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ABSTRACT

The purpose of this project was to determine if the method of subtraction of integers taught to seventh grade students affected their mathematics achievement or retention. Computation, concept, and problem-solving sections of the California Achievement Test were given as pretests and posttests. An investigator-constructed test of the addition and subtraction of integers was also used as a pretest, posttest, and as a retention test one month after completion of the treatments. The project was divided into two studies. The first study involved 140 students and three teachers, and compared the Complement Method of subtraction to the Related Facts Method. Results showed a statistical difference in the area of concepts, favoring the group taught the Related Facts Method. No statistical differences were found in the areas of computation, problem-solving, or addition and subtraction of integers. The second study involved 90 students and two teachers, and compared the Complement Method of subtraction to the Systems Method. Statistical difference on the retention of the subtraction of integers favored the Systems Method. Statistical difference on the retention of the addition of integers favored the Complement Method. No statistical differences were found in any areas of immediate achievement in the second study. (Author/DT)

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Ray C. Sawyer
Eastern Washington State College
Cheney, Washington 99004

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Ray C. Sawyer

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Office of Education
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ABSTRACT

The purpose of this study was to determine if the method of subtraction of integers taught to seventh grade students affected their achievement or retention in addition or subtraction of integers. Their ability to work subtraction problems involving three integers was also measured. Their achievement in the areas of (1) computation, (2) concepts, and (3) problem solving, as measured by the California Achievement Test, level four, sections 3, 4 and 5, was also measured.

Two school Districts participated in the study. In District One there were three teachers and 140 seventh grade students that participated in the study. In District Two there were two teachers and 90 seventh grade students that participated in the study. Each of the teachers taught an experimental group and a control group to control as much as possible the teacher variable and each teacher devoted an equal amount of time to the teaching of the unit in each of their groups.

A pretest and a post-test were administered to each group and their achievement was measured by the change in means on these two tests. One month after the completion of the experiment, a retention test was administered and the groups' retention was measured by the change in means from the post-test to the retention test. The areas measured by these tests were addition of two integers, subtraction of two integers, transfer of Complement Method

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to subtraction of rational numbers, use of whole numbers in subtraction of whole numbers and subtraction of three integers.

The California Achievement Test (1970 Edition) sections 3, 4 and 5, Form A was administered as a pretest and Form B of the test was administered as a post-test. The areas measured by this test were computation, concepts, and problem solving.

In District One the control group was taught the Related Facts Method of subtraction and the experimental group was taught the Complement Method of subtraction of integers. There was a statistical difference in favor of the control group in achievement in the area of concepts as measured by the California Achievement Test (1970 Edition) section 4. There were no statistical differences between groups in achievement or retention in the areas of addition of integers or subtraction of integers. There were no statistical differences between groups in computation or problem solving.

In District Two the experimental group was taught the Complement Method and the control group was taught the Systems Method of subtraction of integers. There was a statistical difference in favor of the control group in retention in the area of subtraction of integers, and there was a statistical difference in favor of the experimental group in retention in addition of integers. There were no statistical differences between groups in achievement in addition or subtraction of integers, or in computation, concepts, or in problem solving.

Acknowledgments

The author would like to thank his family for their patience. He would like to thank his colleagues and the members of his doctoral committee for their constructive criticism. He would also like to thank Dick Fields, Vern Fox, Sue Hatch, Phyllis Newton, John Yaryan, and their seventh grade students for altering their planned activities to participate in this study. Also a great deal of thanks to the administrative officers of the Central Valley School District and the Mead School District for allowing an experiment of this type to be conducted in their Districts. Due to the willingness of these teachers and administrators to try something new, the author was able to carry out this study.

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Chapter 1

THE PROBLEM AND DEFINITIONS OF TERMS

Textbooks presently used in the upper elementary grades use different methods of introducing integers and operations with integers. Professional journals concerned with problems in teaching, regularly contain articles about introducing integers and their operations to students.

There has been a great deal of research into the subtraction of whole numbers but there seems to be very few cases where there has been research into the subtraction of integers. In light of the fact that there has been very little research into the relative merits of the different methods of subtraction of integers it seems plausible that such research should be forthcoming.

THE PROBLEM

Statement of the Problem

It was the purpose of this study to:

1. Compare the achievement of students taught to perform different methods of subtracting two integers.

2. Compare the retention of knowledge and skill possessed by students taught to perform different methods of subtraction of two integers.

3. Determine if the method of subtraction learned affected

the students' ability to work with subtraction problems involving three integers.

4. Determine if students would transfer the use of the subtraction method employed by the investigator to subtraction problems involving positive rational numbers.

5. Determine if the method of subtraction employed by the investigator would affect the students' achievement or retention in addition of integers.

Importance of the Study

There has been research in the area of subtraction of whole numbers,[1] and there has been some research into the question of whether or not a student can do subtraction of whole numbers by learning the "negative number subtraction method."[2] The majority of the studies dealing with subtraction involved subtraction problems where the minuend was larger than the subtrahend and both were whole numbers. However, a study by Coltharp[3] considered different methods of subtracting integers. His study compared the use of ordered pairs with the use of the number line as a model for the integers. In this study, however, the students' overall achievement was measured and there was no mention

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[1]Brownell, W.A. and Moser, H.E., "Meaningful versus Mechanical Learning: A Study in Grade III Subtraction." Duke University Research Studies in Education, No. 8 Durham, N.C., Duke University Press, 1949.

[2]Gran, Eldon Edward, "A Study to Determine Whether the Negative Number Subtraction Method Can Be Learned and Used by Elementary Pupils," Doctoral Dissertation, Ann Arbor, 1967.

[3]Coltharp, F.L., "A Comparison of the Effectiveness of an Abstract and a Concrete Approach in Teaching of Integers to Sixth Grade Students," Doctoral Dissertation, Oklahoma State University, 1968.

of the students' achievement in a specific area such as subtraction. Since Coltharp's study did not investigate the method of subtraction employed by this investigator and advocates of particular methods seem to have chosen their position, relative to a method of subtraction on the basis of conjecture or a confident feeling, research into this area of elementary mathematics instruction was felt to be needed.

DEFINITIONS OF TERMS USED

1. Achievement. Gain in mean score between pretest and post-test.

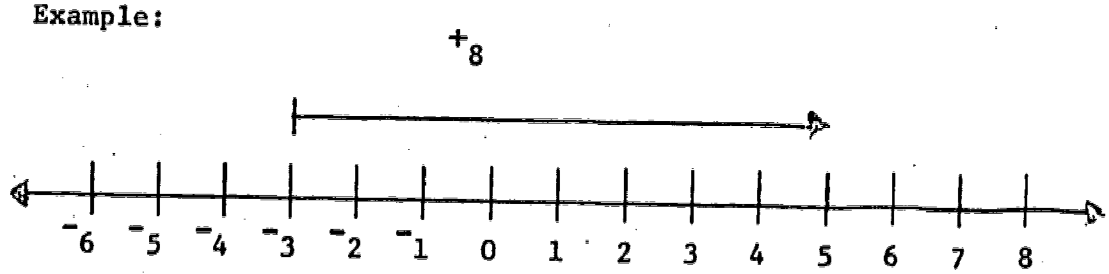
2. Retention. Gain or loss in mean score between post-test and retention test.

3. Complement Method. Method of subtraction by adding the same number to both the minuend and the subtrahend.

Example: $^{+}5 - {}^{-}3 = ({}^{+}5 + {}^{+}3) - ({}^{-}3 + {}^{+}3) = {}^{+}5 + {}^{+}3.$

4. Difference Method. Method of examining the difference between the subtrahend and the minuend.

Example:



On the number line there is a difference of 8 to the right from $^{-}3$ to $^{+}5$. Therefore, $^{+}5 - {}^{-}3 = {}^{+}8$.

5. Ordered Pair Method. A method of subtraction using ordered pair representation of integers.

Example: To work the problem $^{+}5 - {}^{-}3$, the following ordered pairs could be chosen and subtracted.

$$(9,4) - (1,4) = (8,0).$$

Since (8,0) represents the integer $^{+}8$, the answer to the problem $^{+}5 - {}^{-}3 = {}^{+}8$.

6. Pattern Method. A method of subtraction that involves starting with a problem in which the answer is known and making use of a pattern to determind the unknown answer to a more difficult problem.

Example: $^{+}5 - {}^{-}3$. This problem could be worked in the following manner.

$$^{+}5 - {}^{+}4 = {}^{+}1$$
$$^{+}5 - {}^{+}3 = {}^{+}2$$
$$^{+}5 - {}^{+}2 = {}^{+}3$$
$$^{+}5 - {}^{+}1 = {}^{+}4$$
$$^{+}5 - 0 = {}^{+}5$$
$$^{+}5 - {}^{-}1 = {}^{+}6$$
$$^{+}5 - {}^{-}2 = {}^{+}7$$
$$^{+}5 - {}^{-}3 = {}^{+}8$$

7. Related Number Facts Method. Method of subtraction involving the relationship between subtraction and addition.

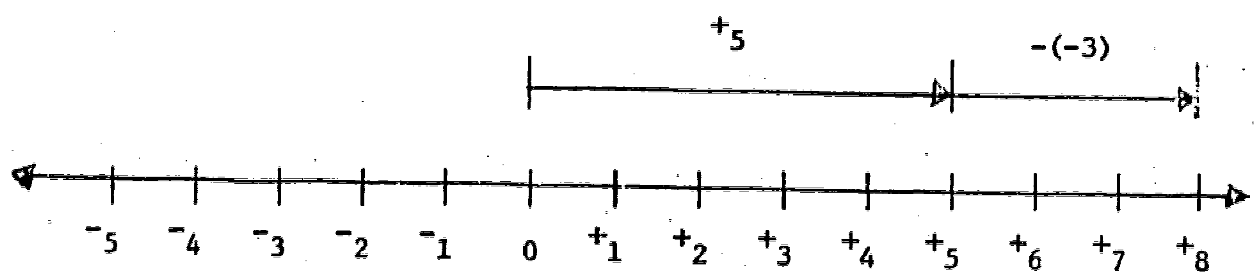
Example: $^{+}5 - {}^{-}3 = N$ if and only if $N + {}^{-}3 = {}^{+}5$; therefore, $N = {}^{+}8$.

8. Systems Method. By examining a modular system the student learns that $x - y = x + {}^{-}y$. This is generalized to the integers.

Example: $7 - 8 = 7 + {}^{-}8 = {}^{-}1$.

9. Take-Away Method. Method of subtraction involving taking the subtrahend "away from" the minuend.

Example:



$^{+}5 - {}^{-}3$

On the number line the last arrow ends above the $^{+}8$, therefore $^{+}5 - {}^{-}3 = {}^{+}8$.

PROPOSED METHOD

The Complement Method of subtraction of integers proposed by the investigator does not rely on the use of the number line, nor does the student need to solve a mathematical sentence by inspection in order to solve the subtraction problem.

The Complement Method relies on a fact that many elementary students know from their work in subtraciton of whole numbers. Many students learn that if $5 - 4 = 1$, then $6 - 5 = 1$; that is, if they add the same number to the subtrahend and minuend the remainder is unchanged. This fact is used by many people when they work the problem $197 - 99 = \square$. They add 1 to both numbers and work the problem $198 - 100 = \square$.

Furthermore the idea of adding the same amount to the subtrahend and minuend, and not changing the answer to the subtraction problem can be reinforced by using models the students see every day. One way is to draw their attention to the odometers on a car speedometer. Some cars have an odometer that indicates total mileage the car has traveled and the other odometer indicates the distance traveled up to 999 miles, and then returns to zero. It is possible to examine the difference of the numbers on the two odometers at certain times. See Appendix D for an example of this method of introduction to subtraction of integers. There are other instances that students are perhaps more familiar with than the instance mentioned above. Consider the choosing of teams for a game of football. Students know that if the difference in the number of students on the two teams is one and if one more student is added to each team the difference is still one.

This idea can be extended to work subtraction of integers, such as $^{+}5 - {}^{+}3 = \square$. There are many numbers that could be added to the minuend and subtrahend in the problem, but the choice of $^{-}3$ will produce the most profound change in the appearance of the problem. Note that by adding $^{-}3$ to both numbers the problem becomes $(^{+}5 + {}^{-}3) - (^{+}3 + {}^{-}3) = \square$ or simply $^{+}5 + {}^{-}3 - 0 = \square$. Clearly if the student understands how to add he can now work the problem. He must also understand, of course, that to subtract zero does not change the answer. The steps of the problem are delineated in the following:

1. $^{+}5 - ^{+}3 = \square$
2. $(^{+}5 + ^{-}3) - (^{+}3 + ^{-}3) = \square$
3. $(^{+}5 + ^{-}3) - 0 = \square$
4. $^{+}5 + ^{-}3 = \square$
5. $^{+}2 = \square$

DELIMITATION OF STUDY

This study was conducted within the following framework:

1. Only students enrolled in the seventh grade during the 1970-71 school year participated.

2. Only students enrolled in the Central Valley or Mead School Districts participated in the study.

3. The study was confined to subtraction of integers, although student achievement and retention in other operations was also measured.

4. Although the students in the experimental groups were taught the Complement Method, it was not required that they write all of the steps in the subtraction problem after they had completed the work in Appendix D.

5. Only teachers teaching at least two sections of seventh grade mathematics were asked to participate in the study.

6. The study was confined to schools that had texts using the systems method or the related number facts method of subtraction of integers.

EXPECTED CONTRIBUTIONS

1. Evaluation of a method in terms of its potential as an initial introduction to subtraction of integers.

2. Evaluation of a method in terms of its appeal to seventh grade teachers.

3. Evaluation of a method of subtraction of integers in terms of its transfer to subtraction of rational numbers.

4. Evaluation of a method of subtraction of integers as to its effects on the students' ability to perform addition of integers.

5. Listing of recommendations relative to further research indicated by the results of this study.

ORGANIZATION OF DISSERTATION

Chapter 2 of this paper includes a review of the literature relative to the problem of subtraction of integers. Chapter 3 deals with the procedures of the study including (1) descriptions of the population, (2) method of training of teachers in the new method, (3) development of the testing instrument, and (4) the description of the treatment administered to each of the groups. Chapter 4 deals with the statistical analyses of the data, and also includes responses of teachers that used the Complement Method. Chapter 5 contains the summary and conclusions reached in the study. The final section of Chapter 5 includes suggestions for further research.

Chapter II

REVIEW OF THE LITERATURE

There are many different proposed methods of teaching integers to children. While most of these proposed methods are logical and mathematically sound there does seem to be a definite controversy as to which method might be best. Indeed, perhaps the methods are all the same as far as the student being able to understand, use and retain them.

The use of a model involving positive and negative particles as an aid in understanding subtraction of integers was advocated by Cotter.[1] For a model of subtraction he considered a field with zero charge but having the same number of positive and negative particles. If two positive charges are removed from the field, the result is a ${}^{-}2$ charge on the field. This illustrates the subtraction problem $0 - {}^{+}2 = {}^{-}2$. For the problem ${}^{+}4 - {}^{-}2$, consider a field having a ${}^{+}4$ charge with other neutrally charged particles in the field. The neutrally charged particles are considered to be made up of one positive charge and one negative charge. When two negative charges have been removed, the field then has a ${}^{+}6$ charge. The mathematical sentence would be ${}^{+}4 - {}^{-}2 = {}^{+}6$.

A method employed by Entwhistle[2] used the following types of examples to illustrate subtraction:

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[1]Cotter, Stanley, "Charged Particles: A Model for Integers," The Arithmetic Teacher, XVI, (May 1969) 349-353.

[2]Entwhistle, Alice, "Subtracting Signed Numbers," The Mathematics Teacher, XLVIII, (March 1955) 175-176.

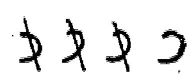
1. The teacher makes a \$15 bet with a student in the class. The teacher loses the bet and has only \$10 to pay the student. Mathematical sentence:

$$^{+}10 - (^{+}15) = {}^{-}5$$

2. Jane owes Mary \$12. She pays Mary \$10, that is, Jane has a debt of \$12 and she subtracts a payment of \$10. Jane now owes a debt of \$2. Mathematical sentence:

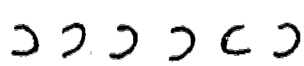
$$^{-}12 - (^{-}10) = {}^{-}2$$

A method using concrete objects was employed by Fremont[3] who used pipe cleaners to represent positive and negative numbers. A pipe cleaner bent in this manner ⊃ represents $^{+}1$ and one opening the other direction represents $^{-}1$. Suppose the problem is $^{+}4 - {}^{+}3$. If subtraction is thought of as a take away process then the problem is worked as follows:

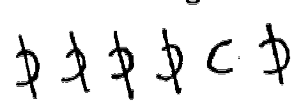


where a slash has been drawn through three of the loops to indicate three have been taken away.

What about the problem $^{+}4 - {}^{+}5$? Obviously we have ⊃ ⊃ ⊃ ⊃ and we wish to take away $^{+}5$. Adding zero will not change the value of $^{+}4$ so zero is added in the form of ⊂ ⊃ and the problem becomes:



Now take away $^{+}5$ as in the following:



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[3] Fremont, Dr. Herbert, "Pipecleaners and Loops--Discovering How to Add and Subtract Directed Numbers," *The Arithmetic Teacher*, XIII, (November 1966) 568-572.

leaving $\subset$ or $^{-}1$.

Henderson used a difference model on the number line. His ideas are illustrated by the following:

> The technique as outlined by W. E. Romig in the January Mathematics Teacher for teaching subtraction of signed numbers is a graphical representation of the mechanical process of changing the sign of the subtrahend and proceeding as in addition. I believe that pupils will find the following method more meaningful; my pupils prefer it to the mechanical method which they find outlined in our text. Too, many teachers that use this plan never introduce the mechanical one at all.
>
> In teaching subtraction of signed numbers, I first draw a number scale. Zero is shown to be the division point between positive and negative numbers. There are only two directions in which one may go upon this scale: The negative direction is right to left from any point on the number scale. Let us first study an example of subtraction in arithmetic: Subtract 4 from 9.
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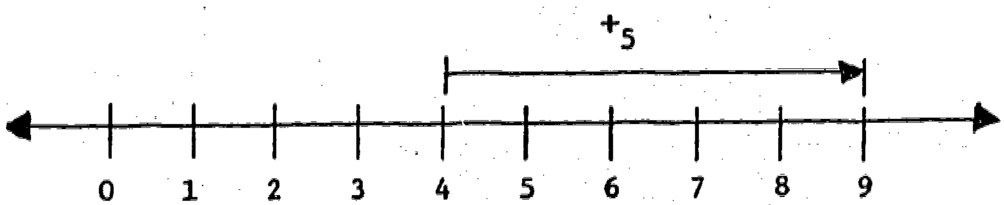
> 
>
>
> Start at 4 and count the units between 4 and 9. In arithmetic the positive direction is always from left to right, so it is logical to picture as negative the direction from right to left. The above problem should be followed up by using 9 as the subtrahend and 4 as the minuend. The absolute value remains the same, but the sign is negative. In other words, when we subtract two quantities, the absolute value of the difference equals the number of units which separate the subtrahend from the minuend, the sign being determined by the direction traveled in going from the subtrahend to the minuend.[4]

The technique advanced by Romig and referred to in the above quote is illustrated by the following:

> Suppose that the pupil has mastered the process of algebraic addition so that he can add correctly such combinations as the following:
>
> | | | | |
> |---|---|---|---|
> | +8 | +8 | −8 | −8 |
> | +4 | −4 | +4 | −4 |

[4] Henderson, L.M., "Alternative Technique for Teaching Subtraction of Signed Numbers," The Mathematics Teacher, XXXVIII (November 1945), 331.

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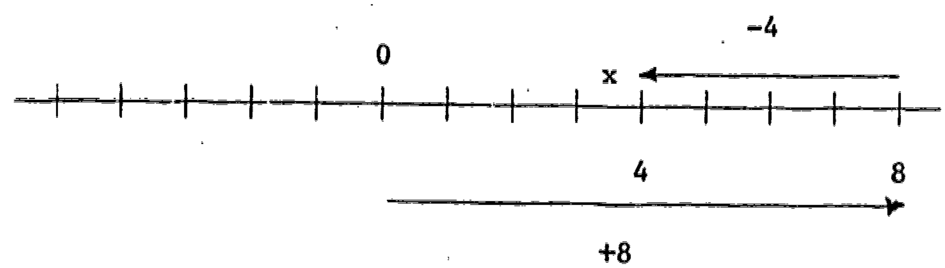
Suppose further that he has learned to add these numbers graphically, thus add

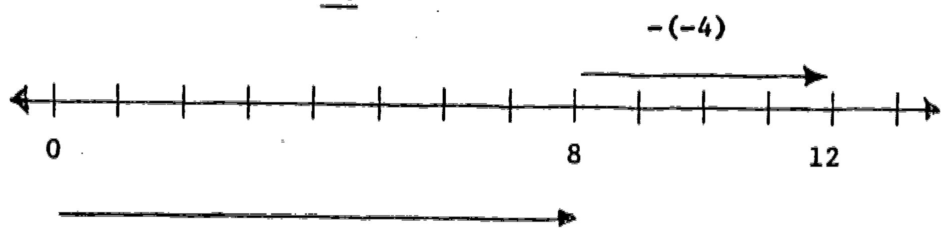
+8
<u>-4</u>



The pupil has been taught to reason, "Starting at zero I count eight units to the right (up) because the sign of 8 is plus, and then four units to the left (down) because the sign of 4 is minus. This places me four units to the right of (above) zero, therefore the sum of +8 and -4 is +4."

