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ABSTRACT

State-space representation of a problem, borrowed from mechanical problem-solving theory, is used to describe a problem's invariant structure formally. Paths within the state-space represent subjects' behaviors, as conventionally distinguished from their "protocals" or "strategies". Its goal is thus to develop the relationship between problem structure and problem-solving behavior. Five specific hypotheses respecting the paths generated by problem solvers (in this case, college students and college educated adults) in the state-space of the 4-ring Tower-of-Hanoi problem were formulated. These hypotheses were motivated by the formal properties of the Tower-of-Hanoi state-space and represented anticipated effects of the problem structure in shaping problem-solving behavior. The experimental results of this study seem to confirm an important role played by features of the problem structure in determining patterns in the problem-solving behaviors of the subjects. The "general" hypotheses were only tested for a single problem, and with a rather limited population of subjects. This research does not take acccunt of the possibility of age, sex or cross-cultural differences which might exist in problem solving behavior. Tables and illustraticns are included demonstrating the problem and structure and its interpretations. (RC)

THE BEHAVIORAL EFFECTS OF SUBPROBLEMS AND SYMMETRIES IN THE TOWER OF HANOI PROBLEM

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THE BEHAVIORAL EFFECTS OF SUBPROBLEMS AND SYMMETRIES IN THE TOWER OF HANOI PROBLEM

1. The Problem

This research employs the state-space representation of a problem, borrowed from mechanical problem-solving theory (Nilsson, 1971) to describe a problem's structure formally. The behaviors of problem-solving subjects are recorded as paths through the state-space representation of the problem corresponding to the succession of steps taken or moves made by the subject. It is suggested that certain features of the problem structure, such as its symmetry and its decomposition into subproblems, might permit the prediction of patterns in subjects' problem-solving behaviors.

In particular, the following questions are asked: Can problem-solving behavior be characterized as "goal-directed" within the state-space representation of a problem? Does problem-solving behavior take account of "subgoal-states" within the problem to facilitate the attainment of the main problem goal? Can a sequence of "intervals" be identified in subjects' behaviors, invariant over a population of problem solvers, corresponding to the solution of particular classes of subproblems? Is there any consistency in a problem solver's behavior when he or she if faced with two different problems of identical (or isomorphic) structure? Does the symmetry structure of a problem affect the solvers' attempts at solution?

These questions were asked in the specific context of the 4-ring Tower-of-Hanoi problem, for a population of college students and college-educated adults.

Banerji (1969), Banerji & Ernst (1972), and Carr (1971) have used the state-space formalism to compare problems and to define problem "isomorphisms," "homomorphisms," and "decompositions" with some precision. Simultaneously, researchers in problem-solving have examined subjects' "strategies," "protocols," "behaviors," and "retrospective accounts" in problem-solving episodes (Newell, Shaw & Simon, 1959; Newell & Simon, 1972; Dienes & Jeeves, 1965, 1970; Branca & Kilpatrick, 1972; Goldin &uger, 1973).

Dienes and Jeeves attempt to catalogue subjects' strategies in solving specific problems, while Branca and Kilpatrick compare subjects' strategies across problems of isomorphic and nonisomorphic structures. Much of this work remains inconclusive: Retrospective accounts of strategies often differ from strategies inferred from observed behaviors. The elements of the problem structure thought to be important are not identified, and the comparison of strategies across problems of different structure is not really possible.

Newell and Simon's (1972) research describes subjects' protocols and seeks to characterize the human problem-solver as an "information-processing system" operating in a "problem space." The problem space is permitted to vary from subject to subject and from task to task. The resulting analysis of necessity has a post-hoc character; and although it leads to an effective mechanical characterization of problem solvers' protocols, it lacks the potential for predictive power in describing human problem-solving.

This paper regards the state-space representation of a problem as descriptive of the problem's invariant structure for all subjects in a population. Paths within the state-space represent subjects' behaviors, as conventionally distinguished from their "protocols" or "strategies." Its goal is thus to develop the relationship between problem structure and problem-solving behavior. It also seeks experimental confirmation of some of the ideas developed in the work of Goldin and Luger (Luger, 1973; Goldin & Luger, 1973).

In this research, five general problem-solving hypotheses respecting paths generated by human problem solvers in the state space of a problem were formulated. These were motivated by the formal properties of a state-space, and represent anticipated effects of problem structure in shaping problem-solving behavior:

Hypothesis 1. (a) In solving a problem the subject generates non-random paths in the state-space representation of the problem. (b) Solution paths tend to be goal-directed and segments of solution paths also form portions of goal-directed paths.

