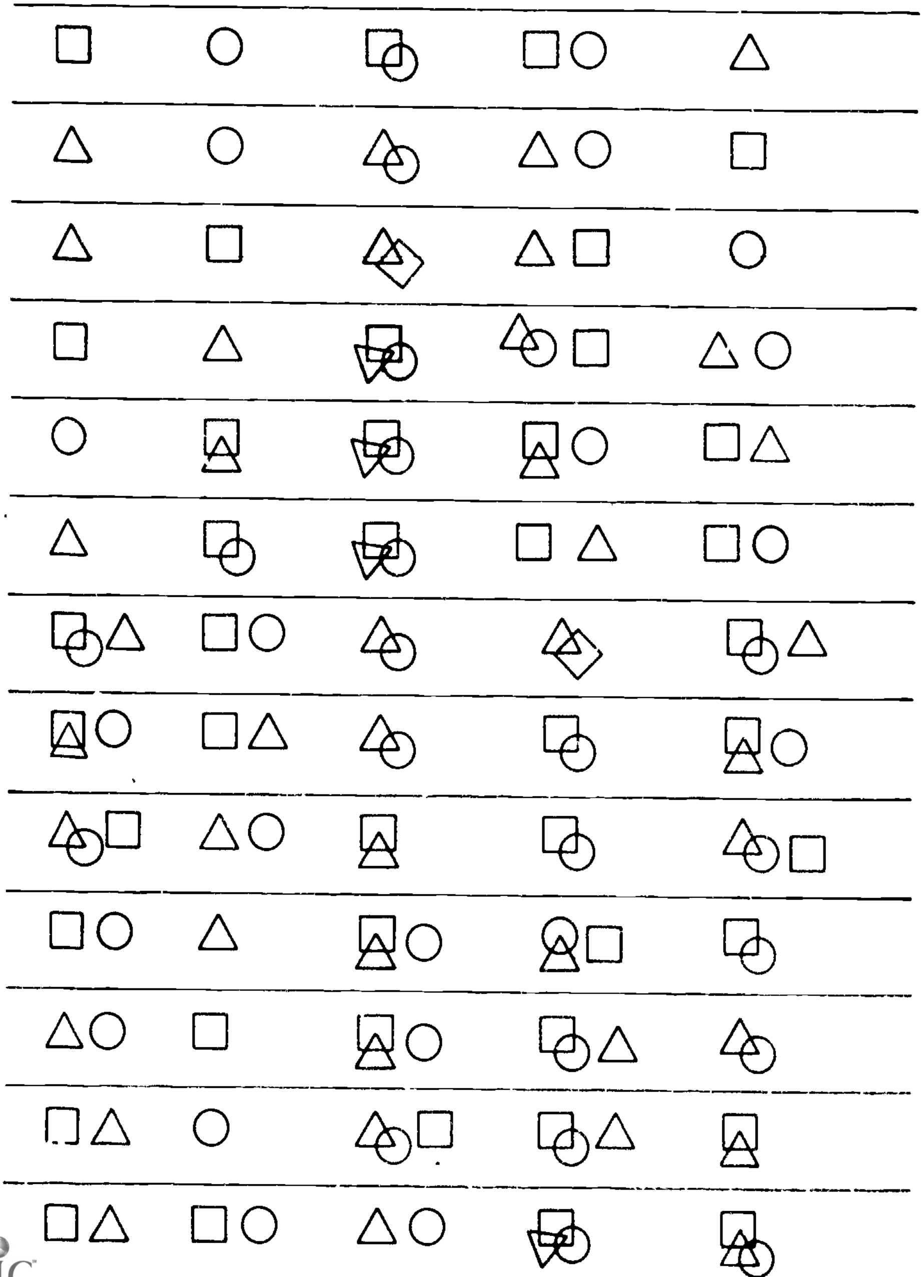
Sets of "black" figures (sets of "points") | "Red" figures ("lines")



Duality in the 13-game

Sets of "black figures (sets of "points")

"Red" figures ("lines")

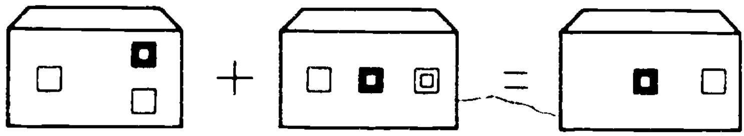


Naturally, for the other games in which there are points, lines and planes, the problem is more difficult. Points have to correspond to planes, and planes to points, and the lines correspond to lines in this duality. So for example, in the fifteen-game each plane or extension as we called it, should correspond to one and only one point, and the name of that plane should be appropriately chosen so the seven points it contains should be able to be translated into the seven planes that pass through seven points which should have the same red names and so on. So, again the duality can be extended from two to three dimensions, from three to four dimensions, and so on. Here is one way of solving the "naming problem" for the fifteen (15) extensions.

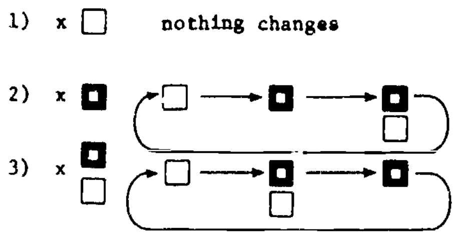
a, b, c, ab, ac, bc, abc	D
b, c, d, bc, bd, cd, bcd	A
a, c, d, ac, cd, ad, acd	B
a, b, d, ab, bd, ad, abd	C
c, d, cd, ab, abc, abd, abcd	AB
b, d, ad, ac, abc, acd, abcd	AC
b, c, bc, ad, abd, acd, abcd	AD
a, d, ad, bc, abc, bcd, abcd	BC
a, c, ac, bd, abd, bcd, abcd	BD
a, b, ab, cd, acd, bcd, abcd	CD
d, bc, bcd, ab, abd, ac, acd	ABC
c, ab, bd, ad, abc, bcd, acd	ABD
b, ac, cd, ad, abc, bcd, abd	ACD
a, bc, cd, bd, abc, acd, abd	BCD
ab, cd, bc, ad, ac, bd, abcd	ABCD

Further Possibilities

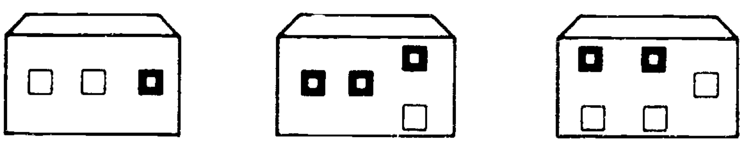
We can make up a 21-point geometry, if vector spaces based on Galois-fields are known. Take for example the 4-element Galois-field and the 64-element three-dimensional vector-space based upon it. We remove the neutral as usual and obtain 63 elements. These can, for example, be "played with" by representing them as 63 houses, as shown in the following figures. In the vector addition two similar windows yield no window. Naturally the left, middle, and right windows have to be taken independently of each other. For example:



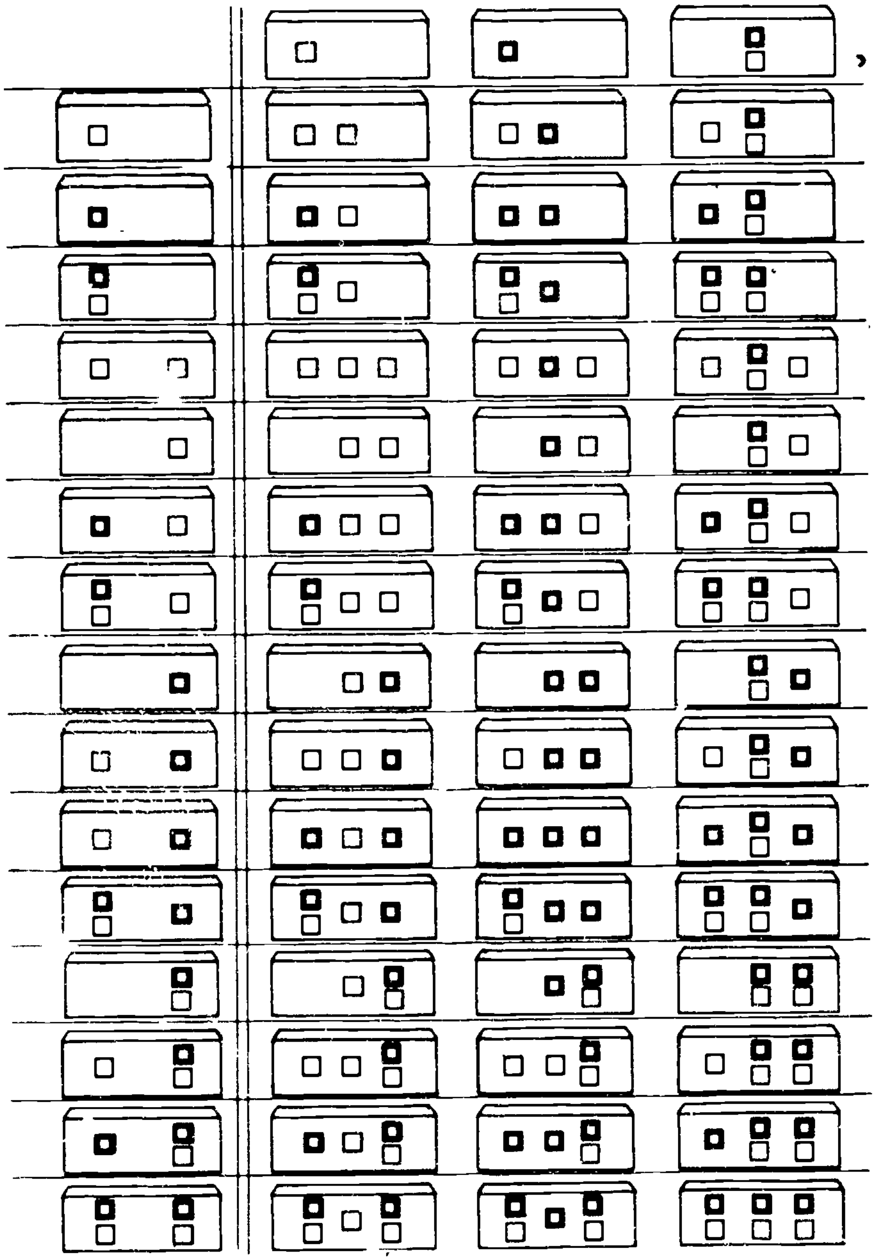
The scalars are:



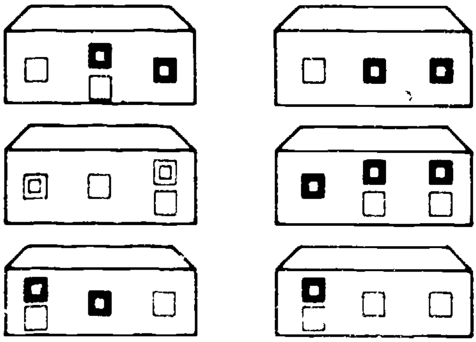
Houses which are "products" of each other, belong to the same village. For example:



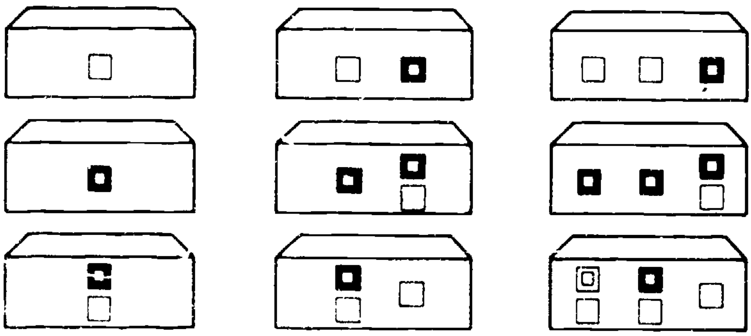
These form one village.



If we pick any two villages, by "adding," we can determine three more villages. A set of five villages so determined could be called a town. A "town" will be a "line" and a "village" a "point." For example, take the following two villages.



By "adding house to house," we obtain these three villages:

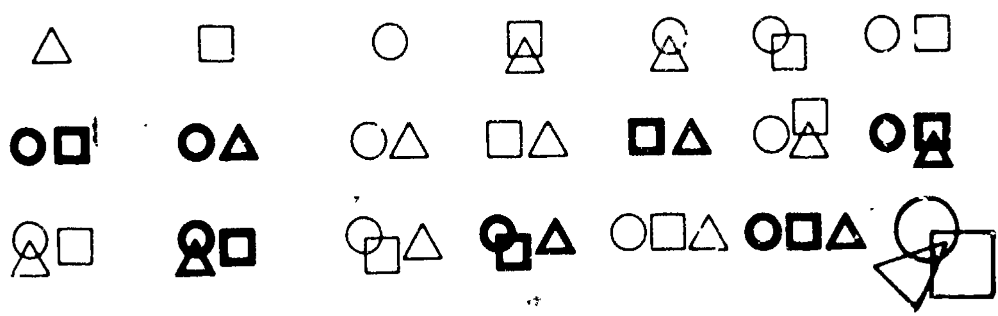


The five villages together form a town.

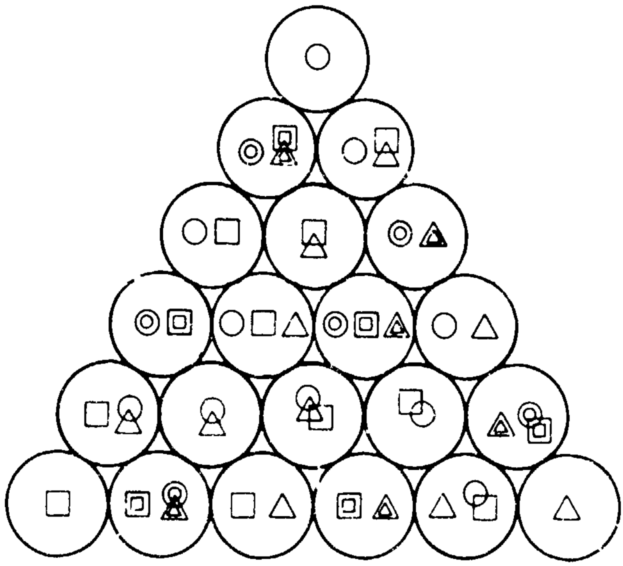
There are 21 such villages. If we denote no window by 0, □ by 1, ▣ by 2 and ▣ by 3, each house will be numbered. Here are the 21 villages: □

001	010	100	011	101	110	120
002	020	200	022	202	220	230
003	030	300	033	303	330	310
210	102	201	012	021	211	311
320	203	302	023	032	322	122
130	301	103	031	013	133	233
121	131	112	113	123	321	111
232	212	223	221	231	132	222
313	323	331	332	312	213	333

Each village can be represented by a figure in order to emphasize that our new units are now villages (i.e., "points") and not houses (i.e., vectors). For example, the following figures can be used.



Representations can be found in many different forms. For example:



Some of the arrows can be put in to show where the "five-somes" or "towns" are. Each village has an "emblem" and each town has five emblems, taken from its villages. Naturally the inhabitants and town governments need to be very cooperative, since every pair of towns has one village in common, and every village pays taxes to five overlapping towns.

Summary

A number of examples have been given to illustrate some of the initial stages of the abstraction and generalization processes. In the abstraction process that leads to the eventual formation of a formalized concept, there are many stages. The first is always a somewhat groping stage, a kind of "trial and error" activity; this is usually described as play. The restrictions in the play lead to rule-bound play or games. This has been well represented in the present paper. The next stage is the identification of many different games possessing the same structure. This is the stage of the search for isomorphisms. When the irrelevant features of the many games have been discarded, we are ready for a representation. Such are the many "link" diagrams suggested. It is only when this stage has been reached that it is fruitful to use a fully symbolic language, the development of which will be a later stage in the abstraction process.

The abstraction process should not be confused with the generalization process. Passing from 3 points in a line to 4 points in a line, or even more so to an infinite number of points in a line is <u>generalizing</u>. Acquiring a deeper understanding of the concepts of point and line through comparisons is <u>abstracting</u>. Bringing in another dimension is <u>generalizing</u>, understanding more deeply what is meant by dimension is <u>abstracting</u>.

It is hoped that through playful activities such as the ones that have been described, these exciting processes could be made available to more and more children.

AN APPLICATION OF PIAGET - COGNITIVE DEVELOPMENT RESEARCH IN MATHEMATICAL EDUCATION RESEARCH

Leslie P. Steffe
University of Georgia

The problems of mathematics and logic are independent (but not separate) from problems of the psychology of mathematical learning. Attempts to solve problems in mathematics or logic by using psychological methods (psychologism) are rejected and, reciprocally, attempts to solve psychological problems solely by using logical or mathematical methods (logicism) are rejected. Psychologists do not, however, advocate a complete separation of mathematics and psychology. On the contrary, a close correspondence exists between certain basic "logical-mathematical" structures and the cognitive operations of the child, where the "logical-mathematical" structures serve as models for the cognitive operations. This correspondence has been elucidated by the Genevans in their developmental analysis of logical-mathematical thought. Aside from the basic empirical work by the Genevans (on number, geometry, space, logical propositions, functions, probability, time, etc.), it is this correspondence and its accompanying theory which has held the fascination of mathematics educators in the United States.

A point which needs emphasis is that while logical-mathematical structures are used by the Genevans to describe the natural thought of the child, these structures "do not correspond to anything as such in the subject's conscious thought (Beth & Piaget, 1966, pp. 167)." A sharp distinction has been drawn between the problems of psychology from a genetic viewpoint and the problems of Education in mathematics.