When the pupil has mastered this process, he is ready to begin subtraction. Explain that the operation of subtraction is the opposite of the operation addition, and numbers to be subtracted must be counted in the direction opposite to that in which they would be counted for addition, thus: subtract

+8
<u>-4</u>



Starting at zero I count 8 units to the right because the sign of 8 is plus. If I were adding, I would count 4 units to the left because the sign of 4 is minus, but since I am subtracting, I must count 4 units to the right, opposite to the direction for addition.[5]

Coltharp[6] examined sixth grade students' ability to work with a "concrete approach" vs. an "abstract approach" working with integers.

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[5] Romig, W.E., "Technique for Teaching Subtraction of Signed Numbers," <u>The Mathematics Teacher</u>, XXXVIII, (January 1945) 36.

[6] Coltharp, F.L., "A Comparison of the Effectiveness of an Abstract and a Concrete Approach in Teaching Integers to Sixth Grade Students," Doctoral Dissertation, Oklahoma State University, 1968.

Coltharp defined the concrete approach as working with the number line as in the Greater Cleveland Mathematics Program, Intermediate Series, Book 7, 1964-65 edition. He defined the abstract approach as the use of ordered pairs. He made the following conclusion:

> Sixth grade students taught manipulation of signed numbers from an abstract approach achieved as well as those taught by means of a concrete, visual approach when students with equated abilities were compared.[7]

The use of patterns to enable the students to understand subtraction of integers was proposed by Magnuson.[8] This method starts with a subtraction problem that the student is familiar with and proceeds to problems that the student could not work before by means of using the pattern.

Example:

$5 - 1 = 4$

$5 - 2 = 3$

$5 - 3 = 2$

$5 - 4 = 1$

$5 - 5 = 0$

At this point the pattern is discussed and then the problems proceed as follows:

$5 - 6 = -1$

$5 - 7 = -2$

$5 - 8 = -3$

.

.

. etc.

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[7] Ibid., p. 50.

[8] Magnuson, Russell C., "Signed Numbers," The Arithmetic Teacher, XIII, (November 1966) 573-575.

Elementary texts use some of the methods described so far in this paper but there are other methods, in addition to the ones mentioned, that are incorporated into texts. A commonly used one is the missing addends method. This method is illustrated by the following from Elementary School Mathematics, Addison and Wesley, 1964, Sixth Grade text. The section dealing with subtraction is started with the following examples:[9]

When you find		
this addend	$n + {}^{-}3 = 0$	you find this
	$0 - {}^{-}3 = n$	difference

After presenting several problems of the type illustrated above, that is, where the sum is zero, the following examples are introduced. The student is to find the missing addend A.

> $2 - {}^{-}5 = A$
>
> If this were zero, the missing addend would be 5. Since this is two more than zero, the missing addend is 2 + 5 or 7.

Another method used in a text series published by Holt,

[9]Eickolz, O'Daffer, et. al., Elementary School Mathematics, Book 6, Reading, Massachusetts: Addison Wesley Publishing Company, Inc., (1964) 291.

Rinehart and Winston[10] is the system approach. The following illustrates this approach.

> When you studied clock arithmetic you learned about subtraction. It is the opposite of addition. In clock arithmetic every element has an additive inverse. We found we could subtract by adding an inverse. For example, in the arithmetic of the five minute clock
>
> $$2 - 3 = 2 + -3$$
>
> In the system of integers all the numbers have additive inverses, so we can subtract this same way. In fact, we can define subtraction to be the addition of the inverse.[11]

The numerous articles dealing with subtraction of integers indicate a diversity of opinion among teachers and other experts in this area. The use of different methods of teaching subtraction that occur in elementary arithmetic series also further indicates that this fact is so. There seems to be no indication in the articles read by the investigator that anyone has researched whether one of the methods is better than the others when the students' achievement and retention is used as a measure of the superiority of the method.

During his years of public school teaching the investigator experienced a great deal of difficulty when trying to teach the students how to subtract integers. The investigator used many of the methods illustrated so far but the students did not respond in a positive manner.

During the 1968-69 school year the investigator was involved in a field test of a new sixth grade text in a new arithmetic series. This series used the Complement Method of subtraction. The students responded quite well to this approach and did not seem to experience

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[10] Keedy, Jameson, Johnson, Exploring Modern Mathematics, Book 2, New York: Holt, Rinehart and Winston, Inc., (1963).

[11] Ibid., p. 23.

many of the difficulties the investigator had observed in his earlier teaching experiences. There are many models for explaining subtraction of integers and many articles written as to the relative attributes of using a particular model. There seems to be no agreement as to which model is most easily used and retained by students. There does seem to be agreement that subtraction of integers is a troublesome area in mathematics as witnessed by the number of articles written on the subject. It seems that, because of the importance of subtraction of integers to the further study of mathematics and the concern of people involved in the area, an investigation of the problem would be very important to the field of mathematics education.

CHAPTER III

PROCEDURES OF THE STUDY

Four school districts were contacted and asked to participate in the study. These school districts were Spokane, Medical Lake, Mead and Central Valley. One of the school districts did not wish to participate and two of the three school districts used the same seventh grade texts. Of the two districts using the same text, one was eliminated by the flip of a coin. The two districts remaining to participate in the study were Central Valley, henceforth referred to as District One, and Mead, henceforth referred to as District Two. The approximate student enrollment of the two districts was 10,000 and 5,000 respectively.

District One used "Modern Mathematics Through Discovery", published by Silver Burdett Company; 1966, and District Two used "Exploring Modern Mathematics", (Book 1) published by Holt Rinehart and Winston; 1963.

In District One, four seventh grade teachers, teaching at least two classes of mathematics, were asked if they would like to participate in the study. Three of the teachers in District One elected to participate. These three teachers had not been teaching concepts of integers previously because it was a section near the end of the seventh grade text and was not ordinarily included in the curriculum at this level. All three teachers graciously volunteered to change their teaching schedule so as to include the study of integers as a special unit.

The two teachers contacted in District Two both agreed to participate in the study. Neither teacher had previously taught concepts of integers in the seventh grade. The text used in the seventh grade did not include a study of integers but it did include the study of clock arithmetic. The investigator prepared ditto sheets, utilizing material from the Holt Rinehart and Winston text (Book 2) used by the eighth grade, that introduced the seventh grade students to integers and the operation of addition of integers.

THE POPULATION

The Teachers

The teachers who participated in this study were teachers regularly employed by their school district to teach seventh grade mathematics. Their teaching background was varied, ranging from a first year teacher to a teacher who had taught 20 years.

There was also a wide variety of academic backgrounds in terms of years of study and degrees held. Since this is typical of seventh grade teachers it was felt that the experiment would be conducted under conditions similar to those in any school districts of comparable size. One thing that did make these teachers different than other seventh grade teachers is that they were teaching at least two classes of mathematics. (See Table I).

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Table I

TEACHERS EDUCATION AND YEARS OF TEACHING EXPERIENCE

District	Teacher	Highest Degree (Date Granted)	Major(s)/Minor(s)	Number of College Level Math. Courses	Years of Education	Years of Experience
1	1	MA (Ed) (1959)	Soc. Studies/ P.E., Science	3	6	20
1	2	B.S. (Ed)	Math	8	4	1st Year of Teaching
1	3	MA (Ed) (1958)	Music, P.E./ English, German	1	5	2
2	1	B.A. (Ed) (1959)	Science/ Math., Chem.	6	5	5
2	2	B.A. (Ed) (1959)	Soc. Studies/ Ind. Arts	3	6	11

The Students

All of the students were seventh graders and none were repeating the grade. There was no ability grouping by the districts and the students were assigned to their mathematics classes in a random fashion. The total number of students participating in the study was 290. Due to the fact that some students missed examinations during the study and others moved away 230 students' data were included in the study.

In District One there were 70 students in the control group and 70 students in the experimental group. Of the students in the control group there were 33 boys and 37 girls. The experimental group was comprised of 30 boys and 40 girls.

In District Two the control group was made up of 21 boys and 24 girls making a total of 45 students. The experimental group in this district also had 45 students of which 19 were boys and 26 were girls.

The determination as to which class was the control group and which was the experimental group, in each district, was made by a flip of a coin. In District One, Teacher One taught Mathematics fourth and fifth period and it was determined by a flip of a coin that his fourth period class would be a control group and his fifth period class would be an experimental group. Since it was the intent of the investigator to control the variable of time, it was then decided that the fifth period class of Teacher Two would be a control group and his sixth period class would be an experimental group. This provided both a control group and an experimental group at fifth period.

Teacher Three taught Mathematics during third and fourth period. By selecting his fourth period class as the experimental group, this provided both an experimental group and a control group at fourth period. Due to the fact that class schedules were set by the district there was no opportunity to provide a sixth period control group to balance with the sixth period experimental group. Also, there was no third period experimental group to balance with the third period control group. However, there was no time period larger than one hour between a control group and an experimental group so the variable of time of day should not be a significant factor in the study.

In District Two there were two teachers. Each teacher taught a third period class and a fifth period class. It was determined by a flip of a coin that Teacher One would teach a control group third period and experimental group fifth period. To control the variable of time, Teacher Two taught an experimental group third period and a control group fifth period. Thus in District Two the variable of time could not be considered significant.

Tables II and III show in tabular form the number of students per group and the time of day that each group met. They also indicate the type of group (control or experimental) at each time of the day.

Table II

TIME OF DAY AND NUMBER OF STUDENTS PER GROUP IN DISTRICT ONE

Period	Teacher One	Teacher Two	Teacher Three
1st			
2nd			
3rd			Control 11 Girls 10 Boys Total 21
4th	Control 12 Girls 13 Boys Total 25		Experimental 12 Girls 12 Boys Total 24
5th	Experimental 12 Girls 10 Boys Total 22	Control 14 Girls 10 Boys Total 24	
6th		Experimental 16 Girls 8 Boys Total 24	

Table III

TIME OF DAY AND NUMBER OF STUDENTS PER GROUP IN DISTRICT TWO

Period	Teacher One	Teacher Two
1st		
2nd	Control 14 Girls 13 Boys Total 27	Experimental 12 Girls 7 Boys Total 19
3rd		
4th		
5th	Experimental 14 Girls 12 Boys Total 26	Control 10 Girls 8 Boys Total 18
6th		

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TEACHER PREPARATION AND TEACHING PROCEDURE

Preparation

The teachers who participated in the study were given the materials to be taught at least two months in advance of the beginning of the study. The investigator then worked with the teachers, on an individual basis, explaining the Complement Method of subtraction of integers. Each teacher was encouraged to ask questions and work the problems in the unit. Other methods of subtracting the integers were also discussed in these training sessions. When the teacher was confident that he could work effectively in the set of integers, the training session was discontinued. The average length of the training sessions was about six hours. During the training session, the procedures of the study were also explained to the teachers.

Teaching Procedure

In District One the teachers started the unit on integers with an introduction to integers and proceeded through addition of integers in both of their groups. See Appendix A for the material taught. At this point subtraction was introduced to both groups. The method the teachers taught to their control group was the related number facts method. (See Appendix B.) The method of subtraction they taught to their experimental group was the Complement Method of subtraction. (See Appendix D.) The final part of the unit was Appendix C and this material was taught to both groups. Thus the only difference between the material taught by the teachers to their control group and their experimental group was the method of subtraction.

District Two teachers started the unit with modular arithmetic, using a five minute clock as their model. (See Appendix E.) After this the integers were introduced (Appendix F) by the teacher to both their control group and their experimental group. Then they proceeded through addition of integers (Appendix G) using the properties of Mathematical systems that had been taught in Appendix E.

At this time the teachers taught their control group the Systems Method of subtraction that is illustrated in Appendix H. They taught their experimental group the Complement Method of subtraction and finished the unit by teaching the material in Appendix C to both of their groups. Thus the material taught to the experimental group was the same with the exception of the method of subtraction.

SELECTION AND DEVELOPMENT OF TESTING INSTRUMENTS

The Mathematics section of the California Achievement Test, 1970 Edition, level 4, grades 6, 7, 8, and 9 was selected to be one of the tests administered. It consisted of three sections: section one, computation; section two, concepts; and section three, problem solving. Form A of this test was administered as a pretest March, 1971, and Form B was administered as a post-test at the end of the school year in June, 1971.

Because of the fact that the California Achievement Test did not contain any problems involving computation in the set of integers, another test with two forms was devised by the investigator. Form A of this test was administered on a pretest, post-test basis, to measure the students' achievement, and Form B was administered one month after the post-test to measure the students' retention. It was felt that two forms of the test were necessary to prevent the students from becoming bored with taking the same test three times and also to avoid the possibility that the students might memorize the answers to the questions.

All tests were administered by the teachers participating in the study. It was felt that if the tests were administered by a special person that this would cause the students to become cognizant of the fact that they were participating in a special study. It was the desire of the investigator to avoid the Hawthorne Effect as much as possible.

Development of Form A and Form B

The test developed by the investigator contained three parts. Part one contained addition problems involving two integers. Part two contained problems dealing with subtraction of two integers and part three contained problems involving subtraction of rational numbers, subtraction of whole numbers and subtraction of three integers. The two forms of this test are contained in Appendix I.

To develop part one and part two of the instrument, the investigator first considered the set of problems $a * b$, where a and b are integers and * stands for either the operation addition or subtraction. Next this set of problems was separated into two subsets; set X, where $|a| < |b|$, and set Y, where $|a| > |b|$. (Figure 1)

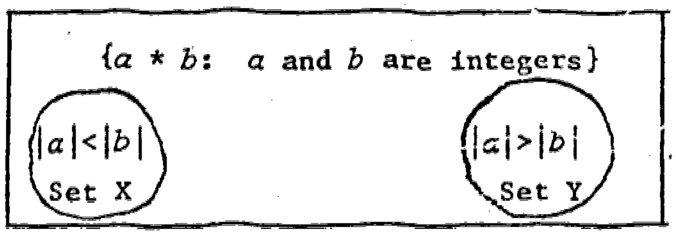

FIGURE 1

SUBSETS OF THE SET OF $a * b$
WITH REGARD TO ABSOLUTE VALUE

It was known that the subsets shown in Figure 1 were not the only subsets possible. There were also the subsets illustrated in Figure 2. These subsets arose from the fact that if an integer is chosen at random it is either positive, negative, or zero. The investigator felt that the inclusion of the integer zero would not be necessary so the subsets shown in Figure 2 resulted.

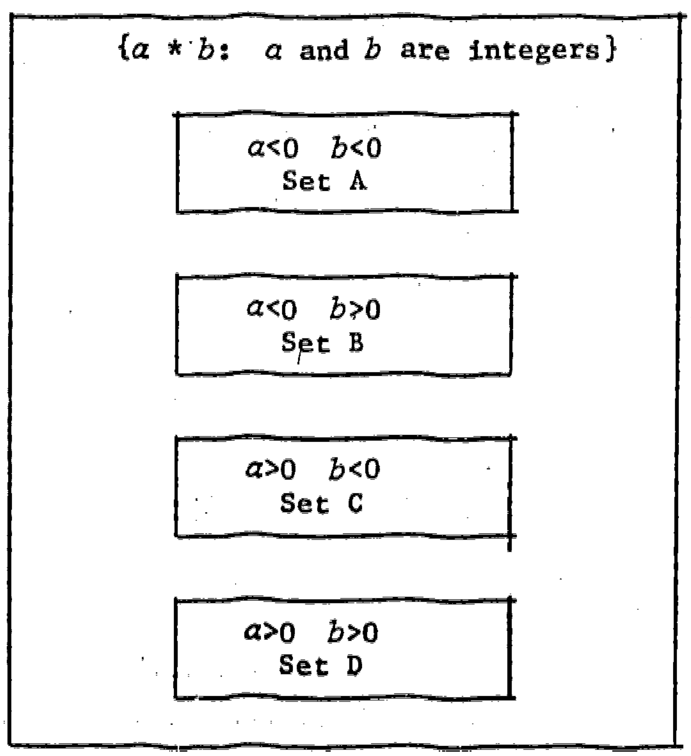

FIGURE 2
SUBSETS OF THE SET OF $a * b$
WITH REGARD TO POSITIVE OR NEGATIVE
VALUES OF a AND b

When Figure 1 was placed over Figure 2 the result was the identification of eight types of problems to be placed on part 1 and part 2 of the test to provide a name for each type of problem. It was decided that the type of problem occurring in the intersection of set A and set X would be called Type AX and for a problem occurring in the intersection of set A and set Y would be called Type AY, etc.

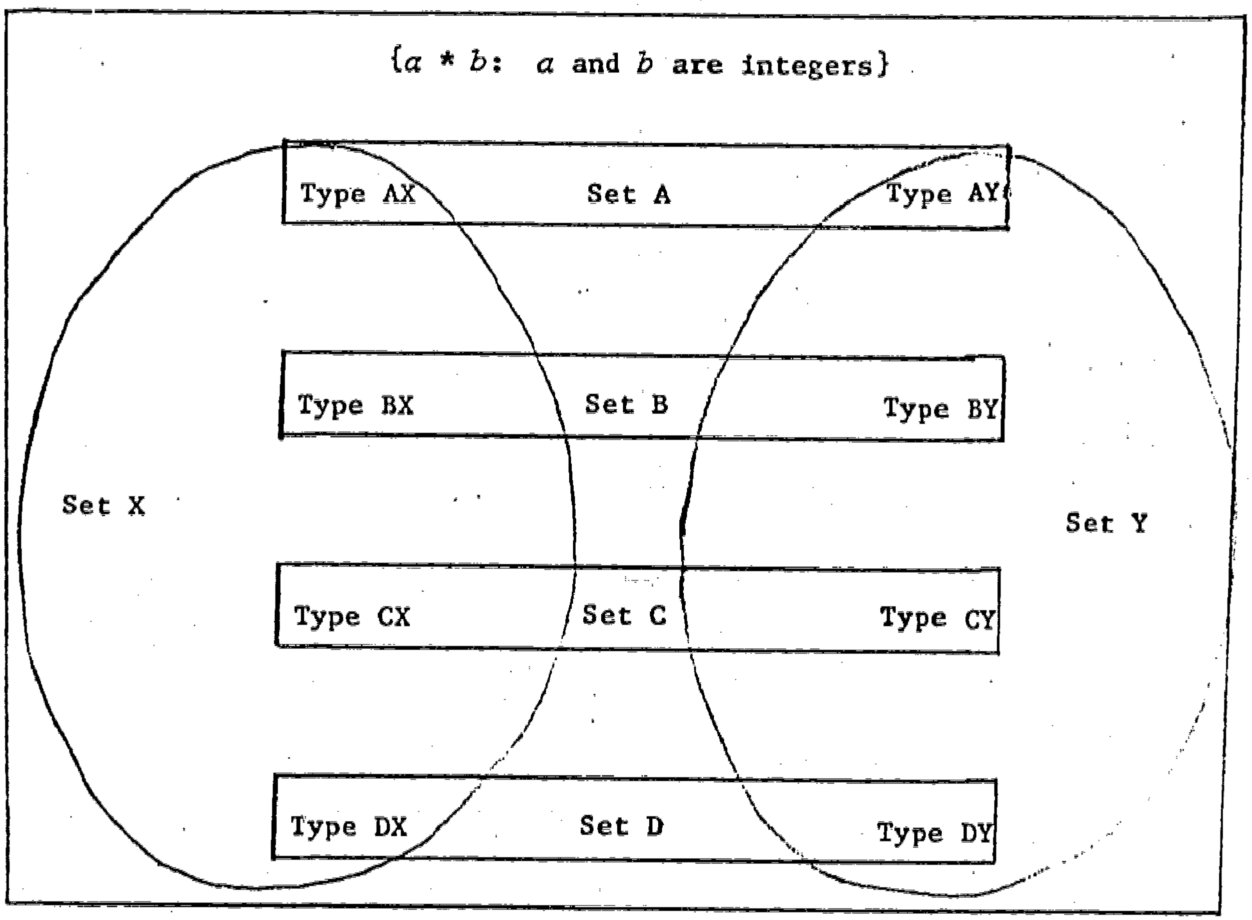

FIGURE 3

TYPES OF PROBLEMS

Figure 3 indicated there were eight types of problems with properties described in Figures 1 and 2 to be placed in part A and part B of the test. Table IV contains examples of each type of problem for the operation addition.

Table IV

EXAMPLES OF TYPES OF PROBLEMS

	Type of Problem	Example
Group one	Type AX Type AY	$^{-}3 + {}^{-}7$ $^{-}7 + {}^{-}3$
Group two	Type BX Type CY	$^{-}8 + {}^{+}12$ $^{+}12 + {}^{-}8$
Group three	Type CX Type BY	$^{+}3 + {}^{-}9$ $^{-}9 + {}^{+}3$
Group four	Type DX Type DY	$^{+}3 + {}^{+}8$ $^{+}8 + {}^{+}3$

It was noted that because of the commutative property of addition, a problem of type BY can be thought of as a problem of type CX. This was determined by examining an example of type BY and type CX and finding that they are the same problem when the commutative property of addition is applied to the problem. Example: $^{-}9 + {}^{+}3 = {}^{+}3 + {}^{-}9$. This fact lead to the definition of four groups of problems as illustrated in Table IV. Thus there were four groups of problems placed in part one of the test, where each group contained one problem of each type that made up the group. Refer to Table IV for definition of groups.

Since subtraction of integers is not commutative, it was not possible to combine types of problems into groups and the result was eight types of problems to be included in part two of the test.

It was determined that two problems of each group (addition) or type (subtraction) should be included in the test because of the possibility that a student might make a careless mistake on an item rather than because of lack of knowledge of how to work the problem. The procedure of including two items of each category also provided a means of testing the reliability of parts one and two of the test.

