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ABSTRACT

This is the first of the NCTM yearbooks to be written by a single author. He traces the history of the terms "variable" and "function" and discusses the meaning of functionality. The logical and psychological bases are examined followed by a general discussion of the psychology of reasoning. Two chapters outline the history of functional thinking in the schools and in textbooks. After a treatment of related mathematical concepts, a detailed examination of a proposed general mathematics course using the function concept as its central theme is given. The appendix includes several tests on mathematical relations and functional thinking. (LS)

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE NINTH YEARBOOK

RELATIONAL AND FUNCTIONAL THINKING IN MATHEMATICS

By Herbert Russell Hamley, Ph.D.

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1934

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EDITOR'S PREFACE

This is the ninth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The titles of the preceding Yearbooks are as follows:

1. A Survey of Progress in the Past Twenty-five Years.
2. Curriculum Problems in Teaching Mathematics.
3. Selected Topics in the Teaching of Mathematics.
4. Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910.
5. The Teaching of Geometry.
6. Mathematics in Modern Life.
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The purpose of the Ninth Yearbook is to present some of the most important ideas connected with relational and functional thinking. The tendency now in American schools is to organize the work around the function idea, and it is hoped that the contents of this volume will be stimulating and helpful to teachers of mathematics in the schools.

I wish to express my personal appreciation as well as that of the National Council of Teachers of Mathematics to Professor H. R. Hamley, for permitting us to publish his contribution as the Ninth Yearbook of a series that is becoming increasingly important and helpful to the field.

W. D. Reeve

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The National Council of Teachers of Mathematics is a national organization of mathematics teachers whose purpose is to:

1. Create and maintain interest in the teaching of mathematics.
2. Keep the values of mathematics before the educational world.
3. Help the inexperienced teacher to become a good teacher.
4. Help teachers in service to become better teachers.
5. Raise the general level of instruction in mathematics.

Anyone interested in mathematics is eligible to membership in the Council upon payment of the annual dues of $2.00. The dues also include suscription to the official journal of the National Council (*The Mathematics Teacher*). All correspondence relating to editorial business matters should be addressed to *The Mathematics Teacher*, 525 West 120th Street, New York City.

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RELATIONAL AND FUNCTIONAL THINKING IN MATHEMATICS

I

INTRODUCTION

The science of mathematics. The science of mathematics has to do with certain fundamental concepts of number and space, with logical propositions derived from these concepts and with practical applications of the propositions so derived to everyday computation and measurement. Thus the science of mathematics is both abstract and concrete. It is abstract in the sense that it is a logical structure based upon certain postulated foundations; it is concrete in the sense that it can be brought into correspondence with our intuitive notions of number and space and with a rational interpretation of the physical universe in which we live.

In recent years there has been a tendency to extend the boundary of abstract mathematics to the utmost possible limit, with the result that it is now impossible to determine where pure mathematics ends and formal logic or philosophy begins. Russell maintains that it is idle to seek a boundary between mathematics and logic, since the two subjects are fundamentally indistinguishable. "If there are still left those who do not admit the identity of logic and mathematics, we may challenge them to indicate at what point, in the successive definitions and deductions of *Principia Mathematica*, they consider that logic ends and mathematics begins. It will then be obvious that any answer must be quite arbitrary."[1] This logical bias of modern pure mathematics is reflected in the definitions that have been given of the term 'mathematics'.

Russell's definition is as follows: "Mathematics is the class of all propositions of the form 'p implies q', where p and q are propositions, containing one or more variables, the same in the two propositions, and neither p nor q contain any constants except logical constants,"[2] a definition that gives us little or no escape from the contention that 'mathematics is deduction by logical principles

[1] Russell, B. *Introduction to Mathematical Philosophy*, p. 194. London, 1924.
[2] Russell, B. *The Principles of Mathematics*, p. 3. Cambridge, 1903. These are the opening words of Russell's classic treatise.

from logical principles." A less complete but more concise definition of mathematics is that of Benjamin Peirce: "Mathematics is the science that draws necessary conclusions."[3] That Peirce conceived this definition in the broadest possible sense is shown by his assertion that "mathematics, under this definition, belongs to every inquiry, moral as well as physical." Whitehead, while giving general assent to Peirce's definition, suggests that we add "from the general premises of all reasoning," and that we define mathematics as "the science concerned with the logical deduction of consequences from the general premises of all reasoning."[4] We may, without undue violence to the spirit of Peirce's statement, modify it in one particular and assert that mathematics is the science of *necessary relations*. This modified form is admittedly less conclusive than the original, but it does suggest the idea of *functionality*—which has been aptly called by Klein the 'soul' of mathematics—and it readily admits a concrete as well as an abstract interpretation.

Any justification of this definition would require a critical analysis of the words 'necessary' and 'relation'. Young[5] has pointed out, in his discussion of Peirce's definition, that what we commonly designate 'logical necessity' is something that can, in some cases, at least, be determined only by its consequences; in other words, that in the final issue the validity of our reasoning appears to rest on no surer basis than the fact that no contradiction has ever been found. This may be an unacceptable doctrine to the pure logician, but it need not cause any concern to the teacher of school mathematics. Klein[6] has urged that we do not regard even the axioms of mathematics as arbitrary assumptions but as *common-sense statements* (*vernünftige Sätze*), which find their justification in their consistency with experience. A similar thought has been expressed by Veblen in his treatment of the validity of hypotheses: "The writer is inclined to believe that the truth of a statement can be determined only by testing all its consequences, so that the real

[3] Peirce, B. "Linear Associative Algebra." *American Journal of Mathematics*, Vol. IV, p. 97, 1881.

[4] Whitehead, A. N. Article on "Mathematics" in the *Encyclopaedia Britannica* (Fourteenth Edition), Vol. XV, p. 87, 1929.

[5] Young, J. W. *The Fundamental Concepts of Algebra and Geometry*, p. 220. New York, 1925.

[6] Klein, F. *Elementarmathematik vom höheren Standpunkte aus*, Band II (*Geometrie*), p. 384. Leipzig, 1925.

test of the validity of the hypotheses of geometry is in the validity of the theorems."[7] We may leave the solution of this problem to the logicians[8] and proceed, on the reliable assumption, that there does exist a substantial correspondence between the commonly accepted axioms of mathematics and the world of our experience.

Concept of relation. The word 'relation' is more easily comprehended than defined. Whitehead attributes to the objects of external reality a 'relational essence', involving the possibility of realization as *relatedness*. "The meaning of the term 'possibility' as applied to *A* [an external object] is simply that there stands in the essence of *A* a patience for relationships to actual occasions."[9] This patience cannot be rewarded by ingression, or this potentiality become actual as *relation*, except through the activity of the mind which relates. This dynamic character of relation is expressed in Spearman's definition, "A relation is an attribute which mediates between two fundaments."[10] According to Spearman a *fundament* is any "character" of things, simple or complex, concrete or abstract; it many even be a relation already determined between other fundaments.

The concept of relation is fundamental to human thought. As Whitehead has remarked: "If anything out of relationship, then complete ignorance as to it."[11] It is through the cognition of relations that the mind transcends the particular and immediate and learns to apprehend and appreciate the abstract and remote. When man first used words to identify things, and numbers to designate quantities of things, he showed that he was capable of thinking in terms of general abstract relations. When he used the word 'spear' to direct attention to a certain type of weapon, he showed a definite capacity for mathematical logic; for from the variable class of objects 'spear', he had abstracted the element 'spear', and symbolized it by a word. Thus were language and mathematical logic born together. Modern mathematical logic has much to do with *classes*, and in particular with 'variable classes'. "This fact, that

[7] Veblen, O. *The Foundations of Geometry*, p. 4. Monographs of Modern Mathematics. New York, 1927.

[8] Ramsay, F. P. "Mathematical Logic." *Mathematical Gazette*, Oct. 1926, p. 185. See also *The Foundations of Mathematics*, p. 356. London, 1931.

[9] Whitehead, A. N. *Science and the Modern World*, pp. 198, 199. Cambridge, 1927.

[10] Spearman, C. E. *The Nature of 'Intelligence' and the Principles of Cognition*, p. 65. London, 1925.

[11] Whitehead, A. N. *Op. cit.*, p. 32.

the general conditions transcend any one set of particular entities is the ground for the entry into mathematics, and into mathematical logic, of the notion of the 'variable,' "[12] and the variable is "perhaps the most distinctly mathematical of all notions."[13] According to Russell, "pure mathematics is the class of propositions, which are expressed exclusively in terms of variables and logical constants."[14]

Relations may be of many different types and may have many different kinds of uses. The fundaments or raw materials of relations consist in whatever entities we choose to include in our discussion. They may be, and often are, relations between other relations. The pure mathematician usually restricts the range of his fundaments to number and space; he may, if he is interested in the endurance of his pattern, add the concept of time. The applied mathematician would at least require the further concept of mass. The most important relation with which we are concerned in mathematics is that of *order*. "There are parts of mathematics which do not depend on the notion of order, but they are very few in comparison with the parts in which this notion is involved." (Russell.[15]) It is important to add that the concept of order, either explicit or implicit, is essential to an adequate theory of variables, and, therefore, to an adequate theory of functions. Cantor has shown this in his *Transfinite Mengenlehre*.

The purpose of mathematical analysis is to investigate the mutual relations existing between certain variable classes and to establish laws pertaining to these relations. In the absence of some arrangement of the quantities concerned, mutual relationship can have no meaning. When we state that a variable x approaches a value a, we assert that every quantity of the variable is nearer a, according to some order of arrangement, than every previous quantity. Such a statement would be devoid of meaning, if the word 'previous' carried no implication of order. The concept of a *limit*, which underlies all higher mathematics, is a serial conception.[16] When

[12] Whitehead, A. N. *Op. cit.*, p. 33.

[13] Russell, B. *The Principles of Mathematics*, p. 80.

[14] Russell, B. "The Philosophical Importance of Mathematical Logic." *The Monist*, Vol. XXIII, p. 487, October, 1913.

[15] B. Russell, in a discussion of the importance of this concept, says: "In former days it was supposed and philosophers are still apt to suppose that quantity was the fundamental notion of mathematics. But nowadays quantity is banished altogether, except from one little corner of geometry." From *Mysticism and Logic*, p. 91. London, 1921

[16] Russell, B. *Introduction to Mathematical Philosophy*, p. 29.

we state that two variables are in functional relationship we assume that they are arranged, although the order of arrangement may differ according to the purpose of our inquiry.[17] It is strange that the notion of order, which lies at the very heart of mathematics and science, should have received so little attention in school mathematics. It is true that number sequence and number grouping are now emphasized in early number teaching but, as a general rule, no further reference is made to order until the student is introduced to directed numbers or progressions. School mathematics in the past has been too closely identified with the sign of equality. School mathematics in the future will deal more often with inequalities; with order, arrangement, and system; classes and groups; correspondence and correlation; similarity and symmetry; and relation, variation, and function. And there is not a term among these which cannot be brought into intimate association with life.

The notion of function. The notion of function, like that of relation, is one of the most elementary in human thought. The rhythm of the seasons and the changing features of the trees must have suggested even to primitive man some kind of correspondence. At all events there came a time when he began to till the soil, confident in the belief that seed-time and harvest had their appropriate seasons. He learned by experience, and learning by experience is, in a real mathematical sense, a functional activity. When the child realizes that fire burns or shows by his movements that he appreciates rhythm, he has succeeded in educing a correspondence between two variables; in other words, he has learned to think and act in functional terms.

The word 'function' has now become almost commonplace in everyday speech, but it is not always used in the mathematical sense. When we speak of 'the function of the teacher' or 'the function of the liver,' we use *function* as synonymous with a duty, or a service, that the teacher or the liver is expected to perform, without imputing to either any necessary association with mathematics. But when we assert that 'the teacher's temper is a function of his liver', we imply a correspondence between the state of the one and the condition of the other, which could be expressed with equal precision of thought by a mathematical equation. It is this meaning of the word 'function' that is to be our particular con-

[17] *Op. cit.*, p. 31. "The fact that a class may have many orders is due to the fact that there can be many relations holding among the members of one single class."

cern in this study. *A function is*, then, ***a correspondence between two ordered variable classes.***

The world is becoming, as Klein has expressed it, "functionally minded." Newspapers and magazines are using, to an increasing extent, not only the functional language but also the functional tools of the mathematician.[18] We read, for example, that public health, the weather, the bank rate, unemployment, the incidence of crime and the condition of trade are all functions of many variables, known or unknown. Such assertions are often accompanied by statistical data and graphical representations of the concomitant rising and falling of related variables entering into the case. Industry and commerce, economics and politics are becoming saturated with functional ideas, so much so, that there is an increasing demand for men with an expert knowledge of 'functional economics'. That being the case, it is our duty to take cognizance of the fact and to reorganize our mathematical teaching so that our youth may receive the knowledge and the discipline that it needs to meet the changing times. As Kilpatrick has reminded us, "We must have a philosophy that not only takes positive recognition of the fact of change but one that includes within it change as an essential element."[19] Functional mathematics will go a long way to meet this demand, for change is the very essence of function, whether we regard the term as an activity or as a mathematical correspondence. A dynamic concept merits a dynamic mode of treatment.

Criticisms of mathematics curricula. In recent years school mathematics has frequently been called upon to defend its right to membership in the society of approved school subjects. It has been asserted that the disciplinary and cultural values of mathematical education have been greatly overestimated. Some critics,[20] while recognizing the dependence of the modern world on the work of the professional mathematician, contend that the average consumer needs only a modicum of mathematics for his daily needs, and that the educational values of school mathematics are incommensurate with the time and effort usually devoted to that subject. Challenges such as these are to be welcomed, if they carry us beyond

[18] See, for example, Adams, James Truslow, "Diminishing Returns in Modern Life." *Harper's Magazine*, April, 1930.

[19] Kilpatrick, W. H. *Education for a Changing Civilization*, p. 41. New York, 1928.

[20] Bowden, A. O. *The Consumer's Uses of Arithmetic*. New York, 1929. See also reviews of Bowden's thesis in the *Mathematics Teacher*, Vol. XXIII, March, 1930.

defense reactions to serious self-examination and reform.[11] When we view the situation dispassionately we have to admit that some of the arguments of our critics are difficult to refute. We cannot deny that school mathematics in the past has been largely concerned with the mechanical and the abstract and almost wholly unrelated to the child's natural interests and future needs. Algebra generally began with arithmetical substitutions, often of a meaningless kind, and continued, by a relentless logical procedure, through the fundamental operations, factors, and the like, to quadratic equations. The treatment of geometry, until quite recently, was just as formal and uninspiring. The Euclidean discipline, excellent as it was for the few who could profit by it, proved in the large majority of cases to be ineffective, partly because it was administered too soon, but largely because it was an abstract study of a static universe.

Status of the teaching of mathematics. Although progress has been slow, it has been none the less sure. Thoughtful teachers are now awakening to the fact that, if school mathematics is ever to meet the demands of modern life, or even to win the respect of the average man of affairs, it must be made more dynamic and functional. They are also beginning to realize that, in endeavoring to satisfy the exigencies of our modern world, they are securing, even more effectively than hitherto, those very objectives which, at first sight, they seemed to be in danger of losing. Broadly considered, the aims of most teachers of mathematics may be summarized under three main headings: utilitarian, disciplinary, and cultural. Functional mathematics affords us an ideal medium through which these aims may be realized. The purpose of this study is to examine the function concept in all its bearings and to justify the claim

[11] See the *Mathematics Teacher* for 1929, 1930, 1931 for the following articles:

Reeve, W. D. "The Universality of Mathematics," Vol. XXIII, p. 71, February, 1930.

Betz, William. "Whither Algebra? A Challenge and a Plea," Vol. XXIII, p. 104, February, 1930.

Langer, S. K. "Algebra and the Development of Reason," Vol. XXIV, p. 285, May, 1931.

Shaw, J. B. "Mathematics as a Fine Art," Vol. XXIII, p. 104. March, 1930.

Judd, C. H. "Informational versus Computational Mathematics," Vol. XXII. p. 187, April, 1929.

Kempner, A. J. "The Cultural Value of Mathematics," Vol. XXII, p. 129, March, 1920.

See also *The Sixth Yearbook, National Council of Teachers of Mathematics*, which is a stirring *apologia* for mathematical education.

that has just been made, by showing that the conception of function may be regarded as the natural coördinating principle of all school mathematics.

Nature of the remainder of this study. In Chapter II we trace the history of the terms 'variable' and 'function' and discuss, from the purely mathematical standpoint, the meaning of *functionality*. In Chapter III we examine the logical and the psychological bases of functional thinking in general. This leads to a general discussion of the psychology of reasoning. Chapter IV is devoted to an outline history of the development of functional thinking in the schools of Europe and America. Chapter V extends the history of the subjects to an examination of modern textbooks in school mathematics. Chapter VI is preliminary to Chapter VII, in which a course of general mathematics, with the function concept as its central theme, is outlined. In Chapter VI some mathematical concepts, other than the function concept, which enter into the course of study developed in Chapter VII, are discussed. In the first part of the chapter a general discussion of these concepts is given. This is followed by detailed examination of the course proposed.

In Appendix A is given a Test of Mathematical Relations which was submitted to several hundred children in English elementary schools, and in Appendix B, a more advanced Test on Functional Relations for secondary schools.

II

VARIABLE AND FUNCTION

Original concept of a variable. Although the germ of the integral calculus is to be found in the work of Archimedes, the concept of the *variable* did not definitely enter mathematical thought until the close of the seventeenth century. The word 'variable' seems to have been first used by Leibniz, who wrote, in the introduction to his *Analyse*:

> Those quantities are called variable which continually increase or diminish and on the contrary those are constant, which remain the same while others change.[1]

This concept of the variable, as a quantity which varies or is a function of duration, was already implicit in Newton's *Method of Fluxions*. Newton's "fluent" which corresponds to the variable of Leibniz, was conceived as a quantity which varied with, or was a function of, real or imaginary time. In 1687, he wrote in his *Principia*:

> Now those quantities which I consider as gradually and indefinitely increasing I shall hereafter call fluents or flowing quantities.[2]

That Newton had the idea of the variable before 1687 is shown in a reference to fluents in the *Quadratura Curvarum*:

> Calling these velocities of the motions or increments Fluxions and the generated quantities Fluents, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the years 1665 and 1666.[3]

Thus a *fluxion* was the rate of change of a *fluent* or variable quantity. To-day Newton's theory of fluxions would be called the Theory of Continuous Functions. Although great progress was made in the

[1] Leibniz, G. W. Quoted by L'Hospital in *Analyse des infiniment petits*, p. 1. Paris, 1696.

[2] Newton, I. *Method of Fluxions*, pp. 20, 6 (translated by John Colson). London, 1738.

[3] Newton, I. *Tractatus de Quadratura Curvarum, 1671.* Introduction (translated by John Stewart). London, 1704.

development of function theory during the two centuries following Newton and Leibniz, the term 'variable' retained the meaning given it by Leibniz—"a quantity which varies."

The changing interpretation of a variable. The first suggestion of a change of any significance is to be found in the work of Biermann:

> We will suppose that certain elements have values fixed once for all; other elements take values of our system of quantities in turns. The former quantities are called constant or unalterable, the latter variable or alterable.[4]

The important feature of this definition is that the variable is not defined as a quantity but as an aggregate of elements selected from a set of existing quantities.

Definitions couched in similar terms are to be found in most of the treatises on analysis published during the following twenty years. Thus René Baire writes, "In mathematics we represent by a letter a number capable of taking different values. We then say that we have a variable,"[5] and Bauer, "We distinguish between constant and variable quantities. The former have a fixed value, the latter can take any assigned values."[6] The same thought is contained in a more complete definition by Burkhardt,

> A number is said to be alterable or variable, when in the course of an investigation one value after another is assigned to it, and to be constant when the value first assigned is retained through the whole investigation.[7]

Harnack introduced a further refinement into the concept of the variable when he urged that we dissociate the variable entirely from its concrete embodiment and look for its essence, not in the thing represented, but in the abstract number values assumed by the variable.

> A quantity is said to be a variable when it is able to assume different numerical values. As in arithmetical investigations we no longer consider the things given in number, so in the conception of a variable quantity we have to free ourselves entirely from considering what this quantity represents (distance, temperature, tension of vapour). Everything measurable in nature can enter into calculation as a variable quantity.[8]

[4] Biermann, O. *Theorie der analytischen Functionem* p. 5. Leipzig, 1887.
[5] Baire. René. *Leçons sur les théories générales de l'analyse.* Vol. I, p. 20. Dijon, 1907.
[6] Bauer, G. *Vorlesungen ueber Algebra*, p. 7. Leipzig, 1910.
[7] Burkhardt, H. *Algebraische Analysis*, p. 37. Leipzig, 1903.
[8] Harnach, A. *Introduction to the Study of the Calculus*, p. 15 (translated from the German). London, 1897.

Many mathematicians of the present day incorporate this condition in their definitions of the variable; others, taking their stand on a broader logical basis than number, regard it as an unnecessary restriction.[9]

According to Molk, Weierstrass was the first to define a variable as a symbol. In a course of lectures given in Berlin between the years 1861 and 1865, he defined a variable as follows:

A real variable is a symbol which represents the different elements of an assemblage of real numbers. Each of these elements is one of the values which the variable can take; the assemblage itself constitutes the domain of the variable.[10]

The first published definition of the variable as a symbol is that given by Pringsheim:

By a real variable we understand a symbol to which is assigned successive different numerical values.[11]

Similar definitions were given by Weber Pierpont, and others. Thus Weber writes, "If x is a symbol for which any number we choose can be put, we then call x a variable,"[12] and Pierpont says, "A symbol which takes on more than one value, in general an infinity of values, is a variable."[13]

None of these definitions imposes any restriction or indicates any specification of the field of selection of the variable. Tannery, however, proceeds more cautiously. He writes:

It is well to modify this notion of a variable . . . . This is not a letter that can take any values whatsoever but any values whatsoever belonging to a certain ensemble. There are functions which are only defined for the integral and positive values of the variable.[14]

The same thought was implicit in one of Tannery's earlier works:

Let us consider an ensemble (X) of distinct numbers and look upon these

[9] Russell, B. *Principles of Mathematics*, Chap. VIII. Cambridge, 1903.

[10] Molk, J. *Encyclopédie des sciences mathématiques pures et appliquées*, Tome 2, Vol. I, Fasicule 1, p. 16. Paris, 1909.

[11] Pringsheim, A. Article on "The Foundations of General Function Theory." *Encyklopädie der mathematischen Wissenschaften*, Band II, p. 8. Leipzig, 1889.

[12] Weber, H. *Encyclopädie der elementaren Algebra und Analysis*, p. 185. Leipzig, 1906.

[13] Pierpont, C. *Lectures on the Theory of Functions of Real Variables*, Vol. I, p. 118. Boston, 1905.

[14] Tannery, J. *De la méthode dans les sciences en mathématiques pures*, p. 11. Paris, 1909.

numbers as values which can be assigned to a letter (x), which we designate a variable.[15]

Writers of modern textbooks on analysis and treatises on function theory seem to have added very little to the definitions that have been quoted. Some prefer to leave the term 'variable' undefined, others to illustrate it by a few selected examples. Young defines a variable as follows:

A variable is a symbol, which represents any one of a class of elements. The elements may or may not be numbers.[16]

The following may be taken as typical of the definitions found in modern textbooks:

A variable is a quantity to which an unlimited number of values can be assigned in an investigation (Granville, Smith, and Longley[17])

When a quantity is permitted to assume different values in a given problem, it is called a variable. (Miles and Mikesh[18])

A letter capable of taking up various values. (Walmsley[19])

A magnitude to which, in the course of any given process, different values, are assigned is said to be a variable. (Lamb[20])

A symbol for a changing quantity. (Czuber[21])

Perhaps the most comprehensive and complete definition to be found in modern mathematical works is that given by Hobson:

If we suppose that an aggregate of real numbers is defined, the aggregate being either enumerable, or of the power of the continuum, such an aggregate is said to be the domain, or field, of a real variable. It is necessary for the purposes of analysis to be able to make statements applicable to each and every number of the aggregate and which shall be valued for any particular number that may be, at will, selected. This is done by employing the *real variable*, denoted by some symbol other than those used to denote real numbers, and the essential nature of the variable consists in its being identifiable with any particular number of its domain.[22]

[15] Tannery, J. *Introduction à la théorie des fonctions d'une variable*, p. 220. Paris, 1904.

[16] Young, J. W. *Lectures on the Fundamental Concepts of Algebra and Geometry*, p. 193. New York, 1912 and 1925.

[17] Granville, Smith, and Longley. *Elements of the Differential and Integral Calculus*, p. 1. Boston, 1929.

[18] Miles, E. J. and Mikesh, J. S. *Calculus*, p. 3. New York, 1930.

[19] Walmsley, C. *Mathematical Analysis*, p. 134. Cambridge, 1926.

[20] Lamb, H. *An Elementary Course of Infinitesimal Calculus*, p. 1. Cambridge, 1924.

[21] Czuber, E. *Vorlesungen über differential-und-integral Rechnung*, p. 13. Leipzig, 1926.

[22] Hobson, E. W. *The Theory of Functions of a Real Variable*, p. 256. Cambridge, 1921.

Historical background of the term 'function'. It is frequently stated in books on function theory that the term 'function' was first used by Descartes to denote *powers* of a variable (x), such as x^2, x^3, etc., but there seems to be no basis for this statement. It is true that the function concept is implicit in Descartes's *Géométrie*, but he does not actually use the term 'function'. He does, however, mention two important functional ideas: one, that "unknown quantities can be expressed in terms of a single quantity"[23] (as powers) and the other, that a curve pictures the dependence of one variable on another. Fermat[24] brought out the concept of dependence even more definitely but he, again, does not mention the word 'function'. It now seems to be agreed that we owe the term 'function' to Leibniz,[25] who used it to denote variable lengths (abscissae, ordinates, tangents, and normals), related in a definite way to variable points of a curve. For example, he asserts that "a tangent is a function of a curve" and that "a function is a fact asserted by an equation."[26] In his earlier writings James Bernouilli used the term in a somewhat similar sense, but in 1718 he took us a step further when he wrote, "We name a quantity composed in any manner whatever of a variable magnitude and constants, a function of the variable magnitude."[27] In 1730, he distinguished for the first time between algebraic and transcendental functions. In these definitions Bernouilli seems to imply that there are three distinct classes of quantities - variable quantities, constant quantities, and functions. As we shall see later, this distinction is not a correct one. The credit of having introduced, for the first time, the familiar $f(x)$ notation goes to Euler. He defines a function as an analytical expression:

> A function of a variable quantity is an analytical expression composed in some way of that variable quantity and of numbers or constant quantities.[28]

[23] Descartes, René. *Géometrie* (livre premier), p. 301. Paris, 1637.
[24] Fermat, P. de. *Varia Opera mathematica*, p. 1. Toulouse, 1679.
[25] Müller, F. *Bibliotheca mathematica*, Vol. II, p. 285. Leipzig, 1901.
See also Pringsheim, A. and Molk, J. *Encyclopedie des sciences mathématiques*, Tome 2, Vol. I, Fasicule 1, p. 1. Paris, 1909.
[26] Leibniz, G. W. *Considérations sur la différence qu'il y a entre l'analyse ordinaire et le nouveau calcul des transcendantes*, 1694. See also *Opera Omnia*, Vol. III, p. 302, Geneva, 1768; and *Leibnizens mathematische Schriften*, Book V, p. 307.
[27] Bernouilli, J. *Par. Mem.*, 1718, p. 106. See also *Opera*, Tome II, pp. 241, 255, Lausanne et Geneve, 1742; and *Opera*, Tome III, p. 174.
[28] Euler, L. *Introductio in analysin infinitorum* (translated by J. B. Labey), Vol. I, p. 4. Lausanne, 1748. See also *Commentaria Academiae Petropolitanae ad annos 1734-5*, Tome VII, pp. 186-87. Petropoli and St. Petersbourg, 1748. "*Si* $f(x/a + c)$ *denotet functionem quamcumque ipsius* $(x/a + c)$."

This definition defines a function as a compound expression of the variable—as we find it, for example, in the statement that $ax^2 + bx + c$ is a function of x—an error that has since been repeated by many writers of textbooks on higher algebra. Burnside and Panton open their treatise on the theory of equations with the words, "A mathematical expression involving a quantity is called a function of that quantity."[29] In his *Introductio in analysin infinitorum*, Euler gave an acute analysis of elementary functions and distinguished for the first time between implicit and explicit functions and uniform and multiform functions. Lagrange, in his *Theorie des fonctions analytiques* (1797), extended Euler's notation by using f, F, ϕ, χ, etc., followed by parentheses to designate functions. He defines a function as "a property of a series of powers of the independent variable."[30]

Development of the concept of functionality. It is an interesting fact that the next developments in this subject came, not from pure mathematicians but from mathematical physicists, who, while working with physical facts, found that the existing mathematical tools were inadequate for their purposes. D'Alembert (1797), in his discussion of the oscillations of strings, and Fourier (1807), in his analysis of trigonometric series, both stressed the need of a more general type of correspondence than any previously propounded. We owe to Fourier[31] the conception of a single function defined in different intervals by different analytical expressions. In 1829 Lejeune-Dirichlet, in a celebrated memoir dealing with the convergence of Fourier's series, gave a definition of a function, which, with slight modifications, has been accepted by mathematicians ever since:

> Let by a and b be understood two fixed values and by x a variable quantity, which gradually assumes all values lying between a and b. Now, if a single finite y correspond to every x, in such a way that while x continuously passes through the interval a to b, $y = f(x)$ likewise varies gradually, then y is called a continuous function of x for this interval. It is quite unnecessary that y in this entire interval should be dependent upon x according to the same law; indeed, we need never think of a dependence expressed in terms of mathematical operations.[32]

[29] Burnside and Panton. *Theory of Equations*, Vol. I, p. 1. Dublin, 1912.

[30] Lagrange, J. L. *Oeuvres de Lagrange*, Vol. IX. Paris, 1881.

[31] Fourier, J. B. J. *La Théorie analytique de la chaleur*, Chap. III, Sec. 6. Paris, 1822. See also *Oeuvres*, Vol. I, pp. 119, 135. Paris, 1889.

[32] Dirichlet, G. Lejeune. "*Ueber die Darstellung ganz willkürlicher Functionem durch Sinus-und Cosinusreihen.*" *Werke*, Vol. I, p. 135. Herausgegeben von L. Kronecker, Berlin, 1889.

Thus, for the first time, "Dirichlet gave the word function a significance independent of any assumption of the possibility of an analytical representation."[33] At the same time he gives the function a definite graphical interpretation, for he continues:

Considering x and y as abscissae and ordinates, a continuous function appears as a connected curve in which only one point corresponds to every abscissa between a and b. This definition does not attribute to the various parts of the curve a common law. We can think of the curve as made up of heterogeneous parts or as described entirely without law. Thus a function is to be regarded as completely determined for an interval, only if it is defined graphically for the whole extent of the interval or is subjected to mathematical laws valid for the several parts of the interval.

In this memoir Dirichlet laid the foundation of the modern Theory of Functions. But, more important for our present purpose, he gave a wider meaning than hitherto to the whole concept of functionality, linking it up on the one hand with physics and on the other hand with geometry. Henceforth physical interpretation and graphical representation became recognized parts of the technique of analyzing functions. Little advantage was taken of this lead, as far as school mathematics was concerned, until Klein,[34] in an article on the graphical representation of functions, subjected the function concept to a searching analysis, and at the same time initiated a movement towards functional thinking which has profoundly influenced the teaching of mathematics in Germany and elsewhere.

Dirichlet's definition has been criticized by later writers,[35] sometimes on the ground that it is inadequate for the needs of modern analysis, and at other times on the ground of its excessive generality. Harkness and Morley write:

This definition, in contrast to those used before Dirichlet's time, errs on the side of excessive generality, for it does not itself confer properties on the functions. The functions so defined must be subject to restrictive conditions before they can be used in analysis. Nevertheless, this definition forms and must continue to form the basis of researches upon discontinuous functions

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See also *Repertorium der Physik, herausgegeben von Heinrich W. Dove und Ludwig Moser*, Band I, p. 152. Berlin, 1837. S. F. Lacroix had already given a somewhat similar definition, but not nearly so general. (*Differential Calculus*, p. 1.)

[33] Dini, U. *Journal für Mathematischen*, Vol. IV, p. 157. See also *Grundlagen für eine Theorie der Functionem einer veranderlichen reelen Groesse*, p. 48. Leipzig, 1892.

[34] Klein, F. "*Ueber den allgemeinen Functionbegriff und dessen Darstellung durch eine willknerliche Curve.*" *Mathematischen Annalen*, Vol. XXII, p. 249, 1883.

[35] Hankel, H. *Untersuchungen ueber die unendlichen oft oszillierenden und unstetigen Funktionem*, p. 5. Tubingen, 1870.

of a real variable. In Dirichlet's sense $f(x)$ is a function of x throughout an interval when, to every value of x within the interval belongs a definite value of $f(x)$. A value of x which makes the function infinity is excluded.[36]

Hobson, however, supports Dirichlet:

An adequate definition of a function for a continuous interval (a, b) must take the form first given to it by Dirichlet.[37]

Modern concept of functionality. These definitions of a function are found in modern textbooks:

If to each value of x of an interval of the x-axis or of a set of points by any prescribed rule, a definite y is made to correspond, then we may say that y is a function of x defined in that interval and write $y = f(x)$. (Knopp[38])

To-day the notion of function is considered to be identical with the notion of correspondence between two ensembles. Suppose a number x runs through the points of an ensemble and the movement of x makes a certain other number y take fixed values, it is said that the variable y is a function of the variable x. (d'Ahhemar[39])

One variable quantity is said to be a function of another, when, other things remaining the same, if the value of the latter be fixed, that of the former becomes determinate. (Lamb[40])

y is a function of x, when, x being given, y is determined. (Appell[41])

If x be a variable which takes on a certain set of values of which the totality may be denoted by $[x]$ and, if y is a second variable the value of which is uniquely determined for each x of the set $[x]$, then y is said to be a function of x defined over the set $[x]$. (Wilson[42])

Let a and b be any two real numbers, where $b > a$. If to every value of x in the interval $a < x < b$ there corresponds a real number y, then we say that y is a function of x in the interval (a, b) and we write $y = f(x)$. (Carslaw[43])

If, to each point of the domain of the independent variable x, there be made to correspond in any manner a definite number, so that all such numbers form a new aggregate which can be regarded as the domain, or field, of a new variable y, this variable y is said to be a (single valued) *function* of y.

In this definition no restriction is made *a priori* as regards the mode in which,

[36] Harkness and Morley. *A Treatise on the Theory of Functions*, pp. 51, 53. London, 1893.

[37] Hobson, E. W. *Op. cit.*, p. 259.

[38] Knopp, K. *Theory and Applications of Infinite Series* (translated by L. C. Young). London, 1930.

[39] d'Ahhemar, R. *Leçons sur les principes de l'analyse*, p. 24. Paris, 1922.

[40] Lamb, H. *An Elementary Course on Infinitesimal Calculus*, p. 13. Cambridge, 1919.

[41] Appell, P. *Eléments d'analyse mathématique*, p. 1. Paris, 1921.

[42] Wilson, E. B. *Advanced Calculus*, p. 40. Boston, 1912.

[43] Carslaw, H. S. *Introduction to the Theory of Fourier's Series and Integrals* (3rd edition). London, 1930.

corresponding to each value of x, the value of y is assigned; and the conception of the functional relation contains nothing more than the notion of determinate correspondence in its abstract form, free from any implication as to the mode of specification of such correspondence. In any particular case, however, the special functional relation must be assigned by means of a set of prescribed rules or specifications, which may be of any kind that shall suffice for the determination of the value of y corresponding to each value of x. (Hobson[44])

Interrelation of mathematics and logic. This survey of the history of the fundamental terms of our subject would be incomplete without some reference to the influence of mathematical philosophy on logic—and the reflex influence of logic on mathematics. All the definitions so far quoted have had a strictly mathematical implication. It should be observed, however, that the function concept need not be restricted to numbers or to those problems which we are accustomed to style mathematical. Thus the logical proposition, 'All men are mortal', could also be expressed, 'If x is a man, then x is a mortal, for all values of x', which is essentially a mathematical form.[45] Here x is a variable in the true mathematical sense, since it may connote a particular man or *any* man belonging to the class 'man'. Again, since two variables are related in the statement, the proposition is, in the mathematical sense, a functional expression. Russell has given the name "propositional function" to a statement of this kind.

By a propositional function we mean something which contains a variable x and expresses a *proposition* as soon as a value of x is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. It agrees with the ordinary functions of mathematics in the fact of containing an unassigned variable; where it differs is in the fact that the values of the functions are propositions.[46]

Elsewhere Russell defines a propositional function as follows:

A propositional function is simply any expression containing an undetermined constituent, or several undetermined constituents, and becoming a proposition as soon as the undetermined constituents are determined. If I say that 'x is a man' or 'n is a number', that is, a propositional function; so is any formula of algebra, say $(x + y)(x-y) = x^2 - y^2$.[47]

The word 'undetermined' used in this definition is to be preferred

[44] Hobson, E. W. *Op. cit.*, p. 257.
[45] Russell, B. *Introduction to Mathematical Philosophy*, Chap. XV. London, 1924.
[46] Whitehead, A. N. and Russell, Bertrand. *Principia Mathematica*, Vol. I, p. 38. Cambridge, 1910.
[47] Russell, B. "The Philosophy of Logical Atomisms." *The Monist*, Vol. XXIX, p. 192, April, 1919.

to the word 'ambiguous' used in the previous extract, for the *indeterminate* character of the variable is its main characteristic, while the characteristic of a mathematical function is that there is a *determinate correspondence* specified by the functional relation between the variables. Thus the characteristic notion of a variable is that it may be identified with *any* term of a particular ensemble: "the notions of *any* and of *denoting* are presupposed in the notion of a variable."[48]

> Let there be some proposition in which the phrase 'any *a*' recurs, where *a* is some class. Then in place of 'any *a*' we may put *x*, where *x* is an undefined member of the class *a*—in other words, any *a*. The proposition then becomes a function of *x*, which is unique when *x* is given.[49]

The example, 'All men are mortal,' will serve to exemplify the rôle of the function concept in formal logic and in everyday speech; it will also serve to show how closely functional thinking is related to intelligent thought in general. This aspect of functional thinking has been the subject of close analysis by Russell, Whitehead, Frege,[50] Royce,[51] and others.[52]

Examination of definitions. Having completed this historical survey of the meaning of the terms 'variable' and 'function', let us now proceed to examine some of the definitions that have been quoted. It will be noted that most of the definitions enunciated up to the time of Biermann, implicitly or explicitly, define a variable as a *quantity*. This is an erroneous conception. Quantities of the same kind constitute a variable class, but the variable itself is neither a quantity, nor a set of quantities composing a class. "If *n* stands for any integer, we cannot say that *n* is 1, nor yet that *n* is 2, nor yet that it is any particular number. In fact, *n* just denotes *any* number and this is something quite distinct from each and all of the numbers."[53] When we state that the variable x is a maximum at the point P, we do not assert that any quantity (x) is a maximum at the point P, but that, at the point P, x has a maximum

[48] Russell, B. *The Principles of Mathematics*, p. 89.

[49] *Op. cit.*, p. 263.

[50] Frege, G. *Die Grundlagen der Arithmetik*. Breslau, 1884. See also *Function und begriff*. Breslau, 1893.

[51] Royce, J. *Encyclopaedia of the Philosophical Sciences*, Vol. I. London, 1913.

[52] Kempe, A. B. "Memoir on the Theory of Mathematical Form." *Philosophical Transactions*, Vol. CLXXVII. See also "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points." *Proceedings of the L. M. S.*, Vol. XXI. See also Peano, G. *Formulaire de mathématiques*. Turin, 1895–1908.

[53] Russell, B. *Op. cit.*, pp. 90, 91.

quantity. Similarly, when we state that the variable x approaches a limit L, we do not assert that any quantity x approaches L as a limit. On the contrary, a limit is a quantity which may or may not belong to the variable. Neither can we define a variable as 'a set of quantities', or a 'variable class', for the relation $y = ax^2$ would then be meaningless; we cannot square a set of quantities. What makes the definition of a variable difficult is the fact that the same word is used to indicate the variable itself and the quantities which compose it. Dirichlet defines a variable as a quantity, but he refers to "a single finite *y*" and "every *x*," showing that according to his conception of the term, a variable is not itself a quantity, but it is *composed* of quantities. Similarly, we cannot define a variable as a number even when we restrict our inquiry to the numbers of pure mathematics. A variable is not *a* number but is constituted of a group of objects possessing the attribute of number.

Again, it will be noted that most of the definitions, from Biermann's to the most recent, define a variable as a *symbol* of some kind. That this is also an erroneous conception may be seen by arguments similar to those that we have just used. The letter x, by which the variable is identified, is a sign, or if taken as a functional instrument, a symbol,[64] to indicate the existence of a variable class of mathematical quantities, just as the word 'chair' is a sign, or a symbol, to designate a variable class of social implements. Mathematical symbols, like words, may be looked upon as linguistic units, acting as functional substitutes for a variable class of a certain kind. But the symbol is not the variable, nor is the variable a symbol. We must distinguish between mathematical symbols and what they denote, just as in everyday speech we distinguish between names and what they denote. Mathematics is a symbolic science, but it is not a science of symbols. Failure to recognize this fact has been responsible for many a wrong attitude towards mathematics.

Definition of 'variable'. The result of this discussion, then, is that the word 'variable', though conveying a definite meaning, must itself remain undefined. We may, however, define a variable, as

[64] We distinguish between sign and symbol and use the latter word, in the Aristotelian sense, as an instrument of communication. The words 'sign' and 'symbol' correspond to Aristotle's *semeion* and *symbola*. The latter word had a distinct social implication.

See Ogden, C. K. and Richards, I. A. *The Meaning of Meaning*. London and New York, 1927.

we define an angle, not by stating what it is, but by specifying its constitution. So we suggest the following statement:

An aggregate of mathematical quantities constitutes a variable. The variable can be identified with any of the number-values of the aggregate, and is usually symbolized by a letter, which may be used as a functional substitute for the variable.

In this definition the word 'aggregate' is used in its technical sense.[55]

Definition of 'function' Although the definitions of the term 'function', which we have cited above, have been framed to suit the particular types of investigation (*e.g.*, Wilson's for the theory of point sets), they show a striking similarity of form. All of them[56] give, implicitly or explicitly, the idea of *determinate correspondence within a certain domain.* Few of them specify any condition of *order*, but leave the order to be implied. The arrangement of the quantities of a variable in order is essential but it is a matter of convention. Any variable can be put into a functional relation with any other variable or variables by adopting a convention that will make the variables alike in order and put the quantities of the variable in determinate correspondence. Correspondence can generally be effected in more than one way.

We set down, then, as the essential characteristics of a functional relationship: *order* and *correspondence.* Thus when we state that the variables y and x are in functional relationship within a certain interval, we assert that, there is an order of the y's and an order of the x's, which may be brought into correspondence, so that when x has a quantity of a certain value, y has a corresponding quantity of a certain value. So we arrive at the following definition:

Two variables y and x are in functional relation when there is a determinate correspondence between the quantities x_1, x_2, x_3, . . . of the x variable and the quantities y_1, y_2, y_3, . . . of the y variable, the order of the arrangement of the quantities of the two variables being alike.

The function is defined, when the domain of the independent vari-

[55] In this definition of a variable we assume that a variable has at least two number-values. Cantor, in his Theory of Aggregates, asserts that a single element may constitute an aggregate.

See Cantor, G. "*Beiträge zur Begründung der transfiniten Mengenlehre.*" *Mathematischen Annalen*, Vol. XLVI, p. 484.

[56] With the exception of such a definition as "any expression containing x is a function of x," which is still popular in textbooks of algebra.

able and the rule which enables us to compute the corresponding values of the dependent variable are specified. It should be noted that we have given a definition of a functional relation rather than that of a function. Although the word 'function' must continue in use, it would make for clearer thinking in schools if the longer expression were used instead.

The specification of order and correspondence as essential qualities of a functional relationship suggests the idea of *correlation*. Russell employs this concept in his treatment of functions, when he states that, in specifying a function, "the independent variable is to be a *series*. The dependent variable is then a *series by correlation*, and may also be an independent series. For example, the positions occupied by a material point at a series of instants form a series by correlation with the instants, of which they are a function."[57] Thus the notion of correlation comes in naturally.

Examples of functional relationship. Before we leave this part of our subject it will not be out of place to insist that our best examples of functional relationship and correlation come through the study of concrete examples of physical change. When we state that the extension of a strained spring is a function of the tension applied, or that the distance traversed by a body falling from rest is a function of the time that has elapsed since it started or that the density of a gas is a function of its volume and pressure, we are expressing functional relationships between certain denominate quantities, which can actually be measured. If there is even an element of truth in Rignano's contention[58] that thinking is the mental execution of a series of experiments, it is important that the experiments should be such that the pupil will readily appreciate or sense them. Again, there is much to be gained by studying the relationships between the variables involved in a physical change, not as examples of cause and effect, as is so often done in the science classroom, but simply as examples of *functional correlation*. Viewed in this way even Mill's[59] canon of Concomitant Variations becomes but the expression of a functional relationship. School mathematics has suffered much in the past by being treated merely as the science of abstract symbolism. We urge that it be studied rather as the science of concrete relations.

[57] Russell, B. *Op. cit.*, pp. 264-65.
[58] Rignano, E. *The Psychology of Reasoning*, Chap. IV. London, 1923.
[59] Mill, J. S. *A System of Logic*, p. 263. London, 1884.

III

THE PSYCHOLOGY OF THE FUNCTION CONCEPT

Shortcomings of mechanistic psychology. A careful survey of the available literature on the psychology of mathematics reveals the fact that, while there is an abundance of reliable material on the abilities and skills of elementary mathematics,[1] there is a dearth of material on the psychology of mathematical reasoning.

Thorndike has analyzed both the nature and the constitution of arithmetical and algebraic abilities with characteristic insight and thoroughness, but his inquiries have been concerned with the acquisition of the technical skills of mathematics, rather than with the conduct of mathematical reasoning. Indeed, he draws no distinction, except of degree, between these two abilities, for he writes:

> Reasoning is not the negation of ordinary bonds, but the action of many of them, especially the bonds with subtle elements of the situation. Some outside power does not enter to select and criticize; the pupil's own total repertory of bonds relevant to the problem is what selects and rejects.[2]

In a later discussion of the same topic he confirms this view:

> These higher powers are in reality the coöperation of many connections of bonds selected and given proper weight for some purpose.[3]

Still later, he says:

> I conclude, therefore, . . . that there exists no fundamental physiological contrast between fixed habits and reasoning.[4]

[1] Thorndike, E. L. *The Psychology of Arithmetic.* New York, 1922. See also *The Psychology of Algebra.* New York, 1928.

See also Rugg, H. O. and Clark, J. R. "Scientific Method in the Reconstruction of Ninth Grade Mathematics." *Supplementary Educational Monographs,* Vol. II, No. 1, 1918.

Everett, J. P. *The Fundamental Skills of Algebra.* New York, 1928.

Symonds, P. M. "The Psychology of Errors in Algebra." *Mathematics Teacher,* Vol. XV, p. 93, 1922.

Schreiber, E. "A Study of the Factors for Success in First Year Algebra." *Mathematics Teacher,* Vol. XVIII, p. 154, 1925.

Judd, C. H. *Psychology of Secondary Education,* p. 107. New York, 1927.

[2] Thorndike, E. L. *The Psychology of Arithmetic,* p. 194.

[3] Thorndike, E. L. *The Psychology of Algebra,* p. 251.

[4] Thorndike, E. L. *Human Learning,* p. 160. New York, 1931.

This conclusion certainly possesses the great virtue of simplicity, but it is not a simplicity that stimulates the imagination. By scientific or logical simplicity we usually mean either simplicity in a generalization or formula, or simplicity in the elements of which the formula is constituted. Thorndike seems to have purchased simplicity in the former at the cost of greater complexity in the latter, for the familiar 'bonds', which were first presented to us as rather patient correlates of habits, now appear as functional agents, having the power, not only of discrimination, but also of coöperation in the production of abstract reasoning. Our elements seem to have grown more complex in our hands.[5] Such 'dispositional plasticity' we are prepared to recognize, but only when associated with a definitely conativistic conception of the mind. But we are at present concerned not so much with the nature or structure of the mind, as with the logico-psychological schema or pattern to which the mind, however we conceive it, conforms, when it is performing its highest functions.[6]

Examination of the concept of functionality by analogy. Let us begin our discussion with a brief examination of the logical bases of our subject. Here we shall make use of an analogy drawn from the main concepts of our thesis.

We have already seen that the concept of functionality has four main components or elements: class, order, variable, and correspondence. We say that two *variable classes* are in *functional relation*, when there is a determinate *correspondence* between the elements of the two classes, these elements being arranged in some prescribed *order*.

Class. In any mathematical discussion the class of the entities with which we are dealing is either specified or implied. When we specify our class to be, for example, 'the class of natural numbers (C)'. we have in mind a definite set of elements which we represent by the symbols: 1, 2, 3, ... We include in this class all natural numbers and exclude everything else; consequently, any

[5] We are reminded of a remark made by Eddington that "whilst it is reasonable to explain the complex in terms of the simple, this necessarily involves the paradox of explaining the familiar in terms of the unfamiliar."—Eddington, A. S. "The Meaning of Matter and the Laws of Nature According to the Theory of Relativity." *Mind*, New Series, No. 116, p. 145, Oct., 1920.

[6] Cf. Koffka, K. *The Growth of the Mind*, pp. 122-25 and Chap. IV. London and New York, 1925.

See also Rignano, E. *The Psychology of Reasoning*, p. 22. London and New York, 1923.

given number N either belongs to the class C or it does not so belong (The Law of the Excluded Middle.) In all classification there is a dichotomous separation, expressed or implied, of the class from the nonclass, or, in psychological terms, of 'figure' from 'ground'. In ordinary speech, however, a discrete classification of this kind is not always possible. When we classify a color as blue, we have in mind a more or less definite concept, which we designate 'blue', but we should have some difficulty in deciding when a range of color shades ceases to be blue, and becomes either indigo or white. A particular shade of blue will occupy a position or grade between these two extremes. Similarly, when we assert that a certain boy is clever, we place him in a class, the extremes of which are obviously difficult to define. In fact, it may almost be said of the boy that he does *and* he does not belong to the 'clever' class. Illustrations of this kind could be multiplied indefinitely, warning us that, when we have given an object a name, we have not necessarily particularized its character. Mathematical education should aim at obviating vagueness; it is a science of close specification as well as of exact procedures.[7]

Order. Now the elements of a class may be arranged in *order*, according to some specific attribute or quality, thus making possible subclasses and gradations of our main class. For example, the class of natural numbers may be arranged in ascending or descending order, or, again, as primes and nonprimes. Similarly, a class of students may be graded in order of merit, as A, B, and C, according to some arbitrary or accepted scale of evaluation. The ordering of a mental series is essential, if we are to reason about the series at all.

Continuity. As a special case of the concept of order we have that of *continuity*, which is fundamental in everyday logic as it is in mathematics. Russell has said, "The notion of continuity depends upon that of order, since continuity is merely a particular type of order."[8] The class of real numbers, the class of points of a line, the class of blues or greys, clever people or stupid people, all suggest the same thought, that of gradation and continuity within a

[7] This is not an argument against the Law of the Excluded Middle but against its thoughtless application. As J. M. Keynes has remarked: "The so-called fundamental laws of thought are to be regarded as the foundation of all reasoning in the sense that consecutive thought and coherent argument are impossible unless they are taken for granted." *Formal Logic*, p. 450. London, 1906.

[8] Russell, B. *Mysticism and Logic*, p. 91. London, 1921.

particular class.[9] We say that an ordered class is continuous, in the mathematical sense, if it is 'dense' and if it satisfies Dedekind's Postulate of Sequence.[10] It may be noted that this concept of continuity, which countenances no gaps, stands in contrast to that of the class itself with its discrete bounds. A continuous set of elements may, of course, be separated into discrete classes at will, as in the case of the *Dedekind schnitt* (Dedekind cut).

