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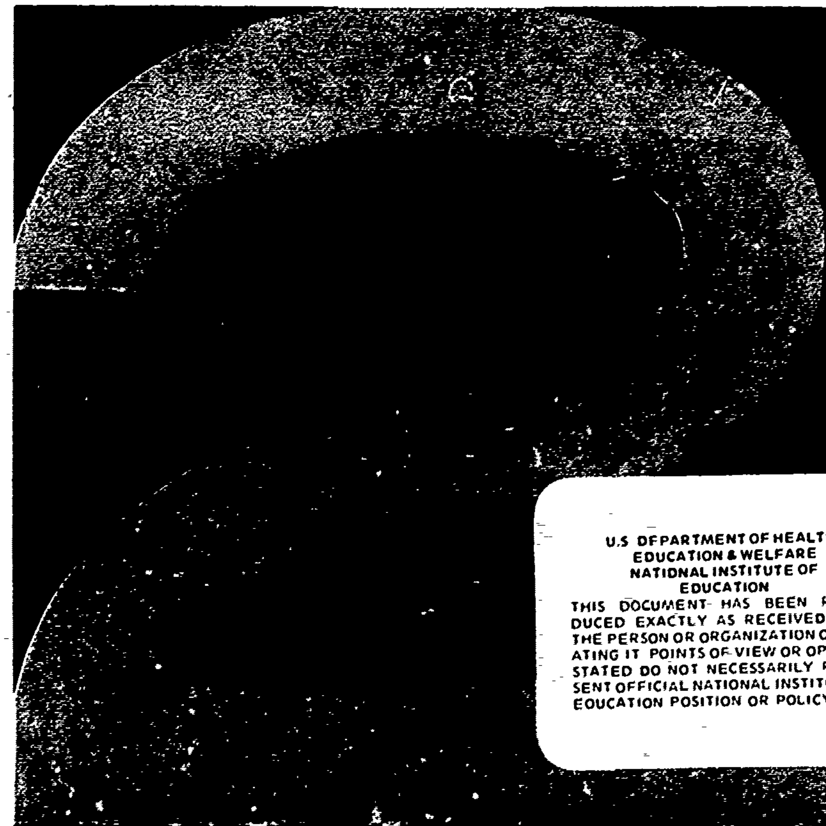
ABSTRACT

A brief survey of the elementary number systems is provided. The natural numbers, integers, rational numbers, real numbers, and complex numbers are discussed; numerals and the use of numbers in measuring are also covered. (DT)

A PORTRAIT OF 2



A BRIEF SURVEY
OF THE ELEMENTARY NUMBER SYSTEMS

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

A Portrait of 2

A BRIEF SURVEY OF THE ELEMENTARY NUMBER SYSTEMS

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I. Introduction

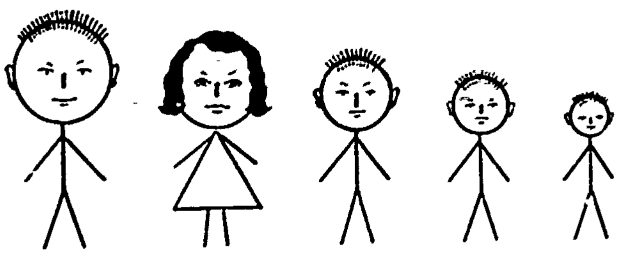
Words and symbols mean different things to different people. What does the word *horse* mean to the reader? What do we think of when the word *horse* is used in casual conversation? What would we think of if we reflected not so casually about the meaning of the word *horse*? Surely the concept *horse* is not a fixed concept the same for all people. Try to imagine what the word means to a three-year-old child, to a farmer, to the owner of a racing stable, to a zoologist, or to a linguist.

Enough about *horse*. We propose to look at 2. We have looked at 2 many times. Maybe it has been a long time since we thought about the meaning of 2. Maybe our concept of 2 is a puny little thing like a three-year-old child's idea of 2. Perhaps a comprehensive view of 2 will provide some nourishment to enhance the growth of our concept of 2.

We propose to look at a modern portrait of 2. Our purpose is not to trace the historical development of number concepts. Rather it is our purpose to present a glimpse of 2 as it appears in the minds of men today. We shall see 2 in an environment of other numbers. In order to understand 2 it is necessary to know something of these number systems. So we shall develop the number systems to such a point as will enable us to see 2 in its proper setting. In addition to looking at 2 as a number we shall look at 2 as a symbol or as part of a composite symbol. We shall see 2 by itself and we shall see 2 in 21, 2 in $\frac{1}{2}$, and 2 in \$2.

II. The Natural Number 2

The simplest of the number systems is the system of natural numbers. or counting numbers if you prefer: 1, 2, 3, ⋯ . We use this symbol to suggest an unending sequence of numbers, the natural numbers in their natural order. Each number in this sequence has an immediate successor and an immediate predecessor except that 1 does not have a predecessor. The mathematician thinks of a natural number in two distinct ways, as an *ordinal* number and as a *cardinal* number. In counting the objects of a collection such as A, B, C, a small child says 1, 2, 3, pointing in succession to A, to B, and then to C. Object B is the second object, object number 2, in this collection of three objects. The youngster has established a one-to-one correspondence between A, B, C and the words 1, 2, 3: A is mated to 1, B is mated to 2, and C is mated to 3. The direct counting of any finite collection of objects amounts to establishing a one-to-one correspondence between the objects in the collection and a set of natural numbers. Thus we think of 2 as a grunt which is part of the ritual of counting. If there is more than a single item in the collection counted, then 2 is the grunt which comes immediately after the grunt 1. Thus our portrait of 2 includes a glimpse of 2 as a counting grunt, as an ordinal number.

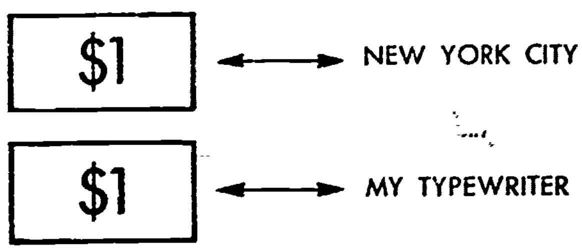


Ordinal: Mary is number 2 in this lineup.
Cardinal: There are 5 persons in this lineup.

Let us now look at 2 as a cardinal number. The cardinal number of a set of objects is a word which conveys the idea of how many objects are in the set. The cardinal number of a set has nothing to do with the arrangement or order of the elements in the set. In fact we may rearrange the elements of a set in any way and the cardinal number of the set remains unchanged. Two sets have the same cardinal number if there exists a one-to-one correspondence between the elements of one of the sets and the elements of the other set. Thus the set S_1 consisting of the letters A and B has the same cardinal number as the set S_2 consisting of the letters C and D. For if we mate A to C and B to D we have a one-to-one correspondence between the elements of S_1 and the elements of S_2—each element in S_1 has a unique mate in S_2 and each element in S_2 has a unique mate in S_1. Any set which has the same cardinal number as the set S_1 consisting of the two letters A and B is said to have the cardinal number 2; or we might say that the cardinal number of the set is 2. Thus the set S which has the city of New York as one element and my typewriter as another element and which has no other elements is a set which has the cardinal number 2.

We have been talking about the cardinal number 2 as something which a set has. The critical reader may say that a set has elements and that is all it has. The elements comprise the set. The set consisting of New York City and my typewriter has two elements: New York City is one and my typewriter is the other. Who may declare that it has something else, a mystic thing called a cardinal number? Of course, no one would say that the cardinal number 2 is a thing. But do we really clarify the situation by saying that 2 is a property—by saying that the set consisting of New York City and my typewriter has the property of twoness? Do we make it clearer by saying that 2 is the property which all pairs have in common? What is a common property? All pairs have several properties in common. For example, all pairs have the following two properties in common. Property 1: Each pair of things is a set of things with more than one element. Property 2: Each pair of things is a set of things with a cardinal number which is less than the cardinal number of the set of fingers on my left hand.

Fortunately the portrait of the cardinal number 2 is not so mystic or abstract as the foregoing comments might indicate. Before stating the modern point of view let us think of the concept of a set for a minute. We shall not attempt to define "set." We think of a set as composed of elements. We consider the words *set*, *collection*, and *class* as synonyms. The elements of a set may be other sets. Thus to avoid confusion we may speak of a class of sets or of a collection of sets, or of a collection of classes. As an example, consider the class of all married couples in the United States. Each element in this class is a set, a set which has as elements a man and a woman. Another example is the set of classes which graduated from Harvard at the June commencements in 1940 to 1950 inclusive. There are 11 elements in this set. Each element is one of the June graduating classes. Of course, each of these classes is a set of elements, each element being a person. And of course each person might be considered as a set of cells, each cell is a set of molecules, and each molecule is a set of particles studied in nuclear physics. But the modern concept of the cardinal number 2 is not quite that complicated. It is simply this: the cardinal number 2 is the class of all pairs; or in other words it is the class of all sets each of which satisfies the condition that its elements can be put into a one-to-one correspondence with the elements of the set, for example, which has New York City as an element, my typewriter as an element, and no other elements. Having defined 2 as this class we must now clarify what we mean when we say that the cardinal number of a set S is 2, when we say that set S has the cardinal number 2. We mean that S is an element of the class which we just defined above as the cardinal number 2. "I have 2 dollars" means that the set which consists of my dollars is an element of the class which we defined as the cardinal number 2. Thus our portrait of 2 includes a glimpse of 2 as a cardinal number, a glimpse which reveals 2 as the class of all pairs.



Let us now consider 2 as an element of the system of natural numbers (the counting numbers 1, 2, 3, · · ·). We have mentioned before that a number system is a set of numbers and certain operations. We concern ourselves for the present with the operations of addition, subtraction, multiplication, and division. In studying the properties of these fundamental operations the mathematician regards them as binary operations. A binary operation is an operation which operates on two numbers. Thus + is a symbol which denotes the operation of addition and 2 + 3 is a composite symbol which indicates that the operation of addition is to be performed or has been performed on the numbers 2 and 3. When an operation takes place there is a result. What is the result of the operation of addition when performed on the numbers 2 and 3? The answer is that the result is another natural number, namely 5. Now a school child looks at 2 + 3 and sees nothing but a job to be done; he does it by writing 2 + 3 = 5. To him the result is denoted by 5 and that is all. But to a mathematician 2 + 3 does not produce an unconquerable urge to write 5. One way of looking at 2 + 3 is that it is a composite symbol (formed from three basic symbols) which denotes the same natural number as is denoted by the symbol 5. Instead of looking at 2 + 3 as an indicated addition, a job to be done, the modern mathematician sometimes looks at 2 + 3 as though the addition were already done and the answer is 2 + 3. From this point of view Grade III arithmetic becomes very easy. Problem: Add 435 and 29. Answer: 435 + 29.

Of course, the teacher would like a statement from the child that (in the Arabic system of enumeration) 464 denotes the same natural number as is denoted by the composite symbol $435 + 29$. Now, getting back to the properties of the fundamental operations in the natural number system, we note that every pair of numbers can be added, that every pair can be multiplied, and that the sum and product are again natural numbers. We say that the system of natural numbers is closed with respect to addition and multiplication. We are using the word *closed* in accordance with the following definition: *A system of numbers is closed with respect to a binary operation if the result of the operation, when performed on any pair of numbers in the system, is a number in the system.* The system of natural numbers is indeed closed with respect to addition and multiplication but it is not closed with respect to subtraction or division. Thus the composite symbols $5 - 7$, $2 - 2$, $4 \div 6$ are meaningless in the system of natural numbers. Of course $7 - 5$, $3 - 1$, $8 \div 4$ are meaningful symbols, meaningful in that they are composite symbols denoting three natural numbers.