Hypothesis 2. Given a decomposition of the state-space of a problem into subproblems then (a) subproblem solution paths tend to be subgoal-directed, and (b) when subgoals states are attained, the paths exit from the respective subproblems.

Hypothesis 3. Identifiable intervals occur during problem solving corresponding to the solution of various subproblems. That is, paths occur during certain intervals which do not constitute the (direct) solution of a problem but which do constitute the solution of all the isomorphic subproblems within the problem.

Hypothesis 4. When two subproblems of a problem have isomorphic state-spaces, the problem solver's paths through these subproblems tend to be congruent.

Hypothesis 5. Given a symmetry group G of automorphisms of the state space of a problem, there tend to occur successive paths congruent modulo G in the state space. Such occurrrences often culminate in the solving of the problem.

It may be that the validity of Hypotheses 2, 3, and 5 depends on the particular way that the state-space of the problem is decomposed into subproblems, since such a decomposition is often not unique.

2. Specific Hypotheses Tested

Five specific hypotheses respecting the paths generated by problem solvers in the state space of the 4-ring Tower-of-Hanoi problem were formulated. These hypotheses were motivated by the formal properties of the Tower-of-Hanoi state-space (see Figures 1, 2), and represented anticipated effects of the problem structure in shaping problem-solving behavior. The following five hypotheses were tested: (1) the nonrandomness and goal-directedness of problem-solving paths through the state-space representation of the Tower-of-Hanoi problem, (2) the special role of subgoal-states; i.e., the subgoal directedness of paths within 2- and 3-ring subproblems, and the automatic exit from the subproblem space once the subgoal state is achieved, (3) the consistent solution of subproblems on the 1-, 2-, or 3-ring level via minimal paths as identifiable "intervals" within the total problem-solving episode, (4) the congruence of subjects' paths through isomorphic subproblem spaces prior to the solution via minimal paths of subproblems on that level, and (5) the special role of problem symmetry in the problem-solving episode, as evidenced by the interruption of problem-solvers' paths and the subsequent generation of the "automorphic image" of the interrupted path.

Criteria for fulfillment of these hypotheses were established as follows:

(1) Paths through the n-ring Tower-of-Hanoi problem were defined as goal directed when they neither reentered an (n-2)-ring subproblem nor moved away from the goal-state by more than an (n-2)-ring subproblem. The nonrandomness of subjects' paths was tested by comparing the number of "corners" (vs. "straight" sections) in subjects' paths with the number of "corners" in randomly generated paths of the same length. Goal-directedness was tested by comparing the percentage of subjects' paths which satisfied the criteria for being "goal-directed" with the percentage of randomly generated paths which satisfied these criteria.

(2) The role of subproblems was investigated by (a) determining the percentage of subjects' paths through all 2- and 3-ring subproblems that met the criteria listed above for goal-directedness; and (b) by determining the percentage of times that subjects' paths, having entered a subgoal state, exit from that subgoal in a manner that also exits from the subproblem space. This percentage is to be compared with the percentage based on random choice (50 percent).

(3) An "n-ring interval" in solving the Tower-of-Hanoi problem was defined as occurring when a subject executed minimal solutions to more than 50 percent of all the isomorphs of an n-ring subproblem for a certain period prior to executing minimal solutions to more than 50 percent of the isomorphs of the (n+1)-ring subproblem. Three such intervals are theoretically possible in solving the 4-ring Tower-of-Hanoi

problem. (See Figure 3; in which a subject demonstrates a "2-ring interval".) It was asked whether a significant number of subjects would evidence at least one of these intervals in their solutions, and whether any subjects would evidence all three such intervals.

(4) Congruence of subjects' paths through isomorphic subproblem spaces was defined as occurring when any one congruence class of nonminimal solution paths predominated in frequency over the others (see Figure 4). It was asked whether such congruences would occur for a significant number of subjects on the 2- and 3-ring levels.

(5) The number of subjects was to be established in which an interruption in a path occurred, followed immediately by the automorphic (symmetric) image of the interrupted path. (See Figure 5, trials 2 and 3 of subject A.) A priori, it was predicted that such interruptions would occur for half the subjects, since probability dictates that 50 percent of the subjects' paths would start towards the goal state, and 50 percent would start in the symmetrically opposite direction.

It should be noted that this study does not rely heavily on conventional statistical tests for establishing the existence of effects in a population of subjects; rather it develops some new techniques for establishing the existence of "patterns" in subjects' behaviors.