Again, we may note that specification within a continuous sequence implies a point of balance between two bounds or extremes. Sometimes these extremes carry the implication of *opposites* or of opposite classes or subclasses within a class, for example, in such terms as rich and poor, high and low, good and bad. Whether we judge a person to be rich or poor, depends, first, on the meaning we attach to these terms, and, second, on the degree to which the person approximates to one or other of the opposite extremes. In the ultimate issue, therefore, our estimate or judgment of a given quality will depend upon usage or experience. Schiller has said: "There is imposed upon every logic which aspires to be *more* than an artificial word-game, a far-reaching and unavoidable dependence on experience."[11] This is certainly true of the logic of life. Failure to recognize the existence of a continuity or a continuous sequence inevitably leads to confusion of thought. As illustrations from the realm of logic and psychology we may mention such antitheses as percept and concept, deduction and induction, analysis and synthesis, judgment and inference, individual and social, subjective and objective, which often lead us to useless disputations as to whether a particular example belongs to one *or* the other, whereas the real question in most cases should be: To what degree does *each* enter?

The importance of the concept of continuity is seldom recognized by writers on formal logic, who generally confine their attention to clear-cut logical classes and ignore the fact that, in the logic of life, such sharp dichotomous divisions are seldom manifested. Among philosophical thinkers Dewey, perhaps more than any other, seems to have taken special care to avoid this error. On almost every page of his *Democracy and Education*, to cite but one of his

[9] Whitehead, A. N. *The Axioms of Descriptive Geometry*, Chap. I. Cambridge Mathematical Tracts, No. 5. London, 1914.

[10] Young, J. W. *Fundamental Concepts of Algebra and Geometry*, p. 82.

[11] Schiller, F. C. S. "The Value of Formal Logic." *Mind*, New Series, No. 162, p. 46, Jan., 1932.

works, he has given practical expression to the concept of continuity, although he has not formulated that concept as a principle. A single quotation will suffice:

Any activity with an aim implies a distinction between an earlier incomplete phase and a later completing phase; it implies also intermediate steps. To have an interest is to take things as entering into such a continuously developing situation, instead of taking them in isolation. . . . The word interest suggests, etymologically, what is *between* — that which connects two things otherwise distant.[12]

Schiller, the most persistent exponent of the 'personal' method in logic, has defined Dewey's attitude as follows:

Dewey shewed that there was a glaring contrast between the theory of Formalism and the practice of the Sciences, between the precepts of Logic as to how men *ought* to acquire knowledge and the methods by which they actually succeeded in doing so.[13]

Limit. Associated with the concept of continuity is that of the *limit*. In this connection it is important to note that the modern conception of a limit does not specify whether or not a variable is *equal* to its limit. All that this conception requires is that the numerical value of the difference between the variable and its limit shall become and remain less than some arbitrarily assigned positive number, in other words, that the variable shall *converge* to the limit. We suggest that the mathematical notion of a limit has its counterpart in psychology. A concept, for example, may be looked upon as the limit of a variable class of percepts, and, being of the nature of a limit, it is not necessarily identical with any particular element of the class. Again, a causal series of events may be taken as an illustration of a doubly-bounded infinite sequence. Consider the causal series involved in the stimulus-response situation: kitty-kitty—meow-meow. Here we have a continuous series of events, spatial and temporal, with end-points in the stimulus and response, respectively. These end-points are mathematical limits of the series $S \rightarrow R$, to which there is no first or last term. What, then, is the first term of the stimulus or the last term of the response series? Again, what *is* the stimulus? Is it a single term of an infinite sequence or its summation? Surely the latter. The same is true of the response. Again, since the $S \rightarrow R$ series is

[12] Dewey, J. *Democracy and Education*, pp. 149, 161. New York, 1929.
[13] Schiller, F. C. S. *Op. cit.*, p. 53.

continuous, where is the point of separation of S and R? A little reflection will show that there is no point of separation. The stimulus is meaningless apart from the response and the response meaningless apart from the stimulus. The total situation $S \rightarrow R$ is a mathematical whole, with mathematical limits as its end-points.

Variable. Our third component of functionality is the *variable*, which we have defined as an aggregate of mathematical elements. Thus, the class of natural numbers constitutes a variable, which may be represented by a symbol identifiable with each and every element of the class. Similarly the word 'man' is a variable symbol, by which any one of the class of men may be identified. Names are symbols of the variables of human discourse. In mathematical logic the variable is identified with elements of a set of quantities, which, in the large majority of cases, vary or differ, but which may also be constant. So, in the logic of everyday use, one term may be used to denote an element of a varied or ordered sequence, another to denote an object of a constant class. Thus the word 'blue', as we have already seen, may do duty for a large variety of shades of blueness, but Wedgwood blue is, so we are told, the same the world over. Thus, to the word 'blue' a *quantitative estimate* may be applied. A particular blue may be estimated a dark or a light blue, somewhere in the interval indigo to Chinese white. This quantitative estimate is applicable to all logical classes which imply an ordered sequence or a continuity.[14,15] Bridges[16] has made practical use of this idea in her judgment of the emotional characteristics of young children. According to her, the emotional condition of an

[14] Ogden, R. M. *Psychology and Education*, p. 314. New York, 1926.
In his discussion of "Thinking and Reason," Ogden has suggested both the idea of gradation and that of quantitative estimate. "Steps will then appear in the gradient. . . . Whenever a serial order of steps appears, we have the possibility of quantification."

[15] Since the above discussion was written, the writer's attention has been drawn to a book entitled *The Technique of Controversy* by B. B. Bogoslovsky (London and New York, 1928). This book is a highly original and practical treatment of the logic of everyday thinking. Bogoslovsky's Principle of Polarity, his Principle of Partial Functioning, and his Principle of Quantitative Indices have a close resemblance to the principles suggested by us. They have been worked out with a wealth of detail that give great force to the main thesis. Bogoslovsky's terms are expressive of their inner meaning and could be applied, with advantage, to the concepts that we have defined.

[16] Bridges, K. M. *The Social and Emotional Development of the Pre-School Child*. London, 1931.

infant at any particular moment may be estimated quantitatively on the scale of excitement, between the opposite extremes distress and delight. Watson uses a similar scale ranging between fear and affection.

Correspondence. Finally, we come to the idea of a *correspondence* between variables, which we designate a *function.* As we have seen, the essential characteristic of a function, beyond that of the variable, is correspondence. Since every element of the variable class of natural numbers can be put into correspondence with the elements of another variable class of numbers, known as the 'squares' of the natural numbers, one class is a function of another. This correspondence is usually symbolized by an equation: $y = x^2$. The notion of function need not, however, be restricted to mathematical classes. As Russell has said: "The notion of *function* need not be confined to numbers, or to the uses to which mathematicians have accustomed us; it can be extended to all cases of one-many relations, and 'the father of x' is just as legitimately a function of which x is the argument as is 'the logarithm of x'."[17]

Some of the correspondences of pure mathematics are definite and precise, others involve the factor of probability, leading us to what is known as 'statistical inference.'[18] Correspondences of both types are met with in life, from the unequivocal statement of fact, to the more hypothetical inductions from experience. Dewey's "complete act of thought" is, in its ultimate analysis, a functional process of the second kind.[19]

The concept of correspondence naturally suggests the relation of *cause* and *effect*, which is not usually considered to be a mathematical relation. Nevertheless, the data upon which judgments of causation are based are strictly mathematical. Newton himself insisted on this point and maintained, in his *Principia*, that when enunciating formulas he was merely expressing correlations of observed facts. Russell maintains that there is nothing more in the notion of cause than a spatio-temporal continuity between sequent events[20] and he, therefore, recommends that the use of the term be abandoned. "The word 'cause' is so inextricably bound up with misleading associations as to make its complete extrusion

[17] Russell, B. *Introduction to Mathematical Philosophy*, p. 46.

[18] Keynes, J. M. *A Treatise on Probability*, p. 327. London, 1922.

[19] Dewey, J. *How We Think*. London, 1928 and 1933. See also *Democracy and Education*, p. 176. New York, 1925.

[20] Russell, B. *The Analysis of Mind*, Chap. V. London, 1928.

from the philosophical vocabulary desirable."[21] While we are unwilling to go to this extreme, we suggest that, as a first rule of scientific method, mathematical relations derived from physical observation should be interpreted as *correlations* rather than illustrations of preconceived relations of cause and effect.

Before we leave this part of our subject, let us note that, just as the concept of continuity brings into our terms of speech an element of uncertainty, or even of ambiguity, so it brings uncertainty into our judgment. The proposition, 'Mathematics is essential to the modern world', may be an acceptable conclusion to most teachers of mathematics, but it involves the variables, *mathematics, essential,* and *modern world*, the exact meanings of which few, if any, feel capable of defining.

One problem of practical interest is whether, in functional thinking, one type of logical pattern is dominant or not. It has been generally accepted in the past that the syllogism is, as Leibniz has expressed it, "a kind of universal mathematics," and that all mathematical thinking is syllogistic. Against this view there have recently appeared many dissentients. Brown, in an article, "Mind and Mathematical Ability," says, "One fact that has been definitely placed beyond doubt, by recent experimental investigation, is that men do not, as a rule, think syllogistically."[22] If, by syllogistic reasoning, he means the syllogistic form of argument, then he is undoubtedly right, but we submit that the syllogism may sometimes be found where its presence is not suspected. There is often an implied syllogism, even in apparently simple statements of fact. The statements: 'Newton was a genius', and 'Even Newton was fallible', will illustrate the point. Ballard seems to be on safer ground when he asserts that "Every attempt to press mathematical reasoning into the syllogistic mould has failed."[23] This is certainly true, for to do so would be to ignore all those mathematical truths that are the products of inductive reasoning. Induction is the method of discovering general truths; deduction the method of expounding them.

Induction and deduction. Several attempts have been made to determine which of the two methods of teaching, the inductive

[21] Russell, B. *Mysticism and Logic*, p. 180. London.

[22] Brown, W. "Mind and Mathematical Ability." *Mind and Personality*, p. 112. London, 1926.

[23] Ballard, P. B. *Teaching the Essentials of Arithmetic*, p. 10. London, 1928.

or the deductive, is the more efficient in the development of concepts. The issue is still in doubt. The evidence given in the studies of Winch,[24] Hull,[25] Fisher,[26] and Fowler[27] is certainly conflicting. The most recent of these studies is that of Fowler, who has maintained the superiority of the deductive method: "The results of the whole series of experiments have shown that the deductive method of teaching, where there is an explanation of the relationship to be taught and immediate reference to particular cases, is much better than the inductive one."[27] So many variable factors entered into this investigation, that the results can hardly be accepted as conclusive. One cannot help feeling that the inductive exercises were too difficult for the subjects to whom they were given.

In functional thinking[28] the deductive and inductive methods of reasoning are naturally and justly blended, for while the main objective is the establishment of a generalization, the process is tested and reinforced at all points by deductive inference.

Functional thinking involves a similar blending of the operations of analysis and synthesis; for functional thinking is creative thinking, and creative thinking cannot proceed without the exercise of both operations. Kant has said, "To separate is to unite." By separating the function into its elements, we tend to make it a unity. In this respect functional mathematics stands superior to the mathematics of the traditional type, which was, at least in its presentation, almost exclusively synthetic. Functional mathematics invites us to discover, rather than to verify truth.

The psychology of functional thinking. Let us now examine the psychology of our subject more closely.

In its ultimate analysis functional thinking is thinking in terms

[24] Winch, W. H. *Inductive versus Deductive Methods of Teaching.* Baltimore, 1913.

[25] Hull, C. L. "Quantitative Aspects of the Evolution of Concepts." *Psychological Monographs*, No. 123. 1920.

[26] Fisher, S. C. "The Process of Generalizing Abstraction." *Psychological Monographs*, No. 90. 1916.

[27] Fowler, H. L. "The Development of Concepts: An Investigation into Methods of Teaching." *British Journal of Educational Psychology*, Vol. I, p. 13, Feb., 1931.

[28] Note that we are using the term 'functional thinking' as the equivalent of 'thinking in terms of fundaments present as *variables* functionally related in the mathematical sense.' Although the view expressed in this chapter has certain resemblances to functional psychology (see Carr, H. "Functionalism." *Psychologies of 1930*. Worcester, Mass., 1930), the term 'functional thinking' is simply another expression for thinking mathematically.

of relations.[29] This does not place functional thinking in any exclusive category, for it may be said that *all* thinking is thinking in terms of relations.[30] The special character of functional thinking is the nature of its relations, which are, as we have seen, concerned with correspondences between variable classes. Functional thinking is not the special privilege of man, for it is true of functional, as of perceptual, thinking, that "the difference between the perceptions of a dog and the thoughts of a sage is a difference not in the nature of the process but in its range and complexity and in the materials with which it works."[31] Many interesting illustrations of this fact have been given in recent years by the exponents of the Gestalt psychology.

In discussing the subject of relational thinking we naturally turn to Spearman, who has made an exhaustive study of cognitive activity in all its aspects. Spearman has shown that all relational thinking can be reduced to the operation of three fundamental principles, which he styles "noegenetic principles."[32] The first principle states that one has an apprehension of one's own experience, or power, to a greater or less degree, to observe what goes on in one's own mind; the second principle (Eduction of Relations) states that "The mentally presenting of any two or more characters, simple or complex, tends to evoke immediately a knowing of relation between them";[33] and the third principle (Eduction of Correlates) states that "The presenting of any character together with any relation tends to evoke immediately a knowing of the correlative character."[34] It is with the second and third of these principles that we are more immediately concerned. Representing the relation R between two fundaments A and B in the symbolic form $A \leftrightarrow R \leftrightarrow B$, the principle of the eduction of relations states that, when A and B are given, R may be educed ($A \rightarrow R \leftarrow B$), and the principle of the eduction of correlates that, when A and R are given, B may be

[29] We may state that A is related to B, when A possesses an attribute which could not exist in the absence of B.

[30] Vailati and Russell have both pointed out that geometrical thinking is relational thinking. See Whitehead, A. N. *The Axioms of Descriptive Geometry*, p. 1. London, 1914.

[31] Nunn, T. P. *Education, Its Data and First Principles*, p. 207. London, 1930.

[32] Spearman, C. *The Nature of 'Intelligence' and the Principles of Cognition.* London and New York, 1923. (Second edition, 1927.) See also *The Abilities of Man: Their Nature and Measurement*, Chap. XI. London and New York, 1927.

[33] Spearman, C. *The Nature of 'Intelligence' and the Principles of Cognition*, p. 63.

[34] Spearman, C. *Op. cit.*, p. 91.

educed ($A \rightarrow R \rightarrow B$). The ability to continue an arithmetical series 2, 4, 6. . . . requires the exercise of both types of eduction. In the first instance, R is a function of A and B and in the second, B is a function of A and R.

The fundaments, or raw materials, of the relations with which we are here concerned are certain mathematical elements and relations derived from these elements. In mathematical thinking these fundaments and their cognized relations become organized into mental patterns, or *schemata*, more or less stable, more or less clearly abstracted. In the lower forms of life schemata are used only in immediate perceptual situations; in the higher forms the schemata have greater stability and can thus be successfully directed towards an absent or remote situation. The ability of man to retain schemata, when removed from their perceived fundaments, may be justly regarded as an index of his intelligence. A child of low intelligence cannot see that, *if* A is as tall as B, and B is as tall as C, *then* A is as tall as C, not because he is unable to cognize the relations between the fundaments in pairs, but because, while cognizing the relationship of B and C, he cannot *carry* and integrate the relationship of A and B into the new schema. It is only when A and C are placed in juxtaposition, either concretely or imaginatively, that the new schema is established.

When two comparable fundaments are presented to us, we immediately proceed to educe a relation of some kind between them. This relation may be educed with varying degrees of abstraction, from the vague cognition that there is a similarity or a difference between the fundaments to conscious discrimination and definition. For example, one child on being presented with the series, 5, 8, 11, . . . , said, "Getting bigger"; another remarked, "Going up by threes." Again, some children, on being presented with the sequence, 1 2 3, 2 3 4, 3 4 5 . . . , merely sensed the rhythm of the number-grouping; others, in addition, cognized the actual number relationships. In the vast majority of relation-schemata various degrees of particularization are possible. This is of the greatest importance in teaching.

It is important to note that relations may be usable although not completely abstracted. This was one of the most interesting results of the tests given in Appendix A. Many of the pupils tested, when asked why they had continued the series, 2, 4, 6, . . . , with the numbers 8 and 10 (in arithmetical progression), and the series, 2, 4, 8, . . . , with the numbers 16, 32 (in geometrical progression), remarked, at first, that they did not know.[33] But they were able to *use* the correct relation. This ability to use relations that have not been fully abstracted has been noticed by Spearman, who writes:

> Coöperation between the preparatory phase of obtaining relations and the applicative phase of educing correlates becomes especially intimate and obvious when the transition from the one phase to the other occurs in a direct manner. This happens whenever the relation—although still, as ever, the vital factor in the whole process—nevertheless does not happen to arrive at the stage of being abstracted from its fundaments (or even perhaps from individual occurrence at all). Despite thus remaining still embodied in certain concrete and particular cases, it nevertheless can already be applied to further cases no less concrete and particular.
>
> The possibility of such direct transference has been already noticeable in all our classes of experimental examples. Most prominent of all in this respect was the test of Analogies. In ordinary life, also, the most conspicuous instances are those designated as 'inference by analogy.' To this class must be ascribed almost all pre-scientific deduction and even conduct. By its means men must have learnt to seek shelter on seeing the sky grow black, countless ages before they could formulate any abstract relations of meteorology.[34]

As a general rule, we may state that when two fundaments are presented and maintained together the relations or schemata tend to undergo progressive clarification. This is analogous, in the psychology of perception, to the Gestalt Law of Prägnanz (pregnancy), that perceptual configurations tend to become more and more sharply defined. The progressive clarification of a relation may imply a form of mental maturation, whereby the relation becomes gradually and unconsciously strengthened, or that the state of abstraction is, in the true mathematical sense, the limit of a series of cognized relations which have become progressively more complex. The interpretation of a graph offers an excellent illustration of progressive clarification, from the elementary cognition of geometrical form to the complete analysis of the equation of the graph. Spearman maintains that "abstraction is the climax of eduction":

> The explanation of the whole matter, then, seems to be that all cognitive growth—whether by eduction of relations or by that of correlates—consists in a progressive clarification; the mental content emerges out of a state of utter indistinguishability and ascends into ever-increasing distinguishability. So soon as any item of mental content has become sufficiently clear and dis-

[33] Examination showed that they had not memorized the geometrical series at any time previously.

[34] Spearman, C. *Op. cit.*, p. 104.

tinguishable, then and then only it admits of being abstracted; that is to say, it can be 'intended' apart from its context. And when this happens, it can be thought of separately and given a name.[37]

The end of abstraction is, then, to reinforce the schema and to convert it into a more stable configuration, thereby enabling it to serve as a fundament for higher schemata. Thus the schema becomes a 'disposition pattern', possessing a certain purposive integrity. In the process of abstraction the concept acts as if it had acquired a momentum in the direction of its clarification. As it continues to function in this direction, it becomes still further clarified, and the fundaments from which it was derived acquire a fuller and fuller meaning. This is what Dewey means when he says that a concept is essentially "operational."

Now, although the process of clarification is unidirectional, it gives rise to a reverse reciprocal action. It is as if the fundaments A and B were held by tensile forces to the relation R in such a way that the abstraction of the relation from the fundaments tended to produce a compensating stress towards them. Suppose, for example, that A and B are related through R according to the formula $A \leftrightarrow R \leftrightarrow B$, and that subsequently the abstracted relation R is applied to new fundaments A' and B' to give us $A' \leftrightarrow R \leftrightarrow B'$, then, not only will the relation F be strengthened in the new situation, but the original fundaments A and B will also tend to be reciprocally strengthened. Such a reciprocal activity may be conceived to be the first half-period of a sustained *oscillation*. Thus, by a process of mental interlacing, the whole series of schemata becomes integrated, each new set of related fundaments acting reciprocally on all that have preceded it.

This principle of reciprocal activity is of fundamental importance in functional thinking. In the endeavor to relate two sets of variables the mind is in a state of tension or indetermination between the elements of the variables and oscillates between them, until the law of correspondence has been abstracted or established. This state of indetermination is not blankness but active and creative thinking. Functional thinking involves a two-fold operation of analysis: one we shall describe as a horizontal analysis, an analysis of the order of the variable itself, and the other as a vertical analysis, or an analysis of the correspondence between the ordered variables. Illustrations of these two procedures will be given later.

[37] Spearman, C. *The Abilities of Man*, p. 216.

So far we have been dealing with the structural aspect of our subject, our main purpose being to study the schematic form of functional thinking. As we have already indicated. schemata are to be regarded not merely as structures having a static configuration but rather as dynamic entities having a directional or purposive integrity. The best physical analogy we can give is that of the gas vortex ring, which not only has a visible structural form, but also possesses intrinsic kinetic energy. This energy becomes manifest only when the ring reacts to an external stimulus. The efficiency of schemata depends partly upon their complexity and stability and partly on their purposive or purposeful energy, or on what McDougall has called "the selectivity of the desire."

Psychological aspect of the reasoning process. We now turn our attention from the schemata themselves to their mode of activity in the conduct of functional thinking.

As we have already indicated, the term 'function' may be used either to denote an *operation* or to express a *correspondence*. In mathematics the word is used almost exclusively to define a correspondence between two or more variables. When we speak of the function of a mathematical sign or symbol, we have in mind the use of such a sign or symbol employed as an operator: when we state that y is a function of x, we are expressing the fact that there is a determinate correspondence between all the number values represented by the symbol y and those represented by the symbol x. In either case, the word 'function' has a dynamic implication. This is obvious when the word is used to indicate an operation. but a little reflection will show that, even when used in the second sense, the notion of activity is implicit, for correspondence requires, first, the *arranging* of the elements of our variable in order. and. second, the *placing* of corresponding elements together to determine their relationship. Function is thus an operation or experiment in thought.

Rignano. in his study, *The Psychology of Reasoning*. maintains that all reasoning is nothing else than a series of operations or experiments performed in the mind, and that the logical process of thought is identical with perceptual reality itself. "To think of making an experiment allows us to perform in imagination, with very great rapidity, not only this single experiment but a very great and practically infinite series of experiments, mentally varied in certain of their conditions and so to verify that they all give the same result."[38] This imagined experiment he calls, following Mach, a "thought experiment" (Gedanken experiment). Reasoning may be regarded, in a very real sense as the process of observing or 'seeing' relationships.

Rignano applies this idea of the thought experiment to the whole range of mathematical reasoning. "Reasoning," he says, "in addition to being a true thought experiment takes further the form of determinate *classificatory operations* (comprising inclusions, adjunctions, intersections, etc., of classes) performed upon materials already produced and presented to the mind by the preceding creative act of the combining imagination. The form of deduction by means of operations performed upon classes to which any reasoning whatever can thus be reduced is nothing else than a kind of cataloguing of the results of determinate experiments *after these laws have already been mentally performed by the combining imagination*."[39] According to Rignano, any form of deduction is a *static* mode of regarding the products of a *dynamic* process.

Miller, nearly fifteen years before, had expressed a similar thought when he regarded "the act of thought as the preliminary imagining in advance of all the results of one of our particular modes of procedure."[40] Others have also emphasized the dependence of pure thought on prior perceptual experience. Mach says: "It is erroneous to assert that the straight line is recognized as the shortest line by *mere visualization*. It is true that we can reproduce in imagination, with perfect accuracy and reliability, the simultaneous change in form and length which the string [line] undergoes. But this is nothing more than a reviviscence of a prior experiment with bodies, *an experiment in thought*."[41] We find the same idea in the work of Hollingworth, who regards thought as the use in the solution of problems of substitutes or symbols for the real objects and processes that belong to the problem. The symbol may be 'an

[38] Rignano, E. *The Psychology of Reasoning*, p. 84. London and New York, 1923. This book is an elaboration of a paper: "Sur la methode d'enseignement des mathematiques et des sciences pour la formation du futur maitre." See also *Revue de metaphysique et de morale*, May, June, 1910.

[39] Rignano, E. *The Psychology of Reasoning*, p. 196.

[40] Miller, I. E. *The Psychology of Thinking*, pp. 133-34. New York, 1909.

[41] Mach, E. *Space and Geometry in the Light of Physiological, Psychological and Physical Inquiry*, p. 62 (translated by T. J. McCormack). Chicago, 1906. For a criticism of Mach's philosophy, see Cohen, M. R. *Reason and Nature*, p. 40. London, 1931.

object, a word, a diagram, a mathematical symbol, or a mental image'.[42]

Dewey has made the thought experiment the basis of his philosophy of instrumentalism. In his *Quest for Certainty*, he says, "All conceptions, all intellectual descriptions, must be formulated in terms of operations, actually or imaginatively possible." And he proceeds:

The significant difference is that of two types of possibility of operation, material and symbolic. This distinction, when frozen into the dogma of two orders of Being, existence and essence, gives rise to the notion that there are two types of logic and two criteria of truth, the formal and the material, of which the former is higher and more fundamental. In truth, the formal development is a specialized offshoot of material thinking. It is derived ultimately from acts performed, and constitutes an extension of such acts made possible by symbols, on the basis of congruity with one another.[43]

Piaget distinguishes between two types of thought experiment. The first is merely productive, being the repetition in imagination of experiences that have actually occurred; the second is productive or creative and involves the abstraction of an extrinsic relation and its application to new and varied cases.[44] But the difference is one of degree rather than of kind. In functional thinking both types of thought experiment are active.

Our purpose in quoting these authorities is partly to uphold the thesis that mathematical reasoning is ideal experiment, and partly to support the view that school mathematics should be taught as the symbolic expression of actual or potential activity: in other words, that school mathematics should be presented as a concrete and dynamic, rather than as an abstract and static, science. We do not think that the function concept can be grasped by the average student in any other way. Mathematics is the projection of *life* upon the plane of human imagination.

Ideal relations. Now, the conduct of experiment involves the following main activities: the collecting of data, the arrangement of the data according to some attribute or quality, the identification of the data by name, and the interpretation of the data so organized in terms of relations or correlations that have been found to subsist between them. In other words, experiment involves the

[42] Hollingworth, H. L. *The Psychology of Thought*, pp. 4, 11. New York, 1926.
[43] Dewey, J. *The Quest for Certainty*, p. 160. New York, 1929.
[44] Piaget, J. *Judgment and Reasoning in the Child*, p. 235. London and New York, 1928.

recognition of the mathematical concepts: class, order, variable, and correspondence. These concepts are implicit in Spearman's "ideal" relations, those relations which we recognize as universal, or which we associate with all types of fundaments. Spearman's ideal relations are those of *likeness, conjunction,* and *evidence.*[45]

The simplest of these ideal universal relations is that of *likeness.* It is important to note, however, that fundaments may show the relation of likeness when taken in one context and unlikeness when taken in another. For example, real numbers are alike in one context (as real numbers) but unlike in another (as rationals and irrationals). Similar triangles are alike as to shape, but, in general, they will be unlike as to size. It is inevitable that the terms 'like' and 'unlike' should be used somewhat ambiguously, but care should be taken, especially in elementary mathematics, to qualify the terms whenever they are used.[46]

The relation of likeness gives us the concept of a *class.* Mathematics begins with the formation of classes of elements that are mutually alike. And just as the mathematical fundaments of which a variable is constituted may be alike or unlike in different contexts, so the same fundaments may, on different occasions, be constituents of different classes. In view of the fundamental importance of the class in mathematics, particularly in the development of the idea of the variable, we have given special attention to it.

The first, and perhaps most important, characteristic of mathematical classes is the possibility of arrangement in *order,* which we have already specified as order of value, order in space, and order in time. The concept of the variable is possible only when ordered sequences exist. Order in time is usually accepted as intuitive. For us, as for Newton, time flows uniformly. But it flows uniformly because events succeed one another uniformly; in other words, because events in space and time are correlated. Guyau claims that the concept of order in time is not temporal but spatial. "In truth, when we localize an event in time, we attach our points of reference to space, and our short-cuts are in reality spatial short-cuts, representations of mental pictures, with the distances vaguely imagined, which receive definiteness by means of number. Our

[45] Spearman, C. *The Nature of 'Intelligence' and the Principles of Cognition,* p. 72.

[46] The relation of likeness between sets of numbers and geometrical forms is brought out in several of the tests, especially in Nos. 5 and 6.

representation even of time itself, our image of time, is in a spatial form."[47] There is much truth in this statement, for as Bergson has said, "Man invariably spatializes time." It is important to note, however, that to conceive time in spatial form requires the ability to conceive space, and also number, in ordered sequence. In the course that we have outlined we begin with the arrangement of numbers or things in order of value or magnitude (tabular arrangement). We then represent this arrangement as order in space (graphical arrangement). At a later stage we represent order in time, both in tabular and in graphical form. Thus tabulation and graphical representation enter as essential processes in the development of the concept of the variable.

Closely connected with the idea of order is that of *rhythm*, which we may define as an ordered sequence of contrasts in number, space, and time. Sonnenschein defines rhythm as follows: "Rhythm is that property of a sequence of events in time, which produces on the mind of the observer the impression of proportion between the durations of several events or groups of events of which the sequence is composed."[48] Whitehead has given an important place to rhythm in the stabilization of reason. "The rhythm of life is not merely to be sought in simple cyclical recurrence. The cycle element is driven into the foundation, and varieties of cycles and cycles of cycles, are elaborated. The cycle is such that its own completion provides the conditions for its own mere repetition."[49] According to Whitehead, the function of rhythm is to delay fatigue and thus to facilitate reason. The attractiveness of rhythm was well illustrated in the remarks of the pupils who took the tests described in Appendix A. In the oral examination with Test I every pupil noticed the rhythmical character of the numbers and several remarked that they were easily able to continue the series when they found that the numbers were "like the beat of a drum," "like dancing," "like music." The satisfaction derived from the detection of the rhythm was evident. In most cases the pattern of the rhythm functioned as a conceptual unit. The apprehension of the groups in rhythmical form seemed to facilitate thought by giving stability to the fundaments. There is no doubt that we have lost much in mathematical teaching by not appealing to rhythm more frequently.

[47] Guyau, J. M. *Genèse de l'idée de temps*, p. 69. Paris, 1922.
[48] Sonnenschein, E. A. *What is Rhythm?* p. 16. London, 1925.
[49] Whitehead, A. N. *The Function of Reason*, p. 17. Princeton, 1929.

In the course that we have outlined we have not specified where advantage may be taken of the child's rhythmical sense. It may be noted, however, that illustrations of rhythm are to be found, not only in the subject matter of mathematics, but also in the mode of mathematical expression. For example: When comparing two similar triangles ABC and XYZ, we write $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$ rather than $\frac{AB}{BC} = \frac{XY}{YZ}$. The former, in addition to being more comprehensive, is more easily grasped, partly because it is horizontally cyclic and partly because it is vertically rhythmic.

The second of Spearman's ideal universal relations is that called *conjunction*, the relation which is expressed in the act of combination and the obverse act of division. This relation is the basis of the concept of quantity, whether concerned in counting or in measuring. It presupposes some relation of likeness, for quantities can only be combined or conjoined, when they possess like qualities. We may attach a meaning to the summation 1 ft. + 1 yd. but we cannot give meaning to 1 ft. + 1 hr., except by an artifice. This restriction to conjunction has to be borne in mind in all number theory, for example, in dealing with complex numbers or in vector algebra. The Hankel-Schubert Principle of Permanence is a recognition of the possibility of such impermanence of operation. The relation of conjunction, as Spearman has remarked, "when taken with its obverse aspect of division, as also its special case of ordination . . . supplies the whole basis of arithmetic (including algebra) and half that of geometry."[50] It is the foundation of all summation and integration, and, incidentally of the concept of the mean. In its obverse aspect it implies the operations of division and differentiation. Thus the relation of conjunction is the basis of mathematics as a calculus.[51]

The third of Spearman's ideal universal relations is that of *evidence*. Spearman is careful to distinguish between the cognizing of a relation *by* evidence and the cognizing of a relation *of* evidence.[52] The former type of cognition is inherent in all relational thinking; the latter, as Spearman has observed, "especially

[50] Spearman, C. *Op. cit.*, p. 72.

[51] We suggest that mathematics should be viewed from three fundamental standpoints: (1) as function, (2) as a calculus, and (3) as a logic.

[52] Spearman, C. *Abilities of Man*, p. 168. See *also The Nature of 'Intelligence' and the Principles of Cognition*, p. 72.

belongs to reasoning." The important character of the cognition of a relation *of* evidence is not the schema employed in reasoning, but the mind's critical analysis of the schema's functioning. In cognizing a relation *of* evidence, the mind is critically examining its own processes. This examination is, in substance, a thought experiment, based ultimately on the inexorable law of nature that two bodies cannot occupy the same position in space at the same time. And if they cannot occupy the same position in space they have an order in space, and an order in space makes both geometry and physics possible. The law of impenetrability, which forbids two bodies to occupy the same position in space is the simplest expression of the law of causation. Claremont has made the direct perception of causation the fundamental factor in intelligence. He writes, "The ultimate intelligence factor is the power to become aware of the necessity in the very nature of things of certain causal relationships,"[53] and, again, "Intelligence is a faculty by which the mind becomes aware of such inexorable interconnections between things and events, and that in the 'intelligent activity' such awareness is made use of (*i.e.*, it is a factor producing or modifying such activity)."[54] A similar thought is contained in a statement of McDougall that "'Insight' is the grasping of, or intuition of, relations, more especially relations of time, space, and causality."[55]

The beginning of, and the incentive to, reasoning is incompleteness. There is a gap to be filled, a deficiency to be made good, an incompatability to be rectified, or an inconsistency to be put right. Such a sense of incompleteness is a call to action, to complete, fulfil, rectify, or put right, as the case may be.

Application of the principles. The bearing of the principles that we have just reviewed on the psychology of functional thinking will now be obvious. Let us suppose that we have two rows of corresponding numbers, which we label x and y rows.

x	1	2	3	4	5
y	1	4	9	16	25

From an examination of the numbers in the x row which we shall designate a horizontal analysis, we educe certain relations, such

[53] Claremont, C. A. *Intelligence and Mental Growth*, p. 25. New York, 1928.
[54] *Ibid.*, p. 31.
[55] McDougall, W. *Modern Materialism and Emergent Evolution*, p. 37. New York, 1929.

as 'increasing order', or 'progression', or 'going up by one's', the degree of particularization depending on the efficiency of our relational cognition. Then, from an examination of the y row (horizontal analysis), we educe other relations, such as 'increasing order', or 'increasing, but not uniformly', or 'all squares'. Now from an examination of the numbers in the same columns, which we shall call a vertical analysis, we educe the relations $4 = 2 \times 2$ or $4 = 2^2$; $9 = 3 + 6$ or $9 = 3^2$; and so forth. Finally, we use the relations themselves as fundaments, and educe the general relation that numbers in the y column are the squares of corresponding numbers in the x column. We express this relation in the form of an equation, $y = x^2$.

At first sight, the above reasoning seems to employ only the Principle of Relation Eduction, but a closer scrutiny of the mental processes involved will show that this is not so. The Principle of Correlate Eduction cannot be eliminated from the examination either of series or of correspondence. When we note, for example, that 4 is a perfect square, we carry this relation in a *predictive* fashion to our next number, 9. Again, when we note that $4 = 2^2$, we anticipate and then verify the fact that the relation $9 = 3^2$ will follow. Thus our procedure conforms exactly with Dewey's description of "a complete act of thought," for at each stage of our reasoning we carry with us a tentative hypothesis, which we verify, elaborate, or modify as we proceed. Into this problem the trial-and-error form of thinking naturally enters, but the hypotheses are so well controlled that the experience may truly be described as reflective thinking.

Before we leave our first example, let us note that the principle of correlate eduction is implied in the generalization, $y = x^2$, for this equation implies the possibility of enlarging our table of corresponding numbers by interpolation and extension. In this way we are able to elaborate new correspondences, all of which follow the law of relationship given in the following data:

x	1.5	2.5		6	7	
y	2.25	6.25		36	49	

Graphical Representation. Second, graphical interpretation of function illustrates relation-correlate thinking. The statement that two variables, x and y, are connected by the relation $y = x^2$ conveys little of its inner meaning to the student who has not been taught to interpret the relation through its graphical form. The graph may be regarded as a *mathematical symbol*, expressing, in its entire form, relations which would otherwise be only vaguely sensed or which would altogether escape notice. Thus the graph exists not merely to illustrate, but to add meaning to, the function which it symbolizes. Like all mathematical symbols, the graph is a functional substitute for a class, usually a large class, of arithmetical relations. That being so, it should be valued for the light it can throw on the properties of functions, rather than for its own intrinsic merit of beauty.

With the aid of the graph we are able, without undue fatigue, to apprehend the functional correspondence of related variables. Just as logical thought may be looked upon as the correlate of a series of experiments, potentially performed, so the graph may be regarded as a concrete entity kinaesthetically sensed. Rignano maintains that this is true of all symbols, "Behind symbolism we can see the content of quantitative, we should almost say, tangible, operations or experiments, thus symbolized."[56] In the experience of most people the graph has a dynamic rather than a static significance; it is a *path* along which the thinker moves as he transfers his attention from point to point. The cognition of graphical relation resolves itself into the kinaesthetic imaging of potential motion.

Motion and the function concept. This leads us to our third example of functional thinking, the study of motion itself. Whenever we envisage the motion of an object, we are thinking in functional terms, for the concept of motion is a two-variable (space-time) concept. It is not difficult to see that the principles of relation and correlate eduction are both active when an observer watches a moving object. He not only notes that the object is 'here' and 'there' at successive instants, but he also anticipates or predicts the position of the object at later instants. When a marksman directs his aim ahead of a moving target, he is thinking in terms of relation-correlate eduction. He fires at the target in the expectation that the space-time relationship already observed and estimated will continue in operation. The study of motion leads us to some excellent examples of functional thinking and forms a natural starting-point for the idea of 'rate of change' or the concept of the 'derivative'.

We have already symbolized the eduction of relations by the

[56] Rignano, E. *Op. cit.*, p. 195.

formula $A \rightarrow R \leftarrow B$ and the eduction of correlates by $A \rightarrow R \rightarrow B$ where A and B are fundaments, and R the relation subsisting between them. We may now symbolize functional thinking by the formula $x \leftrightarrow F \leftrightarrow y$, where x and y are no longer single number-values but variables. This formula indicates that functional thinking is *oscillatory* in character, for not only is y a function of x, but, reciprocally, x is also a function of y. This is true of the process of formulating functions, as will be seen from our first example, for the eduction of the function required, first, the analysis of both sets of numbers as variables (horizontal analysis) and second, the comparison of corresponding numbers of the two variables (vertical analysis). The latter process is a two-way oscillatory activity, for the x and y rows are interrelated. It is this oscillatory character of functional thinking that gives it its peculiar charm.

Examination of the general aims. In the introduction to this study we summarized the aims of mathematical teaching under three main heads—utilitarian, cultural, and disciplinary. We have now to inquire: How far are these aims capable of realization through functional mathematics? Of the utilitarian value of functional mathematics there can be no question. The function is the mathematical correlate of physical change, expressing in symbolic language the relationships that accompany change in the physical world about us. One may say that the utilitarian value of functional mathematics is commensurate with the utilitarian value of physical progress.

The cultural value of functional mathematics can hardly be considered apart from that of mathematics in general. David Eugene Smith has appraised the value of mathematics in an article both eloquent and profound. He shows that mathematics did not come into being to satisfy utilitarian needs. "It seems rather to have had its genesis as a science in the minds of those who followed the courses of the stars, to have had its early applications in relation to religious formalism and to have had its first real development in the effort to grasp the Infinite."[57] That mathematics has beauty has not been generally appreciated. Few who are not devotees would be ready to claim that "the true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in

[57] Smith, D. E. "Mathematics in the Training for Citizenship." *Third Yearbook, National Council of Teachers of Mathematics*, p. 18. 1928.

poetry. . . ."[58] If beauty is an attribute of culture, then cultural mathematics can claim to provide, in a special degree, the food of culture, for order, rhythm, symmetry, harmony, and unity, which are among the accepted qualities of beauty, are also among the popular concepts of functional mathematics.

That mathematics has disciplinary value some are disposed to doubt. They maintain that the days when it was believed that accuracy, judgment, and reasoning were specific abilities that could be developed by appropriate training and discipline have long since passed away. Others, while admitting that the doctrine of formal discipline has been rudely shaken, still cling to the belief that there is something in "the human worth of rigorous thinking." This subject was considered to be a matter of such vital importance that the National Committee on Mathematical Requirements made a special study of "The Present Status of Disciplinary Values in Education," giving the results of experiment and inquiry up to the year 1922. Many experimental studies of this subject have been made since 1901,[59] when Thorndike and Woodworth undermined our faith in the doctrine of transfer.[60] Some of these have been vitiated by faulty technique, others have been discounted because they do not touch the real problem. Those that may be considered valuable and unobjectionable as to procedure seem to point to a conclusion which may be summed up in the words of Burt: "A common element is more likely to be usable if the learner becomes clearly conscious of its nature and of general applicability: active or deliberate transfer is far more effective and frequent than passive, auto-

[58] Russell, B. *Mysticism and Logic*, p. 61.

[59] Whipple, G. M. *Twenty-Seventh Yearbook of the National Society for the Study of Education*, Part II, pp. 186-97. Bloomington, Ill., 1928.

Sandiford, P. *Educational Psychology*, pp. 279-89. London, 1928.

Orata, P. D. *The Theory of Identical Elements*. Columbus, Ohio, 1928.

Inglis, A. *Principles of Secondary Education*. New York, 1918.

Betz, W. "The Transfer of Training, with Particular Reference to Geometry." *The Fifth Yearbook, National Council of Teachers of Mathematics*, p. 149. 1930.

See especially Meredith, G. P. "Conscious of Method as a Means of Transfer of Training." *Forum of Education*, Vol. LXXVII, Feb., 1927. London.

Woodrow, H. "The Effect of the Type of Training upon Transference." *Journal of Educational Psychology*, Vol. XXIII, March, 1927.

Johnson, Elsie P. "Teaching Pupils the Conscious Use of the Technique of Thinking." *Mathematics Teacher*, Vol. XVII, April, 1924.

[60] Thorndike, E. L. and Woodworth, R. S. "The Influence of Improvement upon the Efficiency of Other Functions." *Psychological Review*, Vol. VIII, pp. 247-61; 384-95; 553-64, 1901.

See also Thorndike, E. L. *Educational Psychology*, Vol. II, pp. 350-434.

matic, or unintentional transfer. This seems especially true where the common element is an element of method rather than of material, an ideal rather than a piece of information."[61] In other words, method of procedure, consciously accepted as a desirable ideal, is the key to the problem of transfer.

The bearing of this conclusion on our present problem will be obvious, when we remember that functional mathematics is neither a technical skill nor a formal discipline but *a mode of thinking.* It may, and often will, involve many skills, but its real domain is to be found in the concomitant changes of correlative variables and the relations that subsist between them. If we seek the material of transfer in the common elements of mathematics and life, we find them in the concepts: class, order, variable, relation, correspondence, correlation, and function.

Universality of functional thinking. Although it is the main purpose of this study to discuss the place of the function concept in mathematical education, it will not be entirely irrelevant to suggest that this important concept can be extended to the teaching of other school subjects. The criticisms that have been levelled against school mathematics may be directed, in varying degrees of accuracy, against all the other school subjects. Some of these subjects possess the virtue of being more obviously useful, but it cannot be denied that, as far as methods of instruction are concerned, they are in no better case than mathematics.

The methods of functional thinking are universal in their application. In the teaching of the physical and biological sciences, the applications of the function concept are obvious. In the biological sciences the concepts of class or type and order are fundamental. Biology is the study of dynamic types. We may note, further, that to observe and interpret the corresponding changes in the associated variables of Nature is one of the recognized methods of science. The scientific method is the function concept in action. The boy who observes and records the daily growth of a plant, or who records the concomitant variations in the pressure and volume of a given mass of gas, is thinking in functional terms. In the teaching of geography, the notion of functionality is no less important. Geography is no longer conceived as the study of the earth, but the study of man upon the earth; it is the study of man in relation

[61] Burt, C. L. *Formal Training: The Psychological Aspect*, p. 3. Report of a Committee of the British Association for the Advancement of Science. Bristol, 1930.

to his whole environment, immediate and remote. Here again we have two variables functionally related.

At first sight, history seems to be less accommodating to our general thesis but, if history means anything at all, it surely means the study of the relations between the immediate present and the historical origins of the present. Events on the historical time flux or continuum are functionally related, and should be studied not merely as events, but as events related to other events. The main aim of history teaching should be to study the relationships between the complex present and its less complex origins in the past; in other words, to study our social evolution. Such a study of history might well include "number stories of long ago."[62] Note, too, the reciprocal effect of such a procedure, for the process of reconstructing the past by an examination of the present not only enlarges our conception of the past but intensifies our understanding of the present.

Even less obvious is the application of the function concept to the teaching of the mother tongue, but a little reflection will show that the possibilities of using this concept in the teaching of composition and functional grammar are almost as great as in the teaching of mathematics. As a simple example, let us place side by side two sets of words:

red, glowing, big, sharp, blue, pretty
knife, flower, girl, sky, stick, sun

Here we have two variable classes, adjectives and nouns, which can be associated by a one-to-one correspondence, so that to each adjective there corresponds, though not uniquely, a noun. When we ask the pupil to select adjectives appropriate to each noun, or nouns appropriate to each adjective, we are giving him an exercise in the discrimination and the appreciation of words, which is of the greatest value. Many combinations are possible but there will be only one that the pupil will himself judge to be *the best*.

These remarks will be sufficient to indicate a general principle of instruction, the possibilities of which have never been adequately explored.

[62] Smith, D. E. *Number Stories of Long Ago*. New York, 1919.

IV

THE HISTORY OF THE FUNCTION CONCEPT

Introduction. In this chapter we propose to trace the main currents in the development of the function concept[1] in school mathematics. Although many references to functions are to be found in textbooks and pedagogical journals published in the latter part of the nineteenth century, it was not until the beginning of the present century that functional thinking became recognized as a concept of vital importance. Our discussion might well begin with the year 1900. It may not be unprofitable, however, to review the events that led up to the demand for the function concept in school mathematics. It will be found that, in all the countries with which we shall be concerned, the ground was being prepared for the concept of functionality by the addition of graphs and even of the elements of analytical geometry and the calculus to the school programs.

[1] This chapter is not intended as an exhaustive study of the subject. An effort has been made to select only the personalities and events which seemed to be in the main stream of development. The material for this survey has been gathered chiefly from the following works:

Commission Internationale de l'Enseignement Mathématique, Sous-Commission Française. *Rapport.* See especially Vol. II, "*L'Enseignement secondaire,*" by C. Bioche.

Internationale Mathematische Unterrichts-Kommission. *Abhandlungen über den mathematischen Unterricht in Deutschland.* F. Klein, editor. *Die höheren Schulen in Norddeutschland.* Band I.

Klein, F. *Vorträge über den mathematischen Unterricht an den höheren Schulen.* Leipzig, 1907. Bearbeitet von R. Schimmack. (Usually referred to as Klein-Schimmack.)

Lietzmann, W. *Methodik des mathematischen Unterrichts.* Teil 1. Leipzig, 1906.

Schimmack, R. *Die Entwickelung der mathematischen Unterrichtsreform in Deutschland.* 1907–1910.

See also the following journals:

L'Enseignement mathématique (Paris).

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht (Leipzig).

The Mathematical Gazette (London).

The Mathematics Teacher (New York).

School Science and Mathematics (Illinois).

The first of these journals was recognized as the official organ of the Internationale Mathematische Unterrichts-Kommission, which is referred to by the initials, I. M. U. K.

The Reform Movement in Germany

The idea that the function concept should be made the central theme of school mathematics may be said to have originated with Klein. Others before him had advocated the inclusion of variables and functions in the school program, and even as early as 1873, Oettingen had suggested that "the notion of the function"[2] should form an essential part of all mathematical work in schools, but Klein was the first to press the view that functional thinking (*funktionales Denken*) should be the binding or unifying principle of school mathematics. In this principle Klein had given conscious expression to a thought that had been vaguely conceived by more than one reformer before him.

Beginning of the movement. For the beginning of the Reform Movement in Germany we have to go back to the Prussian Lehrplan of 1816, sometimes referred to as the Süvern Lehrplan,[3] which served for many years as the ideal ultimate standard of school mathematics in Germany. The course outlined in this Lehrplan was an ambitious one and included analytical geometry, trigonometry, and the elements of the calculus. It fell, however, under its own weight, for owing to the low standards then obtaining in the schools, it proved to be quite beyond the powers of any but the ablest students. By a special ordinance of 1834 the Süvern Lehrplan was considerably modified. The course was made less exacting and spherical trigonometry and conic sections were removed altogether.

Schellbach and Balzer. Among the greatest influences for the extension of functional mathematics during the next half-century were the textbooks of Schellbach and Balzer. In 1843 Schellbach published a textbook on conic sections and a collection of mathematical problems,[4] in which he stressed the idea of the variable, and in 1865 Baltzer[5] published his *Die Elemente der Mathematik*, in which the function concept figured prominently. It is worth noting,

[2] Oettingen, A. v. *Über den mathematischen Unterricht in der Schule.* 1873. Festrede zur jahresfeier der stiftung der Universtät Dorpat.

[3] After one of its authors, J. W. Süvern. The initiator of the plan was W. v. Humboldt.

[4] Schellbach, K. H. *Die Elemente der Kegelschnitte.* Berlin, 1843. See also *Mathematische Lehrstunden.* Berlin, 1844. Bearbeitet von A. Bode und E. Fischer, 1860.

[5] Baltzer, R. *Die Elemente der Mathematik.* Band I, *Arithmetik und Algebra.* Leipzig, 1865. Band II, *Planimetrie, Stereometrie, Trigonometrie.* Leipzig, 1883.

however, that neither of these writers represented functions by graphs. Partly through his books and partly through his writings on the pedagogy of mathematics, Schellbach exerted a profound influence on the reform movement during the latter half of the last century. In particular he did much to simplify the methods of the calculus.[6]

Results of Franco-Prussian War. The reform movement in Germany received a great impetus after the War of 1871. Shortly after the termination of the war there arose a public demand for new schools of the type of the *Realschule* and a demand within the schools for the modern spirit in mathematics and science. These demands led to important inquiries concerning the real aim of mathematical teaching in the *Gymnasium* and *Realschule*. In 1873 the Prussian Minister of Education summoned a conference[7] to discuss the question of mathematical education in all its bearings. At this conference Bertrand and Gallenkamp strongly advocated the introduction of analytical geometry and the calculus, partly for their cultural and partly for their disciplinary values. These proposals were regarded as impracticable and idealistic and received the support only of the enlightened few. A few years later, support for the reform movement came from an unexpected quarter. Du Bois Reymond, a public-spirited man of science and an enthusiast for classical training, charged the *Gymnasia* with neglect, asserting that they had fallen far behind the times in mathematical studies. He pointed out that, while analytical geometry and the calculus were becoming more and more popular in the *Realschule*, they had been excluded from the *Gymnasia*. A strong plea was made for mathematics as an instrument of general culture. The representation of functions by curves, for example, offers a new world of ideas, and teaches the use of one of the most productive methods by which the human mind has increased its power. This indictment coming from one who was highly respected both as a scientist and as a man of affairs created a great impression, but there were few who were prepared to support the proposals of Du Bois Reymond actively.

[6] Schellbach, K. H. *Op. cit.* Schellbach had the honor of being the first master of a *gymnasium* to examine in mathematics at a public examination.

See also Klein and Schimmack. *Op. cit.*, p. 88.

Lietzmann, W. *Methodik des mathematischen Unterrichts. Über die Zukunft der Mathematik an unseren Gymnasien*, pp. 18, 21. Berlin, 1887.

[7] *Der Landesschulkonferenz.* Berlin, 1873. See also Klein-Schimmack. *Op. cit.*, p. 88.