We have gone into some detail talking about the natural number system. Is all this necessary for an intelligent glimpse of 2? I believe it is. In relation to composite symbols we see that the natural number 2 may be denoted in many different ways. Thus:

$$2 = 1 + 1 = 3 - 1 = 4 - 2 = 5 - 3 = 6 - 4 = \cdots,$$

$$2 = 2 \times 1 = 2 \div 1 = 4 \div 2 = 6 \div 3 = 8 \div 4 = \cdots.$$

This suggests, too, that 2 may be obtained in an infinite number of different ways as the result of a binary operation performed in the system of natural numbers. Thinking some more about 2 in relation to operations and other numbers, we note that the possibility or impossibility of dividing a natural number by 2 separates all the natural numbers into two mutually exclusive classes. (Mutually exclusive means that these two classes have no common element.) One class consists of all the numbers which can be divided by 2; the other class consists of all the numbers which cannot be divided by 2. Thus the concepts of even number and odd number appear as we study the portrait of 2. Similarly we might think of the class of all natural numbers which can be subtracted from 2 and the class of all natural numbers which cannot be subtracted from 2. The first of the classes consists of one element, namely 1. It is the only natural number less than 2. The numbers which cannot be subtracted from 2 include 2 itself and all numbers which are greater than 2. Of course, the class of all natural numbers greater than 2 is identical with the class of all numbers from which 2 can be subtracted in the natural number system. Enough about 2 as a natural number in an environment of natural numbers.

III. The Numeral 2

Having rested our eyes for a second we are ready for another peek at the portrait of 2. This time we see 2 as a numeral, as a symbol for a number or as part of a composite symbol. We have seen that the numeral 2 considered as one symbol denotes the second of the natural numbers. But what does the numeral 2 mean in the composite symbol 27? What does it mean in the composite symbol 1.324? In the Arabic system of numerals there are ten elements, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Every natural number is denoted by one of these symbols or some combination of them. The combinations of which we speak here, as 373 or 47522, are not composite symbols which include symbols for operations as $8 + 2$, $8 - 2$, 8×2, $8 \div 2$, but are composite symbols formed on the basis of a place-value principle. Thus the immediate successor of 9 in the system of natural numbers is ten, denoted by 10, a combination of two numerals written in a definite order. And we know how these combinations are formed for any integer greater than 10. It is important, however, to have a deeper insight into these symbols than that reflected by the ability to form the symbols. Thus, we teach the elementary-school child that 37 means 3 tens and 7 ones. In more sophisticated language 37 is the cardinal number of the set which is formed when the elements of three sets S_1, S_2, S_3, each of cardinal number 10, and the elements of one set S_4 of cardinal number 7 are collected or combined into one set. It is to be understood here that no object is an element of more than one of the sets S_1, S_2, S_3, S_4; that is, we assume that the elements of these four sets are all distinct. Similarly, 202202 is the cardinal number of the set which is the union of two sets of cardinal 100000, two sets of cardinal 1000, two sets of cardinal 100, and one set of cardinal 2, the elements of these sets all being distinct The system of numeration has been expanded so that using the Arabic numerals 0, 1, and so on to 9, and a decimal point, we may form a symbol for any decimal fraction. Thus 2.022 denotes the rational number (more about these later) which results if one adds the numbers 2, $\frac{2}{100}$, $\frac{2}{1000}$. In concluding this glimpse we note that the innocent-looking little symbol 2, the numeral 2, may have different meanings dependent upon its position in a composite symbol.

IV. The Integer 2

We move next into the system of all integers. This system includes all the positive whole numbers, all the negative whole numbers and zero: $\cdots, -3, -2, -1, 0, +1, +2, +3, \cdots$. Note that one of the elements in this system is $+2$, read positive two, and another -2, read negative two. What is the relationship of these signed integers to the natural number 2? What does the portrait of 2 reveal? Several preliminaries are in order before we are in a position to answer these questions. First, let us review some uses of these numbers in elementary mathematics and in everyday affairs. We note that 0 (read zero) is a number in this system. Yes, we talked about 0 before; we listed it as one of the ten Arabic numerals; we saw it in such symbols as 10 and 202202. But 0 is not one of the counting numbers. The preschool child learns to count, 1, 2, 3, 4, 5; but he has no concept of zero as a number. In school he is confronted with subtraction problems as $2 - 2$ or $3 - 3$; he knows from his experiences with subtraction that $2 - 2$ "should be nothing"; he is taught to write 0. He should learn eventually that subtraction is the inverse of addition, that $3 - 2$ is that number which when added to 2 yields 3, that $2 - 2 = 0$ is mathematically equivalent to $2 + 0 = 2$. We could take all the counting numbers together with zero as a new number system. That is what happens about Grade II or III in the elementary schools. Of course 0 is a very useful number. It appears frequently in inventory lists, balance-on-hand entries, in teachers' grade books, and so on. The other numbers in the system of all integers are useful, too. Think of the stock-market report where the net change in selling price during a 24 hour period is reported as a positive number or a negative number, or the weather reports during a cold snap when many of the cities have had subzero temperatures, or a college grade point system in which $F = -1$, $A = +3$, and so on. But we must move along to a consideration of the integers as a system.

The system of all integers is a set of numbers $\cdots, -3, -2, -1, 0, +1, +2, +3, \cdots$ together with the fundamental operations of addition, subtraction, multiplication, and division. The Grade IX algebra student learns to perform these operations on signed numbers. He knows that $(+2) + (+3) = +5$, $(+2) + (-3) = -1$, $(+2) - (+3) = -1$, $(+2) - (-3) = +5$, etc. The system is closed with respect to addition, multiplication, and subtraction, but it is not closed with respect to division. Subtraction is not always possible in the system of natural numbers, but, in the system of all integers, subtraction is always possible.

The portrait of 2 reveals +2 and −2 as elements of the system of all integers. In this environment we see that +2 and −2 have many representations as composite symbols, for example:

$$+2 = (-15) - (-17) = (-1) - (-3) = 0 - (-2),$$

$$+2 = +1 - (-1) = (+2) - 0 = (+3) - (+1) = \frac{+2}{+1} = \frac{-16}{-8},$$

$$+2 = (-15) + (+17) = (-1) + (+3) = (-1) \times (-2),$$

$$-2 = (-17) - (-15) = (-3) - (-1) = (-2) - 0 = (+6) - (+8),$$

$$-2 = (-1) \times (+2) = \frac{+14}{-7} = \frac{-2}{+1}.$$

Do not be confused by the dual uses of the symbols + and − when working in this system. The symbol may be a part of a composite symbol denoting a signed integer, as in +2 or −2, or it may denote an operation as the midde + sign in the following: (+2) + (+2). Assuming that we have a pretty good picture of the system of all integers let us consider now the relationship of this system to the system of natural numbers. The ease with which a ninth-grader learns to add and multiply in the system of all integers indicates that this relationship is a close one. He does not need to learn a lot of new addition facts or a new multiplication table; all he needs is a rule of signs and a knowledge of addition and multiplication in the system of natural numbers. Thus our ninth-grader might say that (+2) + (+3) = +5 because 2 + 3 = 5. In his mind he is identifying consciously or unconsciously 2 and +2 as the same number. But the portrait of 2 does not reveal the natural number 2 as being identical with the integer +2. What does it reveal? It reveals the integer as a creation of the human mind, as something created from the natural numbers, as something involving the concepts of ordered pair and class of ordered pairs. Intuitively the signed integer −2 is created to give meaning to the composite symbols 1 − 3, 2 − 4, 3 − 5, $\cdots$. You might guess then that −2 is defined as the class of all such symbols, all symbols $a - b$ in which a and b are natural numbers and $b - a = 2$. But the portrait of 2 shows something which has proved to be much better than this, but which has this idea as its basis.

Ordered pairs of numbers appear in elementary work on graphing equations in two letters, say x and y. A point is given in terms of its coordinates as an ordered pair of numbers, such as (3,5). The order of the numbers is significant; (3,5) is different from (5,3). Ordered pairs of numbers are indeed important as coordinates in algebra and in analytic geometry. They are also important in the modern treatments of the nature and structure of number. For our purpose we define an ordered pair of numbers as a composite symbol (a,b) in which a and b are symbols denoting numbers. Thus (7,5), (5,7), (5,10), and $(-17,0)$ are ordered pairs of numbers. We consider (7,5) and (5,7) as different ordered pairs.

Consider then the following ordered pairs of natural numbers: (5,3), (8,6), (17,15), (3,1), (151,149). What common property seems apparent here? Note that $5 - 3 = 8 - 6 = 17 - 15 = 3 - 1 = 151 - 149 = 2$. The portrait of 2 reveals the integer $+2$ as the class of all ordered pairs of natural numbers (a,b) in which $a - b = 2$ and -2 as the class of all ordered pairs of natural numbers (a, b) in which $b - a = 2$. In studying the system of integers, we prefer (5,3) or (7,5) or (100,98) rather than $+2$ as a symbol for the integer, positive two. We define $+2$ to be a certain class of ordered pairs and then we use any element of the class as a symbol for the class. We define equality, consequently, so that $(5,3) = (7,5) = (100,98)$ etc. The formal definition: *If a, b, c, d are symbols denoting natural numbers and if (a,b), (c,d) are symbols denoting integers, then $(a,b) = (c,d)$ if $a - b = c - d$ or $b - a = d - c$.* Thus $(5,3) = (7,5)$ in the system of integers, since $5 - 3 = 7 - 5$ in the system of natural numbers; $(3,10) = (8,15)$ in the system of integers, since $10 - 3 = 15 - 8$ in the system of natural numbers. The other integers are defined in a manner similar to that for $+2$ and -2 above. Thus if n is any symbol denoting a natural number, then $+n$ and $-n$ are symbols denoting integers ($+n$ a positive integer and $-n$ a negative integer): $+n$ is the class of all ordered pairs of natural numbers (a,b) where $a - b = n$ and $-n$ is the class of all ordered pairs of natural numbers (a,b) where $b - a = n$. Thus $+7 = (8,1) = (9,2) = (10,3) = (11,4) = \cdots; -7 = (1,8) = (2,9) = (3,10) = \cdots$. For completeness we define the integer 0 as the class of all symbols (a,a) in which a is a natural number. Thus $0 = (1,1) = (2,2) = (3,3) = \cdots$.

BOSS: What was the net change in Cities Service?

SECRETARY: +2.

BOSS: +2?

SECRETARY: Yes, you know, the class of all ordered pairs of natural numbers (a,b) in which a and b have the property that $a - b = 2$.

We now define addition and subtraction in the system of integers.

Definition.

$$(a,b) + (c,d) = (a + c, b + d),$$
$$(a,b) - (c,d) = (a + d, b + c).$$

Let us see if these definitions "work." To add $+5$ and -3 we first change to ordered pair of natural numbers representations. Thus $+5 = (6,1)$, $-3 = (2,5)$. Then $(+5) + (-3) = (6,1) + (2,5)$. We look at the definition and we think: $a = 6$, $b = 1$, $c = 2$, $d = 5$, $a + c = 8$, $b + d = 6$, $(a + c, b + d) = (8,6)$. We write: $(+5) + (-3) = (6,1) + (2,5) = (8,6) = +2$.

Similarly, $(+5) - (-3) = (6,1) - (1,4) = (10,2) = +8$, (-3) $(+5) = (1,4) - (6,1) = (2,10) = -8$.

Is the system of all integers closed with respect to subtraction? Yes, just look at the definition again, $(a,b) - (c,d) = (a + d, b + c)$. Since a, b, c, d denote natural numbers and since the system of natural numbers is closed with respect to addition, it follows that $a + d$ and $b + c$ are natural numbers, and hence that $(a + d, b + c)$ is an integer. We could go into great detail studying all the operations in this system. But we do not want too many details; we just want enough to get some insight into the nature of 2 as revealed in the portrait. We conclude the discussion of this system, then, by proving one theorem in the theory of integers.

Theorem: *Addition of integers is commutative.* (This means that the result of adding two integers is independent of the order in which they are added.)