3. Conduct of the Investigation

The hypotheses posed above were tested for 58 subjects, 22 male and 36 female. To solve the 4-ring Tower-of-Hanoi problem, each subject was asked to move, in as few steps as possible, four concentric rings (1,2,3,4) from the first to the last of three pegs (A,B,C). Only one ring could be moved at a time, and no larger ring could be placed over a smaller ring. The 4-ring Tower-of-Hanoi problem and its state-space represnetation are pictured in Figure 1. The paths of two typical subjects solving the Tower-of-Hanoi problem are given in Figure 5.

The subjects, college students and college-educated adults, were not overly practiced in mathematical games and puzzles. In particular, they had no prior acquaintance with the Tower-of-Hanoi problem. Each subject was individually interviewed in a well-lighted room, the puzzle was put before him or her on a desk, and paper and pencil were available. The investigator was the only other person present. Once having started the problem, the subject continued to work on it until he or she either gave up or succeeded in moving all the rings from the "begin" to the "goal-state" of the problem in the least possible number of moves. The subject could start over at any time and for any reason that he or she wished. The episode usually lasted 15 to 20 minutes. A tape recorder was kept running continuously to record the subject's verbal responses.

The data for seven subjects were discarded for reasons such as the occurrence of interruptions. Six subjects solved the problem in the minimum number of steps on their first attempt. The behaviors of the remaining 45 subjects were mapped from the tape recordings to the state-space representation. Transcripts of the tapes were also prepared to accompany these representations, which in turn were used to test the hypotheses.

4. Analysis of Data

Tables 1-3 tabulate the data of the two subjects in Figure 5, and also list the totals for all of the 45 subjects mentioned above.

Hypothesis 1: (a) Of 45 subjects' first attempts at solving the problem, 95% met the criterion for nonrandomness, that is they deviated from the random by more than one standard deviation in the occurrence of path "corners", and 78% deviated from the random by more than two standard deviations. Of all 131 attempts by subjects, 97% met the criterion for nonrandomness and 81% deviated from the random by more than two standard deviations. All deviations from the random were in the direction of fewer turns in the paths. (b) Of 45 subjects' first attempts, 87% satisfied the criterion for goal-directedness; and 93% of the subjects' 133 attempts satisfied these criteria.

Hypothesis 2: (a) Subgoal directedness: Of the 685 paths through 2-ring subproblems, 96% met the criteria for subgoal directedness; of the 321 paths through 3-ring subproblems, 91% met the criteria. (b) Subproblem exit: Of the 685 paths through 2-ring subproblems, 96% met the exit criterion; of the 321 3-ring subproblem paths, 98% met the exit criterion.

Hypothesis 3: A maximum of three "intervals" were possible, corresponding to minimal solutions of 1-, 2-, and 3-ring subproblems respectively. Of the 51 subjects, 6 (12%) displayed none of these intervals; 16 (31%) displayed just one interval; 22 (43%) displayed exactly two intervals; and all three theoretically possible intervals were displayed by 7 subjects (14%).

Hypothesis 4: For the 2-ring subproblem, predominance of one congruence class of nonminimal paths occurred for 7 of 45 subjects (16%); non-predominance occurred for 6 subjects (14%); and 32 subjects (70%) permitted no conclusion to be reached (because of insufficiently many nonminimal paths, or an inconclusive distribution of these paths). For the 3-ring subproblem, predominance of one congruence class was shown by 6 subjects (13%), non-predominance by 21 subjects (41%), and 18 subjects (40%) permitted no conclusion.

Hypothesis 5: Of 45 subjects, 44% displayed the predicted effect of the problem symmetry, and 7% exhibited this pattern twice during their problem-solving episodes. This compares reasonably well with the figure of 50% for whom the phenomena was predicted to occur.

5. Evaluation of the Findings

(1) The nonrandomness and goal-directedness of subjects' paths, through the problem and through its subproblems, are established. This is, of course, to be expected from almost any theory of problem solving, and it helps to establish the

credibility of the state-space description of behavior. Also, randomly generated paths may be used as a "standard" against which to observe the direction of deviation of subjects behaviors from the random.

(2) The strength of verification of hypothesis 2b (nearly 100%) is a clear indication of the importance of subgoal states in the problem-solving process.

(3) Intervals corresponding to the solution of classes of isomorphic subproblems occurred in a large majority of the subjects, and point to a role played by the hierarchy of nested subproblems in affecting behavior.