Up to this time the appeal for functional mathematics had been made chiefly on cultural grounds, but towards the end of the century utilitarian considerations began to dominate the situation. Ever since the conclusion of the War of 1871 there had been a gradual development of technical education, carrying with it a demand for mathematics of a more practical type. Graphical methods began to take the place of algebraic analysis, and text-books soon began to reflect the change of outlook. Bardey's *Aufgabensammlung*[8] led the way with an appendix on graphical representation, and other textbooks soon followed the example. In 1882 the new Prussian Lehrplan, which had been awaited with expectancy by the leaders of the reform movement, made its appearance, but it proved to be a disappointment. As Klein has said, "It satisfied nobody."[9] Superficially, it seemed to be an improvement on the curriculum then in force, but actually there was no essential difference. Graphical work was not mentioned and analytical geometry was not included, although analytical methods were permitted in certain theorems in conic sections. In the *Oberrealschule*, analytical conic sections and the calculus were recognized as optional subjects but were not made obligatory. The reformers complained that the new Lehrplan favored the *Gymnasia* at the expense of the *Realschule* and that it did not encourage practical mathematics, which was of such vital importance in industrial life.[10]

Influential periodicals. Among the steadiest influences for the reform of mathematical education about this time was undoubtedly the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, a mathematical and scientific journal for teachers edited by Hoffmann. The editorial articles of Hoffmann were among the strongest and sanest influences for the reform of mathematical teaching in Germany for many years. In an article published in 1887[11] he made an appeal for the development of graphical methods in all the schools and urged that, since official programs are

[8] Bardey, E. *Aufgabensammlung--Arithmetik und Algebra*. Leipzig, 1881. This popular work has been revised many times, first by F. Pietzker and O. Presler in 1908, and in its modern form by W. Lietzmann and P. Zühlke in 1925 and 1930. Bardey's works have exerted a profound influence on the development of functional mathematics in Germany.

[9] Klein-Schimmack. *Op. cit.*, p. 93.

[10] Pietzker, F. "*L'Enseignement mathématique en Allemande*". *L'Enseignement mathématique*, 1901, p. 1.

[11] *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Sept., 1887. The *Zeitschrift* was founded by J. C. V. Hoffmann in 1870.

bound to be conservative, the writers of textbooks should take the lead in the matter. In the discussions that followed the publication of the article frequent references were made to the function concept and it was evident that this concept was gradually being recognized as an important objective in mathematical teaching. Another strong advocate of functional mathematics was Simon, who, at the 1897 meeting of the Förderungsverein,[12] an organization for the advancement of mathematical and scientific studies, urged that the function concept, "the most far-reaching and important idea of algebra," be an integral part of the course from the beginning.

Felix Klein. At the beginning of the present century Felix Klein, Professor of Mathematics in the University of Göttingen, had become the recognized leader of the reform movement in Germany. Klein was equally distinguished as a mathematician and as an authority in the pedagogy of mathematics. In 1893 in an address before the International Congress of Mathematicians (Chicago), he drew the attention of teachers to the vital importance of functional thinking in school mathematics. This theme he further developed in various conferences of teachers in the years that followed, but he received little active support. A carefully considered statement of his views was published in 1902 in one of two articles written in coöperation with Götting,[13] and in a series of studies produced in conjunction with Riecke[14] an abstract of this discussion appeared two years later. In the latter publication Klein developed the whole subject of functional mathematics and asserted that "an elementary treatment of the function concept and an introduction to analytical geometry and the differential and integral calculus ought to be in the regular course of all types of high schools."[15] In 1904 an important conference of mathematicians and scientists was held at Breslau,[16] when Klein again urged the importance of the function concept, claiming that "the function concept graphically presented should form the central

[12] Förderungsverein. *Verein zur Förderung des Unterrichts in der Mathematik und den Naturwissenschaften*, founded in 1891.

[13] Klein, F. "*Über den mathematischen Unterricht an den höheren Schulen.*" *Jahresbericht der Deutschen Mathematikervereinigung*, Vol. XI. p. 128. 1902.

Götting, E. "*Über das Lehrziel im mathematischen Unterricht an höheren Realanstalten.*" *Ibid.*, p. 192.

[14] Klein, F. and Riecke, E. *Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts und den höheren Schulen*, pp. 33, 48. Leipzig, 1904.

[15] *Ibid.*, p. 7.

[16] Naturforscherversammlung. *Vorschläge.* Breslau, 1904.

notion of mathematical teaching, and that, as a natural consequence, the elements of the calculus should be included in the curriculum of all nine-class schools."[17] Klein impressed the conference so strongly that a committee was appointed, known as the Breslauer Kommission, to formulate definite proposals for the reform of mathematical teaching in the schools and special reference was made to the concept of functionality. The proposals of the commission were presented to the general conference at Meran in 1905 in a form now known as the Meraner Lehrplan.[18] These proposals, which may be called the charter of modern mathematics in German schools, set forth the aims of mathematical teaching as follows:

> . . . to bring the course of teaching more closely in line with the natural process of mental development than formerly, to develop as far as possible the faculty for contemplating natural phenomena from a mathematical point of view, and to make the pupil more and more conscious of the continuity of the subject as he passes from stage to stage—a psychological, utilitarian, and didactic principle.[19]

The unifying principle which made this continuity possible was defined to be "education in the habit of functional thinking."[20] To this was added, "the development of the ability for space perception." Thus the Meraner Lehrplan crystallized in a slogan—functional thinking—principles that had hitherto been only vaguely conceived.

Klein followed up the advantage thus gained and for several years conducted a vigorous campaign on behalf of functional thinking. The substance of his addresses at the Meran Conference and afterwards are contained in his *Vorträge über den mathematischen Unterricht* and in the reports of the International Conference of Mathematicians held at Rome in 1908.[21] In these discussions Klein claimed that the function concept was, not simply a mathematical *method*, but the heart and soul of mathematical thinking.

> It is my conviction that the function concept in its graphical form should be *the soul of mathematical study* in the schools.[22]

[17] Klein, F. *Bericht an die Breslauer Naturforscherversammlung über den Stand des mathematischen und physikalischen Unterrichts an den höheren Schulen.*
Appendix to Klein-Schimmack. *Op. cit.*, p. 198.

[18] *Der Meraner Lehrplan für Mathematik* (1905).
See also Klein-Schimmack. *Op. cit.*, p. 208.
Lietzmann, W. *Methodik des mathematischen Unterrichts*, p. 232.

[19] Klein-Schimmack. *Op. cit.*, p. 210.

[20] *Ibid.*, p. 208.

[21] See also *L'Enseignement mathématique*, 1908–1912.

[22] Klein-Schimmack. *Op. cit.*, p. 34.

That he interpreted the term 'function concept' in the broadest sense is evident from the illustrations he gives of its application. He writes:

> The function concept, which was considered to be the central point of mathematical study in the Obertertia, should remain, still, the pivotal idea. Logarithmic theory, trigonometry, etc., all depend, without question, on this concept. It is so with modern geometry. The concept of change as an observable translation of points will convince the student of this generalization. . . . The function concept gives us a splendid opportunity to study figures as continuously varying structures in space.[23]

Finally, he shows that the function concept leads naturally and easily to the calculus:

> There is no reason why this [the calculus] should be considered difficult for the pupil. The elementary principles of the differential and integral calculus certainly do not belong to 'higher analysis' but to elementary mathematics — not only so, they should be an essential element of school mathematics.[24]

Klein repeatedly insisted that an elementary knowledge of the calculus should be regarded as the legitimate goal of school mathematics:

> The elements of the infinitesimal calculus, treated properly, provide far more suitable material for mathematical education at school than that heterogeneous and lifeless subject matter, which nowadays is so repellent to those boys who have no particular ability for mathematics. The calculus would naturally arise from a fundamental and fruitful treatment of the idea of function, an idea admittedly of high importance and deserving of a central position. Further, the calculus is indispensable to a clear comprehension of numerical physical phenomena; and, from the standpoint of mental training, is an essential element of mathematical education.[25]

We quote this passage at length because it is an excellent summary of Klein's views on mathematical education in relation to the function concept, the calculus, the correlation of physics and mathematics, and the doctrine of mental discipline.

In his discussion of the value of functional thinking Klein showed a keen insight into the psychological implications of his subject. He took care, for example, to discriminate the logically from the psychologically simple (elementary) and showed that the idea of functional relationship, being fundamental to life, is therefore psychologically simple and in that sense elementary.[26] As we have

[23] *Ibid.*, p. 128.
[24] *Ibid.*, p. 112.
[25] *Ibid.*, p. 115.
[26] *Ibid.*, p. 111.

already indicated above, Klein also supported the doctrine of mental training,[27] maintaining that the heart of that doctrine was to be found in methods of conceptual thinking—a conclusion that is very close to present-day thought on that subject.

The Meran proposals were further discussed by the Breslauer Kommission and a report was presented to a general conference of the Natural Science Society held at Dresden in 1907.[28] At the conclusion of this meeting the commission was dissolved and a more widely representative committee was formed to continue the work. This committee, known as the Deutsche Ausschuss für Mathematischen und Naturwissenschaftlichen Unterricht (D.A.M.N.U.) was representative of several mathematical and scientific interests.[29] Its original membership included the names of Klein, Gutzmer, Schotten, Stäckel, and Treutlein, all prominent figures in the world of mathematics and science. Of these, by far the most active was Klein, who developed his ideas on mathematical reform in a series of holiday courses for teachers. The substance of these lectures was afterwards embodied in his *Elementar-mathematik*,[30] a work of the highest originality and scholarship.

Influence of the Fourth International Congress. The reform movement entered upon a new phase when the Fourth International Congress of Mathematicians met in Rome in 1908. At this meeting of the Congress, David Eugene Smith, of Teachers College, Columbia University, suggested the appointment of a special committee "to study and compare the tendencies in the teaching of mathematics in various countries and to report to the next meeting of the Congress to be held in Cambridge in 1912." This suggestion was accepted and a committee known as the Internationale Mathematische Unterrichts Kommission (I.M.U.K.) was appointed, with Klein as its chairman.[31] This committee drew up a constitution for

[27] *Ibid.*, pp. 137, 158

[28] Naturforscherversammlung. *Dresdener Vorschläge*, Chap. III, 1907.

[29] *Schriften des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht.* Leipzig, 1907.

[30] Klein, F. *Elementarmathematik vom höheren Standpunkte aus.* Band I, *Arithmetik, Algebra, Analysis*, 1924. Band II, *Geometrie*, 1925.

See also Klein, F. and Gutzmer, A. *Rapport de la commission d'enseignement des naturalistes et medicins.* Dresden, 1907.

[31] *L'Enseignement mathématique*, Oct., 1908. Report on the Internationale Mathematische Unterrichts Kommission.

Atti del IV congresso internazionale dei mathematici Roma, 1908, Vol. III. Rome, 1909.

the commission and stimulated the formation of subcommittees in each country.[32] The French mathematical journal, *L'Enseignement mathématique*, edited by Fehr, was recognized as the official organ of the commission.

With characteristic thoroughness the German subcommittee prepared an elaborate series of reports on the teaching of mathematics in Germany, with a detailed history of its development.[33] Valuable reports were also prepared by the subcommittees of the other nations represented on the commission.[34]

With the publication of these reports the case for functional thinking may be said to have been firmly established in German schools. Since that time considerable progress has been made, partly in broadening the function concept so as to embrace the whole school program and partly in widening its field of application.[35] For several years before his death in 1920, Klein urged that the distinction between pure and applied mathematics should be less rigid and that mathematics and physics should be looked upon as "auxiliary sciences," with the function concept as the natural bond between them.

The Mathematical Congresses of 1910-11. At the meetings

[32] Fehr, H. *Berichte und Mitteilungen verenlazt durch die Internationalen Mathematischen Unterricht Kommission. Vorbericht über Organisation und Arbeitsplan der Kommission.* Leipzig, 1909.

See also *L'Enseignement mathématique*, 1908, p. 326.

[33] *Schriften des Deutschen Unterausschusses der internationalen mathematischen Unterricht-Kommission Abhandlungen über den mathematischen Unterricht in Deutschland.* Herausgegeben von F. Klein. Leipzig und Berlin, 1909–1912.

[34] Commission Internationale de l'Enseignement Mathématique, Sous-Commission Française. *Rapport*, Vols. I-V, 1911.

See also United States Bureau of Education. *Bulletin*, 1911-1917. Washington, D. C.

Board of Education, London. "The Teaching of Mathematics in the United Kingdom." *Special Reports on Educational Subjects*, 1911–1912.

Valuable reports were also received from Austria, Hungary, Spain, Belgium, Switzerland, Russia, and Japan. Altogether twenty-four countries were represented. For the full list see *Proceedings of the Fifth International Congress of Mathematicians.* Cambridge, 1913.

[35] See, for example:

The Württemberg Lehrplan (1912). W. Lietzmann. I. M. U. K. Abhandlungen II.

The Bayer Lehrplan (1914). (*Zeitschrift für mathematischen und naturwissenshaftlichen Unterrichts*, 1915.)

The Prussian Lehrplan or Richtlinien (1925). (Richtlinien für die Lehrpläne der höheren Schulen, Amtliche ausgabe, Beilage zum Zentralblatt für die gesamte Unterrichtsverwaltung in Preussen, 1925, Heft 8. Berlin.) A general survey is given in Lietzmann's *Methodik des mathematischen Unterrichts*, p. 261.

of the Congress of Mathematicians held in Brussels in 1910, and again in Milan in 1911, Klein brought forward proposals for the fusion of the different branches of mathematics and made suggestions for a course of general mathematics for students of physics and the natural sciences.[36] At the latter conference he reported that the various German States had begun to reorganize their teaching of mathematics to bring it in line with the general conclusions arrived at during the meeting of the congress in Rome. He also emphasized the fact that great liberty had been given the teachers in the interpretation of the conclusions of the congress.[37] Representatives of other countries, reporting on the application of the function concept to the work of their schools, stressed the value of concrete illustrations in functional mathematics and the vital importance of early and careful preparation. Ratz, a representative from Hungary, stated that "The notion of function ought to be prepared for with much care, and sufficient time must be allowed the pupil in which to familiarize himself with the new ideas."[38] At this meeting Lietzmann brought forward the question of mathematical *rigor* which had arisen from experience in the teaching of mathematics from the functional standpoint. He cl. ified the prevailing points of view under four heads: (1) strictly logical (Peano, Hilbert), (2) empirical-logical (Euclid, Veronese, Enriques-Amaldi), (3) intuitive-inductive (Borel, Behrendsen-Götting), and (4) intuitive-experimental (Perry). No general conclusions were reached, but opinion seemed to incline towards the inclusion of all types, beginning with the intuitive-experimental in the elementary stages and ending with the strictly logical in the last school year. Lietzmann, who favored this course, was strongly supported by Klein.[39]

Lietzmann's contribution. Before we leave this part of our subject, reference should be made to the conspicuous work of Lietzmann in popularizing the function concept and in systematizing functional methods for school use. This he has done, partly through his textbooks, which, as we shall see, are among the most suggestive that have been published up to the present time, and partly through his writings on the teaching of mathematics. His *Methodik des mathematischen Unterrichts* (two volumes) is a work of outstand-

[36] *L'Enseignement mathématique*, 1911, p. 358.
[37] *Ibid.*, p. 454.
[38] *Ibid.*, p. 456.
[39] *Ibid.*, p. 458.

ing merit, covering the whole field of mathematical teaching.[40] In this work there is perhaps the most exhaustive treatment of the function concept that has yet appeared. An even more immediately useful work is his *Funktion und graphische Darstellung*,[41] a readable and helpful handbook for the practical teacher. Among his published articles special reference may be made to one on the calculus, which formed the basis of an important discussion on the place of the calculus in the schools[42] at the meeting of the Congress of Mathematicians held in Paris in 1914.

The Reform Movement in France

Judging by the articles that have appeared in mathematical and pedagogical journals, the reform movement in France has developed in a different direction from that of the German movement toward functional thinking. In recent years teachers of mathematics in France have been concerned with methods of approach to the teaching of geometry and with questions of mathematical rigor, rather than with the inculcation of functional ideas in algebra or analysis. This does not mean, however, that they have not been alive to the importance of the function concept; they seem, rather, to have accepted graphs and functions as the natural and logical development of the work they had already been doing. We find that graphs were freely employed in French schools before the end of last century.[43] The fact that 'the graphs of certain functions' appeared in the official program of 1902, and were accepted without dispute as an improvement long overdue, is sufficient evidence that teachers of mathematics were prepared for the change and needed no persuasion.

[40] Lietzmann, W. *Methodik des mathematischen Unterrichts*, Teile I, II. Leipzig, 1926.

See also the volumes of the *Abhandlungen* to which reference has already been made.

Stoff und Methode des Rechnunterrichts in Deutschland. Leipzig, 1911.

Die Organisation des mathematischen Unterrichts in den Preussischen Volks und Mittelschulen. Leipzig, 1919. In this book he shows how the function concept has been brought even into elementary school mathematics.

[41] Lietzmann, W. *Funktion und graphische Darstellung*. Breslau, 1924.

[42] Lietzmann, W. "*Die Einführung der Elemente der Differential und Integralrechnung in die höheren Schulen.*" *Zeitschrift für mathematischen und naturwissenshaftlichen Unterrichts*, 1914, p. 145.

See also *L'Enseignement mathematique*, 1914, p. 246.

[43] Klein, F. *Elementarmathematik*, p. 475.

See also *L'Enseignement mathèmatique*, 1903. Articles on the official program of 1902.

The introduction of the concept of motion—Méray. Toward the end of the last century French mathematicians were much exercised over the admission of intuitive methods in geometry, particularly those which involved the concept of motion. Intuitive methods had been employed by Legendre in his famous *Éléments de géométrie*[44] (1794), but they had never been fully accepted by the rigorists. In 1874 Méray published his *Nouveaux éléments de géométrie*,[45] in which he went much further than Legendre by making the idea of motion the basis of his whole conception of geometry. As Borel said later, "Geometry became the study of a group of movements."[46] As an illustration of the application of this idea we may take Méray's definition of parallelism, "Two lines are parallel when a simple translation of the one suffices to superimpose it upon the other."[47] Although Méray succeeded in introducing his method into several normal schools,[48] it was not until 1904, thirty years after the publication of his book, that his methods became officially recognized. In that year the French Association for the Advancement of Science in their meeting at Grenoble recommended Méray's method. A year later the "Instructions" to teachers included in the official program of studies added a note to the effect that teachers "should make a constant appeal to the idea of translation," and the program was modified to suit that idea.[49]

In this struggle for recognition, Méray received the strong support of two eminent mathematicians. In 1903 Borel published a remarkable book on geometry, in which he based his procedure on Méray's concept of translation and added some valuable sections on graphical representation.[50] This book met with instant success and did more than argument would have done to popularize Méray's methods. In 1908 Bourlet followed with an excellent book, which

[44] Legendre, A. M. *Eléments de géométrie.* Paris, 1794.

[45] Méray, C. *Nouveaux éléments de géométrie.* Dijon, 1874. A new edition (enlarged) was published in 1903.

[46] Borel, E. *L'Enseignement mathématique*, 1905, p. 386.

[47] Méray, C. *Op. cit.*, p. 12.

[48] Méray, C. "*Justification des procédes et de l'ordinnance des nouveaux éléments on géométrie.*" *L'Enseignement mathématique*, 1904, p. 89.

Perrin, R. "*La Méthode de M. Méray pour l'enseignement de la géométries.*" *L'Enseignement mathématique*, 1903, p. 441.

[49] *Plans d'études et programmes d'enseignement dans les lycées et collèges de garçons.* Paris, 1905.

[50] Borel, E. *Elémentaire géométrie.* Paris, 1903.

was virtually a revision of the *Nouveaux éléments*.[51] This book, in a revised and modified form, is still a popular textbook in France. In recent years there has been a tendency in some quarters to modify the Méray method, and in others to abandon it altogether.

At first sight this discussion seems to have very little bearing on our main thesis, but this is not so. A little reflection will show that it is all part of a general movement towards a dynamic conception of school mathematics. As we shall see later, the idea of 'motion' makes possible the functional treatment of geometry.

In addition to his treatise on geometry Méray published, about the same time, an important work on analysis,[52] which exerted a powerful influence on the development of functional mathematics. This work proved so successful that it was afterwards expanded into a treatise on analysis of four volumes.[53] Méray's general outlook on the subject of functionality is given in an article published in *La Révue bourguignonne* in 1892. He writes:

> Algebra is not a special form of calculus, . . . it is the theory of rational functions (and also of irrationals). I would open the minds of pupils to these notions relative to variables and functions that are disguised in elementary mathematics.[54]

Laisant expressed the same thought more fully in a very illuminating series of articles on the philosophy and teaching of mathematics, published in 1898:

> Algebra has for its aim the calculus of functions, in contrast to arithmetic, which is the calculus of values. It is, therefore, necessary, before everything else, to present this notion of functionality, which is fundamental to all mathematical theory. The idea of variability, the idea of law, comes spontaneously to our minds in the following way: Things change and these changes are the immediate causes of other changes. For our present object it is important to come down from such generality and to apply this observation to measurable things, that is to say, to quantities to which alone mathematics is related. Among the natural quantities the variable is an excellent example

[51] Bourlet, C. *Eléments de géométrie.* Paris, 1908.
See also Bourlet et Ferval. *Géométrie.* Paris, 1905.
Bourlet, C. *Cours abrégé de géométrie.* Paris, 1906. See also "*Théorie des paralleles basée sur la translation rectiligne.*" *Nouveaux annales de mathématiques*, 1907.

[52] Méray, C. *Nouveau précis d'analyse infinitesimale.* Paris, 1872.

[53] Méray, C. *Leçons nouvelles sur l'analyse infinitesimale et ses applications géométriques.* Parts 1, 2 (1895), Part 3 (1897), Part 4 (1898).

[54] Méray, C. "*Considérations sur l'enseignement des mathématiques.*" *La Revue bourguignonne de l'enseignement supérieur*, 1892.
See also *L'Enseignement mathématique*, 1901. p. 172.

of something about which we have an instinctive conception long before we know how to measure it.[55]

Introduction of the function concept. As we have already indicated, it is not at all certain when functional mathematics began to be taught in French schools. In 1902, largely under the influence of Darboux, graphs and functions appeared in the official programs.[56] Among the topics included in the program of the École Polytechnique are to be found the functions x^n and a^x, the logarithmic function, the limit of $(1 + 1/m)^m$, derivatives, maxima and minima, and the elements of analytical geometry.[57] These programs have been modified several times since, but always in the direction of increased emphasis on functionality. They have also become increasingly more practical, thus obviating an early criticism that they were "too theoretical and abstract."[58]

It seems likely, that in the practical interpretation of the function concept, French mathematicians have been indirectly influenced by Klein and his school, but few references to Klein's work are to be found beyond comments and discussions on the work of the Internationale Mathematische Unterrichts Kommission. Gützmer, in an obituary notice on the death of Klein, wrote:

> Functions were introduced into France in the program of 1902, under the influence of G. Darboux. These notions have now become general in the German States. The reform carried out in France has extended to Germany.[59]

David Eugene Smith made a similar statement in a paper read before the meeting of the International Commission held at Cambridge in 1912:

> Starting in France within the last twenty years and vigorously advocated in Germany within the last decade, it has much to recommend it, if reasonably treated.[60]

[55] Laisant, C. A. *La Mathématique-Philosophie-Enseignement*, p. 46. Paris, 1898.

[56] *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons*, Paris, 31 Mai, 1902. This plan was modified in 1905.

[57] *Les nouveaux programmes de l'école polytechnique de Paris*, Oct. 15, 1902. For a criticism of these programs see *L'Enseignement mathématique*, 1903, p. 77.

[58] Commission Internationale de l'Enseignement Mathématique. Sous-Commission Française. *Rapport* (C. Bioche) p. 11. Paris, 1911. *Sur la place et l'importance des mathematiques.*

[59] Gützmer, A. *L'Enseignement mathématique*, 1920, p. 236.

[60] Smith, D. E. "Intuition and Experiment in Mathematical Teaching." *Proceedings of the Fifth International Congress of Mathematicians*, p. 617. Cambridge, 1912.

Fehr, however, seems to suggest that the real impetus towards functional thinking came from Germany:

The idea [of the function] is not new; it is the order of the day in Germany. . . . One can say that in these days, for the chemist as for the botanist, for the doctor and biologist as for the lawyer, a profound knowledge of the notion of the variable is indispensible, for without it a great number of fundamental properties would remain inaccessible.[61]

In reply to the question as to what should be included in a functional program for schools he cites: coördinates and graphs, statistics, the discussion of the functions ax, ax^2, ax^3, $\frac{a}{x}$, $\frac{1}{x+a}$, $\frac{ax+b}{a'x+b'}$, $\log x$, $\sin x$, $\cos x$, and the elements of the differential and integral calculus. All these topics have since been included in the French official programs. In addition, we find the graphs of such functions as $ax^4 + bx^2 + c$ and $\frac{ax^2 + bx + c}{a'x^2 + b'x^2 + c'}$.

The Calculus in the French secondary school. In France discussions of the concept of functionality have centered round the question of the calculus. It is strange that, although teachers of mathematics had accepted, almost without question, simple graphical methods, they were not nearly so ready to give the same welcome to the calculus, in spite of the recommendations of eminent mathematicians. As Laurent wrote, "Why go through intricate algebraic and geometrical processes, when the calculus will do the same thing much more easily."[62]

Tannery, one of the most highly respected of French writers on mathematics, made a similar plea in an article contributed to the *Revue pedagogique*:

The second procedure, which is excellent, but demands a marked effort, consists in learning some integral calculus before studying the measurement of these volumes. Integral calculus! In the secondary school!! Yes, I am not joking. The effort needed to learn what a derivative is, an integral, and how by means of these admirable tools surfaces and volumes can be evaluated, is certainly less than the effort heretofore demanded of a child to establish the equivalence of oblique and right prisms, of two pyramids (the staircase figure, you know, that is so tiresome to make), then the insupportable volumes of revolution.[63]

[61] Fehr, H. "*La notion de fonction dans l'enseignement mathématique des écoles moyennes.*" *L'Enseignement mathématique*, 1905, p. 177.

[62] Laurent, H. "*Considérations sur l'enseignement des mathématiques dans les classes speciales en France.*" *L'Enseignement mathématique*, 1899, p. 38.

[63] Tannery, J. "*L'Enseignement mathématique.*" *Revue pedagogique*, 1903.

Elsewhere Tannery has discussed the value of functional thinking in a passage that is so broadly comprehensive that it is worth quoting:

> One has not even a slight idea of what mathematics is, one does not suspect its extraordinary scope, the nature of the problems that it proposes and solves, until one knows what a function is, how a given function is studied, how its variations are followed, how it is represented by a curve, how algebra and geometry mutually aid each other, how number and space illustrate one another, how tangents, areas, volumes are determined, how we are led to create new functions, new curves, and to study their properties. Precisely these notions and methods are needed to read technical books in which mathematics is applied.
>
> They are simple and easy so far as essentials are concerned, easier than many demonstrations that we do not hesitate to give to pupils, demonstrations that are long and complicated and that have no bearing beyond what they prove. These methods should penetrate more and more into elementary instruction, both to abridge and to strengthen it.[64]

With the publication of the official program of 1905 the introduction of analytical geometry and the calculus into the schools was no longer open to dispute.[65] That the calculus had secured an established place in French schools by 1914 was evident from the discussion that took place at the meeting of the Congress of Mathematicians held in Paris in 1914.[66] Since then the tendency has been to bring the calculus lower and lower in the school grades, although the latest official program prescribes the calculus only for the *classe de mathématiques*.[67]

Nomography. Before we leave this part of our subject, reference should be made to the growing popularity of nomography in French schools, largely through the stimulus given to that subject by D'Ocagne[68] and Massau. At the meeting of the Internationale Mathematische Unterrichts Kommission held in Cambridge in 1912, Runge made a strong plea for the introduction of nomographical methods, which, as he stated, were taught only in French schools.[69]

[64] Tannery, J. *Notions de mathématiques.* Paris, 1903.

[65] Commission Internationale—Sous-Commission Française. *Rapport aux classes de mathématiques speciales et de centrale des lycées* (E. Blutel), p. 11. 1911.

[66] *L'Enseignement mathématique*, Oct., 1914.

[67] Programmes Officiels du 3 Juin, 1925.—Enseignement secondaire des garçons et des jeunes filles.

[68] D'Ocagne, M. *Traité de nomographie.* Paris, 1899. See also *Calcul graphique de nomographie.* Paris, 1914.

[69] Runge, C. *The Mathematical Training of the Physicist in the University.* Internationale Mathematische Unterrichts Kommission. Cambridge, 1912.

No doubt teachers of mathematics in other countries have lost much by neglecting to explore this branch of graphical mathematics.[70]

THE REFORM MOVEMENT IN OTHER CONTINENTAL COUNTRIES

Austria, Hungary, and Italy. The history of the movement towards functional thinking in other continental European countries has followed lines similar to those that we have already traced in Germany and France. Stated broadly, the reform movement in Austria and Hungary has corresponded to that in Germany, and the movement in Italy to that in France.[71] Switzerland, as one would judge by its position, has been influenced by advances in both countries.[72]

We do not propose to give details of these movements beyond citing the names of a few of those who have made substantial contributions to our subject. In Austria the names of Czuber, Dintzl, Wertlinger, Suppantschitsch, Hočevar, Močnik, Jarosch, and Falk are outstanding.[73] Dintzl was not only one of the original members of the Internationale Mathematische Unterrichts Kommission, but is also the author of several very valuable textbooks. No one has

[70] See Maclean, J. *Graphs and Statistics. Elementary Applications of Mathematical Methods.* Bombay, 1926. An original and suggestive approach to advanced school mathematics of a special type.

[71] "Significant Changes and Trends in the Teaching of Mathematics throughout the World since 1910." *The Fourth Yearbook, National Council of Teachers of Mathematics*, 1929.

See also similar articles in *L'Enseignement mathématique*, 1929, 1930, which are in some cases fuller than the above, although written by the same authors.

Fehr, H. *Commission internationale de l'enseignement mathématique.* Rapport préliminaire sur l'organisation de la commission et le plan général de ses travaux. Genève, 1908.

Stamper, A. W. *A History of the Teaching of Elementary Geometry.* New York, 1909.

[72] Brandenberger, K. *Der mathematischen Unterricht an den Schweizerischen Gymnasien und Realschulen.* Bale, Genève, 1917. The concept of functionality is given great prominence in the chapter, "Methods of Teaching."

See also Fehr, H. *L'Enseignement mathématique en Suisse.* Genève, 1911.

La Notion de fonction dans l'enseignement mathématique des écoles moyennes. Conference held at Zurich in 1904.

Smith, D. E. "Intuition and Experiment." *Proceedings of the Fifth International Congress of Mathematicians*, p. 623. Cambridge, 1912. "In Switzerland the graphical representation of equations and functions is general, as in other countries, and is extended to the idea of limits."

[73] *Commission internationale, sous-commission Autrichien.* Berichte über den mathematischen Unterricht in Oesterrich. Wien, 1912.

See also Simon, O. "*L'Enseignement au gymnase Autrichien.*" *L'Enseignement mathématique*, 1902, p. 157.

contributed more than he has to the broader understanding of functional thinking in Austria. In Hungary we find the names of Beke, Mikola, Szénes, Rátz, Veress, and Goldziher, several of them authors of popular textbooks.[74] Goldziher is joint author with David Eugene Smith of a valuable bibliography of mathematical literature. Teachers of school mathematics in Italy, largely through the stimulus of Peano, Veronese, Enriques and Amaldi, Ingrami, Castelnuovo and de Paolis, have been concerned with the foundations of geometry rather than with functional mathematics in general. About 1880 a reaction set in against the application of rigorous methods to school geometry.[75] Sannio and d'Ovidio, de Paolis, Lazzeri and Bassani, following, it seems, the lead of Helmholtz, advocated the introduction of intuitive ideas, including the concept of motion. On the side of mathematical rigor are ranged the names of Veronese, Enriques and Amaldi, de Franchis and Ingrami.[76] Present tendencies seem to favor the rigorists, for the textbooks of Enriques and Amaldi, de Franchis, Rosati and Benedetti, and Severi are all based on the more rigorous approach. Reference should also be made to the *Algebra* of Marcolongo and the *Nozioni di Mathematica* of Enriques and Amaldi.[77] The latter is one of the soundest and clearest expositions of advanced school mathematics in any language.

Spain and Russia. Of the spread of the reform movement to

[74] Rátz, M. L. *L'Enseignement mathématique*, 1911, p. 456.

See also Goldziher, C. "Austria." *The Fourth Yearbook, National Council of Teachers of Mathematics.* The writer is greatly indebted to Professor Goldziher for detailed information concerning the progress of functional thinking in the schools of Austria, Hungary, Germany, and Italy. Of this information we have been able to include only a small part.

Beke, E. *Über den Jetzigen Stand des mathematischen Unterrichts und die Reformbestrebungen in Ungarn.* Internationale Mathematische Unterrichts Kommission. Roma, 1909.

Beke-Mikola. *Abhandlungen über die Reform des mathematischen Unterrichts in Ungarn.* Leipzig and Berlin, 1911.

Mikola-Rátz. *A függvenyek és az infinitesimális számitások elemei* (Elements of functions and of the calculus). Budapest, 1914.

[75] Loria, G. *L'Enseignement mathématique en Italie.* Paris, 1905.

See also Lietzmann, W. "*Die Grundlagen der Geometrie in Unterricht*" (mit besonderer Berücksichtigung der Schulen Italiens). *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1908, p. 177.

[76] Scorza, G. *Sui libri di testo di geometria per le scuole secondaire superiori.* Roma, 1914.

See also *L'Enseignement mathématique*, 1914, p. 251.

[77] Amaldi, U. and Enriques, F. *Nozioni di Mathematica.* 2 vols. Bologna, 1921.

Spain and Russia[78] very little reliable information is available. That modern Russia is alive to the importance of the concept of functionality is evident from the following extracts taken from books recently published by the state press.

> The development of the concept of functional dependence must penetrate every branch of mathematics of the secondary school and this must be the pivotal concept of the course. . . . Klein has said that 'Elementary mathematics is that mathematics which is within the grasp of a youth of school age of average intelligence.' If that is so, the calculus can claim to be elementary.[79]

The function concept is specifically mentioned in the school programs of 1926 and in the course of study for teacher-training institutions we find the following:

> Special importance to be attached to the development and furthering of the idea of functionality from two points of view and two methods of attack, the analytical and the graphical. . . . The whole program is built upon two ideas, that of the fusion of the different branches of mathematics and on the concept of the function.[80]

Textbooks embodying these ideas have been prepared by Voronetz,[81] Friedman[82] and Boehm.

The Reform Movement in England

It may safely be asserted that in no other country in the world is so much individual liberty allowed the schoolmaster as in England.[83] There are in England no official programs, no fixed courses of study. One consequence of this liberty is an extraordinary vari-

[78] de Galdeano, Z. G. *L'Enseignement mathématique en Espagne.* Commission Internationale, 1911.

Bobynin, V. V. "*L'Enseignement mathématique en Russie.*" *L'Enseignement mathématique*, 1903, p. 247.

Commission Internationale. *Bericht über den mathematischen Unterricht an den Russischen Realschulen.*

[79] Leifert, L. *Pedagogitchesky Sbornik* (Pedagogical Essays), Vol. III., p. 218. Moscow. 1923. The article is on the reform of mathematical teaching after the Revolution. The author surveys the work of Klein, Perry, and Borel. He states that Klein's *Elementar Mathematik* has been translated into Russian.

For the references to and the translation of these passages, the writer is indebted to Mr. Aaron Bakst of Teachers College, Columbia University.

[80] *Projekty Programm* Shkoly II Stupeni; and *Sbornik Materialoff po Pedagogitsheskomu Obrazovaniu* (Utshebnye Planyi Programmy Olya Pedagogitsheskikh Pekhnikumoff. Programs of projects for schools of the second grade: a compilation of materials for pedagogical training.) Published by the U. S. S. R., 1926.

[81] Voronetz, V. *Posobie po Mathematike.* 1926.

[82] Friedman, W. G. *Posobie po Mathematike.* 1927.

[83] See Wolff, G. *Der mathematische Unterricht der höheren Knabenschulen Englands.* Leipzig, 1915.

ety of standards and methods; another is an almost entire absence of anything suggestive of a reform movement. To each of these generalizations it is necessary, however, to make an important qualification: to the first, the natural restriction that comes of conformity to the requirements of University Entrance Examinations;[84] and to the second, a notable exception in the Perry Movement.[85]

Influences at work for reform. For our present purposes we may take the Perry Movement (1901) as our starting-point. Previous to 1901 the course of study for ordinary mathematics consisted of arithmetic, algebra (to the quadratic), and Euclid (Books I to III), and for scholarship mathematics, of higher algebra, trigonometry, Euclid (Books I to XI), geometrical conic sections, and the elements of the calculus. In the older scholarship examination papers we seldom find references to graphs, the word 'locus' being used instead. Thus we have: "Find the locus represented by the equation:

$$y = \sin x + \cos x."$$

And again, "Find the locus represented by the equation:

$$\{y^2 - (a - x)^2\}\{x^2 + (b - y)^2\} = \{(ax + by)^2 - (x^2 + y^2)\}^2."$$

Chrystal's *Algebra*,[86] which was regarded as the indispensable reference book of the specialist, paid considerable attention to the graphical representation of functions.[87]

The Mathematical Association. Perhaps the greatest influ-

[84] In England, secondary school students, as a rule take two examinations; the first, or School Certificate Examination, at the age of about 16 to 17, and the second, or Higher Certificate Examination at the age of 18 to 19.

University scholarships are awarded on the results of a special scholarship examination. The standard of the scholarship examination is extremely high, the object being to select the most promising students for advanced university studies.

[85] Perry, J. "The Teaching of Mathematics." Address at meeting of the British Association for the Advancement of Science, Glasgow, 1901. Published as a separate volume, with full report of the discussion in *Discussion on the Teaching of Mathematics*, British Association, 1901. London, 1902.

See also Perry, J. "The Teaching of Mathematics." *Nature*, 1900, p. 317; 1901, p. 592; 1902, p. 484. "The Rational Teaching of Mathematics." *Nature*, 1901, p. 367. "The Teaching of Mathematics." *Educational Review*, 1902, p. 158.

Jackson, C. S. *Discussion on the Perry Movement*. Internationale Mathematische Unterrichts Kommission, 1912. See also *Mathematical Gazette*, Vol. VI., p. 384, Dec., 1912.

[86] Chrystal, G. *Algebra* (an elementary textbook), Vols. I and II. First published in 1886. This work will remain one of the classics of elementary mathematics.

[87] The more ambitious students also studied Burnside and Panton's Theory of Equations.

ence in the development of mathematical teaching in England has been the Mathematical Association,[88] through its official organ, the *Mathematical Gazette*. In the first number of the *Gazette* (1894), the editor, E. M. Langley, wrote: "But we intend to keep strictly to 'Elementary Mathematics': while not absolutely excluding Differential and Integral Calculus our columns will, as a rule, be devoted to such school subjects as Arithmetic, Algebra, Geometry, Trigonometry, and Mechanics."[89] It was not long before this restriction had to be removed, for the calculus soon began to claim a place among the 'elements.'

Perry's address. In 1901 John Perry in an address before the British Association at Glasgow, put in a vigorous plea for a reform of mathematical teaching in the schools. Perry claimed that school mathematics had failed to arouse "enthusiasm, individuality, and inventiveness" because it was largely a study of abstractions. He strongly urged the adoption of a more practical course of school mathematics, including formulas, logarithms, the use of squared paper, and the calculus. His own words are worth quoting:

> As examples of methods necessary even in the most elementary study of nature I mention: the use of logarithms in computation; knowledge of and power to manipulate mathematical formulae; the use of squared paper; the methods of the calculus. Dexterity in all of these is easily learned by young boys.[90]

Perry's method was what would be termed in America a laboratory, or workshop, method. So accustomed have we become to scale drawing, practical measurements, graphical work, field work and intuitive geometry in the teaching of elementary mathematics, that we are apt to forget the difference between classroom procedure to-day and thirty years ago, when Perry made his appeal.

Criticisms of Perry's premises. The discussion that followed Perry's address was largely concerned with his references to the teaching of geometry. Some criticized his proposals because they were unpsychological,[91] and others because they showed too strong a bias toward engineering.[92] In these discussions Perry received

[88] The Mathematical Association (London) was a development of The Association for the Improvement of Geometrical Teaching, founded in 1871.

[89] Langley, E. M. Editorial. *Mathematical Gazette*, No. 1, London, April, 1894.

[90] Perry, J. *Discussion on the Teaching of Mathematics*, p. 11. London, 1902.

[91] The doctrine of formal training was generally accepted at this time. Euclid was considered to be the medium *par excellence* for such training.

[92] Perry was nimself a professor of engineering.

the strong support of Forsyth, one of the foremost pure mathematicians of England.

During the following year the *Mathematical Gazette* opened its columns to articles on the whole subject of the reform of school mathematics. In one of his editorials the editor described Perry's address as "a picturesque exaggeration," but he found himself in agreement with his main contentions.[93] He urged teachers "to give up Euclid" and to substitute methods that appealed to intuition. Godfrey, Siddons, and other well-known teachers pointed out that some of Perry's suggestions (the use of squared paper, logarithms, etc.) had already been included in their courses.[94] In 1902 the Mathematical Association issued a well-considered report on the teaching of arithmetic and algebra, in which it was recommended "that *graphs* should be introduced as early as possible and should be used extensively."[95] In the same year a special British Association Committee on the Teaching of Mathematics made the following observation: "The general idea of coördinate geometry can be made familiar by the use of graphs and many of the notions underlying the methods of the infinitesimal calculus can similarly be given to comparatively youthful students long before the formal study of the calculus is begun."[96] Henceforth, the history of the movement for graphs and the calculus was that of steady progress. One of the questions that arose was whether the calculus should be treated as a separate subject or as part of the course in algebra. Bryan urged that teachers should "abolish the study of the calculus as a separate subject" and introduce it as a part of algebra and apply it to all the subjects of the mathematical course: trigonometry, geometry (tangents to curves), and mechanics.[97]

Acceptance of Perry's proposals. On reading the history of the teaching of mathematics in England, one is struck by the fact that proposals for reform, apparently revolutionary in character, were so readily accepted by the teachers of mathematics. The reason is, as we have already suggested, that the new methods were

[93] Langley, E. M. "The Teaching of Mathematics." *Mathematical Gazette*, Dec., 1901, p. 105.

[94] See also the discussion on Langley's paper on page 106 of the same issue.

[95] Mathematical Association. "Report of the Committee on Arithmetic and Algebra." *Mathematical Gazette*, March, 1902, p. 183.

[96] British Association. *Report of the British Association Committee on the Teaching of Mathematics*. London, 1902.

[97] Bryan, G. H. "To Teach the Calculus as Early as Possible." *Mathematical Gazette*, Dec., 1903, p. 351.

accepted, not because they had the support or advocacy of an eminent mathematician or because they had behind them the authority of some legislative body, but because they had received the sanction of teachers who had already succeeded in putting them into practice. The movement for the calculus in schools was initiated in the schools themselves. Thus, from the first years of its existence the *Mathematical Gazette* became a clearing house for ideas that had already been developed in the classroom.

In a special report of the Mathematical Association on advanced school mathematics[98] (1904) the introduction of analytical methods in conical sections was recommended and also "an early introduction to the differential and integral calculus and the free use of the same in subsequent work."[99] In a later report (1908), this committee recommended the following subjects for scholarship examinations: analytical geometry, the calculus (total and partial differentiation), integral calculus, plane curves, maxima and minima, curve tracing, and differential equations.[100]

Godfrey's report. Although the word 'function' was freely used in the more advanced textbooks at this time, it was used as the equivalent of an algebraic expression. The concept of functionality, although implicit in much of the work done in schools, had not yet appeared in mathematical literature. The first definite reference to the importance of functionality seems to have been made by Godfrey in a paper, "The Teaching of Algebra," in which he said, "Another fundamental idea is that of functionality." He summed up this part of his subject with admirable conciseness as follows: "Whenever one measurable thing depends on another measurable thing, you have a case for functionality, you have an equation and a graph. To the mathematical eye, life is full of functions and graphs."[101]

In a discussion on a draft report on Godfrey's paper, many members of the association showed that they were fully alive to the

[98] Mathematical Association. "Report of the Mathematical Association on Advanced School Mathematics." *Mathematical Gazette*, May, 1904, p. 53.

[99] It is interesting to note that it was stated that "the use of differentials shall be permitted."

[100] Mathematical Association. "Report of the Committee of the Mathematical Association on the Course for Scholarships." *Mathematical Gazette*, March, 1908, p. 218.

[101] Godfrey, C. "The Teaching of Algebra. What is Educational and What is Technical?" *Mathematical Gazette*, March, 1910, p. 230.

importance of functional thinking. Siddons, in introducing the report, said,

The idea [of functionality] is, we think, the most important that a boy has to acquire in his elementary mathematics.

Nunn supported this view in the following words:

There can be little doubt that the idea of functionality is one of the most important that a boy can carry away from the course of elementary algebra. . . . The various typical curves serve as visual symbols of the abstract relations which the boy has studied in the concrete instances of the physical laboratory of everyday life.[102]

Interest in the concept of functionality, particularly in its relationship to the calculus, was stimulated by the publication of the papers on mathematical education in England prepared for the International Commission on the Teaching of Mathematics (1911-1912). These papers dealt with a variety of topics, but only four or five of them are germane to our present subject.

Godfrey's interpretation of graphic work. Godfrey, in a paper, "The Algebra Syllabus in the Secondary School," gave a very comprehensive picture of the rôle of functionality in mathematics and in life, bringing out very forcibly the dynamic character of functional thinking. On the subject of graphs, he wrote:

We don't want the boy to think merely of a set of spots on the paper, and a nicely-drawn curve . . . , a pattern just sitting quietly on the paper with no life in it; we want him to think *dynamically* rather than statistically; to think of the x as changing, or flowing, continuously, and the x (or $1/x$, or whatever the function may be) flowing *consequently*; to move his pencil point along the curve, and watch the x and y waxing and waning according to the law of their functional relationship.

He then goes on to discuss the importance of the idea of functionality in life:

We live in an atmosphere of functionality. When the study of physics is begun, we have to inquire what is the functional relationship between the length of the spiral spring and the suspended weight, between the pressure and the volume of the enclosed gas, between the inclination of the plane and the force needed to support the weight, between the attraction and the distance apart of two magnetic poles; all these and many other opportunities arise out of correlating physics with mathematics *via* functionality.[103]

[102] Mathematical Association. "Draft Report on the Teaching of Algebra and Trigonometry." *Mathematical Gazette*, March, 1911, p. 232.

[103] Godfrey, C. "The Algebra Syllabus in the Secondary School," pp. 12, 13.

Other articles of importance. In the same series, Barnard examined some of the fundamental concepts of algebra, such as function, limit, continuity, and the like, from the standpoint of school mathematics, and so supplemented Godfrey's paper on the technical side.[104]

One of the most suggestive of these papers was that of Jackson, "The Calculus as a School Subject," in which he examined the whole subject carefully. Among the topics touched upon was that of rigor, in the discussion of which the writer suggested a compromise between extreme opinions on that subject, his criterion of a good method being one that would leave nothing to be "unlearned" by the student later. Among other topics he criticized certain "defective" proofs of the formula for the derivative of x^n, particularly that which depended for its derivation on the Binomial Theorem.[105] In an article contributed a year later to the *Mathematical Gazette*, Jackson gave some valuable suggestions on methods of procedure.[106] This article emphasizes the importance of preparing for the calculus by using finite differences. This article will still repay careful study.

The only other article of the series which calls for special mention is "Examinations for Mathematical Scholarships," by Macaulay and Greenstreet.[107] In this paper specimens of scholarship examination papers set by various colleges of the Cambridge, Oxford, and London Universities in 1910 are given. These examination papers give a better idea of the scope of higher school mathematics in England than could possibly be obtained from textbooks or courses of study.

No. ; of the series of papers prepared for the Internationale Mathematische Unterrichts Kommission. London, 1911.

See also "The Teaching of Calculus in Public and Secondary Schools." *Mathematical Gazette*, p. 235. Jan., 1914. A plea for "the calculus for the average boy."

Godfrey and Siddons. *The Teaching of Elementary Mathematics*, pp. 163, 221. Cambridge, 1931.

[104] Barnard, S. "The Teaching of Algebra in Schools." No. 22 of the series of papers prepared for the Internationale Mathematische Unterrichts Kommission. London, 1912.

[105] Jackson, C. S. "The Calculus as a School Subject." No. 20 of the series of papers prepared for the Internationale Mathematische Unterrichts Kommission. London, 1914.

[106] Jackson, C. S. "The Calculus as an Item in School Mathematics." *Mathematical Gazette*, Dec., 1913; Jan., 1914; March, 1914.

[107] Macaulay, F. S. and Greenstreet, W. J. "Examinations for Mathematical Scholarships." No. 14 of the series of papers prepared for the Internationalen Mathematischen Unterricht Kommission. London, 1912.

These special papers seem to have been the stimulus for the publication of a number of constructive articles and books on functional mathematics by leading professors of mathematics and teachers of school mathematics. Among these we find the names of Whitehead,[108] Nunn,[109] Carson,[110] Neville,[111] Picken,[112] and others. Space will not admit even a survey of the contributions made to our subject by these authorities, but no discussion of the function con-

[108] Whitehead, A. N. "Presidential Address to London Branch of Mathematical Association." *Mathematical Gazette*, March, 1913, p. 89. A contribution of very great importance.

See also "Presidential Address, Mathematical Association." *Mathematical Gazette*, January, 1916.

"The Principles of Mathematics in Relation to Education." *Proceeding of the Fifth International Congress*, p. 449. Cambridge, 1912.

An Introduction to Mathematics. London, 1911.

[109] Nunn, T. P. *The Teaching of Algebra (including Trigonometry)*. London, 1914, 1919, 1927.

See also *Exercises in Algebra.* Part I (1913), Part II (1914).

"The Calculus as a Subject of School Instruction." *Proceedings of the Fifth International Congress*, p. 582. Cambridge, 1912.

See also the following articles in the *Mathematical Gazette:*

"The Arithmetic of Infinites." Dec., 1910; Jan., 1911.

"The Sequence of Theorems in School Geometry." May, 1922.

"The Aims and Methods of School Algebra." Dec., 1911; Jan., 1912.

"The Differentiation of a^x." May, 1926.

"Asymptotes." May, 1929.

[110] *See* the following articles in the *Mathematical Gazette*:

Carson, G. St. L. "Some Unrealised Possibilities in Mathematical Education. March, 1912.

"The Various Uses of Graphs," p. 265. March, 1914. An acute analysis of the problem.

"Intuition." March, 1913.

See also *Mathematical Education.* London and Boston, 1913.

[111] *See* the following articles in the *Mathematical Gazette:*

Neville, E. H. "The Tracing of Conics." Jan., 1921.

"Limits in Geometry." May, 1931.

"The Cubic Equation as a Relation between Complex Variables." March, 1927.

See also many valuable reviews of mathematical books by the same authors.

[112] Picken, D. K. *The Number System of Arithmetic and Algebra.* Melbourne, 1923.

See the following articles in the *Mathematical Gazette* by the same author:

"Ratio and Proportion." Jan., May, 1920.

"The Approach to the Calculus." Oct., 1927.

"Parallelism and Similarity." Oct., 1924.

"Some General Principles of Analytical Geometry." July, 1923.

"The Complete Angle and Geometrical Generality." Dec., 1922.

[113] *See* the following articles in the *Mathematical Gazette:*

Dobbs, W. J. "Coördinate Geometry in Schools." Jan., March, 1920.

Phillips, E. G. "The Teaching of Differentials." July, 1931.

"The Teaching of Analysis." Dec., 1929.

Knowles, W. "The Teaching of Easy Calculus to Boys." March, May, 1914.

cept in school mathematics would be complete without a reference to Nunn's *Teaching of Algebra.*

Nunn's Teaching of Algebra. This great work, worthy of a place among the classics of educational literature, has exerted a profound influence on the teaching of mathematics, not only in England, but also in America. One has only to compare present-day textbooks of mathematics with those of twenty years ago, when Nunn's *Exercises in Algebra* was in the making, to realize the truth of this statement. Formal exercises in substitution, addition, subtraction, multiplication, and division, simple equations, and 'problems leading to simple equations' have given way definitely and finally to formulas, graphical representation, the graphs of statistics, and the graphs of functions, all of which are included in the first section of *Exercises in Algebra.*

Nunn's *Teaching of Algebra* is, in essence, a treatise on the mathematical concept of functionality; it is more, it is a treatise on functional thinking in life:

> Mathematical truths have always two sides or aspects. With the one, they face and have contact with the world of outer realities lying in time and space. With the other, they face and have relations with one another. . . . The history of mathematics is a tale of ever-widening development of both these sides.[114]

And, again:

> Progress has brought about, and, indeed, has required, division of labour. A Lagrange or a Clerk Maxwell is chiefly concerned to enlarge the outer dominion of mathematics over matter; a Gauss or a Cantor seeks rather to perfect and extend the minor realm of order among mathematical ideas themselves. But these different currents of progress must not be thought of as independent streams. One never has existed and never will exist apart from the other. The view that they represent wholly distinct forms of intellectual activity is partial, unhistorical, and unphilosophical.[115]

Nunn does not often use the term 'functionality', but no one could read this volume without comprehending its inner meaning. In the forefront of his discussion he places the *variable.* Some of his remarks on the subject of variables are worth quoting:

> Everyone knows that mathematics is essentially concerned with 'variables'. . . . What is *not* generally noticed is that variables are almost as common outside mathematics as within. Thus, in the statement:'The King of England' is a variable in exactly the same sense as V in the formula $V = Ah$.[116]

[114] Nunn, T. P. *The Teaching of Algebra*, p. 16.
[115] *Ibid.*, p. 16. [116] *Ibid.*, p. 7.