Proof: If we use our definition of addition,

$$(a,b) + (c,d) = (a + c, b + d), \text{ and}$$

$$(c,d) + (a,b) = (c + a, d + b).$$

Since addition is commutative in the system of natural numbers, we have $a + c = c + a$, $b + d = d + b$. From our definition of equality in the system of integers it follows that $(a + c, b + d) = (c + a, b + d)$, which completes the proof.

JOHN: Do you understand that proof, Alexander?

ALEXANDER: Why sure. I'll show you how it works in a special case. Suppose you wish to add $+7$ and -11. Convert to the ordered pair representations: $+7 = (8,1)$, $-11 = (1,12)$. Add in one order and then in the other order:

$$(+7) + (-11) = (8,1) + (1,12) = (9,13) = -4,$$

$$(-11) + (+7) = (1,12) + (8,1) = (9,13) = -4.$$

Since $8 + 1 = 1 + 8 = 9$ and $1 + 12 = 12 + 1 = 13$ in the natural number system, we see that $(8,1) + (1,12) = (1,12) + (8,1)$, that is, $(+7) + (-11) = (-11) + (+7) = -4$.

So we have seen that the integers can be created from the natural numbers using the concept of classes of ordered pairs, and we have indicated that the fundamental operations and their properties for integers can be based rigorously upon the properties of the fundamental operations in the system of natural numbers. In particular, the portrait of 2 reveals +2 as the class of all ordered pairs $(a + 2, a)$, and -2 as the class of all ordered pairs $(a, a + 2)$, in which a denotes a natural number.

We return now to the notion that the natural number 2 and the integer +2 are essentially the same. We accused the typical ninth-grader of having such an idea. Perhaps some of us had such an idea before we started gazing at the portrait of 2. Of course, 2 and +2 are elements of two different number systems. Every integer is a class of ordered pairs of natural numbers. So no integer is the same as a natural number. But some of us may insist that 2 and +2 are alike; at least our intuition says they are alike. How is this likeness reflected in the portrait of 2? To answer this question we must introduce a very sophisticated concept, namely the concept of isomorphism. The structure of the word isomorphism indicates that it ought to mean the property of having the same form. What ought to have the same form as what? We speak of an isomorphism as existing between the elements of one number system and the elements of another number system, or between some of the numbers in one system and all the numbers in another system. We say the systems are isomorphic, or that a part of one system is isomorphic to the other system. But what does it mean?

Definition. A number system S_1 with elements $a, b, c, \cdots$ is isomorphic to a number system S_2 with elements $A, B, C, \cdots$ if there exists a one-to-one correspondence between the elements of S_1 and the elements of S_2 (as suggested by the symbol $a \leftrightarrow A$, $b \leftrightarrow B$, $c \leftrightarrow C$, etc.) which has the property that if a and b are any elements in S_1 and A and B are their mates in S_2 then $a + b \leftrightarrow A + B$, and $a \cdot b \leftrightarrow A \cdot B$.

Thus, if S_1 and S_2 are isomorphic, one might "translate" from one system to the other to perform the fundamental operations. Thus to add a and b in S_1, find the mate of a and the mate of b in S_2, call them A and B respectively. Add A and B in S_2 to get $A + B$. Find the mate of $A + B$ in S_1. This number is $a + b$.

As an example consider two systems A and B of "numbers." Suppose that A consists of the elements a and b, that B consists of the elements α and β, and that addition and multiplication are defined as follows:

$$a + a = a, a + b = b, b + a = b, b + b = a,$$

$$\alpha + \alpha = \alpha, \alpha + \beta = \beta, \beta + \alpha = \beta, \beta + \beta = \alpha,$$

$$a \cdot a = a, a \cdot b = a, b \cdot a = a, b \cdot b = b,$$

$$\alpha \cdot \alpha = \alpha, \alpha \cdot \beta = \alpha, \beta \cdot \alpha = \alpha, \beta \cdot \beta = \beta.$$

We propose to show that the systems A and B are isomorphic. We mate a to α, b to β: $a \leftrightarrow \alpha$, $b \leftrightarrow \beta$. This establishes a one-to-one correspondence between the elements of A and the elements of B. We shall show that this correspondence satisfies the requirement as stated in the definition of isomorphic. Consider the pair a,a and their mates α,α. Note that $a + a = a$, $\alpha + \alpha = \alpha$ and that $a \leftrightarrow \alpha$. Also note that $a \cdot a = a$, $\alpha \cdot \alpha = \alpha$, and that $a \leftrightarrow \alpha$. Consider next the pair a,b and their mates α,β. Note that $a + b = b$, $\alpha + \beta = \beta$, $a \cdot b = a$, $\alpha \cdot \beta = \alpha$. Adding a and b yields b; adding the mates of a and b yields the mate of b. Multiplying a and b yields a; multiplying the mates of a and b yields the mate of a. Similarly we check the pair b,a and their mates β,α, and the pair b,b and their mates β,β. If we add (multiply) any two elements of A (not necessarily distinct elements) and if we add (multiply) the mates of these two elements in B, we find that the two sums (products) are mates. This completes the proof that the systems A and B are isomorphic.

Let S_1 denote the system of all natural numbers and S_2 the system of all integers. Let S_3 be the number system whose elements are all the positive integers in S_2 and in which the fundamental operations are defined as they are in S_2, provided the result is a number in S_3. Then S_3 is the system consisting of the numbers $+1, +2, +3, \cdots$, and these numbers are added, subtracted, multiplied, and divided just as they are in S_2. Now we assert that the system S_1 is isomorphic to the system S_3. For let us mate elements as follows: $1 \leftrightarrow +1$, $2 \leftrightarrow +2$, $3 \leftrightarrow +3, \cdots$, $n \leftrightarrow +n, \cdots$. Then every element in S_1 has a unique mate in S_3 and vice versa; that is, this mating constitutes a one-to-one correspondence between the elements of S_1 and the elements of S_3. Also, we note that if n and m are any two natural numbers, then $n \leftrightarrow +n$, $m \leftrightarrow +m$, $(n + m) \leftrightarrow +(n + m)$, $(n \cdot m) \leftrightarrow +(n \cdot m)$. But addition and multiplication are defined in S_3 so that $+(n + m) = (+n) + (+m)$, $+(n \cdot m) = (+n) \cdot (+m)$. Therefore, $(n + m) \leftrightarrow (+n) + (+m)$, $(n \cdot m) \leftrightarrow (+n) \cdot (+m)$, and the one-to-one correspondence is an isomorphism. Structurally the systems S_3 and S_1 are the "same." In S_1 the sum of 3 and 5 is 8; the product of 3 and 5 is 15. In S_3 the sum of $+3$ and $+5$ is $+8$; the product of $+3$ and $+5$ is $+15$. In conclusion then, 2 and $+2$ are united, united as mates, in the isomorphism which exists between the system of natural numbers and the system of positive integers. We accused the ninth-grader of thinking that 2 and $+2$ are the same thing. Really we should not have accused him at all. Most textbooks and most teachers say the same thing. A textbook has this exercise: $3 + (-4) = ?$ Here is an indicated addition of a natural number 3 and an integer -4. Does this have meaning? No, not on the basis of our theories of the systems of natural numbers and the system of integers. But let us not fight the problem. Let us not be difficult. Let us admit that most people study and use mathematics because it is a useful tool. And if the people who use mathematics say that $3 + (-4) = -1$, we shall agree with them. But we shall explain it to our mathematics friends by saying that they mean $(+3) + (-4) = -1$. So the portrait of 2 is clouded by the practical man who uses 2 as a symbol for the natural number 2 and also as a symbol for the integer $+2$. This does not bother the mathematics student; the picture for him is made clear by a concept called isomorphism.

TEACHER: Add XVIII and XXIV.
JIM: XVIII ↔ 18, XXIV ↔ 24
$18 + 24 = 42$, 42 ↔ XLII. Therefore,
XVIII + XXIV = XLII.

TEACHER: Add $+2$ and $+3$.
MARY: $+2 \leftrightarrow 2$, $+3 \leftrightarrow 3$, $2 + 3 = 5$, $5 \leftrightarrow +5$, $(+2) + (+3) = +5$.

V. The Rational Number 2

So far we have seen 2 as a numeral, as an element of the system of natural numbers, and as an element of the system of integers. We next see 2 as an element of the system of all rational numbers. Later we shall construct the rationals from the integers in terms of classes of ord red pairs—a procedure similar to the one we used to create the integers from the natural numbers. But right now, let us think about rational numbers as they are thought of by most people who use mathematics as a tool, that is, as "simple fractions." But what is simple? What is a fraction? Does a rational number need to be a fraction? Is $\frac{2}{2}$ a simple fraction? Is $\frac{5}{2/3}$ a simple fraction? Of course 2 is not a fraction. But the portrait of 2 reveals 2 as a rational number. So where are we? In the first place a fraction is something with a numerator and a denominator; it is a composite symbol denoting a number, the component parts of this composite symbol being two symbols for numbers and one symbol indicating the operation of division; it is an indicated division or—it is a symbol for the number which results when the numerator is divided by the denominator. So a fraction is a symbol which has a certain form. Actually any number whatsoever may be written in the form of a fraction. For if x denotes any number in any system discussed in this booklet, then $\frac{x}{1}$ also denotes this same number. Well, a rational number may or may not be denoted by a fraction. An elementary definition is: *A rational number is a number which can be written as the quotient of two integers.* Thus we think of $\frac{2}{3}$, $\frac{-3}{4}$, $\frac{10}{12}$, $\frac{2}{1}$, and $\frac{0}{5}$ as symbols denoting rational numbers; each of them is a symbol indicating the quotient of two integers. But 0.5, -3, $\frac{\sqrt{2}}{3\sqrt{2}}$, 3.1416, and $2\frac{1}{2}$ are also rational numbers. For $0.5 = \frac{1}{2}$, $-3 = \frac{-3}{1}$, $\frac{\sqrt{2}}{3\sqrt{2}} = \frac{1}{3}$, $3.1416 = \frac{31416}{10000}$, and $2\frac{1}{2} = \frac{5}{2}$. In this elementary sense, then, 2 is a rational number. For $2 = \frac{2}{1} = \frac{4}{2}$ from our knowledge of the system of integers.

Why do we have rational numbers? Are they important? Of course we can think of situations in which it is desirable and convenient to use rational numbers: $\frac{1}{2}$ an apple, the farm owner and his tenant sharing on a $\frac{2}{5}$ and $\frac{3}{5}$ plan, a weight of $3\frac{1}{4}$ lb., a tolerance of 0.0001 inch. What can we do in the system of rational numbers that we cannot do in the system of integers? Both systems are closed with respect to addition, subtraction, and multiplication. The system of integers is not closed with respect to division; the system of rational numbers is closed with respect to division with the exception that division by 0 is impossible. Thus $(+2) \div (-3)$ is meaningless in the system of integers, but $(+2) \div (-3) = \frac{-2}{3}$, a rational number. Working in the system of rationals we can solve equations that have no solutions in the system of integers. For example, $2x = 5$ cannot be solved in the system of integers; in the system of rationals it has the root $\frac{5}{2}$. Thus we need the rational numbers to give us freedom in performing the fundamental operations. And we see 2 in an environment of rational numbers. We see $2 + \frac{1}{3}$ as an indicated sum of two rational numbers; we see $2 - \frac{1}{3}$ as a composite symbol for a rational number. We see 2 as the rational number which results when $\frac{7}{8}$ is divided by $\frac{7}{16}$, or as $3 \div \frac{3}{2}$, or as $\frac{7}{12} \cdot \frac{24}{7}$. We have a feeling that this rational 2 is closely related to the integer +2 and the natural number 2. We suspect that there is another isomorphism lurking in the shadows, and we are right. But we must reveal the rational number in much sharper outlines before we are ready to explore the isomorphism. This brings us to the high point in the theory of the rational number—the creation of the rational number as a class of ordered pairs of integers.