> No subject, before he has learnt it, has a concept of what a group, lattice, topological homeomorphism, etc. is ..... Thus it is not in the field of reflective thought ... that we shall ask whether these structures are "natural" ... we can thus largely set aside the most awkward factor in the attempt to find a genetic analysis: namely, the factor of educational and verbal transmission (Beth & Piaget, 1966, p. 167).

In their studies, the Genevans claim not to reduce natural thought to formal structures but, instead, to use the formal structures to describe natural thought as it develops in the child--making every effort (a) to be cognizant of limitations of natural thought and (b) to arrive at the most rudimentary structured wholes. These rudimentary structured wholes possess specific laws of combination (e.g. reversibility) as well as exhibit a generality of form across their contents (i.e., the "grouping" structure).

In contrast to these genetic structures are the structures of mathematics proper. Throughout the ages, mathematicians have given definitions of such entities as set, number, point, and line; all objects of mathematics. In modern postulational developments, such entities are left as undefined objects or are defined in terms of other undefined objects. In fact, a clear perception of the necessity of leaving certain objects of mathematics undefined led Courant and Robbins (1941) to comment that 'a dissubstantiation of elementary mathematical concepts has been one of the most important... results of the modern postulational development (p.xix)." As exemplification, consider objects called "vectors." The term "vector" has been defined as a quantity which has direction and magnitude (Phillips, 1933, p.1). This definition is of little algebraic importance because objects exist which do not seemingly fit the definition but yet, along with certain operations, satisfy the properties of a vector space. An example is the set of infinite sequences of elements taken from an arbitrary field. These objects can be considered as vectors, because if an appropriate set of scalars is chosen and appropriate operations defined, infinite sequences satisfy the properties of a vector space. Certainly to view a vector as an infinite sequence is different from viewing a vector as a quantity which has direction and magnitude. While each interpretation of "vector" has associated meaning independent of the structure of a vector space, the ultimate test of whether the objects are classifiable as "vectors" depends on the structure of which they form a part. In abstract treatments of vector spaces, the objects are left undefined.

The learner of mathematics usually does not gain knowledge of vector spaces by studying only abstract structure. The processes involved in acquisition, by the learner, of mathematical structures are complex and, until quite recently, have not been an object of research. Indeed, although the identification of genetic structures by the Genevans is a profound contribution to research in mathematics education, the latter research area contains elements not directly studied by Piagetians. In mathematics education, the student is generally expected to become explicitly (consciously) aware of the mathematics being taught. As already noted, the operational structures of intelligence identified by the Genevans are not present in the mind of the child as conscious structures. While these genetic structures may serve as mechanisms to guide reasoning of the child in the acquisition of mathematical knowledge, the mathematical knowledge may not be at all isomorphic to existing genetic structures. In fact, certain mathematical structures may be more parsimonious models of cognitive operations than are identifiable genetic structures.

Piaget (Beth and Piaget, 1966) has been somewhat explicit about similarities and differences he perceives in mathematical and genetic structures. One difference is that, while the mathematical structures are the object of reflection on the part of the mathematician, the genetic structures are manifested only in the course of the child's behavior. A second crucial difference is that, in the mathematical structures, the form is independent of the content; whereas, in the genetic structures (at least at the concrete operational stage), the form is inseparable from the content. Another difference is that, in the mathematical structures, the axioms are the starting point of formal deduction; whereas, in the genetic structures, the laws are the rules which the child's deductions obey. Similarities also exist. Relations (operations) in mathematical structures[1] correspond to operations in genetic structures and the "conditions" of the relations in mathematical structures correspond to the "laws of combination" in genetic structures. The construction of mathematical entities, then.

---

1 Here Piaget restricts himself to the three basic structures--algebraic, relational, and topological

"is an enlargement of the elements of natural thought and the construction of M (mathematical)[2] structures an enlargement of particular mathematical entities (Beth and Piaget, 1966 p. 189)." This hypothesis is extremely intriguing and certainly deserves rigorous testing. If it can be shown at the outset that certain mathematical structures are more parsimonius models of genetic structures than are grouping structures, the job of testing such an hypothesis would be aided, but not necessarily made easier. In the face of the above stated similarities and differences between genetic structures and mathematical structures, Beilin (1971) has succinctly expressed the belief that "little effort has been expended in testing the relations between the conceptual systems of mathematics and the cognitive systems of the child except in the most limited of circumstances (p. 118)." The remainder of this paper is devoted to an analysis of certain mathematical and genetic structures and a discussion of selected series of experiments.

Genetic Structures and Mathematics Structures

In The Child's Conception of Number, Piaget apparently had two goals. The first was to demonstrate stages in the development of particular concepts and the second was to demonstrate the development of a conceptualizing ability underlying the formation of a host of concepts, i.e., demonstrate existence of genetic structures. While the data presented in this book are "old," the basic theory of the Genevans concerning the development of number in the child has not changed substantially over the last three decades (Piaget, 1970; Beth and Piaget, 1966; Sinclair, 1971). Four main stages have been identified in the development of this conceptualizing ability: (a) sensory-motor, preverbal stage; (b) preoperational representation; (c) concrete operations; and (d) formal operations. Concrete operations are a part of the cognitive structure of children from about 7-8 years of age to 11-12 years of age. Piaget (Beth and Piaget, 1966, p. 172) postulates that such cognitive structure has the form of what he calls "groupings" in which concrete operations are central. While an operation is an interiorized, reversible action always part of a total structure (Piaget, 1964, 32ff), concrete operations are those operations which occur in the manipulation of objects or in their representation accompanied by language (Beth and Piaget, 1966, p. 172). In other terms, "concrete," in a Piagetian sense, means that a child can think in a logically coherent manner about objects that do exist and have real properties and about actions that are possible--a child in the stage of concrete operations can perform mental operations in the immediate absence of the objects. All of the grouping structures known reduce to a single model where the differences in the groupings reside in the various operations which are to be structured.

Grouping I.

In The Psychology of Intelligence, Piaget (1964b) selects special classes for part of the elements in the first grouping discussed (in the context of a zoological classification). These classes satisfy the following pattern: $\phi \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_\beta$ where $\beta \in B$, the index set. This chain of sets constitutes a lattice.[3] In the lattice, the following laws of classes hold.

---

[2] Added by author.

[3] See the appendix for a discussion of mathematical terms used in the paper.

1. $X \cup X = X$: Idempotent Law

2. $X \cup Y = Y \cup X$: Commutative Law

3. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$: Associative Law

4. If $X \subset Y$, then $X \cup Y = Y$: Resorption Law

This lattice structure does not constitute the first grouping. Classes of the form $A_\sigma' = A_\gamma - A_\sigma$, where $A_\sigma \subset A_\gamma$ are also included. The classes A_σ' along with the elements of the lattice are the elements of this first grouping. These elements satisfy the following laws.

1. $A_\sigma \cup A_\sigma' = A_\gamma$: Combinativity

2. If $A_\sigma \cup A_\sigma' = A_\gamma$, then $A_\sigma' = A_\gamma - A_\sigma$: Reversibility

3. $(A_\sigma \cup A_\sigma') \cup A_\gamma = A_\sigma \cup (A_\sigma' \cup A_\gamma)$: Associativity

4. $A_\sigma \cup \phi = A_\sigma$: General Operation of Identity

5. (a)$A_\sigma \cup A_\sigma = A_\sigma$; (b)$A_\sigma \cup A_\gamma = A_\gamma$ where $A_\sigma \subset A_\gamma$: Special Identities

This grouping describes essential operations involved in cognition of simple hierarchies of classes. Proficiency with the use of the class inclusion relation is thereby essential in the establishment of operatory classification. A more general representation of the grouping structure than that given for Grouping I above has been given by Piaget (Beth and Piaget, 1966, pp. 172-173). Although this general representation is not reproduced here, it is important to note that the interpretation given for Grouping I satisfies the requirements of the more general system.

Grouping II

The second grouping discussed is commonly referred to as addition of asymmetrical relations—Piaget's Grouping V. The asymmetrical relations referred to are interpreted as relations which are connected, asymmetrical, and transitive (connected strict partial ordering If A is a set and "<" a connected strict partial ordering defined in A (which well orders A) then the elements of A form a chain, which implies that the ordered set A is a Lattice. If a_1 is the first element of A, a_2 the second, a_3 the third, a_4 the fourth, a_5 the fifth, etc., then it is true that $a_1 < a_2$, $a_1 < a_3$, $a_1 < a_4$, $a_1 < a_5$, etc. If these instances of < are denoted by a, b, c, d, etc., then the diagram in Figure 1 is possible. Combinativity relies on the transitive property of the order relation involved. That is, if $a_1 < a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

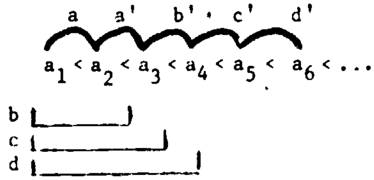


Figure 1

The special notation $a + a' = b$ is also used to represent the same property (Beth and Piaget, 1966, p. 177). The notation of this latter representation seems to be more suggestive of a grouping structure as the following contiguous compositions are highlighted.

1. $a + a' = b$

2. $b + b' = c$

3. $c + c' = d$

4. $d + d' = e$, etc.

A reason for introducing the notation a, b, c, d, etc.; a', b', c', d', etc., other than its being suggestive of a grouping structure, is that it allows the following comparisons-- $a \alpha b \alpha c$, etc. Here, "α" denotes an ordering of instances of the relation "$<$" --a hyperordinal relation (a relation between relations) has been defined where "a" denotes a "smaller" difference than does "b".

Compositions such as $a' + b'$, $b' + c'$, etc., are possible also by virtue of the transitive property, but not $a' + c'$. Elements of the grouping then, are instances of the relation "$<$" (e.g. $a_3 < a_5$, which is just $b' + c'$). Associativity holds naturally because of the transitive property of the relation, but it is a restricted associativity due to the restricted possible compositions. An example is $(a + a') + b' = a + (a' + b')$; or in other terms, $[(a_1 < a_2)$ and $(a_2 < a_3)]$ and $(a_3 < a_4)$ is equivalent to $(a_1 < a_2)$ and $[(a_2 < a_3)$ and $(a_3 < a_4)]$ in the sense that they both imply $a_1 < a_4$. Other grouping properties, however, appear to be rather artificially imposed by the grouping structure. Of the three remaining, reversibility is most viable. Reversibility is distinguished at two levels. The first level of reversibility consists of permuting the terms of an instance of the relation, permuting the relation, or both. In symbols, $R(a_i < a_j) = a_j < a_i$; $R'(a_i < a_j) = a_i > a_j$; and $R''(a_i < a_j) = a_j > a_i$ (Beth and Piaget, 1966, p. 177). The second level of reversibility involves operations concerned with these relations. Piaget (1966, p. 177) combines an instance of a relation with its reciprocal (with R, R', and R") in the following three ways. The first is only possible

1. $(a_i < a_j) + (a_j < a_i) \equiv (a_i = a_j)$

2. $(a_i < a_j) + (a_i > a_j) \equiv (a_i = a_j)$

3. $(a_i < a_j) + (a_j \geq a_i) \equiv (a_i < a_j)$

if the relation is antisymmetric. The second, logically, is just a restatement of the first. The third is certainly logically valid regardless of the properties of the relation. So, (1)-(3) are not taken to express reversibility of asymmetric transitive relations at the second level. The second level of reversibility associated with asymmetrical transitive relations involves instances of the relation $a_i < a_j$ and its reciprocal R". Piaget (1966, p. 178) defines reversibility at the second level by the following two statements.

1. $(a_1 < a_3) + (a_3 > a_2) = (a_1 < a_2)$

2. $(a_1 < a_2) + (a_2 > a_1) = (a_1 = a_1)$

The second statement is interpreted as a suppression of a difference which leads to a relation of equivalence (Beth and Piaget, 1966, p. 178). In the first statement, $a_1 < a_2$ must be known a priori because it is not possible to conclude that $a_1 < a_2$ based on how a_1 is related to a_3 and how a_2 is related to a_3 (one could also say that $a_1 = a_1$ needs to be known a priori). If $a_3 > a_2$ represents a "difference" which is "suppressed," then $a_1 < a_2$ is "left." Now it is apparent that an operation which is distinct from that made possible by virtue of the transitive property has been identified. In effect, the operations exemplified by (1) and (2) directly above assume that the series $a_1 < a_2 < a_3 < \ldots$ has been already produced (or constructed) and is at best a model of how a child, once he has already constructed a series may, for example, start at a_1 and proceed to a_2 and then back to a_1, the starting point. Because the operations associated with reversibility at the second level are distinct from the operations made possible by virtue of transitivity and assume that a series has already been constructed, they are not viewed by me as germane to a model which describes mental operations involving production of a series by a child. In this sense, composition of an instance of a connected, asymmetrical, transitive relation with its reciprocal (R") seems to be a result of an imposition of a general structure. From this position, it is a long way to a wholesale rejection of reversibility by reciprocity. Piaget gives , in at least three different sources (Inhelder and Piaget, 1969, p. 292; Piaget, 1970, p. 29; Piaget, 1964a, p. 130), a discussion of reversibility by reciprocity as it pertains to a seriation of sticks task. Children who display operational seriation find the very shortest stick, then look through the remaining sticks for the shortest one left, etc., until the complete series of sticks is built (assuming no two sticks are of equal length). The reversibility displayed in this task is described as follows: "When the child looks for the smallest stick of all those that remain, he understands at one and the same time that this stick is bigger than all the ones he has taken so far and smaller than all the ones he will take later (Piaget, 1970, p. 29)." Symbolically, if P is the set of sticks taken and Q the set of sticks that remain, then the set of all sticks $S = P + Q$. Moreover, P is a segment of S and is well-ordered by "shorter than." Because "shorter than" well orders S (assuming no two sticks of S are of the same length) there exists a $y \in Q$ where, if $x \in P$, x is shorter than y and if z is any other element of Q not equal to y, y is shorter than z. This is the structure (mathematical) that allows the child to operate as he does. Piaget's reversibility in this context is an understanding by the child that y is longer than any x in P and shorter than any other z in Q. This statement of reversibility has as a necessary condition Piaget's reversibility at the first level of reciprocity. The fact that x is shorter than y is equivalent to y is longer than x involves permuting the terms of the relation as well as the relation (R"). For the child to realize that each x in P is shorter than all y in Q does not seem to demand a knowledge of transitivity because that was precisely how the x's were chosen. However, in choosing the x's the child must use transitivity and possibly reversibility at the first level of reciprocity.

Moreover, R and R' are involved in the statement of the asymmetric property of "shorter than." It seems to me, then, the models of reversibility in the seriation of strings task are given by R, R' and R" and not by models of reversibility at the second level.

It is only prudent to point out at this point that a mathematical model exists which can be used to encompass most of the operations Piaget wants in Grouping V under a unique well-defined mathematical operation. This mathematical model is just the realization of the integers as ordered pairs of natural numbers. Obviously, the natural numbers form a well-ordered set ordered by the connected, asymmetric, transitive relation $<$. In n is a natural number, the segment determined by n ($\{0,1,2,\ldots,n-1\}$) is similar to any of Piaget's Grouping V for some n, so that this discussion is not superfluous. An ordered pair (a,b) of natural numbers is taken to represent the difference b-a. In such case, $(a,b) \oplus (c,d) = (a + c, b + d)$ is the definition of addition of two pairs. The two ordered pairs (a,b) and (b,a) are called inverses of one another because $(a,b) \oplus (b,a) = (a + b, a + b)$, and $(a + b, a + b)$ is just (0,0), the identity element of the system. Moreover, $(a_1,a_3) \oplus (a_3,a_2) = (a_1,a_2)$ and $(a_1,a_2) \oplus (a_2,a_1) = (a_1,a_1)$, which are analogous to the two statements of reversibility at the second level. Also $(a_1,a_5) \oplus (a_7,a_9)$ is an example of a composition not possible in Piaget's Grouping. The assumption is made here that addition of natural numbers exists as well as natural numbers, neither of which is assumed in Grouping V. It is true, however, that whether (a,b) represents a difference or an order $(a < b)$ is innocuous. So, a mathematical model exists for which a neat interpretation of $(a < b) + (b > a) = (a = a)$ is possible. This model, however, is more general than Grouping V in that more than contiguous elements are combinable and the natural number system is assumed. It is the only model known to me which encompasses most of Piaget's Grouping V operations under a unique well-defined operation and makes precise "suppression of differences," but yet does not go way beyond the Grouping V structure.

The fact that Piaget's operations of Grouping V are so well-modelable by addition of integers raises the question of whether reversibility as the second level of reciprocity is any more than operations with integers. This interpretation has sound mathematical foundations because an ordinal number is identified as a well ordered set so that reversibility at the second level of reciprocity may be interpreted as the difference of ordinal numbers.

Questioning the relevance of the second level of reciprocity to seriation tasks leads directly to questioning the relevance of the general identity. The general identity is taken to be an equivalence relation (Flavell, 1963, p. 182), that relation obtained from the second statement of the second level of reciprocity--$(a_1 < a_j) + (a_j > a_1) = (a_1 = a_1)$ (in Piaget's parlance). By substitution in the sum $(a_1 < a_j) + (a_1 = a_1)$, one obtains (through associating terms) $[(a_1 < a_j) + (a_1 < a_j)] + (a_j > a_1) = (a_1 = a_1)$. The first sum is $(a_1 < a_j)$ and the second is $(a_1 = a_1)$ which shows the nonassociativity of special combinations in Grouping V structure. It should be noted that this notation introduces a relation of equivalence $(a_1 = a_1)$ apart from the set A and the order relation $<$.