Nunn gives an important place, indeed, to the variable, when he says:

> The invention of variables was, perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being.[117]

If the concept of the variable was given pride of place among the concepts of mathematics, that of the function, the connection between variables, was made subordinate to it only by the fact that it was rather more complex. The idea of the function dominates the first chapter (on graphical representation); it is no less prominent in the second chapter where the formula is discussed:

> In trying to give an account from the numerical standpoint of the concrete things with which his formulas deal, the young algebraist can hardly fail to notice and to become interested in the fact that 'variables' of widely different character are yet often bound to one another by identical quantitative laws. From that moment onwards it is natural to give an increasing amount of attention to these general forms of connexion between variables. Eventually—under the rather forbidding name of 'functions'—they may become the main object of study.[118]

In other words, the function concept is a strand, which, by holding the variable threads together, unifies and strengthens the whole. A novel feature of the course proposed is the introduction of the concepts of the calculus at a very early stage. Nunn reverses the usual order of presentation of this subject and takes the integral calculus first under the guise of 'area functions.'[119]

Our conclusion, after a careful survey of the relevant literature is that *The Teaching of Algebra* is the most convincing treatise on the function concept in school mathematics that has yet appeared.

THE REFORM MOVEMENT IN AMERICA

The reform movement in America may be said to have begun with E. H. Moore's presidential address to the American Mathematical Society in 1902.[120] Although the function concept was not men-

[117] *Ibid.*, p. 7.
[118] *Ibid.*, p. 7.
[119] *Ibid.*, p. 247.
[120] Moore, E. H. "On the Foundations of Mathematics." *Bulletin of the American Mathematical Society*, Vol. IX, p. 402. 1902.
See also *Science*, Vol. XVII, p. 401.
The First Yearbook, National Council of Teachers of Mathematics, p. 32. New York. 1926.

tioned in that address, many mathematical concepts incidental to functional thinking were strongly emphasized. The idea of functional relationship was brought out by Moore, a few years later, in an illuminating article on the use of squared paper. This article, which describes some unusual methods of graphical representation, will still repay careful study.[121] As in Germany, so in America, we have to go to the meeting of the International Commission of 1908, for the beginning of the movement towards 'functional thinking.' Some years elapsed before the full import of Klein's thesis began to be fully realized in America, but the way was being prepared for its acceptance by the growing attention to graphical work in schools.

The function concept. Among the first exponents of the function concept in America were David Eugene Smith and E. R. Hedrick. The former has always been sympathetic but cautious in his utterances on this subject and has shown an unwillingness to accept any doctrine wholeheartedly until it has proved its worth in practical experience. His observations on the subject of the function concept will be of special value to us, seeing that they come from one who has been for many years in close touch with the theory and practice of mathematical education in other countries. The following is taken from a report presented to the Fifth International Congress held in Cambridge in 1912:

> The second important question relates to the treatment of the function concept. Here the rôle of intuition, in the first steps, is more clearly defined, since we have no well-[illegible]d body of knowledge to be set aside. The chief argument for the elaboration of the function concept seems to be that the calculus has already found a place in the schools under our consideration, and, if it is to hold its place and continue to grow in strength, we must cease to impose it merely from above—we must prepare for it from below. The notions of limit, variability, rate, function, and graph must be so gradually introduced that when the calculus is reached they will be met as we meet familiar friends.[122]

Important contributions to the function concept. Hedrick's first important contribution to the subject appeared about a year

[121] Moore, E. H. "Cross section Paper as a Mathematical Instrument." *School Science and Mathematics*, Vol. VI, p. 421, 1906. See also *School Review*, Vol. VI, p. 317, 1906.

[122] Smith, D. E. Report of an inquiry into "Intuition and Experiment in Mathematical Teaching in the Secondary Schools." *Proceedings of the Fifth International Congress of Mathematicians*, p. 616. Cambridge, 1912. See also *L'Enseignement mathématique*, 1912, p. 514.

earlier than this. In 1911, he wrote an article, "On the Selection of Topics for Elementary Algebra," in which he stressed the vital importance of functional ideas in elementary algebra. Few discussions of this subject have revealed such insight into the real implications of the function concept as this. One is tempted to quote extensively from Hedrick's article, but the following passages will indicate the temper of the whole:

The chief direct value of algebra, in fact the real subject matter of algebra, aside from the rather insignificant chapter of shorthand which I have mentioned, consists of variable quantities, the relations between variable quantities, and the acquisition of the ability to control and interpret such relations.

Algebra emerges strengthened and beautified, no longer needing an apologist, but manifesting itself as a true need of the modern world, which is, both in its manifold scientific enterprises and in its everyday affairs, vitally interested in controlling and interpreting the relations between varying quantities.[123]

It is doubtful whether any more eloquent statement of the value of algebra has appeared in American pedagogical literature since these words were written. Judging by the lack of comment following the publication of this article, one would conclude that teachers of mathematics had missed its real significance. Comparatively few references to the function concept or to functional thinking are to be found in American mathematical literature for the next ten years. It must not be inferred, however, that progress was not being made. Relevant and cognate ideas were being discussed under the titles of graphs and formulas. In many articles, notably those of Dines,[124] Lunn,[125] Kinney,[126] Nyberg,[127] and Jackson,[128] attention was being drawn to the fundamental importance of relational thinking in the treatment of graphs and formulas. Thus teachers were well prepared to receive Hedrick's second article which

[123] Hedrick, E. R. "On the Selection of Topics for Elementary Algebra." *School Science and Mathematics*, Vol. II, p. 7, Jan., 1911.

[124] Dines, L. L. "The Development of the Function Concept." *School Science and Mathematics*, Vol. XIX, p. 99, Feb., 1919.

[125] Lunn, L. E. "A Suggestive Approach to the Use of the Functional Notation." *School Science and Mathematics*, Vol. XVIII, p. 480, May, 1918.

[126] Kinney, J. M. "The Function Concept in High School Mathematics." *Mathematics Teacher*, Vol. XV, p. 484, Dec., 1922.

[127] Nyberg, J. A. "The Teaching of Graphs." *School Science and Mathematics*, Vol. XXI, p. 144, Feb., 1921.

"Teaching Formulas in the Junior High School." *School Science and Mathematics*, Vol. XXI, p. 400, May, 1921.

[128] Jackson, Dunham. "Variables and Limits." *Mathematics Teacher*, Vol. IX, p. 11, Sept., 1916.

appeared in the *Mathematics Teacher* of April, 1922,[129] and for the recommendations of the National Committee on Mathematical Requirements on the function concept, which immediately followed it. Just as the report of the National Committee on Mathematical Requirements[130] is generally recognized as a landmark in the history of American mathema al education, so Chapter VII of the report, "The Function Concept in Secondary School Mathematics," is recognized as the first authoritative statement of the case for functional thinking t . found in American mathematical literature. The first draft of this chapter was prepared by Hedrick himself, and was, in its main essentials, similar to the article to which we have referred. In this epoch-making report the National Committee laid a foundation for mathematical education that will stand the test for many a year to come. It could hardly be otherwise with an aim so broadly conceived:

> The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.[131]

The fundamental importance of the *functional* relation is clearly set forth in the body of the report as follows:

> The one great idea which is best adapted to unify the course is that of the *functional relation*. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course.[132]

The special chapter, "Function Concept," is a comprehensive

[129] Hedrick, E. R. "Functionality in Mathematical Instruction in Schools and Colleges." *Mathematics Teacher*, Vol. XV, p. 191, April, 1922.

See also Webb, Harrison. "Professor Hedrick's Report on the Function Concept." *Mathematics Teacher*, Vol. XV, p. 3(;, Oct., 1922.

[130] National Committee on Mathematical Requirements (a committee of the Mathematical Association of America). *The Reorganization of Mathematics in Secondary Education*, 1923.

[131] *Ibid.*, p. 11.

[132] *Ibid.*, p. 12.

treatment of the whole subject. It begins by emphasizing the importance of functional thinking in life and then proceeds to show that the concept of functionality is implied in any rational treatment of formulas, equations, graphs, variation and proportion, congruence and similarity, and that not only algebra, but *all* the subjects of the mathematical curriculum come within its purview. The vital importance of relational thinking in everyday life is well expressed in the following passage:

> Indeed, the reason for insisting so strongly upon attention to the idea of relationships between quantities is that such relationships do occur in real life in connection with practically all of the quantities with which we are called upon to deal in practice. Whereas there can be little doubt about the small value to the student who does not go on to higher studies of some of the manipulative processes criticized by the National Committee, there can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other.
>
> To attain what has been suggested, the teacher should have in mind constantly not any definition to be recited by the pupil, not any automatic response to a given cue, not any memory exercise at all, but rather a determination not to pass any instance in which one quantity is related to another, or in which one quantity is determined by one or more others, without calling attention to the fact, and trying to have the student "see how it works." These instances occur in literally thousands of cases in both algebra and geometry.[133]

Wrong interpretation of the function concept. It is difficult to understand why, after the publication of such a thoroughgoing discussion of functional thinking, so many writers of school textbooks should have fallen into the error of supposing that the function concept was synonymous with the graphical representation of functions. Yet such has been the case. Few seem to have grasped the idea that the function concept is a mode of thinking rather than a method of illustration. In the ten years that have elapsed since the preliminary report of the National Committee was published in 1922, several writers have pointed out this fact, but none of them have dealt adequately with the psychological or philosophical bases of their subject.

More recent contributions on the function concept. Of the writers who have made outstanding contributions to the literature on this subject in recent years special mention must be made of the work of Georges and Breslich. On the practical side, the work

[133] *Ibid.*, p. 65.

of Swenson[134] in the Wadleigh High School, New York, and of Vevia Blair and Vera Sanford, in the schools associated with Teachers College, seems to be outstanding. To David Eugene Smith, partly through his writings and partly through his guidance of experiments on the teaching of the calculus,[135] we owe much of our knowledge of the deeper meaning of functional thinking.

The writings of Georges are remarkable for their detailed analysis of certain mathematical abilities. Among the contributions that he has made to the subject of functional thinking the most valuable is that contained in an article contributed in 1929 to *School Science and Mathematics.*[136] In this article he gives an exhaustive list of abilities which may be included in the term 'functional thinking'. According to Georges, functional thinking involves three main abilities. First, the ability to recognize mutual dependence between variables and varying quantities; second, the ability to determine the nature of the dependence or relationship between variable quantities; and, third, the ability to express and interpret quantitative relationships.[137] In other words, the *recognition, interpretation, and utilization of relationships are the heart and soul of functional thinking.* The writer goes on to discuss the mathematical implications of these abilities, in the formation of concepts, the acquisition of skills and the development of mathematical habits of thought. He proceeds:

[134] Swenson, J. A. "Selected Topics in Calculus for the High School." *The Third Yearbook, National Council of Teachers of Mathematics*, p. 102.

Swenson, J. A. *A Course in the Calculus for Secondary Schools, with New and Original Treatments of Many Topics (Together with the Records of Seven High School Classes in this Course)*, unpublished Doctor's dissertation, Teachers College Library, New York.

Norgaard, M. A. "Introductory Calculus as a High School Subject." *The Third Yearbook, National Council of Teachers of Mathematics*, p. 93. New York, 1928.

[135] Nordgaard, M. A. *Op. cit.*, pp. 91 92.

Rosenberger, N. B. *The Place of Elementary Calculus in the Senior High School Mathematics.* New York, 1921.

[136] Georges, J. S. "A Supplementary Project in Functional Graphs." *Mathematical Teacher*, Vol. XIX, p. 174, March, 1926. Illustrated with a project on the graphs of cubic equations.

See also the following articles in *School Science and Mathematics:*

"Functional Relations and Mathematical Training," Vol. XXIV, p. 680, Oct., 1926.

"The Properties of Relationships in Elementary Mathematics," Vol. XXX, p. 271, March, 1930.

"On the Nature of Algebraic Language," Vol. XXVIII, p. 135, March 1928.

[137] Georges, J. S. "Functional Thinking as an Objective in Mathematical Education." *School Science and Mathematics*, Vol. XXIX, pp. 508 and 601, May, June, 1929.

The various mathematical concepts, principles, processes, and methods which directly or indirectly contribute to the formation of correct habits of functional thinking, to the acquisition of skills in the manipulation of quantitative method and the development of ability in the expression and interpretation of functional relationships are classified as (1) measurement, (2) representation, (3) variation, (4) relationships, (5) transformations, and (6) generalization.

The term 'transformation' is used in rather a special sense. According to Georges, relationships, whether between abstract mathematical elements or facts of experimentation, are statements of algebraic *transformations* (variations). "The mathematical theory of transformation is the logic of functionality." This article, and others by the same writer, must rank among the most valuable contributions that have been made to the subject of functional thinking.

No less stimulating have been the contributions made by Breslich, who combines a wide acquaintance of classroom technique with an accurate knowledge of mathematical theory. In a valuable article[138] contributed to the *Third Yearbook of the National Council of Teachers of Mathematics*, he analyzed several popular mathematical textbooks for the purpose of ascertaining the extent to which functional ideas are stressed and came to the conclusion that the function concept was not receiving the emphasis it deserved. He attributed this lack of emphasis to the tendency to relegate functional ideas to isolated chapters and sections, and gave some practical suggestions for a more systematic treatment of the whole subject. Breslich has given a much more elaborate discussion of the function concept in his recent book, *Problems in Teaching Secondary School Mathematics*.[139] In this book he goes back a step further than most writers and discusses functional thinking in elementary arithmetic. This discussion contains some very valuable suggestions on tabular arrangement, correspondence, and dependence in their application to arithmetic. The rest of his discussion covers familiar ground, but there is a wealth of illustration which should prove of very great value to the practical teacher. The bibliography at the end of each section of the discussion gives the authors and titles of practically every article on

[138] Breslich, E. R. "Developing Functional Thinking in Secondary School Mathematics." *The Third Yearbook, National Council of Teachers of Mathematics*, p. 42. 1928.

[139] Breslich, E. R. *Problems in Teaching Secondary School Mathematics.* Chicago, 1931.

the subject of functional thinking in American mathematical journals.[140]

Space will not permit us to review the literature on the function concept in detail, but we shall select a few articles which seem to call for special comment. David Eugene Smith has stressed the vital importance of functional dependence in the *Lesson of Dependence*.[141] He maintains that "the dependence of one quantity upon another is a phenomenon which has escaped only the most untutored savage. . . . Our lasting pleasures depend upon success in life, our success depends upon our efforts and upon our inborn qualities and so on through all that enters into our life here and hereafter."

In the collection of essays known as *Monographs of Modern Mathematics,* Bliss has discussed the function concept in its strictly mathematical bearings. Examples of functions of several types, continuous and discontinuous, are given, as well as illustrations of the interdependence of the derivative, the antiderivative and the definite integral. This part of the subject is treated by Bliss in an admirably lucid manner. The article has been written from the point of view of the pure mathematician, rather than from that of the classroom teacher. In the Introduction the author deplores the lack of unity in school mathematics:

> Topics related perhaps inherently but with no indicated relationships follow each other in a confusion of radicals, exponents, progressions, imaginaries, probabilities, and other algebraic conceptions, in a way which must tend to develop a very disjointed understanding on the part of the beginner.[142]

Then he proceeds:

> It is one of the purposes of the present paper to show that this lack of unity may be remedied with the help of a very important conception which is called the function.

[140] Since the above was written, *The Seventh Yearbook, National Council of Teachers of Mathematics* has appeared, with a valuable article by Breslich, "Measuring the Development of Functional Thinking in Algebra."

Other articles in *The Seventh Yearbook* on the function concept are:

"The Function Concept in Elementary Algebra," by N. J. Lennes, a useful survey of the subject, and a very interesting and original article, "*The Function Concept and Graphical Methods in Statistics and Economics,*" by W. Lietzmann, of the University of Göttingen.

[141] Smith, D. E. "The Lesson of Dependence." *Mathematics Teacher*, Vol. XXI, p. 214. April, 1928.

[142] Bliss, G. A. "The Function Concept and the Fundamental Notions of the Calculus." *Monographs on Topics of Modern Mathematics Relevant to the Elementary Field*, p. 264. London, 1927.

In this endeavor he has been, in a certain degree, successful, but he has only particularized a unity that has been recognized ever since the time of Dirichlet. There is very little in the article which would not be known to any teacher of mathematics who had studied the elements of function theory. The author has not included in his discussion those elementary mathematical ideas and skills, which teachers of elementary mathematics have sought to bring under the unifying conception of *functionality*. There is a danger lest we should assume the function concept to be an elementary form of function theory. The report of the National Committee on Mathematical Requirements has warned us against this assumption, "It will be seen that in what follows there is no disposition to advocate the teaching of any sort of function theory."[143]

In recent years there has been an increase in the number of articles of a practical nature emanating from the mathematics classroom. Of these we may mention the contributions[144] of Blank, Booher, Christofferson, and Dresden. Miss Blank attacks her subject in a very broad way and bases her treatment on the idea of variation. She warns us against the danger of treating functionality as a chapter in mathematical study:

> Variability of functionality is, as it were, one of the themes of the symphony coming up again and again, always in a new guise equally interesting, equally novel, related to the former version in the different key, familiar yet almost unique, often coming upon one unawares.

Truly a case has been made for mathematics as a branch of aesthetics! Miss Booher uses Ligda's *Teaching of Elementary Algebra* as the basis of her method of selecting problems for class work. She maintains that the notion of functionality (not the term) should be present in school algebra from the first lesson, for "the ability to see relationships is the very essence of intelligence." Her method includes plenty of oral discussion of formulas, practice in tabulation and in graphical representation. Somewhat simi-

[143] *The Reorganization of Mathematics in Secondary Education*, p. 64.

[144] See the following articles in the *Mathematics Teacher:*

Blank, Laura. "Variability and Functionality in High School Mathematics," Vol. XXI, p. 405, Nov., 1929.

Booher, Eleanor E. "The Use of the Function Concept in First Year Algebra," Vol. XVIII, p. 80, Feb., 1926.

Christofferson, H. C. "The Graph as a Means of Picturing Relationships," Vol. XX, p. 227, April, 1928.

See also Dresden, A. [illegible]e Place of the Function Concept in Secondary School Mathematics." *School Science and Mathematics*, Vol. XXVII, p. 576, June, 1927.

lar suggestions are advanced by Christofferson, who shows how the graphical representation of relationships can be used to vitalize school mathematics: "The use of graphs to show the trend of statistical data, to help scientific experimenter in deriving formulas, to solve maximum and minimum problems, and to picture all sorts of relationships is a powerful tool in mathematics."

No discussion of the function concept in American schools would be complete without a reference to the work of Swenson, who perhaps more fully than any other, has exemplified the spirit of the function concept in his own teaching. The writer has had the privilege of intimate association with him over a period of nearly two years, during which time he followed, with increasing admiration, his work, both in the classroom and on the lecture platform. He has also had the still greater privilege of many hours of profitable discussion with him. His articles, "Selected Topics in Calculus for the High School"[145] and "Graphic Methods of Teaching Congruence in Geometry,"[146] and his doctor's dissertation, *A Course in the Calculus for Secondary Schools, with New and Original Treatments of Many Topics,* give a general idea of his method of approach, but they do not convey a real impression of the vitality of his teaching. In his hands the function concept pulsates with life.

[145] Swenson, J. *The Third Yearbook, National Council of Teachers of Mathematics,* p. 102.

[146] Swenson, J. *The Fifth Yearbook, National Council of Teachers of Mathematics,* p. 96.

V

THE FUNCTION CONCEPT AND THE SECONDARY SCHOOL

A study of representative textbooks. When we seek to determine the influence of any new educational idea upon the life and work of the schools, we usually follow one or more of several courses. We may visit the schools and, by personal observation and inquiry, form an estimate of the extent to which the new idea has been accepted, or study syllabi and programs, records and reports, and accept these as reliable indicators of the best current practice, or examine the textbooks in common use, working on the assumption that what is written in the books will be taught in the schools. We have, from force of circumstances, followed the third of these courses, and have examined, as comprehensively as was possible, the most popular textbooks in use to-day. This method is admittedly less reliable than that of first-hand observation, but it provides us with useful evidence upon which comparisons can be made. When due allowance has been made for the lag of inertia, textbooks do give some indication of the spirit of the age. In some schools, of course, the teaching is far in advance of that of the best textbooks; in others, it is far behind.

In the task of selecting books representative of those in current use in European schools we have received great assistance from eminent teachers in the countries concerned. Not only did these teachers go to considerable trouble to make selections for us, but they also supplied valuable observations on the progress of mathematical teaching in their schools. In this chapter we briefly review some mathematical textbooks extensively used in secondary schools in Europe and America.

Germany

The most modern textbooks of mathematics for secondary schools in Germany show unmistakable evidence of the influence of Klein. This is generally acknowledged in the preface, where statements, to the effect that the function concept as conceived by Klein is the underlying principle of the book, are sometimes to be found. The influence of the Meraner Vorschläge is seen, not only in the emphasis that is now being placed on the notion of function, but also in the obvious endeavor of the authors to relate the mathematics of the school to the practical problems of life. An attractive feature of some of the recent books is the large number of practical examples taken from physics, chemistry, and the social sciences.

It is impossible to give a unified sketch of German mathematical textbooks, since the books show a bewildering variety of treatment. Each type of school, as well as each grade, seems to possess its appropriate textbook. An additional difficulty is that no specified plan of study is obligatory upon all types of schools. The Prussian Department of Education has recently issued a standard Richtlinien, upon which the schools are expected to base their work, but this official program is suggestive rather than prescriptive. The result is that, while most of the books follow the Richtlinien in their general plan, they vary greatly both in content and in their treatment of the subject matter. Roughly, the books may be divided into two types: those which are revisions of standard works by popular authors, and those that have been specially designed to exemplify and inculcate the spirit of the reform movement. Of the books belonging to the two types, those of the second are undoubtedly the more interesting.

From a very long list we select the following:

Lehrbuch der Mathematik mit Aufgaben, by Behrendsen, Götting, and Harnack

I. *Unterstufe Teil I. Geometrie.* 1930.

This book written for students of classes Quarta to Untersekunda (13 to 16 years) is a very interesting and stimulating volume, for in it we find a just blending of the concrete and the abstract, the static and the dynamic, the intuitional and the rigorous. Plane and solid geometry, congruence and similarity, geometrical and algebraic proofs are found side by side to the obvious interest and benefit of the student. The book is not treated in a definitely functional manner, but many functional ideas are included; as, for example, in the discussion on the variation of the third side of a triangle with the variation of the opposite angle, the containing sides being of constant length. An excellent feature of the book is the clear account of conical and orthogonal projection, leading to descriptive geometry. The second half of the book is largely algebraic. In this we find the elements of coordinate geometry and several problems introducing the idea of a limit, e.g., the circumference of a circle as the limit of the perimeter of a polygon. A chapter on trigonometrical functions completes the book.

II. *Unterstufe Teil II. Arithmetik und Algebra.* 1929.

In this book, for classes Unterteria to Untersekunda (14 to 16 years), the function concept is given special attention, but almost entirely as the graphical representation of algebraic functions. A strong foundation is laid in number theory, leading to simple equations of one or two unknowns. This is followed by a very thorough treatment of the graphs of linear functions. The graphs of the quadratic, cubic, and reciprocal functions are also very clearly presented. In this book the function concept is prominent, but it cannot be said to be the central theme of the book.

III. *Oberstufe Teil I.* 1928.

This book has been written as a general textbook of mathematics for the Obersekunda and Prima (18 and 19 years). A great many mathematical ideas are gathered together in one volume: trigonometry, solid geometry (including stereometry), analytical geometry, algebra and complex variables, function theory, and the elements of the differential calculus. A special section is devoted to assurance mathematics and the elements of actuarial science. As one would expect from such a program, the idea of the function figures prominently, but no attempt is made to develop the notion of the function as a definite concept. All the elements are present for a first-rate textbook on the function concept, but the concept itself has not been crystallized. In this book, as in the previous volume, a function is taken to be equivalent to an algebraic expression and its graphical representation.

IV. *Oberstufe Teil II.* 1929.

This is a continuation of Volume I of the series and is intended for the highest class, the Prima. The book is divided into three sections: function theory, analytical geometry, and projective geometry. Function theory includes differential and integral calculus and algebraic series (Binomial, Taylor's, Maclaurin's) and analytical geometry is synonymous with analytical conic sections, including the general conic. The course on projective geometry embraces the main theorems of 'modern geometry', the complete quadrilateral, poles and polars, Pascal's and Brianchon's theorems. The treatment is comprehensive, but it is less thorough than that of representative English textbooks on the same subjects.

The remark that we have made with regard to the first volume of the Oberstufe applies also to this: The function concept is present, but is not fully manifest.

Elemente der Mathematik, by Reidt-Wolff-Kerst.

I. *Unterstufe. Arithmetik und Algebra.* 1930.

This volume covers much the same ground as the corresponding book in the Behrendsen series. A large part of the book is devoted to the formal operations of algebra, but much has been made of graphical representation, both in the appropriate places in the development of the subject (e.g., linear and quadratic equations) and in a special section on statistical graphs at the end of the book. The authors seem to us to reverse the normal psychological and logical order. It is surely more interesting and more educative to lead up to the graphs of functions through statistical graphs, which have the advantage of a natural concrete implication, than to take these subjects in the reverse order. The impression conveyed by this book is that of an effort to revive an old body by the infusion of a new spirit, but the result is still dominantly old.

II. *Unterstufe. Geometrie.* 1929.

This book corresponds to Behrendsen's *Unterstufe I*, and presents the same material in much the same way. Functional ideas, which were to be found in many places in the first series, are not to be found in this. There are, however, other features, such as the determination of area and volume by integration, correspondence in projection, and a discussion of limits, which are very valuable. The book abounds in practical problems and has many interesting concrete teaching devices. In this respect it seems to us to be superior to other German books of the same grade.

III. *Oberstufe. Arithmetik, Algebra und Trigonometrie.* 1928.

In this book we have one of the best examples of functional mathematics in school textbooks. In every chapter we find analytical processes clarified and enforced by graphical and other geometrical illustrations. The first part of the book deals with assurance mathematics and statistical analysis. These have been very clearly presented. The statistical chapter includes sections on probability, the binomial series, and the exponential function. By a series of well-chosen examples the student is finally led to the equation of the normal frequency curve. The next section begins with number theory and leads on to complex functions and the theory of equations (treated graphically). The most satisfying part of the book is that dealing with the differential and integral calculus, where the main concepts are developed in an admirable manner with the aid of graphical illustrations. A chapter on spherical trigonometry completes a most interesting and well-planned book.

IV. *Oberstufe. Geometrie.* 1928.

The first half of this book is devoted to projective and perspective geometry, and the second to analytical geometry. The whole book is characterized by the same careful planning and clear exposition that we have marked in Volume III of the series. The treatment of the first section does not differ in any great degree from the usual: much greater advantage could have been taken of the idea of function in the movement of figures and in places where the notion of limits could have been employed. The second section, by the very nature of the subject, is a treatise on the function concept but the authors seem more concerned to emphasize the facts resulting from the analysis than the concept itself. In this sense the treatment falls short of that of Volume III.

Aufgabensammlung und Leitfaden für Arithmetik, Algebra und Analysis. 1930.

I. *Ausgabe A: für Anstalten realer Richtung —Unterstufe*, by W. Lietzmann.

This book, which is based on Bardey's famous *Aufgabensammlung* comes nearer to catching the real spirit of the function concept than any other modern German textbook. The author has for some years been looked upon as one of the most energetic leaders of the reform movement and this book, although based upon a very old work, incorporates the new spirit most successfully. This is seen, not only in the prominence given to graphical representation (statistical and functional), but also in the excellent problem material, which is taken from mechanics, physics, chemistry, and commerce, as well as from pure mathematics. These examples have obviously been chosen to express the concept of function in a concrete form. The book covers the same ground as others written for this grade, but in a much more satisfying manner. A feature of the book is that the whole argument is carried on by a well-graded series of examples, without any explanations or 'worked examples'. The necessary explanation is given in a *Leitfaden*, or Theory Manual, at the end. This contains all the essentials of the theory in the form of a concise summary.

II. *Ausgabe B: für Anstalten realer Richtung—Oberstufe*, by W. Lietzmann and P. Zühlke.

An extension of the above book, this carries the subject to the Prima stage. The same features, to which we have referred in connection with the more elementary volume, are to be found in this. The book abounds in problems from science and commerce, all systematically arranged under such headings as velocity, work, moments, optics, and electricity. These problems are so selected that they exemplify functional dependence in a practical form. Thus the function concept is inherent in the material of instruction.

Rechnen für höhere Lehranstalten, Teile 1, 2, 3, by G. Wolff and B. Kerst. 1930.

These books have been selected as examples of the application of the function concept to elementary arithmetic. In this respect they are superior to any other books on this subject that have come under our notice in any language. Almost from the first page, exercises are given on the art of tabulation. These lead on to the reading and interpolation of results from tables. This feature of the book is excellent and constitutes a much better exercise in arithmetic than the formal drill that is almost universally found in schools to-day. Geometrical and graphical illustrations are employed freely, not as skills to be acquired but as teaching devices to elucidate the theory. The problem material has been well selected. Many of the examples are taken from real life and show a lack of artificiality that is exhilarating. The idea of approximate estimation is introduced very early and is used constantly throughout the course. The result is that when the subject 'approximations' is dealt with later, the student has all the relevant ideas in readiness. These books are an excellent preparation for functional mathematics in the secondary school.

France

Perhaps in no other country does the work of the schools conform so closely to the official educational program as in France. The same remark may be made regarding the textbooks of mathematics, the most recent of which are almost invariably written "entirely in conformity with the official programs of 3rd June, 1925." In spite of this restriction upon the material that may be included in the course, French textbooks show a great variety of treatment of the subject matter. Some written by university professors follow a direct logical order and abound in problems of an abstract type; others written by professors of *lycées* bring the subject into closer contact with the realities of life, including that of the classroom. The chief characteristic of French mathematical books is the logical development of the subject matter and the rigor of the treatment. Very little is taken for granted. Although, as we have already seen, algebraic and trigonometrical functions have been included in the official programs for many years, the textbooks do not show any definite bias in the direction of functional thinking. Graphs are used, and used effectively, to illustrate the results of analysis, but not as instruments for the development of the thought. The term 'analytical function' occupies a prominent place in French textbooks, but generally as a synonym for an algebraic expression. From the large number of books available, we have selected for review those which, in the opinion of several eminent teachers, were deemed to be most popular. These books are used in *collèges* and *lycées* of secondary grade, which aim at preparing students for the *baccalauréat*, the entrance examination to the University. The secondary course is of seven years (sixth class to first class, followed by a special class known as *classe de mathématiques*).

Algèbre (classes de 3e, 2e, 1e), by Borel et Montel. 1926.

The mathematical textbooks of Borel have been deservedly popular, not only in France, but also in other European countries. Translations of his books have appeared in Germany, Russia, Bulgaria, and even in Japan. This book, which is designed to meet the needs of those who have already learned the elements of algebra, surveys the whole field, from positive and negative numbers to the graphs of homographic functions. The treatment is strictly logical and analytical, but geometrical diagrams and graphs are used to illustrate the argument frequently. The first part of the book deals with positive and negative numbers and the fundamental skills and concepts of algebra. Velocity is introduced to illustrate the product of two signed numbers. The rest of the book is almost entirely devoted to variables and functions, a variable being defined as follows: "We call a number which can take different values a variable." The treatment is dominantly algebraic, but the results obtained by analysis are illustrated, sometimes in a very illuminating way, by graphs. In the theory of quadratic equations, which has been most exhaustively treated, the graphs are particularly effective. Another interesting section is that on homographic functions and their graphs. An attractive feature of the book is the appeal to kinematics for problem material.

Algèbre (*classe de mathématiques*), by Borel and Montel. 1929.

The first half of this book is a résumé, with harder examples, of the *Algèbre* for lower classes, supplemented by a detailed treatment of inequalities, systems of equations, and the variations of functions. The second half of the book is devoted to the elements of the calculus. Here the treatment is masterly. By clear logical steps the pupil is led from the derivatives of polynomials to those of trigonometric functions, homographic functions, rational fractions, and irrational functions. The examples are taken largely from mechanics and are sufficiently difficult to tax the powers of the ablest students. The work is illustrated throughout by graphs.

Précis d'arithmétique (*classes de 6e, 5e, 4e, 3e*) (1928), and *Précis de géométrie plane* (*classes de 4e, 3e*) (1925), by P. Chenevier.

These books are typical of those used in the junior classes of secondary schools in France. In the book on arithmetic, functional notions appear in the discussion on proportion, interest and problems, but the idea is not carried far. The elements of algebra come into the program as generalized arithmetic. The treatment of geometry is, from our point of view, more interesting. The author states, in the preface, that he has abandoned the Méray methods of proof that depend upon the displacement of figures (e.g., in the parallel theorems), but he freely employs the notion of limits (tangents to a circle, the circumference of a circle as the limit of inscribed and circumscribed polygons).

Cours d'algèbre (*classes de 3e, 2e, 1e*,) by P. Chenevier. 1926.

This book covers the same ground as the corresponding work of Borel and Montel. Graphs have been treated in great detail, the author's contention being that the graph is to be looked upon "not as a vague schema of a trend or a useful picture," but as a precise representation of a functional law, from which much may be learned. Special care has been taken to lead up to difficult concepts gradually. Thus the term 'increases indefinitely' is used from the beginning in preparation for the concept 'approaching infinity'. The treatment of logarithms is particularly interesting.

Cours de géométrie plane (*classe de 2e*) (1927), and *Cours de géométrie dans l'espace* (*classe de 1e*) (1928), by P. Chenevier.

These books, taken together, form a comprehensive treatment of the whole of elementary geometry. The author states that he has abandoned "displacement" methods of proof as given by Méray, but he uses the ideas of translation and rotation very effectively, not in demonstrating logical proofs, but in showing how a great variety of cases may be developed as illustrations of one general theorem. Although he does not use superposition in his proofs of congruence, he shows that any triangle, assumed unalterable as to shape may be moved to a new orientation by a single translation and a single rotation. Again, the author makes constant appeal to the idea of limits, his demonstration of the idea that the circumference of a circle may be regarded as the limit between the perimeters of the inscribed and circumscribed polygons being particularly stimulating. Functionality is strongly emphasized in the second of the two volumes, where algebraic proofs are employed.

Cours de géométrie (classe de mathématiques), by P. Chenevier. 1929.

This is an excellent summary of higher geometry for schools, including all that we usually include under the term 'modern geometry', together with conic sections, analytically treated. Functional ideas are to be found, but the book is not designed to bring out this aspect of geometrical thinking.

Arithmétique (classes 6e, 5e) and *Éléments de géométrie plane (classes 4e, 3e)*, by Brachet and Dumarque. 1927.

In the first of these books arithmetic is given a broad meaning, for the book contains arithmetic, algebra, and the elements of practical geometry in about equal proportions. The treatment is clear and the applications practical, but there is no attempt to develop functional ideas. In the *Géométrie*, the authors use motion proofs freely. In this, as in most French books, the parallel postulate assumed is the following: "Two perpendiculars at different points of a given line cannot meet."

Arithmétique (classe de mathématiques), by Brachet and Dumarque. 1929.

The word 'arithmetic' is given a much wider connotation in France than in any other country. *Arithmétique*, as we have it here, covers the whole field of algebraic numbers, their constitution and their operations. The subject is rigorously treated and, while it includes many ideas with which we are familiar in our schools (irrationals, imaginaries, powers, and approximations), it also treats many topics that find no place in our school mathematical studies. The last part of the book, for example, deals with continued fractions, modular classes, congruences, indeterminate equations, and the theory of groups (including Abelian groups). Although the functional notation is employed throughout, the book is not a treatise on function, nor is it designed to develop functional ideas. It deals with mathematical computation or calculus. In this regard it is a necessary supplement to the study of analysis.

Précis d'algèbre (classe de mathématiques), by Brachet and Dumarque. 1930.

This book covers the same ground as the other books for the same class to which we have referred. As the title suggests, the book is a summary of the essentials of higher algebra, rather than a readable treatise. The treatment is concise and, at the same time, thorough. Considerable space has been given to the concept of the variable, which is defined as follows: "If x is a letter capable of taking various numerical values, we say that it is a variable." One of the most interesting chapters is that on the limit of a function of one variable, in which most of the common cases of limits, e.g., $\sin x/x \rightarrow 1$, when $x \rightarrow 0$, which are to be used in later analytical work, are treated.

Hungary

Although, under the stimulus of a popular journal for high school students and under the inspiration of a number of progressive teachers, the movement for mathematical reform has progressed very rapidly in Hungary, textbooks do not seem to reflect the influence of that movement to any marked extent. Graphical methods and some functional ideas have been introduced into elementary books on arithmetic and algebra, but the function concept as we conceive it, and as it seems to be understood by the leaders of the reform movement in Hungary, does not figure prominently. This is probably due to the fact that in Hungary, as in England, analytical geometry and the calculus are studied from separate books on those subjects, whereas in Germany and France the whole of the analytical work is coördinated in one general course of study. More in consonance with the reform movement is an excellent series of elementary textbooks by Professor A. Szenes, and an equally admirable series of advanced books by Professor P. Veress.

Mennyiségtan (számtan—könyvviteltan és algebra—Mértan), Rész I, II, III A, III B, by A. Szenes. 1926-1927.

The first of these volumes deals mainly with the fundamental operations of arithmetic. Numerous examples of great practical interest, culled from actual records of vital statistics, are given to be treated graphically. Comparisons are made between the agricultural and economic condition of the country, its exports and imports, and its transport facilities before and after the war, which suggest an effort to correlate mathematics with national economics. The second part of the book deals with elementary geometry, which is developed, in a very attractive manner, through practical surveying and design.

The second volume takes us through the fundamental skills of algebra, which are very clearly presented by graphical methods. The most interesting part of the book is the treatment of proportion, direct and indirect, which is very skilfully developed through practical examples. Graphs are used with great effect in the treatment of profit and loss, taxes, insurance, and household accounts.

The third volume contains an admirable treatment of positive and negative numbers, somewhat similar to that given by Nunn. Altogether the algebraic section of this book is in accordance with the best modern practice. The section on geometry is mainly a study of mensuration (cylinder, cone, sphere, etc.) and of the main geometrical concepts (point, line, parallels, and symmetry). As in the first volume, much use is made of surveying. The second part of the book deals with more advanced arithmetic and algebra and with the elements of demonstrative geometry.

In this book much attention is given to graphs. Some of these are concerned with railway timetables in the manner suggested by Klein. A feature of these four volumes is their practical outlook. In the earlier books most of the problems are taken from business life and in the later books from physics.

Elemi mennyiségtan (algebra és geometria), a gimnáziumok és reálgimnáziumok. VII Osztalya Számára, by Pal. Veress 1930.

I. *I Kötet* (VII class).

In the introduction to this series the author acknowledges his indebtedness to Klein. He says, "The only works which influenced me during the writing of these books were those of Felix Klein on the reform movement. In the introductory part of my geometry I also had recourse to the textbooks of Borel-Stäckel." The endeavor of the author has been to fuse algebra and geometry with the help of the function concept.

The first volume, intended for the highest class, opens with a section on coördinate geometry (straight line, circle, ellipse, parabola, hyperbola). This is followed by an elementary treatment of the calculus. The concept of functionality is not definitely developed until the question of limits arises in connection with the derivative. Thenceforth functional ideas are brought in freely. The section on integration is excellent. The integral is introduced as the reverse of differentiation and is developed almost immediately after as a summation. The treatment is not rigorous, but the question of rigor is discussed in order that students may realize that the proofs need modification, if they are to stand every test of the rigorist. The last part of the book is devoted to solid geometry but calculus methods are not employed although the ground was ready for them. On the whole, a successful effort has been made to clarify the function concept.

II. *II Kötet* (VI Class).

The book opens with numbers and series (A.P., G.P.) but the usual treatment has been improved by a very thorough study of interpolation. Interpolation from the graph is used later in the chapter on logarithms and exponentials. The discussion then goes to business mathematics (interest, annuities, rent, insurance, and assurance). Here functional ideas are very well illustrated. Graphs are used to clarify what would otherwise be difficult theories. There is again another sudden break into trigonometry (all six ratios), the treatment differing little from the usual. There is, at the end, an interesting chapter on practical trigonometry (surveying, etc.). On the whole, the function concept comes in for attention in this book, but it is not a coördinating theme, for very little coördination is evident.

III. *III Kötet* (Classes IV and V).

This book is a sound, but rather conventional, introduction to secondary school mathematics. The opening section on elementary algebra may be described as formal and old-fashioned. It opens with expressions and substitutions, and proceeds, very much in the style of thirty years ago, through the fundamental rules to equations. The second part of the book deals with elementary geometry. In this part there is an interesting chapter on loci, but otherwise the treatment lacks life. The next part of the book is devoted to graphs, the elements of coördinate geometry, and the applications of graphs to practical problems. This part of the book is excellent, dependency and function being the keynotes of the discussion. The last section includes a miscellaneous section on Pythagoras' Theorem, irrationals, and imaginaries.

There is much good material in this book, and many illustrations of functional mathematics, but it cannot be said to show up the function concept to the fullest advantage.

Austria

As we have already seen, teachers of mathematics in Austria were very quick to catch the spirit of the reform movement in Germany. This has been due partly to the work of the Austrian members of the I.M.U.K., among whom the most active was Dintzl, and partly to the influence of the books of Klein and Lietzmann. The Government Regulations of 1909 contained specific references to the function concept and prescribed a course which definitely kept that concept in view. Teachers were directed to make the connections between algebra and geometry as close as possible, thus facilitating the general application of the concept of dependence. Every encouragement is given in the early study of the subject "to measure, cut, fold, construct, and draw" that the concepts and the rules to be developed from these exercises should be rendered clear.[1]

Austrian textbooks show a marked tendency to follow the German models; in fact, in some schools books by Lietzmann and others have been adapted for use in Austria. The more recent books consist largely of collections of problems, the theoretical material of the ordinary textbook being included in an appendix, called the Leitfaden.[2] This, as we have already observed, is becoming the popular practice in Germany. The aim is to make the formulation and transcribing of principles entirely the work of the class or of the individual student. We have selected for review only one series, by E. Dintzl, which illustrates the concept of functionality admirably.

Mathematisches Unterrichtswerk für Mittelschulen (V and VI Klasse), by E. Dintzl. 1929.

Arithmetik.

This book, based on a popular textbook by Močnik and Močevar, is one of the most successful attempts to base middle school mathematics on the function concept. As the author has stated in the preface: "Especially strong emphasis has been placed on the function concept." The term 'arithmetic' is used in the wide sense common in French and German schools. In this book we find very little of what we should designate arithmetic; we find rather the

[1] Falk, K. "Mathematical Education in Austria" *The Fourth Yearbook, National Council of Teachers of Mathematics*, p. 6.
L'Enseignement mathématique, August, 1930, p. 253.

[2] Lietzmann, W. and Jarosek, J. *Arithmetik*, (*Klasse* 2–8). Wien, 1929. See also *Geometrie.* Wien, 1930.

elements of algebra and of intuitive and of demonstrative geometry. In the section on algebra the function concept occupies a prominent place. At every stage the opportunity is taken to bring out the significance of functionality, and in the special sections on graphical representation the treatment is excellent. Here we find the graphs of the function $y = x^n$, for negative as well as for positive values. The treatment of geometry, which is no less interesting, includes numerical computation, nomography, and the use of millimeter and logarithmic paper. Altogether this is one of the most interesting books on functional thinking that we have examined.

Hilfsbücher für den Mathematik Unterricht an höheren Lehrenstalten, by E. Dintzl. 1930.

Geometrie. (*Erster Teil*).

This series of books for the higher classes of the secondary school is, like the *Arithmetik*, noted above, definitely based on the concept of the function. The book, however, is more than this; it is a remarkably fresh and stimulating treatment of the whole subject of school geometry. It is sound in substance, well connected, and clearly presented. A remark should be made regarding the attractive setting-out of the subject matter. The diagrams, which are most clearly and artistically drawn, are of very great help in the elucidation of the text. In the section on the movement of figures by translation and rotation, the figures have been so clearly drawn, in lines of various thicknesses, that verbal explanation seems almost unnecessary.

The book covers the main essentials of demonstrative geometry, with an abundance of originals for the exercise of the student. The last part of the book is devoted to conic sections. In his treatment of this subject the author combines ordinary geometrical with algebraic methods in a most interesting way.

Arithmetik. (*Zweiter Teil*).

This book is, without a doubt, one of the most satisfactory books on the function concept to be found in any language. From the first page to the last the idea of functionality is evident. Not only is the concept of the function implicit, but the word 'function' is used, so simply and naturally, that it becomes no more difficult to the students than expression or coefficient. In the first paragraph we are told that "the atmospheric pressure is a function of the time," and are then given other examples to show what function really means. The table of contents, "Linear Functions, The Function $y = x^n$ for positive and negative Values of n, The Root Function, The Quadratic Function, The Exponential and Logarithmic Functions, and Complex Numbers," may give some indication of the ground covered in the book, but it gives no adequate idea of the convincing manner in which each topic has been treated. The chapter on exponential and logarithmic functions is particularly interesting, for the functions a^x, e^x, and $\log x$ are thoroughly discussed both algebraically and graphically, *before* the logarithmic rules are deduced.[3] This is, perhaps, slower approach to the use of logarithmic tables than the conventional

[3] See also Nunn, T. P. "The Growth Curve and the Gunter Scale." *Exercises in Algebra*, Vol. I, p. 269.

method, but it makes for a clearer understanding of the whole subject. Any teacher who has tried both methods will not doubt the advantage of the method here followed. At the end of the book there is an Appendix, containing statistical tables of various kinds, excellent material from which to select data for graphical problems.

Arithmetik. (*Dritter Teil*).

The excellent features of the second volume, to which we have referred, are also manifest in this extension of that work. The book is, in substance, an elementary treatise on analysis. The first section deals with series (arithmetic, geometric, logarithmic), and with the subject of interpolation. This leads on to interest and annuities, for which the usual formulas are derived, largely with the aid of graphs. A short discussion on infinite series follows. This is treated graphically and is illustrated by geometrical and physical problems in a very interesting manner.

The major part of the book[3] is devoted to the elements of the differential and integral calculus. After a short discussion on limits, the student is led to an understanding of rate of change through a problem in kinematics, which is worked out in great detail. Only after this discussion of the concrete problem is the differential coefficient introduced. The rest of the book follows the usual course: the derivatives of x^n (positive, negative, fractional), sin x, cos x, tan x, log x, a^n, e^n, are treated simply and clearly. The section on the integral calculus is introduced through a question in mechanics (work), the main concept being developed as a problem in summation. The applications are taken mostly from solid geometry and mechanics. This is followed by a chapter of miscellaneous theorems (Rolle's theorem, Newton's formula for the approximation of the roots, methods of iteration), the solutions of which are based on the work already done in the calculus. The last chapter is a very concise summary of the main ideas of probability and the theory of statistics. These two volumes on *Arithmetik*, taken together, form one of the most satisfying treatments of the function concept that we have found among school textbooks.

ENGLAND

A comparison of mathematical textbooks used extensively in England to-day with those of thirty years ago does not reveal the striking changes that one finds in the textbooks of Germany and America. There are several reasons for this. As a general rule, the teacher of mathematics in England uses a textbook as a source of examples, rather than as a treatise on mathematical procedure. He prefers that his pupils should get their methods from him, rather than from a particular textbook. His chief consideration, therefore, in choosing a book, is the variety and gradation of its problem material. He may, for that reason, prefer to retain a textbook, which provides him with an abundance of graded examples and to supplement it with a small book on some special topic. One finds, for example, Durell and Siddons' ***Graph Book*** used to make up the deficiencies of a textbook of algebra, which seems to have undergone little change in the editions of twenty years. Another reason for this apparent conservatism among English teachers is that, in preparing their pupils for examinations, they are often guided by past examinations, rather than by the published syllabus of the examination. Question papers vary so greatly, especially in the higher examinations, that it would hardly be possible to build a textbook "strictly in conformity with examination requirements." As a consequence, many teachers make their own textbook in accordance with their needs. Again, there is a tendency in England to encourage students to refer to the more advanced treatises for theoretical discussions of their subjects. In this way the school textbook is subordinated to works of a more profound type. It is not an uncommon experience to find that students preparing for the higher school examinations have read fairly extensively into the books which they will use as textbooks in their university courses.

In recent years there have appeared several new books written to meet the needs of the pupils attending the new central and senior schools. These schools are designed for those who are not destined for the university, and for whom, therefore, the matriculation examination has not the same absorbing interest. These new books show a freshness of treatment that is encouraging; intuitive and graphical methods are freely employed and the whole subject is brought into close relationship with life.

From a long list we have selected only those books which have emphasized some aspect of the function concept. We have omitted special books on analytical geometry and the calculus for schools, of which there are many. Some of these aim definitely at developing functional ideas through the subjects considered.

Exercises in Algebra (including Trigonometry), Part I (1913 and 1925), Part II (1914); and the *Teaching of Algebra* (1914), by T. P. Nunn.

We have already referred to these books in our historical survey of the development of the idea of functionality in English schools. They represent, we consider, an outstanding contribution to our problem.

It has been objected by some, that these books are too difficult for the average schoolboy, and by others, that the methods employed are too subtle for any but the superior teacher, but the writer has seen boys of less than average intelligence, in the hands of a mathematics teacher of only moderate ability, so engrossed in their algebra lesson (the Growth Curve and the Gunter

Scale) that they could with difficulty be persuaded to give their attention to another subject. We are convinced that the books are not so difficult as they are unusual.

Exercises in Algebra is not the work of a theorist, but of a practical teacher, who, while possessing a profound knowledge of the philosophy underlying his subject, has nevertheless made the psychology of the pupil his first consideration. The books lack the attractive format of the popular textbook, but they atone for this lack by the inherent interest of their problem material. Although the author does not lay particular emphasis on the function concept, or functionality, these terms being seldom mentioned, there is hardly a page in the whole work, which does not demand functional thinking, in an extremely active form. The titles of the first section of Part I give us a basis for this statement. They are: "graphical representation, formulæ, direct proportion, inverse proportion, proportion of squares and cubes, joint variation, and trigonometrical formulæ." All these topics are treated in a definitely functional manner. Moreover, the topics themselves are knit together by a unifying thought, that of functional relationship.

Space will not permit of a detailed description of the contents of these volumes, but we may refer to some of the sections which seem to us to be of particular value. Among these, we include: "directed numbers, the growth curve and the Gunter scale, area functions and Wallis's Law (Integration), the trigonometry of the sphere (map projections), exponential functions and their derivatives, functions of a complex variable, limits, wave motion and harmonic analysis, and the elementary theory of statistics." All of these subjects have been treated in a highly original and scholarly manner. In only one of these topics, directed numbers, has Nunn's method of treatment been extensively explored by modern textbook writers. This is probably due to the fact that the other topics are outside the scope of elementary mathematics, as the term 'elementary' is generally understood. We venture to suggest that teachers of mathematics can still find much that is worthy of their attention in *Exercises in Algebra*.

Elementary Algebra, Vols. I and II, by Godfrey and Siddons. 1928.

