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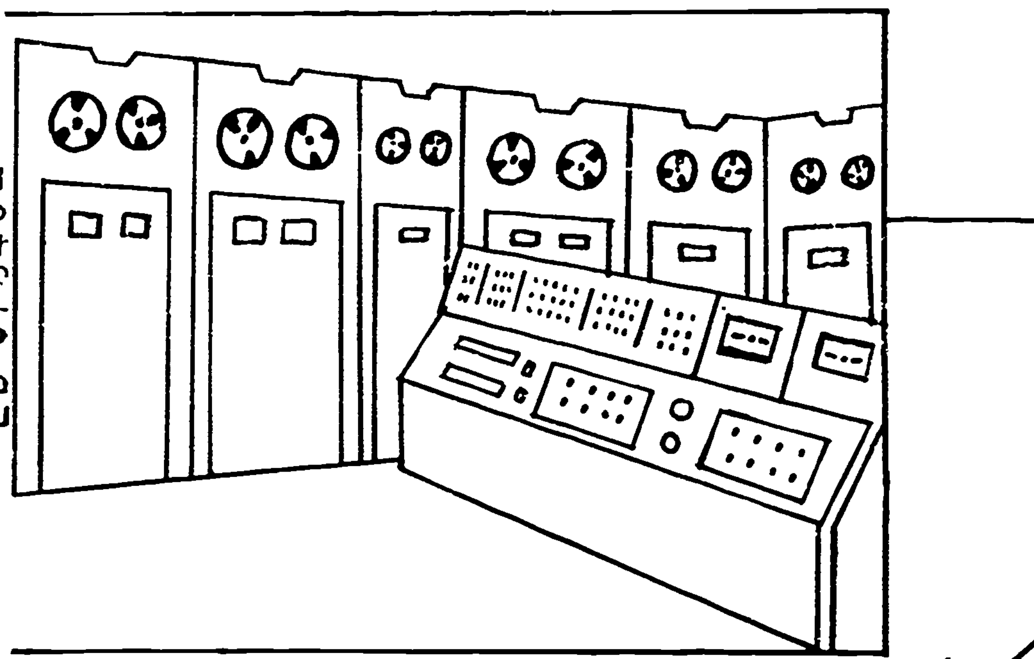
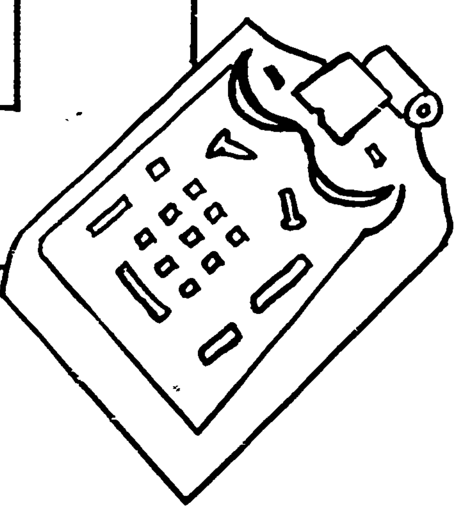
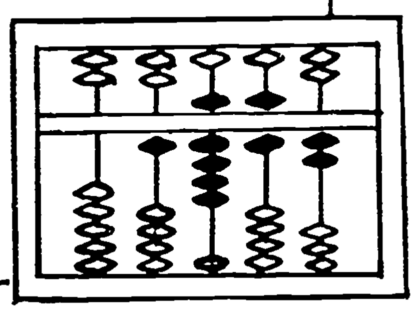
ABSTRACT

This publication is part of a series of guides written by Pennsylvania's Department of Public Instruction and is designed to present some of the unifying concepts in mathematics to elementary teachers. This pamphlet discusses the commutative, associative, and closure properties for addition and multiplication of whole numbers, and the distributive property. Numeration systems, including decimal and non-decimal systems, are briefly described. Short exercise sets are included for each section, along with an answer key. (Related documents are SE 015 951 and SE 015 952.) (DT)