The resulting test then contained eight addition problems, and eighteen subtraction problems in parts one and two respectively. With the additional condition that $|a| \leq 40$ and $|b| \leq 40$ two forms of the test were constructed.

Part three of the test was constructed so as to contain two problems on subtraction of rational numbers, two problems dealing with subtraction of whole numbers, and six problems dealing with subtraction of three integers. The heading "show your work" was placed at the beginning of this part and the intention of this part was to determine if students that had been taught the Complement Method of subtraction would use it when working with rational numbers and with whole numbers. The purpose of the problems involving three integers was to determine if there was a difference, in achievement or retention, between the control group and experimental group when working this type of problem.

Validity and Reliability of the Instrument

> Ascertaining the degree to which an evaluation instrument passes content validity is equivalent to demonstrating the extent to which the content of the instrument adequately samples certain types of situations or subject matter.

> The instrument claiming content validity clearly attempts to include a cross-sectional sample of a large universe of items representing the area in which the pupils' performance is being evaluated.[1]

Because of the fact that the instrument contained problems involving computations with integers and its intended purpose was to measure the students computational ability when working with integers, the instrument was judged to satisfy that part of the above stated criterion for validity. Secondly, the instrument included a cross-sectional sample of a large universe because of the method of determining the types and groups of problems to be included in it.

A measure of the reliability of the instrument was determined by computing the Phi Coefficient of correlation on the items in each group or type classification. The formula used was:

$$\phi = \frac{BC - AD}{\sqrt{(A+B)(C+D)(A+C)(B+D)}}$$

Where A, B, C, and D are the four cell frequencies defined as follows:

		Item 2: Fail	Item 2: Pass
Item 1	Pass	(A)	(B)
	Fail	(C)	(D)

[1]Ackman, Glock, et al. Evaluating Elementary School Pupils. (Boston: Allyn and Bacon, Inc., 1960) pp. 59.

Because $X^2 = N\Phi^2$, we can readily test the significance of Φ by referring to $N\Phi^2$ to a chi-square table with 1 degree of freedom.[2]

Computing the Phi Coefficient gave a measure of whether or not the two items of a particular group or type were correlated or not. That is a statistically significant correlation between the two items indicated they were measuring the same ability. See Table V and Table VI for the Phi Coefficients and their level of significance.

ORGANIZATION AND CONDUCT OF THE EXPERIMENT

At the beginning of the experiment each teacher administered The California Achievement Test (Form A) 1970 Edition, Mathematics section to both of their groups. They also administered Form A of the test devised by the investigator. After these tests were administered, the teachers started the teaching sequence which lasted for approximately two weeks. The time devoted to the unit varied from teacher to teacher, within 2 days, but each teacher devoted the same amount of time to his control and experimental groups. At the end of the unit each group was tested, using Form A again, and one month later all groups were tested using Form B of the test devised by the investigator.

Scope and Sequence of Material Taught in District One

Each teacher in District One started both groups with the

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[2]Ferguson, Statistical Analysis in Psychology and Education. (New York: McGraw Hill, 1966) P. 239.

Table V

PHI-COEFFICIENTS FOR ITEMS OF FORM A; PART 1 AND PART 2

Part 1

Row item	Column item	P row, F	P row, P	F row, F	F row, P	ϕ	$N\phi^2$	p^*
1	5	14	197	7	12	ϕ=.2886	$N\phi^2$=19.1360	p^*<.001
2	3	10	164	55	11	ϕ=.7506	$N\phi^2$=129.5820	p^*<.001
4	8	24	159	36	11	ϕ=.5829	$N\phi^2$=78.1310	p^*<.001
6	7	14	156	48	12	ϕ=.7101	$N\phi^2$=115.9660	p^*<.001

Part 2

Row item	Column item	P row, F	P row, P	F row, F	F row, P	ϕ	$N\phi^2$	p^*
1	4	24	192	9	5	ϕ=.3626	$N\phi^2$=30.2220	p^*<.001
2	3	10	162	46	12	ϕ=.7436	$N\phi^2$=127.1670	p^*<.001
6	10	14	120	83	13	ϕ=.7590	$N\phi^2$=132.48	p^*<.001
7	8	15	125	83	7	ϕ=.8044	$N\phi^2$=148.8100	p^*<.001
5	9	35	91	8	23	ϕ=.4987	$N\phi^2$=57.2010	p^*<.001
11	13	19	112	92	7	ϕ=.7770	$N\phi^2$=138.8510	p^*<.001
12	14	25	131	52	22	ϕ=.5369	$N\phi^2$=66.2860	p^*<.001
15	16	13	118	87	12	ϕ=.7786	$N\phi^2$=139.4260	p^*<.001

*Probability under H_o that $X^2 \geq$ chi square.

Table VI

PHI-COEFFICIENTS FOR ITEMS OF FORM B; PART 1 AND PART 2

Part 1

1 \ 5	F	P
P	12	207
F	5	6

$\phi = .3206$, $N\phi^2 = 24.4260$, $P^* < .001$

2 \ 3	F	P
P	9	182
F	25	14

$\phi = .6279$, $N\phi^2 = 90.6660$, $P^* < .001$

4 \ 8	F	P
P	20	177
F	22	11

$\phi = .5127$, $N\phi^2 = 31.8780$, $P^* < .001$

6 \ 7	F	P
P	31	163
F	27	9

$\phi = .4938$, $N\phi^2 = 56.0740$, $P^* < .001$

Part 2

1 \ 4	F	P
P	25	196
F	7	2

$\phi = .3723$, $N\phi^2 = 31.8780$, $P^* < .001$

5 \ 9	F	P
P	11	64
F	135	20

$\phi = .7051$, $N\phi^2 = 114.3330$, $P^* < .001$

2 \ 3	F	P
P	14	144
F	67	5

$\phi = .8174$, $N\phi^2 = 153.6630$, $P^* < .001$

11 \ 13	F	P
P	20	80
F	119	11

$\phi = .7252$, $N\phi^2 = 120.9570$, $P^* < .001$

7 \ 10	F	P
P	8	82
F	132	8

$\phi = .8539$, $N\phi^2 = 167.6930$, $P^* < .001$

12 \ 14	F	P
P	23	112
F	75	20

$\phi = .6164$, $N\phi^2 = 87.3770$, $P^* < .001$

6 \ 8	F	P
P	12	73
F	139	6

$\phi = .8309$, $N\phi^2 = 158.7690$, $P^* < .001$

15 \ 16	F	P
P	12	65
F	145	8

$\phi = .8028$, $N\phi^2 = 148.2120$, $P^* < .001$

*Probability under H_o that $X^2 \geq$ chi square.

material in Appendix A. This material was the introduction of integers and the explanation of addition of integers. After completing Appendix A the teachers taught their control group subtraction of integers using the Related Facts Method (Appendix B) and at the same time they introduced the Complement Method of subtraction (Appendix D) to their experimental group. At the termination of the study of this material each of the teachers taught properties of subtraction and problem solving (Appendix C) to both of his groups. Thus the only difference in the treatments for each teacher's group was the method of subtraction.

Scope and Sequence of Material Taught in District Two

Each teacher in District Two started both of his groups with clock arithmetic (Appendix E). Again, as in District One, each teacher devoted the same amount of time to the material in each group. After teaching the material in Appendix E, the teacher introduced the integers and addition of integers (Appendix F) to both of his groups. Upon the completion of Appendix F the teacher introduced the Systems Method of subtraction to his control group and the Complement Method of subtraction to his experimental group. The unit of study was terminated with a study of the properties of subtraction and problem solving. (See Appendix C).

Chapter IV

ORGANIZATION AND RESULTS OF ANALYSES OF THE DATA

The first step in the organization of the experiment consisted of an attempt to determine if differences in the selected experimental and control group existed. A one way analysis of variance (ANOVAR) was conducted on each of sections 3, 4 and 5 of the C. A. T. and also on part 1 and part 2 of the instrument devised by the experimenter (Form A). All analyses were conducted at the .05 level of significance. The specific areas tested were:

a. C. A. T. (section 3) - Mathematical computation

b. C. A. T. (section 4) - Mathematical concepts

c. C. A. T. (section 5) - Problem solving

d. Form A (part 1) - Addition of two integers

e. Form A (part 2) - Subtraction of two integers

Since District One and District Two differed in the types of material presented to the control groups, separate analyses were conducted for each district. The control group in District One was taught the Related Facts Method of subtraction while the control group in District Two was taught the Systems Method of subtraction. In both districts the experimental group was taught the Complement Method of subtraction.

It was found that there were no statistically significant differences between control and experimental groups in any of the five analyses conducted for each of the two districts. The results of the analyses for District One are found in Tables VII and VIII, and the results of the analyses for District Two are found in Tables IX and X.

Table VII

PRETEST MEANS AND STANDARD DEVIATIONS
OF CONTROL AND EXPERIMENTAL GROUP (DIST. ONE)

Test	Group	Number	$\overline{X}$	s.d.
C. A. T. (section 3)	Exp.	70	29.4714	8.7256
	Cont.	70	29.1143	617535
C. A. T. (section 4)	Exp.	70	19.8571	3.9131
	Cont.	70	20.0286	4.4133
C. A. T. (section 5)	Exp.	70	7.6286	3.1446
	Cont.	70	7.4143	3.1137
Form A (part 1)	Exp.	70	4.0857	2.4061
	Cont.	70	3.7714	2.3133
Form A (part 2)	Exp.	70	3.8143	1.7218
	Cont.	70	3.9000	2.0638

Table VIII

PRETEST COMPARISONS FOR DISTRICT ONE (ANOVAR)

Test	Source	Mean Square	D.F.	F-Ratio	P
C.A.T. (sect 3)	Total	60.4676	139		
	Groups	4.4375	1	0.073	0.7839
	Error	60.8736	138		
C.A.T. (sect 4)	Total	17.2773	139		
	Groups	1.0234	1	0.059	0.8040
	Error	17.3951	138		
C.A.T. (sect 5)	Total	9.7334	139		
	Groups	1.6055	1	0.164	0.6890
	Error	9.7923	138		
Form A (part 1)	Total	5.4625	139		
	Groups	3.4570	1	0.631	0.5659
	Error	5.4770	138		
Form A (part 2)	Total	3.5046	139		
	Groups	0.2571	1	0.073	0.7840
	Error	3.5282	138		

Table IX

PRETEST MEANS AND STANDARD DEVIATIONS
OF CONTROL AND EXPERIMENTAL GROUP (DIST. TWO)

Test	Group	Number	$\overline{X}$	s.d.
C. A. T. (section 3)	Exp.	45	30.5555	7.3810
	Cont.	45	31.6000	8.6717
C. A. T. (section 4)	Exp.	45	20.7333	4.0530
	Cont.	45	20.4000	5.0739
C. A. T. (section 5)	Exp.	45	8.3556	3.0907
	Cont.	45	8.4889	2.8173
Form A (part 1)	Exp.	45	3.9778	2.5449
	Cont.	45	4.2444	2.6211
Form A (part 2)	Exp.	45	3.8000	1.6705
	Cont.	45	3.6000	2.0845

Table X

PRETEST COMPARISONS FOR DISTRICT TWO (ANOVAR)

Test	Source	Mean Square	D.F.	F-Ratio	P
C.A.T. (sect 3)	Total	64.3876	89		
	Groups	24.5625	1	0.379	0.5469
	Error	64.8402	88		
C.A.T. (sect 4)	Total	22.6753	89		
	Groups	2.5000	1	0.109	0.7411
	Error	22.9046	88		
C.A.T. (sect 5)	Total	8.6512	89		
	Groups	0.3984	1	0.046	0.8260
	Error	8.7450	88		
Form A (part 1)	Total	6.6167	89		
	Groups	1.5999	1	0.240	0.6313
	Error	6.6737	88		
Form A (part 2)	Total	3.5382	89		
	Groups	0.8999	1	0.252	0.6228
	Error	3.5682	88		

Results of Achievement Testing (Form A, parts 1 and 2)

Achievement was defined as the gain in mean score between pretest and post-test on Form A, parts 1 and 2. Achievement was tested by means of a two factor analyses of variance, one factor being repeated measures.

Since the control groups of District One and District Two were taught different materials during the experimental period, it was necessary that separate analyses be conducted for the two districts. However, the analyses for the two districts were related to the same experimental factors.

District One Hypotheses and Results. The null hypotheses for District One consisted of the following:

H_1: There is no difference in achievement, in the area of addition of two integers, between the groups taught the Complement Method of subtraction and the group taught the Related Facts Method of subtraction.

H_2: There is no difference in achievement, in the area of subtraction of two integers, between the groups taught the Complement Method of subtraction and the group taught the Related Facts Method of subtraction.

It was found that there was no significant difference in relation to addition of two integers. The results of the analysis for Hypothesis 1 are found in Tables XI and XII.

Table XI

MEANS AND STANDARD DEVIATIONS FOR EXPERIMENTAL GROUP AND CONTROL GROUP; ACHIEVEMENT IN ADDITION (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\bar{X}$	s.d.	$\bar{X}$	s.d.
70	Exp.	4.0857	2.4061	6.5857	2.1835
70	Cont.	3.7714	2.3133	6.5571	1.7414

Table XII

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN ADDITION (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	6.4032	279		
Between	5.8957	139		
Groups	2.0547	1	0.347	0.5640
Error (G)	5.9235	138		
Within	6.9071	140		
Trials	488.9258	1	141.556	0.0000
G by T	1.4297	1	0.414	0.5282
Error (T)	3.4539	138		

It was also found that there were no significant differences relative to subtraction of two integers. The results of the analysis for Hypothesis 2 are found in Tables XIII and XIV.

Table XIII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN SUBTRACTION (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
70	Exp.	3.8143	1.7218	10.0857	5.2633
70	Cont.	3.9000	2.0638	10.0571	4.3200

Table XIV

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN SUBTRACTION (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	22.9019	279		
Between	15.1557	139		
Groups	0.0547	1	0.004	0.9511
Error (G)	15.2651	138		
Within	30.5928	140		
Trials	2703.2148	1	236.170	0.0000
G by T	0.2266	1	0.020	0.8833
Error (T)	11.4461	138		

District Two Hypotheses and Results. The null hypothesis for District Two consisted of the following:

> H_3: There is no difference in achievement in the area of addition of two integers, between the group taught the Complement Method of subtraction and the group taught the Systems Method of subtraction.
>
> H_4: There is no difference in achievement in the area of subtraction of two integers, between the group taught the Complement Method of subtraction and the group taught the Systems Method of subtraction.

It was found that there was no significant difference relative to addition of two integers. The results of the analysis for hypothesis 3 are found in Tables XV and XVI.

Table XV

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN ADDITION (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	3.9778	2.5449	5.6889	2.4291
45	Cont.	4.2444	2.6211	6.4222	2.2206

Table XVI

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN ADDITION (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	6.9707	179		
Between	8.1376	89		
Groups	11.2500	1	1.388	0.2401
Error (G)	8.1023	88		
Within	5.8167	90		
Trials	170.1367	1	42.666	0.0000
G by T	2.4492	1	0.614	0.5587
Error (T)	3.9877	88		

It was also found that there were no significant differences relative to subtraction of two integers. The results of the analyses for hypothesis 4 are found in Tables XVII and XVIII.

Table XVII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN SUBTRACTION (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	3.8000	2.0845	11.5556	4.5979
45	Cont.	3.6000	1.6705	10.0444	4.4921

Table XVIII

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN SUBTRACTION (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	24.4790	179		
Between	15.0028	89		
Groups	32.9375	1	2.226	0.1354
Error (G)	14.7990	88		
Within	33.8500	90		
Trials	2268.4492	1	263.106	0.0000
G by T	19.3320	1	2.242	0.1340
Error (T)	8.6218	88		

Results of Achievement Testing (C.A.T., Sections 3, 4 and 5)

Achievement was defined as the gain in mean score between pretest and post-test on the C.A.T. section 3 (computation), section 4 (concepts) and section 5 (problem solving). Achievement was tested by means of a two factor analyses of variance, one factor being repeated measures.

Since the control groups of District One and District Two were taught different materials during the experimental period, it was necessary that separate analyses be conducted for the two districts.

District One Hypotheses and Results. The null hypotheses for District One consisted of the following:

H_5: There is no difference in achievement, in the area of computation, between the group taught the Complement Method of subtraction and the group taught the Related Facts Method of subtraction.

H_6: There is no difference in achievement, in the area of concepts, between the group taught the Complement Method of subtraction and the group taught the Related Facts Method of subtraction.

H_7: There is no difference in achievement, in the area of problem solving, between the group taught the Complement Method of subtraction and the group taught the Related Facts Method of subtraction.

It was found that there was no significant difference relative to computation. The results of the analysis for hypothesis 5 are found in Tables XIX and XX.

Table XIX

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
COMPUTATION (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
70	Exp.	29.4714	8.7256	30.3571	9.3017
70	Cont.	29.1143	6.7535	30.8286	8.6727

Table XX

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN COMPUTATION (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	70.5775	279		
Between	127.7635	139		
Groups	0.1875	1	0.001	0.9684
Error (G)	128.6879	138		
Within	13.8000	140		
Trials	118.4375	1	9.071	0.0034
G by T	11.8125	1	0.905	0.6549
Error (T)	13.0562	138		

It was found that there was a significant difference relative to concepts, thus the null hypothesis was rejected. The results for hypothesis 6 are found in Tables XXI and XXII.

Table XXI

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN CONCEPTS (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\bar{X}$	s.d.	$\bar{X}$	s.d.
70	Exp.	19.8571	3.9131	19.9714	4.9518
70	Cont.	20.0286	4.4133	21.5000	5.2688

Table XXII

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN CONCEPTS (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	21.9886	279		
Between	37.0022	139		
Groups	50.5625	1	1.370	0.2421
Error (G)	36.9040	138		
Within	7.0821	140		
Trials	44.0000	1	6.634	0.0107
G by T	32.2500	1	4.863	0.0273*
Error (T)	6.6322	138		

*significant at .05 level

It was found that there was no significant difference relative to problem solving. The results of the analyses for hypothesis 7 are found in Tables XXIII and XXIV.

Table XXIII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN PROBLEM SOLVING (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
70	Exp.	7.6286	3.1446	8.0429	3.2411
70	Cont.	7.4143	3.1137	7.8857	3.2904

Table XXIV

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN PROBLEM SOLVING (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	10.1774	279		
Between	16.8021	139		
Groups	2.4141	1	0.143	0.7077
Error (G)	16.9063	138		
Within	3.6000	140		
Trials	13.7305	1	3.865	0.0484
G by T	0.0508	1	0.014	0.9008
Error (T)	3.5523	138		

District Two Hypotheses and Results. The null hypotheses for District Two consisted of the following:

H_8: There is no difference in achievement in the area of computation, between the group taught the Complement Method of subtraction and the group taught the Systems Method of subtraction.

H_9: There is no difference in achievement, in the area of concepts, between the group taught the Complement Method of subtraction and the group taught the Systems Method of subtraction.

H_{10}: There is no difference in achievement in the area of problem solving, between the group taught the Complement Method of subtraction and the group taught the Systems Method of subtraction.

It was found that there was no significant difference relative to computation. The results of the analyses for hypothesis 8 are found in Tables XXV and XXVI.

Table XXV

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN COMPUTATION (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	30.5555	5.2633	34.0000	4.2116
45	Cont.	31.6000	4.3200	33.6889	3.8860

Table XXVI

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN COMPUTATION (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	72.6299	179		
Between	124.2050	89		
Groups	6.0625	1	0.048	0.8212
Error (G)	125.5476	88		
Within	21.6278	90		
Trials	344.3125	1	19.159	0.0001
G by T	20.7500	1	1.155	0.2854
Error (T)	17.9709	88		

It was found that there was no significant difference relative to concepts. The results and analyses for hypothesis 9 are found in Tables XXVII and XXVIII.

Table XXVII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN CONCEPTS (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	20.7333	4.0530	22.4000	5.5028
45	Cont.	20.4000	5.0739	20.5333	6.2070

Table XXVIII

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN CONCEPTS (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	28.8436	179		
Between	49.2303	89		
Groups	54.4375	1	1.107	0.2958
Error (G)	49.1712	88		
Within	8.6833	90		
Trials	36.5000	1	4.470	0.0351
G by T	26.4375	1	3.238	0.0718
Error (T)	8.1655	88		

It was found that there were no significant differences relative to problem solving. The results and analyses for hypothesis 10 are found in Tables XXIX and XXX.

Table XXIX

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
ACHIEVEMENT IN PROBLEM SOLVING (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	8.4889	2.8173	8.8000	3.4351
45	Cont.	8.3556	3.0907	9.2889	3.2656

Table XXX

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN PROBLEM SOLVING (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	9.9509	179		
Between	15.9573	89		
Groups	1.4219	1	0.088	0.7647
Error (G)	16.1225	88		
Within	4.0111	1	4.520	0.0341
Trials	17.4219	1	1.130	0.2907
C by T	4.3555	88		
Error (T)	3.8548			

<u>Results of Retention Testing (Form A vs Form B)</u>

Retention was defined as the gain or loss in mean score between the post-test and the retention test. The post-test was Form A of the test devised by the investigator and administered at the end of the unit on integers. The retention test was Form B of the test devised by the investigator.

<u>District One Hypotheses and Results</u>. The null hypotheses for District One consisted of the following:

H_{11}: There is no difference in retention, in the area of addition of two integers, between the group taught the Complement Method and the group taught the Related Facts Method of subtraction of integers.