We have already noted in Chapter V that Godfrey was among the earliest advocates of functional thinking in school mathematics. In the Introduction to this book the importance of functionality is stressed: "The idea of graphical solution is certainly important, but this is not the fundamental idea that should underlie the use of graphical representation. The fundamental idea is that of functionality, the continuous change of $f(x)$ as x changes, the interdependence of two variables. This idea should be at the back of the teacher's mind all the time, and the pupil should be led—very gradually—to realise it with increasing distinctness." This fundamental idea has been given due consideration throughout. There is much in the book which may be styled formal, but every opportunity has been taken, in the treatment of formulas, equations, graphs, and variation, to press home the idea of dependence and functionality. The treatment of graphs given in Volume I is particularly thorough. In Volume II there is an unusually full discussion of variation, the various simple types of relationship being treated as cases of functional dependence. In this section is to be found one of the most comprehensive discussions of variation in school mathematical textbooks. This work may almost be described as a treatise on algebra, from the simplest notions of generalized arithmetic to the more difficult concepts of the calculus. No attempt has been made to treat the function concept as the unifying theme of the whole course, but a pupil who works through these books will know what it means to think in functional terms.

A New Algebra for Schools (Parts I and II), 1930, and a *New Algebra for Schools* (Part III), 1931, by C. V. Durell.

The author of these books on school algebra is recognized as one of the most progressive teachers of mathematics in England. One writer has asserted that we may accept the textbooks published by Durell during the past ten or fifteen years as reliable indicators of the best current practice in English schools during that period.

His books are mathematically accurate and show a fine appreciation of the problems and difficulties of the classroom. Always a little in advance of the time, they are sufficiently conservative to satisfy the requirements of those whose main objective is the examination. A reviewer of the first of these two volumes has described it as "the ideal textbook."[4] Whether one would agree with this encomium would depend on one's conception of an ideal textbook. If clarity and interest of treatment and range and gradation of examples be our criteria, then this book may justly claim a high place among modern textbooks. If, however, material for functional thinking be our objective, then this book will need to be supplemented. Graphs and functions find a place, it is true, but they are presented as examples of algebraic technique rather than as instruments of functional thinking. This is unfortunate, for no writer of the present day has the ability to present functional material in a more interesting way than the author of these volumes. This assertion is based upon an examination of the *Graph Book*, by Durell and Siddons, which is one of the most attractive presentations of graphs and their meaning yet published. It may justly claim to be the ideal textbook for graphical work in schools. It is to be hoped that it is the forerunner of the ideal textbook on the function concept, yet to be published.

A New Algebra, by Barnard and Child. 1925.

From the point of view of mathematical rigor this may be regarded as one of the most accurate of school textbooks. Partly for that reason and partly because of its interesting style, the book is one of the most popular in English schools. In their treatment of functions and limits, rational and irrational numbers, rational functions, limits and values, the authors have shown an appreciation of logical rigor which is rarely found in school textbooks. Functional dependence is stressed in the appropriate places, but this thought by no means dominates the book. In the chapters on approximation, variation, graphs, and approximate roots, the idea of dependence is brought out clearly; the chapter on variation being particularly thorough.

The aim of the authors is admittedly the logical development of the subject matter. In pursuing this aim they have been eminently successful. The

[4] *Mathematical Gazette*, Oct., 1931, p. 477.

treatment given to functional dependence may be considered adequate, but one cannot but feel that the function concept could have been given a more prominent place without impairing the logical excellence of the whole.

The student who works through this book will be well prepared to take up more advanced studies in mathematics; moreover, he will have nothing to unlearn. Whether he will have grasped the great lesson of relationship, especially that of mathematics to the world about him, is rather doubtful.

Elementary Algebra, by Baker and Bourne. 1930.

This book, one of the most popular of English textbooks, is the most recent of a number of revisions of a book first issued in 1904. From a superficial examination of the first and last editions, one would be inclined to draw the conclusion that there is little difference between the old and the new, but this is not so. The new material in the book deals almost entirely with functions. Graphical work is introduced much earlier and treated much more fully than formerly; there is much new material on variation; and the algebraic function has been very thoroughly treated in Part II. This book may be said to be representative of the type of algebra most extensively used in English schools. It covers the whole field from the very elements of algebra to fairly difficult work on the calculus. The abundance and variety of its problem material are remarkable. This is one of the reasons for its great popularity. Some of the examples are extremely easy, others are difficult enough to tax the powers of the ablest students. It should be mentioned that the authors of this book were among the first to introduce graphical work into English schools. Some years ago the chapters of their *Elementary Algebra* dealing with graphs were published as a separate volume. This book stood for many years as the standard textbook on the subject of graphs.

School Certificate Algebra, by G. W. Spriggs. 1930.

In the introduction to this book the author shows that he is conversant with recent tendencies in mathematical teaching. "Too much time has been spent in the past on the tricks of simplifying expressions and solving equations, and their application to the actual material of mathematics has been too long deferred—in most cases it is never made. The essence of the study is functionality, and, until this conception has been reached, there is practically no scope for the utilization of the skill that has been so laboriously acquired." In the actual working out of this principle he has shown rather more caution than is necessary. Nevertheless, he has succeeded in infusing something new even into the formal parts of school algebra. In the chapters, "The Formula" and "The Equation, Functions, and Series,' he has given remarkably clear illustrations of functional mathematics. In these chapters his debt to Nunn is obvious. Under "The Gradient of Graphs," he has included a unit of the calculus, in sufficient detail to enable the student to determine algebraic maxima and minima by using the first and second derived functions. In certain respects the book reminds one of some of the more recent German books. The geometrical representation of the convergence and 'sum to infinity' of progressions is similar to that given in modern German texts. The presentation of the subject matter is fresh and stimulating. Had the author been a little less concerned with "the best traditional practice," he would have produced a book of functional mathematics of great value.

Common Sense Algebra for Juniors, by Potter and Rogers. 1928.

This book is one of the most successful elementary textbooks published in recent years. The aim of the authors is very clearly stated in the preface: "This is in consonance with the general aim of the book, which is to emphasise *functionality* and change, an avenue to the methods of the calculus, which is now generally recognised as being the only real mathematical approach. Graphs and graphic methods are introduced at an early stage, at first with statistics and undirected numbers and quantities, and later with directed numbers." The result is a very attractive and stimulating book. The authors have had a wide experience of mathematical work in elementary and secondary schools, one as an inspector, and the other as a teacher. This is evident, not only in the development of the subject matter, which is psychological rather than logical, but also in the many excellent teaching devices. Students beginning algebra could not fail to be interested in the subject as it is presented in this book. There is much in the book that may be considered formal, but the functional idea is given a very prominent place. In another book, *Graphs and Their Applications* by Potter and Larrett (1931), we have one of the most successful attempts to develop functional ideas through the graph. Column graphs, travel graphs, conversion graphs, and graphs of functions are treated with admirable clearness.

A School Algebra, by A. M. Bozman. 1931.

In the Introduction to this book the author discusses the place of functions in school mathematics and gives it as his opinion "that those absorbing branches of mathematical study—analytical geometry and the calculus—are not subjects appropriate to the school course in algebra." He strongly advocates a preparation for these subjects by attention to graphs and gradients. The book is developed in accordance with this conclusion. There are some excellent chapters on graphs, statistics, travel graphs (loci), and functions. These are among the best that we have seen in English books. The rest of the book does not call for special comment.

A First Course in Algebra, by W. G. Borchardt. 1924.

This book follows what is now being recognized as a modern course: formulas, equations, generalized arithmetic, graphs (statistical and functional), directed numbers, quadratic equations (graphically illustrated), and formal algebra of a more difficult type. No functional aim has been specified, but the book would readily lend itself to functional treatment. The material is ready for a first-rate book on the function concept, but that concept has not been consciously developed.

A School Algebra, by H. S. Hall. 1926.

No review of English mathematical textbooks would be complete without some reference to the works of H. S. Hall. Many successful teachers of to-day learned the foundations of their subject through Hall and Knight's *Higher*

Algebra and Hall and Stevens' *Geometry*, both standard works in their day. *A School Algebra* is included in this review, not because of its functional ideas, but because it represents one of the first attempts (first published in 1912) to meet the demand for a fuller treatment of graphs and functions. The point of view of the author is given in the Preface: "Graphs are interwoven with the text, not so much for their own sake as for the purposes of illustration." Thus it follows that graphs are treated as a separate subject, rather than as a method of explicating algebraic equations or of developing the idea of functionality.

America

Even to the casual reviewer of school textbooks, the influence of the report of the National Committee on Mathematic Requirements on American mathematical textbooks will be apparent. Practically all revisions of works published before 1923, and all new works published since then, bear the impress of that epoch-making report. In some cases the influence of the report has been a superficial one; chapters on graphical representation, variables, and functions have been added in response to a demand, but their inclusion has not changed, in any essential feature, the formal character of the whole. In other cases, the influence has been more marked, for lately there have appeared books of entirely new form and embodying a new spirit. This is particularly true of books published since 1928, which, from the point of view of functional thinking, are a distinct advance on those of the older type. The impression created by a close study of American textbooks is that in many cases the authors have tried to satisfy too many demands. While endeavoring to follow the recommendations of the National Committee, they have at the same time tried to meet the requirements of the New York Regents' Examination and the examination of the College Entrance Board. The consequence is that the spirit of the National Committee's recommendations is in danger of being stifled by the formal exigencies of college examinations. Recent revisions of the requirements of these examinations lead one to hope that this conflict of objectives will, in the future, be obviated.

Although functional ideas are to be found in most of the textbooks published before 1923, we are confining our attention to those published since that date. Of the large number of books examined, comparatively few can be reviewed here. These have been divided into two classes: First, popular textbooks which, through several revisions, have stood the test of time; and, second, textbooks which have not such a wide influence as the above, but which seem to make a definite contribution to the subject of functional thinking in school mathematics.

It should be noted that we shall be concerned mainly with the presence of functional ideas in the textbooks under review. We have omitted all reference to other excellent features which have made some of these books deservedly popular. When we designate a book as superior, we have in mind its superiority from the point of view of its functional program.

New Elementary Algebra, by Wells and Hart. 1928.

This book, the latest of a series of revisions of a popular textbook, is an illustration of the effort to graft new teaching on an old stem. Bar graphs and other functional ideas are introduced in the first chapter, and the graphs of elementary functions are fully treated later, but the bulk of the book is devoted to the formal skills of algebra. The book contains functional notions, but it is not permeated with them.

Modern Algebra, by Wells and Hart. 1929.

In this book the authors are rather more successful than in the earlier work noted above. In the introduction they state: "Attention is called to the chapter on *Functional Relationship*. The desire to place in the hands of teachers and pupils a satisfactory treatment of this subject, which has come to be stressed in recent years, was one of the chief reasons for writing this new text." True to this desire the authors have given an exposition of functional relationship, which reveals a clear understanding of the difficulties involved, and obvious acquaintance with class technique. The excellence of this section of the book only serves to bring out more clearly the more formal character of the rest of the book. In this book the function concept certainly finds recognition, but it is incidental rather than constitutional.

New Mathematics, by J. C. Stone. 1929.

In the Introduction the author states that 'the underlying purpose of a course in elementary algebra is to develop the power to represent quantitative relationships by formulas and equations; the power to interpret such expressions of relationship; and to develop the skills needed in the computation which is required in using formulas and equations.' This aim has been satisfactorily realized. In the first part of the book the author treats the formula, its interpretation, and its graphical representation, clearly and exhaustively. The idea of functionality is developed b problems in which concomitant changes in variables are thoroughly discus l. The author then passes over to the treatment of the formal skills of algebra and returns to graphical representation in a later chapter. Had the spirit of the opening chapters been maintained the result would have been a work of great value.

A Second Course in Algebra, by Stone and Mallory. 1931.

In this the authors get much nearer to the functional ideal. In the Introduction they claim that "the idea of functional relationships is the unifying featu of the text," but we do not feel that they have substantiated the claim. The idea of functionality has been well brought out in the chapters on linear systems, linear and quadratic graphs, ratio, proportion and variation, but it can hardly be called a unifying feature of the text.

Elementary Algebra, by Edgerton and Carpenter 1929.

This is a commendable attempt to graft the new upon the old. The book opens with a chapter on graphical representation by bar, segment, and connected line graphs. The treatment is thorough, so much so that the material seems almost sufficient for a whole term's work. This is followed by an almost equally thorough treatment of the formula, leading to a discussion of the function and dependence. We pass from these rather searching chapters to the formal rules of algebra which, from the point of view of difficulty, cannot be compared with the earlier chapters. The book is badly balanced. A rearrangement of the material would add greatly to the value of the book.

A First Book in Algebra, by Durell and Arnold. 1928.

This book is also based on an earlier edition of a book of extremely formal type. The authors claim that "the Function Concept, or dependence, is demonstrated and emphasized throughout," but it is difficult to see how this claim can be substantiated. Even in the chapter on the formula, the idea of dependence is not strongly emphasized. Separate chapters are devoted to the graphs of functions and dependence, and here the treatment is clear and incisive, but the book is, on the whole, devoted to the formal skills of algebra, rather than to its functional ideas.

Modern Algebra. First Course, by Schorling and Clark. 1929.

The editor of the series states that, besides being in agreement with the recommendations of the National Committee, "this textbook is strictly in accord with the teaching of modern psychology." The particular conclusion of modern psychology, by which the authors have been guided, is that pupils learn facts in which they are most interested and which are related to their everyday needs. In applying this principle they have given us a book of considerable originality in which the formal processes of algebra are introduced through practical problems and developed in such a way as to lead to an appreciation and an understanding of mathematical ideas. This is shown in several places; for example, where the formula, equations, and graphs are developed through statistical problems. On the whole, a successful attempt has been made to exemplify functional ide s by means of practical problems, but the function is not the theme of the book.

Modern Algebra. Second Course, by Schorling, Clark, and Lindell (1929).

In this volume the authors have tried to satisfy the needs of candidates for the College Entrance Examinations and at the same time to preserve the psychological features of the *First Course*. The result is disappointing, for the freedom and spontaneity of treatment which characterized the first volume have been almost obliterated in the more formal and conventional material demanded by the colleges. The concept of functionality, so generously illustrated in the more elementary work, has been relegated to the chapters on graphical representation.

Elementary Algebra. First Year Course (1915) and *Second Year Course* (1917), by Cajori and Odell.

These books have been included in our list, although published more than ten years earlier than those discussed above, because they were definitely in advance of their time. In the preface the authors acknowledge their debt to Nunn and state, in the preface to the series, that "the concept of a *function* does not receive isolated and abstract treatment; it is presented as a fundamental idea in proportion, variation, and graphics. Its connection with problems of everyday life is firmly established." Although the books lack the attractive appearance of the modern textbook, the fundamental concepts of the subject have been treated more clearly than in any other American textbook that has come under our notice. Functionality may be said to be, if not the only, at least the main, theme of the book. Attention may be drawn to the treatment of the logarithmic function, which has been developed, in a most effective manner, by a graphical method.

A New First Course in Algebra, by Hawkes, Luby, and Touton. 1926.

This book is a popular one, and justly so, for it is clear and accurate and abounds in good teaching devices. But the authors have not caught the functional spirit. In the Introduction they state: "A new feature is Chapter II on graphs, where the fact is recognized that graphs have an informative use, which is widely exemplified in recent periodicals and the daily press." The result is that graphs are looked upon as illustrations, rather than essential mathematical symbols from which relationships may be interpreted and even deduced.

A New Second Course in Algebra, by Hawkes, Luby, and Touton. 1926.

This book possesses some very good features, but these relate to the method of exposition, rather than to the philosophy underlying the subject. The authors seem to have gone out of their way to dissociate mathematics from the physical phenomena which mathematical symbols represent. "In the study of algebra we are concerned with the mathematical formulas in terms of which the physical relations are expressed rather than the actual phenomena themselves." This is, we maintain, an entirely wrong attitude. School mathematics divorced from reality leads to futility. The function concept has not been given the prominence that its importance deserves. Functions and graphs are not introduced until almost the middle of the book (Chapter XII). The attention given to the function concept is, on the whole, rather perfunctory.

First Course in Algebra (1924) and *Second Course in Algebra* (1926), by J. A. Nyberg.

These are among the most popular of mathematical textbooks, but they are written on conventional lines. The author does not claim to have given any special attention to the function concept, but graphs are introduced as illustrations of statistical data and as representations of algebraic functions.

First-Year Algebra, by Milne and Downey. 1924.

In the Introduction the authors state that "the graph is represented with a view to giving the pupils an adequate understanding of the various types in common use." Just as undue emphasis on scientific facts results in the neglect of scientific methods, so undue attention to graphical types tends to a neglect of their functional significance, as it has in this case.

Essentials of Algebra, by Smith and Reeve. 1924.

It may be confidently asserted that, when this book was published, it represented the most comprehensive treatment of the function concept to be found in American textbooks. Its influence has been great, not only upon the schools in which it has been used, but probably even more upon later writers of mathematical texts. The *Essentials of Algebra* is a course of unified mathematics and stresses, as its title suggests, the essentials of algebra—formulas, graphs, directed numbers, and equations. The treatment throughout is fresh and stimulating. A scholarly presentation of the fundamental concepts is combined with a fine appreciation of sound classroom procedure. In its general plan this book is more like the best German books of that time, both in spirit and in content, than any other American books that we have examined.

General High School Mathematics, by Smith, Reeve, and Foberg. 1925.

The same excellent features which we have noted in the *Essentials of Algebra* are to be found in this more advanced textbook of general mathematics. The book is a unified course of algebra and geometry, in which every opportunity is taken to bring out the notion of correspondence, in such subjects as formulas, similarity, symmetry, and the graphical representation of function. This must rank as one of the earliest and at the same time most successful attempts to unify mathematics.

Beginners' Algebra (1922) and *A Second Course in Algebra* (1924), by Comstock and Sykes.

Although the authors would probably not now claim that "the function is presented explicitly as the central and controlling idea," the books represent a commendable attempt to infuse the function concept into the dry bones of formal algebra. The chapters on graphs and functions are particularly good and embody the fruits of practical experience in teaching those subjects. As in the large majority of books published about the same time, graphs are used to illustrate rather than to develop the idea of function. In the chapter on logarithms, for example, the graph of the logarithmic function is presented after the algebraic treatment has been completed and not before, as we find it in Nunn's *Algebra*.

As we have already indicated, American mathematical textbooks published since 1928 show a striking contrast to those published before that date. In the most recent works the function concept which had previously been largely incidental, began to show signs of becoming *constitutional*. It may be claimed by the writer of the modern textbook, with greater justice than formerly, that the function concept is the unifying principle of the whole course. This improvement we may attribute to several influences, all of which may ultimately be traced to the report of the National Committee on Mathematical Requirements. Not only has this report profoundly influenced the thought of teachers and textbook writers throughout the country, but more important for the dissemination of new ideas, it has also influenced those who are responsible for the mathematical syllabi of the great public examinations. No better exemplification of the spirit of progress in mathematical studies could be given than a comparison of the College Entrance Requirements of 1920 with those of 1930-31, in spite of the fact that some of the later syllabi were admittedly a compromise between the old and the new. The attitude of the Regents of the University of the State of New York is expressed in the following extract from the syllabus of 1930:

> And while it would be foolish to minimize the indispensable nature of skills and of sound habits, it is equally true that the teacher who ignores the tremendous significance of meanings, attitudes, and ideals will at best secure satisfactory examination results but will accomplish very little for the permanent education of her pupils. Hence a modern teacher is constantly urged to give due attention to the real message and cultural significance of the subject she is teaching.[5]

This extract is typical of the broad vision and liberal spirit of the whole. We venture to suggest that those responsible for the syllabi of College Entrance Examinations should go a step further and prepare, perhaps as alternatives to the conventional examinations, question papers exemplifying more fully the modern spirit in mathematical teaching.

Among the books published since 1928 which are not revisions of older texts, those of Betz, Breslich, Strayer and Upton, and Engelhardt and Haertter deserve special notice. In each of these a praiseworthy attempt has been made to develop the function concept as the central theme of the book.

Algebra for To-day. Second Course, by W. Betz. 1931.

This book is a continuation of a more elementary book, *Algebra for To-day, First Course*, which does not call for special comment. In the Introduction to the *Second Course*, the author states that "the functional program of algebra, in its three-fold aspect, is stressed from the very beginning and is made the central theme of the entire course." In his endeavor to present the function

[5] The University of the State of New York. *Syllabus in Elementary Algebra*, p. 5. Albany, 1930.

concept as essentially a mode of thinking the author has been most successful. The keynote of the whole book is dependence, which is stressed in practically every section of the work. Formulas equations, and graphs are all functionally treated. Graphs are shown to be not mere illustrations but rather expressions of thought. At the end of the book there is an excellent summary entitled, The Study of Relationships. In this chapter all the main types of functional method are called in review: tables, formulas, graphs, equations, variation, and functions, including trigonometrical functions.

Senior Mathematics, Book III, by E. R. Breslich. 1929.

Although this book contains much that may be styled formal, it presents the formal material in quite a fresh way. As one would expect from an author who has contributed so much to the existing literature on functional thinking, the function concept has been given special prominence. One feels, however, that opportunities have been missed. There are places in the book where the treatment could have been improved by an appeal to the principle of dependence. Attractive features of the book are its clarity and its freedom from the artificiality that mars so many books on elementary mathematics. The sections on the graphs of functions are treated in a masterly way.

Modern Algebra (Ninth Year). Strayer-Upton Junior Mathematics, by C. B. Upton. 1930.

This must be ranked as one of the best treatments of the function concept in elementary textbooks. In the Introduction the authors state: "The study of relationships between quantities as expressed by the formula, the equation, and the graph is an important feature of this text. In the treatment of each of these topics particular care has been taken to show the dependence of one quantity on another and to bring out the idea of a mathematical law." In this endeavor the authors have been eminently successful. The idea of dependence is emphasized on almost every page and is reinforced constantly by an appeal to graphs. A feature of the book is the way in which mathematical function has been related to physical function, in other words, to life. In this book the formal skills of algebra are given their due weight. They are treated as means to an end, not as ends in themselves. The *end* is functional thinking.

First Course in Algebra (1926) and *Second Course in Algebra* (1929), by Engelhardt and Haertter.

These books make a rather striking contrast. In the *First Course* there are chapters on graphs, ratio, proportion, and variation all effectively treated from the functional standpoint, but the rest of the book is developed along conventional lines. In the *Second Course* we have one of the best efforts yet made to knit school algebra together by means of the function concept. The book opens with the formula and statistical graphs, including the frequency histograph. There is an excellent chapter on function and variation, in which the results of the previous chapters are gathered together. Here we get the quadratic, the cubic, and the inverse function all introduced through problems from life and illustrated by the appropriate graphs. Later in the book the main principles of analytical geometry are developed. The graph of the logarithmic function is used in a most effective manner in dealing with interpolation between known values. This book must be ranked as one of the most satisfying of available textbooks of functional thinking.

High School Mathematics, by J. A. Swenson. 1923.

Although this is not among the most recent of the books designed to exemplify the function concept, it is included in this list, because it is the first fruits of the labor of one who has done much to establish functional thinking in American schools. Superficially viewed this book covers much the same ground as the conventional textbook, but a closer inspection will show that the examples have been carefully designed to bring out the concept of functionality. In the early part of the book the idea of corresponding change is brought out through carefully graded examples. This is reinforced, a little later, by the graphical representation of related variables. This part of the subject has been treated very clearly. Of particular interest is the chapter on variation, which contains enough material for a substantial treatise on that subject. Here, as elsewhere, the treatment shows the skill of the practical teacher. This book contains much of value that could be copied with advantage by other writers of mathematical textbooks.

VI

THE FUNCTION CONCEPT IN PRACTICE

The function concept and elementary thought and practice. Before we enter upon the final stage of our study, that of outlining a course of school mathematics, embodying the function concept as its central principle, we shall discuss certain implications of that concept in their relationship to elementary mathematical thought and practice. In the proposed course an attempt has been made to fuse the subjects, algebra and geometry, into a mathematical unity; in other words, to develop a course of general mathematics. Such a synthesis is possible, not because the two subjects possess any natural similarity of content, but because they evoke a natural similarity of concept. For numbers and points at least have this in common: similar relationships can be found to subsist between them. The Dedekind-Cantor axiom is a simple recognition of this fact. Many other illustrations will be found in an article by Huntington, "The Fundamental Propositions of Algebra,"[1] which, as far as logical principles are concerned, could equally well be read as an exposition of the fundamental propositions of geometry. It may be objected that the course, as outlined, does not make sufficient provision for exercise in the fundamental skills of algebra and the elementary disciplines of geometry. The position taken in this matter is that the formal skills and disciplines of mathematics should be treated as means to an end, not as ends in themselves; that, in a course which arouses the interest of the student and challenges his thinking, the technical skills may almost be left to take care of themselves. Where there is no felt need, there is no sense of value. The course itself is a modification, in the light of further observation and study, of one worked out in

[1] Huntington, E. V. *Young's Monographs on Topics of Modern Mathematics*, Chap. IV. London, 1927.

Cf. Veblen, O., "A System of Axioms for Geometry." *Transactions of the American Mathematical Society*, Vol. V, p. 346, with Huntington, E. V., "A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude." Transactions of the American Mathematical Society. Vol. III, p. 264.

the secondary school classroom, over a period of several years. Experience showed that proficiency in algebraic techniques was quickly acquired, when such proficiency was essential to the development of the subject in hand. Not only so, it was found that the more important formal skills were given deeper meaning in the process. It seemed that the temper of the workman's tools could be fully appreciated only in their use.

Practical and concrete nature of the course. It will be found that the course shows a strong bias towards the practical and the concrete. Many of the concrete problems have been taken from mechanics or from some other branch of physical science. Many others, in ways not actually specified, readily lend themselves to concrete treatment. Mechanical problems not only admit a wide choice of problem material, but also insure a clearer understanding of functional correspondence than would otherwise be possible. Mechanics is functionality in concrete form. As Whitehead has said, "We should civilize and clothe ideas" by investing them in concrete form, always remembering, of course, that "mathematical training consists in making these ideas precise and the proofs accurate."[2] Klein has said that "for a thorough and fruitful treatment of the function concept, the fundamentals of mechanics may be taken as necessary material."[3] With this opinion we are in full agreement. It is not clear whether Klein restricts the term 'mechanics' to *kinematics*, or whether he also includes *kinetics*, which involves the concept of mass. In our course we have confined our attention almost exclusively to kinematics, not because of any unwillingness to include kinetics, but because the space-time concept provides us with all the functional material we need.

The concepts of time and space. There are many reasons, philosophical and practical, why the concept of time should be an integral part of school mathematics. The similarities and contrasts between space and time have been pointed out by many prominent writers. David Eugene Smith,[4] in an interesting summary of Schopenhauer's discussion of the duality of space and time, tells us that space is homogeneous and continuous, infinitely di-

[2] Whitehead, A. N. "The Principles of Mathematics in Relation to Elementary Teaching." *Proceedings of the Fifth International Congress of Mathematicians*, Vol. II, p. 453. Cambridge, 1912.

[3] Klein, F. and Shimmack, R. *Vortrage über den Mathematischen Unterricht*, p. 113.

[4] Smith, D. E. "Time in Relation to Mathematics." *Mathematics Teacher*, Vol. XXI, p. 253. 1928.

visible, infinitely extended, empty and indeterminate; so also is time. Space is permanent and static; time is transient and progressive. Space makes geometry possible; time makes algebra possible. The last statement suggests a remarkable paper by Sir William Rowan Hamilton, in which he contrasts geometry, as the science of pure space, with algebra, as the science of pure time. "Now the notion or intuition of *order in time* is not less but even more deep-seated in the human mind than the notion or intuition of order in space. A mathematical science may be founded on the former as pure and as demonstrative as the science founded on the latter."[5] Modern scientific thought carries abstraction even further, when it merges space and time in a single concept.[6] "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."[7] If we are right in maintaining that the great concepts of higher mathematics should find a place in school mathematics, if only in a rudimentary form, we are certainly right in including *time*, as one of the most fundamental of them. As Nunn has expressed it, "Motion is simply 'geometry *plus* time', and any reason which justifies the study of geometry as a branch of mathematics must justify the inclusion of kinematics."[8]

Since we have decided to treat of concrete bodies in space, rather than of space in the abstract, we shall be concerned often with the study of real things in motion. Geometrical figures will be presented, not merely as abstract and static entities, but as the geometrical correlates of solid bodies, capable of motion, as solid bodies are. Points will appear as the geometrical correlates of particles, capable of motion, as particles are. This dynamical conception of geometry seems to have been held by Newton, to whom a curve was "a nascent entity awaiting generation in thought." We use the same idea in common speech when we say that a point 'moves along a curve' or 'passes through a maximum or a minimum value'. Theoretically there should be little objection to such a conception of a point, since it fulfils two generally recognized con-

[5] Hamilton, W. R. "The Theory of Conjugate Functions, or Algebraic Couples with a Preliminary Essay on Algebra as the Science of Pure Time." *Transactions Royal Irish Academy*, Vol. XVIII, p. 296. 1837.

[6] Broad, C. D. *Scientific Thought*, Chap. II. London, 1927.

[7] Minkowski, H. "*Raum und Zeit.*" *The Principles of Relativity*, p. 75. London, 1923. (Edited by A. Einstein).

[8] Nunn, T. P. *The Teaching of Algebra*, p. 171.

ditions, one, that points must bear to each other the relations that geometry ordinarily requires, and the other, that points must be so related to lines, areas, and volumes that the ultimate analysis of lines, areas, and volumes into sets of points can be rationally interpreted. Our conception is Whitehead's "extensive abstraction" in an elementary form.[9] The purist may object that the concrete particle will be so potent that its less substantial correlate will be ignored. The answer to this objection is that no great harm will be done. The student will appreciate the more abstract concept when, and only when, he feels the need of it. To proceed 'from the concrete to the abstract' is a fairly safe maxim.

Relations between the elements of a single class. The function concept may be defined in general terms as a *determinate correspondence between variable classes*. This suggests that we begin our study with the notion of a class. Young has pointed out that "in the abstract formulation of any mathematical science the notion of a *class* (or *set*) and *belonging to a class* are fundamental."[10] A class is constituted when from objects of a group, at least two in number, a common characteristic or resemblance is abstracted. We commonly ascribe to a class of objects *attributes*, to distinguish each such mode of resemblance. Color, size, and form are attributes of classes. Classes of objects also possess the attribute of *number*, which is as truly an attribute of a class as color or size. "Number is what is characteristic of numbers, as *man* is characteristic of men."[11] In mathematics we are concerned with *abstract* classes (numbers and points) and with concrete or *denominate* classes (3 books, 3 meters, 3 grams). Both abstract and concrete classes may be further divided into *subclasses*. Thus the class of positive integers may be subdivided into the class of even integers and the class of odd integers, or the class of primes and the class of nonprimes, while the class of mathematics books may be subdivided into arithmetic, algebra, and geometry or into ancient and modern. When we partition classes into subclasses we form a *manifold classification*.[12] The operation of class formation or classification, of great importance in statistical analysis, is also of fundamental importance in science, medicine, and sociol-

[9] Whitehead, A. N. *The Concept of Nature*, p. 75. Cambridge, 1921.
[10] Young, J. W. *Fundamental Concepts*, p. 59.
[11] Russell, B. *Introduction to Mathematical Philosophy*, p. 11.
[12] Yule, G. U. *An Introduction to the Theory of Statistics*, Chap. V. London, 1927.

ogy; indeed, in all departments of life. It is the person with the 'tidy' mind who knows where things are to be found. In view of its great importance we have included manifold classification as a definite topic in the course. The formulation of classes by attributes and the further estimation of classes as *values* lead us to the concept of the variable, the type or representative of any or all of the elements of a class. The elements regarded as quantities constitute a *variable class*. One of the most important facts about variables is the *order* of the quantities of which the variable is constituted. Some arrangement of order, potential or actual, is necessary before we can recognize the mutual relationships of the quantities of the variable. As we have seen, there may be for any class more than one possible order. For the purposes of pure mathematics all that is necessary is that an order should be recognized, but for school mathematics the arrangement should be made a practical exercise in ***tabulation***. So we present the student with sequences in which arrangements can be tabulated according to their order of *value*, order in *space*, or order in *time*. We introduce him not only to the symbols expressing 'greater than', 'equal to', and 'less than' ($>$, $=$, $<$) but also to 'after', 'simultaneous with', 'before', 'succeeds', 'abreast of', and 'precedes'. This leads us quite naturally to the idea of a *series*, in which the mode of arrangement is defined, and, at a later stage, to the concept of a *limit*. Incidentally, we become familiar with such terms as bounded, unbounded, first term, last term, lying between, infinite, and converging series. The familiar notation $a > x > b$, to express either the fact that x lies between a and b or the fact that x has a value intermediate between a and b, is introduced early in the course.

Correspondence between the elements of two classes. Having discussed relations between the elements of a single class, we then proceed to the *correspondence* between the elements of two classes, of which one-to-one, or one-one, correspondence, is a special case. The elementary idea of correspondence leads to analytical *functions* and *correlation*, on the one hand, and to geometrical *similarity* and *symmetry*, on the other. If one of our classes is a value-class and the other a time-class, we shall have as our functions *velocity* and *acceleration*. If one of our classes is a number-class and the other a point-class, we shall be led to the idea of an *array*. In the early study of correspondence, therefore, we include such topics as: the correlation of value-classes, geometrical similarity (including congruence), kinematics, graphical representation, and statistical arrays.

Some debatable topics in geometry. A few remarks may be ventured at this point on some debatable topics in geometry. In our scheme, we have, in agreement with the best modern thought, omitted all reference to *superposition*. The case against superposition has been very clearly expounded in the report of the Mathematical Association (British), *The Teaching of Geometry in Schools*.[13] Russell has summed up the modern view as follows: "The apparent use of motion is deceptive; what in geometry is called a motion, is merely a transference of our attention from one figure or set of elements to another. Actual superposition, which is nominally employed by Euclid, is not required." Why, then, continue to use a method of proof which is not only no easier than others, but is also admittedly inferior to them? Our method is to accept as a postulate the Principle of Congruence that "any figure (plane or solid) may be exactly reproduced anywhere,"[14] and to proceed on the logical truism that, if a figure is determined, without ambiguity, by a particular set of specifications, then any figure of the same kind conforming to the same specifications is congruent with it. Thus, if two sides and the included angle of a triangle are sufficient data to determine a triangle without ambiguity, then any two triangles having two sides and the included angle correspondingly equal are congruent. Laisant, as long ago as 1899, expressed the same idea as follows: "Two figures constructed with the same data, in the same manner and on the same scale are congruent. This is equivalent to a theory of congruence; dare we say, the only reasonable theory of congruence."[15]

It may be objected that our criticism of the method of superposition is hardly consistent with our dynamic conception of mathematics in general, in which we include the motion of figures. This inconsistency is only apparent, for, as the report to which we have referred states: "This argument leaves it open to us to use the motion of figures. To the mathematician, a moving body is a body whose positions are correlated with those of a real variable and a moving point *is* a correlation between some class of positions in

[13] Mathematical Association. *The Teaching of Geometry in Schools*, p. 28. London, 1923.

[14] *Ibid.*, p. 35.

[15] Laisant, C. A. "*Congruence et similarité.*" *L'Enseignement mathématique*, Vol. I, p. 342. Paris, 1899. Reflections sur le premier enseignement de la geometrie.

space and some range of values of a real variable. If the *name* of 'time' be given to the variable, that is only in Newton's phrase 'for the sake of perspicacity and distinction,' and no logical dependence on physical motion is involved even remotely."[16] Motion is simply a one-to-one correspondence between a certain class of points and certain time-intervals. All that we do in following a motion is to transfer our attention from one figure or a set of figures to another.

We now come to the much-debated question of a *parallel-postulate*. This is not the place for a lengthy discussion of this important topic but the issue may be narrowed, as far as school mathematics is concerned, to three possibilities. We may accept, as in England and other European countries, the time-honored postulate of Euclid, which is usually modified to the form known as Playfair's Axiom; we may, if we wish, adopt the form usually adopted in American schools: "One, and only one, straight line can be drawn from an external point parallel to a given straight line";[17] or we may accept the recommendation of the Mathematical Association, following the lead of Nunn, and postulate the Principle of Similarity: "Any figure can be reproduced anywhere on an enlarged or diminished scale."[18] For reasons that have been clearly stated in the report and for others in line with the main contentions of this thesis, we have adopted the third of these postulates. Our experience has convinced us that many advantages accrue from the early introduction of similarity into the school course, not the least of them being that questions concerning parallels come out of the work on similar triangles quite naturally (Ex. 132). In this way parallelism is seen to be only another case of correspondence. Although, in the treatment given in the report, the conception of ratio is not definitely involved, with the advantage that the question of commensurability does not arise, we

[16] Mathematical Association. *Op. cit.*, p. 34.

[17] Or, if we prefer it, the axiom more popular in France: "One, and only one, perpendicular can be drawn from an external point to a line."

[18] Nunn, T. P. "The Sequence of Theorems in School Geometry." *Mathematical Gazette*, May, 1922, p. 65, or, *Mathematics Teacher*, Oct., 1925. These articles are worthy of careful study. See also Mathematical Association, *The Teaching of Geometry in Schools*, p. 35, and the discussion on the report, *Mathematical Gazette*, May, 1924, p. 73.

Picken, D. K. "Parallelism and Similarity." *Mathematical Gazette*, Oct., 1924, p. 195.

Hill, M. J. M. "The Postulate of Parallels." *Mathematical Gazette*, Jan., 1925, p. 271.

Birkhoff, G. D. and Beatley, R. "A New Approach to Elementary Geometry." *The Fifth Yearbook, National Council of Teachers of Mathematics*, p. 86, 1930.

have included ratios and intercepts in our program. It will be noted that in this, as in other topics, examples from solid geometry are placed beside examples from plane geometry (Ex. 135 to 137), the contention being that no distinction should be drawn between plane and solid geometry, when the two can be treated similarly.

The graph. We have already seen that the function concept is often assumed by textbook writers to be synonymous with graphical representation. These writers seem unaware of the fact that it is possible to treat the subject of graphs in such a mechanical way that the real notion of function may be entirely overlooked. Although we have given the function concept a much broader interpretation than the writers referred to, we have to recognize that the graph is one of the most useful instruments of functional thinking. But the conception of the graph, like that of the function, is not a subject that can be disposed of in a lesson or even a series of lessons; it is essentially a mode of thinking, and as such should be an integral part of the whole course. In our suggested course we have approached the subject gradually, so that, when the student comes to deal with the *graphs of functions*, he will have at his command all the elementary concepts and the technical skills that he needs. In the variation array, the frequency histograph, the curve of best fit, and in the discussion of direct and inverse variation, all the essential ideas of graphical representation have been presented. It is important that the graph of a function should be looked upon, not as a line connecting a number of plotted points, but as a *functional whole* which expresses, in its entirety, certain generalized concepts of number and space. Among these concepts is that of a correspondence between variables. So the graph is treated as an entity functionally related to the axes of reference. If the graph is changed relative to the axes, the relationship must of necessity be changed; or, if the axes are changed relative to the graph, the relationship must likewise be changed. This method of treating functions has been followed by the writer for many years, and always with gratifying results.[10] The basis of the method is the well-known operation of the transformation of coördinates. By a succession of simple illustrations, with the graphs of $y = ax$, $y = x^2$, $y = x^3$, $y = 1/x$, and $x^2 + y^2 = r^2$, the student is led to the general theorem that the graph of the equation

[10] The idea was first suggested in 1908 by Professor E. J. Nansen, Emeritus Professor of Mathematics, Melbourne University, in a course of lectures.

$y - b = f(x - a)$ may be derived from that of $y = f(x)$ by (1) a translation of the graphical form a distance $+a$ in the x direction and (2) a further translation a distance $+b$ in the y direction. The same concept is brought out later in the course, when the wave equation $y = a \sin (x - vt)$ is derived from the wave form $y = a \sin x$ by a translation of the wave form at a uniform velocity $+v$ along the x-axis. There is danger, of course, that these operations, like the drawing of graphs through plotted points, may become mechanical, but the student will at least grasp the idea that all functions of the same form may be represented by graphs of the same configuration, and that all these graphs are members of a common *family*.

The formula. No discussion of the function concept in school mathematics would be complete without some reference to the *formula*. We do not propose to dwell at length on this aspect of elementary mathematics, however, because its importance now seems to be generally recognized. Most modern textbooks follow the lead first given by Nunn and place the formula in the forefront of their teaching. "Formal work in algebra . . . is here planned to begin with lessons intended to cultivate the formula as an instrument of mathematical statement and investigation."[20] The formula is preëminently a functional instrument, for it expresses in symbolic language a relationship between two or more variables, one of the variables being determined, when all the others are known. The cultivation of the formula, like that of the graph, involves two main processes, one of *formulation* and the other of *interpretation*. And the knowledge derived from the interpretation often exceeds that apprehended in the formulation. The formula and the graph may both be regarded as generalized expressions of relationships, capable of a general as well as of a particular interpretation. This is not generally recognized. Just as we lose the functional concept of the graph as a whole, if we confine ourselves to the consideration of particular points, so we lose the functional meaning of the formula, if we restrict our attention to the substitution of particular values. The formula, like the graph, is a *functional whole*. So, in the suggested problem material we find, for example, that a squared term is more *potent* in its influence upon the whole, than a factor of the first degree. This we regard as a very important part of the training given by the study of the formula. In the area formula $A = l \cdot w$, the factors

[20] Nunn, T. P. *Op. cit.*, p. 63.

l and w have equal potency; an increase of 10 per cent in either would produce an increase of 10 per cent in the area. This is not so in the case of the formula $v = \pi r^2 h$ where the factor r is more potent than the factor h. Similarly, in dealing with the formula $C = E/R$, we may discuss the *direct* and *inverse* potency of E and R in determining C. Such a discussion of a formula should precede all mechanical exercise in substitution; it should also precede any formal treatment of the *equation*, as a relation of equality between variables. The general character of the equation, as distinct from the particular, has been strongly emphasized by Thorndike, who makes "a clear distinction, almost a contrast, between the equation as an organization of facts to find some unknown or hidden fact and the equation as an expression of a relation between variables."[21] In the opinion of Thorndike these two aspects, which are by nature different, should be kept distinct; he advises that "the two aspects of the equation should be kept distinct from the start and to a large extent throughout; that they should, other things being equal, be given different names, taught at different times and in different ways and with different applications." With this we do not agree. Our conviction is that the particular and the general should as far as possible be fused together; that the student should be led through the particular to the general, and that an analysis of the general should precede further applications to particular cases.

Carson expresses a similar view in an article on the uses of graphs in which he stresses the importance of general concepts in mathematical teaching:

> Although it is not of great difficulty, this discussion of shape is of the utmost importance, and deserves full and careful treatment. Apart from a vital application in the drawing of graphs, it forms a true origin for the concept of a function, for the sequence of changes is exhibited as one *whole*, and appears as a distinct entity, namely, a function.[22]

Summary. Let us now consider the course we have outlined in a little more detail.[23] We begin, as we have stated, with the idea

[21] Thorndike, E. L. and others. *The Psychology of Algebra*, Chap. IV, p. 126. New York, 1923.

[22] Carson, G. St.L. "The Various Uses of Graphs." *Mathematical Gazette*, March, 1914, p. 266.

[23] The references in this chapter are to the typical problem material in the next chapter (A Course of Study based on the Function Concept). It should be noted that only one example of each type of problem is given.

of a *class*, which we illustrate by a number of simple examples, no formal definition of a class being desired at this stage. Through Ex. 1 we learn to subdivide and tabulate classes according to particular attributes (color and subject), and so to form a *manifold distribution.* We now arrange two classes, weight and length, in ascending or descending *order* and get our first notion of *correspondence* (Ex. 2). Even at this early stage we may prepare for the concept of the variable by using letters to represent the class. Thus we may write *H* for height (*any* height included in the class) and *L* for length. The idea of correspondence is brought out more definitely in the related numerical series of Ex. 4. When dealing with such series we may, if we so desire, refer to them as arithmetical or geometrical series or sequences. In Ex. 5, 6, 7, we have the correspondence of points and lines, giving us *similarity* and *symmetry.* (Colored crayons help to make these correspondences more easily understood.) The idea of correspondence, which is fundamental to our subject, is still further developed in Ex. 8. In this exercise the boys are arranged in an *array*, tallest on the right, shortest on the left, facing the observer, and from the numbers and the graph obtained, some important statistical concepts are derived: the *variation array* (or ogive),[24] *median, mean,* and *deviation from the mean.* Thus we get, early in the course, an introduction to *graphical representation* through a problem which naturally suggests the form o the graph. In studying this example the student may confine his attention to the changes in only one of the variables (the ordinate), since the other variable is a set of equally spaced intervals. This example does us valuable service, for it is used again in Ex. 11 and 12, through which we approach the subject of *directed numbers.* The idea of growth, which is also functional, is brought out in Ex. 9, when the whole graph of Ex. 8 is raised two inches. More difficult examples of growth would, of course, follow.

Concept of the mean. The concept of the *mean* is of fundamental importance. Not only is the elementary idea important in itself, but the generalized concept is found in the formula for the area of a trapezium, the Prismoid formula, Pappus' Theorem, the formulas for the Center of Gravity and the Center of Pressure.

[24] We prefer the term 'variation array' to the more usual 'ogive', since it suggests an array of quantities of varying values.

Closely related to the idea of the mean is that of *integration*, which we introduce at a very early stage in our treatment of areas. A word may be said with regard to the method of introducing the mean. We suggest that it be approached through the *guessed mean*. Suppose, for example, we require the mean of three heights (in inches), 54, 57, 60. We guess the mean, or average, as 57 and show that the *deviations* above and below this guessed mean balance. We now take 54, 57, 61 and again guess 57 to be the mean. In this case we notice that our positive deviation exceeds our negative deviation by 1. It seems natural to conclude that our mean is greater than 57. A little reasoning would lead us to conclude that the mean is 57 1/3 and we verify this conclusion as we did before. We further note that any guessed mean would lead us to the same result. For example, we may take our guessed mean to be the lowest number of all, 54, and still get the same result; we may even take our deviations from zero and arrive, finally, at the usual method of computation, which is epitomized in the formula $M = \Sigma N/n$. By approaching the subject in this way, we are led to a real understanding of the principle involved, and we prepare the way later for a discussion of errors.

Directed numbers. We now come to the important topic of *directed numbers*,[25] using for our introduction the figures of Ex. 8. The procedure is as follows: Instead of numbering the array from the end, we number it from the middle (a common drill-procedure). We then label the boys on the right *plus* and those on the left *minus*, and read their numbers: positive 1, positive 2, . . . , and negative 1, negative 2, and so on. The labels thus given to these positive and negative numbers naturally suggest right and left *directions* or *opposites*. These terms are now introduced, and exercises in their use given (Ex. 12, 13, 14). Finally we develop the meaning of the plus and minus signs as *operators* (Ex. 15). This could be done with the aid of Ex. 8, but not as convincingly as with Ex. 15. When we combine an operation with a direction, we are faced with the necessity for a *rule of signs*. Thus we find the rule of signs illustrated for the first time in connection with the addition and subtraction of directed numbers. The procedure is fully exemplified in Ex. 16. Our second illustration of the rule of signs follows al-

[25] The term 'directed number' we owe to Sir T. P. Nunn. The treatment given in this section is, in the main essentials, the same as that given in his book, *The Teaching of Algebra*, Chap. XVIII.

most immediately in connection with Ex. 19 and 20, in which we introduce *parentheses.*

Generalized arithmetic. Having surveyed the above important concepts and processes, we are now in a position to generalize our arithmetic. This we proceed to do in Ex. 21, using problem material similar to that already used in arithmetical form (Ex. 22, 27). It is important, in this connection, to stress the fact that an algebraic letter not only represents *a* number, but *any* number of a particular ensemble. When the student has acquired a little facility in the use of letters to represent numbers, he may be introduced to that powerful instrument of functional thinking—the *formula.* Here, as we have already indicated, it is important to bear in mind, first, that the letters represent quantities of a certain *variable* class, and, second, that all the variables are *factors* entering into a balanced situation. Thus the formula $A = l \cdot w$, for the area of a rectangle, not only tells us that if the length is 5 ft. and the width 4 ft., the area is 20 sq. ft., but it also tells us that the length and the width are *factors* (equally potent) upon which the area *depends*, that 'the greater the length, the greater the area, the width being constant', and 'the greater the width, the greater the area, the length being constant'. and that 'the area depends on the length and the width to equal degrees'. Without this conception of functional dependence, substitution exercises of the conventional type are of little educative value. As stated elsewhere, the concept of *dependence* is of the first importance and we cannot begin too early to lead the student to acquire it. This concept cannot be acquired in a single lesson or a course of lessons. It must be woven into the fabric of the course. Examples 31 to 36 indicate some of the ways in which this may be done. The range of topics could, of course, be multiplied almost indefinitely.

Derivation of formulas. Up to the present our formulas have been taken from various departments of life, but no attempt has been made, except in very general terms, to derive them from first principles. It is not wise, however, to pursue this policy exclusively. Geometry and mechanics provide us with many extremely useful formulas, which can easily be built up at this elementary stage. We therefore leave this part of our subject for the time being and take up the study of geometry.

The study of geometry. Here we begin with the ideas of *translation* and *rotation* and with geometrical *form* (Ex. 37 to 39).

In discussing these terms the opportunity is taken to include such interesting and useful exercises as Euler's Theorem, the point-line formula, correspondence between an area and a line, curves of pursuit, and the mechanical tracing of curves, the chief purpose being to enable the student to become acquainted with some important geometrical terms (Ex. 40 to 43). *Conical projection* also enters to illustrate further the idea of correspondence. This we consider a most valuable method of developing the perception and intuition of space relations.

Example 45 is an illustration of an interesting exercise in section-drawing which requires some degree of 'space intuition'. Experience seems to show that, when this type of exercise is omitted from early mathematical teaching, the ability to visualize sections or contour lines of solid forms may remain undeveloped.

We are now ready to introduce *time* as a variable. We begin with some very simple problems on linear motion and calculate for certain defined intervals the change of distance ($\triangle s$) corresponding to a certain change of time ($\triangle t$). From the numbers so obtained we derive the *rate of change* of distance with time ($\triangle s / \triangle t$) in Ex. 48. Thus the notation of finite differences is introduced very early in the course.

We now come to the definition of an *angle*. In Ex. 51 to 53 it is shown that the angle is a figure, but not only so, it is a figure which may be conceived as having been generated by the rotation of a line, just as the line was conceived as generated by the movement of a point. In the Course of Study we have drawn a distinction between the angle as a figure and the measure of an angle and have used the term 'rota' of an angle to correspond with the 'length' of a line in linear measurement. It should be noted that, in this part of the work, the class is taught to use a circular protractor. It would make for much clearer thinking if a circular protractor could be used exclusively in elementary work. An interesting group lesson with angles is suggested in Ex. 54. In this example practice is given in measuring lines and angles and the ideas of scale measurement and graphical representation are reinforced. Some interesting *field work* (Ex. 55) may now be done, the way thus being prepared for trigonometry which follows later. Another important idea, that of the *locus*, may be conveniently brought in at this stage (Ex. 59), as part of the field work. Incidentally, Ex. 59 serves as a preparation for the *s.s.s* = *s.s.s.* case of congruence (Ex. 113c.). The relationships between angle, arc, and radius are now discussed. In Ex. 60 we show that the arc is proportional to the radius when the angle is fixed; in Ex. 61 we show that the arc is proportional to the angle when the radius is fixed. On the analogy of the formula, $A = l \cdot w$, we assume that $A = \theta \cdot r$ or $\theta = A/r$, and reserve the complete discussion of this equation until we introduce radian measure. Analogous to linear velocity we get *angular velocity*, as an interesting and very important extension of the idea of rate of change. The idea of angular velocity is usually considered to be too difficult for elementary work, but we have not experienced this difficulty. On the contrary, pupils of the junior high school level are keenly interested in computing the number of revolutions per minute of a turning body. It should be noted that angles are not restricted to 360°.

The next section on *orthogonal projection* may seem, at first sight, an unnecessary digression, but it is introduced at this point for the following reasons: (1) It keeps alive the notion of correspondence and supplements the work on conical projection. (2) It provides material for the study of some interesting geometrical forms (ellipse, parabola, etc.). (3) It prepares the student for the trigonometrical ratios, sine and cosine, through horizontal and vertical projection. (4) It facilitates the discussion of the angle between two planes, and the angle between a line and a plane (Ex. 68 to 74).