The student of elementary algebra knows that (*) $\frac{4}{2} = \frac{-6}{-3} = \frac{8}{4} = \frac{16}{8} = \frac{14}{7}$ and that (**) $\frac{2}{4} = \frac{-3}{-6} = \frac{4}{8} = \frac{8}{16} = \frac{7}{14}$. On the basis of the development in this booklet and an interpretation of $\frac{a}{b}$ as an alternate way of writing $a \div b$, we note that the symbols in the set (*) of the preceding sentence are meaningful in the system of integers. Each of the five composite symbols in that set is a symbol for the integer, positive two. On the other hand, the symbols in the set (**) are meaningless in the system of integers. For 2 cannot be divided by 4, −3 cannot be divided by −6, etc., in the system of integers. In creating the rational numbers we give meaning to the symbols in the set (**), and simultaneously we give new meaning to the symbols in the set (*).

We could define the class of all symbols $\frac{a}{b}$ in which a and b are integers and $b = 2a$ to be the rational number $\frac{1}{2}$. (This same rational number might also be denoted by $\frac{2}{4}$ or $\frac{-3}{-6}$ or any other symbol in the class.) Similarly, then, the class of all symbols $\frac{a}{b}$ in which a and b are integers and $a = 2b$ should be the rational number 2. Some confusion might result if we defined rational numbers in this manner. Should we say that $\frac{6}{3}$ is a symbol for the rational number 2 or a symbol for the integer +2 or a symbol for the natural number 2? The portrait of 2 avoids this confusion by defining a rational number as a class of ordered pairs of integers. And, although it does not appear generally in the classical literature on number systems, we shall use brackets in writing these ordered pair symbols. We do this to eliminate any confusion which might arise due to our definition of an integer in terms of ordered pairs of natural numbers.

Definition. The rational number 2 is the class of all ordered pairs of integers [a,b] in which a and b are integers, $b \neq 0$ and $a = 2b$. (The symbol $\neq$ is read *is not equal to.*)

Any one of these pairs, as [2,1], [6,3], or [−8,−4], is a symbol denoting the rational number 2. In our formal development of the rational number system we prefer this ordered pair symbol. Later we shall use again the ordinary symbols of the scientist and the engineer.

We have defined the rational number 2. Let us now define rational numbers generally. *If a and b are any two integers and if $b \neq 0$, then [a,b] is a symbol for a rational number. This rational number is the class of all ordered pairs of integers [c,d] in which $d \neq 0$ and $ad = bc$.* And any one of these symbols [c,d] denotes the same rational number as is denoted by the symbol [a,b]. Implicit in this definition of rational number is the following explicit definition of equality for rational numbers: $[a,b] = [c,d]$ *if and only if* $ad = bc$. Thus [3,4] and [6,8] are symbols denoting the same rational number $\left(\text{the fifth-grader writes it as } \frac{3}{4}\right)$ since $3 \cdot 8 = 4 \cdot 6$. Similarly, [+3,−2] and [−75,+50] are ordered pair symbols for the rational number which the ninth-grader writes as $-\frac{3}{2}$.

We proceed to the definitions of the fundamental operations in the rational number system:

$$[a,b] + [c,d] = [ad + bc, bd],$$

$$[a,b] - [c,d] = [ad - bc, bd],$$

$$[a,b] \cdot [c,d] = [ac, bd],$$

$$[a,b] \div [c,d] = [ad, bc].$$

In the symbols $[a,b]$, $[c,d]$, the letters a, b, c, d are symbols denoting integers and it is understood that $b \neq 0$, $d \neq 0$. (This understanding follows from our definition of rational number.) Take a look at the symbols on the right-hand sides of the four equations in the definition of the fundamental operations. Are they meaningful? Is each of them a symbol for a rational number? Note that each of them is an ordered pair of integers, for, since the system of integers is closed with respect to addition, multiplication, and subtraction, it follows that $ad + bc$, $ad - bc$, ac, ad, bc, and bd are six composite symbols denoting integers. One other point needs to be checked. In an ordered pair symbol for a rational number, the second integer in the ordered pair must not be 0. In the definition of addition, subtraction, and multiplication, the second integer in the ordered pair symbol is bd. As stated above, $b \neq 0$, $d \neq 0$. From our knowledge of the system of integers we know then that $bd \neq 0$. It follows that addition, subtraction, and multiplication are always possible in the system of rational numbers. More precisely, the system of rational numbers is closed with respect to these three operations. How about division? Is the system of rational numbers closed with respect to division? Let us look at the definition again. $[a,b] \div [c,d] = [ad, bc]$. It is understood that $b \neq 0$, $d \neq 0$. Does this insure that $bc \neq 0$? No, it does not. For if $c = 0$, then $bc = 0$. And, if $bc = 0$, then $[ad, bc]$ is not a rational number. Now, if $c = 0$, then $[c,d]$ is a symbol for the rational number 0. On the other hand, if $c \neq 0$, then $[c,d]$ is not the rational number 0 and the division $[a,b] \div [c,d]$ is defined. For, if $c \neq 0$, then $bc \neq 0$ and $[ad, bc]$ is a symbol for a rational number. We conclude that in the system of rational numbers division is always possible with one exception—division by 0 is impossible. Thus the system of rational numbers is closed with respect to all four of the fundamental operations with the exception that division by 0 is impossible.

Let us try our definition of the fundamental operations on several examples.

$$\frac{1}{4} + \frac{2}{3} = [1,4] + [2,3] = [1\cdot3 + 4\cdot2, 4\cdot3] = [11,12] = \frac{11}{12},$$

$$\frac{1}{4} - \frac{2}{3} = [1,4] - [2,3] = [1\cdot3 - 4\cdot2, 4\cdot3] = [-5,12] = \frac{-5}{12},$$

$$\frac{1}{4}\cdot\frac{2}{3} = [1,4]\cdot[2,3] = [1\cdot2, 4\cdot3] = [2,12] = [1,6] = \frac{1}{6},$$

$$\frac{1}{4} \div \frac{2}{3} = [1,4] \div [2,3] = [1\cdot3, 4\cdot2] = [3,8] = \frac{3}{8},$$

$$\frac{5}{6} \div 2 = [5,6] \div [2,1] = [5\cdot1, 6\cdot2] = [5,12] = \frac{5}{12}.$$

Does rational 2 plus rational 2 yield rational 4? Using [2,1] as a symbol for the rational number 2 and using our definition of addition we have $[2,1] + [2,1] = [2\cdot1 + 1\cdot2, 1\cdot1] = [4,1]$; and [4,1] is a symbol denoting the rational number 4. How about a rational 1? Yes, we have one; [1,1] is an ordered pair symbol for the rational 1. Note that, according to our definition of multiplication, $[a,b]\cdot[1,1] = [a\cdot1, b\cdot1] = [a,b]$. This proves that the product of any rational number r and the rational number 1 is the rational number r. Similarly the zero element in the rational number system is [0,1]. Thus $[a,b] + [0,1] = [a\cdot1 + b\cdot0, b\cdot1] = [a + 0, b] = [a,b]$ and $[a,b]\cdot[0,1] = [a\cdot0, b\cdot1] = [0,b] = [0,1]$; we add $[a,b]$ and our zero [0,1] and we get $[a,b]$; we multiply $[a,b]$ by our zero [0,1] and we get [0,1]. Let us prove a theorem.

Theorem. Addition of rational numbers is commutative. (This means that the result of adding two rational numbers is independent of the order in which they are added.)

Proof: Let a, b, c, d denote integers and the ordered pair symbols denote rational numbers. Then from our definition of addition of rational numbers we have:

$$[a,b] + [c,d] = [ad + bc, bd]$$

$$[c,d] + [a,b] = [cb + da, db].$$

But $ad + bc = cb + da$ and $bd = db$ from our knowledge of the system of integers. Therefore $[ad + bc, bd] = [cb + da, db]$ and $[a,b] + [c,d] = [c,d] + [a,b]$.

We have had a glimpse of the rational number system; we have seen 2 as a rational number, as a class of ordered pairs of integers; and we have indicated that the fundamental operations as applied to rational numbers may be defined and studied in terms of the ordered pair notation. All this appears in our portrait of the rational number 2. And, finally, as we promised at the beginning of our discussion of rational numbers, we note that the portrait of 2 identifies the rational number 2 with the integer 2 using the concept of isomorphism. Thus the mathematician and the engineer clasp hands again.

Here is the isomorphism. Let S_1 denote the system of all integers and S_2 the system of all rational numbers. Let S_3 be the system of all rational numbers each of which, in the ordered pair of integers notation, can be denoted by a symbol $[a,1]$. In S_3 the fundamental operations are as they are in S_2. Thus S_3 has the elements $\cdots, [-3,1], [-2,1], [-1,1], [0,1], [1,1], [2,1], [3,1], \cdots$. We mate these elements to the elements of the system S_1 of all integers as follows: $\cdots, [-3,1] \leftrightarrow -3, [-2,1] \leftrightarrow -2, [-1,1] \leftrightarrow -1, [0,1] \leftrightarrow 0, [1,1] \leftrightarrow 1, [2,1] \leftrightarrow 2, [3,1] \leftrightarrow 3, \cdots$. (We are using the unsigned symbols 1, 2, 3 to denote integers here, not natural numbers.) In general, if n is any integer, then we mate n with $[n,1]$. This establishes a one-to-one correspondence between the elements of S_1 and the elements of S_3. We assert that it is an isomorphism. Let a and b be any two integers. Then $a \leftrightarrow [a,1]$, $b \leftrightarrow [b,1]$, $a + b \leftrightarrow [a + b,1]$, $a \cdot b \leftrightarrow [a \cdot b,1]$. In order for our correspondence to be an isomorphism we should have $a + b \leftrightarrow [a,1] + [b,1]$ and $a \cdot b \leftrightarrow [a,1] \cdot [b,1]$. This we shall have if $[a,1] + [b,1] = [a + b,1]$ and $[a,1] \cdot [b,1] = [a \cdot b,1]$. This we do have in view of our definitions of addition and multiplication. Let us take an example. Suppose we wish to add the integers -5 and $+3$. Let us translate to the system of rationals, then add, and then translate back. We should get -2. Let us see. $-5 \leftrightarrow [-5,1]$, $+3 \leftrightarrow [3,1]$, $[-5,1] + [3,1] = [-5 \cdot 1 + 1 \cdot 3, 1 \cdot 1] = [-5 + 3,1] = [-2,1]$, $[-2,1] \leftrightarrow -2$. As another example suppose we wish to multiply: $[-5,1] \cdot [3,1]$. The result should be $[-15,1]$ according to our rule for multiplication. Let us see if we can translate to integers, then multiply, and then translate back. $[-5,1] \leftrightarrow -5$, $[3,1] \leftrightarrow 3$, $-5 \cdot 3 = -15$, $-15 \leftrightarrow [-15,1]$.

The portrait of 2 reveals the rational number 2 as something different from the integer positive 2. The mind of man has created the rational number 2 as a class of ordered pairs of integers. But the same mind has produced the isomorphism which reveals in its simplest form the relationship of the fundamental operations in the two systems. What does $2 + \frac{1}{2}$ mean? Are we adding an integer and a rational number? The easiest explanation is that $2 + \frac{1}{2}$ means rational 2 plus rational $\frac{1}{2}$, that the symbol 2 in this context must denote rational 2. (If we were to give a rule for adding an integer and a rational number, it would probably be: $a + [b,c] = [a,1] + [b,c] = [ac + b,c]$. Note how a is replaced by

$[a,1]$ and recall that $a \leftrightarrow [a,1]$ in the isomorphism.) Finally, what does $2 + 2 = 4$ mean? Does it indicate an addition of natural numbers? of positive integers? or of rational numbers? It could indicate any one of these. It means what we want it to mean. And, if we want it to mean a particular one of these three, then we shall have to rely on the context or we shall have to write something more definite, perhaps an added remark or perhaps different notation.