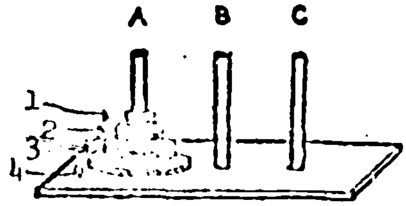
(4) The expected congruence of nonminimal solution paths across isomorphic subproblems was not confirmed by the data.

(5) However, the effect of symmetry in the problem situation was clearly observed to be in line with our expectations.

The author sought to establish a framework for studying the effects of problem structure on problem-solving behaviors. The experimental results of this study seem to confirm an important role played by features of the problem structure in determining patterns in the problem-solving behaviors of subjects. In particular, the goal-directed behavior within subproblems and immediate exit from the subproblem space once the subgoal was achieved, indicate the problem solvers' effective "decomposition" of the problem in attempting its solution. The intervals within the problem solution indicate the effect on the problem solver of the isomorphic structure of the subproblems. Finally, the symmetry structure within the problem was reflected in the problem solver's interrupted paths. These interruptions often culminated in the solution of the problem.

We may speculate that a subject's cognitive structures ought to be defined to include the symmetry operations and subproblem decompositions that the subject can employ during problem-solving. These in turn determine the states of the environment that the subject is able to treat as distinct and those treated as equivalent in problem solving. These structures of the subject may change during the course of problem-solving, and so lead to an effective "reduction" of the state-space (Goldin & Luger, 1973).

In conclusion, it should be noted that the suggested "general" hypotheses were only tested for a single problem, and with a rather limited population of subjects. Each of the hypotheses (section 1, above) ought to be tested in several specific problem-solving situations perhaps beginning with a problem whose structure is isomorphic to the Tower-of-Hanoi, such as the tracing of the Hamiltonian paths on a hypercube. Finally, the present research does not take account of the possibility of age, sex, or cross-cultural differences which might exist in problem-solving behavior. Hopefully further work in this area will broaden the domain both of subject populations and of problem situations considered.



4-ring Tower of Hanoi board in its initial state.