H_{12}: There is no difference in retention, in the area of subtraction of two integers, between the group taught the Complement Method and the group taught the Related Facts Method of subtraction of integers.

It was found that there was no significant difference relative to retention of addition. The results of the analyses for hypotheses 11 are found in Tables XXXI and XXXII.

Table XXXI

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
RETENTION IN ADDITION (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
70	Exp.	6.5857	2.1835	7.2000	1.3683
70	Cont.	6.5571	1.7414	6.7714	3.2512

Table XXXII

RESULTS OF ANALYSIS OF VARIANCE;
RETENTION IN ADDITION (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	3.2626	279		
Between	4.7430	139		
Groups	3.6563	1	0.770	0.6142
Error (G)	4.7508	138		
Within	1.7929	140		
Trials	12.0156	1	7.021	0.0089
G by T	2.7969	1	1.634	0.2004
Error (T)	1.7115	138		

It was found that there was no significant difference relative to retention of subtraction. The results of the analyses for hypothesis 12 are found in Tables XXXIII and XXXIV.

Table XXXIII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
RETENTION IN SUBTRACTION (DIST. ONE)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
70	Exp.	10.0857	5.2633	5.9429	4.2116
70	Cont.	10.0571	4.3200	6.7571	3.8860

Table XXXIV

RESULTS OF ANALYSIS OF VARIANCE;
RETENTION IN SUBTRACTION (DIST. ONE)

Source	Mean Square	D.F.	F-Ratio	P
Total	23.3712	279		
Between	25.8638	139		
Groups	10.8047	1	0.416	0.5272
Error (G)	25.9729	138		
Within	20.8964	140		
Trials	969.4336	1	68.830	0.0000
G by T	12.4219	1	0.882	0.6484
Error (T)	14.0844	138		

<u>District Two Hypotheses and Results</u>. The null hypotheses for District Two consisted of the following:

H_{13}: There is no difference in retention, in the area of addition of two integers, between the groups taught the Complement Method and the Systems Method of subtraction.

H_{14}: There is no difference in retention, in the area of subtraction of two integers, between the groups taught the Complement Method and the Systems Method of subtraction.

There was a significant difference at the .05 level relative to retention of addition. The change in means for the experimental group was $^+1.1333$ and the change was $^+0.1111$ for the control group. Thus hypothesis 13 was rejected. The results of the analysis for hypothesis 13 are found in Tables XXXV and XXXVI.

Table XXXV

MEANS AND STANDARD DEVIATIONS FOR EXPERIMENTAL AND CONTROL GROUPS; RETENTION IN ADDITION (DIST. TWO)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	5.6889	2.4291	6.8222	1.9339
45	Cont.	6.4222	2.2206	6.5333	2.1489

Table XXXVI

RESULTS OF ANALYSIS OF VARIANCE;
RETENTION IN ADDITION (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	4.8927	179		
Between	6.4135	89		
Groups	2.2227	1	0.344	0.5661
Error (G)	6.4611	88		
Within	3.3889	90		
Trials	17.4219	1	5.558	0.0195
G by T	11.7500	1	3.749	0.0530*
Error (T)	3.1344	88		

*significant at the .05 level

It was found that there was a significant difference at the .05 level relative to retention of subtraction. The change in means was $^{-}1.6445$ for the experimental group and $^{+}1.4445$ for the control group. Thus the null hypothesis was rejected. The results of the analyses for hypothesis 14 are found in Tables XXXVII and XXXVIII.

Table XXXVII

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
RETENTION IN SUBTRACTION (DIST. TWO)

		Post-test		Retention test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
45	Exp.	11.5556	4.5979	9.9111	5.4295
45	Cont.	10.0444	4.4921	11.4889	4.7940

Table XXXVIII

RESULTS OF ANALYSIS OF VARIANCE;
RETENTION IN SUBTRACTION (DIST. TWO)

Source	Mean Square	D.F.	F-Ratio	P
Total	23.1271	179		
Between	30.1376	89		
Groups	0.0469	1	0.002	0.9677
Error (G)	30.4796	88		
Within	16.1944	90		
Trials	0.4492	1	0.029	0.8588
G by T	107.3359	1	6.998	0.0094*
Error (T)	15.3377	88		

*significant at the .05 level

Results of Testing Subtraction of Three Integers

Since all of the groups participating in the experiment studied the properties of subtraction (Appendix C), the data from the two control groups were combined and the data from the two experimental groups were combined together to form one control group of 115 subjects and an experimental group of 115 subjects. The experimental group consisted of the experimental group from District One combined with the experimental group of District Two, while the control group consisted of the control groups from District One and District Two.

The following two null hypotheses were formed:

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H_{15}: There is no difference in achievement in the area of subtraction of three integers, between the group taught the Complement Method and the group taught the Related Facts Method or the Systems Method of subtraction.

H_{16}: There is no difference in retention, in the area of subtraction of three integers, between the group taught the Complement Method and the group taught the Related Facts Method or the Systems Method of subtraction.

It was found that there was no significant difference at the .05 level relative to achievement of subtraction of three integers. The results of the analyses for hypothesis 15 are found in Tables XXXIX and XXX.

Table XXXIX

MEANS AND STANDARD DEVIATIONS FOR EXPERIMENTAL AND CONTROL GROUPS; ACHIEVEMENT IN SUBTRACTION OF THREE INTEGERS (COMBINED DISTRICTS)

		Pretest		Post-test	
Number	Group	$\overline{X}$	s.d.	$\overline{X}$	s.d.
115	Exp.	0.1913	0.5443	2.1739	2.2370
115	Cont.	0.2174	0.7347	2.3217	2.2466

Table XXXX

RESULTS OF ANALYSIS OF VARIANCE;
ACHIEVEMENT IN SUBTRACTION
OF THREE INTEGERS (COMBINED DISTRICTS)

Source	Mean Square	D.F.	F-Ratio	P
Total	3.7309	459		
Between	2.9148	229		
Groups	0.8694	1	0.297	0.5929
Error (G)	2.9238	228		
Within	4.5435	230		
Trials	480.2173	1	194.008	0.0000
G by T	0.4258	1	0.172	0.6821
Error (T)	2.4752	228		

It was found that there was no significant difference, at the .05 level, relative to retention of subtraction of three integers. The results of the analyses for hypothesis 16 are found in Tables XXXXI and XXXXII.

Table XXXXI

MEANS AND STANDARD DEVIATIONS
FOR EXPERIMENTAL AND CONTROL GROUPS;
RETENTION IN SUBTRACTION
OF THREE INTEGERS (COMBINED DISTRICTS)

Number	Group	Pretest $\overline{X}$	Pretest s.d.	Retention test $\overline{X}$	Retention test s.d.
115	Exp.	2.1739	2.2370	1.5304	2.2838
115	Cont.	2.3217	2.2466	1.8174	2.2344

Table XXXXII

RESULTS OF ANALYSIS OF VARIANCE;
RETENTION IN SUBTRACTION
OF THREE INTEGERS (COMBINED DISTRICTS)

Source	Mean Square	D.F.	F-Ratio	P
Total	5.1052	459		
Between	7.1585	229		
Groups	5.4346	1	0.758	0.6112
Error (G)	7.1661	228		
Within	3.0609	230		
Trials	37.8382	1	12.976	0.0007
G by T	0.5564	1	0.191	0.6670
Error (T)	2.9191	228		

Transfer of Complement Method.

Problems 1 and 2 of part 3 were problems dealing with subtraction of rational numbers. The numbers were named by mixed numerals and were of the type that can be worked quickly using the Complement Method of subtraction. As an example problem 1 of part 3 Form A could be worked in the following manner:

$$2\ 1/4 - 3/4 = (2\ 1/4 + 1/4) - (3/4 + 1/4)$$
$$= 2\ 2/4 - 1$$
$$= 1\ 2/4$$

Problems of this type were not worked during the experiment. The investigator wished to determine if students would use the Complement Method of subtraction of their own volution.

To determine if the students used the Complement Method, their work on the post-test for problems 1 and 2, Form A, part 3, was examined. In no single case was work shown that indicated the students had used the Complement Method as illustrated above.

There were four basic ways the subjects worked the problem. They are illustrated below.

a. $3\ 1/8 - 1\ 5/8 = 25/8 - 13/8 = 12/8$

b. $3\ 1/8 - 1\ 5/8 = 2\ 9/8 - 1\ 5/8 = 1\ 4/8$

c. $3\ 1/8 - 1\ 5/8 = (3\ 1/8 + {}^{-}1\ 5/8) - (1\ 5/8 + {}^{-}1\ 5/8)$

$= 3\ 1/8 + {}^{-}1\ 5/8 - 0$

$= 2\ 9/8 + {}^{-}1\ 5/8$

$= 1\ 4/8$

The example c above is using the Complement Method of subtraction but the work involved is essentially the same as that of example b.

One student had an interesting method of working the problem. His work follows:

d. $3\ 1/8 - 1\ 5/8 = 2\ {}^{-}4/8$

His answer could be considered correct if it is interpreted as $2 + {}^{-}4/8$, but the form is not very conventional.

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Use of Complement Method in Subtraction of Whole Numbers

Problems 3 and 4 of Form A, part 3, were (a) 325 - 99 and (b) 342 - 198. Problems of this type were worked by the experimental groups and it was illustrated to them that the problem could be worked using the Complement Method. An example would be as follows:

$$325 - 99 = (325 + 1) = (99 + 1)$$
$$= 326 - 100$$
$$= 226.$$

Since problems of this type were worked during the experiment by the experimental groups, they could not be considered transfer type problems. But, it was deemed important to determine if students would use the Complement Method in working problems of this type. The work of all of the students was checked and the results are in Table XXXXIII.

Table XXXXIII

PERCENT OF STUDENTS USING COMPLEMENT METHOD IN SUBTRACTION OF WHOLE NUMBERS

Number	Group	Percent
115	Exp.	4%
115	Cont.	0%

Justification of Use of Analysis of Variance

To remove at least partially the confusion surrounding this useful technique, it should be stated at the very outset that analysis of variance, in its most basic form, is nothing more than a clever statistical method of testing for significant differences between means of two or more groups. Typically, the performance of these groups can be considered to represent results of the treatment by an independent variable whose possible relationship to a dependent variable is being studied.[1]

Early in this exposition one important point should be clarified. When a researcher uses the analysis of variance statistical model he is primarily interested in mean differences rather than variance of differences.[2]

In the Mathematical development of the analysis of variance a number of assumptions are made. Questions may be raised about these assumptions and the extent to which the failure of the data to satisfy them leads to the drawing of valid inferences.

One assumption is that the distributions of the variables in the populations from which the samples are drawn are normal. For large samples the normality of the distributions may be tested using a test of goodness of fit although in practice this is rarely done. When the samples are small, it is usually not possible to rigorously demonstrate lack of normality of data. Unless these is reason to suspect a fairly extreme departure from normality, it is probable that the conclusions drawn from the data using an F test will not be seriously affected . . .

A further assumption in the application of analysis of variance is that the variances in the populations from which the samples are drawn are equal. This is known as homogeneity of variance . . .[3]

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[1]Popham, Educational Statistics, Use and Interpretation. (New York: Harper and Row, Publishers, 1967) P. 164.

[2]Ibid, p. 165.

[3]Ferguson, Statistical Analysis in Psychology and Education. (New York: McGraw-Hill Book Company, 1966) P. 294.

Since it must be assumed that the variances within the subgroups of the analysis are not significantly different, it is a good practice to check variance homogeneity at the outset of the analysis. A simple first test of homogeneity of variance may be made by calculating the individual variances of the subgroups and dividing the smallest S^2 into the largest S^2. The quotient of this division is an F value which is interpreted for statistical significance by a Table of F. [4]

With crude normal distribution of homogeneous variance the probability of Type I error is very close to that associated with normal theory F-test. With samples of 11 cases or more, no adjustment appears necessary in the tabled values of F needed for significance at the 10%, 5%, and 1% levels.

When considering data with a limited number of score values, analysis of variance techniques have an advantage over the X^2 test of independence when the sample size is very small, when the study involves more than one factor, or when the primary interest is in the differences among means rather than the variance of the population. [5]

Box (1953, 1954)proved mathematically that the distribution of mean square ratios is little affected by the non-normality and heterogeneity of variance when the samples are equal size, even when they are relatively small say 5 to 10. [6]

The data indicated that the variances were within the limits specified by Popham.

Since it was the desire of the investigator to look at differences in means and the data indicated that the variances were within the limits specified by Popham, the data were analyzed using analysis of variance as indicated in the first section of this chapter. All of the analysis of variance analyses were conducted on a UNIVAC SPECTRA 70/46 computer making use of programs taken from Donald J. Veldman, Fortran Programming for the Behavioral

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[4] Ibid, Popham, p. 181.

[5] Hsu Tse-Chi, Feldt, and Leonard, "The Effects of Limitations on the Number of Criterion Score Values on the Significance Level of the F-test", American Education Research Journal, VI (November, 1969) P. 526.

[6] [illegible]

Sciences (pp. 246-80).

Responses of Teachers to Methods of Subtraction

One year after the completion of the experiment, the investigator contacted the teachers that participated in the experiment and asked them if they were teaching about integers in the seventh grade. All but one of the teachers was teaching about integers.

All of these teachers were teaching the Complement Method of subtraction. One of the teachers indicated that he was also using some other methods. When the investigator asked the teachers why they were using the Complement Method they said it was because they felt they had more success with it. That is, they felt the students responded better to the Complement Method and it seemed easier to teach than the method they had used with their control groups.

Chapter V

SUMMARY AND CONCLUSIONS

It was the purpose of this study to:

1. Compare the achievement of students taught to perform different methods of subtracting integers.

2. Compare the retention of knowledge and skill possessed by students taught to perform different methods of subtraction of two integers.

3. Determine if the method of subtraction learned affected the students' ability to work subtraction problems involving three integers.

4. Determine if students would transfer the use of the subtraction method employed by the investigator to subtraction problems involving positive rational numbers.

5. Determine if the method of subtraction employed by the investigator would affect the students' achievement or retention in addition of integers.

SUMMARY

Since District One and District Two differed in the type of material presented to the control groups, as pertaining to the operations addition and subtraction of integers, separate analyses in these areas were conducted for each district. In the area of subtraction of three integers, all groups had studied the properties of subtraction. It was determined that the data from the two districts' groups would be combined resulting in one control group and one experimental group.

Since District One and District Two differed in the type of material presented to the control groups, as pertained to the method of subtraction of integers, separate analyses were conducted for each district. All groups studied properties of subtraction, that is, the facts that subtraction is not commutative or associative. For this reason the data of the control groups for both districts was grouped together and the data from the experimental groups for both districts was grouped forming one control group and one experimental group for the combined districts. Another area that was investigated was the students' use of the Complement Method and it was deemed appropriate to consider this data for the combined districts rather than each individual district.

Therefore, the summary section contains five subsections. The first subsection summarizes the findings in District One relative to achievement and retention as measured by Form A and Form B. This subsection also contains a summary of results relevant to achievement in the areas measured by the California Achievement Test. The second subsection contains the summaries in the same areas as described above for District Two.

The third subsection summarizes the results of subtraction of three integers. The fourth subsection contains a summary of the students' use of the Complement Method in subtraction of whole numbers and the fifth subsection contains a summary of the transfer of the Complement Method of subtraction of rational numbers.

Summary District One

In this district the control group was taught the Related Facts Method of subtraction and the experimental group was taught the Complement Method. There were no statistical differences between the groups in achievement (See Table XI, pp. 39) or in retention of addition (Table XXXI, pp. 52) as measured by Form A and Form B. There were no statistical differences between the groups in achievement in the areas of computation (Table XX, pp.45) and problem solving (Table XXIV, pp. 47) as measured by section 3 and 5 of the California Achievement Test, form A and form B. However, as can be determined from Table XXII, page 46, there was a statistical difference between groups in the area of concepts as measured by section 4 of the California Achievement Test, form A and form B. The changes in means for the experimental group was $^{+}$.1143 and $^{+}$1.4714 for the control group. Thus the control groups' performance was statictically better than the experimental groups'.

Summary for District Two

In this district the control group was taught the Systems Method and the experimental group was taught the Complement Method of subtraction. There were no statistical differences between the groups in achievement in the area of addition (Table XVI, pp.42) or subtraction (Table XVIII, pp. 43) as measured by Form A of the test devised by the investigator. There also was no statistical difference between the groups in achievement in the areas of computation (Table XXVI, pp. 49) or concepts (Table XXVIII, pp. 56) or problem solving (Table XXX, pp. 51) as measured by sections 3, 4 and 5 of the California Achievement Test.

As can be seen from Table XXXVI, page 55, there was a statistical difference in retention of addition between the two groups. The change in mean for the experimental group was $^{+}$1.1333 and the change was $^{+}$0.1111 for the control group, indicating that the experimental groups' performance was superior to the control groups'. Table XXXVIII indicates that the control group did significantly better than the experimental group in retention of subtraction. The change in means was $^{+}$1.4445 for the control group and $^{-}$1.6445 for the experimental groups.

Subtraction of Three Integers

There were no statistical differences between groups in the area of subtraction of three integers. The performance of the groups was quite poor as indicated by the mean scores. There were six problems that tested the ability to subtract three integers yet the highest mean score for the groups was just slightly higher than 2, not even 50%, on the post-test. This fact indicated that very few students were successful in this area.

Use of the Complement Method in Subtraction of Whole Numbers

The students that studied the Complement Method of subtraction were introduced to the concept that 178 - 99 = 179 - 100. Problems 3 and 4 of part 3 of Form A and Form B asked students to work problems of this type and show their work. Only 4% of the experimental group used the Complement Method in the manner illustrated above to work these problems. The remainder of the students used the standard algorithm, of regrouping and taking away, to work the problems.

Transfer of the Complement Method

The students were not introduced to the idea of subtracting rational numbers using the Complement Method. An example of using the Complement Method in subtraction of rational numbers would be as follows:

$$\begin{aligned} 1\,3/8 - 7/8 &= (1\,3/8 + 1/8) - (7/8 + 1/8) \\ &= (1\,4/8) - (8/8) \\ &= 1\,4/8 - 1 \\ &= 4/8 \end{aligned}$$

Problems 1 and 2 of Form A and Form B, part 3 were problems of the type indicated above. When the students' work was examined it was found that none of the students used the Complement Method in a meaningful manner. Students either used the standard algorithm when working this problem or the method illustrated below:

$$\begin{aligned} 3\,1/8 - 1\,5/8 &= (3\,1/8 + {}^{-}1\,5/8) - (1\,5/8 + {}^{-}1\,5/8) \\ &= 3\,1/8 + {}^{-}1\,5/8 - 0 \\ &= 2\,9/8 + {}^{-}1\,5/8 \\ &= 1\,4/8 \end{aligned}$$

This method differed only slightly from the standard algorithm illustrated below:

$$3\ 1/8 - 1\ 5/8 = 2\ 9/8 - 1\ 5/8 = 1\ 4/8$$

Thus, it was not considered as using the Complement Method in a meaningful way.

It should be noted that one student worked the problem as follows:

$$3\ 1/8 - 1\ 5/8 = 2\ {}^{-}4/8$$

This answer to the problem is correct but not in a standard form.

CONCLUSIONS

Since there were no statistical differences in achievement, in the areas of subtraction and addition of integers, between the groups taught the Related Facts Method and the Complement Method of subtraction, it is felt that either of these methods can be taught to students. The fact that the group studying the Related Facts Method performed significantly better in the area of concepts than the experimental group that studied the Complement Method, leads to the conclusion that studying the relation between subtraction and addition causes an increase in the student's knowledge of concepts as measured by the California Achievement Test, section 4. The group using the Related Facts Method studied the idea that in order to work $4 - 10 = N$ the problem $N + 10 = 4$ must be solved. Thus, they studied the relation of addition to subtraction while the experimental group did not study this concept. Since this was the only difference in the treatment of the two groups, the difference in the groups must be attributed to this difference in treatment.

The statistical differences in the area of addition between the groups taught the Systems Method and the Complement Method might be attributed to the fact that the experimental group (Complement Method) actually worked more addition problems that the control group. That is, in working the problem 5 - 8 the experimental group worked two addition problems (1) $5 + {}^{-}8$ and (2) $8 + {}^{-}8$ before arriving at the step of $5 + {}^{-}8$ while the control group merely worked the problem as $5 - 8 = 5 + {}^{-}8$.

The group taught the Systems Method performed better in the area of subtraction of integers than the group taught the Complement Method. This is probably due to the fact that the Systems Method was consistent with the material they studied in the unit. That is, the knowledge they learned about clock arithmetic was applied to addition of integers and then to subtraction of integers in a consistent manner. The experimental group, however, did not have this consistent presentation. They used their knowledge of clock arithmetic to work addition of integer problems but did not rely on this background to work subtraction of integers.

It was felt that the Complement Method might adversely affect a student's ability to work addition problems because in pilot studies of use of the Complement Method students were observed to use the Complement Method in addition problems. This of course leads to the incorrect answer to the addition problem. The following problem illustrates a student's incorrect useage of the Complement Method.

$$^{+}5 + {}^{+}3 = ({}^{+}5 + {}^{-}3) + ({}^{+}3 + {}^{-}3)$$
$$= ({}^{+}5 + {}^{-}3) + 0$$
$$= {}^{+}2$$

It was feared by the investigator that the Complement Method of subtraction might be to addition of integers as "cancel" is to reduction of fractions. However, an examination of the results of the students' achievement and retention in the area of addition reveals that the group taught the Complement Method performed as well as the control group in District One and better than the control group in District Two. Thus the Complement Method had no adverse affect on the students' ability to add integers.