The special identities are of the form $(a_1 < a_2) + (a_1 < a_3) = (a_1 < a_3)$, where $a_2 < a_3$. Even if this statement is taken to mean that if $a_1 < a_2$ and $a_1 < a_3$, then $a_1 < a_3$ whenever $a_2 < a_3$, it is rather innocuous to the relational structure of $<$. Empirical evidence is quite scanty that special identities model any sort of thought in the child. In fact, Flavell (1963, p. 193) does not even mention them in his summary statements concerning empirical evidence for existence of Grouping V nor does Beilin (1971) mention them in his review of training studies concerning logical thinking. In such training studies, experimenters generally focus on conservation, transitivity, class inclusion, or reversibility (Beilin, 1971). Hence, similar statements may be made regarding the general identity and associativity. Piaget apparently assumes, as noted by Flavell (1963), that "where reasonable evidence for one or two components is found, the existence of the grouping structure as a whole can be inferred (p. 190)."

In view of the foregoing discussion, there seems to be little reason to go beyond the relational structure per se in the case of connected, asymmetrical, transitive relations for a model of intellectual operations modeled by Grouping V. Piaget (1964a) himself has commented, "The criterion for the psychological existence of relations is the ... construction of their logical transitivity (or, if they cannot become transitive, the justification for their non-transitivity) (p. 11)." One should not misinterpret the assertion that there seems to be no reason to go beyond the relational structure per se in the case of connected, asymmetrical transitive relations to find a model of intellectual operations concerning those relations, to mean that the grouping cannot be a model. The simple fact that it has been applied as a model counteracts such an interpretation. In that application, however, one has to be willing to accept the conditions of the application. By not accepting all of the conditions, new problems are opened in mathematics education, problems which may be important not only in mathematics education, but also in cognitive development theory.

Before elaborating more on these problems, it is necessary to further discuss Genetic Structures. Three remaining grouping structures are of central interest: Grouping IV, VI, and VIII--Bi-Univocal Multiplication of Classes, Addition of Symmetrical Relations, and Bi-Univocal Multiplication of Relations.

Grouping VI

The symmetrical relations dealt with are not necessarily reflexive or transitive. This fact complicates Grouping VI and introduces special restrictions on combinativity. For example, if aR_bb means that a and b are brothers and aR_gb means that a and b have the same grandfather, then from asserting that aR_bb and bR_gc, it can be concluded that aR_gc. Thus, it is possible to "combine" two relations which are distinct. The general identity of Grouping VI is analogous to the general identity of Grouping V in that it is denoted by $a = a$ (a is in an identity relation with itself)--that is, logical identity. Reversibility is given by the symmetric property--if aR_bb then the reciprocal bR_ba is taken to be the inverse and is analogous to R" (or R') of Grouping V. Flavell (1963, p. 138), in his analysis of Grouping VI, identifies reversibility at the second level of reciprocity (also analogous to that in Grouping V) as $(aR_bb) + (bR_ba) = (a = a)$. This general identity also behaves according to the statement $(a = a) + (aR_b b) = (aR_b b)$. Special identities are given by tautology $(aR_b b) + (aR_b b) = (aR_b b)$ and what is called resorption $(aR_b b) + (aR_g b) = (aR_g b)$.

Logical Identity and Grouping VI

At this point, it is necessary to draw a sharp distinction between logical identity and equivalence relations in general. Following Tarski(1954), the statement "x = y" is defined as follows; "x = y if and only if x and y have every property in common (p. 55)." From this assertion, one can conclude that (a) everything is equal to itself (x = x); (b) if x = y, then y = x; (c) if x = y and y = z, then x = z (thus "=" is an equivalence relation); and (d) if x = z and y = z, then x = y (two things equal to the same thing are equal to each other) (Tarski, 1954, pp. 56-57). Logical identity, however, far from exhausts equivalence relations. When two planar point sets--segments, triangles, pentagons, etc.--are called congruent, what is meant intuitively is that one can be made to fit exactly on the otner. While more formal definitions for congruence can be given (Gans, 1969, p. 20), it is only necessary to note that congruence for planar point sets is an equivalence relation. It is not, however, an equivalence relation in the sense of logical identity. For in a triangle whose sides are congruent, one would not say that the sides are identical. There are cases, however, where it a queation of the logical identity of two geometric entities. Such cases may arise as special cases of two distinct, but overlapping general properties--the altitude and median to a base of a triangle are logically identical in the case where the triangle is isosceles.

Other examples of the distinction between logical identity and equivalence relations important in this development are set equality (an example of logical identity) and set equivalence (an example of an equivalence relation but not of logical identity). Equal sets are equivalent but it is not necessary for equivalent sets to be equal. Equality of ordered sets and set similarity is another example of logical identity and of an equivalence relation which is not an example of logical identity. Two equal ordered sets are certainly similar but two similar sets need not be equal (to be equal, they would have to contain the same elements). In the example above concerning congruence of planar point sets, if two congruent point sets also contain the same points, then the sets of points are not only congruent, but are also equal (in the sense of set equality) and hence are logically identical as well as congruent.

That the general identity of the grouping structures can be interpreted as an aspect of logical identity is no exaggeration. Not only does Piaget hypothesize that a fundamental grouping of equalities occurs in disguised form as a special case in all other groupings (Flavell, 1963, p. 187), but, in his Heinz Werner lectures (Piaget, 1969), he also analyzes the development of identity in the child in which he included a partial discussion of how identity is related to grouping structures. In the case of Grouping I, "in an additive classification, ... we do have A = A, B = B, etc. but only on the condition that A - A = 0 and A + 0 = A. The identity, A = A depends on a regulator, the "identical operation" of a grouping, that is ± 0; from this point of view, identity has become operational only because it has been integrated into a system of operations (Piaget, 1969, p. 21)." Identity, then, can apparently be view as an interiorized operation at the level of concrete operations

The above quotation can be interpreted as a condition for psychological existence of the reflexive property of logical identity of set equality. Logical identity is transitive, however, as well as symmetric. While both of these properties are dealt with by Piaget (1969), the first is in the context of an experiment dealing with the growth of a plant and the second is in the context of an experiment dealing with beads in two states--as a necklace and spread out in a box. An experiment dealing with logical identity in the context of set equality would have to deal with the three properties of set equality. So, only an aspect (reflexive property) of logical identity is dealt with in this first grouping, and it is very limited in that "the child of 7-8 years may be said to understand the operation $+A - A = 0$, insofar as he knows that adding A, then taking it away, is equivalent to doing nothing ... (Inhelder and Piaget, 1969, p. 146)." The null class apparently is a later construction developing at 10-11 years of age due to the fact that the concrete operations assume the objects do exist. Since Piaget (1966, p. 176) clearly states that the general identical element of Grouping I is the empty class, the relatively late development of the empty class (10-11 years) seems to be inconsistent with the status of Grouping I.

"Identity" appears as a preoperational notion but is not to be taken as a source of the groupings. Instead, it is to be considered as an interiorized operation--a part of the grouping. In terms of Groupings V and VI, identity must be interpreted more broadly than set equality. As already noted, the elements of these two groupings may be considered as instances of the relations which are organized. However, the general identity of the two groupings is neither an order relation in the case of Grouping V nor an instance of the symmetrical relation being considered in the case of Grouping VI but is identified in each case as an identity--and is now interpreted as nothing more than an aspect of logical identity.

Equivalence and Grouping VI

Set equivalence, set similarity, and congruence for planar point sets are all examples of equivalence relations (symmetrical relations) which are not examples of logical identity. In all three cases, one need not incorporate logical identity in a statement of Grouping VI properties. If "$\sim$" denotes any equivalence relation defined on some set A, then a statement of combinativity, "If $a \sim b$ and $b \sim c$, then $a \sim c$," is just a statement of transitivity. Reversibility becomes "If $a \sim b$ and $b \sim a$, then $a \sim a$"; associativity is expressed as "[($a \sim b$ and $b \sim c$) and $c \sim d$] is equivalent to [$a \sim b$ and ($b \sim c$ and $c \sim d$)]," because each is equivalent to $a \sim d$; the general operation of identity is "if $a \sim a$ and $a \sim b$, then $a \sim b$." A statement of the special identities is not very interesting, but, if the form given by Flavell (1963, p. 183) is adhered to, "If $a \sim b$ and $a \sim c$, then $a \sim c$" is a statement of special identity through "tautology", that is to say, the statement is always true. From the statement of the antecedent, one can conclude that $b \sim c$, which to me is a stronger conclusion that $a \sim c$. At any rate, in the statement of the grouping properties, no recourse to logical identity is necessary. It is to be emphasized that the reflexive property of an equivalence relation does not express the same thing as does the reflexive property of logical identity. To say, for example, that a set is similar to itself is different than saying a set is equal to itself. Similarity implies one-to-one correspondence and order, whereas set equality, in the sense of ordered sets, implies only order and element membership--one-to-one correspondence is not necessary to set equality.

The statement of the Grouping VI properties given above in the case of equivalence relations were made possible not only because of the properties (reflexive, symmetric, and transitive) of equivalence relations, but are also a modification of the statements given for Grouping VI in the psychological literature. That modification is more than a modification of statement forms is quite apparent due to the fact that the statement of the general operation of identity does not include a statement involving logical identity. In the case of Grouping V (that concerned with connected, asymmetrical, transitive relations), it has been already pointed out that reciprocity at the second level (and hence, the general identity) is viewed as not being germane to a model which describes mental operations involved in seriation activities by young children. This view is stronger when considered in the context of logical identity, for logical identity is quite distinct from the order relations described.

Partial Orderings and Grouping VIII

Mathematically, two distinct relational structures have been discussed in this section--equivalence relations and connected, asymmetrical, transitive, order relations (connected strict partial orderings). Logical identity was considered as a special equivalence relation. One remaining relational structure worth mentioning is a connected partial ordering; i.e., a connected relation "$\underline{\alpha}$" which is reflexive, antisymmetric, and transitive. This relational structure can be thought of as being more general than those of either equivalence relations or connected strict partial orderings in that it "contains" both of them as substructures. The relationships among the three structures are quite simple. Consider the set of all living people. This set can be partitioned into equivalence classes using the equivalence relation "same height as." If a subset of the people is chosen in such a way that one person is chosen from each equivalence class, this subset is orderable using the connected, strict partial ordering "shorter than" (or "taller than"). One can consider the equivalence classes as having been ordered. The ordering of the equivalence classes could have been accomplished not by successive application of two relational structures, but instead by application of the relation "shorter than or the same height as," which is a connected partial ordering defined on the set of people. This relation not only partitions the set of people into equivalence classes, but also orders the classes by virtue of ordering the individuals of the classes. Special characteristics of the relation exist. One is that if a is shorter than b and if a is the same height as c, then c is shorter than b. In fact if A is the set of all people the same height as a, any individual who is a member of A may be substituted for a in the statement "a is shorter than b", the result being a true statement. Generally, connected partial orderings are not dealt with in any of the grouping structures discussed thus far. The fact that it is a viable candidate for a model of active intellectual operations is strengthened by consideration of the relation "fewer than or just as many as" defined on finite sets in terms of one-to-one functions. What this relation does is to partition finite collections into equivalence classes and then order the classes in the same manner that the equivalence classes of people were ordered by "shorter than or the same height as." Any finite collection can be considered as representing the class to which it belongs, or as representing a cardinal number (where the cardinal number is defined as the class to which the finite collection belongs).[4] One may think that Grouping VIII (Bi-Univocal Multiplication of Relations) would be general enough to encompass connected, partial order relations but instead, it seems to me that this grouping assumes them as Flavell (1963, p. 184) applies Grouping VIII in the context of asymmetrical relations.

As an example of Grouping VIII, let A denote a set on which two connected partial orderings are defined. If A is taken to be a collection of bundles of sticks and the two relations are taken to be "shorter than or just as long as" ($\leq$) and "more than or just as many as" ($\leq'$) then A can be depicted schematically as in Figure 2. If a row and column entry is considered as a

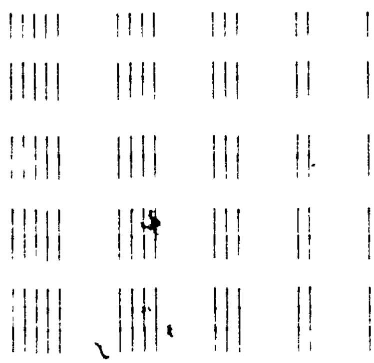


Figure 2 Matrix of Sticks

bundle of sticks, then $\leq$ orders A by rows and $\leq'$ orders A by columns. Obviously, the bundles of sticks in Figure 2 were chosen so all row and column positions would contain entries. This requirement is not necessary; that is, some row and column positions could be without entries. The relation $\leq$ is asymmetrical if it is applied to rows and it is in this sense that Grouping VIII involves asymmetrical relations. The relation defined by "shorter than or just as long as and more than or just as many as"

[4] Note that any set may be used as a standard set. In particular, the counting set {1,2,...n} may be used as a standard set. Whether the counting set or the equivalence class is considered as the cardinal number of a finite set theoretically makes little difference.

(deonted by $[\leq, \leq']$) is a partial ordering on bundles of sticks, but it is not a connected ordering. If S_{ij} is a bundle of sticks in the (i,j) position (the ith row and jth column), then if $S_{ij}[\leq, \leq']S_{k\ell}$, it will always be true that $i \leq k$ and $j \leq \ell$. In the following discussion, if for some (i,j) and (k,j), $S_{ij}[\leq, =']S_{kj}$ and if for some (i,j) and (i,ℓ), $S_{ij}[=, \leq']S_{i\ell}$, the shorthand notation $S_{ij}[\leq]S_{kj}$ and $S_{ij}[\leq']S_{i\ell}$ is used.

Compositions are really at two levels of complexity. The first is depicted by "If $S_{ij}[\leq]S_{kj}$ and $S_{kj}[\leq']S_{k\ell}$, then $S_{ij}[\leq, \leq']S_{k\ell}$." This composition simply means that if the sticks in S_{ij} are shorter but equal in numerosity to the sticks of S_{kj} and the sticks of S_{kj} are just as long as but more numerous than the sticks of $S_{k\ell}$, then the sticks of S_{ij} are shorter but more numerous than the sticks of $S_{k\ell}$. Also, "If $S_{ij}[\leq, \leq']S_{k\ell}$ and $S_{k\ell}[\leq, \leq']S_{mn}$ then $S_{ij}[\leq, \leq']S_{mn}$" denotes a possible composition at a more general level of complexity than before. An interpretation is that if the sticks in S_{ij} are more numerous but shorter than the sticks in $S_{k\ell}$ and the sticks in $S_{k\ell}$ are more numerous but shorter than the sticks in S_{mn}, then the sticks in S_{ij} are more numerous but shorter than the sticks in S_{mn}. Interpreting Flavell (1963, pp. 185, 186), the inverse operation could be depicted as follows: "If $S_{ij}[\leq, \leq']S_{k\ell}$ and $S_{k\ell}[\geq, \geq']S_{ij}$, then $S_{ij}[=, =']S_{ij}$, where the general identity would be $S_{ij}[=, =']S_{ij}$, which is not an expression of logical identity but is nothing more than reflexivity of $[\leq, \leq']$. This expression of reversibility is analogous to the expression of reversibility of Grouping V at the second level of reversibility. Here, it also seems to be an unnecessary addition to the structure. Certainly, one would want a child to know that if the sticks of S_{ij} were shorter but more numerous than those of $S_{k\ell}$, then the sticks of $S_{k\ell}$ are longer but less numerous than those of S_{ij}, which is analogous to reversibility at the first level of reciprocity. One can give a legitimate interpretation of reversibility at the second level or Grouping VIII in terms of movement from position to position in the matrix, but this interpretation is different than that given for relations considered immediately above, and the two should not be confused.

There exists, then, the relational structure of $[\leq, \leq']$, which is a partial ordering. This relational structure seems sufficiently rich to encompass the behavioral analogues observed in the child, just as the relational structure of connected strict partial orderings seemed sufficiently rich to encompass the behavioral analogues observed in the child in the case of Grouping V, without recourse to all of the Grouping properties. More discussion on this point is given after a discussion of one remaining genetic structure--that of Bi-Univocal Multiplication of Classes.

Grouping IV

Starting with an example, if a class C is partitioned into two subclasses, say A_1 and A_2, and two other subclasses, say B_1 and B_2, then $C = A_1 \cup A_2 = B_1 \cup B_2$. Moreover, $C = (A_1 \cup A_2) \cap (B_1 \cup B_2) = (A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (A_2 \cap B_2)$. If C is taken to be a set of marbles, then A_1 could be blue marbles, A_2 green marbles; B_1 glass marbles, and B_2 steel marbles. The set of marbles then, is partitioned into blue glass marbles, green glass marbles, blue steel marbles, and green steel marbles. In other words, a matrix or double entry table of four cells has been generated with the component classes along each dimension. Rather than just 2 x 2 tables, general m x n tables may be generated through appropriate partitionings. The grouping structure is supposed to describe essential mental operations and interrelationships among mental operations involved in partitioning a given class into subclasses and in constructing simpler partitionings given a multiple partitioning.

Composition is described as just the intersection of classes; either the intersection of two classes (e.g. $A_1 \cap B_1$) or the intersection of unions [e.g. $(A_1 \cup A_2) \cap (B_1 \cup B_2)$]. Other properties are straightforward except, perhaps, for reversibility. If $A_1 \cap B_1 = E$, then a "division" of classes is introduced so that $A_1 = E:B_1$ and is to be interpreted as an abstraction of the class A_1 from $A_1 \cap B_1$. That is, if from the class of blue glass marbles the class of glass marbles is abstracted, the class of blue marbles remains. The general identity element is taken to be the most general class of the system which is, in the example, the class of marbles. If, from among the marbles, the steel ones are singled out and then if their "steelness" is disregarded, they are only regarded as marbles. Symbolically, this process is represented by $B_2 = B_2 \cap C$, so $C = B_2:B_2$. This sort of reversibility is distinct from that of Grouping I in that in Grouping I the elements of a class were subtracted rather than a property of them being abstracted. Both of these processes of reversibility are to be considered as <u>inversions</u> "which makes an inverse operation t^{-1} correspond to an operation t, which, combined with it, ends by annulling it (Piaget, 1966, p. 176)."