Our concept of 2 as a number is growing. It started as a natural number; now it is a natural number or an integer or a rational number. The natural numbers are isomorphic to the positive integers. The integers are isomorphic to the rational numbers $[a,1]$, in which a is an integer. One way to view the growth of the number concept is to see: natural number, positive integer, all integers, rational numbers $[a,1]$ with a an integer, then all rational numbers. We have seen 2 in each stage of this development; we have seen it in our ever-increasing environment of other numbers. And we have seen the isomorphisms which relate the 2's in the different systems. In symbols we have $2 \leftrightarrow 2 \leftrightarrow 2$ or $2 \leftrightarrow +2 \leftrightarrow [2,1]$, meaning (natural 2) $\leftrightarrow$ (integer 2) $\leftrightarrow$ (rational 2). Two more links on this chain of matings will conclude our portrait. We shall see 2 as an element of the system of real numbers, and finally we shall see 2 as an element of the system of complex numbers. Before proceeding with our discussion of the real number 2, however, we wish to insert at this point some remarks on the approximate or measurement number 2.

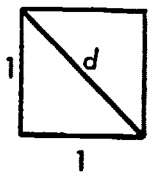
VI. The Measurement Number 2

When we count a finite set the result is a natural number; when we measure something by counting the number of times a standard unit is contained in the quantity being measured, we frequently find that the result is not a natural number. Laying off a yard stick along the edge of a room we may find that its length is more than 5 yards and less than 6 yards. We need a number between 5 and 6. For measuring purposes we need many numbers between 5 and 6; we need the rational numbers. Now measurement numbers are not exact numbers. When we say that the diameter of a piston is 3.564 centimeters, we mean that the diameter is between 3.5635 cm. and 3.5645 cm.—we mean that 3.564 cm. is the diameter to the nearest .001 of a centimeter. In contrast to exact numbers (the natural numbers or integers which result from the operation of counting) measurement numbers are sometimes called approximate numbers. As we gaze at the portrait of 2, then, we see 2 as a measurement number—as an approximate number. We see such measurements as 2 inches, 2 gallons, 2 kilowatts, 2 grams, 2 seconds. But we also see such measurements as 2.0 miles, 2.00 kilograms, and 2.000 cubic feet, in which the zeros are *significant*; they indicate the accuracy of 2 as a measurement number—nearest tenth, nearest hundredth, or nearest thousandth.

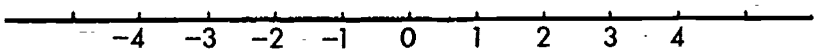
VII. The Real Number 2

The real number 2, as revealed by its modern portrait, is a creation of the human mind. You might guess that modern man has created the real numbers from the rational numbers, that a subset of the real numbers is isomorphic to the set of all rational numbers, and that in this isomorphism, (real 2) ↔ (rational 2). But before we look at the modern creation, let us consider the need for real numbers; let us see how real numbers are used.

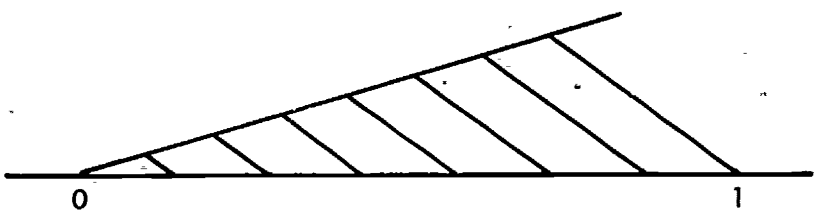
The system of rational numbers is closed with respect to addition, subtraction, multiplication, and division (except that it is impossible to divide by 0). As we have explained before, this means that these operations can be performed in this system; it means that the sum of every pair of rational numbers is a rational number; likewise the difference, the product, and the quotient of two rational numbers are rational numbers (except that 0 cannot be the divisor in a quotient). The system seems to be complete from a mathematical point of view. What else would we like to do with numbers besides add them, subtract them, multiply them, and divide them? Perhaps some of us are thinking: What about the restriction of not dividing by 0? Well, the modern mathematician does remove this restriction in some situations, in some theories. But that is another story. Division by 0 is impossible in any of the elementary number systems, the ones which we are discussing here. In the elementary number systems $0 \cdot a = 0$ for every a in the system, and $a \div b = c$ means $a = bc$. Multiplication comes first; then division is defined as the inverse operation. Note that $10 \cdot 3 = 30$ and $30 \div 3 = 10$; thus $(10 \cdot 3) \div 3 = 10$. Also $(10 \div 3) \cdot 3 = 10$. The one "undoes" the other. Now suppose division by 0 were possible. In other words suppose $a \div 0 = b$ where a and b are numbers in some system. We should then have $a = 0 \cdot b = 0$, and thus $a = 0$; that is, $a \div 0 = b$ is impossible if $a \neq 0$. But suppose $\frac{0}{0} = b$. This is all right from the standpoint that division is the inverse of multiplication provided b is such a number that $0 = 0 \cdot b$. But this is true regardless of the value of b. Hence $0 \div 0$ could be any number. But the result of adding two rational numbers is unique; there is just one answer. Similarly there is just one answer if two numbers are subtracted, or multiplied, or divided (with divisor different from 0). For this and other reasons mathematicians have long agreed that $0 \div 0$ is indeterminate or meaningless. The point here is that the possibility or impossibility of dividing by 0 has nothing to do with the need for creating real numbers. This brings us back to the question of what defects, if any, are present in the rational number system.



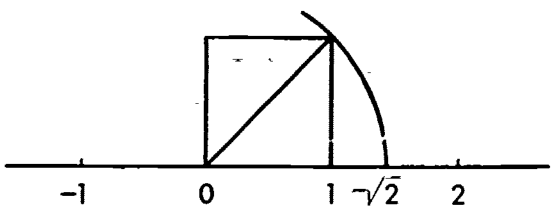
Consider the following problem in elementary geometry: find the length of the diagonal of a square of side length 1 unit. To be definite, let us suppose that the length of each side is exactly 1 unit and that each angle is exactly 90°. If we call the length of the diagonal d and if we recall a theorem due to Pythagoras, we conclude that $d^2 = 2$. (The theorem states that the square of the hypotenuse of a right triangle is the sum of the squares of the legs.) So what is the value of d? We say that d is the square root of 2; we write $d = \sqrt{2}$. But suppose the only numbers are rational numbers. We say d is approximately 1.41; 1.41^2 is a little less than 2 and 1.42^2 is a little greater than 2. But what is d exactly? Is there a rational number whose square is 2? The answer is no. To show this, suppose there were a rational number $[a,b]$, (a and b are integers), such that $[a,b]\cdot[a,b] = [2,1]$. We may assume that the symbol for the rational number $[a,b]$ is such that a and b have no common positive integral (whole number) divisors except 1. (We are assuming that the fraction $\frac{a}{b}$ is in lowest terms.) Then, from our definition of multiplication in the system of rational numbers, we have $[a^2,b^2] = [2,1]$ and from our definition of equality (really from our definition that a rational number is a class of symbols related in a certain way) we have $a^2 = 2b^2$. Now we may transplant ourselves into the system of integers. We note that $2b^2$ is an even number. Hence a^2 is even. Hence a is even (for the square of an odd integer is odd). Hence there is an integer c such that $a = 2c$. Hence $(2c)^2 = 2b^2$. Hence $4c^2 = 2b^2$. Hence $2c^2 = b^2$. Hence b^2 is even. Hence b is even. Hence a and b have the common divisor 2. But we started with a and b having no such common divisor. Hence the assumption that d is rational leads by logical reasoning to a pair of integers which do not have 2 as a common divisor and which also do have 2 as a common divisor. The only way to avoid this dilemma is to conclude that the assumption is false—to conclude that d is not a rational number. But we need a number d whose square is 2. The ancient geometers needed it; they lost a lot of sleep worrying about the fact that there was no such number; they thought this was a flaw in their otherwise beautiful theory. The modern mathematician does not worry about a need for a square root of 2. He knows there is no rational number whose square is 2. He needs this square root, however, so he creates it. The thing created is not a rational number; it is a real number. Later we shall look at this creation structurally; we shall see how 2 is made from rational numbers in several modern theories. Right now we recognize the existence of this number and the symbol commonly used for it, $\sqrt{2}$. The important thing about this number right now is that $(\sqrt{2})^2 = 2$. Now we can solve the equation $x^2 = 2$ and get an exact answer; $(\sqrt{2})^2 = 2$ exactly!



A useful mathematical device is the number scale—numbers corresponding to points along a line in a very familiar way. The numder scale we wish to think about is one which exists only in the mind. We start with a straight line (perfect, you know) which we think of as running from left to right. An arbitrary point on it is labelled 0; a point to the right of it is labelled 1. Assuming now that we can lay off equal distances, we use the segment from 0 to 1 to mark the successive integers in both directions. Going to the right from 0 we have 1, 2, 3, $\cdots$; to the left we have $-1, -2, -3, \cdots$. In the mind every integer appears as the label of some point on the line. Next we label all the points which correspond to rational numbers. Where does the label [47,7] go? Well, [47,7] as a symbol for a certain rational number has served its purpose. So we shall use the more familiar symbols $\frac{47}{7}$ or $6\frac{5}{7}$. We may recall from elementary geometry a ruler and compass construction for dividing a



segment into 7 equal parts. Having divided the segment from 0 to 1 into 7 equal parts, we lay off segments of length $\frac{1}{7}$ and label eventually not only $6\frac{5}{7}$ but $1000\frac{6}{7}$, $-2\frac{3}{7}$, and so on. So now we imagine that all the rational numbers have found their places as labels for points on our number scale. The big question is: Have all the points on the line been labelled? It seems logical that the answer might depend upon our notion of line, of point, of the relationship between points and lines. We did not say anything that would help us here when we said that a straight line exists in the mind. We have a feeling that whatever a line is, it ought to have unlabelled points, even after all the rational numbers have found their places. For we feel that $\sqrt{2}$ ought to be the length of a segment with the point labelled 0 as its left end point. The right end point ought to be labelled $\sqrt{2}$. Can this point be constructed with ruler and com-



passes? Yes, easily. Construct a square with the segment from 0 to 1 as one side. Swing an arc with center at 0 and length equal to the diagonal of the square. This arc will cut the number line in a point to the right of 1, the point which we label $\sqrt{2}$.

The foregoing paragraphs may have led you to believe that the need for $\sqrt{2}$ is the only reason for creating the real numbers. More likely they have not lead you to believe any such thing. For as you know, there are infinitely many needs similar to this need. We need $\sqrt{3}$, $\sqrt{10}$, $\sqrt[3]{4}$, $\sqrt[4]{0.517}$, $\sqrt[3]{-1.2}$, $\sqrt[3]{1 - \sqrt{3}}$, a number usually denoted by π, and many, many others. We do not need these numbers in the factory when we are recording diameters of pistons; we do not need them in finance accounts, and we do not need them in measuring timbers for a bridge. But in the realm of ideas and theory, in the area of creation and design, we need them. It is true that in the final computations an engineer may use rational approximations for numbers which he has determined. But his initial toe-hold on some important number may well have been made by solving equations whose solutions depended on some of the pure mathematical properties of irrational numbers. Mathematically, these numbers are indispensable. We need them to make mathematics simpler; we need them to make mathematics beautiful; we need them as labels on our number scale; we need them so that the full force and power of our knowledge of numbers can be effectively used in our study of geometry (analytic geometry).