Finally the facts that (1) the Related Facts group performed better in the area of concepts than the Complement Method group; (2) the Complement Method group's retention was better than the Systems Method group's in the area of addition of integers; and (3) the Systems Method group's retention was better than the Complement Method group's, leads to the conclusion that some mixture of the Methods might be the best solution to the problem of subtraction of integers.

SUGGESTIONS FOR FURTHER RESEARCH

In light of the findings of this report the following questions are asked and suggest areas of further reaearch.

(1) Is there a mixture of methods of subtraction that will produce the best results?

(2) In light of the fact that no students used the Complement Method in subtraction of whole numbers, would the introduction of the Complement Method at the time subtraction of whole numbers is introduced result in a better performance of a group taught the Complement Method of subtraction of integers?

(3) Would the use of the Complement Method of subtraction of integers allow for the introduction of operations on integers at a grade level lower than the seventh grade?

(4) Does the Complement Method lead to a better understanding of subtraction of integers than other methods of subtraction of integers?

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Appendix A

Introduction to Integers (Dist. One)
and Addition of Integers

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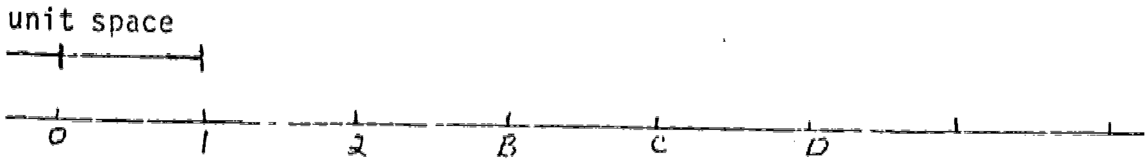
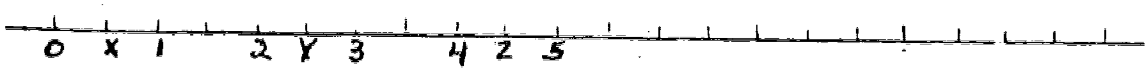
THINKING ABOUT THE NUMBER LINE



1. Throughout this course, you have made use of a "number line." Often the drawing, or model, may have looked like the model above. To draw a number line, we first decide upon a unit segment, or unit space. Then we lay off this unit space, or copy it, in succession along a line, as indicated in the drawing. What number corresponds to B? to C? to D?

2. Suppose we divide each unit space in half, as in the drawing below. What number corresponds to X? to Y? to Z?



3. Using 1 inch as the unit space, make a drawing of a number line. Mark the points that correspond to the whole numbers 0, 1, 2, 3, 4, and 5. Now divide each unit space in half and mark the points that correspond to 1/2, 3/2, 5/2, 7/2, and 9/2.

Without using your ruler, indicate on the number line about where the points that correspond to 1/4, 7/4, 9/8, and 11/8 are.

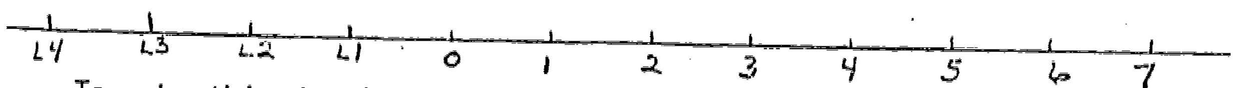
4. Consider this set of numbers: .1, .2, .3, .4, .5, .6, .85, .93 . For each number, is the corresponding point on the number line between 0 and 1?

Now study this set of numbers: 1.3, 2.7, 8.6, 1.01 . Is the point corresponding to each of these numbers to the right of 1?

Think of the fractional number 10001. Would the point corresponding to this number be to the right of 0? Would the point corresponding to 100,000 be to the right of 0?

5. Think back over your work in this book. Is every non-zero number we have used associated with a point that is to the right of 0? Indicate on your number line about where the point that corresponds to π is.

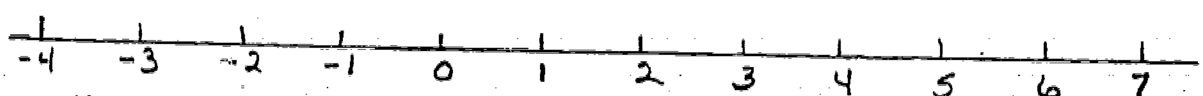
6. Now look at your model of a number line. You know that a line extends indefinitely in both directions. But so far, we have used only the part of a line that begins at the point associated with 0 and that extends indefinitely to the right. That is, we may say we have used a number ray. Is it possible to invent numbers to associate with the points to the left of 0? The drawing below shows one possible way of associating new numbers with points to the left of 0.



To make this drawing, we chose any point on a line and marked it 0. Then we decided on a unit space and marked off unit spaces to the left and to the right of the 0 point. We have assigned L1 to the point that is one unit space to the left of 0; L2 to the point that is two unit spaces to the left of 0, and so on.

Draw a number line and lavel the point corresponding to each of the following: 1; L1; 2; L2; 1/2; L1/2; L1 1/2; L7.

7. Usually, in mathematics, numbers such as -1, -2, -3, and -7 1/2 are associated with points to the left of 0.



We read "-1" as <u>negative one</u>, "-2" as <u>negative two</u>, and so on. How would you read "-5"? "-2 1/2"? "-1 1/4"? The n-mbers associated with the points to the left of 0 are called <u>negative numbers</u>. The numbers to the right of 0 are called <u>positive numbers</u>. Sometimes the positive numbers are represented as "+1," "+2," "+5 1/2," and so on.

THE SET OF DIRECTED NUMBERS

We have introduced positive and negative numbers, using a number line. But such numbers are used in many situations.

1. Mary was given a share of stock for her birthday. At the end of a week, her father handed her a slip of paper that had this listing on it. Her father explained that the listing told how many dollars the stock increased in value or decreased in value each day. To indicate an increase of one dollar in the value, a + 1 is used. What does -1 indicate? +1 1/2? -1/2? -1 1/2?

2. In the same situation where +1 indicates an increase of one dollar, what would +4 indicate? -4? Do both -4 and +4 indicate a change in value of four dollars? Do they also tell the direction in which the change occurred? Are the changes in the same direction, or in opposite directions?

The set of numbers consisting of zero, all positive numbers, and all negative numbers is called the set of directed numbers.

3. Jim kept a record of his gain or loss in weight, month by month, over a four-month period. To indicate a loss of three pounds in January, he used a -3. What does -2 indicate for February? What does +1 indicate? -1?

Is the loss of one pound the opposite of a gain of one pound?

4. Many things in life can be thought of as opposites. What is the opposite of a gain of 2 1/2 yards in a football play? The opposite of a withdrawal of $100 from a bank account?

5. Is a gain of $5 the opposite of a loss of $5? If we think of a profit of $5 as +5, how would we think of a loss of $5? We say the directed number +5 is the opposite of -5, and -5 is the opposite of +5. What is the opposite of -3? the opposite of +3?

What is the opposite of -2 1/2? the opposite of +11? the opposite of + 1/4? the opposite of -1/3?

6. Suppose in a football game we represent a loss of three yards on a play as -3. How would we represent a loss of five yards? Now imagine that a team has a loss of three yards, and then a loss of five yards. Would the total loss be the sum of the two losses? If we use N to represent the total loss, we may write the equation below. In this equation, parentheses are used to avoid confusion of the negative sign of the numeral with the plus sign of addition.

$$-3 + (-5) = N$$

What is the correct replacement for N?

7. On first down, a team lost four yards. On the next down, the team lost eleven yards. Does this equation tell about the problem?

$$-4 + (-11) = N$$

What is the correct replacement for N? What is the total loss on the two plays?

8. As was mentioned earlier, positive numbers can be represented as +1, +2, +3, and so on, or simply as 1,2,3. In this illustration of the net yardage gained or lost in a football game, what does 155 indicate? What does -29 indicate?

	Boonevile	Taft
Rushing yardage	155	-29
Passing yardage	182	-7

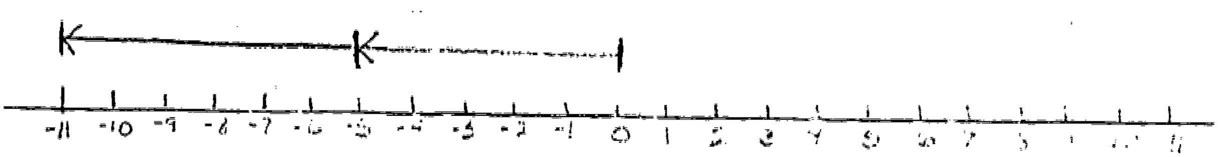
Addition on the Number Line

1. As you know, we can use the number line to illustrate an addition sentence such as 5 + 4 = N. To shew the addtion of 5 and 4 on the number line, we begin at 0 and draw an arrow that shows a move of five spaces to the right. Then, from 5, we draw another arrow showing a move of four spaces to the right. At what numberal does the second move end? What is the sum of 5 and 4?

2.a. Draw a number line and represent the addition of 2 and 9.

b. Did you begin at 0 and draw an arrow that represented a move of two spaces to the right, and then another arrow that represented a move of nine spaces to the right?

3. Just as we draw arrows to the right to represent a sentence such as 5 + 4 = 9, we draw arrows to the left to represent addition involving negative numbers. For instance, to represent the addition sentence -5 + (-4) = N, we begin at 0 and draw an arrow to show a move of five spaces to the left. Then, from -5, we draw an arrow to show a move of four spaces to the left.



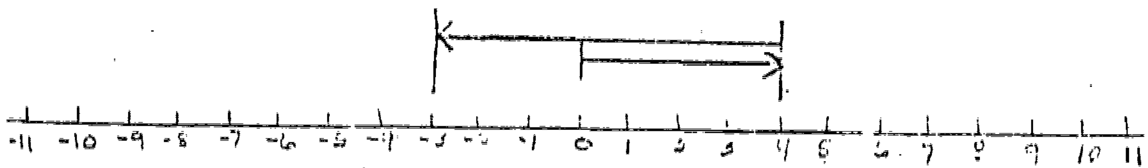
At what numberal does the second move end? Then what is the correct replacement of N?

4.a. Now draw a number line and represent this addition sence: -3 + (-6) = N.

b. Did you begin at 0 and represent a move of three spaces to the left, and then, from -3, represent a move of six spaces to the left?

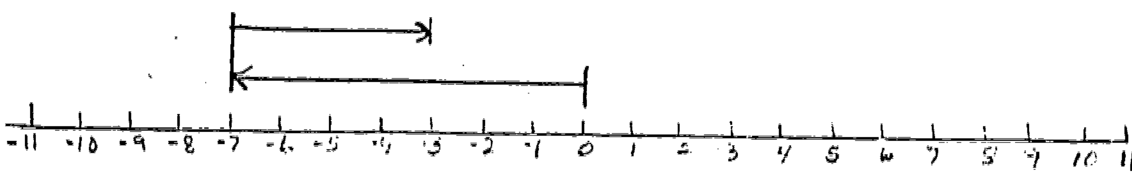
Then what is the correct replacement for N?

5. Suppose we wish to find the correct replacement for the frame in this sentence: $4 + (-7) = \square$.

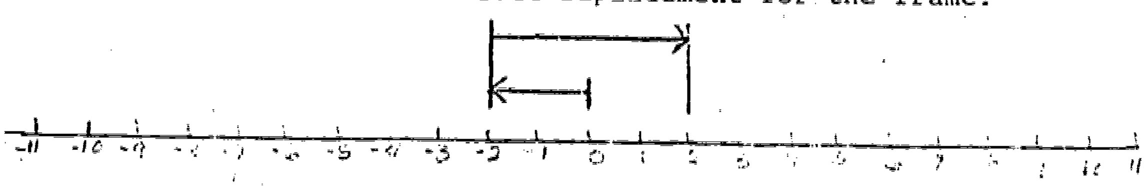


Starting at 0, we draw an arrow to show a move of four spaces to the right; then, from 4, we represent a move of seven spaces to the left. At what number does the second move end? Then what is the correct replacement for the frame?

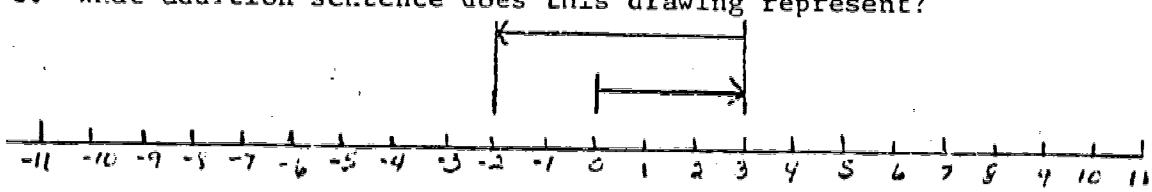
6. To represent the sentence $-7 + 4 = \square$, we start at 0 and draw an arrow to represent a move of seven spaces to the left. Then, from the point for -7, we represent a move of four spaces to the right. What is the correct replacement for the frame?



7. Does the drawing below represent this addition sentence: $-2 + 4 = \triangle$? What is the correct replacement for the frame?



8. What addition sentence does this drawing represent?



Represent each sentence on a number line. Find the sum.

9.a. $-1 + 3 = N$ b. $7 + (-5) = N$

10.a. $1\ 1/2 + 1 = N$ b. $-1\ 1/2 + (-1) = N$

Addition of Directed Numbers

In each of the following, a pair of positive numbers is given. Find the sum.

	a	b	c
1.	+8, +7	12, 15	4.5, .3
2.	3, 1 1/4	+6, +1	+2.0, +.5

Do you add positive numbers just as you have always added numbers in arithmetic? Does it matter whether we use the number +8 or 8 to name the positive number eight?

Is the sum of any two positive numbers a positive number?

In each of the following, a pair of negative numbers is given. Find the sum.

3.	-2,	-6, -1/2	-1/5, -2/5
4.	-3, -11	-2 1/2, -1/2	-.3, -.6

Is the sum of any two negative numbers a negative number? Without using a number line, tell what each of the following equals: -3 + (-8); -1 + (-1 1/2); -4.1 + (-.3).

In each of the following, find the correct replacement for N. Use a number line if necessary.

5.	7 + (-2) = N	-4 + 6 = N	3 + (-5) = N
6.	1 + (-9) = N	9 + (-1) = N	7 + (-8) = N

In Exercises 5 and 6, you found sums of positive and negative numbers. Were the sums always positive? Always negative? Was every sum a directed number? Do you think the set of directed numbers is closed under addition?

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7. Sentence by sentence (in Exercises 5 an 6), study the two numbers that were given and the answer you obtained. Do you see a pattern you can use to obtain the correct replacement for N in this sentence? If not, use a number line to help you.

8. Now try this sentence. Without using a number line, predict which of the following is the correct replacement for N: 12, -12, 6, or -6? Use a number line to check your answer.

For each of the following, predict which of the replacements given is correct. Use a number line to check your answer.

	a	b	c
9.	4 + (-7) = N	-2 + 8 = N	1/2 + (-2) = N
	[11, -11, 3, -3]	[10, -10, 6, -6]	[2 1/2, -2 1/2, 1 1/2, -1 1/2]
10.	-4 + 7 = N	2 + (8) = N	-1/2 + 2 =N
	[11, -11, 3, -3]	[10, -10, 6, -6]	[2 1/2 -2 1/2, 1 1/2, -1 1/2]

Study the examples in 9a and 10a; in 9b and 10b; in 9c and 10c. Then find the sum for each of the following. Try to do the work without using a number line.

11.	6 + (-8)	-7 + 12	9 + (-6)
12.	-1/4 + 1	2 + (-2 1/2)	11 + (-3)
13.	1 1/2 + (-1/2)	6 + (-8 1/4)	-5 + 8 1/3

14. If you are not sure of the pattern for adding a positive number and a negative number without using a number line, make up examples of your own until you see the pattern.

15. Find the sum for each of the following examples.

a. -8 + (-4) b. -4 + (-8)

c. 14 + (-2) d. -2 + 14

e. 4 1/2 + 1 3/4 f. 1 3/4 + 4 1/2

g. -1/2 + (-2 1/2) h. -2 1/2 + (-1/2)

Study the results of the pairs of examples in a and b; in c and d; in e and f; in g and h. Does it seem that the commutative principle of addition is true for directed numbers?

16. Find the sum for each of the following examples. Perform the operation indicated in brackets first.

a. {5 + (-7)} + (-3)

b. 5 + {-7 + (-3)}

c. {-9 + (-8)} + 6

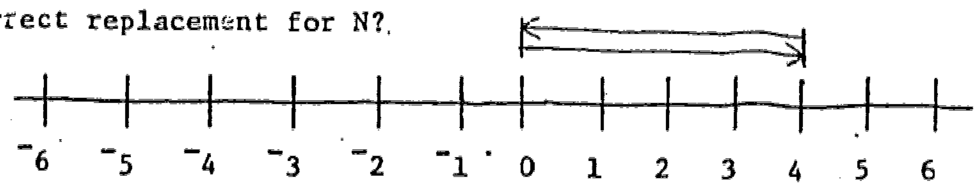
d. -9 + {-8 + 6}

e. {-1 1/4 + 3} + (-2)

f. -1 1/4 + {3 + (12)}

Does it seem that the associative principle of addition is true for directed numbers?

1.a. The arrows in the drawing below represent the addition sentence 4 + (-4) = N. At what number does the second move end? Then what is the correct replacement for N?



b. Using a number line, represent this addition sentence: -4 + 4 = □. What is the correct replacement for the frame? What is the sum of 4 + -4?

2. Find the sum for each of the following.

a. -9 + 9 = △

b. 11 + (-11) = □

c. 1 1/2 + (-1 1/2) = ◇

d. -1/4 + 1/4 = △

Study the sentences in Exercises 1 and 2. In each case, were we adding a number and its opposite? Is the sum 0 in each case?

3. In your work with the set of fractional numbers (non-negative numbers of arithmetic), was 0 the identity element for addition? Now think of the set of directed numbers. What does -2 + 0 equal? 0 + 8? 0 + (-6)?

1/2 + 0? -6 1/2 + 0? 0 + (-6 1/2)?

Is zero the identity element for addition in the set of directed numbers?

Using what you know about the commutative and associative principles of addition, the idea of opposites, and the identity element for addition, find the sum for each of the following. Do the work mentally.

4.a. 13 + (-5) + 5 b. -7 1/2 + 14 3/4 + 7 1/2

5.a. 7/8 + 0 + (-7/8) b. -5 1/5 + (-3 1/3) + 5 1/5

6.a. 2 + (-2 1/2) + 1/2 b. -1/4 + 9/4 + (-2)

7. Now try this sentence: 2 1/3 + 1 1/3 + 1/6 + ☐ = 0. What is the correct replacement for the frame?

Practice

1. Draw a number line and mark points corresponding to the following set of numbers: {0, 1, -1, 2, -2}. Then divide each unit space into tenths, and mark the points corresponding to these numbers: .1, -.1, .2, -.2, and so on, to 1.9 and -1.9.

a. Where is the point associated with -1/2? -4/5? -2/5? -11/10? -7/5? Write the fraction beneath the decimal that names the same number.

b. On a number line, can each unit space be divided into hundredths and can numbers be associated with these points to the left and to the right of zero? Can a unit space be divided into thousandths or hundred thousandths and numbers associated with these points?

2. Each of the sentences below asks you to find the sum of the same six numbers. Using each sentence, find the sum.

a. 1 1/2 + (-3 1/4) + 2 + (-8) + (-6 1/4) + 2 1/4 = N.

b. {1 1/2 + 2 + 2 1/2} + {-3 1/4 + (-8) + (-6 1/4)} = N.

Did sentence b make the work easier?

3. Find the sum in any way you think is convenient.

a. $9 + (-73) + (-56) + 17 + (-19) + 13$

b. $-1\ 3/4 + 3\ 1/8 + (-2\ 1/2) + (-3\ 1/4) + 7\ 3/8 + (-1\ 1/2)$

c. $1.7 + (-.8) + (-1.1) + .9 + (-2.6) + (1.3)$

4. Suppose you are asked if this sentence is true. What is $-9 + (-9)$? What is $5 + (-23)$? Then are "$-9 + (-9)$" and "$5 + (-23)$" names for this same number? We say that the simplest name for the number is -18. Is the sentence true?

For each of the following sentences, write True or False. If the sentence is true, write the simplest name for the number represented by each expression on either side of the equal sign.

a. $-3 + 13 = 31 + (-21)$

b. $-3.5 + 3\ 1/2 = -1.25 + 5.4$

c. $14 + (-5) = 17 + (-9)$

d. $-3\ 1/5 + (-2\ 4/5) = -9.7 + 6.7$

e. $-4\ 3/8 + 35/8 = 5.6 + (-5.6)$

f. $-4\ 1/4 + 3\ 1/2 = 8\ 3/4 + (-7)$

Problems

1. Over a five-day period, a certain stock had the following changes in value (in dollars): $+1\ 1/2$, $+\ 3/4$, -2, $-1/2$, and $-\ 7/8$. To find the total, or net, change in value for this period, could you use the following equation?

$$+1\ 1/2 + (+3/4) + (-2) + (-1/2) + (-7/8) = N$$

What was the net change in the value of the stock?

2. At the beginning of the five-day period, the stock had a value of 71 1/8 dollars. What was its value at the end of the five-day period?

3. In a football game, Bob carried the ball six times. The chart at the right lists the numbers that tell how many yards Bob gained or

lost each time. The numeral 7 indicates a gain of 7 yards. What does -3 1/2 indicate?