Graphical representation. We now enter upon a more thorough study of *graphical representation*. In our first example (Ex. 75) the graph is regarded as the pictorial or geometrical representation of a verbal statement, having the formula as its counterpart in the realm of numbers. Example 75 should be carefully discussed, for from it the following concepts may be derived: corresponding change, rate of change, slope, gradient, constant and variable gradients, interpolation, maxima and minima, point of inflexion, the graph as a ready reckoner, and the graph as the representation of a function. Following this are examples of most of the conventional graphical forms: the bar graph (Ex. 80, 81), the sector graph (Ex. 84), the line of best fit (Ex. 85). The bar graph is not studied as an end in itself, but as a means to an end—the *frequency histograph* or *frequency distribution* of statistics. This we consider to be a very important part of the work, which should be linked up quite definitely with the work already done on the variation array. Even more important for elementary teaching is the *line of best fit*, which is fundamental to the physical and statistical sciences. This part of the work should be done in the physics laboratory. The opportunity should be taken at this stage to discuss *liability to error*, and to impress upon the student the fact that the line of best fit is an average in geometrical form, indicating a general trend, and that it need not pass through any one of the plotted points.

Areas and volumes. Up to the present we have been mainly concerned with quantities of a single dimension. We now pass on to *areas* and *volumes*, which we introduce through the already familiar conception of the *generator* and *directrix*. This method of approach naturally suggests *integration* as the operation of continuous summation (Ex. 99). As a very interesting application of integration, we include the summation of an *arithmetical progression*. The method suggested is that due to Nunn.[26] The area of a triangle may be regarded as an important special case of such a progression. We are now quite prepared to extend our ideas of direction to include the notion of *directed areas* or length-length products, and to follow a third illustration of the rule of signs (Ex. 102). Incidentally, the algebraic identities $a(b+c) = ab + ac$, $(a+b)\ (a-b) = a^2 - b^2$, and $(a+b)^2 = a^2 + 2ab + b^2$ are illustrated graphically (Ex. 103). The last of these identities is useful in discussing the derivation of the *square root* of a number.[27] Our next illustration of a product will be a *rate-time* product, leading to the formula $S = vt$, which, when the relevant factors are all directed quantities, again illustrates the rule of signs (Ex. 108 to 110).

Before we proceed to study *congruence* and *similarity*, we discuss the intersections of three lines forming a trilateral or triangle, and of four lines forming a quadrilateral or quadrangle (Ex. 112). It is important that this broader conception of triangles and quadrangles should be given when the terms are first presented. The same remark applies to the introduction of the parallelogram (Ex. 133). The three ordinary cases of congruence are illustrated in Ex. 113. This part of the work will naturally be followed by simple originals depending upon congruence (Ex. 114 to 119). Ex-

[26] Nunn, T. P. *Op. cit.*, Chap. XIX, p. 199. See also Lietzmann-Zühlke. *Aufgabensammlung und Leitfaden Oberstufe, Leitfaden*, p. 3. Leipzig, 1930.
[27] Nunn, T. P. *Op. cit.*, Chap. XVI.

ample 116 deserves special consideration. It is a valuable exercise for developing a functional attitude towards the subject of congruence. The three-angle case (*a,a,a*) leads us immediately to the idea of *similarity* as correspondence of form. The essentials of the similarity of figures are clearly brought out in Ex. 123 and 124. Incidentally, the opportunity is taken to discuss proportion—first, as the equality of ratios; and, second, as the equality of rates (Ex. 128 to 131). We now proceed to the study of *parallels* (Ex. 132 to 136), using the equal-angle theorem of similarity as our fundamental theorem. It will be noticed that the examples on parallelism are taken from solid, as well as from plane, geometry.

Trigonometrical functions. Our next section is a short unit on the three main *trigonometrical functions*, which we treat in a manner that has almost become conventional. We have taken care, however, to introduce the subject in such a way that the value of the new symbolism in the economy of thought will be appreciated (Ex. 138 to 140 and 151 to 153). The Theorem of Pythagoras, which we have not so far needed, falls into its natural place in this section of the work and is treated in several different ways. The fundamental equation $\sin^2 \theta + \cos^2 \theta = 1$ is then presented as the Theorem of Pythagoras in trigonometrical form

Variation. We now take up, much more thoroughly than is usual in elementary mathematics, the study of *direct* and *inverse* variation. Direct variation will already be familiar to the student, but it must now be studied, geometrically and algebraically, as an example of functional relationship. The two problems suggested (Ex. 146 and 147) are designed to bring out this relationship both concretely and analytically. From the tables of values of x and y, it is seen that, whereas in the case of direct variation the two sets of numbers increase or decrease in the same sense, in the case of inverse variation, they vary in a contrary sense (Ex. 147).

Graphs of algebraic functions. The general line of approach to the graphs of algebraic functions has already been indicated. We first of all draw, as carefully as possible, the graph of the function $y = x^2$ and, cutting along the line of the graph, make a tracer, or parabolic edge. By appropriate translations this tracer may be employed to give us the graph of, say, $y - 2 = (x - 3)^2$ (Ex. 164). Further, if we keep the x units to their original length, and alter the y units to half their previous length, the trace of $y = x^2$ will now represent $y = 2x^2$ (Ex. 165). Retaining these new units we may translate the tracer and so derive the graph of the function $y - 2 = 2(x - 3)^2$. It is important to note, although this has not been suggested in the examples, that the graph of $y = 2x^2$ could be obtained from that of $y = x^2$ by a uniform elongation of the ordinates of the latter in the y direction. This may actually be demonstrated with a sheet of india rubber. Similarly, by a contraction of the ordinates, the graph of $y = x^2$ could be made to fit the function $y = \frac{1}{2}x^2$.

A similar method of treatment may be applied to the graphs of the cubic function $y = x^3$ and of the rectangular hyperbola $y = 1/x$. Finally, we derive from first principles the equation of the circle, and from it that of the ellipse (Ex. 198). It will be noticed that the form of the ellipse is obtained from that of the circle by the expansion or contraction of the ordinates in a definite ratio. The equation of the ellipse may be derived from that of the circle by expressing this operation of deformation in algebraic form. For if we replace y by $\frac{a}{b}y$ in the equation, $x^2 + y^2 = a^2$ we derive $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This method of treatment brings out very clearly the essential character of the ellipse as a 'deformed circle', or, if we prefer it, as the projection of a circle on a plane. We suggest that, for elementary mathematics, the ellipse be defined in this way.

VII

A COURSE BASED ON THE FUNCTION CONCEPT

The function concept the central theme. In the following pages we have outlined a course of school mathematics embodying the function concept as its central theme. The first two years' work is specified in some detail, as this constitutes the basis of the whole course. The suggested course has been supplemented by definition and brief explanations of the mathematical terms employed and by typical problems for school work.

The purpose of these Definitions and Concepts is partly to indicate the general philosophy underlying the course and partly to show that the function concept is not the only concept to be considered in a mathematical course. It has already been suggested that the main themes or disciplines of elementary mathematics are *functionality, calculus,* and *logic.* The concept of the calculus includes those of the mean, differentiation, and integration. These concepts, in their elementary form, have been given a place even in the work of the first year. The purpose of the problems, given under Typical Problem Material, is to indicate methods of approach not usually employed. Many of the problems will be found to be stereotyped and even commonplace. They have been included to show how easily the old material may be incorporated in a modern structure. No indication has been given of the number of problems of a particular type that will normally be required. This will depend partly on the needs of the individual and partly on his interests. In the statement of the definitions mathematical rigor has frequently been sacrificed to immediate needs. The professional mathematician would probably be unwilling to accept the definitions given of the point, the line, and the plane, but he may be willing to concede that pupils taught on these lines will have no more to unlearn than those taught by other methods. Definitions should not be memorized by the pupil; they are intended as a general background for the teacher. The suggested course is well within the capacity of a normally intelligent class.

Suggested Course	Definitions and Concepts
The notion of a class. Element of a class.	A number of entities or things having a specific attribute or quality can be said to belong to a certain class, set, or assemblage. We shall use the first of these terms. The natural numbers form a class; so do the points of a line. A class may consist of *concrete* quantities (10 books), *denominate* quantities (10 cm., \$10), and *abstract* quantities (10, ½). Each item of a class is called an *element* of that class.
Correspondence between classes. One-to-one correspondence.	Two classes are said to be in *one-to-one correspondence* when each element of the one class can be paired with one, and only one, element of the other class, and vice versa. Two classes may be put into one-to-one correspondence in more than one way.
Order as a relation: (a) In value. (b) In space. (c) In time. Linear order and cyclic order.	The *order* of a class is specified as a *relation* of some kind among the members of the class. Right, left; sooner, later; preceding, following; etc., are examples of such relations. We may have order in *value*, order in *space* or *position*, and order in *time*. Order may be linear or *cyclic* (reentrant).
The concepts 'greater than' and 'less than.' Series. Simple serial relations.	The concepts 'greater than' and 'less than' can be applied only to quantities of the same kind. When successive elements of a class are connected by the same kind of relation they form a *series* or *sequence*. Each element of a series is called a *term*.[1]

[1] For a detailed discussion see, Knopp, K. *Theory and Application of Infinite Series*. (Translated from the German by R. C. Young.) London, 1928.

COURSE

ELEMENTARY MATHEMATICS

Typical Problem Material

1. In a library there are 30 books on mathematics, 40 books on history, and 25 books on geography. Of the mathematics books 10 are red, 12 green, and the rest blue; of the history books 20 are green, 9 blue, and the rest red; of the geography books 11 are blue, 9 red, and the rest green. Tabulate the books according to color classes. How many books belong to the red class?

Arrange as follows:

	Mathematics	History	Geography
red			
blue			
green			

This is called a 'manifold distribution'.

(a) How many green books were neither on mathematics nor on geography?

(b) How many books were neither green nor historical?

For further examples see Appendix B, Ex. I and II.

Many other examples may be taken from business statistics, e.g., cost of different qualities of materials.

2. The weights of eight boys were 81, 80, 92, 100, 95, 118, 137, 120 (pound), and their corresponding heights 52, 49, 57, 60, 54, 64, 68, 62 (inch). Arrange these figures in two corresponding columns: (a) the order of increasing heights, and (b) the order of decreasing weights. The measurements could be taken from the pupils themselves. Repeat with other physical measurements.

3. In a final examination in history and geography it was found that 10 boys were placed in the same *order of merit*. The scores were:

History: 87, 23, 47, 58, 79, 63, 52, 72, 37, 68
Geography: 25, 43, 91, 68, 77, 37, 56, 82, 61, 50

Rearrange these numbers so that they correspond in decreasing order of merit.

4. Can you find laws or relations for the following series of numbers:

(a) 1, 2, 3, 4, 5 and 3, 6, 9, 12, 15?
(b) 3, 5, 7, 9, 11 and 1, 2, 4, 8, 16, 32?

Do the series have correspondence? If not, complete the correspondence. Continue each series for three more terms.

For further examples, see Appendix B, Ex. V.

Suggested Course	Definitions and Concepts
Similarity and symmetry as correspondence.[2] First ideas of geometrical form.	Two classes are said to be *similar* when they are of the same kind and when there is a one-to-one relation which correlates the terms of the one class each with terms of the other class.
An array. Numbers as distinguishing marks or labels of things.	Things arranged in order form an *array*. The items in an array may be distinguished and identified by numbers. Numbers may thus be used as *labels*.
Dedekind-Cantor Axiom: Correspondence between the points of a line and the domain of real numbers.	*Dedekind-Cantor Axiom:* "Any real number can be represented in a unique manner by a point on a line, and conversely."
Elementary statistical concepts. The variation array and variation curve. (a) Average or mean. (b) Median. (c) Upper and lower limit.	Measures of any trait of natural objects selected at random and arranged in order of magnitude form a *variation array*. The *mean* and the *median*.
Positive and negative numbers. (a) Symbols of order.	Positive and negative numbers may be used as symbols of *order*. A sequence of five things may be labelled: 1, 4, 3, 4, 5, or −1, −2, −3, −4, −5, *or* −5, −4, −3, −2, −1, or by any other scheme

[2] Young, J. W. Article on "Symmetry." *Fifth Yearbook, National Council of Teachers of Mathematics*, 1929, p. 145.

Typical Problem Material

5. Draw sketches of two similar houses of different sizes. (The word 'similar' is taken as intuitive; its meaning will be clarified in the performance of the exercise.)

Color some corresponding parts with the same color. Join some corresponding points. Make also cardboard or paper models of similar houses.

6. Draw two geometrical figures on the blackboard, similar in their main features but lacking certain correspondences. Ask members of the class to complete the figures. Similar exercises may be taken with algebraic forms.

7. Draw a symmetrical figure on the blackboard and ask the class to show the correspondences. Develop the idea of symmetry and illustrate with reference to architecture, ornaments, etc. Show that a symmetrical figure may be made to correspond part to part by folding. Give plenty of practice in making geometrical designs, similar or symmetrical.

8. The heights of 21 boys in a scout troop were measured as follows: 64, 63, 62, 61, 62, 60, 45, 52, 72, 62, 61, 69, 66, 63, [illegible], 59, 58, 54, 56, 49, 57 (inch). Arrange these heights in increasing order, shortest on the left, tallest on the right. For example:

1	[illegible]	3	4	
45	49	52	54	

(This may be done with the class itself as a playground exercise.)

Find the average or *mean height*. [For procedure see Chap. VI, page 121.]

9. Using graph paper, draw a line, which we shall call an *axis* and mark equally spaced points on it: 1, 2, 3, 4, . . . Erect at each of these points vertical lines to represent the heights of the boys of Ex. 8 drawn to scale. Join the tops of these lines by a curve (variation curve). What is the height of the middle boy of the array (median)?

If each boy grows 2 in. taller in a year, what will the variation curve look like then?

10. Similar examples: weights of boys in class, marks in examinations, lengths of leaves on a plant.

Show that natural objects selected at random often approximate to the same form of variation-array. Discuss cases in which this does not apply.

11. The 21 boys of Ex. 8 are renumbered from the middle, or median, boy (11). If 11 becomes 1, 12 becomes [illegible], and so on, how shall we number the old numbers 10, 9, 8, 7, . . . ?

(We give 9 the number 1 but put a minus sign (−) in front of it to distinguish him from 11, whom we call + 1.)

The old and new numberings will now appear as follows:

Suggested Course	Definitions and Concepts
	of identification, as long as we specify the numbers by giving them definite meanings. Thus, we may say: 3 is on the right of 2, or 3 is greater than 2, or −3 is on the right of −2, or −3 is on the right of −4. No necessary suggestion of direction is implied. The class of numbers is thus divided into two subclasses, the class of ***positive numbers*** and the class of ***negative numbers***.[3]
(b) Symbols of direction.	Plus and minus signs may be used as signs of 'direction', e.g., right, left; North, South; hence, ago. The position or number which separates the plus numbers from the minus numbers is called the ***origin*** or ***point of reference*** (zero, equator, now). The number *zero* may be looked upon as a member of both the class of positive and the class of negative numbers.
Directed numbers.	Numbers which carry the conception of direction as well as of magnitude are called ***directed numbers***.[4]
Vector.	A quantity which can be represented in magnitude and direction by a line is called a *vector quantity* and the line so representing it is called a *vector*.
(c) Symbols of opposites.	The origin or point of reference separates a class into two *opposite* subclasses. Plus and minus signs are used to distinguish opposites.
(d) Symbols of operations.	Plus and minus signs are used to indicate the algebraic operations of addition (+) and subtraction (−). Two or more quantities of the same kind may be compounded by addition. The quantities so compounded are called *components* and the result of compounding them is called the *resultant*. Thus $(+3) + (+5) = (+8)$; $(+3) + (-5) = (-2)$

[3] See Picken, D. K. *The Number System of Arithmetic and Algebra*, Chap. III. Melbourne, 1924.

[4] Nunn, T. P. *The Teaching of Algebra, including Trigonometry*, Chap. XVIII. London, 1927.

Typical Problem Material

Old	1	2	3	4	5	6	7	8	9	10	11
New	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	+1

Old	12	13	14	15	16	17	18	19	20	21
New	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11

How would a work-party previously numbered (4, 8, 12, 16) now be numbered?

How many places or paces are there between -7 and $+3$; -4 and $+8$?

12. (Referring to Ex. 11.) If we say that the plus numbers are to the right and the minus numbers to the left of zero (the origin), is -3 to the right of -6? Is -3 to the right of -1? Is -7 to the right of $+7$?

(When numbers like -3, -7, $+7$ have the implied notion of direction, they are written (-3), (-7), $(+7)$ and are called *directed numbers.*)

13. Arrange the following temperatures in increasing order: 61, 81, 71, 56, 43, 91, 69, 52, 75, 84, 49. Find the average, or mean, temperature and the difference, or deviation, of each reading from the mean, with the appropriate sign. Add the plus and minus signs to the deviations.

14. Write down the opposites of: right, up, above, ascent, North, East, more, add, credit, profit, hence, ahead, raise, to, in, early, quick, forward, advance, Anno Domini, high, elevation, heavy, dense, condense, acceleration, attraction, clockwise, and arrival. Give simple numerical exercises on these terms and their opposites.

15. A boy, starting from a stairway landing, made the following four journeys:

(a) He went up 20 steps, and then went up 12 steps.
(b) He went up 20 steps, and then down 12 steps.
(c) He went down 20 steps, and then up 12 steps.
(d) He went down 20 steps, and then went down 12 more steps.

How many steps was he from the landing in each case?

Suggested Course	**Definitions and Concepts**
	When the resultant and all but one of the components are given, the remaining component can be found by *subtraction*, the complement of addition.
	Thus $(+8) - (+3) = (+5)$ $(-2) - (-5) = (+3)$
	The above examples illustrate the Rule of Signs.[5] The ways in which such operations are performed depend on the nature of the physical facts from which the quantities are derived.
Association and commutation.	To the operations of addition and subtraction the laws of *association* and *commutation* apply. Plus and minus signs are also used to indicate vector addition and subtraction:
	Thus $\overline{AB} + \overline{BC} = \overline{AC}$ $\overline{AB} - \overline{AC} = \overline{CB}$
Parentheses.	*Parentheses* may be regarded as envelopes or enclosures within which quantities are grouped for convenience. An operation affecting one quantity within the parentheses affects the whole.
	Thus $2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $\frac{3 + 5}{2} = \frac{3}{2} + \frac{5}{2}$
Generalized arithmetic.	The letters a, b, c; A, B, C; etc., used in algebra are simple forms of verbal statements about physical quantities measurable in terms of certain arbitrarily chosen units.[6] Since algebraic letters represent linguistic units involving numbers, they are subject to the laws and operations both of logic and of arithmetic.

[5] Young, J. W. *The Fundamental Concepts of Algebra and Geometry*, p. 113. New York, 1925. "Let it be emphasized that there can be no such thing as an *a priori* proof of these laws of signs, but that they are pure conventions, finding their justification on the logical side in their consistency with previous assumptions and on the practical side in their serviceableness."

[6] Nunn, T. P. *Op. cit.*, p. 6. "Such symbolism as $(a + b)^2 = a^2 + 2ab + b^2$, . . . , may be regarded as verbal statements about numbers expressed for a special purpose in a conventional form."

Typical Problem Material

16.(a) A boy, starting from home, went 5 mi. East, and then 3 mi. East. How far was he from home at the end?

(b) The boy went 5 mi. East, and then 3 mi. West. How far was he from home?

(c) The boy started out for a place 5 mi. East. In the first hour he went 3 mi. East. How much further did he have to go?

(d) The boy started out for a place 5 mi. East. In the first hour he cycled 7 mi. East. How much further did he have to go?

(e) The boy started for a place 5 mi. East. In the first hour he went, by mistake, 3 mi. West. How much further did he have to go?[7]

17. Further suggested topics: temperature changes, profit and loss, time lines, mechanics, baseball scores; also problems using vector notation.

18. How many years are there between 500 B.C. and 500 A.D.? Between 500 A.D. and 1930 A.D.? Express the result in mathematical form.

19.(a) The length of a sheet of paper is 9 in. and the width 7 in. Write the length of the perimeter in parenthetical form.

(b) Thirty boys went for a hike, taking in their haversacks some sandwiches. Each boy had 2 bread-and-butter, 3 jam, and 4 egg sandwiches. Express the total number in parentheses.

20. Find the value of $7(50 - 32)$, $8(50 + 32 - 87)$, $3(169 - 156)$ in two ways.

21. Generalized arithmetic should not be regarded as a new chapter in school mathematics, but as an integral part of the development of the subject. Any statement which can be expressed in the numbers of arithmetic can also be expressed in the symbols of algebra.

For example, Ex. 16 may be written:

(a) A boy, starting from home, went a mi. East, and then b mi. West. How far was he from home at the end?

(b) The boy went x mi. East and then y mi. West. How far was he from home?

Again, Ex. 19 could be written:

(a) The length of a sheet of paper is l in. and the width b in. Write the length of the perimeter in parenthetical form.

22. Find the value of $2(3a + 5b)$, $6(7a + 3b) - 4(6a + 2b)$.

[7] Nunn, T. P. *Op. cit.*, Chap. XVIII.

Suggested Course	Definitions and Concepts
Nondirected and directed numbers.	As we have nondirected and directed numbers in arithmetic, so we have nondirected and directed numbers in algebra. For example, $S = (+a) + (+b)$ is the result of compounding two directed numbers $(+a)$ and $(+b)$. The signs within the parentheses are signs of 'direction' and those between the parentheses are signs of 'operation'.[8]
Opposites.	The numbers $(+a)$ and $(-a)$ are opposites, if $(+a) + (-a) = 0$.
The formula.	A formula is the symbolic expression of a verbal statement of equality or inequality. For example, $a > b$ may express in shortened form a statement such as: The number a is greater than the number b or the point a is to the right of the point b. Again, $A = l \cdot w$ may symbolize the statement: "The area of a rectangle is the product of its length and its width."
Functional relation.	A formula may be looked upon as the expression of a particular relation and of a general or *functional relation*, of which the particular relation is a special case.
Dependency.	A term of a formula is said to *depend* on, or be a function of, other terms of the formula when, the latter terms being known, the former is thereby determined.
The subject of a formula.	That term in a formula which *depends* upon other terms in the formula in such a way that, when the independent terms are known, the dependent term can thereby be determined, is called the *subject* of the formula.[9]
Changing the subject of a formula.	It is customary to place the subject of a formula alone on the left-hand side of the sign of equality or inequality. Any term may, according to our convenience or need, be taken to be the subject of the formula. Thus the formula, cost = price × quantity (or $c = p \cdot n$), may also be written $p = c/n$ or $n = c/p$.

[8] For the origin of the term 'directed numbers' see Nunn, T. P., *op. cit.*, Chap. XVIII.

[9] Nunn, T. P. *Op. cit.*, p. 77 and Chap. X.

Typical Problem Material

23. In four successive hours a man motored (in miles): $(+a)$, $(+b)$, $(+c)$, $-d)$. How many miles did he motor altogether?

24. If $S = (3a + b) - (2a - b)$ and $a = +3$, $b = -2$, find the value of S.

25. Add $(3a + b + 2c)$, $(a - b + 3c)$, $(a + 2b - c)$.

26. Subtract $5a - 4b + 2c$ from $2a - 7b - 3c$.

27. It is given that $P = 3p - 4$. Put in two columns the values of p and P when $p = 0, 1, 2, 3, 4, 5$. Illustrate graphically and interpolate other values.

28. If $f(x)$ is short for $3x + 4$, find $f(1), f(2), f(3), f(4), f(5)$. Graph the values.

29. What is the opposite of S ft. up, d yd. West, t yr. ago, a lb. lighter?

30. Show that S ft. up is the same as $-S$ ft. down. Use concrete device.

31. The cost of a consignment of jam is given by the formula $C = np$, where p is the price per jar and n is the number of jars in the consignment. Draw up a table and a graph showing the cost of 1, 2, 3, . . . 12 jars, when the price is 20c. a jar. What is the price per jar if 10 jars cost $2.50? How many jars can be bought for $2.00, if the price is 18c. per jar?

32. (a) The interest I get from a certain sum of money placed in the bank depends upon or is a function of . . . and . . . .

(b) The cost of painting a cylindrical tank depends upon . . . , . . . , and . . . .

33. Discuss the factors upon which the following depend:
   (a) The weight of a rectangular block of metal.
   (b) The price of certain mining shares.
   (c) The exchange rate between America and England.
   (d) The speed of an automobile or an airplane.
   (e) The health of any boy in the school.
   (f) The yield of wheat in a field.
   (g) The distance a ball would fall from rest in a given time.

Write your conclusions as follows: The weight of a rectangular block of metal is a *function* of the length, width, and height of the block, and the density or specific gravity of the metal.

Which of the factors you have mentioned tend to increase, and which to decrease, the thing you are discussing?

Discuss the degree to which the factors enter in Ex. 33 (a), (d), and (g). (See also Ex. 35.)

Suggested Course	Definitions and Concepts
Formula building: (a) Variables. (b) Dimensions. (c) Relations — direct and indirect.	The cultivation of the formula involves not only the facility to interpret and apply formulas but also the ability to collect materials for formula building. This includes the discussion of possible *factors* or *variables*, the degree to which they enter into the formula, i.e., the *dimensions*, and the type of relationship, whether *direct* or *inverse* (the more . . . the more and the more . . . the less). Thus the mass of a cylinder depends upon the radius (to the second degree), the length (to the first degree), and the relative density.
Motion of a material body. (a) Translation. (b) Rotation.	When all parts of a physical body have like motion relative to fixed axes of reference, the body is said to have motion of *translation*; when different parts of the body have different motions, the body is said to have motion of *rotation*.
Definitions of (a) Particle. (b) Point. (c) Line and direction. (d) Curve and direction. (e) Angle.	(a) A *particle* is the smallest conceivable material body. A particle can have a motion of translation, but not of rotation. (b) The geometrical correlate of the physical body is the geometrical *figure*; the geometrical correlate of the particle is the *point*. (c) Any two points A and B determine a *straight line* (AB) and a *direction* $\overline{AB}$. The concepts 'straight line' (AB) and 'direction' $\overline{AB}$ are thus directly associated. One implies the other. A direction $\overline{AB}$ implies an opposite direction $\overline{BA}$. A straight line may be limited (*line segment*) or unlimited (infinite). An infinite half-line, that is, a line of unlimited length terminated in one direction is called a *ray*.

Typical Problem Material

34. The simple interest on a principal P for n years at r % per annum is given by the formula: $I = \frac{P.r.n}{100}$.

Make a table showing the simple interest on $300.00 for any number of years up to 10, at 4% per annum. Draw a graph representing the interest for any year up to 10 years.

Find a formula giving the rate per cent per annum, when the principal and the simple interest are given.

35. Find the value of the subject of each of the following formulas, the values of the other terms being given:

(a) The *airplane* formula: $R = KSV^2$, when $K = .005$, $S = 25$, $V = 120$. Check the result by making S the subject of the formula. If V be doubled, what will be the change in R?

(b) The *falling body* formula: $S = \frac{1}{2}gt^2$, when $g = 32$ and $t = 9$. Check the result by making g the subject of the formula.

(c) The *power shaft* formula: $M = \frac{d^3r}{125}$, when $d = 5$ and $r = 80$.

Check the result by making r the subject of the formula.

36. (a) Write down a formula giving the area of the four walls of a room when the length, width, and height are given. If the length be doubled, how will the area be affected?

(b) Write a formula to enable a man to calculate his salary after n years, if his salary begins at x dollars, and increases at the rate of y dollars a year after the first year.

37. Give illustrations of bodies which have
(a) Translation without rotation.
(b) Rotation without translation.
(c) Translation and rotation.

38. The following exercise is intended as a class demonstration:
What kind of geometrical shape would you get in each of the following cases?
(a) Rotating a rectangle of wire about one side.
(b) Rotating a rectangular disc about one edge.
(c) Rotating a triangle about one side.
(d) Rotating a circle about a diameter.
(e) Translating a circle perpendicular to its own plane.
(f) Translating a triangle perpendicular to its own plane.
(g) Translating a square along a line not perpendicular to its own length.

(The geometrical terms 'rectangle', 'perpendicular', and 'plane' need not be mentioned but should be indicated by action.)

Suggested Course | **Definitions and Concepts**

A line segment is measurable in terms of a unit or standard of length. The *measure* of the line segment is a real number.

The study of geometry may be approached from the *nonmetrical* or descriptive, or from the metrical standpoint. In the former case the most useful approach is through a study of geometrical forms, solid and plane, and through 'projection'.

A curve as the path of a moving point-particle.

Any line, straight or curved, may be regarded as a continuous set of points, specifiable as the path of a moving particle. Continuity is here accepted as intuitive.

(d) A straight line (hereafter called a line) has the same direction at all its points; two directions belong to a curve at any point, except an end-point.

Projection:
(1) Conical projection.
(2) Section.
(3) Similarity.

$P_1, P_2, \ldots$ is a system of points in a plane S. O is any fixed point outside S; the lines $OP_1, OP_2, \ldots$ meet a second plane S' at the corresponding points $Q_1, Q_2, \ldots$. Then the system of points $Q_1, Q_2, \ldots$ on S' is the *conical projection* of the system of points $P_1, P_2, \ldots$ on S with respect to the point O, which is called the *vertex of projection*.

The lines $OP_1Q_1, OP_2Q_2, \ldots$ form a *pencil* or a sheaf of rays. The figure formed by a plane cut-

Typical Problem Material

39. Examine models, preferably those prepared by the class as a project, of the following geometrical forms:

(a) Prism on triangular, quadrilateral, and pentagonal bases.
(b) Pyramid on triangular, quadrilateral, and pentagonal bases.
(c) Combination of prisms and pyramids.
(d) Combination of prisms and prisms (e.g., octahedron).

Make three columns for faces (F), vertices (V), and edges (E).

Add the number of faces and vertices together ($F + V$) and note the relationship to the number of edges (E). Verify Euler's Theorem: $F + V = E + 2$.

Discuss the case of the triangle and rectangle. (The names of geometrical solids, except prism and pyramid, need not be introduced.)

40. (a) Draw a number of lines on paper and count the number of points of intersection. Enter in your book: number of lines (L), number of points (P). Show that number of points $(P) = \frac{L(L-1)}{2}$.

(b) Mark a number of points on paper and join them with lines. Enter in your book: number of points (P) and number of lines (L). Show that number of lines $(L) = \frac{P(P-1)}{2}$.

Note here as elsewhere, the idea of *duality*.

41. Fold a square sheet of paper so as to make 4 squares. Put a point inside each square. Connect all the points in succession by a broken line without allowing the line to cross itself. Repeat with the square divided into 9, 12, and 16 parts. (See Young, J. W., *Fundamental Concepts*, page 168.)

42. (a) A boy was cycling along a straight road; a dog in a neighboring field saw the boy and ran directly towards him. What kind of curve did the dog follow in his pursuit of the boy? Draw the positions of the boy and the dog after, say, 1, 2, 3, 4, 5, 6 sec. This curve is called a *curve of pursuit*.

(b) Three dogs A, B, C are at three different points in a field. A sees B and chases him, B sees C and chases him, C sees A and chases him. Describe the path of pursuit of one of the dogs.

43. Drawing certain well-known curves mechanically, e.g., circle, ellipse (string method), parabola, spirals (unwinding thread on cotton reel), cycloid, epicycloids, and the like. Make mechanical contrivances to demonstrate some of these.

44. (a) Using a small electric torch, project upon a wall shadows of plane and solid objects (rectangle, circle, cube, sphere, etc.). Note the form of the figures produced in various positions. By means of strings, or thin india-rubber cords, show the directions of the rays through corresponding points of the object and shadow.

(b) Make a small spherical frame of wire to represent the meridians of longitude and the parallels of latitude of the earth. Place a small electric lamp at the center and draw the projections of the meridians and parallels on plane and cylindrical sheets of paper. (Map projection)

Suggested Course | **Definitions and Concepts**

ting the pencil is called a plane *section*. There is a one-to-one correspondence between the points and lines of the sections of a pencil of rays.

When the planes S and S' are parallel, the geometrical figures so obtained are *similar*.

Motion of a particle—
Translation.
(a) Direction.
(b) Speed.

The motion of a particle requires (a) direction and (b) speed for its complete specification.

The direction of motion of a particle at any instant is one of the directions belonging to the line of motion at the point corresponding to the given instant.

The speed, or velocity, of a particle is the time-rate of change of position.

Thus $v = \frac{s}{t}$ or $s = v \cdot t$., and $v = \frac{s - s'}{t - t'} = \frac{\Delta s}{\Delta t}$.

Velocity may be constant or variable.

Ratio and rate.

A *ratio* is a relation between two quantities of the same kind; a *rate* is a relation between two quantities of different kinds.[10]

Thus $\frac{1 \text{ yd.}}{1 \text{ in.}}$ is a ratio; $\frac{1 \text{ yd.}}{1 \text{ sec.}}$ is a rate.[11]

Velocity is a rate.

'Two quantities of different kinds have neither sum, nor difference, nor ratio.[12]

Rate of change.

The *rate of change* of a variable y with respect to a variable x within a given interval is obtained by dividing the change in y (Δy) by the corresponding or concomitant change in x (Δx).

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[10] Wallis, J. *Treatise on Algebra*, 1685. "Quantities are of the same kind if they are comparable, i.e., if it can be proved that one is greater than, equal to, or less than the other."

[11] Henderson, J. B. "The Stroud System of Teaching Dynamics." *Mathematical Gazette*, May, 1924, p. 99.

[12] Picken, D. K. "Ratio and Proportion." *Mathematical Gazette*, Jan and May, 1920; May, 1924.

Typical Problem Material

45. Examine the form of plane sections of a sphere, cube, cone, and pyramid. Ask the class to draw figures of imaginary plane sections of other figures (contour lines).

46. As a class project make similar models of a single-room house, using as the unit of measurement 1 in., 1½ in., 2 in., etc., to the foot, show that measures of corresponding parts are in the same ratio.

47. Simple problems on the speeds of vehicles (automobiles, airplanes, ships), and the distance traversed in a given time. Use line diagrams drawn to scale to illustrate the problems.

48. A boy cycled from A to X passing certain villages B, C, D, E, and F on the way. The distances of each village from A and the time of arrival (starting at noon) are given in the following table:

Place	Distance from A (mi.) s	Time from noon (hr.) t
A	0	noon
B	30	1.30
C	50	2.30
D	60	3.30
E	80	4.30
F	100	5.00
X	120	5.30

Find the boy's rate of cycling during each stage of the journey, thus:

Make two new columns headed Δs (change in distance) and Δt (change in time), and fill in the columns, Δs and Δt, for each interval AB, BC, etc. Find the rate of cycling $\frac{\Delta s}{\Delta t}$ for each interval.

49. A boat moves along a river, with velocity V (miles per hour) relative to the water. The velocity of the water relative to the bank is v mi. per hour. Show that the distance traversed by the boat in time t is $(V + v)t$ or $(V - v)t$.

Calculate the distance traversed upstream in 4 hr., when $V = 10$ mi./hr. and $v = 1/2$ mi./hr.

50. The following formula gives the velocity (v) of a stream at the bottom of a river when the velocity (V) at the surface is given: $v = V - 2\sqrt{V} + 1$.

Calculate the velocity of the water at the bottom of the river when that at the surface is 144 ft./min.

Suggested Course	Definitions and Concepts
An angle.	(e) When a physical body rotates, i.e., moves in such a way that different parts of the body have different motions, it is said to move through an angle. If we take an indefinitely thin rod as the simplest form of extended physical body, the line is its geometrical correlate. Any two lines in the same plane constitute an *angle*. The term 'angle' is here defined as a geometrical figure having no necessary association with magnitude.
Pasch's Axiom.	"Let A, B, C, be three points, not lying in a straight line, and let L be a straight line in the plane ABC and not passing through any of the points A, B, or C. Then, if L passes through a point of the segment AB, it must also pass through a point either of the segment BC or of the segment AC." (Pasch's Axiom) Again, "Let A, B, C be three points, not lying in a straight line, and let L be a straight line passing through A and not passing through the points B or C. Then L lies *between* AB and AC, if it cut the segment BC." From this axiom we derive the concepts 'greater than' and 'less than' as applied to angles.
Direction.	An elementary conception of *direction* may be regarded as intuitive. This is implied in the statement that a straight line has the same direction at all its points. Any two directions in a plane determine an angle. Just as the length of a line is measured in terms of a unit length span between two arbitrarily chosen points, so the magnitude of an angle is measured in terms of a unit angle span between two arbitrarily chosen directions. To find the measure of a line, we 'translate' our unit; to find the measure of an angle, we 'rotate' our unit.
The cycle or complete rotation.	The unit angle is the *degree*, which is 1/360th part of a complete rotation, or cycle (C).
Right angle.	The right angle (R) is one-fourth of the cycle. Thus $C = 360° = 4R$.

Typical Problem Material

51. Cut out a paper circle of 2-in. radius. Fold the paper dividing the circle into 2, then 4, then 8 equal parts. Mark the creases with a pencil, thus making 8 equal angles at the center. Now divide the circle into 6 equal parts (by spacing the radius round the circumference). Now divide the circle into 24 equal parts (60°− 45°). Discuss the division of the circle into 360 equal parts. Measure the angles so drawn with a protractor.

52. Cut out two circles, each of 2-in. radius, and with the aid of a protractor, divide one of them into parts so that it may serve as a *circular* protractor. Make radial cuts to both circles (along the zero line in the case of the circular protractor) and fit them together so that the sizes of the sectors may be varied at will. Turn over to the unmarked side, guess the magnitude, or 'rota', of the angle of the sector, and write the guess down. Turn to the marked side and test your guess. Enter your work thus:

Guess	Test	Difference (with proper sign)

53. (a) Draw a number of angles on your book; guess the number of degrees in each, and test your guesses with a protractor. Write down the difference between the guess and the test in each case, with the appropriate sign, thus:

Guess	Test	Difference

(b) Make a clock-face protractor with two hands marked in degrees. Draw an angle on the blackboard. Turn the hands of the clock until the angle between them is, according to estimation, equal to the angle drawn on the blackboard. Compute the angle of the clock-face by taking the difference of the 'readings'. Test the estimate by measuring the given angle with an ordinary protractor. Repeat with angles of various sizes (up to 360°).

Repeat Ex. 53 (a) using a rotating wheel. Take angles greater than 360°.

Suggested Course	Definitions and Concepts
The 'rota' of an angle.	There is no term for an angle corresponding to the length of a line. We suggest the use of a term, such as 'rota', which suggests 'amount of rotation'. Thus, the rota of any angle of an equilateral triangle is 60°.
Fundamental assumptions regarding angles.	*Fundamental assumptions* regarding angles are:
(1) Right angles.	(1) All right angles are equal.
(2) Vertically opposite angles.	(2) When two lines intersect, the vertically opposite angles are equal.
(3) Directed angles.	(3) The rota of an angle may be finite or infinite, positive or negative. Angles greater than 360° may be conveniently represented on a Riemannian Surface.[13]
(4) The complete angle.	(4) Two directions determine eight angles in all.[14]
The circle.	If a finite line OA rotate round a fixed point O, the point A traces a *circle*. The line OA is called the *radius*, or the *radius vector*.
Points of the compass.	The points of the compass should be studied practically. (See Ex. 54.)

[13] The Riemann Surface may be illustrated with a soap film on a spiral frame. See Boys, C. V., *Soap Bubbles*, London, 1912.

[14] Picken, D. K. "The Complete Angle and Geometrical Generality." *Mathematical Gazette*, Dec., 1922.

Typical Problem Material

54. The Points of the Compass. (A group lesson.)

Divide the class into groups of three or four pupils each. Procure a number of light blackboards (or drawing-boards with paper), one for each group. Place the boards upon the workbenches or desks. Procure a number of small compass needles. Ask the class to draw a number of lines North–South and East–West, dividing the board into a set of rectangles or squares. (The idea of parallelism should be brought out here.) Draw a few lines in the N.E.–S.W. direction, and others in the N.W.–S.E. direction. Problems like the following may be given:

(a) A scout went 3 mi. North, 2 mi. East, and then 3 mi. South–East. Show, by a scale drawing, his final position. Measure his distance from his starting-point and indicate his direction from the North–South line.

(b) A ship sailed 50 mi. in the direction N. 32° E., changed its course and sailed 40 mi. in the direction N. 20° W. Find its position relative to its starting-point. (The above classroom technique may be frequently used in this work.)

55. Field Work.[15]

Use of plane table and alidade to make a map or plan of the school ground. Compare the plans drawn by various groups and show that they are similar (i.e., the angles are equal and the sides proportional).

The map may be used as an illustration of conical projection.

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[15] Shuster, C. N. "The Use of Measuring Instruments in Teaching Mathematics." *Third Yearbook, National Council of Teachers of Mathematics*, 1928, p. 215.

Swainson, O. W. *Topographic Manual*, p. 38 ff. United States Coast and Geodetic Survey. Washington, 1928.

Suggested Course	Definitions and Concepts
Locus.	The circle is the *locus* of all points in a plane equidistant from a given point.
Angle and corresponding arc.	The angle at the center of a circle is proportional to the subtended arc. The circumference of a circle is proportional to the radius of the circle. ($C = 2\pi r$.) The arc subtended by a given angle is proportional to the radius of the circle.
Angular motion.	Analogous to the velocity of a particle along a line, we have the angular velocity of a line (rod) round a point. Thus $\omega = \frac{\theta}{t}$ or $\theta = \omega t$. And $\omega = \frac{\theta - \theta'}{t - t'} = \frac{\Delta\theta}{\Delta t}$. Angular velocity may be variable or constant.
Angle of inclination.	Two rays OX and OY divide the plane which contains them into two regions. These rays determine two angle rotas which are in general unequal and which are known as the two *angles of inclination* of the two rays. The angles of inclination of two co-terminal rays are *supplementary*.

Typical Problem Material

56. Draw two lines AOA', BOB', such that $AO = OA'$, and $BO = OB'$. Join AB and $A'B'$ and measure them. Show that $AB = A'B'$. Show that the greater the angle (up to 180°) between the lines, the greater is the length of the line AB.

57. Draw two intersecting lines and see how many different angles you can find. Measure them.

58. Exercises on circles. Designs with circles.
(The purpose of these exercises is partly to give the pupil experience in the use of compass and partly to show the connection of geometry with design.)

59. A treasure was hidden in a field 40 ft. from one tree and 30 ft. from another. The trees were 25 ft. apart. Draw a plan showing possible positions of the treasure. If the trees were in a direct North—South line and the treasure was to the East of the trees, where was it situated?

60. Draw four concentric circles. Make two columns, Radius and Corresponding Arc, and show that for a given angle at the center, the arcs are proportional to the radii, i.e., $s_1/s_2 = r_1/r_2$.

61. Draw a circle. Measure some angles at the center and their corresponding arcs. ($s_1/s_2 = \theta_1/\theta_2$.) Measure the *rate of change* of angle with arc in each case.

62. How many times would a wheel of 14-in. diameter turn when traveling 1 mi. along a road?

63. A gear wheel of 10-in. radius turns at the rate of 100 r.p.m. A second wheel of 5-in. radius bears on the first. How many revolutions does the second wheel make in 5 min.?

64. The pedal wheel of a bicycle of 5-in. radius is connected by a chain to the rear sprocket wheel of 1½-in. radius. The radius of the rear wheel is 14 in. How far would the bicycle go in 10 min., if the pedal is turned one revolution every 2 sec.?
If the pedal wheel were increased from 5 in. to 6 in. in radius, how much further would the bicycle go in the same time?

65. Show that $(\alpha + \beta) + (2\alpha - \beta) = 3\alpha$ is true of angular motion. (Use the divided circles of Ex. 52.)

66. The angle through which a wheel revolves in a certain time is given in the following table:

Angle (θ) (deg.)	30	60	90	110	130
Time (t) (sec.)	1	2	3	4	5
Angle (θ) (deg.)	140	175	225	280	600
Time (t) (sec.)	6	7	9	11	20

Suggested Course	Definitions and Concepts
Projection. Orthogonal projection.	P_1, P_2, . . . is a system of points in a plane S. Q_1, Q_2, . . . are the feet of the perpendiculars from P_1, P_2, . . . to a line XY (or a plane S'). Then the system of points Q_1, Q_2, . . . is called the *orthogonal projection* of the systen. P_1, P_2, . . . on the line XY (or the plane S').
Horizontal and vertical projections.	The orthogonal projections of a system of points P_1, P_2, . . . on horizontal and vertical lines (or planes) are called horizontal and vertical projections of the system of points.
Perpendicular to a plane.	A line is perpendicular to a plane when it is perpendicular to any line which it meets in the plane.
Angle between a straight line and a plane.	If ON be the projection of the line OP on the plane OXY, then the angle between the line OP and the plane OXY is the angle PON.
Angle between two planes.	The angle between two planes is the angle between two intersecting lines, one in each plane, each perpendicular to the line of intersection of the planes. This angle is called the *angle of inclination* of the two planes.
Graphical representation. (a) The graph as a pictorial representation of a verbal statement.	Graphs may be used as pictorial representations of verbal statements. A graph has its own 'story'.

Typical Problem Material

Draw two more tables showing change in angle ($\Delta\ \theta$) and the corresponding change in time ($\Delta\ t$) for each interval. Find the ratio $\frac{\Delta\ \theta}{\Delta\ t}$ in each case.

67. Calculate the angular velocity (in degrees per second) of a top which makes 600 r.p.m.

68. (a) Imagine a shower of rain (or the sun's rays) falling vertically. What would be the shape of the dry patch (or the shadow) made by a circular disc, a square disc, an ellipse, a sphere, etc., when placed at certain inclinations to the ground?

(b) Make cardboard models, with projection lines of thread, to illustrate projection.

69. Show that the sum of the orthogonal projections of a number of vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, . . . $\overline{MN}$ on a line XY is equal to the projection of the vector $\overline{AN}$ on the same line.

70. Draw a circle and make angles (POX) 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180° from an initial radius OX. Measure the projection ON of the radius vector OP upon the line OX for each position of OP. Tabulate the results in two columns, Angle and Projection.

Find the angle for which the projection is 0.4, 0.5, 0.6 of the length of the radius vector.

71. A ship sailed 50 mi. in the direction 32° E. of N. How far was it North of its starting-point, and how far East?

72. A ship sailed 80 mi. in the direction N.30° E. on the first day of its passage. 85 mi. N.35° E. on the second day, and 90 mi. N.40° E. on the third day. Find the total distance traversed as measured by the log, the effective distance from the point of departure and the total 'northing' and 'easting'.

73. With a straight stick and a number of threads (or with an umbrella frame) make a framework to represent a right circular cone. Find the angle between the generating lines and the base.

74. Take the models of houses used in Ex. 46. Find the inclination of the roof and of the roof edge to the ground.

75. The Graph of a Story.

A boy motorcycled from A to X, passing certain villages A, B, C, D, E, F on the way. The distances of the villages from A, their heights above sea level, and the boy's time of arrival at them are given in the following table:

Suggested Course	**Definitions and Concepts**
Interpolation from the graph.	When a graph shows a general trend it is possible to estimate, to a certain degree of approximation, values of the variables between the given data. This process of estimation is called *interpolation*. A graph may be regarded as a 'ready reckoner', since unknown values of the variable may be interpolated between ascertained or given values.
Graphs representing rate of change.	Examples illustrating rates of different kinds are: speed $\frac{\Delta s}{\Delta t}$ and slope (gradient) $\frac{\Delta h}{\Delta s}$.
	Example illustrating constant rate and constant gradient.

Typical Problem Material

Place	Distance from A (mi.)	Height above Sea Level (ft.)	Time Taken (sec.)
A	0	10	noon
B	30	50	1.30
C	50	200	2.30
D	60 } *s*	600 } *h*	3.30 } *t*
E	80	1,000	4.30
F	100	900	5.00
X	120	700	5.30

(a) Make a scale drawing of the journey on graph paper, showing the distances between the villages horizontally (20 mi. to the inch), the corresponding heights above sea level vertically (800 ft. to the inch).

(b) Tell the story of the journey from the point of view of the cyclist. How far from *A* would you expect the boy to be at a height of 400 ft. above sea level? Where would the journey be most difficult? Where easiest?

(c) Make from the data three new columns, Change of Distance (Δs), Change of Altitude (Δh), and Change of Time (Δt) for each interval.

76. Using the data of Ex. 75, work out for each interval the following:

(a) $\dfrac{\text{Change in height}}{\text{Corresponding change in distance}}$ or $\dfrac{\Delta h}{\Delta s}$.

(b) $\dfrac{\text{Change in distance}}{\text{Corresponding change in time}}$ or $\dfrac{\Delta s}{\Delta t}$.

(c) $\dfrac{\text{Change in height}}{\text{Corresponding change in time}}$ or $\dfrac{\Delta h}{\Delta t}$.

$\dfrac{\Delta h}{\Delta s}$ is the *rate* of change of height with distance (gradient) between the points considered.

$\dfrac{\Delta s}{\Delta t}$ is the *rate* of change of distance with time (speed) between the same points.

$\dfrac{\Delta h}{\Delta t}$ is the *rate* of change of height with time between the same points.

77. Water flows into a tank of 50-cu. ft. capacity at the rate of 5 cu. ft. per minute.

(a) Make a table with columns showing the volume of water in the tank (V) for each second of time (t) up to the time the tank overflows.

(b) Draw a graph showing how much of the tank is filled each minute.

(c) Find, from the graph, how much water was in the tank at the end of 6½ min.

Suggested Course	Definitions and Concepts
Growth curve.	Example illustrating normal growth: since growth is cont' 'ious and gradual the curve of growth will be ; smooth curve. Reference should be made to automatic recorders of meteorological phenomena, crescographs (e.g., J. C. Bose's magnetic crescograph).
Graphical representation. (b) Bar graph.	A *bar graph* or *column diagram* may be looked upon as a broadened line graph. In a bar graph the bases are equal, the relative magnitudes of the quantities represented being proportional to the lengths or to the areas of the bars.

Typical Problem Material

(d) Draw two new columns, Change in Volume (ΔV) and Change in Time (Δt), and find the rate of change of volume with time.

78. The following measurements of the height of a plant were taken every Monday morning for nine weeks:

Date	June 1	June 8	June 15	June 22	June 29
Height (inch)	1.0	2.2	4.4	7.2	11.8

Date	July 6	July 13	July 20	July 27
Height (inch)	13.[illegible]	15.0	15.6	16.0

(a) Draw a graph showing the height of the plant each week.
(b) Draw a graph showing the *rate of growth* of the plant each week.
(c) When was the height 9 in.? When was the rate of growth greatest?

(This example may be prepared for by observing for several weeks the growth of a tulip, or othe quick-growing plant. Further examples may be taken from the medical statistics of the school, when such are available and from tables giving normal weights of babies during the first year of life, and the heights of children during school age.)

79. Make a formula and draw a graph to represent the simple and compound interests on $200 at 4% per annum for 6 yr.

80. During a severe rain storm a rain gauge measured the fall in inches as follows:

Time	9 a.m.	10 a.m.	11 a.m.	12 noon	1 p.m.
Inches	0	1	1½	2	2¼

Time	2 p.m.	3 p.m.	4 p.m.	5 p.m.
Inches	3	4¼	5	6

Draw a bar graph to represent the day's fall.

81. The following figures show the growth of trade in the United States between the years 1850 and 1910. Make a bar graph or column diagram to represent these statistics. (The money is in millions of dollars.)

Suggested Course	Definitions and Concepts
Frequency distribution. Class interval. Frequency histogram. Frequency polygon. Frequency curve.	The *frequency histogram* is a special form of the comparison bar or column graph. It should be given special attention in the school course, since it is widely used in economics, politics, and education. The frequency polygon is a polygon drawn through the middle points of the tops of the bars of the frequency histogram.

Typical Problem Material

Year ...	1850	1860	1870	1880	1890	1900	1910
Imports	180*	260	440	645	795	855	1,610
Exports	130	230	445	845	870	1,410	1,780

* $180,000,000.

In each case draw a bar graph to represent the *rate of increase* of import and export trade.

82. On an examination the number of candidates obtaining scores between 0 and 10%, 10% and 20%, and so on, were given as follows:

Percentage	0–10	10–20	20–30	30–40	40–50
Frequency	2	12	30	47	54

Percentage	50–60	60–70	70–80	80–90	90–100
Frequency	48	29	13	3	1

Draw a 'frequency histogram' of the bar graph to represent the results. (Use a base-line class interval of one-half of an inch.)

Which parts of the graph would you mark: excellent, very good, good, only fair, bad, or would you grade A, B, C, D, and E?