Let us look at the number scale again. There are two basically different ways of looking at the numbers in this scale. In the one we see the evolution of the number concept—from natural number to real number. In the other we see the numbers on the number scale as comprising the system of real numbers. In the one we see steps: Step 1. We see the points labelled 1, 2, 3, $\cdots$, and we think of them as natural numbers. Step 2. We see all the points labelled $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$, and we think of them as integers. Step 3. We see all the points labelled with rational numbers; we see $\frac{1}{2}, 5\frac{2}{3}, -2\frac{5}{7}$, 3.145; but we also see -13, -2, 0, 4, 73, and so on, this time as symbols for rational numbers. Step 4. We see all the points which are labelled as real numbers; and when we see all these points, we are looking at all the points on the line. Of course this step reveals a point labelled $\sqrt{2}$ and many others we have not seen before. But it also reveals every point we saw in steps 1, 2, 3. We see $\frac{1}{2}$ again; we see -2 again; and we see 2 again. This time we see them all as elements of the real number system. We may look at the number scale and, if we are not in a hurry, we may see all the steps, we may see all the number systems which we have discussed and the isomorphisms which tie them together. On the other hand we may look at the number scale and see only the real numbers; we may see a one-to-one correspondence (established through the labelling) between the points on the line and the numbers in the real number system.

This leads us to a system of notation for real numbers, the symbols in terms of which most users of mathematics think of real numbers, decimals. Suppose x is a real number. If x is also a rational number (you know, through isomorphism; or call it a rational real, if you wish), then the decimal symbol for x can be obtained as follows. Write $x = [a,b]$ using the ordered integer pair symbol; express a and b in the decimal system using Arabic numerals; divide a by b using the division rule learned in the elementary school. It is easy to prove that the division will either come out even (the quotient being a finite decimal), or the division will not come out even, in which case the digits in the quotient will eventually appear in repeating blocks. For example, $\frac{5}{4} = 1.25, -\frac{3}{7} = -0.428571\overline{428571}$, and $5\frac{62}{165} = \frac{887}{165} = 5.3757575 \cdots = 5.3\overline{75}$. (In the last two examples the superscore indicates the block of digits which repeats.) If the decimal terminates or if it repeats, it is a symbol for a rational number; an infinite nonrepeating decimal is a symbol for a real number which is not a rational real number. Consider the example, $x = 1.010010001 \cdots$. We cannot write all the digits in an infinite decimal but we can describe it so that it can be written in the mind. In this example there is (from left to right) a first 1, a second 1, and so on;

and so on means an nth 1 for $n = 1, 2, 3, \cdots$ (through all the natural numbers). For $n = 1, 2, 3, \cdots$, the nth 1 is followed by a block of n 0's (the first 1 is followed by one 0, the second 1 is followed by two 0's, and so on). We have deliberately described a decimal which never repeats; of course the 0's and 1's are repeated over and over, but there is no block of digits which successively repeats itself. So the number x is not a rational real number; it is an irrational real number. Where is it on the number scale? It lies between 1 and 2; it lies between 1.01 and 1.02; it lies between 1.01001 and 1.01002; and so on. The number scale as an object of thought contains exactly one point which satisfies all the requirements of the last sentence; that point is the one which receives the label $1.010010001 \cdots$.

One theory of real numbers is a theory based on real numbers as infinite decimals. It is possible to define the fundamental operations of addition, subtraction, multiplication, and division, and to develop their properties, using the infinite decimal concept of real number. If an engineer thinks about a real number at all, he probably visualizes it as a decimal. The portrait of 2 reveals a real number 2; we see it as a point on the number scale and we see it as an infinite decimal symbol, $2.000 \cdots$. Later we shall see that each rational real number has two infinite decimal representations. In this connection the portrait of 2 reveals it not only as a real number with the representation $2.000 \cdots$ but also with the representation $1.999 \cdots$. This theory of the real number is pretty much down to earth. The real number is defined as a symbol and the operations are defined in terms of these symbols. The whole theory rests upon a system of notation using Arabic numerals; it rests upon an extension of a class of symbols for rational numbers, an extension from certain types of decimals to all infinite decimals. It is not an extension of the rational number system based upon the intrinsic properties of the rational numbers themselves. We propose then to look in upon two modern theories which create the reals from the rationals using procedures which are independent of the rational number symbolism.

First, let us consider the theory of Dedekind. In this theory each real number is created as a pair of infinite classes of rational numbers. To describe these classes we return to the number scale and look at it from Step 3; we see all the rational numbers as labels of points on the number scale. The relative position of these numbers on the scale establishes an order, a linear order, in the system of rational numbers. If we look at any pair of rational numbers, two different rational numbers, we see that one of them lies to the left of the other one, that one of them is less than the other. This order relationship, which is revealed so clearly on the number scale, can be described rigorously without the aid of our geometrical crutch—the line which is the basis of the number scale. It is convenient for this purpose to use the ordered integer pair notation for rational numbers. We say that $[a,b]$ is a positive rational if a and b are both positive integers or both negative integers; we say that $[a,b]$ is a negative rational if a and b are integers of opposite sign. Thus $\frac{3}{4} = [3,4] = [-3,-4]$ is positive while $-\frac{3}{4} = [-3,4] = [3,-4]$ is negative. We say that the positive rational numbers are greater than 0, that the negative rational numbers are less than 0. We say that $[a,b]$ is greater than $[c,d]$ and write $[a,b] > [c,d]$ if $[a,b] - [c,d] = [ad - bc,bd]$ is a positive number; we say that $[c,d]$ is less than $[a,b]$ and write $[c,d] < [a,b]$ if $[a,b] > [c,d]$. Of course, if $[a,b] - [c,d] = [0,0]$, then $[a,b] = [c,d]$.

Let us now create the real number 2 as it appears in the Dedekind theory. Let A denote the class of all rational numbers which are less than or equal to the rational number 2; let B denote the class of all rational numbers which are greater than 2. Then we define the real number 2 to be the ordered pair of classes $\{A,B\}$; we write $2 = \{A,B\}$, or, for clarity, (real 2) $= \{A,B\}$. On the number scale we see A as the set of all rational numbers lying to the left of (rational 2) including (rational 2) itself; B is the set of all rational numbers lying to the right of (rational 2). Intuitively, we see (real 2) as a partition of the rational numbers into a lower segment and an upper segment, the number (rational 2) being the largest in the lower segment. In general, if r is any rational number, let A_r denote the set of all rational numbers s less than or equal to r; let B_r denote the set of all rational numbers t greater than r; and define (real r) $= \{A_r,B_r\}$. These real numbers we might call the rational reals; the real numbers which we have not created yet are the irrational reals. Later we shall show that the rational reals are isomorphic to the rationals.

Now for the construction of the irrational reals using the device of the Dedekind partition. To construct the number $\sqrt{2}$ we take classes A and B as follows: A contains all negative rational numbers, 0, and all positive rational numbers r having the property that $r^2 < 2$; B contains all rational numbers which are not in A; $\{A,B\}$ is the real number ordinarily denoted by $\sqrt{2}$. In general, *if A and B are two classes of rational numbers having the properties* (i) *every rational number is either in A or in B*, (ii) *every element of A is less than every element of B*, (iii) *there is no smallest element in B, then $\{A,B\}$ is a real number*. Note that if (i) and (ii) are satisfied for a given pair of classes A,B, then it is impossible for A to have a largest element and B to have a smallest element. For if r is the largest rational number in A, and s is the smallest rational number in B, then $r < s$ by condition (ii) and $\frac{r+s}{2}$ is a rational number which is greater than r and less than s. By condition (i) we see that $\frac{r+s}{2}$ is either in A

or it is in B. It cannot be in A since r is the largest element in A and it cannot be in B since s is the smallest element in B. Therefore if A and B satisfy conditions (i) and (ii), and if B has a smallest element, then A has no largest element. But if we moved this smallest element of B into class A, then A and B would satisfy all three conditions. So if A and B satisfy (i), (ii), and (iii), that is, if $\{A,B\}$ is a real number, then either A has a largest element or it does not. If A has a largest element r, then $\{A,B\}$ is the number (real r); if A has no largest element, then $\{A,B\}$ is an irrational real number. Consider again the number $\sqrt{2}$. As you will recall, there is no rational number r such that $r^2 = 2$. In view of the linear order of the rationals we have $r^2 > 2$ or $r^2 < 2$ for every rational number r. Suppose r_1 is any positive rational number in the lower segment A which we described when we defined $\sqrt{2}$. Then $r_1^2 < 2$. Regardless of what r_1 we start with in A, we can find a larger rational number in A. Given an r_1 in A, let r_2 denote the rational number $\frac{4r_1}{r_1^2 + 2}$. Since $2 > r_1^2$, we have $4 > 2 + r_1^2$, $4r_1 > r_1(2 + r_1^2)$ and $r_2 = \frac{4r_1}{2 + r_1^2} > r_1$. On the other hand, we have $2 - r_1^2 > 0$, $(2 - r_1^2)^2 > 0$, $4 - 4r_1^2 + r_1^4 > 0$, $4 + 4r_1^2 + r_1^4 > 8r_1^2$, $2(2 + r_1^2)^2 > (4r_1)^2$, $2 > \frac{(4r_1)^2}{(2 + r_1^2)^2} = r_2^2$. We have shown that the rational number r_2 is larger than r_1 and, since $2 > r_2^2$, that r_2 is in A. So it is impossible for A to have a largest element. What is the largest rational number whose square is less than 2? Answer: There is none. For if r_1 were the largest one with this property, then the above proof shows that r_2 is a larger one with this property. But this contradicts the assumption that r_1 is the largest; so there cannot be a largest. Of course the eighth-grader learns how to find larger and larger r's with the property that $r^2 < 2$. He finds 1.4, 1.41, 1.414, 1.4142, $\cdots$. Actually he does not carry the process very far; none of his problems requires more than five or six places. But suppose he carried it out to 50 places; or suppose he hired an electronic computer to calculate 1000 places; would he then have the largest rational number with square less than 2? The answer is no; there is no such largest rational number. Now suppose that this eighth-grade square root process has been carried out to give an infinite number of places. Someone may say that is impossible. Actually, yes. But let us imagine that it has been done. In the mind, then, we have an infinite decimal and we can describe the process of determining each digit in this decimal. Someone may guess that this is the largest rational number with square less than 2. No, no! The answer is no on two counts. First, this infinite decimal is not a rational number, it is one symbol for a real number. Secondly, its square is not less than 2, its square is exactly 2. Of course, the infinite decimal is another symbol for $\sqrt{2}$.

We have discussed at some length the creation of the real number as a Dedekind partition of the rational numbers. We see (real 2) as an ordered pair $\{A,B\}$, where B is the set of all positive rational numbers each of which exceeds 2 and where A is the set of all other rational numbers. We see $\sqrt{2}$ as an ordered pair $\{A,B\}$, where B is the set of all positive rational numbers each of whose squares exceed 2, and A is the set of all other rationals. In general a real number is a pair $\{A,B\}$ where A and B are sets of rational numbers satisfying three conditions, which we listed. The real number system is the set of all these numbers and the fundamental operations for combining real numbers.