PROPERTIES OF NUMERATION SYSTEMS

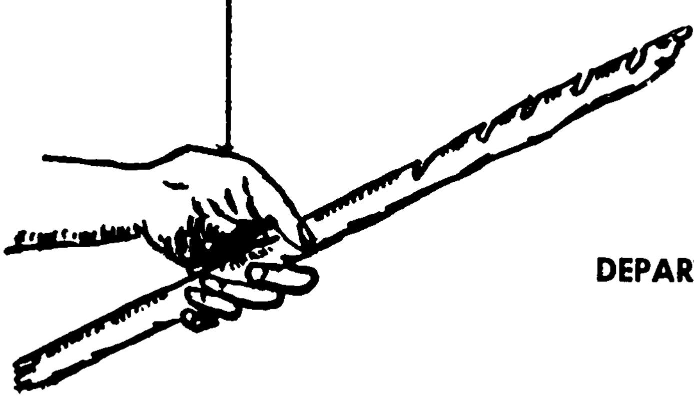






Mathematics Series No. 1

PENNSYLVANIA CURRICULUM DEVELOPMENT PROGRAM



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MESSAGE FROM THE SUPERINTENDENT OF PUBLIC INSTRUCTION

"The moving finger writes; and, having writ, moves on," noting changes which are occurring inexorably in our environment. Our technology changes in an ever more revolutionary fashion, and we must so educate our youth that they can make continual adjustments. In school mathematics this requires more emphasis on structure and generalizations to furnish a broad background which a capable individual can use in making adaptations.

Appropriate changes in emphasis in the secondary school mathematics program necessitate both alteration and acceleration in the program of the elementary schools. Experimental programs have shown that much more mathematics can be taught to younger children than had previously been thought possible. The arithmetic program can be taught more efficiently; algebraic and geometric concepts can be introduced effectively as early as the primary grades; and modern approaches can be used to provide both motivation and understanding. Another area of great importance is the combined approach to science and mathematics in the elementary grades. Steps are being taken to effect a unified approach along these lines in our Commonwealth. This combination makes it absolutely necessary to introduce in the elementary grades many mathematical concepts formerly developed in classes in mathematics on a more advanced level.

For the most part, we do not wish to make wholesale, sweeping changes. Rather we hope to develop materials which will enable the teacher to make transitions gradually, hoping in this way to build a strong foundation for further improvements.

George W. Hoffman

Acting Superintendent of Public Instruction

PREFACE

With this publication the Department of Public Instruction initiates a series of guides designed to present to elementary teachers the golden threads of mathematics education which begin when mathematics is first studied and extend through the realm of graduate research.

Fundamental among these threads is the study of the number system. Of paramount importance is the knowledge of the properties of numeration systems, such as the commutative, associative, and distributive laws. While everyone admits the existence of these laws, an emphasis on their meaning and applications leads to much more understanding of and proficiency in operations with numbers. The study of the number line affords an introduction to a geometric model of the number system which lends itself to greater understanding of algebraic equations. Working with numbers written to bases other than ten opens up new areas of application of mathematics and also leads to a better comprehension of our decimal system.

Prominent among trends in mathematics education at the elementary school level is the acceleration of the traditional topics of arithmetic as well as an earlier and more intensive introduction of many algebraic and geometric principles. The use of set notation can be a most valuable generalizing concept. There must also be an emphasis on problem solving at all levels, with due attention paid to applications of scientific principles.

These guides will show you how many of these threads can be woven into the course which you are now evolving to meet the demands of tomorrow.

PROPERTIES OF NUMERATION SYSTEMS

"Properties," "numeration," and "number system" -- what do they mean? And what do they have to do with the study of arithmetic? Are there other technical terms that challenge the arithmetic teacher?

These questions can be disturbing. Fortunately, the concepts are easily grasped. What is even more satisfying is the fact that these terms, and others associated with them, are basic to the very structure of arithmetic. Studying them is intriguing. If an arithmetic teacher comprehends the significance of the meaning of these terms she can guide her pupils with both skill and confidence, for their implications extend deep into the body of mathematics. If a pupil is led to see the applications of these terms to his study of arithmetic he will not need to revise his thinking as he climbs from rung to rung to the top of the mathematics ladder!