Find the net yardage gained or lost by Bob in these six times.

4. Helen said, "I am thinking of a number. If I add this number to the sum of -2 and -5, I get 0 as the result." To find the number that Helen was thinking of, can you use this equation?

$$[-2 + (-5)] + N = 0$$

What is the number?

5. Betty said, "I am thinking of a number. It is equal to the sum of -6 1/4, -2, and the opposite of -2." What is the number?

6. Don kept a record of his gain or loss in weight, month by month. On January 31, he wrote a "2" in the record to indicate a gain of 2 pounds during January. What does the -1 1/4 for February indicate?

What was his net gain or loss in weight for the entire year? Use a numeral for a positive number or a negative number to represent the gain or loss.

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Appendix B

Related Facts Method of Subtraction

Permission to reprint the material in this appendix has been granted by The Silver Burdett Company. The following is acknowledgment of the text it was reprinted from.

Myron F. Rosskopf, Robert See Morton, H. Stewart Moredock, Glenn A. Gilbert, Modern Mathematics Through Discovery I, (pp. 335-337) © 1966 Silver Burdett Company. Reprinted by permission of General Learning Corporation.

ERIC Full Text Provided by ERIC

SUBTRACTION

You know that subtracting is the inverse of adding. For any subtraction sentence such as 8 - 5 = 3, we can write a related addition sentence 5 + 3 = 8 (or 3 + 5 = 8). Similarly, for the subtraction sentence in a, at the right, we can write the addition sentence in b. Will the replacement for N that makes b true also make a true?

Suppose □, △, and N represent any numbers. In mathematics, we can describe the relationship between subtraction and addition as follows:

$$\square - \triangle = N \text{ if and only if } \triangle + N = \square$$

1. Consider the sentence at the right. We read the sentence, "7 minus negative 2 equals what number?" or simply, "7 minus negative 2 equals N." Notice that the first "-" is the symbol for subtraction, but the second "-" is part of the symbol for a negative number.

To find the correct replacement for N in the subtraction sentence, we can use the related addition sentence that is shown here. Think: What number must we add to -2 to obtain 7?

Then what is the correct replacement for N in the sentence 7 - (-2) = N?

2. Read the subtraction sentence in a. To find the correct replacement for N, we can use the related addition sentence in b.

What number must be added to -3 to obtain -8? Then what is the correct replacement for N in the sentence in a?

Now read the subtraction sentence in c. To find the correct replacement for N, can we use the related addition sentence in d?

What number must be added to -1 to obtain 6? Then what is the correct replacement for N in the sentence in c?

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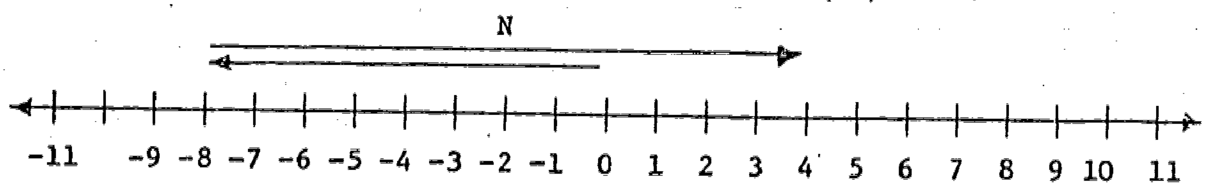
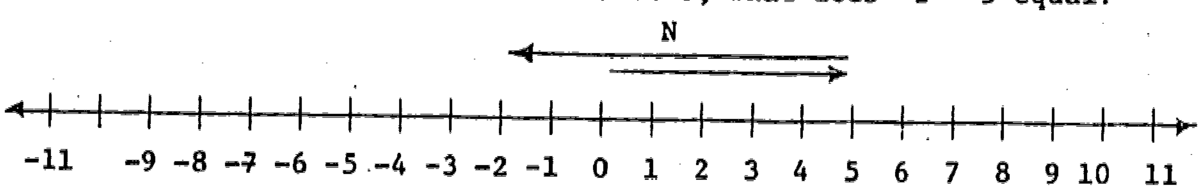
3. We can use a number line to help us find the difference between two numbers. For instance, consider the subtraction sentence 4 - (-8) = N. The related addition sentence is -8 + N = 4. On a number line, we start at 0 and make a move of eight spaces to the left to represent -8. Now we have to make a move (of N spaces) that will end at 4 on the number line. Will we have to move to the right, or to the left, to get to 4? How many spaces will we have to move? Have we found that -8 + 12 equals 4? Then is this sentence true: 4 - (-8) = 12?



4. To find what -2 minus 5 equals, we can write sentence b and use a number line. Starting at 0, we make a move of five spaces to the right to represent 5. Now we have to make a move (of N spaces) that will end at -2 on the number line. In what direction must we move--to the right or to the left--to get to -2? How many spaces must we move? Then have we found that 5 + (-7) = -2? If this is true, what does -2 - 5 equal?



For each of the following, write a related addition sentence and find the correct replacement for N. Use a number line if necessary.

5.a. 9 - (-3) = N	b. 5 - (-6) = N	c. -7 - 1 = N
6.a. 2 - (-2) = N	b. -1	c. -3 - (-2) = N
7.a. 5 - (-3) = N	b. 2 - (-3) = N	c. -5 - 5 = N

WORKING WITH SUBTRACTION

1. Below are six subtraction examples. Find the answers.

a. 6 - 1 b. 6 - 0 c. 6 - (-1)

d. 6 - (-2) e. 6 - (-3) f. 6 - (-4)

Study the examples in 1.a-f and their answers. Do you see a pattern? Then tell what 6 - (-5) equals; 6 - (-6); 6 - (-6 1/2).

2. Below are six subtraction examples. Find the answers.

a. 2 - 1 b. 2 - 2 c. 2 - 3

d. 2 - 4 e. 2 - 5 f. 2 - 6

Study the examples in 2.a-f and their answers. Then answer these questions. What does 2 - 7 equal? 2 - 8? 2 - 9? 2 - 10 1/2? 2 - 11 1/2?

3. Find the answer for each of the following examples.

a. -3 - 2 b. -3 - 1 c. -3 - 0

d. -3 - (-1) e. -3 - (-2) f. -3 - (-3)

g. -4 - 4 h. -3 - (-6) i. -3 - (-8)

Look for a pattern in the answers for 3.a-i. Then tell what -3 - (-10) equals; -3 - (-12); -3 - (-8 1/2).

4. Find the answer for each of the following examples.

a. -4 - (-6) b. -4 - (-5) c. -4 - (-3)

d. -4 - (-2) e. -4 - 0 f. -4 - 2

g. -4 - 4 h. -4 - 6 i. -4 - 8

What does -4 - 10 equal? -4 - 12? -4 - 12 1/2?

Find the answer for each example.

5.a. 3 - 5	b. 3 - (-1)	c. 3 - (-4)
6.a. 9 - 10	b. 9 - 15	c. 9 - 100
7.a. -1 - 2	b. -1 - 4	c. -1 - 8
8.a. -4 - (-7)	b. -4 - (-4)	c. -4 - (-1)

A PATTERN FOR SUBTRACTING

1. What number is named by "9 - (-5)"? What number is named by "9 + (+5)"? Then is the following sentence true?

$$9 - (-5) = 9 + (+5)$$

How is +5 related to -5? Are +5 and -5 opposites of each other?

2. What number is named by "-1 - (+2)"? What number is named by "-1 + (-2)"? Then is the following sentence true?

$$-1 - (+2) = -1 + (-2)$$

How is -2 related to +2?

In each of the following, find the number named in a; then find the number named in b.

3.a. 8 - (-3)	b. 8 + (+3)
4.a. 19 - (-5)	b. 19 + (+5)
5.a. -2 - (+4)	b. -2 + (-4)
6.a. 31 - (-1)	b. 31 + (+1)
7.a. 16 - 2	b. 16 + (-2)

What did you find? For each exercise, does the expression in b name the same number that the expression in a names?

In each of the following sentences, find a replacement for the frame that will make the sentence true.

8. $17 - (-3) = 17 + \triangle$

9. $10 - (-12) = 10 + \square$

10. $-6 - (-4) = -6 + \diamond$

11. $27 - 4 = 27 + \triangle$

12. $8 - 14 = 8 + \square$

Study Exercises 1-12. Do you think that for any subtraction example we can write an addition example that will give the same result? Suppose you are given the subtraction example $\square - N$. Can you use the relationship stated below?

$$\square - N = \square + \text{Opposite of } N$$

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Appendix C

Properties of Subtraction

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Myron F. Rosskopf, Robert See Morton, H. Stewart Moredock, Glenn A. Gilbert, Modern Mathematics Through Discovery I, (pp. 338-340) ©1966 Silver Burdett Company. Reprinted by permission of General Learning Corporation.

PROPERTIES OF SUBTRACTION

1. If the only numbers you could work with were positive numbers and zero, could you find the number named by any of the following?

a. 1 - 4 b. 7 - 13 c. 1 1/2 - 2

Then, is the set of positive numbers and zero closed under subtraction? But if you use the set of directed numbers--the negative numbers, zero, and the positive numbers--then is there a number that is named by 1 - 4? 7 - 13? 1 1/2 - 2? Tell what number is named in each case.

Do you think that by inventing the negative numbers we have made subtraction always possible? Is the set of directed numbers closed under subtraction?

2. Find the number named by each of the following.

a. 9 - 12 b. 12 - 9 c. 1 - 5 d. 5 - 1

Do the expressions in a and b name the same number? Do the expressions in c and d name the same number? Is subtraction commutative?

3. Find the number represented by each of the following. Do the operation indicated in the brackets first.

a. [7 - 3] - (-2) b. 7 - [3 - (-2)]

c. [-10 - 4] - 9 d. -10 - [4 - 9]

Do the expressions in a and b name the same number? Do the expressions in c and d name the same number? Then is subtraction associative?

In the set of directed numbers, subtraction is always possible. But, unlike addition, subtraction is neither commutative nor associative.

Subtract

4.a.	19 - 31	b.	4 - 8.7	c.	-6 1/2 - (-3 1/2)
5.a.	8 - (-1.3)	b.	9 - (-11 1/5)	c.	-16 - 8 3/4
6.a.	-3 3/5 - 4	b.	-8.4 - (-8.4)	c.	3 1/2 - 9 1/4
7.a.	-2.8 - (-3.5)	b.	-1.2 - 3.4	c.	7.1 - (-7.1)

PROBLEM SOLVING

The altitude, or elevation of places on earth is measured in relation to sea level. An elevation listed as "1650" indicates that a place is 1650 feet above sea level. When the elevation of a place is listed as "-15" what does it indicate?

What does a temperature stated as "-2° F" indicate? What does a temperature stated as "2° F" indicate?

1. This table lists the elevation of four different places. Suppose you are asked to find the difference in elevation between Mt. Everest and Death Valley. Could you use this equation?

$$29{,}028 - (-282) = N$$

Find the correct replacement for N.

	Elevation (in feet)
Mt. Everest	29,028
Mt. McKinley	20,320
Death Valley	-282
Qattara Depression (Egypt)	-436

What is the difference in elevation between Mt. Everest and Death Valley?

2. What is the difference in elevation between Death Valley and the Qattara Depression?

3. What is the difference in elevation between Mt. McKinley and Death Valley?

4. A temperature of -126.9° F in Antarctica was the lowest temperature recorded anywhere in the world. A temperature of 136° F in Libya was the highest temperature ever recorded. What is the difference between these two recorded temperatures?

5. During a "cold spell," an announcer stated that the temperature had risen 2.5 degrees and the temperature was now -4.5° F. What was the temperature before the rise?

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Appendix D

Complement Method of Subtraction

SUBTRACTION

Bill and his parents are going on a vacation. Bill is interested in the number of miles his family will travel during their trip. The speedometer has two odometers on it. One records the total number of miles traveled and the other records distances up to 999 miles. When Bill's family started on their trip, the number on one odometer was 38138 and the number on the other was 79. See Figure I.

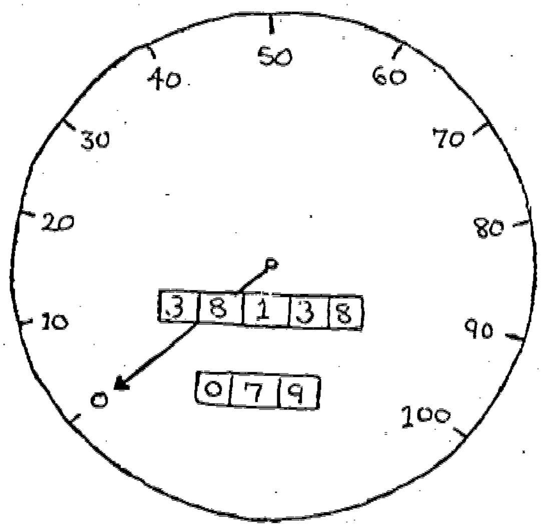

Figure I

Oral Exercises:

1. What is the difference between the two readings on the odometers?
2. After Bill's family has traveled 31 miles, what are the numbers on the two odometers? What is the difference of these two new numbers?
3. When Bill's family has traveled 431 miles from home what are the numbers on the odometers? What is the difference of these numbers?

4. What happens to the answer to a subtraction problem if you add the same number to both of the numbers in the problem?

This new idea will be called the complement method of subtraction. Let's see how this idea can help us to do certain subtraction problems.

Written Exercises:

Let's use this idea to work the following subtraction problems.

Example:

(1) $$\begin{array}{r} 231 \\ -19 \\ \hline \end{array} = \begin{array}{r} 232 \\ -20 \\ \hline 212 \end{array}$$

(2) $$\begin{array}{r} 173 \\ -85 \\ \hline \end{array} = \begin{array}{r} 178 \\ -90 \\ \hline \end{array} = \begin{array}{r} 188 \\ -100 \\ \hline 88 \end{array}$$

1. Find the missing number.

a) $$\begin{array}{r} 245 \\ -78 \\ \hline \end{array} = \begin{array}{r} 248 \\ -\quad \\ \hline \end{array}$$

b) $$\begin{array}{r} 157 \\ -48 \\ \hline \end{array} = \begin{array}{r} \quad \\ -50 \\ \hline \end{array}$$

c) $$\begin{array}{r} 231 \\ -78 \\ \hline \end{array} = \begin{array}{r} 233 \\ -\quad \\ \hline \end{array} = \begin{array}{r} 253 \\ -\quad \\ \hline \end{array}$$

2. Subtract the following using the complement method of subtraction.

a) $$\begin{array}{r} 291 \\ -25 \\ \hline \end{array}$$

b) $$\begin{array}{r} 384 \\ -27 \\ \hline \end{array}$$

c) $$\begin{array}{r} 573 \\ -199 \\ \hline \end{array}$$

d) $$\begin{array}{r} 752 \\ -387 \\ \hline \end{array}$$

e) $$\begin{array}{r} 237 \\ -95 \\ \hline \end{array}$$

f) $$\begin{array}{r} 531 \\ -89 \\ \hline \end{array}$$

SUBTRACTION OF DIRECTED NUMBERS

The complement method of subtraction helped us sometimes when we were subtracting whole numbers. Let's see how it can help us when we subtract directed numbers. We need to review some ideas we have learned and then we will be ready to subtract directed numbers.

Oral Exercises: Find the missing numbers.

1. 5 - 3 = (5 + 7) - (3 + ____) = ____
2. 19 - 8 = (19 + ____) - (8 + 2) = ____
3. 23 - 17 = (23 + ____) - (20) = ____
4. 12 - 5 = (17) - (____) = ____
5. 21 - 18 = (23) - (____) = ____
6. 31 - 7 = (31 + ____) - (7 + ____) = 24
7. 5 - 0 = ____
8. 8 - 0 = ____
9. N - 0 = ____
10. 5 + $^{-}5$ = ____
11. 7 + $^{-}7$ = ____
12. $^{-}8$ + 8 = ____
13. N + ^{-}N = ____

Let's use the ideas from the oral exercises to help us learn how to work the following problem: 7 - $^{-}3$. We know that the complement method works for subtraction so we know 7 - $^{-}3$ = (7 + ____) + ($^{-}3$ + ____). We also know it is easy to subtract zero from a number so let's put $^{+}3$ in the blank. The problem now becomes 7 - $^{-}3$ = (7 + $\underline{^{+}3}$) - ($^{-}3$ + $\underline{^{+}3}$).

But this is just $(7 + {}^+3) - 0 = 7 + {}^+3 = 10$ so the answer to the problem $7 - {}^-3$ is 10.

Exercise set:

1. Fill in the blanks so the following sentences will be true.

a) $9 - ({}^-3) = (9 + \underline{\quad\quad}) - ({}^-3 + \underline{\quad\quad})$

$= (9 + \underline{\quad\quad}) - (0)$

b) $2 - ({}^-2) = (2 + \underline{\quad\quad}) - ({}^-2 + \underline{\quad\quad})$

$= (2 + \underline{\quad\quad}) - (0)$

c) $5 - (3) = (5 + \underline{\quad\quad}) - (3 + \underline{\quad\quad})$

$= (5 + \underline{\quad\quad}) - (0)$

d) $5 - (6) = (5 + \underline{\quad\quad}) - (6 + \underline{\quad\quad})$

$= (5 + \underline{\quad\quad}) - (0)$

e) ${}^-1 - 1 = ({}^-1 + \underline{\quad\quad}) - (1 + \underline{\quad\quad})$

$= ({}^-1 + \underline{\quad\quad}) - (0)$

f) $2 - ({}^-3) = (2 + \underline{\quad\quad}) - ({}^-3 + \underline{\quad\quad})$

$= (2 + \underline{\quad\quad}) - (0)$

g) ${}^-7 - {}^-1 = ({}^-7 + \underline{\quad\quad}) - ({}^-1 + \underline{\quad\quad})$

$= {}^-6 - 0 = {}^-6$

h) ${}^-3 - ({}^-2) = (\underline{\quad\quad} + \underline{\quad\quad}) - (\underline{\quad\quad} + \underline{\quad\quad})$

$= (\underline{\quad\quad} + \underline{\quad\quad}) - (0)$

$= \underline{\quad\quad\quad\quad\quad\quad}$

i) ${}^-5 - 5 = (\underline{\quad\quad} + \underline{\quad\quad}) - (\underline{\quad\quad} + \underline{\quad\quad})$

$= (\underline{\quad\quad} + \underline{\quad\quad}) - (0)$

$= \underline{\quad\quad\quad\quad\quad\quad}$

2. Work the following subtraction problems:

a)	$^{-}3 - 2$	b)	$^{-}3 - 1$	c)	$^{-}3 - 0$
d)	$^{-}3 - (^{-}1)$	e)	$^{-}3 - (^{-}2)$	f)	$^{-}3 - (^{-}3)$
g)	$^{-}3 - (^{-}4)$	h)	$^{-}3 - (^{-}6)$	i)	$^{-}3 - (^{-}8)$

3. Find the answer for each of the following:

a)	$3 - 5$	b)	$3 - (^{-}1)$	c)	$3 - (^{-}4)$
d)	$9 - 10$	e)	$9 - 15$	f)	$9 - 100$
g)	$^{-}1 - 2$	h)	$^{-}1 - 4$	i)	$^{-}1 - 8$
j)	$^{-}4 - (^{-}7)$	k)	$^{-}4 - (^{-}4)$	l)	$^{-}4 - (^{-}1)$

Appendix E

Clock Arithmetic

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Permission to reprint the material in this appendix has been granted by the Holt, Rinehart, and Winston, Inc. The following is acknowledgment of the text it was reprinted from.

Keedy, Jameson, and Johnson, Exploring Modern Mathematics, Book I, (pp. 191-210) © 1963 Holt, Rinehart, and Winston, Inc.

WHERE DOES MATHEMATICS COME FROM?

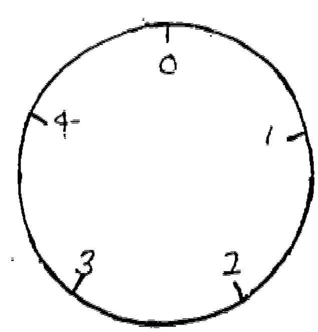
Arithmetic is one kind of mathematics. You have studied it for several years. How do you suppose mathematics is invented? Sometimes we do it by getting ideas from the world around us. We see how things fit together and figure out rules for working problems. Afterward we can find out things about the world around us by working mathematics problems. Let's see how this is done. We can make up mathematics of our own. Let's start with something simple.

ARITHMETIC OF A FIVE-MINUTE CLOCK

Perhaps you have never seen a five-minute clock, but it is easy to imagine one. The hand takes 5 minutes to go around once. Let's make up a new kind of arithmetic. It will be an arithmetic to fit this kind of clock.

5.1 Let's Explore

Draw a picture of a five-minute clock on your paper, and put the numerals '0', '1', '2', '3', and '4' on it, like the picture at the right.



Now answer these questions:

1. If the hand starts at '0' and moves 3 places forward, where will it be?

2. If the hand starts at '1' and moves 3 places forward, where will it be?

3. If the hand starts at '3' and moves 2 places forward, where will it be?

4. If the hand starts at '4' and moves 3 places forward, where will it be?

5. If the hand starts at '2' and moves 0 places forward, where will it be?

6. If the hand starts at '3' and moves 6 places forward, where will it be?

7. If the hand starts at '2' and moves 8 places forward, where will it be?

8. What is the difference if the hand goes 6 places or 1 place? Does the hand stop at the same place?

9. Does the hand stop at the same place if the hand goes 0 places, 5 places or 10 places?

10. Can you see that we will need only five numbers in our new arithmetic?

ADDITION

Let's decide how to add in our new arithmetic. Let's agree that '1 + 2" means this: The hand starts at '0'. It goes 1 place, and then 2 more places. The hand will be at '3'. We will say that 1 + 2 = 3. Let's take another example: '3 + 4' will mean that the hand starts at '0', goes 3 places and then 4 more places.