It is not our intent to develop this real number theory, but we shall taste a little of it. How shall we define addition? We are assuming that we have complete knowledge of the fundamental operations in the rational number system. Let x and y be two real numbers. Then $x = \{A,B\}$, and $y = \{C,D\}$; x and y are Dedekind partitions of the rational numbers; for x and y the lower segments in these partitions are A and C, respectively; the upper segments are B and D, respectively. We define the sum of x and y to be a real number $z = \{E,F\}$ where the sets E,F are formed as follows. For every pair of rational numbers r and s, r in A and s in C, put the rational number $r + s$ into E. F is the set of all rational numbers not in E. It can be shown that $\{E,F\}$ is a real number. Similarly we can define subtraction, multiplication, and division of real numbers. And we can develop the properties of these operations in the real number systems using these definitions. For example, it is easy to prove that addition is commutative in the real number system. When we defined the sum of x and y above, we formed E as the class of all rational numbers $r + s$ where r is in A and s is in C. If we follow this definition and add y and x, adding in the other order, we should form E as the class of all rational numbers $s + r$ where s is in C and r is in A. Since addition is commutative in the rational number system, it follows that $s + r = r + s$, that the class E is the same class for $y + x$ as for $x + y$, and hence that $x + y = y + x$. Can we establish a linear order for the real numbers? Let x and y, $x = \{A,B\}$, $y = \{C,D\}$, be two real numbers. We say that $x = y$ if A and C are identical sets; that is, if A and C are symbols denoting the same set of rational numbers. Otherwise we say that x is different from y; we write $x \neq y$. In the latter case, one of them ought to be greater than the other one. How shall we define this order relation? Suppose that $x \neq y$. Then A and C are different classes of rationals. This means that some rational number r is in one of these two sets but not in both of them. To be definite, suppose there is a rational number r in A and that r is not C. Then we define x to be greater than y; we write $x > y$. The order relation thus established in the real number system is the same as the one which one would naturally define in terms of infinite decimals. How about the number scale? Yes, we can think of each point on the number scale being labelled with some symbol for a real number x. This symbol may be one in ordinary use, as 2, or it may be the ordered pair symbol $\{A,B\}$ where A,B are classes of rationals forming a Dedekind partition. And we can pull the rationals and the rational reals together through an isomorphism as we have suggested previously. We mate each rational number r to the real number $\{A,B\}$, which we called (real r) above, where A is the set of all rational numbers not exceeding r and B is the set of all other rational numbers. Let us see if we can add by "translation," as required in the definition of isomorphism. Suppose we wish to add (rational r) and (rational s). Then (rational r) $\leftrightarrow$ (real r), (rational s) $\leftrightarrow$ (real s); (real r) + (real s) $= \{E,F\}$, in which the largest element in E is the rational number which is the sum of the numbers r and s; and $\{E,F\} \leftrightarrow$ (rational $r + s$). This completes our glimpse of the Dedekind real numbers.

Rational		Real
r	$\longrightarrow$	$r = \{A,B\}$
s		$s = \{C,D\}$
		$\{A,B\} + \{C,D\} = \{E,F\}$
$r + s$	$\longleftarrow$	$r + s = \{E,F\}$

The portrait of 2 reveals a real number 2 as envisaged by Dedekind. It reveals 2 as a Dedekind partition of the rational numbers. It reveals 2 in a system of numbers where each element is such a partition and where the operations are defined and developed in terms of these partitions. But it also reveals another real number 2; it reveals 2 as an element of another number system which we shall now discuss.

In the theory of real numbers due to Georg Cantor, a real number is a class of sequences of rational numbers, the class having certain properties which we shall state later. In the meantime we must talk about sequences, in particular about sequences of rational numbers. An infinite sequence of numbers is a correspondence which mates a number with each of the natural numbers. A symbol for an infinite sequence which suggests the nature of the sequence idea is: $a_1, a_2, a_3, \cdots, a_n, \cdots$. In this composite symbol a_1 denotes the number in the sequence which is mated to 1, a_2 denotes the number in the sequence which is mated to 2, and so on; a_n denotes the number in the sequence which is mated to n. Briefly we say that a_1 is the first element in the sequence, a_2 is the second element, and so on. Now, of course, it is impossible to write an infinite sequence of numbers. We can picture it in the mind; we can communicate our knowledge of an infinite sequence using a finite number of words or symbols. Consider the sequence in which $a_n = \frac{1}{n}$ for every natural number n. The same sequence is suggested if we say: consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n} \cdots$. Other examples of infinite sequences are: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2n}, \cdots$; $1, -1, 1, -1, \cdots, (-1)^{n+1}, \cdots$; and $1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \cdots$. Note that in the definition of an infinite sequence, the correspondence is not required to be a one-to-one correspondence. Each natural number n appears exactly once as a subscript in the symbol. $a_1, a_2, a_3, \cdots$; but the a_n themselves need not be distinct. In fact $1, 1, 1, \cdots$ is a sequence in which every a_n is the same number: $a_1 = 1, a_2 = 1, a_3 = 1$, and so on.

For our purposes, we are particularly interested in convergent sequences of rational numbers. We shall give two definitions for a convergent sequence and then discuss the relationship between the two. The intuitive idea in the first definition is that a_n *approaches* closer and closer (it may be there or it may get there in some examples) to some fixed number as n gets larger and larger. Thus the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$, is convergent since $\frac{1}{n}$ is as close to 0 as we please for all n sufficiently large. In this example we say that 0 is the limit of the sequence, and we say that the sequence is convergent by the external criterion. It is an external criterion since the convergence depends upon the relationship of the numbers in the sequence to a number called the limit; the limit number need not be an element of the sequence. Other examples are as follows:

(1) $$1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \cdots, 0, \frac{1}{n}, \cdots.$$

This sequence is convergent with limit 0.

(2) $$1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \cdots, \frac{2n-1}{n}, \cdots.$$

This sequence is convergent with limit 2.

(3) $$1, 0, 1, 0, 1, \cdots, 0, 1, \cdots.$$

This sequence is not convergent; we call it divergent. So much for the intuitive idea behind the definition; here is the definition itself.

Definition. A sequence $a_1, a_2, a_3, \cdots$ is convergent (by the external criterion) with limit a if corresponding to every positive rational number r

there is a natural number N with the property that $a - r < a_n < a + r$ for every natural number n which exceeds or equals N.

In the intuitive idea, we want a_n close to a for all large n. In the formal definition the *how close* comes first; we want a_n and the limit a to differ by less than r. In the formal definition the "for all large n" is made precise next; there must be a natural number N so that something will be true for all $n \geqq N$, so that a_n will lie between $a - r$ and $a + r$ for every $n \geqq N$. Consider again the example, $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \cdots, \frac{2n-1}{n}, \cdots$. Let us see how the definition works on this example. Suppose that $r = \frac{1}{10}$ then we want $\frac{2n-1}{n}$ to be between 1.9 and 2.1 for all large n. Since $\frac{2n-1}{n} < 2$ for all n, we need not be concerned with the 2.1. What we need is $\frac{2n-1}{n} > 1.9, 2n - 1 > 1.9n, .1n > 1, n > 10$. So corresponding to $r = \frac{1}{10}$ we take $N = 11$. Then if $n \geqq 11$, we have $\frac{2n-1}{n} = 2 - \frac{1}{n} < 2 < 2.1$ and $n > 10, .1n > 1, 2n - 1.9n > 1, 2n - 1 > 1.9n, \frac{2n-1}{n} > 1.9$. Hence $\frac{2n-1}{n}$ lies between 1.9 and 2.1 for all $n \geqq 11$. We have not proved that the sequence is convergent with limit 2; we have illustrated the meaning of the definition with a particular r. To prove convergence we must show that every positive r works. So let a positive rational number r be given. We want to describe a procedure for determining the N which will work regardless of the value of r. For all $n \geqq N$ we want $2 - r < \frac{2n-1}{n} < 2 + r$. Since $\frac{2n-1}{n} < 2 < 2 + r$ for all n, the requirement $\frac{2n-1}{n} < 2 + r$ places no restriction on the value of N. So all we need is $2 - r < \frac{2n-1}{n}$. This will be true if $2n - 1 > 2n - nr$, if $nr > 1$, if $n > \frac{1}{r}$. Let N be the smallest natural number with $N > \frac{1}{r}$. Then if $n \geqq N$, we have $n > \frac{1}{r}$, and $2 - r < \frac{2n-1}{n} < 2 + r$. This proves that $1, \frac{3}{2}, \frac{5}{3}, \cdots, \frac{2n-1}{n}, \cdots$ is convergent to the limit 2.

The intuitive idea in the second definition is that $a_1, a_2, \cdots, a_n \cdots$

is convergent (by the internal criterion) if, given a specified closeness (a positive number), there is some element in the sequence beyond which the two elements of every pair differ by less than the specified closeness. In the external criterion the elements of the sequence are required to be close to some number, which might be "external to," that is, which might not be an element of the sequence. In the internal criterion the closeness requirement is specified internally—close to each other. The formal definition is as follows:

Definition. The sequence $a_1, a_2, a_3, \cdots$ is convergent by the internal criterion if, corresponding to every positive rational number r, there is a natural number N with the property that $-r < a_n - a_m < r$ for every pair of natural numbers n, m, each of which exceeds or equals N.

There is a theorem which says that a sequence which is convergent by the external criterion is also convergent by the internal criterion. Indeed, if a_n and a_m are both close to a, then they are close to each other. If $a_1, a_2, a_3, \cdots$ is convergent to a and if we want a_n and a_m to differ by less than r, then we take N in the first definition so that $a - \frac{1}{2}r < a_n < a + \frac{1}{2}r$ for all $n \geqq N$. Then if $n \geqq N$ and $m \geqq N$, we have $a - \frac{1}{2}r < a_n < a + \frac{1}{2}r$ and $a - \frac{1}{2}r < a_m < a + \frac{1}{2}r$. So a_n and a_m both lie on the number scale in the interval with end points $a - \frac{1}{2}r$ and $a + \frac{1}{2}r$. Since this interval is r units long, it follows that a_n and a_m differ by less than r.

A good question at this point is: Is there a theorem which says that a sequence which is convergent by the internal criterion is also convergent by the external criterion? If we are talking about sequences of rational numbers and if the limit a in the external criterion definition is required to be a rational number, then the answer to the question is no. If we are working in the system of rational numbers, then there are convergent sequences which do not have limits. A simple example is the sequence 1, 1.4, 1.41, 1.414, $\cdots$ in which the terms are determined by the elementary-school process for finding a decimal approximation to $\sqrt{2}$. Inasmuch as this sequence is convergent in the system of real numbers to the real number $\sqrt{2}$ and since a sequence can have only one limit, it follows that the sequence cannot have a "rational real" limit. Let us look at it another way. If the sequence had a rational limit L, then the infinite decimal $1.414 \cdots$ would be another symbol denoting the same number as L. But this is impossible since the infinite decimal 1.414 ⋯ is nonrepeating while the decimal representation for a rational number is finite or repeating.

Now what has all this business of sequence to do with creating the reals from the rationals? One way to look at it is this. There are many sequences of rational numbers which are convergent by the internal criterion. Some of them have limits and some of them do not. The intuitive idea is that we shall create a limit for each of them that does not have a limit. And, of course, this limit will be a real number. Before we do this let us recall how we created the rational numbers from the integers. We looked at the system of integers and we saw that the system was not closed with respect to division. In the system of integers the symbol $\frac{5}{3}$ is meaningless. So we created the rational numbers and $\frac{5}{3}$ is a symbol which denotes one of these new numbers. We have a similar situation in connection with the internally convergent sequences of rational numbers. Consider again the sequence of decimal approximations for the $\sqrt{2}$:

$$1, 1.4, 1.41, 1.414, \cdots .$$

Let us think of this whole sequence as one symbol, a symbol for a real number. Just as [8,5] is one representation for a certain rational number, so also 1, 1.4, 1.41, ⋯ is one representation for a certain real number. Just as [8,5], [16,10], [−32,−20] $\left(\text{or } \frac{8}{5}, \frac{16}{10}, \frac{-32}{-20}\right)$ are different representations for the same rational number so a real number has many representations as a convergent sequence of rationals. In Cantor's theory two convergent sequences of rational numbers,

$$a_1, a_2, a_3, \cdots, a_n, \cdots,$$

$$b_1, b_2, b_3, \cdots, b_n, \cdots,$$

are representations for the same real number if the sequence

$$a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n, \cdots,$$

is convergent to the limit 0. In this case we say that the sequences a_1, a_2, a_3, ⋯ and b_1, b_2, b_3, ⋯ are equivalent sequences. Just as a rational number is defined as the class of all its representations as an ordered pair of integers, so a real number is defined as the class of all its representations as a convergent sequence of rational numbers. Thus (real 0) is the class of all sequences of rational numbers which converge to (rational 0); (real 1) is the class of all sequences of rational numbers which converge to (rational 1); $\sqrt{2}$ is the class whose elements are the sequence 1, 1.4, 1.41, 1.414, ⋯ and all other sequences equivalent to this sequence.