83. A teacher gave the following scores to a class on an examination:

Pupil	Score for Each Question (maximum 10)							
A	2	1	3	3	4	4	5	6
B	4	5	5	7	6	7	8	8
C	4	2	3	4	7	8	6	8
D	5	4	5	6	7	8	8	6
E	5	6	7	5	6	9	7	9
F	3	5	6	5	6	7	6	4
G	2	3	4	5	7	6	7	6
H	4	5	6	6	6	7	6	5
I	5	3	4	6	7	7	8	7
J	5	6	5	7	8	6	7	8
K	3	4	5	6	7	6	7	9

Make a frequency histogram of the scores and draw a frequency curve to represent them.

Suggested Course	Definitions and Concepts
Graphical representation by angles or sectors.	Angles or sectors of circles may be used to represent magnitudes relatively.
Graphical representation. (c) The statistics of experiment.	In all experimental data derived from measurements, errors of various kinds are inevitable. Some of these are due to variations in the material measured, others to 'personal' errors or errors of observation.
Line of best fit.	The line of best fit to experimental data is the line which best represents, in graphical form, the general trend of the data obtained in an investigation.
The frequency curve as the line of best fit.	The frequency curve is the line of best fit to the vertices of a given frequency polygon.[16]
Automatic recorders.	When a variable changes with time, the values of the variable may often be recorded automatically by a self-recording device. Examples of such devices are found in self-recording crescographs, barometers, thermometers, rain gauges, etc.
Surfaces.	A *surface* may be looked upon as a continuous set of points marking the boundary between two continuous regions of space. It is the geometrical correlate of the indefinitely thin sheet. At each point of a surface there is, in general, an unlimited number of directions belonging to the surface.
A plane.	A *plane* is a surface such that at any point of the surface all the directions lie wholly on the surface. (a) Two intersecting lines, (b) Three points not lying on the same line, (c) A line and a point not lying on the line } determine one and only one plane.

[16] Kelley, T. L. *Statistical Method*, Chap. VII. New York, 1928.
Elderton, W. P. *Frequency Curves and Correlation*. London, 1906.

Typical Problem Material

84. Draw an angle or sector graph to show the proportion of calcium, carbon, and oxygen in a sample of chalk (calcium carbonate) weighing 100 gm., which contains calcium (40 gm.), carbon (12 gm.), and oxygen (48 gm.).

85. An india-rubber band, 40 cm. long, was stretched by weights and its length was measured for each weight applied, thus:

Weight suspended (gm.)	0	5	10	15	20	25	30
Length of band (cm.)..	40.0	41.8	43.4	45.6	47.2	49.0	50.8

(a) Make a *line of best fit* among the plotted points of these readings and show that it bears out, approximately, Hooke's Law: "The extension is proportional to the weight applied."

(b) What weight would you expect to stretch the band to 48 cm.?

(c) What will be the length of the band when 27 gm. are suspended?

86. Draw a frequency curve for the data given in Ex. 79.

87. Attach a piece of thread to the top of a growing plant, pass it over a fixed pulley and suspend a small weight at the other end to keep the thread tight. Fix a light pointer radially to the pulley and note the position of the end of the pointer daily. This is a simple crescograph.

88. Discuss the type of surface obtained in the following cases:

(a) Translating a thin stick along a line perpendicular to its length.

(b) Translating a circular ring along a line perpendicular to the plane of the circle.

(c) Translating a triangle, rectangle, etc., along a line not in the same plane.

(d) Rotating a line (*i*) round a circle perpendicular to its length, (*ii*) round a circle not perpendicular to its length.

(e) Rotating a line about a point outside the line.

(f) Rotating a line about one end (circle and cone); a circle about a diameter (sphere).

(The pupil should be asked to imagine the result before he sees it demonstrated in concrete form.)

89. Discuss the type of surface obtained in the following cases:

(a) Translating a line along a line perpendicular to itself, the line 'gradually' decreasing in length.

(b) Translating a circle (or a triangle) perpendicular to itself, the circle 'gradually' decreasing.

Suggested Course	Definitions and Concepts
Plane generated by the movement of a line. (a) Generator. (b) Directrix.	Since any two intersecting lines determine a plane, a plane may be 'generated' by the translation of a line (generator) in any direction (directrix) except that of its own length.
Area.	The measure of a surface in terms of a unit is its *area.* The area of a rectangle is given by the product of its length and its width or $A = lw$.
Indices.	The formula $A = a \cdot a$ for the area of a square of side a may be written $A = a^2$.
Algebraic multiplication. (a) Nondirected numbers.	The following identities may be illustrated graphically: $a(b+c) = ab + ac$; $a(b-c) = ab - ac$; $(a+b)(a-b) = a^2 - b^2$; $(a+b)^2 = a^2 + 2ab + b^2$
Square root.	To calculate square root geometrically in simple cases.[17]
Difference series. Arithmetic progression.	A series is an expression of the form $u_1, u_2, \ldots u_n$, where the terms $u_1, u_2, \ldots u_n$, obey some *law of progression*. In the simplest case where $u_1, u_2, u_3 \ldots$, etc., differ by a constant quantity, the series is called a difference series, or an arithmetic progression. An arithmetic progression may be conveniently illustrated by a bar graph having equal steps in height. The sum of an arithmetic progression may be computed in terms of the area of the graph.[18]
Area by integration.	Since a plane may be generated by the translation of a line in any direction except that of its own length, an area may be computed by a process of *integration*.

[17] Nunn, T. P. *Op. cit.*, Chap. VIII.
[18] *Op. cit.*, Chap. XIX.

Typical Problem Material

90. Illustrate with a soap film on a rectangular frame with one movable side.

91. Draw an irregular closed figure on square paper and calculate its area by counting squares. Find the area of a circle in the same way.

92. Write a formula for the area of the four walls of a room given the length, width, and height of the walls.

What will the formula be if the length and width are increased by 10%?

93. Write formulas for the areas of rectangular figures of various shapes (*L* shape, *T* shape, cross, etc.).

If the length and the width of a rectangle are both increased (or decreased) 10%, by how much is the area increased (or decreased)?

94. The length of a rectangle is 26.54 in. to the nearest .01 in., and the width 17.67 to the same degree of accuracy. Find the area.[19]

95. Factors:

$$9a + 6b = 3(3a + 2b)$$
$$22 \times 18 = (20 + 2)(20 - 2) = 400 - 4 = 396$$
$$(a + b)^2 - c^2 = (a + b + c)(a + b - c)$$
$$(a + b)^2 = a^2 + 2ab + b^2$$

96. (a) Complete the following expression so that it becomes a perfect square: $x^2 + 20x$. Illustrate with a diagram.

(b) One side of a rectangle was 20 in. longer than the other side. The area was 500 sq. in. Find the dimensions of the rectangle.

Show that $(5.3) > \sqrt{27.5} > 5.2$.

(c) Find the length of a side of a square when the area is 30 sq. ft.

97. A man dug a trench 2 ft. wide at the following rate: 40 ft. on the first day, 42 ft. on the second day, 44 ft. on the third, and so on for 6 days. Draw a bar graph (with the trenches side by side) to illustrate the amount of ground dug. Find the total length dug and show that it is given by the formula: $S = \frac{n}{2}(a + l)$, where S is the total length, n the number of days, a the length dug on the first day, and l the length dug on the last day.

98. Find the sum of: 5, 8, 11 . . . to 20 terms.

Find the sum of the first n natural numbers.

99. (a) Take a number of strips of wood of the same width and length, and with them form a rectangle. Keeping the bottom fixed, 'shear' the rest so as to form (1) a parallelogram; (2) irregular 'parallelograms' with curved sides.

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[19] Goodwin, H. M. *Precision of Measurements and Graphical Methods*. New York, 1920.

Jones, H. S. *Modern Arithmetic*, p. 260. London, 1930.

Suggested Course	Definitions and Concepts
	Consider a rectangle as made up of a number of thin rectangular strips of the same width integrated together. The area is the sum of the areas of these strips, i.e., is equal to the product of the generatrix and directrix. Similarly, the area of a parallelogram may be derived from that of the rectangle by 'shearing' the strips. It will then be seen that the area of the parallelogram is equal to the area of the rectangle of the same base and height.
Area of a triangle.	The area of a triangle may be determined in three different ways: (a) As half the area of the corresponding rectangle or parallelogram. (b) As the integration of a number of thin strips of the same width and gradually decreasing lengths. (c) As the *average* length of strip multiplied by the height. $A = \frac{1}{2}lh$.
	The area of a trapezoid is equal to the *average* of the parallel sides multiplied by the height.
Space.	Space may be looked upon as the aggregate of all points. Space holds within its domain all points, lines, and surfaces.
Volume.	The measure of space, in terms of a unit, is *volume*.
Parallelopiped or cuboid.	The volume of a rectangular parallelopiped or *cuboid* is given by the formula $V = lwh$. A volume may be generated by the translation of an area. As in the case of the area of a rectangle, the volume of a cuboid may be computed by dividing the cuboid into a number of thin rectangular discs and *integrating* them.
Algebraic multiplication. (b) The product of directed numbers. (1) Directed areas or length-length products.	Since a length may be treated as a directed quantity, an area or length-length product may, by a suitable convention as to signs, be treated as a directed quantity. $(+A) = (+a)(+b)$ $(-A) = (+a)(-b)$ $(-A) = (-a)(+b)$ $(+A) = (-a)(-b)$

Typical Problem Material

Show that the area is the same in each case: $A = lw$, where l is the length of the base and w the width.

(b) Draw a triangle with base 3 in. and vertical height 2 in. Divide the triangle into strips by drawing a number of equally spaced lines (about a dozen) parallel to the base. Show that the area of the triangle is equal to the height of the triangle multiplied by the average length of the parallel lines.

100. Show, by using a number of strips of wood of the same width but decreasing lengths, that the area of a trapezoid is equal to the average length of the parallel sides multiplied by the height.

Find the area of trapezoids in simple arithmetical cases.

101. Illustrate the volume of a cuboid as the integration of a number of rectangular cardboard discs of the same area. Shear these discs and obtain the volumes of other figures of the same area of base and vertical height.

102. If a rod $(+a)$ be moved perpendicular to its length a distance $(+b)$, it may be said to trace out a positive area $(+A)$. What sign would you give the area in the following cases?

(a) $(+a)$ moving a distance $(-b)$
(b) $(-b)$ moving a distance $(+a)$
(c) $(-b)$ moving a distance $(-a)$
(d) $(-a)$ moving a distance $(-b)$
(e) $(-a)$ moving a distance $(+b)$

Show that $(+A) = (+a)(+b) = (-a)(-b)$.
And $(-A) = (-a)(+b) = (+a)(-b)$.

Suggested Course | **Definitions and Concepts**

Rule of signs.

These equations illustrate the Rule of Signs.

Algebraic fractions.

Graphical representation of fractions by dividing lines, areas, volumes.

Proof that $\frac{m}{n}$ of $A = \frac{1}{n}$ of $mA = \frac{mA}{n}$.

(2) Rate-time products.

The product of a nondirected and a directed number is a directed number, e.g., momentum mass × velocity. The product of two directed numbers of different kinds is a directed number, e.g., distance = velocity × time, i.e., $s = vt$.

$$(+s) = (+v)\,(+t)$$
$$(-s) = (-v)\,(+t)$$
$$(-s) = (+v)\,(-t)$$
$$(+s) = (-v)\,(-t)$$

The rule of signs.

These equations again illustrate the Rule of Signs.

The quotient of two directed numbers. Ratio and rate.

The quotient of two like directed numbers is a *ratio.* The quotient of two unlike directed quantities is a *rate.*

The equations

$$(+s) = (+v)\,(+t) = (-v)\,(-t)$$

and

$$(-s) = (+v)\,(-t) = (-v)\,(+t)$$

suggest the equations

$$\frac{(+s)}{(+v)} = (+t) = \frac{(-s)}{(-v)}$$

and

$$\frac{(-s)}{(+v)} = (-t) = \frac{(+s)}{(-v)}.$$

From which follows also:

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{(-a)}{(+b)} = -\frac{(+a)}{(-b)}$$

Typical Problem Material

103. Apply the idea of directed area to the equations:

$$a(b+c) = ab + ac$$
$$(a+b)(a-b) = a^2 - b^2.$$

104. Draw a rectangle to represent a rectangular block of ground of area A. Show that two-thirds of a block is the same as one-third of two blocks or two blocks divided by three. Thus $\frac{2}{3}$ of $A = \frac{1}{3}$ of $2A = \frac{2A}{3}$.

Repeat with circles.

Show that $\frac{m}{n}$ of $A = \frac{1}{n}$ of $mA = \frac{mA}{n}$.

105. When one-third of a bottle of a medicine A was added to one-fourth of a bottle of the same size of another medicine B, the mixture was found to measure 3½ oz. Find the number of ounces of medicine A and B will each hold when full.

106. If $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, prove that $f = \frac{uv}{u+v}$.

If $C = \frac{nE}{R + nr}$, express E in terms of C, R, and n.

107. Reduce the following to a single algebraic fraction: $\frac{a}{a+b} + \frac{b}{a-b}$.

108. Two automobiles, P and N, are moving in opposite directions at the rate of 30 mi. per hour (½ mi. per minute). They pass a certain mile-post at the same time. Assuming that they maintain a constant rate, and using the formula $s = vt$ to determine the distance traversed, find

(a) Where P was 6 min. before passing the mile-post.
(b) Where P will be 6 min. after passing the mile-post.
(c) Where N will be 6 min. after passing the mile-post.
(d) Where N was 6 min. before passing the mile-post.

Show that $(+3) = (+6)(+\frac{1}{2}) = (-6)(-\frac{1}{2})$.
And $(-3) = (-6)(+\frac{1}{2}) = (+6)(-\frac{1}{2})$.

109. A man, named A, saves money at the rate of $20.00 a month; another man, named B, loses money at the same rate. Compare their financial positions five months ago and five months hence with their positions to-day. Write down a formula to represent each person's financial position.

110. A man, named A, saves money at the rate of $20.00 a month, and another man, named B, loses money at the same rate. When was A $100.00 worse off; when will he be $100.00 better off? When was B $100.00 better off; when will he be $100.00 worse off?

Suggested Course	**Definitions and Concepts**
Triangles or trilateral.	Three noncollinear points A, B, C determine a *triangle*. Since any two points determine a line, three noncollinear points (A, B, C) determine three lines (a, b, c) through the points A, B, C taken in pairs. The line segments AB, BC, CA are the *sides* of the triangle, and the angle segments BCA, etc., the angles of the triangle. Three nonconcurrent lines determine a *trilateral*.
Principle of Duality.	A triangle has three sides. A trilateral has three vertices.
Quadrangle and quadrilateral.	Four points, A, B, C, D, no three of which are collinear, determine a complete *quadrangle*; four lines a, b, c, d, no three of which are concurrent, form a complete *quadrilateral*. These examples illustrate the Principle of Duality. A complete quadrilateral has, in general, six vertices and three diagonals.
Congruence.	Two figures A and B are *congruent* when the points of A and B have one-to-one correspondence and the distance between any two points of A is equal to the distance between the corresponding points of B.
Principle of Congruence.	In the case of plane closed figures, the areas are also equal. "Any figure (plane or solid) can be exactly reproduced anywhere" (Principle of Congruence).[20] Two figures which can be 'constructed' to the same specifications in only one way (i.e., uniquely) are congruent. Equal lines, equal angles, vertically
Examples of congruence.	opposite angles, rectangles having equal length and breadth, cuboids (parallelopipeds) having equal dimensions, etc., are congruent.[21] *Assumption.* Through a point on a line there exists one and only one perpendicular to the line. *Axiom.* If PQ is a given line-segment and OX a given ray, we can find one, and only one, point A on OX such that $OA = PQ$. *Axiom.* If PQR is a given angle and OX a given ray in a given plane, we can find one, and only one, ray OA in the plane and on a given side of OX, such that angle AOX = angle PQR. *Assumption.* Two circles will intersect, if at all, in two points, one on either side of the line joining their centers.[21]

[20] Mathematical Association, *The Teaching of Geometry in Schools*, p. 35. London, 1925.

[21] Forder, H. G. *The Axioms of Geometry*. For a discussion of congruence based on the congruence of point couples, see *Mathematical Gazette*, March, 1929, p. 321.

Typical Problem Material

If they both had $500.00 in the bank six months ago, tabulate in two columns their banking accounts for the following twelve months, the present date being July 1. Draw a graph to represent the monthly variations.

111. The formula $s = vt$ gives the distance s traversed by a cyclist, when the velocity and time are known.

(a) If the velocity is increased 10%, by how much is the distance increased?

(b) If the time is increased 10%, by how much is the distance increased?

(c) If both velocity and time are increased 10%, by how much is the distance increased?

112. Show that a complete quadrilateral has six vertices and three diagonals.

113.(a) Copy the triangle ABC, given AB, BC, and the angle ABC. Measure and compare the remaining side and angles and the area of the original triangle and the copy.

(b) Copy the triangle ABC given AB and the angles ABC and CAB. Show that the two triangles are congruent.

(c) Copy the triangle ABC when the three sides are given and show that the two triangles are congruent.

(d) Copy the triangle ABC, right-angled at B, when the hypotenuse and one other side are given.

114. Make a prism having edges 3′, 4′, 5′ at the base. and 4′, 5′, 6′ at the apex. Compare results obtained by the class and find out how many different forms are possible.

115. If two equal lines are drawn to a line or a plane from a point outside, they are equally inclined to the perpendicular from the point.

116. Draw a number of triangles, ABC, $A'BC$, $A''BC$, etc., having a common base BC and equal sides AB, $A'B$, $A''B$, etc. Measure the third side, AC, $A'C$, $A''C$, etc. Show that the greater the angle ABC, the greater is the third side. Draw up a table of your measurements of angles and third sides and graph them.[22]

117. Draw a triangle ABC having given the angle B equal to 60°, and the sides AB and BC in the ratio of 3:5. How many such triangles can be drawn? Increase each of the given sides by 25%; what difference does this make to the shape of the triangle?

[22] Swenson, J. A. "Graphic Methods of Teaching Congruence in Geometry." *Third Yearbook, National Council of Teachers of Mathematics*, 1929, p. 9.6

Suggested Course	Definitions and Concepts
Congruence of triangles.	*Fundamental Congruence Theorem.* If two triangles ABC and $A'B'C'$ have $AB = A'B'$, $AC = A'C'$, and angle BAC = angle $B'A'C'$, the triangles are congruent, i.e., $BC = B'C'$, angle ACB = angle $A'C'B'$, angle CBA = angle $C'B'A'$, and the area magnitudes of the two triangles will be equal. (*s*, *a*, *s*) The following additional cases of congruent triangles may be exemplified practically (*a*, *s*, *a*), (*s*, *s*, *s*), (*s*, *s*, *a*, ambiguous case). As an important special case of the *s*, *s*, *a* theorem we consider two right-angled triangles, which have their hypotenuses and one other side in each given equal. *Isosceles Triangle Theorem.* If two sides of a triangle are equal, the angles opposite these sides are equal, and conversely. The image of a geometrical figure in a plane mirror is symmetrically congruent with it. This is true also of non-Euclidean geometry.[23]
Loci: (a) Points equidistant from two given points.	*Loci.* (1) The locus of a point equidistant from two given points is a line (the perpendicular bisector of the line joining the points).
(b) Points equidistant from two given lines.	(2) The locus of a point equidistant from two intersecting lines is a line (the bisector of the angle made by the two intersecting lines).
Similarity. Principle of Similarity.	"Any figure can be reproduced anywhere on any enlarged or diminished scale" (Principle of Similarity).[24]
Similar figures.	Two figures, *A* and *B*, are similar when the points of *A* and *B* have one-to-one correspondence, and the distance between any two points of *A* bears a constant ratio to the distance between the corresponding points of *B*. Two triangles are similar, when they have their corresponding angles equal, and the sides about the corresponding angles proportional.

[23] Fletcher, W. C. "A Method of Studying Non-Euclidean Geometry." *Mathematical Gazette*, March, 1923, p. 261.

[24] Mathematical Association, *The Teaching of Geometry in Schools*, p. 35. London, 1925.

Nunn, T. P. "The Sequence of Theorems in School Geometry." *Mathematics Teacher*, Oct., 1925. See also *Mathematical Gazette*, May, 1922.

Typical Problem Material

118. Scout or surveying problems capable of being solved by assuming the main congruence theorems, e.g., computing the width of a river or the height of a building.

119. Simple originals, including the common theorems relating to circles, e.g.:

(a) Equal chords of a circle are equidistant from the center, and, conversely. Chords nearer the center are greater than those more remote. Show by measurement.

(b) The angle in a semicircle is a right angle.

(c) Symmetry—point and line symmetry.

(d) Simple constructions, e.g., to bisect a line, to bisect an angle, to draw a circle passing through three noncollinear points.

120. Show that we can draw as many circles as we like through one or two points, but only one through three points.

121. A treasure was buried [illegible] yards from each of two trees or 50 yards from each of two intersecting fences Where was it? How many possible cases are there?

122. Illustrate similar figures with reference to scale drawings, maps, models of houses, etc., of various sizes.

A photograph 4 by 5 in. was enlarged to 10 by $12\frac{1}{2}$ in. How tall would a figure 1 in. high become when enlarged? How large would an area of 3 sq. in. become on enlargement?

123. Cut out four rectangles exactly equal in size; piece them together so as to make a larger rectangle. Show that the larger rectangle is similar to the original.

124. Cut out four triangles equal in all respects to a triangle ABC. Piece them together so as to make a triangle $A'B'C'$ *similar* to ABC. Repeat with nine such triangles instead of four. Compare the sides and areas of $A'B'C'$ and the original triangle ABC.

Suggested Course	Definitions and Concepts
Similar triangles.	(a) Given two (or three) angles in each triangle equal. If ABC and $A'B'C'$ are two triangles having angle A = angle A' and angle B = angle B', then angle C = angle C' and the sides $a:b:c = a':b':c'$ (a, a, a). (b) Given one angle in each triangle equal and the sides about the given angles proportional $(s.a.s)$. (c) Given the sides proportional $(s.s.s)$.
Proportion.	If $P_1, P_2, \ldots P_n$, a set of quantities of one kind, correspond to $Q_1, Q_2, \ldots Q_n$, another set of quantities also of one kind, so that $P_1 : P_2 : \ldots : P_n = Q_1 : Q_2 : \ldots : Q_n$, then $P_1, P_2, \ldots P_n$ are said to be proportional to $Q_1, Q_2, \ldots Q_n$. Proportion is, therefore, a relation of equality between *ratios*. Again, if the P's and Q's are quantities of different kinds and $P_1 : P_2 = Q_1 : Q_2$, then $P_1/Q_1 = P_2/Q_2$. Thus proportion is also a relation of equality between *rates*.
Parallel lines.	Hilbert's Axiom of Parallels. If L is any straight line and A a point not on L, then there exists in the plane S determined by L and A one and only one straight line L', which contains A but does not meet L. The line L' is said to be *parallel* to L.[26]
Parallel planes.	If S is a plane and A a point not on S, then there exists one and only one plane S', which contains A but does not meet S. The plane S' is said to be parallel to S.
Corresponding angle theorem.	If two co-planar lines be cut by a transversal in such a way that the corresponding angles are equal, the two lines are parallel, and, conversely, if two parallel lines are cut by a transversal, the corresponding angles are equal.
$A + B + C = 180°$.	The sum of the three angles of a triangle is 180°.
The same direction.	Two co-planar lines have the *same direction* if they are equally inclined in the same sense to any transversal crossing them.

[26] Hilbert, D. *Grundlagen der Geometrie*, p. 29. Seventh edition. Leipzig, 1930.

Typical Problem Material

125. ABC is a given triangle, $A'B'$ is a giv line. On $A'B'$ draw a triangle equiangular to ABC. Show that the sides about the equal angles are proportional.

126. Draw two triangles, ABC and $A'B'C'$, having given that the angle A is equal to the angle A' and the sides about the equal angles are proportional. Compare the other sides and angles.

127. Draw a triangle ABC with its sides 3, 4, 5 in. Draw an enlargement of this triangle with its sides $1\frac{1}{2}$ times those of ABC. Compare the angles.

128. Draw a pentagon and make a 3 : 2 enlargement of it without using a protractor: (a) similarly situated and (b) symmetrically situated.

129. Three partners divide the profits of their business amounting to $24,000.00 in the ratio of 3 : 4 : 5. How much did each receive?

130. A train traveling at a uniform rate goes 160 mi. in $3\frac{1}{2}$ hr. How long will it take to go 300 mi? State in the form: $300/160 = x/3\frac{1}{2}$.
A train traveling at a uniform rate goes s mi. in t hr. How long will it take to go $5s$ mi.? ns mi.? $\frac{ns}{2}$ mi.? y mi.?

131. If $a : b = c : d$, show that $a : c = b : d$, and $ad = bc$.
If $a + b : a - b = c + d : c - d$, show that $a : b = c : d$.
Illustrate this with a geometrical diagram.

132. Draw two similar triangles BAC, $B'AC'$ having the angle A common and the other angles corresponding. Produce BC and $B'C'$ and note that they do not meet when produced (parallel lines).

133. Draw two parallel transversals to two parallel lines (parallelogram). How many equal angles, can you find? How many equal line-segments can you find?

134. The theorems known as: 'the alternate angle theorem', 'the two right-angle theorem', 'the angle-sum theorem' should be deduced from the fundamental assumption.

135. The theorem of equal intercepts for (a) parallel lines, (b) parallel planes Keep one of the transversals fixed and 'move' the other into various positions, including that in which a triangle is formed with the fixed transversal and the parallels.

Suggested Course	Definitions and Concepts
Parallelogram.	A parallelogram is the figure formed at the intersections of two pairs of parallel lines.
Equal intercepts theorem: (a) Parallel lines. (b) Parallel planes.	A set of parallel lines cutting a transversal at equal intervals, will cut any other transversal at equal intervals. Similar theorem for a set of parallel planes.
Proportional intercepts theorem: (a) Parallel lines. (b) Parallel planes.	A set of parallel lines cutting one transversal at intervals in the ratios $a:b:c: \ldots$, will cut any other transversal in the same ratios.
Indirect measurement.	Indirect measurement is facilitated by the use of trigonometrical ratios.
Trigonometrical ratios.	Determination of the height of an inaccessible object by the length of the shadow, using
(a) Tangent of an angle.	(a) Similar triangles, and (b) The tangent of the angle of elevation.
Gradient.	The angle between a line drawn in any given direction and the horizontal is called the *slope* of the line. The *gradient* is defined as the tangent of the angle of slope. The gradient of a straight line with reference to another line is constant at all points on the line. The gradient of a curved line varies from point to point. If ABC is a triangle right-angled at C, AC being horizontal, then $\tan A = \frac{BC}{AC}$ or $BC = AC \tan A$. Thus $\tan A$ is the factor by which we multiply the 'horizontal advance' AC to get the corresponding 'vertical rise' BC.
Table of tangents.	Practice should be given in the use of a *table of tangents* and in estimating tangents from a graph.

Typical Problem Material

136. If one of the parallel lines (of Ex. 135) is now translated, still parallel to the others, we have the theorem of proportional intercepts for (a) parallel lines; (b) parallel planes.

137. Divide a given line in the ratios 3 : 4 : 5.

138. The shadow of a tall chimney measured 80 yd. At the same time of day the shadow of a vertical stick 10 ft. high measured 16 ft.

(a) Find, by similar triangles, the height of the chimney.

(b) Measure the 'angle of elevation' of the chimney, or the altitude of the sun, from the end of the shadow (32°).

Repeat as a 'project' with a number of trees, telegraph poles, etc.

139. The angle of elevation of the top of a building from a point on the ground, 160 ft. from the building, was 32°. Find the height of the building.

140. An inclined plane was built at an angle of 32° with the horizontal. Find the quotient of the 'vertical rise' and the 'horizontal advance' or the *gradient* of the plane. The ratio $\frac{\text{vertical height}}{\text{horizontal distance}}$ varies with the angle, and is called the tangent of the angle. (tan 32° = 5/8, or 0.625.)

141. Two boys, using a clinometer, proceeded to measure the height of the top of the school flagstaff as follows: They selected a point on the playground and found that the angle of elevation of the staff was 35°. On walking 50 yd. towards the flagstaff, they found the angle of elevation to be 42°. Find the height of the flagstaff above the ground, given that tan 35° = 0.7, tan 42° = 0.9, and that the observer's eye is 5 ft. above the ground. Verify your answer by making a scale drawing on graph paper.

142. On graph paper draw a circle of unit radius OA (1 in. or any length to represent a unit). Draw a line AP perpendicular to OA. Mark off angles 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, and measure the corresponding perpendiculars PA. Arrange the angles and the perpendiculars in a table.[26]

The graph or the table may be looked upon as a *ready reckoner* to give the tangent of any angle less than 90°. (Note that 90° has no tangent.)

143. A cathedral spire is situated due North of a point P. From a point Q, 120 yd. due East of P, the spire lies in a direction N.18°W. Find the distance from P to a point on the ground immediately below the top of the spire. Find the height of the spire, given that its angle of elevation from P is 14°. Make a model.

[26] Strayer and Upton. *Modern Algebra*, p. 250. New York, 1930.

Suggested Course	Definitions and Concepts
The gradient of two planes.	The gradient of two planes is the tangent of the angle of inclination of the two planes.
Tangent of an angle greater than 180°.	The tangent of an angle in the second quadrant (between 90° and 180°) is negative.
The Theorem of Pythagoras.	The Theorem of Pythagoras states: "The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides." Thus $a^2 + b^2 = c^2$.
Trigonometrical ratios. The sine and the cosine of an angle.	The *sine* and the *cosine* of an angle may now be introduced: $\sin A = b/c$, $\sin B = a/c$.

Typical Problem Material

144. Explain with the aid of a model or a diagram, why a horse, when pulling a load up a steep hill, walks in a zigzag path.

145. OP is a vector revolving from an initial line $X'OX$. ON is the projection of OP on $X'OX$. Make four columns giving: angle POX, PN, ON, PN, ON, and fill in the columns when the angles are 0°, 30°, 60°, 90°, 120°, 150°, 180°. Since PN is always on the same side of the line $X'OX$, we may mark it +; the sign of ON will depend on whether it is on the OX or the OX' side of O.

146. Draw a number of triangles having sides (a, b, c) of lengths (cm.): 3, 4, 5, or 5, 12, 13, or 8, 15, 17, etc. Measure the angle C in each case. Write in three columns the values of a^2, b^2, c^2. See if you can detect any relationship between them $(a^2 + b^2 = c^2)$. Find other sets of numbers that will show the same relationship. ($a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, where m and n are integers, will give other sets.)

147. Draw a triangle having sides equal to 3, 4, 5 in. Draw squares on the three sides (a^2, b^2, c^2). See if you can, by cutting one or more of the squares a^2 and b^2, fit them so that they together equal c^2 in area.

148. Draw a perpendicular from the angle C to the hypotenuse AB of the right-angled triangle ABC. Show by similar triangles that $a^2 + b^2 = c^2$. (Theorem of Pythagoras). (Ex. 146, 147, 148 represent three stages in the development of Pythagoras' Theorem. They have been set down here in consecutive order but in practice Ex. 148 would be postponed a little.)

149. A scout troop walked 6 mi. direct East from their base O, and then 8 mi. direct North to a place A. How far was the troop from the base at the end?

150. Using Ex. 141, find the distances between the top of the flagstaff and the observer's eye, when the two angles were being read.

151. Referring to Ex. 149, draw a sketch and find the 'bearing' of A from O. Write the answer in two ways: North so many degrees East, and East so many degrees North.

152. On the first day out a scout troop walked 6 mi. E. and then 8 mi. N., and on the second day 5 mi. E. and then 6 mi. N. Draw a sketch of the journey and find (a) the distance of their final position from the base; and (b) their final bearing from the base.

153. A scout troop of eight boys, taking their bearings from the base O, were told to spread out, between East and North, in a fan-like formation, each a distance of 100 yd. (r). The bearings of the eight boys from O were 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80° North of East. Draw a sketch of the formation and measure the distance of each boy East (x) and North (y) of the base O. Put your results down in columns:

Suggested Course	Definitions and Concepts
Trigonometrical identities.	The fundamental identities of elementary trigonometry: $\tan A = \sin A/\cos A$, $\sin^2 A + \cos^2 A = 1$.
Projections: The cosine law.	The projection of a plane area A on a plane P is $A \cos \theta$, where θ is the angle between the two planes A and P.
Proportional variation: (a) Direct variation. (b) Inverse variation	If P represent a set of quantities $P_1, P_2, \ldots$, all of one kind and Q a set of quantities $Q_1, Q_2, \ldots$ also of one kind, and if the P's and Q's taken in pairs be related in such a way that $P_m/P_n = Q_m/Q_n$, we then say that P *varies directly* as Q, and write $P \propto Q$. Again, if $P_m:P_n = Q_n:Q_m$, we say that P *varies inversely* as Q, and write $P \propto 1/Q$. Since P and Q are types of sets of quantities they are called *variables*. (For the definition of a variable, see page 20.) Now, in the case of direct variation, we have $P_m/Q_n = P_m/Q_n$. Putting each of these expressions equal to k, a number dependent on the units of P and Q, we have $P_m = kQ_m$ or $P = kQ$. The statements $P \propto Q$ and $P = kQ$

Typical Problem Material

Angle	x	y	x/r	y/r

The ratios x/r and y/r are called the *sine* and the *cosine* of the angles concerned. Look up a book of tables of the *sines* and *cosines* of angles and check the accuracy of your results.[27]

154. Find the angles whose sines are 0.500, 0.707, and 0.866: (a) by a drawing and (b) from tables.

155. If a, b, c are the sides of a right-angle triangle, the hypotenuse of which is C, show that $\sin A = \cos B$, $\cos A = \sin B$, $\tan A = \sin A \cdot \cos A$, $\tan B = \sin B/\cos B$, $\sin A/a = \sin B/b$.

156. Draw a circle of 1-in. radius. Draw two radii OA and OB having an angle α between them. From B drop a perpendicular BN to OA; from A erect a perpendicular AP to OA to meet OB produced in P. Using Pythagoras' Theorem, prove the following identities:

(a) $\sin^2 \alpha + \cos^2 \alpha = 1$

(b) $\tan \alpha = \sin \alpha / \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \cos^2\alpha} / \cos\alpha$

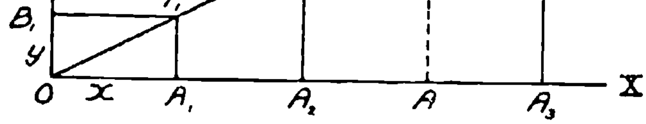
(c) $\sin \alpha = \tan \alpha / \sqrt{1 + \tan^2\alpha}$

157. Using squared paper, mark equally-spaced points along an axis to represent the angles 0°, 10°, 20°, . . . 90°. Draw at each of these points, ordinates corresponding to the sines of the angles taken from a sine table. Connect the ordinate points and so make the 'curve of sines'.

158. (a) Show that the projection of a plane area A on a plane P is $A \cos \alpha$, where α is the angle between the planes A and P. (Note: Use a square or a rectangle to begin with.)

(b) Find the area of an ellipse by the method of projection.

159. Direct Variation.



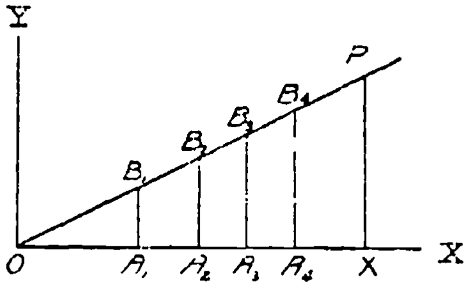
[27] For interesting extensions of these ideas to problems in navigation see Nunn, T. P. *Op. cit.*, Chap. XIII.

Suggested Course	Definitions and Concepts
	are equivalent. Similarly, the statements $P \propto 1/Q$ and $P = k/Q$ are equivalent.
Direct variation. Physical problems.	The functional character of direct variation is best studied through physical problems, in which the relationship 'the more . . . the more' and 'the less . . . the less' are illustrated concretely.
Graphical representation. (d) Graphs as ready-reckoners.	The graph of direct variation is a straight line. The graph of inverse variation is an hyperbola.

Typical Problem Material

Draw a number of rectangles $OA_1P_1B_1$, $OA_2P_2B_2$, etc., each having the OB side equal to half the OA side. Cut these rectangles out and place them so that they have one right angle common (at O). Show that the corners P_1, P_2, P_3, . . . lie on a line through O. Draw any other rectangle $OAPB$ from P and show that $OB = \frac{1}{2}OA$, or $y = \frac{1}{2}x$, where $OB = y$ and $OA = x$. Any one of the rectangles (say $OAPB$) may be used as a *type of a very large number of similar rectangles. Note that the sides and area of these rectangles vary, but the ratio of the sides y/x does not vary;* in other words, y/x is *constant* $= k$ say; then $y = kx$. This is called *direct variation*, the sides of the rectangles being in *direct proportion*. The sides OA and OB (x, y) being types of many possible pairs of sides, are called *variables*.

160. OP represents a straight inclined plane supported by stakes A_1B_1, A_2B_2, A_3B_3, etc., and in each case $AB = \frac{1}{2}OA$.



(a) Show that the length of the stake varies as the distance of its foot from O in each case.

x	y
OA'	$A'B'$

(b) Make two columns x and y and fill in the figures for each stake. Show that in each case $A'B'/OA'$ = constant, or $y = \frac{1}{2}x$.

(c) Make two more columns Δy and Δx and find $\Delta y/\Delta x$.

161. Example of direct variation.

Write the following in the two forms, $P \propto Q$ and $P = kQ$:

(a) The circumference of a circle varies directly as its radius.

(b) The interest on a sum of money varies directly as the rate per cent and also as the time.

(c) The area of a triangle of given altitude varies directly as the length of the base.

(d) The weight of a body of given size varies directly as the density of the material.

(d) Graphs and ready-reckoners (*continued*).

Typical Problem Material

(e) The speed of a falling body varies directly as the time it has been falling.

(f) The stretch of an elastic spring varies directly as the force applied.

(g) The amount of water flowing into a bath varies directly as the time, if the pressure is constant.

(h) The length of arc of a circle varies directly as the angle subtended at the center.

162. Graphs as ready-reckoners.

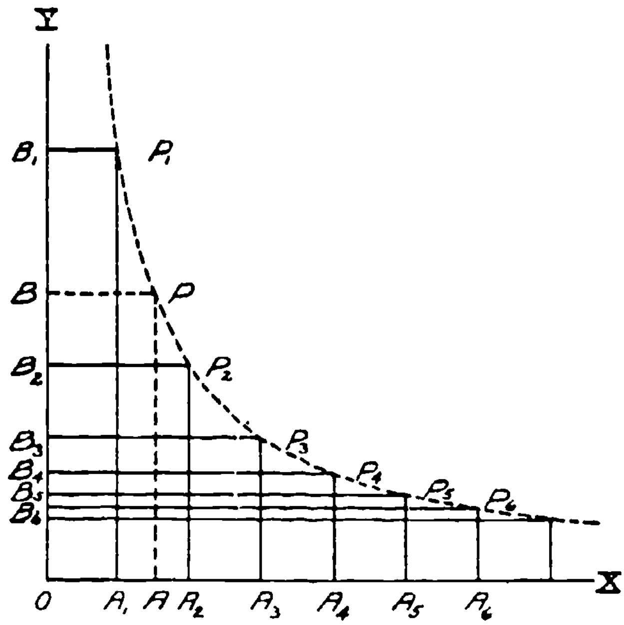
(a) Example 160 shows that the graph of direct variation is a straight line. Draw a graph to enable you to read off quickly the circumference of a circle when the radius is given.

(b) Draw a graph to enable you to reckon the interest on a sum of money at the rate of 5% per annum for any number of years up to ten.

(c) Draw a graph to represent the formula $v = 32t$, where v is the speed of a falling body (feet per second) and t is the time (second) that the body has been falling.

163. Inverse variation.*

You are given that the area of a rectangle is 6 sq.in., but nothing more about the sides OA and OB. How many rectangles can you make having this area?



*This problem should be given immediately after Ex. 160, before Ex. 161 and 162 are given. Direct and inverse proportion should be introduced together and treated as a whole. Numerous examples of both types may be found in physical experiments.

Suggested Course	Definitions and Concepts
Inverse variation. Physical problems. Examples of inverse variation.	The functional character of inverse proportion, as in the case of direct variation, is best studied through physical problems in which the relationship 'the more . . . the less' is seen concretely

Typical Problem Material

Make three columns as follows:

x	y	xy
OA	OB	Area
1	6	6
2	3	6
3	2	6
etc.	etc.	etc.

Cut out rectangles $OA_1P_1B_1$, $OA_2P_2B_2$, . . . and place them so that they have one right angle common (at O). Show that the corners P_1, P_2, P_3, . . . lie on a curve (called a hyperbola). Take any point P on the hyperbola and make a rectangle $OAPB$ which may be regarded as a type of a very large number of rectangles all of area 6 sq. in.

Note that the sides of these rectangles vary, but the product of the sides of area xy does not vary; in other words, xy is constant (i.e. $= k$); so $y = k/x$.

This is called *inverse variation*, the sides are in *inverse proportion*.

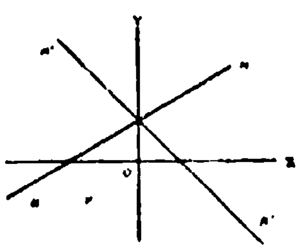
Make two columns as follows:

x	$1/x$
6.0	$1/1 = 1.000$
3.0	$1/2 = 0.5000$
2.0	$1/3 = 0.333$
1.5	$1/4 = 0.250$
1.2	$1/5 = 0.200$
1.0	$1/6 = 0.167$
etc.	etc.

Draw a graph of the corresponding numbers in the two columns, x and $1/x$. The graph shows that x and $1/x$ are directly proportional.

164. The following experiment was made with an automobile tire pump. A pressure gauge was attached to the open end of the pump, and the pressure read off for various positions of the piston. The volume of air and the corresponding pressure are recorded as follows:

Volume of air (cu.in.)	3	6	9	12
Pressure (atmospheres)	8 00	4 00	2 67	2 00
Volume of air (cu.in.)	15	18	21	24
Pressure (atmospheres)	1 60	1 33	1 14	1 00

Suggested Course	Definitions and Concepts
Graphical representation. (e) Graphs of functions. Functions of a single variable.	A set of quantities belonging to a certain domain constitutes a *variable*. The variable symbol may be indentified with any particular member of the domain. Such a variable is called an *independent variable*. If, to each element of the domain of an independent variable x, there corresponds, in any manner a definite quantity, y, that all such quantities constitute a new set, which can be regarded as the domain of a new variable y, then y is said to be a *function* of x. The variable y is called a *dependent variable*.
	Two variables y and x are in *functional relation* when there is a correspondence between the quantities x_1, x_2, . . . of the x variable and the quantities y_1, y_2, . . . of the y variable and this correspondence follows a prescribed rule.
The linear function $y = ax$.	The equation $y = ax$ expresses the simplest form of functional relationship and is represented graphically by a line passing through the intersection of the axes of coördinates.
	The function $y = ax$ is called a *linear function* of x. The constant a is the tangent of the slope or the gradient of the line relative to the x-axis.
	If Δy is the 'vertical rise' corresponding to the 'horizontal advance' Δx, then $a = \Delta y / \Delta x$. When ϕ, the angle of slope, is less than 90°, (position AB), tan ϕ is positive; when ϕ is greater than 90°, and less than 180° (position $A'B'$), tan ϕ is negative. Thus when the line ascends from left to right (positive slope), a is *positive*; when it descends from left to right (negative slope), a is *negative*.

Typical Problem Material

Draw a graph of these readings and find the law of variation between them.

165. Examples of inverse variation. Express each as a formula:

(a) The time taken to travel a certain distance varies inversely as the rate.

(b) The number of days required to dig a trench varies inversely as the number of men employed.

(c) The number of books that can be bought for $100.00 varies inversely as the price of each book.

(d) The time required for a given sum to realize $1,000.00 interest varies inversely as the rate per cent.

(e) The force required to lift a heavy weight with a lever varies inversely as the length of the lever; similarly, for wheel and axle.

(f) The length of wire that can be drawn from a given quantity of copper varies inversely as the area of cross-section of the wire.

166. A boy had a trolley running on rails OX (or a block of wood sliding on a table) from a starting-point O. He erected a vertical pole on the trolley and by means of pulleys and string made a device enabling him to lift a weight (W) through a distance (y) always equal to half the distance (x) traversed by the trolley along OX. Draw a graph of the path traced out by W relative to a fixed background. Could you make an arrangement to demonstrate this?

Since the *vertical rise* (y) of the weight *depends* upon the *horizontal advance* (x) of the trolley or block, y is called the *dependent* and x the *independent* variable.

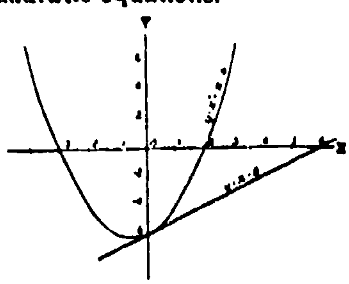
167. Draw the graphs of $y = \frac{1}{2}x$, $y = 2x$, $y = -\frac{1}{2}x$, $y = -2x$. In each case make columns showing change in $x(\Delta x)$ and change in $y(\Delta y)$ for six points on the graph. Calculate $\Delta y/\Delta x$.

168. Draw the graphs of the equations $y = \frac{1}{2}x$ and $y = \frac{1}{2}x + 5$. Show that they are parallel. Consider the graph of $y = \frac{1}{2}x$ to be a thin rod. How could you get that of $y = \frac{1}{2}x + 5$ from it? Show that it passes through the point (0, 5).

169. Draw the graphs of $y = \frac{1}{2}x + 10$ and $y = \frac{1}{2}x - 8$.
Draw the graph of $y = \frac{1}{2}x - 4$ or $y = \frac{1}{2}(x - 8)$ and show that it is a line parallel to $y = \frac{1}{2}x$, passing through the point (8, 0).

170. Draw the graphs of the functions $y = \frac{2}{3}x + 6$ and $y = \frac{2}{3}x - 4$.

Suggested Course	Definitions and Concepts
Translation of the graph of $y = ax$, giving (1) $y = ax + b$ by vertical and (2) $y = a(x - c)$ by horizontal translation.	Since the equation $y = ax$ expresses y as a function of x, the equation $y = ax + b$ also expresses y as a function of x. The graph of $y - b = ax$ or $y = ax + b$ is obtained by translating the graph of $y = ax$, a distance $+b$ in the positive y direction (vertical translation). Similarly, the graph of $y = a(x - c)$ is obtained by translating the graph of $y = ax$, a distance $+c$ in the positive x direction (horizontal translation). The graph of $y - b = a(x - c)$ or $y = a(x - c) + b$ is derived from $y = ax$ by performing each of the above operations in turn in either ord Thus $y = ax$, $y = ax + b$, $y = a(x - c)$ are parallel lines.
Solution of linear equations by graphs.	Since x and y can have any values we please, we may find y when x is given, and x when y is given.
The parabolic function $y = ax.^2$ Derivation of the graphs of $y - b = ax^2$ and $y = a(x - c)^2$ from the graph of $y = ax.^2$	The graph of the function $y = ax^2$ is a parabola; that of $y = ax^2 + b$ is the same parabolic form translated in the positive y direction, a distance $+b$, and that of $y = a(x - c)^2$, the same parabola translated in the positive x direction a distance $+c$. A geometrical 'trace' of the function $y = x^2$ drawn to given units, when cut out, will fit the graphs of the functions $y = x^2 + b$, $y = (x - c)^2$ and $y = (x - c)^2 + b$, drawn to the same scale.
Symmetry of the parabola.	The parabola $y = x^2$ is symmetrical about the y-axis.
Derivation of the graph of $y = ax^2$ from that of $y = x^2$.	The 'trace' of the function $y = x^2$ drawn to given units will serve to represent the graph of $y = ax^2$, if the original x unit is unaltered, and the y unit is changed in the ratio of $1:a$. The graph of $y = -x^2$ may be looked upon as the mirror image of $y = x^2$ in the x-axis.
Graphical solution of quadratic equations.	The graph of a parabolic function may be used to solve quadratic equations.



Typical Problem Material

171. Put the equation $3x - 4y = 6$ in the form $y = ax + b$ and draw its graph.

172. Draw the graphs of $y = -2x$, $y = -2x + 5$, $y = -2(x - 4)$, on the same graph paper.

173. Solve the equations $2 = \frac{1}{5}x$, $3 = \frac{1}{5}x + 10$, $6 = \frac{1}{5}(x - 8)$, by using the graphs of Ex. 156.

174. A boy had a trolley running on rails OX, from a starting-point O. He erected a vertical pole on the trolley and placed a pulley at the top. He then passed a piece of string over the pulley and fixed two unequal weights at the ends. He started the trolley off from O with uniform speed and arranged so that the vertical rise (y) of the smaller weight was exactly equal to the square of the horizontal advance (x) of the trolley. Draw a graph to show the path of the rising weight relative to a fixed background. Could you make an arrangement to demonstrate this?

175. Draw the graphs of $y = x^2$, $y = x^2 + 5$, $y = x^2 - 5$, and show that a translation of the first will lead to the other two.

176. Draw the graph of $y = (x - 3)^2$ and show how it is related to that of $y = x^2$. Draw the graph of $y = (x - 3)^2 + 5$.

177. Cut out a 'trace' of the function $y = x^2$ and show that it will fit the graph of the equation $y = (x - 4)^2 - 4$.

178. Cut out a 'trace' of the function $y = x^2$ to given units. Now draw the graph of $y = 3x^2$, using the same x unit as before, and the length of the y unit one-third of its previous value. Show that the 'trace' of the function $y = x^2$ will now fit the new graph.

179. Draw the graph of $y = 3(x - 2)^2 - 5$, using the trace of $y = x^2$. Draw the graphs of $y = -x^2$, $y = -x^2 + 5$, $y = -(x - 3)^2$.

180. Solve by graphs: $4 = (x - 4)^2$, $6 = (x - 4)^2$, $10 = (x - 4)^2$.

181. Solve graphically: $x^2 - 3x + 2 = 0$, $x^2 - 3x - 4 = 0$.

Suggested Course	Definitions and Concepts
The graph of $y = ax^2 + bx + c$.	Since the expression $y = ax^2 + bx + c$ can be put in the form $y = ax^2 + (bx + c)$, it is possible to build its graph by drawing the line $y = bx + c$ and adding at each point of the line so obtained ordinates appropriate to $y = ax^2$. For example, to draw the graph of $y = x^2 + x - 6$, we first draw the graph of $y = x - 6$, and add, at each point of the line, ordinates appropriate to $y = x^2$. Thus we get the parabola shown above.
The line $y = bx + c$ is a tangent to the parabola $y = ax^2 + bx + c$.	Since the parabola is everywhere above the line, except at the point A, the line is *tangent* to the parabola at the point A. Thus the line $y = x - 6$ is a tangent to the parabola $y = x^2 + x - 6$ at the point $(0, -6)$.
Factors of trinomials.	It will now be advantageous to give some exercise in finding the factors of simple trinomial expressions.
The Factor Theorem.	An elementary idea of the Factor Theorem should now be given: If, in the expression $ax^2 + bx + c$, α be substituted for x and the expression vanishes, then $(x - \alpha)$ is a factor of the expression.
The cubic function.	The graph of a cubic equation may be introduced through a practical problem. The general method of procedure is already known to the class.
Graph of $y = ax^3$.	The graph of the function $y = ax^3$ has a point of inflexion at the origin. The graph of $y - b = a(x - c)^3$ may be derived from that of $y = ax^3$ by two 'translations' of the latter.
The graph of $y = ax^3 + bx + c$.	The line $y = bx + c$ is an inflexional tangent to the graph of the function $y = ax^3 + bx + c$.

Typical Problem Material

182. The time of swing of a pendulum is related to its length by the formula $l = 9.8\ T^2$ where l is the length of the pendulum in inches, and T the time of a complete swing in seconds. Draw a graph which will enable you to calculate the time of swing when the length is given. Verify practically.

183. Draw the graph of the equation $y = x - 6$.

At each point of the line so obtained, add ordinates corresponding to $y = x^2$. The result will give the graph of $y = x^2 + x - 6$.