General Discussion of Grouping Structures

Not only are the genetic structures discussed to be considered as models of cognitive operations essential to the cognition of hierarchical classifications, multiple classifications, seriations, and multiple seriations, but they are also to be considered as making quantification and conservation possible. Although definite distinctions can, and should, be made between mathematical and genetic structures, there are no a priori reasons to believe that some mathematical structures could not be shown to be models of cognitive operations in the same sense as the grouping structures are. In fact, it has been indicated that when the grouping structures are specialized to encompass mathematical structures of interest, certain antimonies occur which are not resolvable on logical grounds. Such is the case for Grouping V, Grouping VI, Grouping VIII, and, to a lesser extent, Grouping I.

As noted, the structure of connected, asymmetrical, transitive relations seems sufficiently rich to encompass behavioral analogues observed in seriation behavior of children, without recourse to reversibility at the second level of reciprocity, the general identity, and special identities. An application of a model for the integers was made to series of Piaget's Grouping V, but I feel that the model for the integers is too general in that it introduces differences, and differences are not logically essential to the relational structure of concern. The model did, however, allow for a neat interpretation of the grouping operation. The fact remains that as far as I can detect at this time, the structure of connected, asymmetrical, transitive relations is, as a logical model of seriation behavior, more parsimonius than Grouping V.

However, I do not consider the question answered; further empirical work, in the spirit of Piaget, needs to be done whose goal is to ascertain which model is the more parsimonious one While this research should incorporate all aspects of each model, the general identity, reversibility at the second level of reciprocity, and the special identities should be of special concern. The ordered pair model for the integers should not be ignored, because if reversibility at the second level of reciprocity is observed in seriation behavior, the child may already be operational with "number." Because the general identity has been shown to be an integral part of the grouping structures, its development needs more explication. There is an excellent prospect that logical identity is a developmental phenomenon. Then developmental studies should also be undertaken, whose goal is to trace the development of logical identity and its relation to developmental aspects of seriation and classification behavior.

The extent to which partial orderings may be considered as a genetic structure also needs investigation. In such an investigation, it would be necessary to relate the grouping structure to partial orderings to obtain the more parsimonious model of active intellectual operations concerned with relations. Moreover, the role of logical identity also needs explication here.

Because partial orderings contain equivalence relations as a substructure, a partial ordering may encompass classifications. But rather than equivalence relations being discussed in the context of partial orderings, they are being singled out as a special entity of concern. When Grouping VI was specialized to equivalence relations, the relational structure was sufficiently rich to imply the grouping properties--recourse to logical identity was not needed. This fact suggests that equivalence relations in general may have a developmental history of their own separate from that of logical identity or symmetrical relations which are not equivalence relations. It has not been pointed out that the notion of contiguous elements, so central to the grouping structures is quite blurred in the context of equivalence relations. The notion is distinct for seriations, but it is not for equivalences. For example, given a collection of sticks which a child is to classify on the basis of "the same length," what sticks are to be considered as contiguous or "next to each other?" Any two will do. Simply because the model (equivalence relations) does not require step by step combinations does not mean that children will not use such combinations. If children do combine elements step by step, a model of the phenomenon should account for it. While the grouping structures account for contiguous combinations, specializing Grouping VI to equivalence relations does not lead to any requirement that an element occupy a unique place in a classification, as did Grouping V for a seriation. Because of this fact, children may operate differently with equivalence relations than with order relations.

In the above example, another aspect of equivalence relations is brought out--for every equivalence relation defined on a set of elements, there exists a partition and for every partition there exists an equivalence relation. Hence, there is a one-to-one correspondence between the equivalence relations defined on a set and the possible partitions of the set. So, one would hypothesize a close relationship between behavioral manifestations of equivalence relations and classificatory behavior of children This possible fundamental relationship has not been fully investigated and, in my estimation, is not clearly accounted for by Piaget's Groupings. I must be quick to point out that Piaget contends that classes and relations develop synchronously. Then should one infer partitioning behavior from relational behavior, or vice versa?

Grouping Structures and Number

Even though the relevance of the total Grouping Structure to cognition of relations has been questioned, the Genevan literature concerning the development of number (and measurement) can be understood only in the context of the Grouping Structures. The Genevans view the genetic construction of the natural numbers as being brought about through a "synthesis" of Grouping I and V (Beth and Piaget, 1966, p. 175). In fact, Grize has gone so far as to show that, starting from his general presentation of the Grouping Structure, certain modifications can be made which lead to a structure which he shows to be that of the natural numbers (Beth and Piaget, 1966, pp. 268-270). Showing logically that such a modification is possible does not prove, however, that the restrictions made on the Grouping Structure correspond to any developmental process in the child. That such a possibility exists cannot be ignored or taken lightly. But rather than dwelling on possible research studies which would shed light on a potential convergence between Grize's formalization and the development of number, the manner in which Groupings I and V are "synthesized" is outlined.

Piaget (1964, pp. 183, 184) gives two essential conditions for the "transformation" of classes into numbers. In the discussion, it must be assumed some hierarchical system $\subset A_1 \subset A_2 \subset \ldots \subset A_i$ of classes exists, each complement of which contains a singular element. For example, A_1 could be a bead, A_1' a cube, A_2' a bean, etc., where $A_2 - A_1 = A_1'$, $A_3 - A_2 = A_2'$; etc. The first condition given is that all elements must be regarded as equivalent (all qualities of the individual elements are eliminated). But, if condition one holds, then A_2 would not be a class of two elements, but instead of only one, for $A_1 \cup A_1' = A_1$ if $A_1' = A_1$--which is to say that the quality of the elements is eliminated. If the differences of A_1 and A_1' are taken into account, then they are no longer equivalent to one another except with respect to A_2. This brings the second essential condition into focus. In effect, the equivalent terms must somehow remain distinct but that distinction no longer can have recourse to qualitative differences. Given an object (the bead), then any other object is distinguished from that object by introducing order--by being placed next to, selected after, etc. "These two conditions are necessary and sufficient to give rise to number. Number is at the same time a class and an asymmetrical relation .. (Piaget, 1964, p. 184)." In qualitative logic, objects cannot be, at one and the same time, classified and seriated, since addition of classes is commutative whereas seriation is not (Piaget, 1952, p. 184). If the qualities of the elements are abstracted, then the two groupings (I and V) no longer function independently, but necessarily merge into a single system. The only way to distinguish A_1, A_1', A_2', ... is to seriate them $A \rightarrow A \rightarrow A \rightarrow \ldots$, where $\rightarrow$ denotes the successor relation (Beth and Piaget, 1966, pp. 266, 67). Clearly, Piaget considers each A to be a unit-element, at once equivalent to, but distinct from, the others, where the equivalence arises through the elimination of qualities and the distinctiveness arises through the order of succession.

In Piaget's system, then, number is not to be reduced to one or another of the groupings, but instead is a new construction--a synthesis of Groupings I and V. Elements, from the point of view of their qualities, are either considered from the point of view of their partial equivalences and are classified, or are considered from the point of view of their differences, and are seriated. It is not possible to do both at once unless the qualities are abstracted (or eliminated), and then it is necessary that both are done simultaneously--one cannot help it!

It is now possible to understand the development of one-to-one correspondence. Qualitative correspondence is correspondence which is based only on the qualities of corresponding elements, whereas numerical correspondence is correspondence in which each element is considered as a unit element. Intuitive correspondence is correspondence based entirely on perception and, consequently, is not preserved outside the actual field of perception; but operational correspondence has as its distinctive characteristic the fact that it is preserved independently of perception (Piaget, 1964, p. 70). Qualitative correspondence, then, can be either intuitive or operational but numerical correspondence is essentially operational. Children pass through three stages regarding one-to-one correspondence; the first is essentially no correspondence (up to approximately 5 years of age), the second is intuitive qualitative correspondence, and the third is operational or numerical correspondence. Essentially, then, operational one-to-one correspondence assumes number (as viewed by Piaget).

Set similarity is also a developmental phenomenon. Piaget (1964, p. 97) differentiates between qualitative correspondence between two seriations and numerical correspondence between two series. The construction of a single series and that of finding a one-to-one correspondence between two series amounts to the same thing insofar as Piaget's behavioral analyses show. Children again go through three stages with regard to set similarity--no conception of the possibility of seriation, or similarity, seriation or similarity based on perceptual processes; and then numerical correspondence between two series.

The notion of a unit is central in Piaget's system and is not deducible from the Grouping Structures, but rather is the result of the synthesis already alluded to. Once reversibility is achieved in seriation and classification, "groupings of operations become possible, and define the field of the child's qualitative logic (Piaget, 1964, p. 155)." Here, operational seriation has as a necessary condition reversibility at the first level of reciprocity. "A cardinal number is a class whose elements are conceived as 'units' that are equivalent, and yet distinct in that they can be seriated, and therefore ordered. Conversely, each ordinal number is a series whose terms, though following one another according to the relations of order that determine their respective positions, are also units that are equivalent and can therefore be grouped in a class. Finite numbers are therefore necessarily at the same time cardinal and ordinal ... (Piaget, 1964, p. 157)." The development of classes and relations does not, as it may seem from the above quotations, precede the development of number in Piaget's theory, but those developments are simultaneous. Without knowledge of the quantifiers "a," "none," "some," and "all," which implicitly involve cardinal number, the child is not capable of cognition of hierarchical classifications. A genetic circularity consequently exists in the developmental theory of classes, relations, and numbers.

Subtleties exist in the notion of ordered sets which sometimes are obscured by physical embodiments. If a connected, asymmetrical, transitive relation α is defined in a set A, then one may think of the elements of a as being ordered according to α. A physical embodiment is the case where a collection of sticks, no two of which are the same length, is ordered by "shorter than." This relation α completely determines a particular order on A. If a relation α' distinct from α, but nevertheless a connected strict partial ordering, is defined on A, an ordering of A exists distinct from the former. Such an ordering in the case of the sticks could be an ordering based on, say, diameter (where, of course, appropriate conditions on the diameters hold). If A represents the sticks ordered by α and A' by α', then A is similar to A' but the two are not necessarily equal ordered sets, which would be the case if and only if α and α' ordered A in the same way. If A contained n sticks, then a similarity mapping could be established between the standard set $\{1,2,3,\ldots,n\}$ and A ordered by α. Of course, a similarity mapping could be established between the standard set $\{1,2,\ldots,n\}$ and A, ignoring α.

In his study of ordination and cardination, Piaget (1964, chap. VI) employed three experimental situations, one involving seriation of sticks, one seriation of cards, and one seriation of hurdles and mats. In the seriation of sticks experiment, the child was asked to seriate ten sticks from shortest to longest and then was given nine more sticks and was asked to insert these into the series already formed (the material was constructed in such a way that no two sticks were of the same length). He was then asked to count the sticks of the series after which the sticks not counted (or sticks the child had trouble counting) were removed, apparently along with one or two he did not have trouble counting. The experimenter then pointed to some stick remaining and asked how many steps a doll would have climbed when it reached that point, how many steps would be behind the doll and how many it would have to climb in order to reach the top of the stairs formed by the sticks. The series was then disarranged and the same questions as before were put to the child, who would have to reconstruct the series in order to answer the questions.

There is no question that aspects of ordinal number and cardinal number were involved in the above experiment. Any conclusions drawn with regard to number, however, by necessity are a function of a capability to construct a series of sticks based on a connected asymmetrical relation having little to do with ordinal number. To demonstrate my concern more concretely, an eight year old child was asked by me which, of a collection of books on a table, would be the third one. He answered, "What do you mean, any one could be third!" Piaget's experiment with the staircase, then, was more an experiment concerning similarity between a set of n sticks ordered by "shorter than" and the standard counting set $\{1,2,\ldots,n\}$ than it was an experiment concerning ordination and cardination. A similar analysis holds for the seriation of the cards experiment. While no analysis of the hurdles and mats experiment is given, suffice it to say that it too involves specific relations.

In the mathematical development of cardinal and ordinal number, no analogue of Piaget's "arithmetical unity" exists except for elements of sets. "Set" is taken as an undefined object and relations, cardinal number, and ordinal number are defined in terms of sets. Such a procedure is logically impeccable, although Piaget (1970, p. 37) is of the opinion that to define cardinal number and ordinal number in such a way is to introduce number into the definition of number. This opinion is based on the different types of one-to-one correspondence identified in developmental theory--operational one-to-one correspondence assumes number. But, as already noted, Piaget's formulations lead to a genetic circularity among classes, relations, and number, a circularity of definition avoided in mathematics. Such a circularity does not inherently invalidate the results of developmental research on cardinal and ordinal number; but the question arises whether other theoretical analyses are possible for the same data, and, if so, would this alternate analysis lead to new empirical research?

Because logical identity is an equivalence relation, there exists an accompanying difference relation "not identical to." This symmetrical difference relation seems to be quite important in classification, because if objects are classified together they share common properties, but they are also different one from the other, even if this difference is no more than their distinctness. Moreover, even if objects are different one from the other, it does not necessarily follow that they are orderable on the basis of those differences. To say that two objects are different only implies that a symmetrical relation exists between them. Surely a seal is different from a dolphin, but who would try to order a seal and a dolphin on that basis? It appears to me, then, that it is quite feasible for a child to view a class of objects as being equivalent in some aspects but yet different in others, where no order is necessarily implied in such a realization of differences. Because an excellent possibility exists that logical identity is a developmental phenomenon, and because set equality is an example of logical identity, an excellent possibility exists that a four year old child, say, would not maintain the invariance of class membership under spatial transformation of the elements, thus having formed only graphic collections; but an eight year old child, say, would maintain class membership under the same spatial transformation simply because the concept of logical identity is an operational concept for the eight year old but not yet for the four year old. Why it is necessary for "number" to develop before operational classification is possible is not entirely clear. Logical identity being applied to the rearrangements of the members of a class of objects is quite analogous to logical identity being applied to plant growth. In either case it seems that recourse to number is not necessary. It would not be surprising if, at some point in time, logical identity was used by a child as justification for a numerical conservation. On the other hand, it also would not be surprising if logical identity was an earlier development than number, either cardinal or ordinal. That does not mean that I consider logical identity as a necessary and sufficient condition for the psychological existence of cardinal and ordinal number. Nothing could be farther from the actual case. It would be rather surprising, though, if a child had a well developed concept of both cardinal and ordinal number but not of logical identity.

If one does not consider number to be necessary for operatory classification, how is one to account for the development of operational one-to-one correspondence? If a child sets up a qualitative one-to-one correspondence between two classes and one or both of the classes is rearranged, there is no hope that the correspondence would be maintained without logical identity. Following Van Engen (1970, pp. 34-52), if a number (e.g., four) is regarded as a particular set in the member-of-a-class meaning, then logical identity is surely a logical prerequisite to number, but one-to-one correspondence is not.

One-to-one correspondence is a logical prerequisite, however, to the class meaning of cardinal number where one-to-one correspondence is taken as an equivalence relation. An ordinal number can also have a member-of-a-class meaning in that it can be regarded as a particular ordered set, which implies the existence of a connected, strict, partial ordering. The class meaning, of course, involves one-to-one correspondence in the context of set similarity.

Not only are developmental studies concerning the objects called cardinal number and ordinal number desirable, where the developmental studies take into consideration logical identity, classes, and relations, but such developmental studies, concerning addition, multiplication, subtraction, and division, are also necessary. It should be clear that the Genevan theory concerning the development of number is not being rejected in the absence of developmental data concerning the foregoing conceptual framework dealing with cardinal number, ordinal number, relations, and classes. Experiments need to be done, however, designed so that judgments can be made concerning viable theoretical interpretations of the data. A priori decisions are not possible.

Some Experiments

Up to this point the only thing offered in this paper is an analysis of developmental theory as it applies to developmental phenomena concerning relations, classes, and number and a suggestion of directions that research can take in light of that analysis. A good start has been made toward the collection of facts necessary to the construction of a theoretical position concerning the development of mathematical concepts. These data are incomplete as they do not answer even the questions posed in this paper, and at times are directed toward answering questions other than those raised here. While no apologies are offered for the present state of existing data, it is only prudent to acknowledge the present state of data collection. With this acknowledgement in mind, two restricted series of studies are discussed below. While considerably more data exists than is presented, the two series are selected because they give quite different perspectives on closely related phenomena.

Conservation and Transitivity--Status Data

The first series of experiments involves a study of transitivity across relational types (order and equivalence relations) and relational content (matching, length). These studies are important to mention due to the centrality of transitivity as a criterion for psychological existence of a relation in developmental theory and to the importance of transitivity to equivalence and order relations and, ultimately, to number and measurement.

Divers (1970) conducted one of the first experiments in this series. The subjects for his experiment were 49 kindergarten children, of whom 26 were black, and 47 first grade children, of whom 27 were black. The remaining children were caucasian. The age range for the kindergarteners was 65 to 76 months with mean age 71 months, and the age range for the first graders was 78-96 months with mean age 85 months. The relations dealt with were "same length as," "longer than," and "shorter than"--two order relations and an equivalence relation. The ascertainment of the influence of three contextual situations on transitive reasoning was of interest. It was felt that a situation in which no apparent perceptual conflict was present but in which the physical objects were actually present would facilitate transitive reasoning to a greater extent than either of the two situations in which (a) the objects were not visually present and (b) the objects presented an obvious perceptual conflict. Moreover, it was predicted that children for whom evidence was present of conservation of the relations involved would be more likely to engage in transitive reasoning than would children for whom little or no evidence was present of such conservation.