In general, if $a_1, a_2, a_3, \cdots$ is any convergent sequence of rational numbers, then the class of all sequences of rational numbers which are equivalent to this sequence is defined to be a real number. Each one of these sequences is called a Cantor representation of the real number.

How do we operate with Cantor's real numbers? We give here a definition for the addition of real numbers in Cantor's theory. Let x and y be two real numbers with Cantor representations as follows:

$$x = \{a_1, a_2, a_3, \cdots, a_n \cdots\},$$

$$y = \{b_1, b_2, b_3, \cdots, b_n \cdots\}.$$

Define the sum $x + y$ to be the real number represented by the sequence

$$a_1 + b_1, \; a_2 + b_2, \cdots, \; a_n + b_n, \cdots.$$

Similarly we can define the other fundamental operations and from them develop a theory of real numbers. As an example of a theorem in this theory we have: *Addition of real numbers is commutative.* For with x and y as above we have:

$$x + y = \{a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots\},$$

$$y + x = \{b_1 + a_1, b_2 + a_2, \cdots, b_n + a_n, \cdots\}.$$

If we subtract these representations for $x + y$ and for $y + x$ term by term, we get the sequence 0, 0, 0, $\cdots$. And since 0, 0, 0, $\cdots$ converges to 0, the representations for $x + y$ and for $y + x$ are representations for the same real number. Hence $x + y$ and $y + x$ are symbols for the same real number. This completes the proof of the theorem.

How can we establish a linear order among the real numbers? Given x and y with representations as above, define $x > y$ if

(i) the sequence $a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n, \cdots$ does not converge to 0,

(ii) the sequence $a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n, \cdots$ is convergent, and

(iii) there is a natural number N such that $a_n - b_n > 0$ for all $n \geqq N$.

How are the real numbers of Dedekind related to the real numbers of Cantor? On the basis of their definitions they are entirely different. But intuitively they are quite similar. Actually, it can be proved that the two systems of numbers are isomorphic.

Having created the real numbers as Dedekind partitions of rationals and as classes of equivalent convergent sequences of rationals, one might be tempted to create a new system of numbers by forming Dedekind partitions of real numbers or by taking classes of (internally) convergent sequences of real numbers. This can actually be done. But the results are not interesting. Each of the new systems of numbers which results is uninteresting since it is isomorphic to the system of real numbers.

When we defined the real number as a Dedekind partition of the rationals, we created, corresponding to each rational number r, a real number which naturally mated itself with that r. This real number $\{A,B\}$ is the one in which r is the largest element in A. But we created many real numbers which did not correspond in that way with the rationals; there are infinitely many real numbers $\{A,B\}$ in which A has no largest element. In this sense the real number system is much larger than the rational number system.

When we defined the real number as a class of equivalent sequences of rational numbers, we created, corresponding to each rational number r, a real number which naturally mated itself with that r. This real number is the class of all sequences of rational numbers which converge to r. But we also created many real numbers which do not correspond in that way to a rational number; there are infinitely many real numbers with representations which are internally convergent sequences of rational numbers but not externally convergent. In this sense Cantor's system of real numbers is much larger than the system of rational numbers.

If we attempt to enlarge the system of real numbers by forming Dedekind partitions of reals or classes of convergent sequences of reals and calling them, say, *superreals,* we fail since as stated before, the resulting systems are isomorphic with the real number system. The reason for this is embodied in two theorems, a climax theorem in the Dedekind theory and a climax theorem in the Cantor theory, which we state without proof.

Theorem. If $\{A,B\}$ is any Dedekind partition of real numbers (three conditions analogous to our three conditions for a Dedekind partition of rational numbers), *then there is a largest real number in A.*

Theorem. If $x_1, x_2, \cdots, x_n, \cdots$ is any sequence of real numbers convergent by the internal criterion; then there is a real number x such that $x_1, x_2, x_3, \cdots$ is convergent to x by the external criterion.

Before we leave the real number system we must attend to some unfinished business. The reader may recall our discussion of real numbers as decimals. It was stated that the real number 2 has two representations as an infinite decimal, namely:

$$(\text{real } 2) = 2.000 \cdots \text{ and}$$

$$(\text{real } 2) = 1.999 \cdots .$$

One explanation of this can be given now. In the real number system the theory of limits of sequences reveals the number $\sqrt{2}$ as the limit of the rational real numbers 1, 1.4, 1.41, 1.414, $\cdots$. Similarly (real 2) is the limit of the sequence 2, 2.0, 2.00, $\cdots$. (Each symbol in this sequence denotes the real number 2 and so this assertion is trivial.) But (real 2) is also the limit of the sequence of real numbers 1, 1.9, 1.99, 1.999 $\cdots$. Hence we write $2 = 1.999 \cdots$.

Perhaps the following argument might be more appealing to the reader. Since $\frac{1}{3}$ has the infinite decimal representation $0.33 \cdots$, we can multiply "through" by 3 to get $1 = 0.999 \cdots$, and add equals to equals to get $2 = 1.999 \cdots$. Similarly every finite decimal is equal to an infinite decimal as illustrated by the examples: $0.25 = 0.24999 \cdots$, $0.0523 = 0.0522999 \cdots$, and $17.3 = 17.2999 \cdots$.

This concludes our discussion of real numbers. The portrait of 2 reveals the real number 2 of Dedekind as a partition of the rational numbers, the real number 2 of Cantor as a class of equivalent convergent sequences of rational numbers, the real number 2 of the applied mathematician as $2.000 \cdots$ or $1.999 \cdots$, and the real 2 of the engineer as something which for his purpose is the same as the natural number 2, the integer 2, and the rational number 2.

VIII. The Complex Number 2

The system of real numbers seems quite complete as regards mathematical operations. It is closed with respect to the fundamental operations; the lower segments in its Dedekind partitions have a largest element; its internally convergent sequences have limits. But there is one very important defect in the real number system. Some very simple equations cannot be solved in the real number system. In other words, there are simple equations whose roots are not real numbers. One such equation is $x^2 + 1 = 0$. If x is a real number, then x^2 is positive or zero and $x^2 + 1$ is positive; hence it is impossible that $x^2 + 1 = 0$. Yes, we can go through the motions of solving the equation. We can write $x^2 + 1 = 0, x^2 = -1, x = \pm \sqrt{-1}$. When we write these things we are writing something which has form but no substance, something which is meaningless in the real number system. To solve $x^2 + 1 = 0$ we need a number whose square is -1. To solve $x^2 + 2 = 0$ we need a number whose square is -2. To solve $x^2 - 2x + 2 = 0$ we need a number which can be decreased by 1 to leave a remainder whose square is -1. There are no such real numbers. So we create them; we call them complex numbers. And when we create them we have the mathematical equipment for solving not only such simple equations as those listed above but every polynomial equation whose coefficients are elements of the complex number system. This is truly a notable instance of the fact that the creation of new numbers simplifies mathematics and makes it a thing of beauty. Indeed, the modern theory of complex numbers is a high point in the intellectual achievements of man.

Now that we are convinced of the need for complex numbers and have paid them such high compliments, we proceed to create them from the real numbers. We created the integer as a class of ordered pairs of natural numbers, the rational number as a class of ordered pairs of integers, the real number as a Dedekind partition of the rationals and as a class of sequences of rational numbers. We might expect something more complicated for our last creation. Actually, it is simpler. The modern concept of a complex number is based on the following definition.

Definition: A complex number is an ordered pair of real numbers.

If a and b are any real numbers, then $((a,b))$ is a symbol denoting a complex number. This is not the symbol used by the practical man. It is a convenient symbol to use in the development of the theory of complex numbers. The traditional symbol will be given later. We noted in our discussion of the rational numbers that different ordered pairs of integers might denote the same rational number, as $[3,4] = [6,8]$. In the rational number system we defined equality: $[a,b] = [c,d]$ if and only if $ad = bc$. The situation in the complex number system is much simpler: $((a,b)) = ((c,d))$ if and only if $a = c$ and $b = d$. Of course it may happen that $((a,b))$ and $((c,d))$ are different symbols even though $((a,b)) = ((c,d))$. But in this case, the difference is due to the fact that a and c are different symbols for the same real number or that b and d are different symbols for the same real number. Thus $((1 + 2,5)) = ((3,5))$, and $((x,y)) = ((2,3))$ implies that $x = 2, y = 3$.

The fundamental operations are easily defined and studied using the ordered pair notation. Definition: $((a,b)) + ((c,d)) = ((a + c,b + d))$, $((a,b)) - ((c,d)) = ((a - c,b - d))$, $((a,b))\cdot((c,d)) = ((ac - bd,ad + bc))$, and if $c^2 + d^2 \neq 0, ((a,b)) \div ((c,d)) = \left(\left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)\right)$. Using these definitions it can be shown that the complex number system is indeed an extension of the real number system. For the system of all complex numbers $((a,0))$, where a is a real number (and 0 is the real 0), is isomorphic to the real number system. As you would guess, we mate $((a,0))$ with a to establish this isomorphism. For example, $0 \leftrightarrow ((0,0))$, $1 \leftrightarrow ((1,0))$, and $2 \leftrightarrow ((2,0))$.

Yes, we can solve $x^2 + 1 = 0$ in the complex number system. Recall the isomorphism and consider the 1 and the 0 in the equation as meaning $((1,0))$ and $((0,0))$ respectively. Substituting $x_1 = ((0,1))$ and $x_2 = ((0,-1))$ for x we find $x_1^2 = ((0,1))\cdot((0,1)) = ((0 - 1,0 + 0)) = ((-1,0))$, $x_1^2 + ((1,0)) = ((-1,0)) + ((1,0)) = ((0,0))$; $x_2^2 = ((0,-1))\cdot((0,-1)) = ((0-1,0 + 0)) = ((-1,0))$, $x_2^2 + ((1,0)) = ((-1,0)) + ((1,0)) = ((0,0))$.

In traditional symbols, the complex number $((0,1))$ is written as i, $((0,-1))$ is written as $-i$, and, in general, $((a,b))$ is written as $a + bi$. In the special case of $((a,b))$ with $b = 0$ it is customary to write $((a,0)) = a + 0i = a$; it is customary in many situations to consider the complex numbers $((a,0))$ as special complex numbers called real numbers. Confusing? Not really. Perhaps the vocabulary could be improved. But the idea is clear in view of the isomorphism mentioned above.

As a final remark in this brief encounter with the complex numbers we mention the beautiful situation as regards roots in this system. Every complex number, except 0, has two square roots, three cube roots, four fourth roots, and so on. For example, the three cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}i}{2}$, and $-\frac{1}{2} - \frac{\sqrt{3}i}{2}$. Check them if you can, using the definitions of the fundamental operations as we listed them.

In this last section we have seen 2 as a complex number. We have seen it as the ordered pair of real numbers ((2,0)), as a symbol $2 + 0i$, and as the symbol 2 again. We have seen 2 in an environment of complex numbers, as an element of a system which is one of the most beautiful achievements of the mind of man.

Perhaps the reader has wearied in this study of the portrait of 2. Perhaps he feels that 2 is not really as complicated as the author thinks. Perhaps he feels that the author has strayed from his subject and forced a lot of modern number theory upon him. But the author is not concerned about that now. The fact that the reader is reading these words indicates that the author has achieved his purpose. It has not been his purpose to present a treatise on any subject. Rather, it has been his purpose to enlarge the reader's concept of number, to give him some insight into the nature of number as a creation of the human intellect. To achieve this end the author has described the modern portrait of 2 as he sees it.

IX. Suggestions for Further Readings

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