Before we begin explaining the terms and their uses it should be pointed out that we will limit their application to the so-called natural numbers, or counting numbers, represented by the sequence 1, 2, 3, ..., 99, 100, ..., 1000, 1001, ... and to the two operations of addition and multiplication.

PROPERTIES

What is a "property?" A "property" is a characteristic of number usage which is true in the light of our experience with real things. For example, if we have a set of seven apples we can separate them into two smaller sets of either four and then three apples, or three and then four apples. If a teacher desires to know the enrollment of a certain class, she can add the number of girls enrolled to the number of boys enrolled or she can add the number of boys enrolled to the number of girls enrolled. Using numerals,* the separations of the set

*Numeral versus number. A numeral is a symbol for a number. The numeral, therefore, may be a streak of graphite left on a sheet of paper by writing on it with a pencil, or a streak of chalk dust left clinging to a "board" after writing on it with a chalk crayon, or a streak of ink left by drawing a pen over a sheet of paper. But no one has ever seen a number. Numbers are mental concepts and therefore cannot be seen, felt, or touched. The number five is the mental understanding or comprehension of fiveness - five fingers, five dogs, five rugs, for example. Numerals for the number five are V and 5.

of seven apples into two equivalent smaller sets could be written as $4 + 3 = 3 + 4$; the two ways of writing the total class enrollment could be (using arbitrary numbers) $17 + 13 = 13 + 17$.

Such experiences suggest that when any two numbers are added the sums are independent of the order in which the numbers are added. That is, numbers have the property that when any two of them, such as 5 and 7, are added in the order of $5 + 7$ the sum is always the same as though they were added in the order of $7 + 5$. This property is called the commutative property of addition. The term "commutative" can be associated with the term "commuter," one who travels back and forth.

The commutative property of addition is the basis for having confidence in the method of checking addition exercises by adding upward if the original adding was done downward.

8	↓	8	19
6	14	6	11
3	17	3	5
2	19	2	↑
19		19	

Exercises. By using the commutative property state equivalent expressions:

1. $6 + 2 = ?$

2. $9 + 12 = ?$

3. $a + b = ?$

4. Can application of the commutative property reduce the number of addition and multiplication facts? To what extent and how?

Another property of the addition of numbers applies to the addition of three or more numbers. For example, if we wish to determine the number of pets belonging to a certain family we can add the number of dogs to the number of cats and then add that sum to the number of canaries. The same number, however, can be determined by adding the number of cats to the number of canaries and then adding that sum to the number of dogs. This idea can be expressed with numerals and parentheses, using the arbitrary numbers of 4, 5, and 6, as follows: $(4 + 5) + 6 = 4 + (5 + 6)$. This property of numbers is called the <u>associative property of addition</u>. It is used in addition exercises as follows:

(a). If two exercises of the form $\begin{array}{r}7\\6\\\underline{6}\end{array}$ and $\begin{array}{r}6\\6\\\underline{7}\end{array}$ are given to a pupil the associative property can suggest that the first exercise should be added upward but the second downward, thus first adding the pair of 6's in each case.

(b). If an exercise of the form $3 + 8 + 2 + 7 + 6 + 4$ is given to a pupil he may notice that $8 + 2 = 10$, $6 + 4 = 10$ and $3 + 7 = 10$ and thus know, on the basis of the associative property, that the three tens or 30 is the same sum he would have found had he added the numbers in the order in which they appeared in the exercise.

Exercises. By using parentheses indicate applications of the associative property.

1. 10 + 11 + 12 = 10 + 11 + 12

2. a + b + c = a + b + c

There is one other very important property of the natural numbers. It uses both operations. It is called the distributive property of multiplication over addition. It is illustrated by the relationships 3 (4 + 5) = 3 (4) + 3 (5) = 12 + 15 = 27. This property assures the pupil that if he is thinking of garden plots of different lengths but equal widths he can find the total area for purposes of determining how much fertilizer is needed to fertilize all of them by adding their different lengths and multiplying once by the common width instead of multiplying all lengths by their widths and adding the larger products. Using a common width of 75 and lengths of 50, 75, 82 and 93, the pupil will know by using the distributive property that

75 (50 + 75 + 82 + 93) = 75 x 50 + 75 x 75 + 75 x 82 + 75 x 93

75 (300) = 3750 + 5625 + 6150 + 6975

22,500 = 22,500

Exercises. Rewrite the following expressions using the distributive property:

1. 12 x 8 + 12 x 9 = ?