On the basis of a preliminary knowledge of terms test, 35 per cent of the black kindergarteners and 7 per cent of the black first graders were eliminated from further study and 13 per cent of the caucasian kindergarteners and none of the caucasian first graders were eliminated from further study. These children were eliminated to decrease the possibility of falsely diagnosing children as not being able to engage in transitive reasoning. Two tests were administered to the children remaining in the study, a conservation of length relations test and a transitivity test. The conservation of length relations test consisted of nine items, three for each relation. In any group of three items written for a relation, one involved a screened stimulus, one a conflictive stimulus, and one a neutral stimulus. In any item, after the initial comparison and transformation took place, (that is, after the sticks were placed in their final position) three questions were asked of the child--one for each relation--so that a child had to know which relation still held after transformation, as well as which ones did not, in order to score an item correctly. A child was classified at a high level of conservation if he scored at least two items correctly for each relation, as a low conserver if there was not more than one relation on which he scored two or more items correctly and a medium conserver otherwise. Table 1 contains the number of children within each of the conservation categories by grade. The table reflects the internal consistency reliability of .75 on the conservation test in that substantial frequencies occurred in each category. The transitivity of length relations test consisted of 27 items, nine for each relation. For each relation three of the nine items involved a neutral stimulus, three a conflictive stimulus, and three a screened stimulus. In case of the screened stimulus, the experimenter compared a red and blue stick after which the child was asked "Is the red stick the same length as (or longer than or shorter than, depending on the relation) the blue stick?" The red stick was then covered with an opaque cloth. The same procedure was followed with the blue stick and a green stick after which the green stick was covered with an opaque cloth and the blue stick removed from the experimental setting. Three questions similar to the preceding question were then asked of the child to which he had to respond "yes" once and "no" twice in order to answer the item correctly. For example, if the red and green stick were actually of

Table 1

Number of Children by Grade and Conservation Level

	Conservation Level		
Grade	High	Medium	Low
K	10	10	17
1	20	14	11
Total	30	24	28

the same length. the child had to respond "yes" to the question "Is the red stick the same length as the green stick?" and "no" o the two others in order to answer the item correctly. The transitivity test had a mean score of 12.4, a standard deviation of 6.2 and an internal consistency reliability of .87, all computed on the responses of 82 children.

The statistical design employed is called a mixed design (Kirk, 1968) with two between subject variables and two within subject variables. The ANOVA computed on 60 randomly selected subjects from the 82 completing both tests is given in Table 2. Both between subject variables were significant as was the within subject variable "Stimulus Condition." No significant interactions were present. The mean scores for the between subject variables are presented in Table 3. Conservation level was a much stronger between subjects variable than was grade level, although some overall improvement was noted for the first graders over the kindergarteners. Because a child could obtain a score of 11 per cent based on chance responses, the means reported in Table 3 are spurious because they are not corrected for guessing. The only significant within subject variable was Stimulus Condition. The means for this variable are contained in Table 4. Essentially, no differences occurred between the screened and conflictive stimulus, the variability thus occurring between the neutral stimulus and the two others. No differences were observed between the black and caucasian children on either of the conservation of relations test or on the transitivity of relations test.

The above experiment was essentially replicated (with modification) by Owens and Steffe (1972), using matching relations rather than length relations. The three matching relations were defined operationally for 51 caucasian middle class children enrolled in two denominational kindergartens in Athens, Georgia, rather than eliminate children from the study on the basis of a lack of knowledge of terminology. Even after seven 20-30 minute instructional sessions, 16 of the 51 children were not able to display adequate knowledge of terminology. These 16 children repeated selected activities after which seven still did not display knowledge of terminology and were subsequently eliminated from further study.

Table 2

ANOVA for Transitivity Scores

Source of variation	df	MS	F
Between Subjects			
Grade Level (A)	1	12.15	4.12*
Conservation (C)	2	47.07	15.96**
AC	2	1.24	< 1
Subj. W. Groups	54	2.95	
Within Subjects			
Relations (B)	2	.87	< 1
AB	2	.12	< 1
CB	4	1.62	1.22
ACB	4	2.36	1.77
BXSubj. W. Groups	108	1.33	
Stimuli (D)	2	8.98	14.03**
AD	2	.28	< 1
CD	4	.77	1.20
ACD	4	.06	< 1
DXSubj. W. Groups	108	.64	
BD	4	.91	1.82
ABD	4	.45	< 1
BCD	8	.75	1.50
ABCD	8	.40	< 1
BDXSubj. W. Groups	216	.50	

* ($p < .05$)
** ($p < .01$)

Table 3

Mean Scores on Transitivity: Grade Level by Conservation Level (Nearest Percent)

	Conservation Level			
Grade	High	Medium	Low	Total
K	67	44	24	45
1	71	58	40	56
Total	69	51	32	51

Table 4

Mean Scores for Stimulus Condition (Nearest Percent)

Stimulus	Neutral	Screened	Conflictive
Mean	56	43	42

A conservation of matching relations test was administered in conjunction with the knowledge of terms test and was used to classify children as high conservers and low conservers of matching relations. Six items were constructed for each relation for a total of 18 test items. All items involved only the conflictive stimuli identified by Divers (1970). The internal consistency reliability was .94 on the test. This led to categorizing the children into two conservation categories--high and low. A child was classified as a high conserver provided he conserved the relation on four of the six items on each relation and as a low conserver otherwise. All children had scores above the criterion or appreciably below except one; his score was slightly below the criterion level. This child and another child, who was of legal age to be in grade one, were eliminated from the data analysis; 21 boys and 21 girls were left as subjects. Twenty-seven were high conservers and 15 were low conservers. Only kindergarten children were used in this experiment; in Divers (1970) study age was not a strong between subjects variable. Conservation was used as a between subjects variable as were the three relations. Stimulus condition was used as a within subjects variable: six transitivity items were written for each stimulus condition. Table 5 contains the analysis of variance. Since an interaction occurred between conservation level (C) and relations (R), the main

Table 5

ANOVA for Conservation Levels and Relations

Source of Variation	df	MS	F
Conservation (C)	1	28.14	11.14**
Relation (R)	2	18.74	8.64**
CXR	2	14.22	6.56*
Subj. W. Groups	36	2.17	
Stimuli (S)	2	5.64	5.26^{+}
CXS	2	1.47	1.37
RXS	4	.42	< 1
CXRXS	4	1.56	1.46
CXSubj. W. Groups	72	1.07	

* ($p < .05$)
**($p < .01$)
($p < .05$), Conservative Test

effects of C and R cannot be discussed per se. A Newman-Kuels post hoc test was performed on the simple effects of the CXR interaction to determine the source of variation. Within the high conservation levels the mean of 77% for the relation "as many as" differed from the mean of 51% for the relation "more than" ($p < .05$) as well as from the mean of 46% for the relation "fewer than" ($p < .05$). The only relation on which high and low conservers differed significantly ($p < .05$) was the equivalence relation "as many as." In the case of the within subjects variable, Stimulus Condition, the means as percents were 59, 51, and 47 for the neutral, screened, and conflictive stimulus, respectively. The neutral stimulus differed from both the screened and conflictive, which did not differ statistically.

As indicated by Beilin (1971, p.88), it is important to explicate what constitutes the mechanism underlying conservation behavior and distinguish it from an operational definition for conservation. The same comments also pertain to transitivity. In the spirit of Piaget (1964, p.42), who takes conservation to be a behavioral manifestation of the existence of a grouping structure, in the two experiments reported on (in part) above, it was assumed that conservation of the relations involved would be a behavioral manifestation of the presence of the relational structures defined on sets of concrete material. In both studies, transitivity of three particular relations, one equivalence relation and two connected, strict, partial order relations, was operationalized and was also taken as psychological existence of the relational structures of interest--at least the existence of the particular relational structures. It was legitimate, then, to use conservation as a blocking variable, because those children who were high conservers also should have performed quite well on transitivity of the relations of interest. It was assumed that conservation and transitivity of the relations were just reflections of the existence of relational structures of which transitivity was a part. Those children who were low conservers should also have done relatively poorly on transitivity of the relation for the same reason. Perfect relationships were not expected because, by necessity, the variables, conservation of relations and transitivity of relations, were given operational definitions (which were thought to be strong).

The variable Stimulus Condition was interesting because concrete operations are taken to mean that a child can think in a logically coherent manner about objects that do exist and actions that are possible either with objects or in the immediate absence of objects. Hence, for children who were categorized as high conservers, the screened stimulus should have presented no more difficulties for the children in transitive reasoning than would the neutral stimulus where the objects were present. In fact, the screened stimulus should have forced the child to focus on the only information available--the two hypotheses. It was anticipated that the conflictive stimulus would present special difficulties because children would be more apt to reason using nontransitive hypotheses. For those children in the low category of conservation, Stimulus Condition should not have been significant, for such children theoretically should not be in possession of a genetic relational structure.

In the case of Divers (1970) data, Conservation was highly significant and did not interact with any other variable. So, for the length relations, no statistical contradiction was present that conservation could be considered as a behavioral manifestation for psychological existence of relational structures insofar as transitivity goes. Of course, exact relationships were not obtained--only statistical relationships. But the statistical relationships were indeed strong and, in my judgment, were within tolerance of an operational definition of the variables of concern. Divers (1970, p.73) constructed contingency tables from which more exact relationships between conservation and transitivity could be aacertained. Only 12 of 94 responses were categorized as being nonconservation responses but as being also successful transitive responses. Of these responses, only one child showed no evidence at all of being a conserver. The others did display conservation in some cases. There were children, however, who displayed no avidence of being able to engage in transitive reasoning but were categorized as high conservers. So, conservation of length relations cannot be said to be a necessary and sufficient condition for transitivity of length relations, but only a necessary condition. It would seem, then, that conservation of relations can appear before symptoms of a relational structure can be found, but once such a symptom is found (transitivity), conservation is almost certain to follow. It is the case, then, that a good possibility exists that transitivity of length relations is a sufficient condition for conservation of length relations, which does not contradict the theoretical assumption that once a relational structure becomes operational for a child, conservation should be present. In the case of length relations, however, conservation may appear before transitivity.

Even though Stimulus Condition was significant in favor of the neutral stimulus, the variable was not strong enough to warrant any serious theoretical speculation; however, children do engage in transitive reasoning in the immediate absence of concrete objects to the same extent that they engage in transitive reasoning in the presence of perceptual conflict. That slightly greater mean scores were observed for the neutral stimulus than for the two other stimulus conditions only suggests that children obtained cues from the neutral stimuli which they did not obtain from the two other stimulus conditions. The evidence was against the hypothesis that children engaged in solution by nontransitive hypotheses in the case of the neutral stimulus; a two-by-two contingency table constructed (using conservation by transitivity) for each stimulus condition did/not contradict the hypothesis that transitivity of length relations is sufficient for conservation of length relations for any stimulus condition.

Horizontal differentials are well accepted for developmental data (Lovell, 1972, p.169). The above two studies suggest that development of transitivity of order relations, in the case of matching relations, lags behind the analogous development for length relations. This expectation is in contrast to the results reported by Sinclair (1971, p.153) that length is a later achievement than number, lagging six months to a year in development. Conservation of length lags even farther behind conservation of number--two to three years (Sinclair, 1971, p.153). Sinclair rightly considered length as a product of measurement so that no contradiction is necessarily present concerning achievement of length and matching relations and of length and number, as reported by Sinclair.

Because the samples were different in the above two studies, the observed time lag was only suggestive. Data from two different studies (Steffe and

Carey, 1972; Owens, 1972) confirmed that no such lag existed in transitivity in either an all caucasian, middle to upper class, kindergarten sample or in an essentially all black kindergarten and first grade sample. In the case of conservation of matching and length relations, conservation of length seemed to precede conservation of matching, but the trend was not strong enough to be of any consequence. One is forced to conclude, then, for first grade and kindergarten children, length relations and matching relations develop in about the same way, but in most cases, one cannot infer the presence of one from the presence of the other.

Although the data of the above four experiments did not shed light on a parsimonious model for active intellectual operations concerned with matching and length relations, disparities were observed between theoretical analysis of conservation of relations and of relational structures; cases existed where children were classified as high conservers, although no evidence was present that they could use the transitive property. There should be no question concerning my theoretical interpretation of the relation between conservation and transitivity. Piaget (1964b) has related "without the grouping there could be no conservation..." (p. 42). Smedslund (1963) has also found children who pass conservation tests and fail transitivity tests concerning length relations; so the phenomenon is not particular to our way of operationalizing the constructs. The data do raise questions concerning necessary mechanisms underlying conservation of length and matching relations and lend some credibility to problems brought out in earlier analyses of application of the grouping structure to equivalence and connected strict partial order relations.

Multiple Classification and Relations

Up to this point, the data have been status data regarding conservation of relations and transitivity of relations across relational types (equivalence and order) and relational content (matching and length) for kindergarten and first grade samples only. Three experiemnts have been done--each involving multiple classes or relations in some way. These experiments are mentioned because of certain contrasts they present in the development of multiple classifications and relations. The first experiment of the three was done by David C. Johnson (1971). He constructed 18 items, six 3 x 3 matrix items, six 2 x 2 matrix items, and six intersecting ring items. These items contained no special mathematical content. The content can be classified as perceptible in the sense of Olver and Hornsby (1966). In the matrix items, the child was instructed to select, from four possible choices, the object which would go in the one cell left empty (which was always a corner cell). The experimenter first focused the child's attention on the matrix by saying "look at all the things here. They form a pattern." The experimenter then pointed to the empty cell and said, "The thing that was supposed to be right here was left out." The experimenter then pointed to the four choices and said, "One of these things is supposed to be here. Which one is the one left out?" Three of the 3 x 3 matrix items involved multiple classification only (shape by color) and three involved a relation and classification (bigger than by shape, bigger than by color, and more than by color). The latter three items each involved a partial order relation, whereas the former three involved only equivalence relations. The six 2 x 2 matrix items were strictly analogous to the 3 x 3 matrix items. In each intersecting ring item, each of the two intersecting rings contained two objects and the child was instructed to find which of four objects belonged in the intersecting region. For example, one ring contained a red and a yellow circular region and the other a blue square region and blue triangular region. To be correct, the child had to choose a blue circular region for the intersection from the choices of a blue circular region, a red circular region, a blue square region, and a green triangular region. Notice each distractor possesses something in common with some object in one of the rings.

Johnson chose kindergarten and first grade children for his study, where the children had a chronological age (CA) either in the interval (64-76) or (77-89) months and an IQ in the intervals (80-100) or (105-125). The two variables CA and IQ were used as classification variables; 20 children were in each of the four defined categories. These children were randomly assigned to a treatment or control group where the treatment consisted of experiences in classification. The treatment lasted 17 consecutive school days for 25 minutes per school day. The treatment for the control children consisted of regular school instruction. The treatment did not involve multiple classification or relational activities per se, except for an intersection activity. Results of the ANOVA's run are summarized in Table 6. The mean scored for the TxI interaction are presented in Table 7. It was apparent that while the higher IQ children profited more from the treatment than the low IQ children, both groups profited. While such a result is educationally significant, its psychological significance is blurred by two factors. First, evidence was present that the control children regarded the overlapping region of the two rings as forming a distinct region separated from the two original rings.

Table 6

F-values for ANOVA's

Type variation	3 x 3 Matrix	2 x 2 Matrix	Intersecting Rings
Treatment (T)	8.25**	5.11*	38.84**
Age (A)	< 1	< 1	< 1
IQ (I)	2.28	4.45*	4.69*
TxA	< 1	< 1	2.07
TxI	2.40	2.64	6.65*
AxI	< 1	< 1	1.09

* (p < .05)
** (p < .01)

Table 7

Mean Scores: T X I Interaction for Intersecting Rings

T / IQ	Exp	Con
High	57	12
Low	31	11

The second concerns the type of content of the items. The mean scores for the matrix items are presented in Table 8. Because direct instruction was not given on matrix items, it was encouraging that so much apparent improvement

Table 8

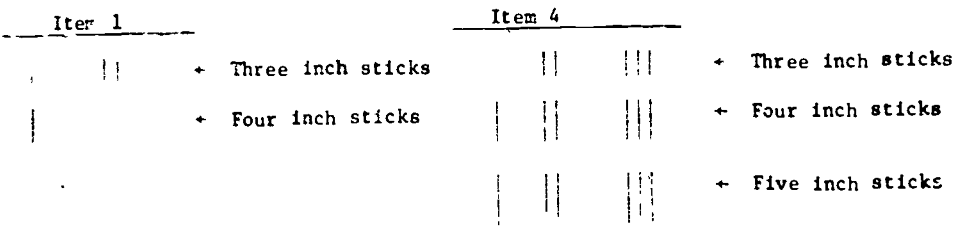
Mean Scores: Matrix Items

T / IQ	3 X 3 Matrix		2 x 2 Matrix	
	Ex	Con	Ex	Con
High	74	45	68	43
Low	52	43	42	37

could be attributed to the treatment, especially due to the vast array of pictorial data children are subjected to in mathematics instruction in the first two grades. In any event, the experiment was educationally significant for the matrix items. Because Piaget (1966) makes such a distinction between physical knowledge and logical mathematical knowledge, the apparent improvement could be questioned because the items required mainly physical knowledge for their solution. Consequently, the cognitive structure of the children may not have been altered, but rather their discriminatory powers pertaining to physical characteristics of the objects of the items may have been improved. Such a possibility was heightened by the results of a class-inclusion test also administered by D. Johnson (1971). No differences were detected for this test between the experimental group and the control group; and generally low mean scores were obtained (less than 40 percent was the greatest mean score obtained for any group)

In light of the results of the above experiment (which is only partially reported), Martin L. Johnson (1971) chose quite different content for an experiment involving matrix items. He constructed six items, which in my estimation required logical mathematical knowledge for solution to a greater extent than did D. Johnson's. The item layouts were deceptively simple. Two of the items involved the partial ordering "shorter than or just as long as" and "fewer than or just as many as," discussed earlier under the auspices of Grouping VIII. One of these items was a 2 x 2 matrix item and the other a 3 x 3 matrix item. The layouts of these items are given in Figure 3. The child was asked, of course, to complete the matrices. In the remaining four items, only single sticks were placed in each cell so no recourse to numerosity

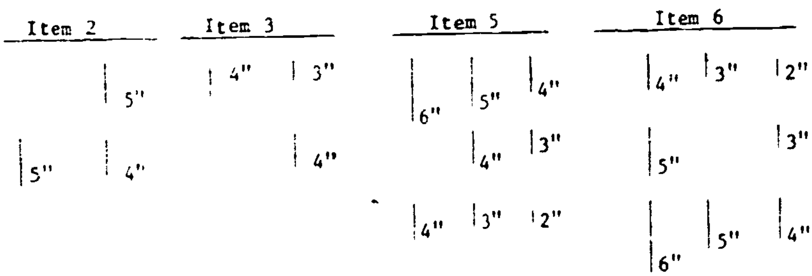
Figure 3



or relations thereof was necessary. The ordering was "shorter than or just as long as"; in these cases a connected partial ordering. The four item layouts were as depicted in Figure 4. The ordering proceeded from a corner cell with sticks on some diagonals being of the same length. The strategies used by children to complete the matrix layouts could vary.