Where will it be? We see that 3 + 4 = 2.

5.2 Let's Explore

1. Do these additions in our new arithmetic.

   a. 1 + 3
   b. 2 + 3
   c. 4 + 4
   d. 4 + 1
   e. 3 + 2
   f. 0 + 0
   g. 0 + 3
   h. 3 + 3
   i. 4 + 3

2. How many numbers are there in the arithmetic of the five-minute clock? What are they?

3. We can make an addition table to show how to add any two numbers in this arithmetic. Here is part of it. It shows

+	0	1	2	3	4
0					
1				4	
2					
3			0		
4	4				

that 1 + 3 = 4

also 3 + 2 = 0

also 4 + 0 = 4.

We always take the first number on the left and the second number along the top.

Copy the table and finish it.

Now that we have an addition table we do not need the clock anymore. We can study the properties of addition from the table. Do you suppose this kind of addition is something like addition for whole numbers of natural numbers?

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5.3 Let's Explore

1. Use your addition table to do these additions. Look for a pattern.

a. 1 + 2	d. 3 + 2
b. 2 + 1	e. 4 + 2
c. 2 + 3	f. 2 + 4

2. What property of addition does this seem to suggest?

3. Use your addition table to do these additions. Look for a pattern.

Examples:

$$1 + (3 + 2) = 1 + 0 = 1$$

$$(1 + 3) + 2 = 4 + 2 = 1$$

a. 3 + (1 + 4)	e. 4 + (2 + 4)
b. (3 + 1) + 4	f. (4 + 2) + 4
c. (2 + 3) + 2	g. (3 + 4) + 1
d. 2 + (3 + 2)	h. 3 + (4 + 1)

4. What property of addition does this seem to suggest?

5. Use your addition table to do these additions.

a. 0 + 2	b. 4 + 0
c. 1 + 0	d. 3 + 0
e. 0 + 0	

6. What do you know about the number zero in this arithmetic?

You have probably seen that the order in adding makes no difference in our new arithmetic. This means that for any numbers a and b in the arithmetic, $a + b = b + a$. Do you remember that this is called the *commutative property* of addition? This same property holds for whole numbers.

Did you see that grouping in addition makes no difference? This means that for any numbers a, b and c in the arithmetic, $a + (b + c) = (a + b) + c$. Do you remember that this is called the *associative property* of addition? This property also holds for whole numbers. Remember, since addition is associative we do not need to write parentheses in these cases.

Do you remember that the set of whole numbers is closed under addition? This means that when we add whole numbers we always get a number in the set of whole numbers. Is the set of numbers in our new arithmetic closed under addition? Is the result of any addition in the set? We see that it is, so the set of numbers in this arithmetic is closed under addition.

Did you also see that when we add 0 to any number we get that same number? In other words, for any number a, $a + 0 = a$. Do you remember that we call 0 the *additive identity* because it has this property? Zero is also the additive identity for whole numbers.

EXERCISES

---

1. Use your addition table to do these additions. Add the numbers several times using a different order or grouping each time. Check

your answers using the clock.

a. 3 + 4 + 1

b. 2 + 4 + 3 + 1 + 4 + 2

c. 3 + 4 + 1 + 2 + 2 + 4

d. 1 + 4 + 2 + 3 + 2

2. Make an addition table for an arithmetic of a four-minute clock.

3. What properties does addition have in the arithmetic of a four-minute clock?

4. How does addition in the arithmetic of a four-minute clock compare with addition in the arithmetic of a five-minute clock?

5. How can you tell an *additive identity* from a table?

6. How can you tell the *commutative property* from a table?

7. How can you tell the *associative property* from a table?

8. How can you tell *closure* from a table?

SUBTRACTION

For whole numbers we said subtraction was the opposite of addition. Let's make up subtraction for the clock in the same way.

5.4 Let's Explore

If we want to do the subtraction 2 - 3, we will look for some number which when added to 3 gives us 2. Use your table for the five-minute clock. Find '3' on the left. Move your finger across until you find '2'. Now look at the top heading. You find '4'. This means that 3 + 4 = 2. Then we know that 2 - 3 = 4.

1. Use your addition table to do these subtractions.

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a. 4 - 3

b. 3 - 4

c. 0 - 2

d. 1 - 3

e. 2 - 4

f. 4 - 1

g. 1 - 4

h. 0 - 4

2. How is this kind of subtraction different from subtraction of whole numbers?

We made up subtraction without the clock. We used only our table. What do you suppose subtraction can tell us about the clock?

When we made up addition, we thought about moving the hand clockwise. Subtraction is the opposite. Perhaps it can tell us what will happen if we move the hand the opposite way.

3. We found that 2 - 3 = 4. If the hand of the clock goes to '2' and then backward 3 places where will it be? Check all your subtractions in exercise 1 on the clock.

4. What does subtraction tell us about the clock?

We have learned something important about mathematics. We made up addition using the clock. Then we made up subtraction without using the clock. We simply used our addition table and our imagination. Yet subtraction also tells us something about the clock.

This is the way of mathematics. We sometimes make it up from the world around us. Then we can work problems in mathematics, without thinking of the world. But our answers may tell us something about the world anyway.

EXERCISES

---

1. Make an addition table for a seven-minute clock.

2. Do these subtractions using your table. Do not look at the clock.

a. 6 - 3	d. 4 - 1	g. 0 - 6
b. 3 - 5	e. 2 - 5	h. 3 - 6
c. 5 - 6	f. 3 - 0	i. 4 - 5

3. Think of subtraction as moving backward around the clock. Check each of your subtractions of exercise 2, using the clock.

4. What properties does addition have for the seven-minute clock?

5. What properties of subtraction do you find in this mathematical system?

HOW TO MAKE AND STUDY MATHEMATICAL SYSTEMS

We have studied arithmetic based on a clock. This tells us something about how to make up mathematical systems. What do we need?

1. We need a set. It might consist of numbers or perhaps other things.

2. We need to make up one or more operations.

3. We need to find out the properties of the operations (such as the commutative property and the associative property).

4. We need to find special members of the set (such as identities).

ERIC

WHAT IS AN OPERATION?

Do you remember what an operation is? We studied operations for whole numbers. When given two members of our set we have a way of finding at most a third one, we say we have an *operation*. Sometimes there may be no third member, but there is never more than one if we have an operation.

In whole numbers subtraction is an operation. We can write names of whole numbers using the subtraction symbol. These are names of whole numbers:

'8 - 2', '10 - 3', '5 - 1'

These are not names of whole numbers, because we cannot do the subtractions and get whole numbers for the answers:

'4 - 5', '3 - 7', '8 - 11'

We cannot use a table to define subtraction for whole numbers. The set is too large. Instead, we have a *rule* to define the operation. We say that '$a - b$' is a name for the whole number c, when $c + b = a$. Of course there may not be such a number c. Then '$a - b$' is not the name of any whole number.

5.5 Let's Explore

1. Study the five tables on the previous page. Tell which of them are tables of an *operation*.

Here are some sets and some rules for combining their members.

We shall see that there are other things to look for also.

For small mathematical systems, like clock arithmetic, we can make operation tables. We cannot study bigger systems very well with tables. Here are some tables for small mathematical systems. We shall use them to learn more about mathematical systems. The letters are not variables here. They are names for members of the systems.

1.

*	*a*	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*a*	*c*
c	*c*	*b*	*a*

2.

o	*m*	*p*	*q*	*r*
m	*q*	*r*	*m*	*p*
p	*r*	*m*	*p*	*q*
q	*m*	*p*	*q*	*r*
r	*q*	*p*	*r*	*s*

3.

△	*t*	*w*	*p*
t	*w*	*t* or *p*	*t*
w	*t*	*w*	*w* or *p*
p	*p*	*t*	*p* or *w*

4.

□	*f*	*g*	*h*	*k*
f	*k*	*h*	*g*	*f*
g	*h*	*f*	*k*	*g*
h	*g*	*k*	*f*	*h*

5.

○	✓	□	X	△
✓	X	✓	△	□
□	✓	□	X	△
X	△	X	□	✓
△	□	△	✓	X

i. [3, 4, 5, 6, 7, 8, 9]	The symbol '$x * y$' is another name for x (the first one)
ii. [7, 14, 12, 19, 4, 29, 35, 21]	The symbol 'x O y' is another name for y (the second one)
iii. [1, 2, 3, 4, . . .]	The symbol '$x \square y$' names the L. C. M. of x and y
iv. [0, 2, 4, 6, 8, 10, 12]	The symbol '$x \triangle y$' means the same as '$2 \cdot x + y$'.
v. [1, 3, 7, 8, 10, 16, 17, 24, 29, 31, 33]	The symbol '$x \triangle y$' means the same as '$x + 3 \cdot y$'.
vi. [1, 2, 3, 4, 5, 6, . . .]	The symbol '$x \mid y$' means the same as 'x^y'. For example, '$2 \mid 3$' means 2^3, or 8.
vii. [1, 2, 3, 4, 5, 6, . . .]	The symbol 'x @ y' means either x or y .

2. Using rule i, find:

a. $4 * 7$
b. $9 * 8$
c. $6 * 3$
d. $4 * 5$
e. $9 * 4$
f. $6 * 7$
g. $7 * 3$
h. $8 * 4$
i. $5 * 9$

Is * an operation?

3. Is rule ii an operation? Do whatever exploring you wish.

4. Using rule iii, find:

a. $2 \square 3$
b. $3 \square 5$
c. $2 \square 4$
d. $9 \square 12$
e. $15 \square 5$
f. $35 \square 15$

Is this an operation?

5. using rule iv, find:

a. $2 \triangle 6$
b. $6 \triangle 4$
f. $6 \triangle 2$
g. $18 \triangle 4$

c. 6 ∠ 10

d. 12 ∠ 2

e. 4 ∠ 6

h. 8 ∠ 8

Is this an operation?

6. Using rule v, find:

a. 1 △ 3

b. 8 △ 3

c. 10 △ 3

d. 10 △ 7

e. 3 △ 10

f. 16 △ 8

Is this an operation?

7. Using rule vi, find:

a. 3 | 2

b. 3 | 3

c. 5 | 2

d. 2 | 4

e. 2 | 5

f. 4 | 2

Is this an operation?

8. Using rule vii, find:

a. 2 @ 3

b. 4 @ 16

c. 8 @ 9

d. 6 @ 27

Is this an operation?

You have no doubt discovered how easy it is to tell an operation when it is described in a table. If there are no double entries in a table, then it describes an operation. When a rule for combining is not given by a table it is not quite so easy to tell. Then we must somehow figure out whether there can be more than one answer.

EXERCISES

1. Make up two tables which describe an operation.

2. Make up two tables which do not describe an operation.

3. In the set of natural numbers, is O an operation if x O y is the greatest common factor of x and y?

4. In the set of whole numbers, is $\bigtriangledown$ an operation if '$x \bigtriangledown y$' means y^x?

5. In the set of whole numbers is ! an operation if 'x ! y' means $5 \cdot x - 2$?

6. Make up two new operations in the set of natural numbers.

THE CLOSURE PROPERTY

We studied closure for whole numbers. We said the set of whole numbers is *closed under addition*. This means that *for any* pair of whole numbers, their sum is also in the set. The set of whole numbers is not closed under subtraction because we can find whole numbers a and b such that $a - b$ is not a whole number.

5.6 Let's Explore

1. Study the tables on page 121.
   a. Which of them describe operations?
   b. Which of the sets are closed under the operation?

2. You have found that rules i through vi on page 122 are operations. Which of the sets is closed under the operation?

EXERCISES

1. Make up two tables of operations under which the given set is closed.

2. Make up two tables of operations under which the given set is not closed.

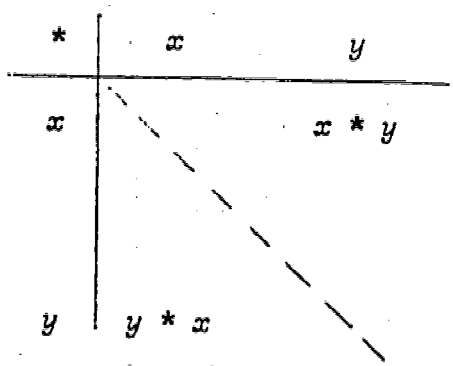
3. Given the set, [1, 3, 5, 7, 9, 11, 13, . . .]:

   a. Make up an operation under which this set is closed.

   b. Make up two operations under which this set is not closed.

THE COMMUTATIVE PROPERTY

Perhaps you have already discovered how to tell from a table whether an operation has the commutative property. If not, you might think of it this way. Imagine a table for an operation *. Now to find $x * y$ you find 'x' on the left and 'y' along the top. To find $y * x$ you find 'y' on the left and 'x' along the top, like this. Can you see that $x * y$ and $y * x$ are across the diagonal (dotted line) from each other? We say they are *reflections* of each other. If $x * y =$



$y * x$ then we say that the table is *symmetric with respect to the diagonal.*

If an operation is commutative, then the upper right half of the table must be just like the lower left half. If you folded the paper along the dotted line, the two havles would match. This is an easy way to tell from a table whether an operation is commutative.

If a table of an operation is symmetric with respect to the diagonal, then the operation described is commutative.

Suppose an operation isn't given a table. Then how can we tell if it is commutative?

5.7 Let's Explore

1. Look at rule i on p.122. It describes an operation. Find

a. 3 * 4	c. 5 * 5	e. 6 * 8
b. 8 * 6	d. 4 * 3	f. 5 * 5

Is this operation commutative?

2. Look at rule iii on p. 122. Find:

a. 2 □ 3	d. 3 □ 5	g. 9 □ 12
b. 3 □ 2	e. 4 □ 8	h. 12 □ 9
c. 5 □ 3	f. 8 □ 4	

Is this operation commutative? How can you tell?

We see that if an operation isn't given by a table, it is sometimes harder to tell whether the operation is commutative. We must figure out some way to tell. If we can find one case where the operation fails, then we know that the operation is not commutative. If we can find 10 million cases where it does work, we still may not be sure. In exercise 2 above we could tell, since the operation is taking the L. C. M. of two numbers. We know the order doesn't affect the result, no matter which pair of numbers we choose.

EXERCISES

---

1. Which of the tables on pp. 121 shows a commutative operation?
2. Which of the operations on p. 121 is commutative?
3. Make up a table for an operation which is commutative.
4. Make up a table for an operation which is not commutative.

THE ASSOCIATIVE PROPERTY

We found an easy way to tell a commutative operation when it was described by a table. How can we tell whether an operation has the associative property? There is no easy way to decide by looking at a table. We must look for other ways to tell.

5.8 Let's Explore

1. Look at rule i on p. 122. Find

   a. (3 * 5) * 4
   b. 3 * (5 * 4)
   c. (9 * 4) * 8
   d. 9 * (4 * 8)
   e. (3 * 5) * 7
   f. 3 * (5 * 7)
   g. (4 * 3) * 4
   h. 4 * (3 * 4)
   i. (5 * 5) * 8
   j. 5 * (5 * 8)
   k. $(x * y) * z$
   l. $x * (y * z)$

   Is this operation associative?

2. Look at rule iv on p. 122. Find

a. 2 Δ (4 Δ 2)

b. (2 Δ 4) Δ 2

c. (4 Δ 6) Δ 2

d. 4 Δ (6 Δ 2)

e. (4 Δ 4) Δ 4

f. 4 Δ (4 Δ 4)

g. (8 Δ 6) Δ 10

h. 8 Δ (6 Δ 10)

Is this operation associative?

In exercise 2 you have found that the operation does not have the associative property. There are examples which show this. In parts a and b we saw that 2 Δ (4 Δ 2) ≠ (2 Δ 4) Δ 2. This is one example. One example like this is enough to show that the operation does not have the associative property.

In exercise 1 we studied several examples. None of them showed that the operation is not associative. When you did parts k and l you found the answer to be x both times. What does this mean? It means that no matter what numbers 'x', 'y' and 'z' represent, the answer is the number represented by 'x'. Or we could say: For *any* numbers x, y and z in the set, $x * (y * z) = (x * y) * z$. This is the associative property. You have really proved it when you used letters, because they are variables which can represent *any* numbers in the set.

EXERCISES

---

1. In table 1, p.121, find

a. $(b * a) * c$

b. $b * (a * c)$

c. $(c * a) * c$

d. $c * (a * c)$

e. $(b * c) * a$

f. $b * (c * a)$

g. $c * (a * b)$

h. $(c * a) * b$

Do your answers prove that the operation has the associative property?

2. In table 2, p.121, find

a. $r \circ (p \circ m)$

b. $(r \circ p) \circ m$

c. $(m \circ q) \circ p$

d. $m \circ (q \circ p)$

e. $(r \circ q) \circ m$

f. $r \circ (q \circ m)$

Is this operation associative? Why?

3. Using rule ii, p. 122, find

a. $7 \circ (12 \circ 14)$

b. $(7 \circ 12) \circ 14$

c. $(29 \circ 4) \circ 35$

d. $29 \circ (4 \circ 35)$

e. $(19 \circ 21) \circ 14$

f. $19 \circ (21 \circ 14)$

Do your answers prove that the operation has the associative property? Why?

4. See if you can prove that the operation of rule ii, p.122 is associative. Hint: Use variables.

5. Is the operation of rule iii on p. 122 associative? Can you prove your answer?

6. Is the operation of rule v on p. 122 associative? Can you prove your answer?

7. Is the operation of rule vi on p. 122 associative? Can you prove your answer?

IDENTITY ELEMENTS

When we studied the whole numbers we talked about identity elements, or "identities." We said that 0 is the additive identity, because when we add it to any whole number we get that same number. We said that 1 is the multiplicative identity, because when we multiply any whole number by 1 we get that same number.

Other mathematical systems may have identities too.

5.9 Let's Explore

1. Look at table 1 on p.121. Find

   a. $a * a$ c. $a * c$ e. $c * a$

   b. $a * b$ d. $b * a$

2. What can you say about the element a?

3. Look in the table. Find the column headed 'a'. How does it compare with the column of table headings?

4. Look in the table. Find the row headed 'a'. How does it compare with the top row of table headings?

5. Can you see a way to tell from a table whether an operation has an identity?

Did you discover how to find an identity in a table? You look for a column and a row which are the same as the table headings. If you find them, the heading for that row and column names the identity for the operation.

If an operation table has a row and a column exactly like the table headings, then it has an identity element.

ERIC

1. Look at table 2 on p.121. Is there an identity element for the operation O? If so, what is it?

2. Look at table 4 on p.121. Is there an identity element for the operation □? If so, what is it?

3. Look at table 5 on p.121. Is there an identity element for the operation O? If so, what is it?

4. Make up a table for an operation having an identity element.

5. Make up a table for an operation not having an identity element.

INVERSE ELEMENTS

The idea of inverse elements in a mathematical system is probably new to you. If we perform an operation on two elements and get the identity, we say those elements are *inverses* of each other for that operation.

5.10 Let's Explore

1. Look at table 2, p.121. What is the identity for the operation O?

2. Solve these equations for this system.

   a. m O □ = q

   b. p O Δ = q

   c. □ O r = q

3. Look at table 4, p.121. What is the identity for the operation □?

4. Solve these equations for this system. (x, y and z are the variables.)

a. $g \square x = k$

b. $x \square g = k$

c. $h \square y = k$

d. $y \square h = k$

e. $f \square z = k$

f. $z \square f = k$

In exercise 4, you found that g and h were inverses of each other for the operation $\square$. This means that $g \square h = k$ and $h \square g = k$, where k is the identity element.

5. What is the inverse of m for the operation O?

6. What is the inverse of h for the operation $\square$?

7. What is the inverse of f for the operation $\square$?

For an operation *, if performing that operation on two elements gives us the identity then we say the elements are *inverses of each other*. (Or: a and b are called inverses of each other with respect to * if and only if $a * b = i$ and $b * a = i$, where i is the identity for *.)

EXERCISES

For exercises 1-5 look at table 5 p. 121.

1. What is the identity element for the operation O?

2. Is there an inverse for the element $\square$? If so, what is it?

3. Is there an inverse for the element $\triangle$? If so, what is it?

4. Is there an inverse for the element $\times$? If so, what is it?

5. Is there an inverse for the element $\checkmark$? If so, what is it?

6. In the system of whole numbers, what is the additive identity?

7. Which whole numbers have additive inverses (inverses for the operation of addition)?

8. In the system of whole numbers, what is the multiplicative identity (identity for multiplication)?

9. Which whole numbers have multiplicative inverses (inverses for multiplication)?

10. In a table for an operation, how can you recognize when two elements are inverses of each other?

INVERSES IN CLOCK ARITHMETIC

In the arithmetic of the five-minute clock, we can name inverses like this:

The symbol '$^-3$' means the additive inverse of 3.

Then $3 + {}^-3 = 0$.

The symbol '$^-1$' means the additive inverse of 1.

Then $1 + {}^-1 = 0$.