2. 28 + 12 = ?

3. a b + b d = ?

Still another observable property of addition is that the sum of any two numbers always yields a unique number. Thus, if we add the two natural numbers of 4 and 6 we find the sum is the natural number 10 and that the sum is unique. That the sum of two natural numbers is a natural number is regarded as a property of natural numbers called, in technical language, the closure property for addition. The sum is unique since it can never be any number other than 10. It tells the pupil that every pair of numbers has one and only one sum, assuming, of course, that the set of numbers being considered (such as the natural numbers in this case) has the closure property for addition.

All of the above-named properties of the natural numbers for the operation of addition hold, or apply, equally satisfactorily for the operation of multiplication.

Examples:

(a). $3 \times 5 = 5 \times 3$ Commutative property for multiplication.

(b). $(3 \times 4) \times 5 = 3 \times (4 \times 5)$ Associative property for multiplication.

(c). $3 \times 5 = 15$, a unique natural number. Closure property for multiplication.

Thus far in this article we have considered only the set of natural numbers. It should be pointed out, therefore, at this time, that not all of the properties described above apply to all operations and to all sets of numbers.

For example, if we are working with the set of odd numbers we can neither name nor write the sum of $5 + 3$. That is, addition does not have the property of closure using the set of odd numbers. If we are working with the set of natural numbers we can neither name nor write the difference represented by $6 - 6$, nor the quotient of $7 + 5$. To be able to name and write the difference between 6 and 6 or the quotient of 7 divided by 5 we need to extend or enlarge the set of numbers we are using to include zero and fractions. This idea of extending, or enlarging, the set of numbers as needed and noticing the additional properties that result, gives the mathematician power to cope confidently with many different types of problems.

We shall not elaborate further on the topic of properties. In conclusion, let it be pointed out that in the theory of mathematics these properties are generalized for several sets of numbers and assumed to hold for all of the numbers even though the sets are infinite in extent. At this stage of mathematical development the "properties" become "assumptions," "axioms," or "postulates," depending on the word choice of the writer. As the teacher leads the pupils from a conscious use of the properties of numbers to the conscious use of the generalized assumptions, she has truly led the pupils from the field of arithmetic into the field of mathematics.

NUMERATION

What is meant by "numeration?" For the moment we will use "system" as an undefined term. The phrases "system of notation," "system of numeration," and "number system" are all related and sometimes used erroneously as synonyms. Each phrase, however, should be understood clearly by every teacher.

A system of notation in connection with arithmetic is a system of representing numbers with written symbols, that is, making numerals. Examples with which we are well acquainted are the Roman and Hindu-Arabic systems. In the former, we write seven as VII; in the latter, 7. In the Mayan system of notation seven would be written as $\underline{\cdot\cdot}$. Systems of notation have certain principles or rules. In the Roman system, for example, if two numerals are placed side by side the one representing the smaller number when placed to the left is subtracted from the one to its right; when the smaller number is placed to the right it is added to the larger number. Thus IX equals ten minus one or nine; XI equals ten plus one or eleven. In the Hindu-Arabic system of notation the principle of position gives, in the numeral 345, the 3 a positional value of hundreds, the 4 a positional value of tens, and the 5 a positional value of ones. Thus, in the Hindu-Arabic system each 3 in the numeral 333 represents a different value but in the Roman system each X in the numeral XXX represents precisely ten.

A system of numeration is a system for expressing numbers in words. It is not concerned with the numerals (symbols) as such, nor with the number properties. An outstanding characteristic of a numeration system involves the use of a base, or radix. This is esentially a way of grouping. In the Hindu-Arabic system, for example, we group by tens, hundreds, thousands, etc. In Biblical times grouping was done by scores. The phrase "four score and ten" is representative of this type of grouping. In the binary system of numeration the base is two; in the duodecimal system, twelve.