The subjects for the study were 72 children, 24 kindergarten, 24 first graders, and 24 second graders. Twelve of each were randomly assigned to an experimental group and twelve to a control group. The children in the experimental group were given 13 instructional sessions, each about 20 minutes in

Figure 4



duration, with the following activities being covered. Experiences were provided (a) in making comparisons between objects and developing a strategy for determining the length relation that holds between any two linear objects, (b) in classifying linear objects on the basis of the equivalence relation "as long as", (c) in seriating linear objects from longest to shortest using an operational procedure consistent with Piaget's stage three seriation behavior, (d) in combining classification and seriation, and (e) in multiple seriation. The children in the control group received instruction in the context of the regular classroom. After the above experiences, the following mean scores on the six matrix items were obtained. These means are quite low and hardly exceed chance responses. The item difficulties range from .14 to .26, the most difficult being item 1 and the least difficult, item 4.

Table 9

Age	Exp	Con
5	27	20
6	15	5
7	25	20

As already indicated, a viable hypothesis for the disparity of the data on the matrix items of the above two studies is the type of knowledge required for item solution. The items were structurally close enough so that one would expect fairly consistent performance, other factors being held constant. Because children performed so poorly on the matrix items requiring logical mathematical knowledge for solution, an immediate question arose concerning children's measurement behavior involving polygonal paths, because the content of polygonal paths is so close to that of matrix items constructed by M. Johnson. Not only was there a question concerning children's measurement behavior involving polygonal paths, but also concerning whether partial orderings are viable candidates for models of genetic structures--especially for length relations and for children in the age range of 5-8 years.

It is possible for children to compare the length of two polygonal paths by knowing how many segments are in each path and the relation between any two segments, one from each path if the segments of each path are homogeneous with respect to length. Under the latter condition, four logical possibilities exist; the segments of one path are longer than the segments of the other but equal in number, the segments of both paths are equal in length but unequal in number, the segments of one path are longer than the segments of the other but fewer in number, and the segments of both paths are equal in length and number. In the first two cases, one segment is longer than the other, in the third case no comparison is possible based on the information given, and in the fourth case, both paths are of the same length. Terry Bailey (1973) administered four tasks, one of each of the above types to 40 first, 40 second, and 40 third grade children who were randomly selected from a larger population of lower middle class to middle class children in April and May of 1972. Of these 120 children, conclusive evidence was present that only four children could establish the length relation between two polygonal paths, and these four children were all third graders. Conservative criteria were employed to classify a child as being able to establish a length relation between two paths (explanations had to be given using both number and length relations). Even so, the third graders were late eight year olds or early nine year olds, so that the relational structures (genetic) so necessary for solution of the problems should have been manifested in more than 10 percent of the third graders. The data, however. is consistent with that of M. Johnson's. Bailey's and M. Johnson's data, coupled with that of Carpenter (1972) are serious enough in consequence to warrant mounting a massive set of developmental and experimental studies concerning measurement processes of elementary school children.

Final Comments

Data exist other than those discussed above. This data were collected in studies with two overriding purposes. The first was ascertainment of convergence of logical thinking and the second was ascertainment of structural aspects of logical thought under controlled experimental conditions--that is, would one observe the same structural aspects of logical thinking after intervention of planned experiences as one would observe in the absence of such planned intervention? The results of the data obtained to this point (Owens, 1972; D. Johnson, 1971; M. Johnson, 1971; Steffe and Carey, 1972; Lesh, 1971), while not without contradictions, suggest that one can observe radically different interrelationships after intervention of planned experiences than would be the case without such intervention. These data and future similar data are quite important in view of similarities hypothesized by Piaget (1966, pp 187-89) to exist between mathematical structures and genetic structures.

APPENDIX

Some Mathematical Structures

Cardinal Number

Hausdorff (1962), in his classic work Set Theory, commented that "this formal explanation says what the cardinal numbers are supposed to do, not what they are...we must leave the determination of the 'essence' of cardinal number to philosophy (pp.28-29)." Although Hausdorff's point of view is consistent with modern postulational developments in mathematics, it does not lessen the importance of his work on cardinal (and ordinal) number for research on acquisition of mathematical knowledge. For the structures which characterize the mathematical knowledge the child is asked to acquire seldom, if ever, correspond exactly in form to structural aspects of the child's natural thought. It is truely the case that Hausdorff is not concerned with the nature of cardinal (and ordinal) number and leaves the determination of their "essence" to philosophy, and ultimately to psychology as well. Not only is there a difference in the way in which the objects called cardinal and ordinal numbers are viewed in mathematical structures as discussed by Hausdorff and in genetic structures as discussed by Piaget, but there are formal differences in the structures and these differences are profound.

In the following exposition, only "naive" set theory is dealt with. In this theory, such constructions as "the set of all cardinal numbers" lead to antimonies. For a theorem is provable which leads to an unbounded sequence of cardinal numbers--which means that for any set of cardinal numbers, there is still a greater one. Consequently, "the set of all cardinal numbers" is not conceivable even though it would appear to be so. In axiomatic treatment of set theory, these obvious contradictions have been removed (Kelly, 1955, pp.250-81). As the theory does not allow for unlimited construction of sets--the object $\{x: x \text{ is a cardinal number}\}$ and $\{x: x \text{ is an ordinal number}\}$ are not sets. A distinction is made between a class and a set in that a class is undefined, whereas a set is a class which is a member of another class. That is, a class x is a set if and only if there is a class y so that x is a member of y. Using this special restriction, cardinal and ordinal numbers are defined to be sets of a special kind. Rather than follow this axiomatic treatment of the development of cardinal and ordinal number, the treatment of "naive" set theory given by Hausdorff is adhered to because of its intuitive appeal.

Ordered Systems

During subsequent discussion, occasion arises to employ general ordered systems, the basic concept of which is that of a partially ordered set. By definition, a relation $<$ defined in a given set P partially orders P if it is both transitive and antisymmetric. The relation $<$ is (a) transitive in P if whenever x, y, z are in P and $x < y$ and $y < z$, then $x < z$; and (b) antisymmetric in P if whenever x and y are in P and $x < y$ and $y < x$, then $x = y$. In the latter definition, equality is taken in the sense of logical identity.

A ready example of a partially ordered set is the set of subsets P(X) of a given set X ordered by the set inclusion relation "$\subset$." The set inclusion relation has the additional property of reflexivity (for each set A, $A \subset A$) but does not have the property of being connected in P(X) (i.e., for any two sets A and B of P(X), it is not necessarily true that $A \subset B$ or $B \subset A$).

If P is a partially ordered set and E a subset of P, then an element x of P is called an upper bound for E if for every $e \in E$, $e < x$. An element x_* is the least upper bound for E if for any other upper bound $y \in P$, $x_* < y$. Analogous definitions can be given for lower bounds and the greatest lower bound of E. A lattice is a partially ordered set L for which every two element subset {x,y} of L has a least upper bound and a greatest lower bound. Examples of lattices are P(X) ordered by set inclusion and the positive integers ordered by "a divides b." The least upper bound of any two sets A and B of P(X) is $A \cup B$ and the greatest lower bound is $A \cap B$; and the least upper bound of any two positive integers is their greatest common divisor and the greatest lower bound is their least common multiple.

A chain in a partially ordered set P is a subset C of P in which < is connected (that is, a subset C where if x, $y \in C$, $x < y$ or $y < x$). Any such subset C of P is partially ordered by < and is a lattice as well as a chain. The set of natural numbers ordered by < is an example of a chain. It is important to note that < is transitive and asymmetric (if $x < y$, then $y \nless x$). Yet, it is a partial ordering because the antisymmetric property is satisfied vacuously.

Relations of equivalence also exist as well as relations of order. By definition, a relation R defined in a set X is an equivalence relation if R is reflexive, symmetric, and transitive. Set equivalence is a ready example of an equivalence relation as are the congruence relation for point sets and set similarity.

Cardinal Number

Hausdorff (1962) assigns objects, called cardinal numbers, to sets in such a way that if object a corresponds to set A and object b corresponds to set B, a = b if and only if A is equivalent to B. It is important to note that the set A to which the cardinal number a is assigned may or may not be an ordered set. Two cardinal numbers may be compared by comparing the sets to which they are assigned. $a \leq b$ means that $A \sim B_1$ where $B_1 \subset B$. It may be that $A = B_1$ in which case $A \subset B$. Subleties exist concerning comparison of any two cardinal numbers in that it is, in fact, true that the comparability of any two cardinal numbers relies on Zermelo's well-ordering theorem, which states that any set can be well-ordered. This theorem is necessary (in Hausdorff's development) to show that there do not exist two incomparable sets, i.e., that it is never the case that there exist no A_1 and no B_1 so that $A_1 \sim B$ and $B_1 \sim A$.

The sum and product of cardinal numbers determine their arithmetic. "The sum a + b of two cardinal numbers is the cardinality of the set theoretic sum $A \cup B$*, where A and B are any two disjoint sets having the cardinalities a and b respectively (Hausdorff p.33)." This definition is justified because

* "$\cup$" has been substituted for "+".

if $A \sim C$ and $B \sim D$ where D and C are disjoint, then $C \cup D \sim A \cup B$, so that the cardinality of $C \cup D$ is equal to that of $A \cup B$.

The product of two cardinal numbers a and b is defined as follows. "The product ab of two cardinal numbers is the cardinality of the set theoretic product A x B, where A and B are any two sets with cardinalities a and b respectively (Hausdorff, p.35)." The product of a and b is invariant of the particular choice of the sets A and B just as was the sum except that in the sum, A and B had to be disjoint. That is, if $A \sim C$ and $B \sim D$, then $A \times B \sim C \times D$, so that the cardinality of C x D is equal to that of A x B. The commutative, associative, and distributive laws hold for the processes just defined, and depend directly on the commutative, associative, and distributive laws for set operations.

Ordinal Number

Just as set equivalence is a basic notion for cardinal number, set similarity is a basic concept for ordinal numbers. For clarity, the order relations discussed below are asymmetric and transitive (strict partial orderings) as well as being connected, which means that any two elements are related. Two ordered sets are called similar if there exists a one-to-one correspondence between their elements that preserves order. That is, if $a,b \in A$ and $c,d \in B$ where $a \leftrightarrow c$ and $b \leftrightarrow d$, and if $a < b$, then $c <' d$, where $<$ and $<'$ are the orderings in A and B, respectively. In symbols, "A is similar to B" is denoted by "$A \simeq B$." Set similarity is an equivalence r ation just as is set equivalence.

As mentioned earlier, Zermelo's well-ordering theorem states that any set may be well-ordered. A set A being well-ordered by a relation $<$ means that any subset A_1 of A has a first element (an element a_0 such that $a_0 < x$ for any x in A_1). Hausdorff (1962, p.51) assigns order types to ordered sets in such a way that similar sets, and only similar sets, have the same order type assigned. In symbols, $r = s$ means $R \simeq S$. If a set is well-ordered, then its order-type is called an ordinal number.

In general, the arithmetic of order types is not isomorphic to the arithmetic of cardinal numbers. For if A and B are disjoint ordered sets, then the set theoretic sum of A and B (A + B) is a new ordered set such that the order of the elements of A is retained, the order of the elements of B is retained, and every $a \in A$ precedes every $b \in B$. If a is the order type of A, b the order type of B, then a + b is the order type of A + B. That $a + b \neq b + a$ in general can be seen by the following example. Let $A = \{1,2,3,\ldots,n\}$ and $B = \{n + 1, n + 2,\ldots\}$. The order type of A is n, the order type of B is ω, and of A + B is $n + \omega$ or ω(ω is the order type of the natural numbers). But the order type of B + A is $\omega + n$ which is not ω because $B + A = \{n + 1, n + 2,\ldots,1,2,\ldots,n\}$ contains a last element (A + B does not). So $\omega + n \neq n + \omega$. Because n and ω are ordinal numbers and, in general, the sum of two ordinal numbers is not commutative, the arithmetic of ordinal numbers is not isomorphic to the arithmetic of cardinal numbers. Nevertheless, two sets with the same ordinal number necessarily possess the same cardinal number.

As pointed out earlier, a set A which is ordered by an order relation which is connected, asymmetric, and transitive satisfies the conditions for a chain. In particular, if A is well-ordered by <, A is a chain. An intuitive example of a chain important to subsequent discussion is as follows: Let A be a well-ordered set. Then A has a first element, say a_0; $A - \{a_0\}$ has a first element, say a_1; $A - \{a_0,a_1\}$ has a first element, say a_2; etc., so that $A = \{a_0,a_1,a_2,a_3,...\}$. The notation used here is that the index of every element is the ordinal number of the set of elements preceding it. For a_3, "3" is the ordinal number of $\{a_0,a_1,a_2\}$, which is called a segment of A determined by "a." In more general terms, each element a of A determines some segment P where $P = \{x \in A: x < a\}$. If $Q = \{x \in A: x \notin P\}$, then $A = P + Q$. Note that $a \notin P$ because < is irreflexive, so a is the first element of Q. A result of this definition is that a well-ordered set is never similar to one of its segments, which leads to the fact that for any two ordinal numbers a and b, either $a < b$, $b < a$, or else $a = b$. In particular, $a < b$ means that A is similar to a segment of B. Of course, if it were possible for B to be similar to one of its segments, then it would be true that $a = b$ as well as $a < b$.

As indicated above, the elements of a set A which is well ordered can be indexed by successive ordinal numbers. This assertion can be shown more definitely without difficulty. Just let $O(\alpha) = \{$ordinal numbers β such that $\beta < \alpha\}$. $O(\alpha)$ can be represented as $\{0,1,2,3,...,\sigma,...\}$ where $\sigma < \alpha$ (Hausdorff, 1962, p.70). Moreover, if A is a well-ordered set of type α, then it is possible to represent A as $\{a_0,a_1,a_2,...,a_\sigma...\}$ where $\sigma < \alpha$ and α is the ordinality of A and the index of each element of A is just the ordinal number of the segment belonging to it. If A is a finite set, then $A = \{a_0,a_1,a_2,...,a_{n-1}\}$ and n is the ordinality of A where 0 is the ordinality of the empty set. Because any ordering of a finite set is a well-ordering, it is impossible to distinguish the orderings with reference to the ordinal number of the set; i.e., all orderings give the same ordinal number. Thereby, the ordinal and cardinal numbers of finite sets correspond, and it is possible to find the cardinal number of a set by a process of counting, that is, by indexing the elements of the set A by the ordinal numbers $\{0,1,2,...,n-1\}$ by virtue of successive selection of single elements. (Select some a_0, then some a_1, etc., until the last one a_{n-1} is selected.) Then n is called the cardinal number of the set. This process is often referred to as rational counting.

The notion of equivalence classes of finite sets is implicit in the above discussion because $\approx$ is an equivalence relation. This observation has led to the definition of an ordinal number as an equivalence class of well-ordered sets and a cardinal number as an equivalence class of sets without regard to order (Barnes, 1963, p.194). The set $\{0,1,2,...,n-1\}$ of cardinality (and ordinality) n can be considered as the standard set of an equivalence class of sets each of cardinality n. It must be explicitly pointed out that the arithmetics of cardinal numbers and ordinal numbers of finite sets are, in fact, isomorphic.

To view a cardinal number as a class of sets should be no more foreign to mathematics educators than to view the objects of a finite field formed by the integers modulo a prime as classes of sets. Of course, to tell a five year old child that a number is an equivalence class of sets is absurd. The identification of a number as a set of objects, however, is a natural way to think about cardinal and ordinal number. In the well-known "empty hat" (Van Engen, 1970, pp.38-39) approach to cardinal number, "0" is defined to be the empty set, "1" is defined to be the set containing 0 as an element, etc. More formally, $0 = \emptyset$; $1 = \{0\}$; $2 = \{0,1\}$; $3 = \{0,1,2\}$; $4 = \{0,1,2,3\}$; ...; $n = \{0,1,2,\ldots,n-1\}$. This approach relies on the representation of $0(\alpha)$ as $\{0,1,2,3,\ldots\ ,\ldots\}$, such that $\sigma < \alpha$, already discussed and made possible through the well-ordering theor of Zermelo. Thus, "4" is the ordinal number of the segment $\{0,1,2,3\}$ and is identified with the segment itself. Because cardinal and ordinal numbers are indistinguishable, it is also the cardinal number of the set.