5.11 Let's Explore

1. What numbers of the arithmetic of a five-minute clock do these symbols represent?

a. $^-1$

b. $^-0$

c. $^-3$

d. $^-4$

e. $^-2$

ERIC

2. Add:

a. $2 + {}^{-}2$

b. ${}^{-}2 + 2$

c. $4 + {}^{-}4$

d. $3 + {}^{-}3$

e. $0 + 0$

f. $1 + {}^{-}1$

3. Add:

a. $3 + {}^{-}2$

b. ${}^{-}4 + {}^{-}2$

c. ${}^{-}1 + {}^{-}4 + 3$

d. ${}^{-}0 + 2 + {}^{-}3$

In clock arithmetic, the additive inverse of a number x *is named* '${}^{-}x$'. *(This is read "the additive inverse of* x.*")*

EXERCISES

In the arithmetic of a five-minute clock,

1. Add:

a. $3 + {}^{-}3$

b. ${}^{-}4 + 4$

c. $2 + {}^{-}4$

d. $2 + {}^{-}0$

e. $3 + {}^{-}1 + {}^{-}2$

f. ${}^{-}4 + 3 + {}^{-}2$

g. ${}^{-}3 + {}^{-}4 + {}^{-}1$

h. ${}^{-}3 + {}^{-}3 + {}^{-}3$

2. Solve these equations

a. ${}^{-}1 + 2 = \quad$

b. ${}^{-}3 + {}^{-}4 = x$

c. $\quad + {}^{-}2 = 1$

d. ${}^{-}4 + y = 3$

e. ${}^{-}\quad + 2 = 1$

f. ${}^{-}x = 3$

g. $4 = {}^{-}y$

h. ${}^{-}x + 3 = 4$

i. $3 + {}^{-}0 = {}^{-}z$

j. ${}^{-}4 + {}^{-}x = {}^{-}2$

Now that you know about additive inverses, you will need to know more about symbols for them.

5.12 Let's Explore

In the arithmetic of the five-minute clock,

1. What is the additive inverse of 3?

2. What is the additive inverse of the number you found in exercise 1?

3. The symbol '$^-3$' is a name for 2. Write an equation which says this.

4. The symbol '$^-(^-3)$' means the additive inverse of the additive inverse of 3. Replace the name '$^-3$' here by the name '2'.

5. What number is named by '$^-(^-3)$'?

6. What number is named by '$^-(^-4)$'?

7. What number is named by '$^-(^-1)$'?

8. What is the additive inverse of the additive inverse of 2?

9. What is the additive inverse of the additive inverse of 0?

10. Copy and complete this statement.

    For any number p in the arithmetic of the five-minute clock,

$$^-(^-p) = \underline{\qquad}$$

5.13 Let's Explore

In the arithmetic of the five-minute clock, the additive inverse of 2 + 1 can be named as '$^-(2 + 1)$', or '$^-3$'.

1. What numbers do each of these symbols represent?

a. $^-(1 + 3)$	f. $^-1 + {}^-4$
b. $^-1 + {}^-3$	g. $^-(0 + 3)$
c. $^-(4 + 2)$	h. $^-0 + {}^-3$
d. $^-4 + {}^-2$	i. $-(4 + 3)$
e. $^-(1 + 4)$	j. $^-4 + {}^-3$

2. What do these answers suggest to you?

No doubt you have discovered two interesting facts about additive inverses in clock arithmetic. As you study other number systems later you will find they also have these properties. Let's remember them.

For any number p, $^-(^-p) = p$.
(The additive inverse of the additive inverse of p, *is* p.*)*
For any numbers p, q, $^-(p + q) = {}^-p + {}^-q$
(The additive inverse of a sum is the sum of the additive inverses)

EXERCISES

All of these exercises refer to the arithmetic of the five-minute clock.

1. Find the simplest name for each of these numbers.

a. $^-4$	i. $^-(3 + {}^-2)$
b. $^-3$	j. $^-(4 - 2)$
c. $^-1$	k. $^-(^-1 + {}^-4)$

d. ${}^{-}2$

e. ${}^{-}0$

f. $3 + {}^{-}4$

g. ${}^{-}({}^{-}3)$

h. ${}^{-}({}^{-}0)$

l. ${}^{-}(1 + {}^{-}3 + 4)$

m. ${}^{-}3 + {}^{-}(4 + {}^{-}2)$

n. ${}^{-}1 + {}^{-}(1 - {}^{-}3)$

o. $2 - {}^{-}(4 + {}^{-}2)$

p. ${}^{-}(1 + {}^{-}3) - {}^{-}1$

2. Which of these sentences are true?

a. ${}^{-}(1 + 3) = {}^{-}1 + {}^{-}3$

b. ${}^{-}2 + {}^{-}4 = {}^{-}(2 + 4)$

c. $3 + {}^{-}1 = {}^{-}(3 + 1)$

d. ${}^{-}(4 + {}^{-}2 + 3) = {}^{-}4 + {}^{-}({}^{-}2) + {}^{-}3$

e. ${}^{-}(4 + {}^{-}2 + 3) = {}^{-}4 + 2 + {}^{-}3$

f. ${}^{-}4 + 2 + {}^{-}3 = {}^{-}(4 + {}^{-}2 + 3)$

g. ${}^{-}2 + 3 + {}^{-}2 = {}^{-}(2 + 3 + 2)$

h. ${}^{-}({}^{-}2) + {}^{-}3 = {}^{-}({}^{-}2 + 3)$

3. Find the solution set of each of these equations.

a. ${}^{-}(2 + 3) = {}^{-}\quad + {}^{-}3$

b. ${}^{-}1 + {}^{-}x = {}^{-}(1 + 3)$

c. ${}^{-}({}^{-}x) = 4$

d. $x + x = 1$

e. $y + {}^{-}2 = 0$

f. $4 + {}^{-}y = 2$

g. ${}^{-}(2 + x) = {}^{-}2 + {}^{-}x$

h. $1 + {}^{-}1 = {}^{-}2 + y$

i. ${}^{-}(3 + y) = {}^{-}3 + 2$

j. ${}^{-}(1 + {}^{-}3 + 4) = {}^{-}1 + y + 4$

k. ${}^{-}2 + {}^{-}1 + x = {}^{-}(2 + {}^{-}4)$

Appendix F

Introduction to Integers (Dist. Two)

ERIC

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Keedy, Jameson, and Johnson, Exploring Modern Mathematics, Book II, (pp. 20-24) © 1963 Holt, Rinehart, and Winston, Inc.

INTRODUCTION TO INTEGERS (DIST. TWO)

To this point we have been working with a clock. Let's now look at a system where the numbers are positioned on a line instead of a circle.

Let's Explore

1. Draw a line on your paper.

2. Fold your paper in half. Mark the point where the crease crosses your line.

3. Label this point "0".

4. Use a unit this long _____. Label points for whole numbers to the right of "0".

5. Where shall we label points for the additive inverses?

6. One unit to the left of "0" make a dot. Label it $^{-}1$.

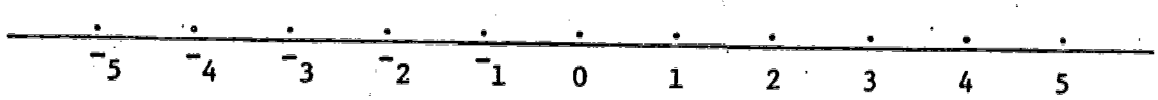
7. One unit to the left of '$^{-}1$' make a dot. Label it $^{-}2$.

8. Continue to the edge of your paper.

9. You now have a number line for the integers. If your paper were big enough, how far in each direction would it go?

<u>POSITIVE AND NEGATIVE INTEGERS</u>

Your number line should look like this



The point 0 divides this line. The numbers to the right of 0 are the natural numbers. Those on the left side of 0 are their additive inverses.

The numbers on the right are also called positive integers. Those on the left are called negative integers. Sometimes we name the positive integers like this:

'$^{+}3$', '$^{+}45$', '$^{+}178$',

we read '$^{+}7$' as "positive seven". Symbols like these name Natural numbers. We name them this way sometimes for emphasis, so that we won't confuse them with negative integers.

EXERCISES

---

1. Name the additive inverse of each integer.

a. 2
b. 4
c. 17
d. $^{+}35$
e. 0
f. $^{-}3$
g. $^{-}8$
h. $^{-}24$
i. $^{-}30$

2. Name the additive inverse of each of the following integers in two ways. Example: The additive inverse of $^{-}4$ is $^{-}(^{-}4)$ or 4.

a. $^{-}5$
b. $^{-}7$
c. $^{-}35$
d. $^{-}3$
e. $^{+}36$
f. $^{+}17$

3. Find the following sums.

a. $^{-}5 + 5$
b. $^{+}6 + {}^{-}6$
c. $^{-}5 + {}^{-}(^{-}5)$
d. $7 + {}^{-}7$
e. $25 + {}^{-}25$
f. $^{-}36 + 36$

ADDING NEGATIVE INTEGERS

How shall we find the sum ^-m and ^-n, where m and n are natural numbers? We shall wish to do it so that addition is commutative and associative, if that is possible. From our work in clock arithmetic we see the definition of ^-m and $^-n = {}^-(m + n)$. When we work the problem $^+5 + {}^+6 + {}^-5 + {}^-6$ using the commutative and associative property we have $^+5 + {}^-5 + {}^+6 + {}^-6 = 0 + 0 = 0$. Since we know that $^+5 + {}^+6 + {}^-({}^+5 + {}^+6)$ also equals zero it is seen that $^-5 + {}^-6 = {}^-({}^+5 + {}^+6)$. The ideas that we learn in our clock arithmetic also work for this new set of numbers.

EXERCISES

---

1. Name the additive inverse of each of the following sums of integers in two ways.

Example: The additive inverse of $^+4 + {}^+7$ is $^-({}^+4 + {}^+7)$ or $^-4 + {}^-7$.

The additive inverse of $^-4 + {}^+8$ is $^-({}^-4 + {}^+8)$ or $^-({}^-4) + {}^-8$.

a. $^+5 + {}^+8$

b. $^+6 + {}^+3$

c. $^-3 + {}^+5$

d. $^-7 + {}^-6$

e. $^+3 + {}^+7$

f. $7 + 10$

2. Rename the following negative integers as the sum of two negative integers in two ways.

Example: Rename $^{-}7$: $^{-}7 = {}^{-}3 + {}^{-}4$ or $^{-}7 = {}^{-}5 + {}^{-}2$

a. $^{-}8$

b. $^{-}6$

c. $^{-}5$

d. $^{-}3$

e. $^{-}15$

f. $^{-}25$

Appendix G

Addition of Integers (Dist. Two)

ADDITION

How will we find the sum when one number is a negative integer and the other is a positive integer? Let's use the ideas we have so far and explore.

Let's Explore

1. Can you find the missing number?

a. $5 + {}^{-}3 = 2 + \_\_\_\_ + {}^{-}3 = 2 + 0$

b. $8 + {}^{-}5 = 3 + \_\_\_\_ + {}^{-}5 = 3 + 0$

c. $15 + {}^{-}7 = 8 + \_\_\_\_ + {}^{-}7 = 8 + 0$

d. ${}^{-}15 + {}^{+}8 = {}^{-}7 + \_\_\_\_ + {}^{+}8 = {}^{-}7 + 0$

e. ${}^{-}11 + {}^{+}6 = {}^{-}4 + \_\_\_\_ + {}^{+}6 = {}^{-}4 + 0$

f. ${}^{-}8 + {}^{+}10 = {}^{-}8 + \_\_\_\_ + {}^{+}2 = 0 + {}^{+}2$

g. ${}^{+}7 + {}^{-}12 = {}^{+}7 + \_\_\_\_ + {}^{-}5 = 0 + {}^{-}5$

EXERCISES

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1. Find the following sums.

Example: $6 + {}^{-}8 = 6 + {}^{-}(6 + 2)$

$= (6 + {}^{-}6) + {}^{-}2$

$= 0 + {}^{-}2$

$= {}^{-}2$

a. ${}^{-}7 + {}^{+}18$

b. ${}^{-}5 + {}^{+}2$

c. ${}^{+}3 + {}^{-}17$

d. ${}^{+}7 + {}^{-}3$

e. $20 + {}^{-}3$

f. $17 + {}^{-}8$

REVIEW EXERCISES

1. Find the sums of the following integers.

a. $^{-}8 + {}^{-}5$

b. $^{-}3 + {}^{+}4$

c. $^{-}7 + {}^{+}2$

d. $^{-}5 + {}^{-}8$

e. $^{+}13 + {}^{-}14$

f. $13 + 5$

g. $^{+}5 + {}^{-}3 + {}^{+}5$

h. $^{-}4 + {}^{+}7 + {}^{+}4$

i. $^{-}5 + {}^{-}3 + {}^{+}10$

j. $9 + {}^{-}5 + {}^{-}1$

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Appendix H

Systems Method of Subtraction

ERIC

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Keedy, Jameson, and Johnson, Exploring Modern Mathematics, Book II (pp. 20-24) © 1963 Holt, Rinehart, and Winston, Inc.

ERIC

[illegible]

[illegible] additive inverses [illegible] other mathematical systems [illegible]

[illegible]

1. Do [illegible]

a.	[illegible]	f.	[illegible]
b.	$3 + {}^-2$	g.	[illegible]
c.	[illegible]	h.	[illegible]
d.	[illegible]	i.	[illegible]
e.	[illegible]	j.	[illegible]

2. What connection do you see between additive inverses and [illegible]

3. Do these subtractions and additions. See if your [illegible] correct.

a.	[illegible]	h.	[illegible]
b.	$2 + {}^-({}^-4)$	i.	[illegible]
c.	$3 - {}^-2$	j.	[illegible]
d.	$3 + {}^-({}^-2)$	k.	[illegible]
e.	[illegible]	l.	[illegible]
f.	$4 + {}^-({}^-2)$	m.	[illegible]
g.	[illegible]	n.	$2 + 4$

4. Do these [illegible] additions. See if your [illegible] still works correct.

a. $^-3 - 2$

b. $^-3 + {}^-2$

c. $^-4 - {}^-3$

d. $^-4 + 3$

e. $^-2 - {}^-4$

f. $^-2 + {}^-({}^-4)$

g. $^-4 - {}^-2$

h. $^-4 + 2$

Did you find out that we can always subtract by adding? If we want to subtract a number we can do it by adding its inverse. To subtract 4 from 2, we can add $2 + {}^-4$. If we want to find $3 - {}^-2$ we can add the inverse of $^-2$. So the problem can be written '$3 + {}^-({}^-2)$' or '3 + 2'. This is an important property. Let's remember it.

> In the arithmetic of a five-minute clock, the subtraction $x - y$ is the same as the addition $x + {}^-y$. (To subtract, we may add the additive inverse.)

EXERCISES

1. Do these subtractions in five-minute clock arithmetic. Do them by adding the additive inverse.

a. $2 - 4$

b. $2 - 0$

c. $0 - 3$

d. $^-3 - 4$

e. $^-2 - 2$

f. $^-1 - 3$

g. $4 - {}^-1$

h. $4 - {}^-3$

i. $2 - {}^-3$

j. $3 - {}^-4$

2. Which of these equations in five-minute clock arithmetic are true?

a. $3 - 2 = 3 + {}^-2$

b. $^-2 + 4 = {}^-2 - {}^-4$

c. $^-1 - 3 = 1 + {}^-({}^-3)$

d. $3 - {}^-2 = 3 + {}^-({}^-2)$

e. $3 - (2 - 3) = 3 + {}^-(2 - 3)$

f. $3 + {}^-(2 - 3) = 3 + {}^-(2 + {}^-3)$

g. $3 + {}^-(2 + {}^-3) = 3 + {}^-2 + 3$

h. $3 - (2 - 3) = 3 + {}^-2 + 3$

3. Find the solution sets of these equations in five-minute clock arithmetic.

a. $x - 3 = 2 + {}^{-}3$

b. $4 + {}^{-}2 = 4 + {}^{-}x$

c. $3 - 4 = 3 + {}^{-}x$

d. $4 - {}^{-}x = 4 + 2$

e. $x - {}^{-}3 = x + 3$

*4. Is $2 - 3$ the additive inverse of $3 - 2$? (In other words, is it true that $2 - 3 = {}^{-}(3 - 2)$?)

SUBTRACTION

When you studied clock arithmetic you learned about subtraction. It is the opposite of addition. In clock arithmetic every element has an additive inverse. We found we could subtract by adding an inverse. For example, in the arithmetic of the five minute clock

$$2 - 3 = 2 + {}^{-}3$$

In the system of integers all numbers have additive inverses, so we can subtract this same way. In fact, we can define subtraction to be the addition of the inverse.

For any integers x, y the subtraction $x - y$ is defined to mean the same as the addition $x + {}^{-}y$. (To subtract y from x, we add the inverse of y to x.)

Examples:

$$4 - 3 = 4 + {}^{-}3 = 1$$

$${}^{-}5 - 2 = {}^{-}5 + {}^{-}2 = {}^{-}7$$

$$7 - {}^{-}3 = 7 + {}^{+}3 = 10$$

$${}^{-}6 - {}^{-}8 = {}^{-}6 + {}^{+}8 = 2$$

EXERCISES

Subtract, writing the simplest name for your result.

1. $14 - 7$
2. $^{-}8 - 3$
3. $2 - 10$
4. $13 - {}^{-}7$
5. $2 - {}^{-}5$
6. $^{-}20 - 4$
7. $^{-}18 - {}^{-}7$
8. $^{-}4 - {}^{-}9$
9. $^{-}12 - {}^{-}6$
10. $17 - {}^{+}9$
11. $^{+}11 - {}^{+}15$
12. $^{-}6 - {}^{+}23$
13. $^{+}13 - {}^{-}16$
14. $^{-}15 - 2$
15. $12 - 3$
16. $^{-}5 - 20$
17. $^{-}18 - {}^{-}8$
18. $^{-}13 - 13$
19. $^{-}20 - {}^{-}20$
20. $^{-}24 - {}^{-}16$

Appendix I

Form A and Form B of Evaluation Instrument Developed by Investigator

Appendix I

TEST. FORM A.

Part I. Addition.

1. $5 + 7 =$ ______ 2. ${}^{+}5 + {}^{-}8 =$ ______ 3. ${}^{-}10 + {}^{+}7 =$ ______

4. ${}^{-}3 + {}^{-}8 =$ ______ 5. ${}^{+}7 + {}^{+}3 =$ ______ 6. ${}^{+}8 + {}^{-}2 =$ ______

7. ${}^{-}8 + {}^{+}11 =$ ______ 8. ${}^{-}12 + {}^{-}7 =$ ______

Part II. Subtraction.

1. $4 - 3 =$ ______ 2. $5 - 7 =$ ______ 3. $9 - 17 =$ ______

4. $25 - 18 =$ ______ 5. ${}^{-}24 - {}^{+}18 =$ ______ 6. ${}^{+}15 - {}^{-}16 =$ ______

7. ${}^{+}8 - {}^{-}6 =$ ______ 8. ${}^{+}15 - {}^{-}19 =$ ______ 9. ${}^{-}36 - 15 =$ ______

10. ${}^{+}23 - {}^{-}21 =$ ______ 11. ${}^{-}17 - {}^{-}25 =$ ______ 12. ${}^{-}36 - {}^{-}8 =$ ______

13. ${}^{-}16 - {}^{-}24 =$ ______ 14. ${}^{-}27 - {}^{-}19 =$ ______ 15. ${}^{-}8 - {}^{+}15 =$ ______

16. ${}^{-}21 - {}^{+}36 =$ ______

Part III. Show your work for each problem below.

1. $2\ 1/4 - 3/4 =$

2. $3\ 1/8 - 1\ 5/8 =$

3. $325 - 99 =$

4. $342 - 198 =$

5. $^-5 - (^+7 - ^-3) =$

6. $(^+5 - ^-8) - ^+7 =$

7. $5 - (^-3 - ^+8) =$

8. $(^+7 - ^-5) - ^+8 =$

9. $^+5 - ^-3 - ^+4 =$

10. $^-7 - ^+5 - ^-3 =$

TEST. FORM B.

Part I. Addition.

1. $6 + 8 =$ ______
2. $^{+}6 + ^{-}7 =$ ______
3. $^{-}10 + ^{+}6 =$ ______
4. $^{-}4 + ^{-}7 =$ ______
5. $^{+}7 + ^{+}2 =$ ______
6. $^{+}6 + ^{-}4 =$ ______
7. $^{-}7 + ^{+}12 =$ ______
8. $^{-}14 + ^{-}3 =$ ______

Part II. Subtraction.

1. $5 - 2 =$ ______
2. $6 - 9 =$ ______
3. $8 - 15 =$ ______
4. $23 - 17 =$ ______
5. $^{-}23 - ^{+}18 =$ ______
6. $^{+}14 - ^{-}17 =$ ______
7. $^{+}7 - ^{-}5 =$ ______
8. $^{+}17 - ^{-}18 =$ ______
9. $^{+}36 - 14 =$ ______
10. $^{+}24 - ^{-}22 =$ ______
11. $^{-}18 - ^{-}24 =$ ______
12. $^{-}32 - ^{-}7 =$ ______
13. $^{-}14 - ^{-}21 =$ ______
14. $^{-}26 - ^{-}17 =$ ______
15. $^{-}7 - ^{+}14 =$ ______
16. $^{-}22 - ^{+}35 =$ ______

Part III. Show your work for each problem below.

1. $3\ 1/5 - 4/5 =$
2. $2\ 1/7 - 1\ 4/7 =$
3. $375 - 98 =$
4. $343 - 199 =$
5. $^{-}6 - (^{+}7 - ^{-}2) =$
6. $(^{+}7 - ^{-}8) - ^{+}6 =$
7. $7 - (^{-}5 - ^{+}9) =$
8. $(^{+}8 - ^{-}3) - ^{+}7 =$
9. $^{+}5 - ^{-}8 - ^{+}7 =$
10. $^{-}6 - ^{+}4 - ^{-}2 =$