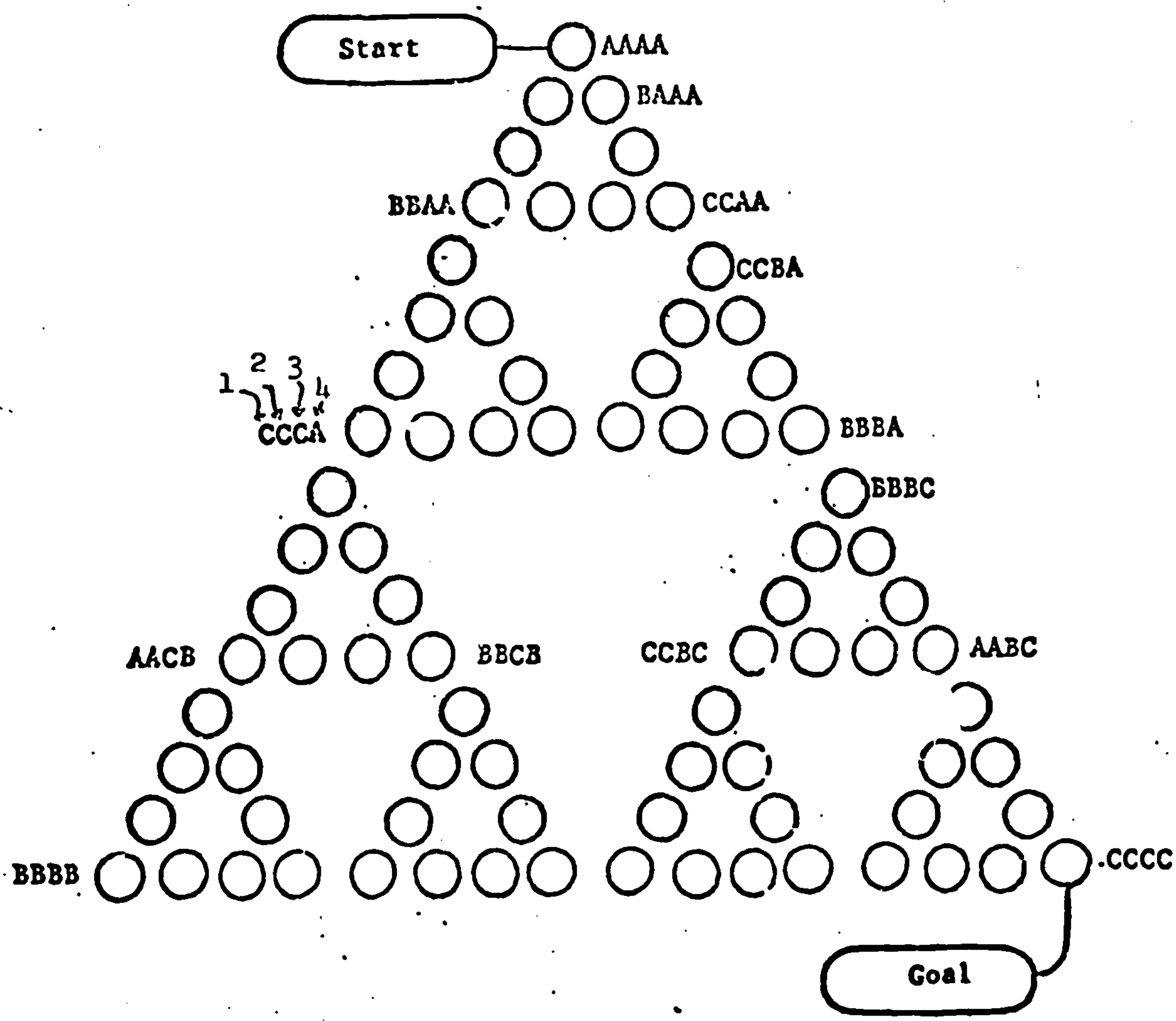
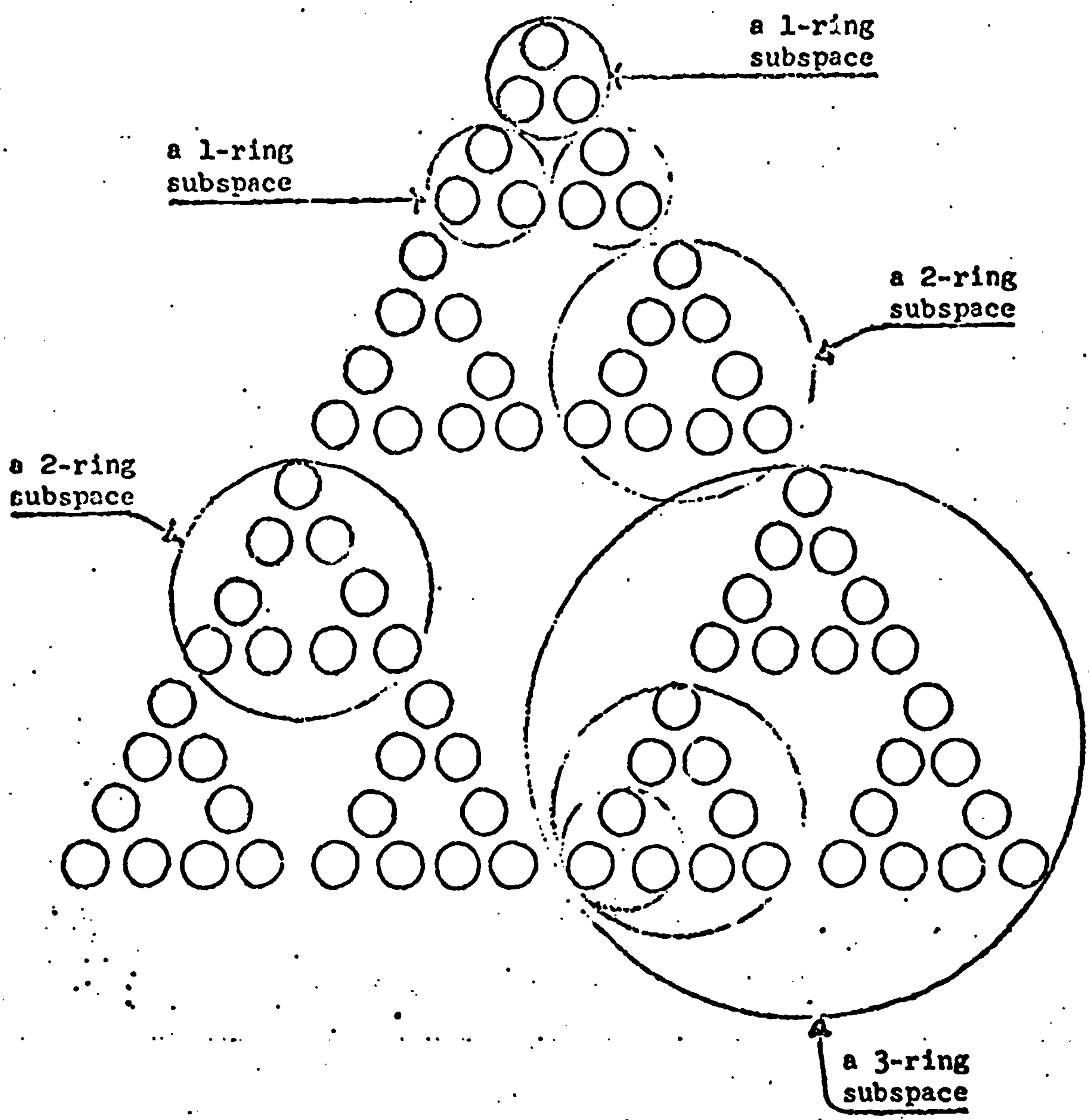