Concretely, if A is a finite set to be counted, then by successive selection of elements, successive segments of set A are determined and a chain of ordered sets is formed. "One," in the selection of the first element has both cardinal and ordinal characteristics in that "one" tells how many elements have been selected and also that the first one has been selected. A subset of the collection A of one element has also been determined. "Two" in the selection of the next element also has both cardinal and ordinal characteristics in that "two" tells how many elements have been selected and also that the second one has been selected. The segment corresponding to "two" is an ordered set, is a subset of the collection A, and contains the set consisting of the first element. It is ordered by the relation "precedes," which is transitive and asymmetrical (and is thereby a strict partial ordering). If this counting process is continued until A is exhausted, then $A = \{a_1,a_2,\ldots,a_n\}$ has been well-ordered by the relation "precedes." A chain of sets has been established in that if $A_1 = \{a_1\}$, $A_2 = \{a_1,a_2\}$, etc., then $A_1 \subset A_2 \subset \ldots \subset A_n$. In this sense, one can say that one is included in two, two is included in three, etc. If A is counted in a different way, $A = \{a_1*,a_2*,a_3*,\ldots,a_n*\}$. It must be noted that while a_i* may not be the same element as a_i, nevertheless a_i* is the ith element and also the cardinal number of $A_i* = \{a_1*,a_2*,\ldots,a_i*\}$ where $i \leq n$. While A_i and A_i* are similar (and therefore equivalent), they are not necessarily equal sets.

References

Bailey, T. On the measurement of polygonal paths by young children. Unpublished manuscript.

Barnes, W.F. Introduction to abstract algebra. Boston: D.C. Heath, 1963.

Beilin, H. The training and acquisition of logical operations. In M.F. Rosskopf, L.P. Steffe, S. Taback (Eds.) Piagetian cognitive-development research and mathematical education. Washington, D.C.: NCTM, 1971, pp. 81-124.

Beth, E.W. & Piaget, J. Mathematical epistemology and psychology. (Translated from the French by W. Mays.) Dordrecht, Holland: D. Reidel, 1966.

Carpenter, T.P. The role of equivalence and order relations in the development and coordination of the concepts of unit size and number of units in selected conservation type measurement problems. Technical report no. 178, Wisconsin research and development center for cognitive learning. Madison: University of Wisconsin, 1971.

Courant, R. & Robins, H. What is mathematics? New York: Oxford University Press, 1941.

Divers, B. The ability of kindergarten and first grade children to use the transitivity property of three length relations. Unpublished doctoral dissertation, University of Georgia, 1970.

Flavell, J. The developmental psychology of Jean Piaget. New York: D. Van Nostrand, 1963.

Gans, D. Transformations and geometries. New York: Appleton-Century Crofts, 1969.

Hausdorff, F. Set theory. (2nd English ed.) New York: Chelsea Publishing Company, 1962.

Inhelder, B & Piaget, J. The early growth of logic in the child. New York: The Norton Library, 1969.

Johnson, D. An investigation in the learning of selected parts of a boolean algebra by five and six year old children. Unpublished doctoral dissertation, University of Georgia, 1971.

Johnson, M. Effects of selected experiences on the classification and seriation abilities of young children. Unpublished doctoral dissertation, University of Georgia, 1971.

Kelly, J.L. General topology. New York: D. Van Nostrand, 1955.

Lesh, R. The generalization of Piagetian operations as it relates to the hypothesized functional interdependence between class, series, and number concepts. Unpublished doctoral dissertation, Indiana University, 1971.

Lovell, K. Intellectual growth and understanding mathematics. Journal for Research in Mathematics Education, 1972, 3, 164-182.

Owens, D.T. The effects of selected experiences on the ability of disadvantaged kindergarten and first grade children to use properties of equivalence and order relations. Unpublished doctoral dissertation, University of Georgia, 1972.

Owens, D.T. & Steffe, L.P. Performance of kindergarten children on transitivity of three matching relations. Journal for Research in Mathematics Education, 1972, 3, 141-154.

Phillips, h.B. Vector analysis. New York: John Wiley & Sons, Inc., 1933.

Piaget, J. The child's conception of number. (3rd impression) London: Routledge and Kegan Paul, 1964. (a)

Piaget, J. The psychology of intelligence. (4th impression) London: Routledge and Kegan Paul, 1964. (b)

Piaget, J. On the development of memory and identity. Barre, Massachusetts: Clark University Press with Barre Publishers, 1969.

Piaget, J. Genetic epistemology. New York: Columbia University Press, 1970.

Sinclair, H. Number and measurement. In M.F. Rosskopf, L.P. Steffe, S. Taback (Eds.) Piagetian cognitive development research and mathematical education. Washington, D.C.: NCTM, 1971, pp. 149-159.

Smedslund, J. Concrete reasoning: a study of intellectual development. Monographs of the Society for Research in Child Development, 1963, 24, (2, Whole No. 93).

Steffe, L.P. Carey, R.L. An investigation in the learning of relational properties by kindergarten children. A paper presented at the Annual 1972 AERA meeting, Chicago, Illinois.

Tarski, A. Introduction to logic. New York: Oxford University Press, 1954.

Van Engen, H. Epistemology, research, and instruction. In M.F. Rosskopf, L.P. Steffe, S. Taback (Eds.) Piagetian cognitive-development research and mathematical education. Washington, D.C.: NCTM, 1971, pp.34-52.

Future Research in Mathematics Education: The View from Developmental Psychology[1]

Harry Beilin
City University of New York/Graduate School

Predictions of the future that go beyond extrapolations of the present are characteristically products of fantasy and imagination. I will confine my prognostications to delineating features of contemporary research in mathematics education, indicating how some current research does not deal adequately with the problems being addressed and will suggest alternatives that may be the basis for future research.

In assessing the present and future of mathematics education I would hold that any educational program that ignores available knowledge of the child's intellectual development is likely to be only partially successful. This assertion rests on the assumption that mathematical learning as a type of cognitive learning is under the control of the child's developing cognitive capacities. The source of this thesis is principally, although not exclusively, the work of Jean Piaget.

Piaget and Mathematical Education

Piaget's theory is having considerable impact upon research in mathematics education and will probably continue to, although other psychological developments will become increasingly important even in the near future. For the present, Piagetian research has been far from fully explored and the application of the theory to educational practice is still in its infancy.

Piaget's views of education, and mathematical education in oarticular (Piaget, 1972), are based upon the following assumptions:

1. Learning is under the control of the child's development and not the reverse. That is, experience alone is not sufficient for learning; it requires an organism whose cognitive structures are of a level of development that will enable the products of experience to be integrated with them.

2. Logico-mathematical structures are spontaneously and gradually constructed as the child develops. These structures are considered by Piaget to have a natural relation to those of modern mathematics and knowledge of these relationships is felt to be a necessary condition for the teacher to foster creative learning.

3. The origin of mathematical thought is in the actions of the child and not in his language.

Each of these assumptions will be considered in light of the research problems they raise as well as other developments in psychology that have a bearing upon the same issues.

[1]Presented at the "Cognitive Psychology and the Mathematics Laboratory" Symposium at The School of Education, Northwestern University, February, 1973.

1. Mathematical learning and development.

Until quite recently, the acquisition and development of mathematical skills and concepts was considered to be a problem for the psychology of learning and problem solving. In the view of empiricist psychologists these concepts were acquired through the manipulation of external sources of stimulation and reward. The products of such external control combine internally in a simple or complex association under the impetus of some form of motivation. Gestalt psychologists have traditionally considered the same processes to be fundamental forms of reasoning and thought, acquired according to the laws that organize the processes of perception.

Both empiricist and gestalt psychologists have shared the belief that the origin of knowledge is in perceptual and sensory processes. Experience, for the empiricist, is the critical element in learning. while for the gestaltist, organization derived from the biological properties of the organism accounts for change resulting from experience.

With Piaget, the elements of both empiricist and gestalt views integrate in a theory based upon the autoregulation of development, by virtue of which active experience and internal organization provide the materials out of which intellectual structures are constructed. The acquisition of knowledge becomes possible only as the developing intellectual system enables the active experience of the child to become assimilated to it. Experience itself does not ensure learning. The theory holds that the defined course of development controls learning, that is, learning does not occur except as a function of the state of the organism. The state of the organism, in turn, is established by a relatively fixed sequence of stages through which the child progresses as he moves toward full intellectual competence.

Piaget and his colleagues (Inhelder and Sinclair, 1969) provide evidence for the claim that the state of the child's development affects what he is capable of acquiring through experience, in a number of experiemnts in which children at different cognitive levels were trained with the same tasks. Great difficulty in learning is reported for children who had not reached a defined cognitive level. While studies designed to train children at different ages with the same tasks have been limited, there have been a number of attempts with both Piagetian tasks (Beilin and Franklin, 1962), and non-Piagetian learning tasks (Gollin, 1965). The results of these studies support a developmental conception of learning.

A large number of other studies that set out to establish that logical and mathematical concepts are trainable have not closed the question as to whether learning can occur at ages prior to the development of particular facilitating developmental stages (Beilin, 1971). An answer to the question requires an experimental procedure that unequivocally tests a particular reasoning process at an age when one could assume that some elements of that process had not been acquired spontaneously or naturally. A few experimental studies have been conducted with very young children (about 4 years of age) that critically embarrass Piagetian theory on this issue. The Bryant and Trabasso (1972) study on early transitive inferences is one such instance, but the question appears to be far from settled.

a. Stages in development

Claims for the developmental control of learning have been questioned in various ways. One persistent source of difficulty is the theory of stages, which has been under attack for at least four decades. Suppes (1972) has added his voice to the chorus, reiterating the view that the idea of stages is too uncritically accepted, that the concept at least as proposed by Piaget is too imprecise, and that only detailed experimentation and quantitative test will determine whether the existence of stages should be accepted as fact.

Suppes defines the central issue as that of differentiating stages from continuous development. Piaget does not deny that development is continuous. He points out, in fact, that whether development is continuous or discontinuous depends upon the scale that one uses to measure behavior. Suppes seemingly rejects Piaget's argument, but then uses it himself. He says, "The problem is to find out for the given scale at which experimentation is conducted whether the process is all-or-none or incremental, and whether there are microscales, for example, at which the process is continuous even if the data indicate all-or none learning at the ordinary scale of experimentation." Piaget couldn't agree more with that statement since it presents no problem for stage theory, in fact, it supports the view that there may be contrasting data concerning continuity depending upon the scale.

Suppes says further, "There is also no reason to think that when concept formation and mastery of novel concepts are evident that learning is necessarily to be characterized in terms of stages than incrementally." Piaget would probably agree with this statement too since he differentiates between different types of concept construction, some that may be acquired incrementally and others on an all-or-none basis depending on the process of formation that is involved.

The claim that "precise" and detailed experimentation might lead to a test of the existence of stages is itself imprecise, and hopefully Suppes has a definition of precision that would not exclude most experimentation in the social sciences, including psychology. Precise data as such are not in themselves sufficient since scientific data rarely stand uninterpreted. The same data, in fact, are often integrated into theories that are incompatible. Even with precise data it is unlikely that clear-cut support for either continuity or discontinuity theories will be readily forthcoming, since confirmation of such general theories often depends upon many experiments to provide sufficient data to elucidate the ramifications of a theory and to enable differential confirmation among competing theories.

As Suppes suggests, the theory of stages requires evidence of generalization across common sets of concepts; he holds, however, that there is "little" evidence on this point. It is difficult to tell what is meant by "little" in this context, but the issue of the so-called horizontal décalage, or generalization across concept domains has been extensively discussed in the Piagetian literature, and a number of studies by non-Genevans are addressed to this issue. As the data show, there is both generalization and variability; sometimes variability within a stage takes a consistent form, sometimes not. The issue is a difficult one for stage theory and investigations sympathetic and unsympathetic to stage theory will be dealing with it for a long time. There may not be enough data to settle this issue, but the problem has been far from ignored.

In total, Suppes' claim that stage theory has been uncritically accepted on the basis of little evidence is itself based on limited evidence. The issues are not new and have received both theoretical and empirical attention. Whether the data justify far-reaching policy decisions in education is another issue, but then again the claims for alternative views, including Suppes', are equally open to question.

b. Cognitive structures and strategies

The stage question does not exhaust the issues concerning learning and development. Until recently, Piaget has dealt very little with the nature of learning. His conception of learning appears to be identified with the behavioristic conceptions of Hull-Spence, Skinner, and Pavlov. Although undoubtedly aware of recent research and theory in cognitive learning, he uses classical behaviorism as a backdrop against which to contrast his equilibration theory of development. He characterizes behavioristic conceptions as based primarily upon the response to external stimulation as the causal determinant of learning and development. His own conception, on the other hand, is based upon the autoregulation of internal and external behavioral input. Modern learning theory is not as simplistic as Piaget makes it out to be. Most contemporary conceptions of learning include some conception of internal, mediating, or symbolic cognitive processes. Learning research in turn has shifted considerably in its orientation and focus. study of discrimination learning, for example, so long considered by iorists as the cornerstone of behavioral processes, now includes analysis linguistic and attentional mediation, and some neobehaviorists define these mediational phenomena as symbolic. In addition, problem solving strategies that used to be the exclusive concern of cognitive theorists are now identifiable in discrimination learning (Levine, 1966; Gholson et al., 1972). Parallels between the problem solving strategies in discrimination learning and those in Piagetian tasks are also being studied (Gholson et al., in press). In a sense, the Piagetian attitude that learning is only an aspect of the processes of development, and the traditional alternative that development is under the control of learning are moving rapidly to an integration. The differences among a number of contemporary neobehavioristic and cognitive theories are often difficult to distinguish.

In Piagetian research, emphasis until recently was on understanding the development of cognitive structure. A shift is taking place to discover the ways in which structures function. The influence of both structural and functionalist views on research in mathematical education is not new; it was evident, for example, in the work of Dienes and Jeeves a decade ago (Dienes, 1963, Dienes and Jeeves, 1965). While the study of structure has affected research in mathematics education more than the work on strategies, the immediate future may see a reversal in emphasis.

Research on cognitive structures and strategies[1] is particularly significant in mathematics education for two reasons: First, in mathematical reasoning and problem solving, children at different levels of development acquire different ways of dealing with the same type of problem. Secondly, at any stage of development quite different strategies may be utilized by different children for the same problem, and different strategies may be utilized by the same child for different problems. While problem solving strategies appear to be significant to learning and development, it is surprising that so little is known of their nature and how they are acquired.

One feature of strategies receives considerable attention in mathematics education research, yet little attention among cognitive and developmental psychologists. This concerns the use of algorithms in problem solving and thinking. Algorithms, in essence, are a special type of strategy used in reasoning. Their utility in mathematics is obvious, but their function in thought was not as obvious until the development of computer models for simulating intelligence.

The instructional value of algorithmic methods is much discussed in mathematics education. Their efficacy is usually contrasted with that of "true concept learning," and their use is often a matter of issue in educational policy. The difficulty over the use of algorithmic methods stems in part from the lack of differentiation between <u>conceptual algorithms</u> and <u>instructional algorithms</u>. Instructional algorithms are devices, usually symbolic, that provide standardized ways of approaching the analysis or solution of problems and are essentially pedagogical instruments. The most important question about them is whether they work as well or better than other approaches within a defined set of instructional objectives. While it may be more advantageous or desirable to create or re-create novel personal solutions to problems from an understanding of fundamental principles, it may be more facilitating and economical, at least in some contexts, to have ready-made solution strategies.

Although practical considerations are important in considering the value of algorithms, even more important is the need to determine what is essential for thought and problem solving to occur. If thinking occurs naturally with the use of conceptual algorithms, that is, standardized routines or subroutines employed within a reasoning process, then they cannot be abolished by educational edict.

Algorithms, thus, are not simply arbitrary devices for solving school problems but enter into the very nature of the processes by which cognition

---

[1] Many kinds of strategies have been identified. In general, a strategy is a consistent approach taken by a subject in solving a problem. These strategies can be defined by a rule that is independent of the content of the problem being solved, such as an alternating strategy, in which a child alternates from left to right and then repeats the sequence in a two choice discrimination problem. A cognitive structure, on the other hand, is defined by the regularities in the behavior of the subject that suggest a rule or logic that is intimately tied to the content of a problem or a class of problems.

develops. They may serve as instructional devices as well, but developments in computer simulation of thinking show that algorithms serve a much more serious and necessary function in reasoning and learning. Rather than banish these structures in the sometime romanticized search for creative thought, more adequate knowledge of the role of these structures in thinking is required. The task for mathematics education is to develop instructional algorithms whose structure and content will articulate most adequately with the structure and nature of conceptual algorithms.

c. <u>Information processing approaches to mathematical educational research</u>

Research in learning and thinking is being reinterpreted by a number of contemporary cognitive psychologists in terms of the ways in which information is processed. The original interest in information processing came from recognizing that the human learner, viewed as part of a communication system, has limited capacity to store, process and transmit information. Much has recently been learned of the way information is coded so that it is understood, acted upon, and learned effectively. More is now known too of the components of information storage and processing. Most of this research has been addressed to the nature of memory, cognition, language, and perception, and at that to fairly limited aspects of these. Recent efforts have also been directed to constructing information processing models of cognitive development and mathematical reasoning. This research is concerned with the nature of mathematical reasoning and problem solving interpreted in terms of the organization of information and the systems required to process such information. Because of the nature of the information to be processed mathematical reasoning lends itself most favoredly to an information processing type of analysis.

The relatively rapid application of cognitive psychology and cognitive development theory suggests the transition into a new period of development for research in mathematics education.