Examples: The last numeral written in each set names the same number. The 12 in (a), 1100 in (b), 10 in (c), and 22 in (d) name the number of elements in a dozen.

(a). Hindu - Arabic (Decimal) 1 2 3 4 5 6 7 8 9 10 11 12...

(b). Binary 1 10 11 100 101 110 111 1000 1001 1010 1011 1100...

(c). Duodecimal 1 2 3 4 5 6 7 8 9 D L 10...

(d). Base five 1 2 3 4 10 11 12 13 14 20 21 22...

Pupils can learn a great deal about the decimal system of numeration by working with systems having bases other than ten.

Exercises:

1. Write the six table using the duodecimal system of numeration.

2. Describe the multiplication tables we would have if we used a numeration system with base five.

3. Name the addition facts a pupil would need to learn if he used only the binary system of numeration.

NUMBER SYSTEM

Finally, what is a "number system?" A number system is comprised of any collection of objects for which two operations called addition and multiplication have closure for any pair of objects and such that addition is commutative and associative, multiplication is commutative and associative, and multiplication is distributed over addition. There are many number systems. One that we considered earlier in this article is the natural number system. We assume that for (any) two natural numbers the operations of addition and multiplication had closure, commutativity, and associativity and, further, that multiplication was distributed over addition.

Examples.

(a). $8 + 12 = 20$ and $9 \times 6 = 54$. Closure property.

(b). $8 + 4 = 4 + 8$ and $8 \times 4 = 4 \times 8$. Commutative property.

(c). $8 + (4 + 2) = (8 + 4) + 2$ and $8 \times (4 \times 2) = (8 \times 4) \times 2$. Associative property.

(d). $4 (6 + 8) = 4 (6) + 4 (8) = 24 + 32 = 56$. Multiplication is distributed over addition.

Exercises:

1. Does the set of even natural numbers comprise a number system?
2. Does the set of unit fractions (1/1, 1/2, 1/3, 1/4,...) comprise a number system?
3. Does the set of negative integers comprise a number system?
4. Does the set of negative integers and positive integers comprise a number system?
5. Does the set of positive integers and positive fractions comprise a number system?
6. Does the set of negative integers, positive integers, positive fractions and zero comprise a number system?
7. Does the set of natural numbers 1 to 10 inclusive comprise a number system?

8. Does the set of numbers 1 and 0, used in the binary system, comprise a number system?

9. Can any finite set of numbers comprise a number system?

ANSWERS TO EXERCISES

Page 3, No. 1, 2 + 6; No. 2, 12 + 9; No. 3, b + a; No. 4, Yes, 3 + 4 and 4 + 3, for example, are identical addition facts.

Page 4, No. 1, (10 + 11) + 12 = 10 + (11 + 12); No. 2, (a + b) + c = a + (b + c).

Page 4, No. 1, 12 (8 + 9); No. 2, 4 (7 + 3); No. 3, b (a + d).

Page 7, No. 1,

6 x 1 = 6	6 x 7 = 36
6 x 2 = 10	6 x 8 = 40
6 x 3 = 16	6 x 9 = 46
6 x 4 = 20	6 x D = 50
6 x 5 = 26	6 x L = 56
6 x 6 = 30	

No. 2, We would have only three tables. The entries in the two times table would number four beginning with 2 x 1 = 2 and ending with 2 x 4 = 13.

No. 3, 0 + 0 = 0; 0 + 1 = 1; 1 + 1 = 10; 1 + 0 = 1.

Page 8, No. 1, Yes, all of the properties are satisfied. No. 2, Addition does not have closure (1/2 + 1/3 = 5/6 and 5/6 is not a unit fraction). No. 3, No, multiplication does not have closure. (-2) x (-3) = 6 and 6 is not a negative integer. No. 4, addition does not have closure: (6 + (-6)) = 0 and 0 is not a member of a given set. No. 5, Yes, all of the properties are satisfied. No. 6, Yes, all the properties are satisfied. No. 7, No. neither addition nor multiplication has closure. No. 8, No, addition does not have closure: 1 + 1 = 10 and 10 is not a member of a given set. No. 9, No, addition can not have closure because the sum of the first number and the last number will never be included in a given set.

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