FIGURE 1. State-space representation of the 4-ring Tower of Hanoi problem.

The four letters labelling a state refer to the respective pegs on which the the four rings are located. Legal moves effect transitions between adjacent states.

FIGURE 2. Subproblem decomposition of the Tower of Hanoi state-space.



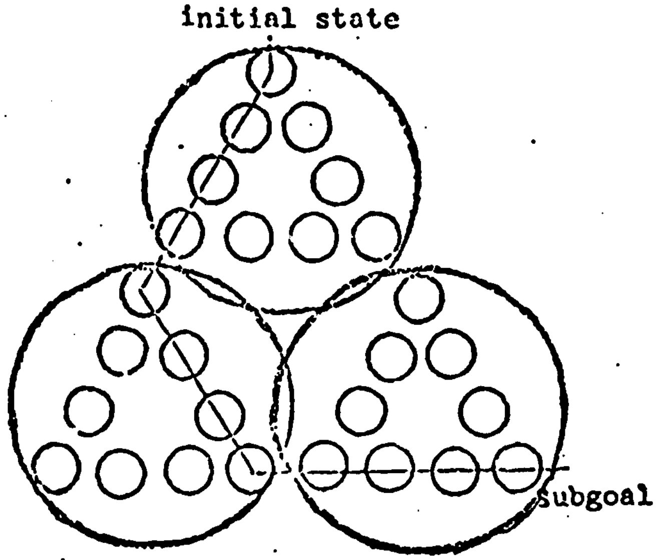


FIGURE 3. An interval in problem-solving.

The 2-ring subproblem is consistently solved in the minimum number of steps, while the 3-ring subproblem is not. The state-space has been effectively reduced modulo its 2-ring subproblem decomposition.

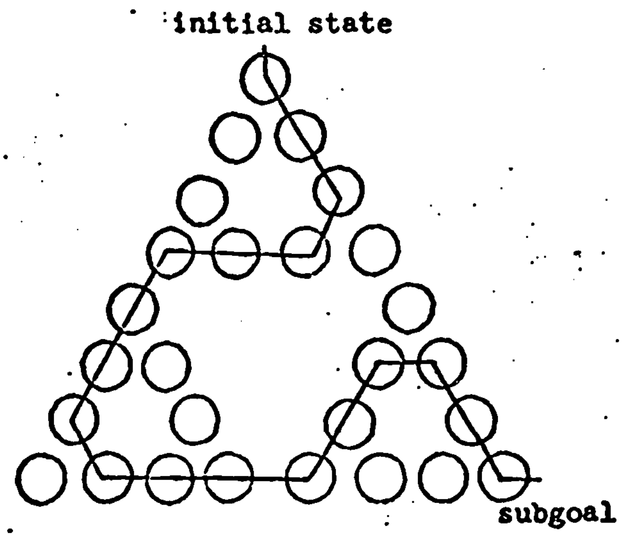


FIGURE 4. Congruent paths through isomorphic subproblems.

All three paths through the 2-ring subproblems in Figure 16 are congruent to each other.

FIGURE 5. Paths of two subjects through the state-space of the Tower-of-Hanoi problem.