The recent revolution in mathematics and mathematics education can be characterized by three periods. First, came the striking and significant changes in the conception of modern mathematics. These changes in the theory of mathematics led to great pressure for concomitant change in mathematics education. The social, political and economic climate of the so-called sputnik era provided the occasion for rapid changes in mathematics curricula that brought them into greater accord with the newer approach to mathematics. It was accompanied by changes in the technology of mathematics instruction, principally at the pre-elementary and elementary levels. The result was the widespread introduction of instructional aids (such as the Cuisenaire rods, Dienes blocks, Montessori materials, etc.) designed to foster the comprehension and learning of fundamental concepts in mathematics. We now appear to be moving into a third era, characterized by the psychologizing of mathematics education, based upon the notion that curriculum organization should be mapped onto the psychological processes of the developing child.[1] The

---

1 We have really only spiralled around. Parallel developments occurred in the 1920s and 1930s when it was realized that psychology had something to offer to education.

Piagetian influence in this recent movement has been the most evident, but it is clear that in the future it will only be one force among many, as other developments in psychological theory and technology rapidly infuse the thinking of mathematics educators.

2. The logical structure of mathematics and the logical structure of thought

The logicist interpretation of mathematics, by which mathematics is reduced to a system of logic (Bennaceraf & Putnam, 1964), is after all only one possible approach to mathematical theory. It appears, however, to be the one to which Piaget relates his own psychological logicism. He argues that contemporary mathematics with its emphasis upon logical structure, principally as characterized by the Bourbaki group, shows a "natural" relation to the logical structures of intelligence. Three types of logical structure in intelligence are said to relate to Bourbaki "mother structures." First, are the algebraic structures, the system of logical classes. Second, are the order structures that characterize the system of (ordering) relations, and finally, there are the topological structures based upon ideas of continuity, neighborhood and separation. These three "elementary" structures later combine and form the logical groups (e.g., the "four-group") that enter into propositional logic and the combinatorial system.

The formal similarity of the logical system of mathematics to the logical nature of thought suggests to Piaget that teachers should be critically aware of the nature of child and adolescent thought development, even though Piaget does not specifically suggest that the course of mathematics instruction should parallel the course of logical thought development. Piaget, perhaps wisely, offers no program as to how mathematics curricula should be developed. He does, however, offer some specific suggestions (Piaget, 1972) concerning the focus of such curricula:

1. The Piagetian developmental scheme proposes that the structures of intelligence arise out of the actions of the child. Two kinds of knowledge are distinguished: physical knowledge and logico-mathematical knowledge. "Physical knowledge," derives from the physical experience of acting upon objects (e.g., comparing weights, densities, etc.) in order to discover the properties of objects themselves. "Logico-mathematical knowledge" derives from a type of experience that garners its information, not from the physical properties of objects, but from the coordination of actions on objects. The source of logico-mathematical knowledge is a particular aspect of action, but nevertheless from action carried out by the child himself. When information is given the child by others, the child in effect has to reinvent or reconstruct that knowledge for himself in order to achieve true understanding.

This leads Piaget to suggest that the teacher should always be aware that the child and adolescent is far more capable of indicating his understanding through "doing," and "in action" than he is in any other form of expressing himself, including verbally. The expression of thought occurs in action long before the child is consciously aware of his thought and long before he can represent it linguistically. With sensitivity on the part of the teacher, the child may be made more aware of his understanding through discussion in what would be equivalent to a Socratic dialogue. Through the organization of work with partners his own age and with a sophisticated older child as leader, the child could also be led to appropriate verbalization and "awareness."

2. For Piaget a new logical structure comes into being in at least two ways. First, the generalized forms of action upon objects are assimilated to existing cognitive schemes (generalized structures) to form new structures. Secondly, existing schemes are integrated and composed into new structures in response to problem solving needs. Some form of internal change has to take place in both instances for real understanding to occur. The mathematics teacher, however, provides the child with ready-made logico-mathematical structures and expects the child to understand them. The child's ability to repeat a notion and even to apply it in limited ways often gives the impression of knowledge. True understanding of teacher-given knowledge requires reconstruction of the idea by the child. The test of understanding is the spontaneous application of the idea by active generalization. To ensure understanding the teacher has to go beyond his "lessons" and "organize situations that give rise to curiosity and solution seeking (Piaget, 1972)." Difficulties in understanding, says Piaget, should not lead to "feedback" procedures in which a solution or correction is given directly. Instead the teacher should utilize an active method in which counter-suggestion leads to new exploration so that the child is able to correct himself. This involves the application of Piaget's so-called "clinical method" to pedagogical practice. Unfortunately, it is not the kind of application that is easily made by a teacher who is charged with the instruction of a large group of pupils.

3. The early logical thought of the child arises out of a great deal of active experimentation with objects. Simple and complex coordinations between perceptions and actions are made during the very first year of life, prior to the symbolic representation of such coordinations. In a real sense, formalization through symbolic representation follows action. This general model applies to all periods of development, and Piaget notes, unhappily, that mathematics teachers are tempted all-too-often to reverse the procedure and prematurely provide formalization prior to active experimentation. Representations or models should correspond to the natural logic of the child's level of thought, with systemization and formalization to follow the kind of knowledge that "intuitively" comes through action. Piaget argues that mathematicians should not eschew such "intuitive" knowledge, since mathematical intuition is essentially operational (i.e., logical), and the nature of operative thought is to dissociate form from content. There is no harm, in fact it may be necessary, to at first encounter experience in which both form and content are intuitively grasped, before they are formally separated.

On an a priori basis alone it would seem that Piagetian theory would relate well to early (preschool and elementary) mathematics. It is not self-evident, however, as to how mathematics curricula should be organized or how mathematics should be taught, since curriculum organization derives both from the inherent logic of the subject matter and the intellectual competencies of the child. The relation of mathematics to other subjects, the experience of the child, and the cultural context in which teaching occurs are also relevant.

One particular difficulty is that the logical organization of a subject may not parallel psychological development. Piaget implies that for mathematics there is such congruence, as in the logical relations among topological, projective and euclidean geometries; but there is relatively little evidence, except for certain areas of mathematics, to confirm this. Investigations of school curricula in present use show critical disparities between curriculum sequences and empirically determined sequences of intellectual development (see L. Beilin, 1973, for one such example). These studies indicate that the particular curriculum sequences studied cannot be justified either in terms of their logical organization or their accord with psychological development. Some mathematics curricula in present use, particularly those more recently developed, seem to have a better logic for their organization, but as yet very little evidence exists of their relation to the cognitive development of the child.

Piaget's suggestions for activity-based mathematical learning may take a child so far into mathematics, although how far is not clear. Those areas of mathematics (geometry, arithmetic, etc.) in which learning may be accompanied by the active manipulation of objects is not specified by Piaget although he implies that even advanced mathematical understanding may be fostered by such manipulation. Dienes' demonstrations in this Symposium suggest that there may be many applications even for advanced forms of mathematical reasoning. While Piaget proposes that formalization should proceed in its own time it would appear that considerable experimentation will be needed to establish such timetables.

Educational psychologists and others are eager to know how teachers can be aided to acquire understanding of the intuitive relation between mathematics and intellectual development well enough to carry on an instructional dialogue with children. Even more, educators are concerned with how to organize learning so that the teacher can accomplish this with large numbers of children. Present knowledge is inadequate to provide the answers.

3. Mathematics as a language

Piaget is at great pains to declare that language is not the critical source of thought and knowledge. Language functions instead to represent and communicate thought. This is particularly relevant for mathematics since it is commonly held that mathematics is a language, or has properties in common with natural language. In Piaget's theory, logical and mathematical structures are said to define the nature of the thought process; language, on the other hand, is a socially-created conventional system for symbolically representing the products of thought. As a consequence, Piaget considers that educators, particularly mathematics educators, have placed much too much emphasis upon the linguistic aspect of education out of proportion to its role in mathematical thinking and learning. For mathematics educators the issue is important since so much of mathematics instruction is symbolically formalized, although much effort has gone into making early mathematics instruction less so. Again, the focal question concerns how knowledge is acquired and how thinking occurs. If Piaget is correct that the development of thought, particularly in its early manifestations, is achieved through activity, then an educational policy may be required that places greater emphasis upon activity and less upon linguistic forms of instruction. While Piaget recognizes that progressive linguistic formalization and model building is necessary, he emphasizes again the need to ensure that understanding accompanies linguistic formalization.

The relation between language and thought is not as clear-cut as Piaget proposes. The by now well-known work of the generative-transformational linguists (Chomsky and others) has shown that at an early age (2 to 4 years) the child acquires a relatively small set of linguistic rules from which he can create a vast verbal output. Considerable controversy exists as to the components of this linguistic rule system and how it functions, but in spite of a number of differences there is consensus as to the great power of that generative system. Nevertheless, in spite of the achievements of contemporary linguistics, little is known of the nature of logic and mathematics as special or formal languages. In addition, few investigators of language acquisition have been concerned with whether mathematics and logic are acquired in the same manner that natural language is acquired, or whether, as Piaget implies, they are not acquired as languages at all but as systems of thought. The distinction may not be important if language acquisition is under the control of developing cognitive structures. If, on the other hand, language development is autonomous and has an internal logic that differs from the structures of thought, then rather different practical and theoretical consequences ensue. The issue, at the moment, has relatively little available research data to decide it. What evidence there is, even from the Genevan group, indicates that natural language acquisition cannot be accounted for solely by available knowledge of cognitive development (Sinclair, 1971). Even less is known of the acquisition of mathematical knowledge and its representation in formal languages.

A number of attempts have been made recently to analyze mathematics in linguistic terms although not necessarily as a generative-rule system. Some research studies have been concerned with the comprehension of mathematical statements, not so much as logically formulated propositions, but as mathematical propositions embedded in natural language contexts. These investigations are designed to determine whether comprehension is fostered or impeded by the form or complexity of the natural language contexts in which mathematical data are presented. In other studies, the order of sentence constituents is altered to determine the effects of sentence and problem order on the [illegible]ition of problems. These studies parallel those done by psycholinguist[illegible] [illegible]on-mathematical contexts. While studies in psycholinguistics are ordinarily addressed to theoretical questions concerning the comprehension of surface structure characteristics of the grammar, those performed in a mathematical context have been addressed to problems of solution efficiency. Although these mathematical studies could easily illuminate some more general issues they seem, for the present at least, to be much more limited in focus.

Studies in the generative aspects of language and cognition has led to considerable interest in the nature of rules and rule structures and their place in thinking and learning. Because of the obvious relation between rule-related thought and mathematics, a fair amount of research is being pursued in the learning of mathematical rules. It is not my impression, however, that mathematical reasoning is being examined as a generative system to any appreciable extent. That is, study is not directed to the acquisition of mathematical rule as instruments of creative problem solving, or for use in the construction of mathematical ideas. Nor are the strategies by which such rules are functionally related to problems under solution being investigated to any extent. Rather, it would seem as though most of the effort is directed to studies of success or failure in learning rules. This appears to be a much more sterile enterprise than what the study of mathematics learning could be.

A recent effort to apply the generative-transformational linguistic model to the rule structure of mathematics is seen in the work of Scandura (1971). His position is that mathematical knowledge and mathematical "behavior" are "rule-governed." He distinguishes two conceptions of rule structure. One involves the idea of generative procedures composed of rules, and the second, the idea of rule-governed behavior. Rule-governed behaviors are those that are produced by a common algorithmic (generative) procedure. Traditional conceptions of concepts and associations are said to be special cases of such rules. Scandura goes on to detail the form such rules take to satisfy the recursive functions of a generative theory. Decoding, transformation, encoding and selectional rule types are specified that suggests a combination of both information processing theory and generative-transformational linguistic theory.

Mathematics as such is not concerned with rule-governed behavior but with the rules themselves. Mathematics educators, however, are very much concerned with both since there is an apparent relation between rule-governed behavior and mathematical rules. As already suggested, Piaget is also very much concerned with rule-governed behavior and its relation to mathematical rule structure, in fact it is part of his central thesis. There is thus considerable commonality between the fundamental assumption of the structural learning group (represented by Scandura) and the Piagetians, although there are some important differences between them as well.

Not all mathematicians, mathematics educators, or psychologists are optimistic about the adequacy of generative-transformational (Chomskyan) theory as a model for mathematical reasoning. Among mathematicians, Suppes (1972) considers generative-transformational theory to be a very inadequate explanation of even linguistic performance. To make the Chomskyan argument seem absurd he suggests as an analogy the relation of first-order logic to all current mathematical ideas. From these relations, he says, one can enumerate the theorems of a mathematical subject by enumerating the proofs. The enumeration of the proofs, he holds, is equivalent to the deep structure rules of Chomsky's grammar. But no one, says Suppes, would seriously claim that knowing such proofs can provide adequate knowledge of how students discover elementary proofs, or how mathematicians discover new and complex proofs, which he holds a competence theory should do. In spite of Suppes' argument it is not clear that the constructive process of generating sentences is equivalent or even analogous to the discovery processes in deductive reasoning.

Suppes' own approach to a theory of how children learn mathematical concepts is through the development of models of performance in simple mathematical tasks, such as learning to use the (instructional) algorithms of addition. His first models were linear regression models applied to a small number of performance characteristics. A regression model that predicts response probabilities, however, does not in itself postulate a specific process by which students apply problem solving algorithms. Subsequently developed models were process models specifically designed to satisfy the information processing requirements of algorithmic tasks. These models were based upon finite automata, although they were soon superceded by probabalistic automata models. The probabalistic automata models were seen to have limitations as well, lacking perceptual processing components, and so further developments suggested various advances over the automata models. These involve "register machines" that process perceptual information through a series of subroutines that combine different algorithms. Suppes' models appear to have come closer to the approach of Minsky and Papert (1972) whose simulations of intelligence involve models derived from theories based upon very different psychological assumptions. The Papert position is, in fact, much more Piagetian.

There are many properties of a natural language that mathematics does not appear to share. This is particularly so in regard to semantics, that is, in the way meaning is treated. The terms of a mathematical system or theory are not interpreted in the same way lexical entries in natural language are. In fact, one reason mathematics is referred to as a formal language is that its terms are ostensibly content free; those of a natural language are not. It seems dubious, however, that as a consequence mathematics is all syntax with no meaning. Instead, meaning appears to take a different form, or may be said to have a different significance. It is thus not enough to say that mathematics is a system of abstract forms and uninterpreted terms that represents the abstract relations ordinarily represented in natural language. In any case, it appears that the relations between formal languages such as mathematics and natural language will receive a great deal of attention in the years ahead from linguists, logicians, psychologists and mathematics educators for these relations may have important bearing upon mathematical reasoning and problem solving.

To sum up, I have tried to show that mathematics education research gains its strength from the infusion of theories and models from various disciplines. It also carries the burdens of these disciplines as well as those of its own.

The era of dramatic curriculum change and technological innovation appears to be over. A new era concerned with the psychological basis for mathematics learning and reasoning is already fully entered upon. It is being fed by psychological theories of very diverse origin. Mathematics education research is now a very lively intellectual and scholarly arena in which mathematics, philosophy, psychology, linguistics and computer technology are converging on the solution of some very real problems.

References

Beilin, H. Developmental stages and developmental processes. In D.R. Green, M.P. Ford & G.B. Flamer (Eds.) Measurement and Piaget, New York: McGraw-Hill, 1971.

Beilin, H. & Franklin, I. Logical operations in area and length measurement: age and training effects. Child Development, 1962, 33, 607-618.

Beilin, Lois A. A Piagetian model for the development of curriculum sequence. Presented to the meeting of the American Educational Research Association, New Orleans, 1973.

Bennaceraf, P. & Putnam, H. Philosophy of mathematics, New York: Prentice-Hall, 1964.

Bryant, P.E. & Trabasso, T. Transitive inferences and memory in young children. Nature, 1971, 232, 456-458.

Dienes, Z.P. An experimental study of mathematics learning, London: Hutchinson, 1963.

Dienes, Z.P. & Jeeves, M.A. Thinking in structures, London: Hutchinson, 1965.

Gholson, B., Levine, M. & Phillips, S. Hypotheses, strategies, and stereotypes in discrimination learning. Journal of Experimental Child Psychology, 1972, 13, 423-446.

Gholson, B., Phillips, S. & Levine, M. Effects of the temporal relationship of feedback and stimulus information upon discrimination-learning strategies. Journal of Experimental Child Psychology (in press).

Gollin, E. A developmental approach to learning and cognition. In H.W. Reese and L.P. Lipsitt (Eds.) Advances in Child Development and Behavior, 1965, 2, 159-186.

Inhelder, B. & Sinclair, H. Learning cognitive structures. In P.J. Mussen, J. Langer & S Covington (Eds.) Trends and Issues in Developmental Psychology, New York: Holt, Rinehart & Winston, 1969, 2-21.

Levine, M. Hypothesis behavior by humans during discrimination learning. Journal of Experimental Psychology, 1966, 71, 331-338.

Minsky, M. & Papert, S. Artificial intelligence: Progress Report. Memo no. 252, Cambridge, Mass: Massachusetts Institute of Technology, 1972.

Piaget, J. Comments on mathematics education. International Congress on Mathematical Education, Exeter, England, 1972.

Scandura, J.M. A theory of mathematical knowledge: can rules account for creative behavior? Journal for Research in Mathematics Education, 1971, 2, 183-196.

Sinclair, H. Piaget's theory and language acquisition. In M.F. Rosskopf, L.P. Steffe & S. Taback (Eds.) Piagetian cognitive-development research and mathematical education, Washington: National Council of Teachers of Mathematics, 1971.

Suppes, P. Facts and fantasies of education. Address to American Educational Research Association, 1972. (Also Phi Delta Kappan, in press.)