Place the trace of the graph $y = x^2$ (to the same units) upon the new graph and note that it is congruent with it. Note also that the new parabola meets the line in one and only one point. The line and parabola *touch* at this point. The line is said to be a *tangent* to the parabola.

184. Draw the graph of $y = x^2$. Take a number of points on the graph and estimate their 'coördinates'. Make columns as follows:

x	y	Δx	Δy	$\Delta y/\Delta x$

Show that $\Delta y/\Delta x$ is not constant.

185. Repeat Ex. 171 using the graph of $y = x^2 + x - 6$.

For a number of points find Δy, Δx, and differences between these Δ's, called second differences, $\Delta^2 y$ and $\Delta^2 x$. Note that $\Delta^2 y\ \Delta^2 x$ is constant in each case.

186. Find the roots of the equation $x^2 + x - 6 = 0$.

187. Find the factors of the expression $x^2 + x - 6$.

188. I hold in my hand a rectangular block of wood, the cross-section of which is a square. The volume of the block is 34 cu.in. I saw off a small cube from the block and I find that the remaining piece is 6 in. long. Find the dimensions of the block.[28]

189. Draw the graphs of

$$y = x^3,$$
$$y = x^3 + 5 \text{ or } y - 5 = x^3,$$
$$y = (x - 2)^3 + 5 \text{ or } (y - 5) = (x - 2)$$

190. Draw the graph of the line $y = x - 10$. At selected points of this graph add ordinates corresponding to the function $y = x^3$ and so derive the graph of $y = x^3 + x - 10$. (Note that when x is negative, x^3 is also negative. The points of the graph for negative values of x will, therefore, lie below the line $y = x - 10$. Note also that the line $y = x - 10$ is an inflexional tangent to the graph of $y = x^3 + x - 10$.)

[28] This problem was set the writer's class, in their first year of algebra, by Professor John Perry.

Suggested Course	Definitions and Concepts
The hyperbolic function. The graph of $y = \frac{a}{x}$.	The graph of $y = a/x$ has already been studied. The graphs of $y = a/x + b$ or $y - b = a/x$, and $y = \frac{a}{x - c}$ may be derived from that of $y = a/x$ by simple translations.
The equation of a circle.	The equation of a circle of radius r with origin as center is: $$x^2 + y^2 = r^2.$$
	The equation of a circle of radius r, having its center at the point (a, b) is: $$(x - a)^2 + (y - b)^2 = r^2.$$
The equation of an ellipse.	The equation of an ellipse of semidiameters a and b, referred to the origin as center, is: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Typical Problem Material

191. Draw the graph of $y = 1/x$, and cut it out. Using this trace, draw the graphs of $y = 1/x + 8$ or $y - 8 = 1/x$, $y = 1/(x - 3)$ and $y = 1/(x - 3) + 8$ or $y - 8 = 1/(x - 3)$.

192. Draw the graph of $y = \frac{x - 2}{x - 3} = 1 + \frac{1}{x - 3}$ or $y - 1 = \frac{1}{x - 3}$.

193. Draw the graph of $y - 10 = \frac{1}{x - 5}$ or $(x - 5)(y - 10) = 1$.

194. Refer back to Ex. 153. Using the figures obtained for x, y, and r, work out x^2, y^2, r^2, and show that $x^2 + y^2 = r^2$.
(Most books on trigonometry give a table of squares.)
Can you give any reason why this equation must necessarily be true?

195. Extend the above problem and allow the scout troop to radiate from the base O in all directions like the spokes of a wheel. Let the easting be marked $+x$ and the westing $-x$, the northing $+y$ and the southing $-y$. Then show that in all cases $x^2 + y^2 = r^2$. This is called the equation of the *circle* of radius r, having O as its center.
Check this equation by taking a number of points on it.

196. Draw a circle of 1-in. radius with its center at the origin. Draw another circle of 1-in. radius with its center at the point (2, 0) or imagine the first circle translated a distance 2 in. along the x-axis. Show that the equation of this circle is $(x - 2)^2 + y^2 = 1$.
Draw another circle of radius 1 in. with its center at the point (0, 3) or translate the original circle 3 in. along the y-axis.
Show that the equation of this circle is $x^2 + (y - 2)^2 = 1$.

197. Draw the circle $(x - 2)^2 + (y - 3)^2 = 1$.

198. Draw a circle and any diameter $A'OA$ to it. Erect a number of perpendicular ordinates P_1Q_1, P_2Q_2, . . . to $A'OA$ cutting the circle at P_1, P_2, ... Bisect these ordinates at R_1, R_2, ... Join R_1, R_2, ... by a smooth curve. This curve is called an *ellipse*. Repeat by doubling the ordinates.

199. As in Ex. 198, draw ordinates P_1Q_1, P_2Q_2, . . . and divide these ordinates in the ratio of 1:2 at points R_1, R_2, ... Join these points and get another ellipse. Repeat with the ratios 2:3 and 3:2, etc.

200. Draw a circle $x^2 + y^2 = a^2$, and change all the y's in the ratio a/b. Show that the equation of the ellipse so formed is $x^2/a^2 + y^2/b^2 = 1$. (For most classes this exercise would come later.)

APPENDIX A

TESTS OF ELEMENTARY MATHEMATICAL RELATIONS

These tests, involving simple arithmetical and geometrical relations, were designed to discover how far elementary school children, without any previous teaching or practice, could cognize mathematical relations and educe their correlates. No information or suggestions were given the pupils to whom the tests were administered beyond the instructions at the head of each test. These instructions were read by the teacher. The time allowed for Tests 1 to 5 was three minutes, for Test 6, fifteen minutes, and for Test 7, five minutes.

The tests were administered to 240 school children attending London elementary schools. The average age was 11 years 2 months and the average I.Q. was 102. None of the children had ever had any algebra or graphical work before taking the test. They had studied a little intuitive geometry in the form of geometrical drawing.

As the tests were designed chiefly for the purpose of throwing light on the pupil's method of educing relations, a personal introspective inquiry was conducted with forty of the subjects. The test was given in the ordinary way, but at the end of each test the pupils were asked to describe the method they had followed in arriving at their solutions. No comments or corrections were made by the experimenter. The remarks were taken down in detail and a note was also made of eye movements, to ascertain whether the relations were "held" at the first reading or not.

A summary of the observations made by the pupils is given at the end of each test. This by no means gives a complete account of the many interesting observations made. The most important general observations were the following:

1. Rhythm was sensed in all the tests with series. Frequently the rhythm was likened to music.

2. Relations were *used*, although not consciously abstracted. In some cases the relations were completely abstracted with difficulty, but the pupil

was convinced of the correctness of his correlates in spite of his inability to state the relation satisfactorily.

3. The more intelligent student abstracted the relation more quickly than the less intelligent and carried the relation more clearly as he used it.

4. Algebraic expressions did not present the difficulty that had been anticipated. It was evident that likeness of algebraic form was readily understood.

5. In most cases the graphs were interpreted with great facility, although the pupils had never received any instruction in graphical representation.

6. The pupils, without exception, remarked that they found the tests interesting. Several suggested that their school arithmetics should contain problems of a similar kind. Many of the pupils seemed to derive satisfaction from the fact that they could test their own conclusions without recourse to a set of answers.

Statistical Analysis of the Results

The tests were scored as follows:

Test	*Score*	*Total*
1	One point for each question	10
2	One point for each question	12
3	One point for each question	8
4	One point for each question	10
5	One point for each connection	23
7	Half of a point for each part of each question	17
		80

Several solutions were possible in Test 5.

Means and Probable Errors of the Tests

Test	*Mean*	*Possible Score*	*P.E.*
1	7.01	10	1.47
2	7.62	12	1.61
3	5.52	8	1.08
4	6.80	10	1.01
5	17.78	23	3.54
7	11.70	17	1.58

Correlations with Intelligence

Test	*Correlation with Intelligence*	*P.E.*
1	.32	.039
2	.41	.036
3	.49	.033
4	.59	.028
5	.81	.015
7	.30	.050

MATHEMATICAL RELATIONS

Test 1

Look at the following rows of numbers. Notice that there are blank spaces on the right showing where some numbers have been rubbed out. You have to write the correct numbers in the blank spaces.

1. 2 3 2 3 2 3 — — — —
2. 2 2 3 3 2 2 3 3 — — — —
3. 3 3 3 4 4 3 3 3 4 4 — — — —
4. 0 1 0 2 2 0 3 3 3 0 — — — —
5. 1 2 0 2 3 0 3 4 0 4 5 — — — —
6. 5 0 5 0 0 5 0 0 0 5 0 — — — —
7. 6 0 6 0 0 6 0 6 0 0 6 0 — — — —
8. 5 1 5 0 4 9 4 8 4 7 — — — —
9. 9 1 8 2 7 3 6 4 — — — —
10. 3 2 1 4 3 2 5 4 3 6 5 4 — — — —

Test 2

In the following lines are two blank spaces, showing where two numbers have been erased. Write the correct numbers in the blank spaces.

1. 2, 4, 6, —, —
2. 5, 8, 11, —, —
3. 19, 15, 11, —, —
4. 26, 21, 16, —, —
5. 2, 4, 8, —, —
6. 64, 32, 16, —, —
7. 1, 3, 9, —, —
8. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, —, —
9. 3, 7, —, 15, —
10. 4, 9, —, —, 24
11. 2, 6, —, 54, 162, —
12. $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, —, —

Test 3

From the four numbers given on the right of each of the following lines, select *one* number that will go best in the blank space. Write it in the blank space.

1.	2, 10, —, 6, 8	13, 12, 3, 5
2.	3, 9, 18, —, 21	15, 14, 7, 11
3.	7, 21, 35, 14, —	4, 12, 28, 11
4.	13, —, 33, 43, 63	23, 12, 7, 14
5.	$\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{8}$, $\frac{4}{16}$, —	$\frac{4}{32}$, $\frac{5}{16}$, $\frac{5}{32}$, $\frac{5}{64}$
6.	$\frac{x}{y}$, $\frac{x^2}{y^2}$, —, $\frac{x^4}{y^4}$, $\frac{x^5}{y^5}$	$\frac{x^3}{y^2}$, $\frac{x^3}{y^3}$, $\frac{x^2}{y^3}$, $\frac{x^3}{y^4}$
7.	$x + 2y$, $2x + 3y$, $3x + 4y$, $4x + 5y$, —	$5x + 4y$, $5x + 6y$, $6x + 5y$, $6x + 6y$
8.	$a + d$, $a - d$, $2a + 3d$, $2a - 3d$, $3a + 5d$, —	$3a + 4d$, $3a + 6d$, $3a - 5d$, $3a - 6d$

TEST 4

In each of these lines there is one wrong number. Cross it out.

1. 1, 4, 7, 11, 13
2. 1, 2, 4, 9, 10
3. 17, 15, 11, 8, 5
4. 27, 21, 15, 9, 4
5. 3, 9, 27, 80, 243
6. $\frac{1}{2x}, \frac{2}{4x}, \frac{3}{4x}, \frac{4}{5x}, \frac{5}{6x}$
7. $\frac{x}{y}, \frac{x}{2y}, \frac{x}{4y}, \frac{x}{8y}, \frac{y}{16x}$
8. $x + 2y$, $2x + 3y$, $3x + 5y$, $4x +$ [illegible]y, $5x + 6y$
9. $x + y$, $x - y$, $2x + 3y$, $2x - 3y$, $3x + 5y$, $3x + 6y$
10. $2(a + b)$, $3(2a + b)$, $4(3a + b)$, $5(4a + 2b)$, $6(5a + b)$

TEST 5

Look at the following rows of figures arranged in pairs. Each number on the top row has a partner on the bottom row, but not usually directly underneath. These partners have some feature in common, for example, 15 and 25 both end in 5. You have to draw lines between the partners that are somewhat alike, thus:

15
╲
25

1.	15	21	24	11	7	13	
	36	25	37	23	42	22	
2.	20	7	34	45	22	54	64
	17	11	49	16	30	56	25
3.	x	$5a$	$3y$	$3x$	$2y$	$7b$	
	y	$2x$	$4x$	$3b$	$6a$	$5y$	
4.	$(x + y)$	$(x^2 + 2y^2)$	$(x + y + z)$	$(x - 2y)$	$(x^2 + 2xy)$		
	$(x^2 + y^2)$	$(x - y)$	$(2x + 3y)$	$(x^2 + xy)$	$(x + 2y + 3z)$		

TEST 6

1. The heights of ten boys were taken (in inches) as follows:

A	B	C	D	E	F	G	H	I	J
65	53	62	47	56	51	60	64	49	57

(a) Arrange the boys *by letter* on the line below, putting the tallest on the right and the shortest on the left.

. . . .

(b) Without doing any calculation *guess* the *average* height of the ten boys.
Answer: ______

2. Ten boys obtained the following scores in English and Arithmetic:

	A	B	C	D	E	F	G	H	I	J
English	72	70	68	64	62	60	58	54	51	48
Arithmetic	81	77	74	68	50	62	60	58	68	51

(a) Examine the figures and find which boy did much better in Arithmetic than in English compared with the other boys. Answer: ——

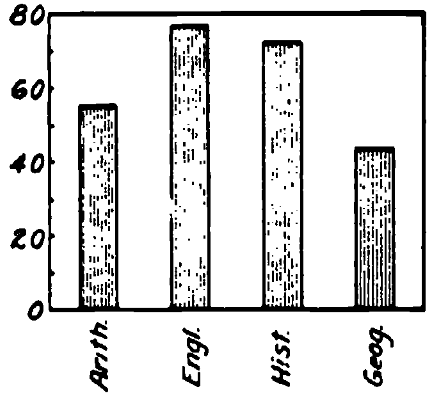
(b) What score would you have expected this boy to get? Answer: —

(c) Which boy did not do as well in Arithmetic as in English compared with the others? Answer: ——

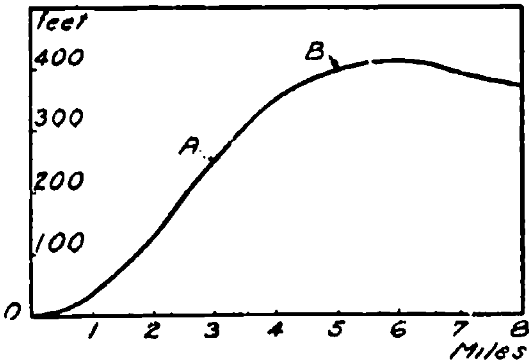
(d) What score do you think this boy should have obtained in Arithmetic? Answer: ——

3. After his final examination Tom made the drawing given here to show his score in each subject. Write down, as nearly as you can, his scores in

(a) Arithmetic: ——
(b) English: ——
(c) History: ——
(d) Geography: ——



4. This is a sketch of a road over a hill. The place marked *O* is at sea level and the heights above sea level are marked, in feet, on the scale at the side. The distances of places on the road are marked horizontally in miles.



(a) How high is *A* above sea level? Answer: ____
(b) How high is *B* above sea level? Answer: ____
(c) How far is *A* from *O* in miles? Answer: ____
(d) How far is the highest point above sea level? Answer: ____
(e) How long would a man take to go from *A* to *B* at the rate of 3 mi. an hour? Answer: ____

5. Look carefully at the following rows of figures marked *x* and *y*. In each case the number in the *y* line is obtained from the number above it in the *x* line by a simple calculation. Can you find out how it was done? If so, fill in the spaces under 8 and 10.

(a)

x	1	2	3	4	5	6	7	8	9	10
y	2	4	6	8	10	12				

(b)

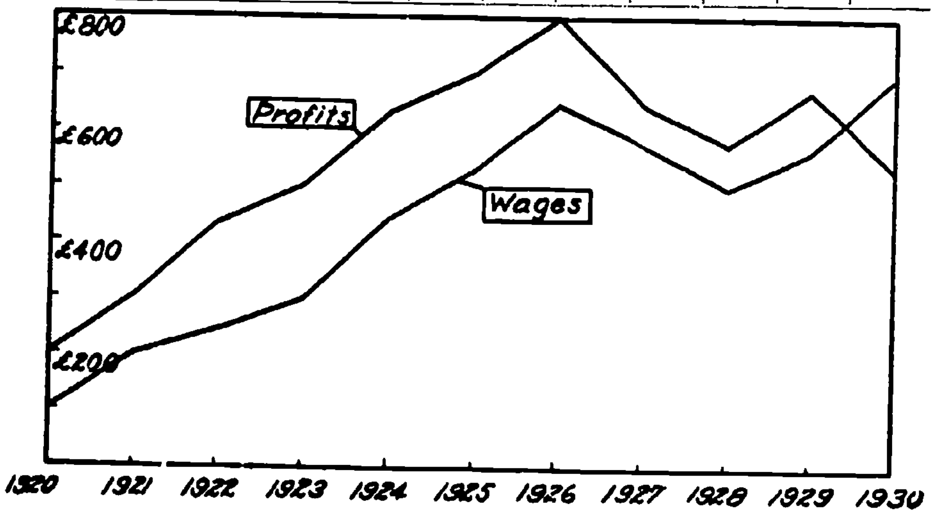
x	1	2	3	4	5	6	7	8	9	10
y	1	4	9		25	36				

(c)

x	1	2	3	4	5	6	7	8	9	10
y	3	5		9	11	13				

(d)

x	1	2	3	4	5	6	7	8	9	10
y	2½	3	3½	4	4½	5				



6. The above chart shows you the wages paid and the profits made by a grocer from 1920 to 1930. Answer the following questions:

(a) How much was paid in wages in 1921? (   ) In 1924? (   ) In 1927? (   )

(b) What were the profits in 1922? (   ) In 1927? (   ) In 1930? (   )

(c) In which year were the wages greatest? (   )

(d) In which year were the profits greatest? (   )

(e) Look at the chart carefully and find out which of the following statements are true and which false. If true, write *true*; if false, write *false*.

(i) Generally speaking, better wages result in better profits. Answer: ______

(ii) Good profits *always* go with good wages. Answer: ______

(iii) Generally speaking, profits exceed the wages paid. Answer: ______

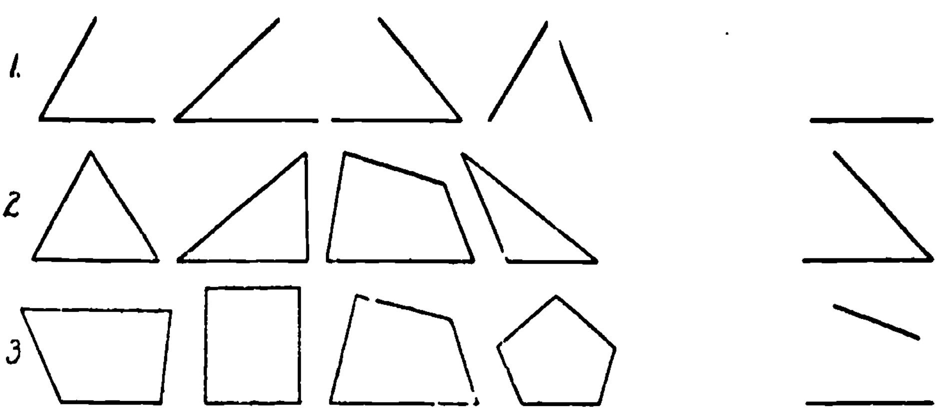
(iv) Profits *always* exceed the wages paid. Answer: ______

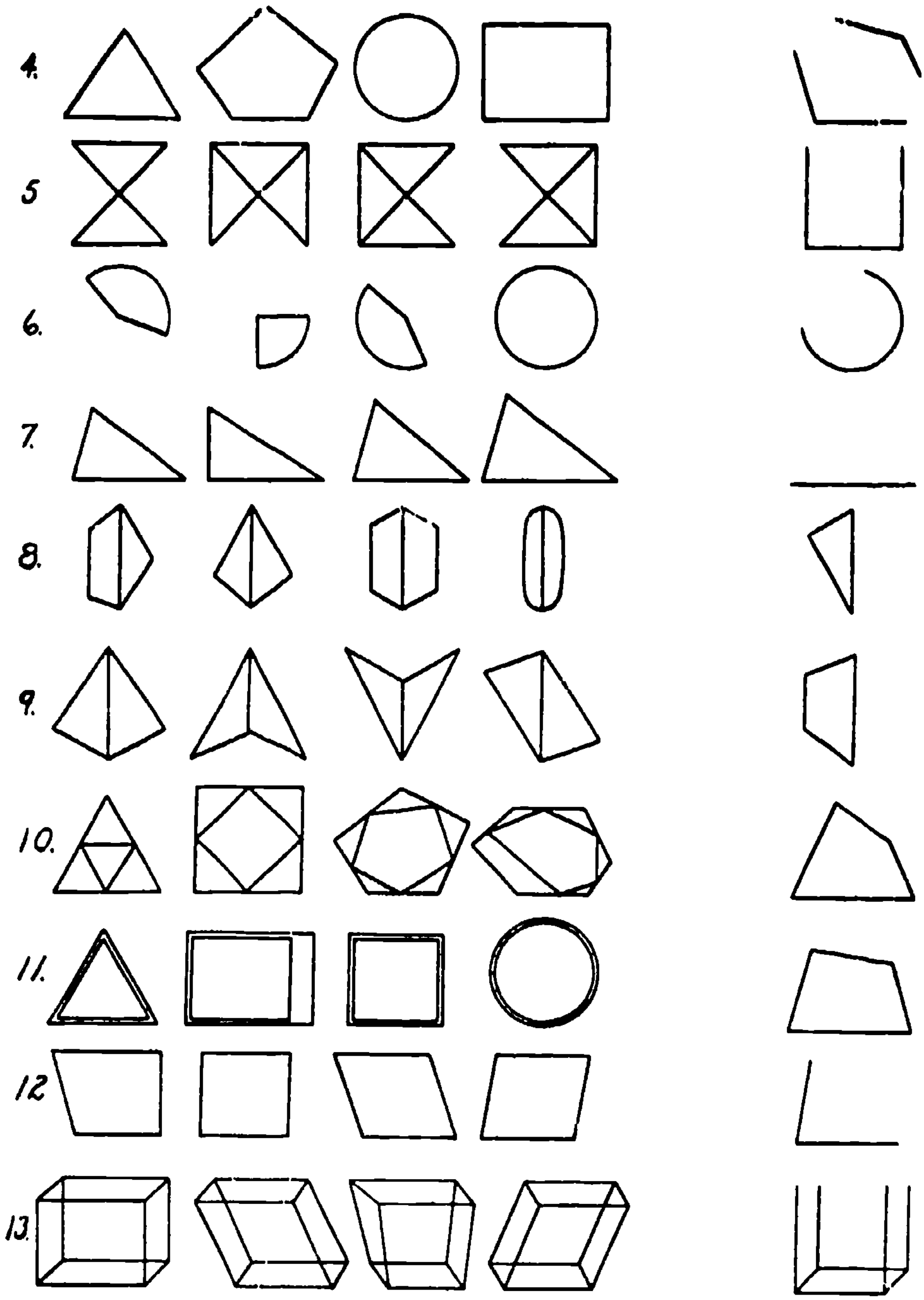
(v) The year of greatest profits was the year of greatest wages. Answer: ______

TEST 7

(a) In each of the following rows there are four figures, one of which is out of place because it does not possess the feature that the other three have. Cross out the wrong figure.

(b) Having done that, look to the right-hand side of the page and note the figures given there. In each case part of the figure has been rubbed out. Complete the figure so that it will be like the three that you have left on the left-hand side of the page.

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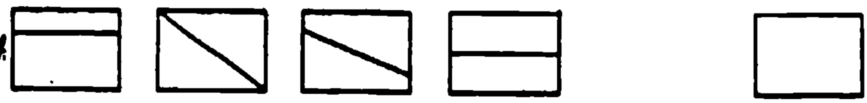
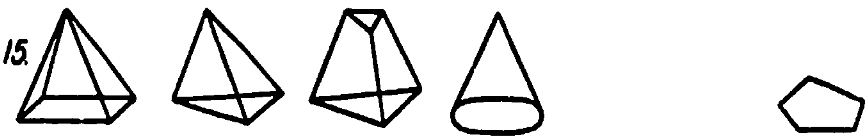
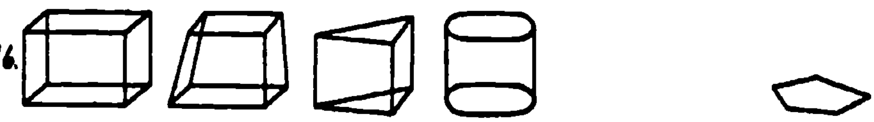
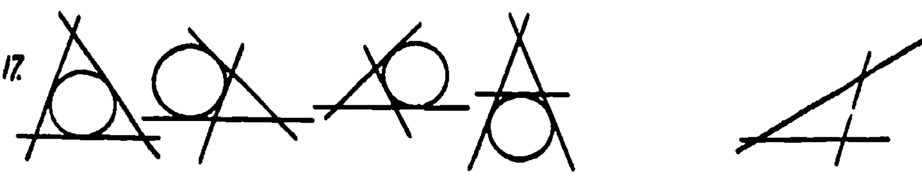


15.


16.


17.


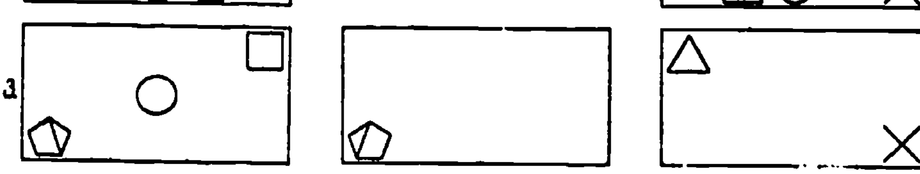
TEST 8

The following pairs of figures were drawn exactly alike, but some parts were afterwards rubbed out. Draw the missing parts so that figures will again be alike.

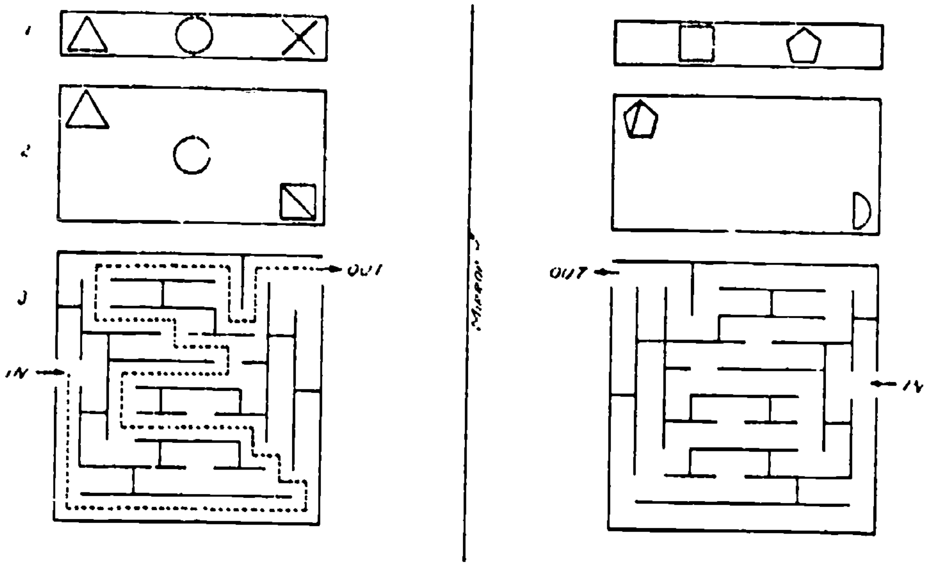
(*Specimen Examples Only*)

1.


2.


3.


Imagine the line drawn down the middle of the page to be a mirror, and the figures on the right to be reflections of the figures on the left as seen in the mirror. Draw in the missing parts so that the figures will be alike.



OBSERVATIONS ON THE TESTS

INTROSPECTIVE EXAMINATION

TEST 1

This test was designed to test the ability of the pupil to cognize groups of numbers arranged in serial order.

Observations

1. In all cases of correct response the numbers were visualized as groups, e.g., (120), (230), (340). In most cases these groups were connected rhythmically. In No. 3, for example, the numbers were given as (333 - 44), the three 3's taking about the same time to repeat as the two 4's. In practically every case the rhythm had to be sensed before the series could be continued.

2. Many interesting interpretations were given of the relations of the numbers in No. 9. As a general rule, the pupils saw that $9 + 1 = 10$, $8 + 2 = 10$, $7 + 3 = 10$, and so on. In several cases they noted that $9 - 1 = 8$, $9 - 2 = 7$, $9 - 3 = 6$. In only a few cases did they read 9, 8, 7, 6 in descending order and 1, 2, 3, 4 in ascending order.

3. Judging by eye movements, the only series read twice were 6, 7, and 9. The more backward had to read these lines several times.

TEST 2

The purpose of this test was to examine the pupil's ability to detect relations and to educe correlates to the relations.

Observations:

1. A striking feature of the responses, especially to the first six examples, was the insistence that the answers came 'in a flash', or 'almost without thinking.' When questioned, all the pupils gave the correct relations, but insisted that they had not thought them out before.

2. Examples 1 and 5 made an interesting comparison. In each case the blanks were filled in without hesitation. When asked how they did No. 1, most replied: 'Went up by twos', and No. 5, 'Doubled each time'. When asked why they had added 2 in the one case and doubled in the other, they replied: "It couldn't be anything else." On further questioning they admitted that they had "adding" in mind at first but they quickly changed to "doubling."

3. Trial-and-error methods were dominant in No. 9, 10, and 11. There was conscious effort to find a relation, which was followed by great satisfaction when they 'got it'. Many different methods of arriving at the results were shown.

4. All stated that the problems were 'interesting', because once they had 'found the connection', they could fill in the blanks easily. Satisfaction was felt by the pupils that they were arbiters of their own work. They knew that their results were correct.

TEST 3

The purpose of this test was to give exercise in relation finding of several kinds. Several examples involving algebraic expressions were given to test the ability of the pupils to detect algebraic configurations. None of the pupils examined orally had previously had any algebra.

Observations:

1. Several who had done only moderately well in the earlier tests scored full points on this test. They attributed their success to the fact that they had a definite and limited choice of responses.

2. Although none had done any algebra, all except one did No. 6, 7, and 8 correctly. They cognized the expressions as wholes, as like configurations.

3. The word 'like' was used in various ways, as: 'all odd numbers', 'all even numbers', 'all ending in the same number', 'all divisible by 7'.

TEST 4

This is similar to the other tests but requires a different kind of response.

Observations:

1. In reply to No. 1, several crossed out 4, because all the other numbers were 'primes'; the majority crossed out 11, taking the numbers as a progression.

2. Several observed that the series of No. 5 "got bigger quickly" and so decided that they must "multiply."

3. To No. 3 the most popular response was to cross out 8 "because all the other numbers are odd." Only a few crossed out 15.

4. The examples involving algebra (6 to 10) gave little trouble to the non-algebra pupils. As in Test 3 the algebraic expressions were treated as configurations.

TEST 5

This is a simple matching test, involving similarities of form and structure.

The difficulty lay not in detecting similarities in single cases but in getting the best possible set of connections for the whole.

Observations:

1. This test did not appeal to the less intelligent pupils who made whatever connections seemed obvious and then left the rest unconnected. The more intelligent pupils found the test very interesting, because in some cases more than one connection was possible. For example, in No. 1 it was possible to connect 7 and 42 or 7 and 37, but the selection of 7 and 42 would have left 21 and 37 unconnected.

2. For No. 2 more than one solution was possible, $\left\{\text{e.g., in } \begin{matrix} 54 & 64 \\ 18 & 56 \end{matrix}\right\}$ but in practically every case 54 and 18 were connected, then 64 and 56. The reason given for this combination was that the divisors 9 and 8 were 'better' then 2 and 2.

3. Although the pupils had not learned algebra, x^2 and y^2 were read as 'x squared' and 'y squared'. When questioned they remarked that "if 3^2 is 3 squared, then x^2 must be x squared." They had had the notation 3^2 in work on prime numbers.

4. It was remarkable that many pupils did No. 4 without hesitation, although they had never before seen expressions of the kind. When they were asked to add $x + 2y$ to $2x + y$ many were successful.

TEST 6

We have given *samples* from a large number of questions of a similar type. The purpose of the test was to find out whether, without any previous formal instruction or training, children of elementary school grades are able to detect serial and functional relationships expressed in tabular or graphical form.

Observations:

The results of the test were most instructive, for they showed that

1. Even pupils of inferior intelligence experienced little difficulty in interpreting the graphs. In some cases they estimated the values with considerable accuracy. For example, in No. 3 they gave one result as 'somewhere between 190 and 195 feet', showing that they had given limits to their estimates.
2. In most cases kinaesthetic imagery was employed. Observations such as the following were made: "I imagined myself on the line," "I imagined myself drawing the line," "I stood between the points," and "I balanced the two numbers."
3. The pupils did the test with obvious interest and enjoyment and suggested that their school arithmetic should include problems of this type.
4. The responses were given so quickly and with such confidence that one regretted having in the past spent much time teaching the obvious. Exercises in the interpretation of graphs should come before exercises in construction.
5. In questions 2(a) and 2(b) the answers were sometimes given very quickly but the rationalization of the answers took a long time. Replies like the following were given: "It must be 'I', it couldn't be anyone else." When asked to explain why 'I' had done better than one would have expected, the first reply was: "He did better than H"—not "He did better than H and J."

The descending order of the scores in English was not mentioned, but the fact was carried over to the series for arithmetic. In not a single case did the pupil think it necessary to state that the English scores were in descending order, but all insisted that the scores in arithmetic should have been in descending order. There was an obvious transfer of meaning and of attention from the English to the arithmetic series. Yet when questions 2(b) and 2(d) were discussed the scores in English were used as a basis. It was obvious that the serial character of the English scores was carried in the mind as a concept.

Test 7

This test of form discrimination presented little difficulty even to the less intelligent.

Observations:

1. Only the more intelligent crossed out the fourth figure of No. 7. The key to the problem was 'symmetry'. In most cases the ellipse was crossed out. Very few crossed out the third figure of No. 8. Again, the key to the solution of this question was 'symmetry'.

2. Most pupils crossed out the second figure (triangle) of No. 10, the reason being that the triangle did not 'look like' the other figures.

3. It was surprising that so many saw that the key to No. 14 was 'bisection of the area'. When this problem was given to a class of adults, several failed to give the correct answer.

4. The solid figures of No. 15 and 16 proved no more difficult than plane figures. In all cases of error the figures with circular bases were crossed out.

Test 8

We have given a few samples from a large number of problems on similarity and symmetry. The restoration of similar figures presented little difficulty but the symmetrical figures proved a stumbling-block even to the more intelligent. In some cases the pupil folded the paper and worked out the correspondences by an obvious 'thought experiment'. When questioned afterwards many stated that they put the left-hand figure face down on the right-hand figure 'in imagination'.

Some interesting results were obtained by comparing the first and second parts of the test. In the first part the pupil was asked to make the two figures *alike*, for example:

This did not prove a difficult exercise.

In the second part of the test the pupil had to unite the two configurations into a composite figure to be filled in a third blank space. This proved a stumbling-block to many, the difficulty being that the positional relations had

to be 'held' and carried to a third position, whereas the first exercise consisted simply in marking a one-to-one correspondence. The materials of the two parts were the same.

APPENDIX B

MORE ADVANCED TESTS OF MATHEMATICAL RELATIONS

TESTS OF MATHEMATICAL RELATIONS

Test 1: Classes

I. In a certain elementary school there were 131 boys altogether. There were 34 boys in Standard I, 32 in Standard II, 32 in Standard III, and the rest were in Standard IV. Of the boys in Standard I, 10 had blue eyes, 12 had grey eyes and the rest had brown eyes; of the boys in Standard II, 9 had grey eyes, 11 brown and the rest blue; of the boys in Standard III, 10 had brown eyes, 8 blue, and the rest grey. In Standard IV there were equal numbers of boys with blue, grey, and brown eyes.

Fill in the following table and write at the right-hand side how many boys altogether had blue eyes, how many had grey, and how many had brown.

Note: Standard IV is the highest class.

	Standard I	Standard II	Standard III	Standard IV	Total
blue					
grey					
brown ..					
Total ...					131

1. How many boys did not have blue eyes? ______
2. How many boys in Standard II had neither blue nor grey eyes? ______
3. How many boys above Standard II had brown eyes? ______
4. How many boys below Standard III did not have grey or brown eyes? ______

II. One hundred and sixty boys sat for a scholarship examination in English and arithmetic. The results were published in the grades *A*, *B*, and *C*; *A* being the highest grade. It was found that in English 26 boys obtained *A* and 89 obtained *B*, and that in arithmetic 31 boys obtained *A* and 85 obtained *B*. It was also found that 12 boys had *A* in both subjects, 65 had *B* in both, while 10 boys had *A* in English and *B* in arithmetic and 11 boys had *A* in arithmetic and *B* in English.

Fill in these numbers on the following chart and answer the questions given below:

		English A	English B	English C	Totals
Arithmetic	A				
	B				
	C				
	Totals				160

1. How many boys obtained *C* in English? _____
2. How many boys obtained *C* in arithmetic? _____
3. How many boys obtained *C* in both subjects? _____
4. How many boys obtained *A* in English, but did not obtain *A* in arithmetic? _____
5. How many boys did not obtain *B* in both subjects? _____

III. The heights of a number of boys (measured in inches and tenths-of-an-inch), were given as follows:

58.2,	64.3,	57.8,	68.7,	50.0,	67.3,	51.7,	57.2,
50.0,	55.0,	60.7,	62.8,	65.0,	53.4,	56.7,	60.0,
62.3,	66.7,	54.3,	54.9,	64.9,	51.7,	63.5,	68.2,
53.5,	57.3,	56.4,	59.9,	51.3,	57.5,	62.4,	69.9.

The boys were divided into 4 classes: from 50 to 55 (but not including 55), from 55 to 60 (but not including 60), from 60 to 65 (but not including 65), from 65 to 70 (but not including 70).

Write on the line below the number of boys in each of these classes.

50 55 60 65 70

1. How many boys were more than 60 in. in height? _____
2. How many boys were not above 55 in. in height? _____
3. How many boys were more than 55 in. and less than 65 in. in height?

Test 2: Correspondence

IV. In a term examination in mathematics it was found that 10 boys were placed in exactly the *same order of merit* in algebra and geometry. The percentages obtained were:

Algebra: 42, 91, 67, 49, 74, 58, 51, 61, 72, 84

Geometry: 71, 46, 53, 82, 90, 65, 77, 63, 85, 57

Draw lines connecting the corresponding scores of each boy.

V. The following rows of figures marked x and y correspond to one another; that is, for each number in the x line there is a number in the y line, and for each number in the y line there is a number in the x line In each line there are blanks showing where certain numbers have been rubbed out. Fill in the blanks. Having done that, write at the side, in words or in equation form, the *law* or *relationship* between the corresponding numbers of the two lines.

1	x	1		3		5	6	7
	y	1	4	9	16		36	

2	x	1	2	3		5	6
	y	3	5		9		13

3	x		3	5			11
	y			16	22	28	

4	x	2	5		11	14		20
	y	5	26	50			290	

VI. Two sets of numbers x and y are related according to a given rule, so that, when the x number is known, the y number is determined. The x numbers are 1, 2, 3, 4, 5. Write down the y numbers corresponding to these x numbers when y is:

	Answer
1. Twice the x number increased by 1.	1. ________
2. Twice the square of the x number.	2. ________
3. Twice the square of the x number decreased by 2.	3. ________
4. Three times the cube of the x number increased by 2.	4. ________
5. The reciprocal of the x number decreased by 1.	5. ________
6. The square of the x number increased by twice the x number.	6. ________
7. The x number added to its reciprocal.	7. ________
8. The square of the x number added to the square of its reciprocal.	8. ________

Test 3: Corresponding Changes

VII. On the left-hand side of this page there are certain geometrical figures. In these figures certain changes are made, the rest of the figure remaining the same. You are asked to note the corresponding changes in the other ways specified.

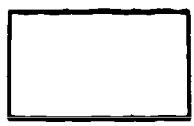
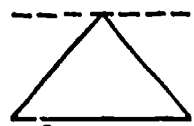
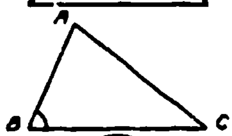
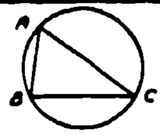
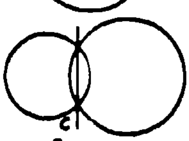
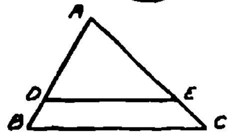
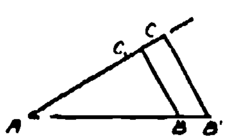
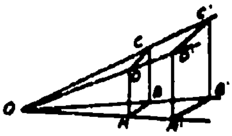
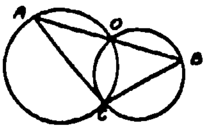
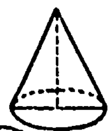
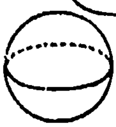
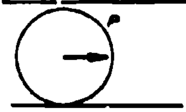
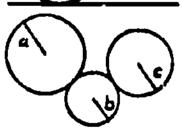
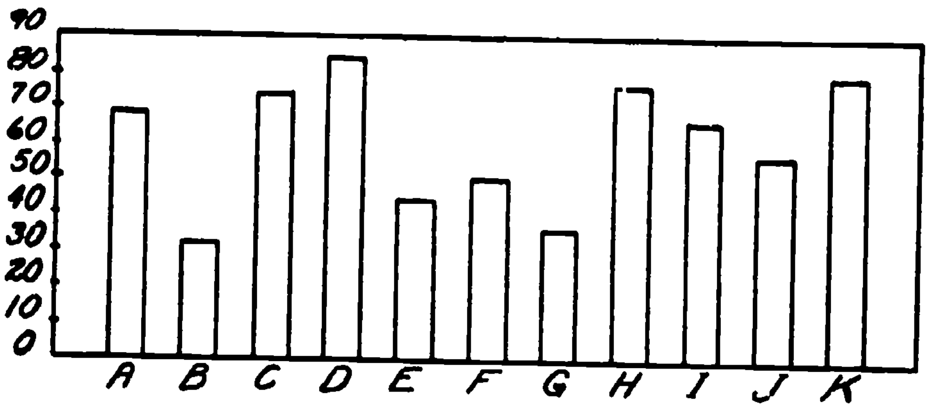
Figure		*Change Made*	*Find the Corresponding Change in*	*Answer*
1.		Height doubled.	Area	
2.		Sides doubled.	Area	
3.		Vertex moves along a line parallel to base.	Area	
4.		Angle *B* increases.	Side *AC*	
5.		Point *A* moves along circumference.	Angle *BAC*	
6.		Circles moved until they touch.	Common chord *C*	
7.		(*DE* is parallel to *BC*.) *C* moves along line *BC*.	Ratio *AE*: *EC*	
8.		(*BC* is parallel to *B′C′*) *B′C′* is moved from *BC* to position at which $BB' = \frac{1}{2}AB$.	*B′C′*	

Figure		Change Made	Find the Corresponding Change in	Answer
9.		($A'B'C'D'$ is parallel to $ABCD$) $A'B'C'D'$ moves from $ABCD$ to position at which $OA' = 2OA$.	Area $A'B'C'D'$	
10.		(BC is a diameter) A moves along circumference.	$BA^2 + AC^2$	
11.		(O is the mid-point of AB) P moves from B to O.	AP, PB	
12.		AB revolves round O.	Angle ACB	
13.		Radius of base increased 10%.	Volume of cone	
14.		Radius of sphere increased 10%.	Volume of sphere	
15.		C moves so that $AC + CB$ is constant.	Position of point C (draw curve)	
16.		Circle rolls along a line, touching it.	Position of point P (draw curve)	
17.		(Gear wheels bearing on one another) wheel A turns through angle θ.	Angle through which wheel C turns	

TEST 4: THE INTERPRETATION OF GRAPHS

VIII. The marks obtained by 11 boys in an examination in mathematics were represented diagrammatically as follows:

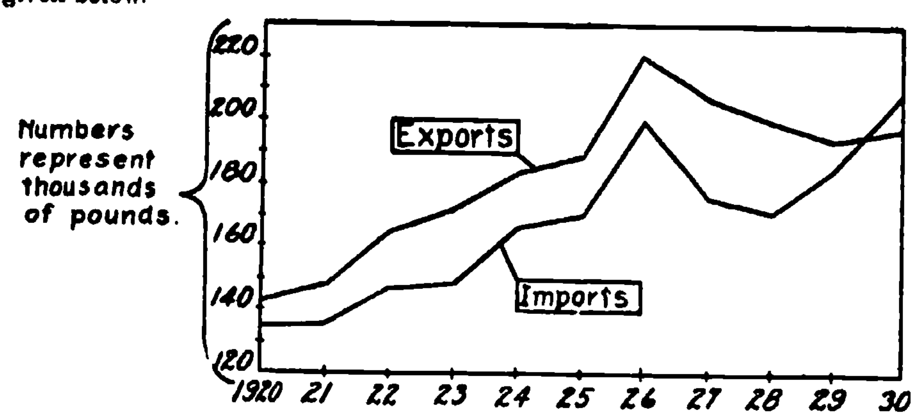


Write down the names of the boys (by letter) in order of merit, and write underneath the marks of each.

Order of merit: ____________________

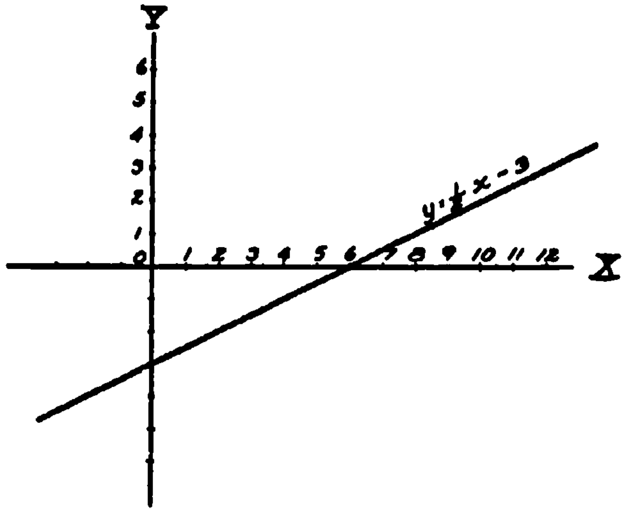
Marks: ____________________

IX. The diagram below shows the exports and imports of a particular port from 1920 to 1930. Examine the diagram carefully and answer the questions given below.



1. When were the exports greatest? ______
: When were the imports least? ______
3. When was the yearly rise in exports greatest? ______
4. When was the rise in exports and imports the same? ______
5. What were the exports in 1923? £______,000.
6. What were the imports in 1927? £______,000.
7. What were the imports, when the exports were £120,000? ______
8. What were the exports, when the imports were £80,000? ______

X. The graph given below represents the equation $y = \frac{1}{2}x - 3$.



1. Put a cross on the graph where $x = 2$. Find the corresponding value of y:

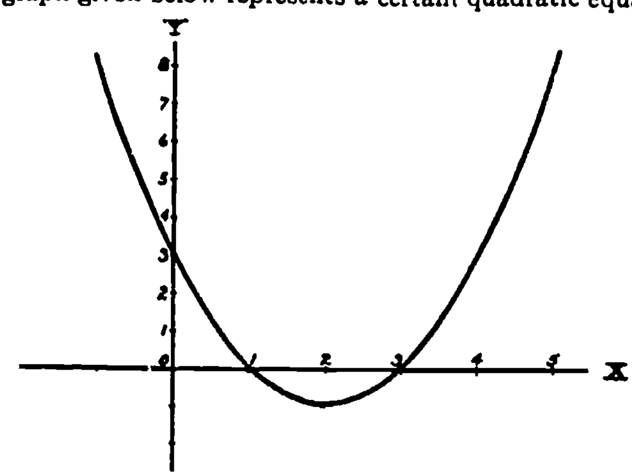
2. Put a small circle on the graph where $y = 1$. Find the corresponding value of x:

3. Draw a line parallel to the graph through the point (2, 0). What is the equation of this graph?

4. Take any two points on the graph. For these two points find the *change in x value* and the *corresponding change in y value*, and write down the value of the quotient

$$\frac{y \text{ change}}{\text{corresponding } x \text{ change}} = \_\_\_\_\_\_.$$

XI. The graph given below represents a certain quadratic equation.



1. What is the equation of the graph?

2. Put crosses on the graph where $(x - 1)(x - 3) = 3$. What are the corresponding values of x? ______

3. As x varies from 0 to 1, y varies from __ to __.
4. As x varies from 2 to 4, y varies from __ to __.
5. The value of the function y is zero, where $x =$ __.

Test 5: Formulas

XII. Write down the formulas to find:

1. The area (A) of the four walls of a room, given the length (l), wi lth (w), and height (h). 1. __________

2. The area (A) of a rectangle, given one side (a) and the diagonal (c). 2. __________

3. The area (A) of the whole surface of a circular cylinder, given the radius of the base (r) and the height (h). 3. __________

4. The volume (V) of a right circular cone, given the radius of the base (r) and the height (h). 4. __________

5. The dista ce (d) traversed by a train, which goes at the rate of v_1 mi. per hr. for the first n_1 hr. and v_2 mi. per hr. for the next n_2 hr. 5. __________

6. The distance (d) between two cars which start from the same place and travel for t hours with speeds v_1 and v_2 mi. per hour: (a) in the same direction; and (b) in directions at right angles to each other. 6. (a) __________ (b) __________

7. The depth (d) of water in a cylindrical tank (radius of base = r ft.) after it has been supplied with water for n hours from a tap which gives V cu. ft. per hour. 7. __________

8. The nth term (N) of the series 4, 7, 10, 13, ... 8. __________

9. The nth term (N) of the series 2, 6, 18, 54, ... 9. __________

10. The nth term (N) of the series 2, 5, 10, 17, 26, ... 10. __________

XIII. In each of the following formulas the quantity on the left-hand side is called 'the subject of the formula.' You are asked to find the effect on the subject of the formula, when certain changes are made in some of the other quantities.

Formula	*Change Made in Term*	*Change Made in the Subject*
1. $C = 2\pi r$	r is doubled.	1. __________
2. $s = \frac{1}{2}gt^2$	t is trebled.	2. __________
3. $A = \frac{\sqrt{3}}{4}a^2$	a is halved.	3. __________
4. $E = R.C$	R and C are both doubled.	4. __________
5. $t = 2\pi\sqrt{\frac{l}{g}}$	l is doubled.	5. __________

6. $Y = \frac{Wl^4}{4bd^3l}$	W is doubled and l, b, d halved.	6. ________
7. $R = KSV^2$	(a) S is increased 10%. (b) V is increased 10%.	7. (a) ________ (b) ________
8. $f = \frac{uv}{u - v}$	u and v are both increased 10%.	8. ________
9. $H = \frac{RC^2T}{J}$	R and C are increased 10%.	9. ________

Test 6: Relations

XIV. We frequently find that one quantity depends on a number of other quantities. For example, the interest obtained from an investment depends on the *amount* invested, the *rate per cent.*, and the *time* the money has been invested. Complete the following sentences showing the dependence in each case.

1. The circumference of a circle depends on ______.
2. The area of a rectangle depends on ______ and ______.
3. The volume of a rectangular box depends on ______, ______, and ______.
4. The weight of a rectangular block depends on ______, ______, ______, and ______.
5. The volume of a sphere depends on ______.
6. The price paid for a bag of wheat depends on ______ and ______.
7. The wages received by a workman depend upon the number of ______ he works and the ______ per day.
8. The distance traversed by a ball in falling from rest depends on ______ and ______.
9. The amount of expansion of an iron rail when heated depends on ______, ______ and ______.
10. The area of a triangle depends on the ______ when the ______ is constant.
11. The volume of a given mass of gas depends on its ______ and ______.

XV. In the following examples there are several dependent factors entering into the case. Full marks will be given for any two correct factors.

1. The speed of a motor car: ______, ______.
2. Good health: ______, ______.
3. The yield of wheat from a field: ______, ______.
4. The amount paid in income-tax: ______, ______.
5. The exchange rate between England and America: ______, ______.
6. The cost of running a motor-car: ______, ______.
7. Success in school work: ______, ______.
8. The price of certain mining shares: ______, ______.
9. The bending of a beam fixed at one end: ______, ______.
10. The premium paid on a life insurance policy: ______, ______.