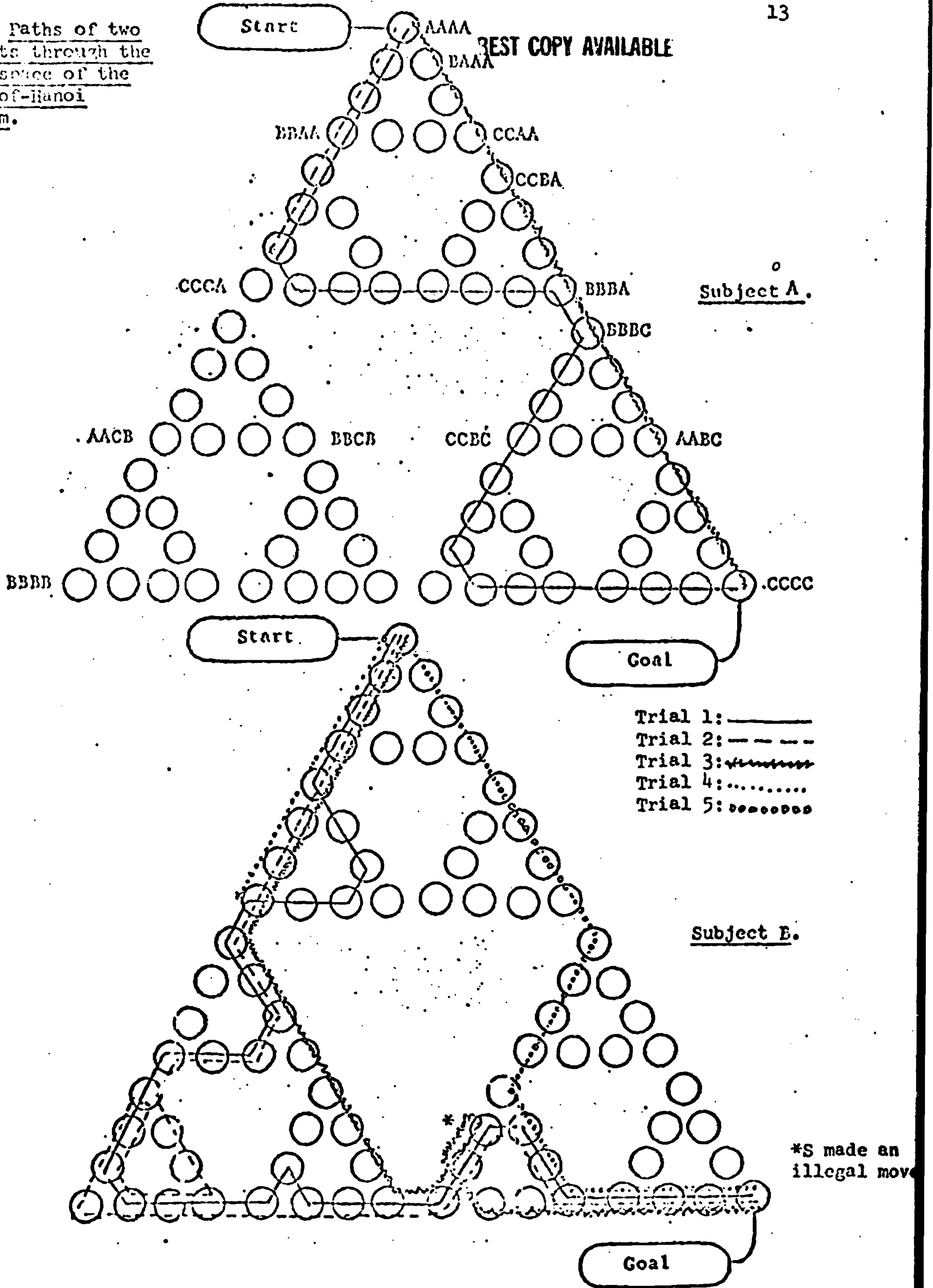


Table 1

Non-randomness and Goal directedness of Paths

Key:

N = number of states entered in a path.
f = fraction of corners in a trial. f = C/N-1 where C is the number of corners.
GD = { 1 for direct solution path (also goal directed)
+ for goal-directed path
- for nongoal-directed path }

Trial	1			2			3			4			5		
	N	f	GD	N	f	GD	N	f	GD	N	f	GD	N	f	GD
Subject A	27	.23	+	6	.0	+	15	.0	1						
Subject B	36	.49	+	42	.34	+	27	.19	+	7	.0	+	19	.17	+

Totals for all 45 subjects:

f for a random path is .67. A standard deviation in f for a random path is ± .10. Of subjects' 45 first trials, 43 (95%) satisfy the non randomness criteria i.e., f .57. Of 131 Total trials 127 (97%) met this criteria.

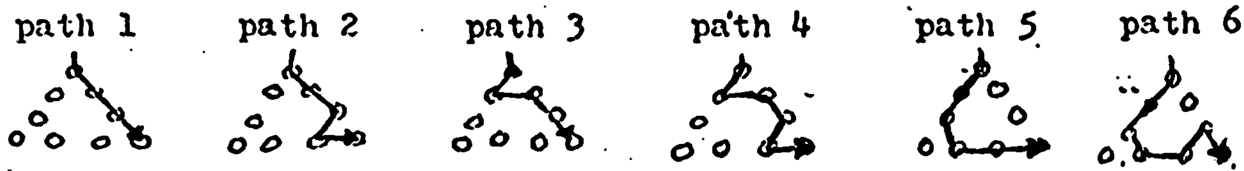
Of random paths less than 66% were goal directed. Of subjects' first attempts 39 of 45 (87%) were goal-directed. Of 131 total trials, 121 (93%) were goal directed. (The "total trials" in both instances above do not include direct solution paths (1) which are, of course, also goal-directed.)

Table 2

Paths through 2-, and 3-ring subproblems.

Key:

1,2,3, ... = congruence classes of goal directed paths. For example, there are six congruence classes of goal-directed paths through 2-ring subproblems:



A similar decomposition exists for goal-directed paths through 3-ring subproblems. "1" will always refer to the solution path of fewest steps.

x = non goal directed path.

$1 - f_x$ = function of goal-directed paths (Hypothesis 2a).

\- = failure to exit from subproblem goal state

f_- = fraction of paths which at the end fail to exit from the subproblem (Hypothesis 2b).

Congruence (Hypothesis 4) =
- N for noncongruence of paths through isomorphic subproblems
- C for congruence of paths through isomorphic subproblems
- I for insufficient evidence to determine either C or N

	Distribution	Congruence	$1- f_x$	f_-
Subject A:				
2-ring paths:	[1 2 1 1 2 1 1 1 1 1 1 1	C	1.0	.0
3-ring paths:	[2 2 1 1 1	C	1.0	.0
Subject B:				
2-ring paths:	[1 2 2 2 3 2 1 1 1 1- 2-1-1 1 2 1 1 1 1 1 x 1 1 1 1 1 1 1 1	C	.97	.1
3-ring paths:	[2 x 3 1 x-1 3 1 1 x 1 1 4	N	.77	.08
Totals:	2-ring paths	7C, 6N, 32I	.96	.04
	3-ring paths	6C, 21N, 18I	.91	.02

Table 3

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Intervals in solving isomorphic subproblems

Key:

1 = most direct solution path for subproblem

D = not the most direct solution path for subproblem

first row: sequence of 2-ring subproblems entered

second row: sequence of 3-ring subproblems entered

third row: sequence of trials on 4-ring problem

! = acquisition of most direct solution, defined to occur at the point after which more than half of any future sequences of subproblems (of the appropriate level) are solved by the most direct path (Hypothesis 3).

* = point between successive paths through 4-ring problem congruent by virtue of symmetry automorphism (Hypothesis 5)

Subject A:

```
subproblems
                                *
   2-ring:  1 0 ! 1 1 0 1 ! 1 1 ! 1 1 1 1
   3-ring:    0       0   !  1  !  1   1
   4-ring:        0          0  !    1
```

intervals: 1-ring, yes; 2-ring, yes; 3-ring, yes.

Subject B:

```
subproblems                                                               *
   2-ring:  1 0 0 0 0 0 ! 1 1 1 0 1 0 1 1 1 0 1 ! 1 1 1 1 1 1 0 1   1 1 1 1 1
   3-ring:   0   0   0    1    0    1   0  !  1   1   1   0      1   0
   4-ring:       0               0                  0       0        0
```

intervals: 1-ring, yes; 2-ring, yes; 3-ring, yes.

Totals: occurrence of 1-ring interval: 21 subjects yes, 30 subjects no
occurrence of 2-ring interval: 37 subjects yes, 14 subjects no
occurrence of 3-ring interval: 23 subjects yes, 23 subjects no

88% of all subjects had at least one interval
57% of subjects had at least two intervals
13% of subjects had all three theoretically possible intervals

Twenty of 45 subjects (44%) displayed the predicted effect of symmetry in the problem solution. Three subjects (7%) displayed this twice, and four other subjects came within 1 or 2 states of displaying the phenomenon (i.e., the path following the "break" lacked 1 or 2 states of being the automorphic image of the interrupted path).

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