DOCUMENT RESUME

ED 068 329 SE 014 840

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TITLE Meaningful Instruction in Mathematics Education.
INSTITUTION ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, Columbus, Ohio.
PUB DATE Jun 72
NOTE 73p.

EDRS PRICE MF-$0.65 HC-$3.29
DESCRIPTORS Algorithms; *Elementary School Mathematics; *Instruction; *Learning Theories; Literature Reviews; *Mathematics Education; *Research Reviews (Publications); Teaching Procedures

ABSTRACT

This monograph focuses on the influence of meaning theory on elementary school mathematics programs and on mathematics instruction. Theories of arithmetic instruction up through 1935 are described, and the philosophy of meaning theory, contributions and discussions by mathematics educators, and applications to actual instruction are delineated for the period from 1935 through 1960. A discussion of the implicit concern for meaning in the modern mathematics movement from 1960 to the present concludes the first section. The second section summarizes studies that establish and affirm the importance of and need for meaning and studies that explore the effect of teaching various procedures with meaning. The final section of the paper briefly discusses the implementation of mathematically meaningful instruction. An extensive bibliography is included. (DT)

ED 068329

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SE 014 840

MATHEMATICS EDUCATION REPORTS

MEANINGFUL INSTRUCTION IN MATHEMATICS EDUCATION

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June, 1972

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Preface

Much of the mathematics curriculum reform of the past decade, and the mathematics curriculum development of today is based on the assumed superiority of meaningful instruction and learning. Few mathematics educators question this concept, despite the fact that no extensive overview of the development of meaningful instruction has been presented.

This paper presents such an overview. It is important, not only because it establishes an historical perspective for meaningful instruction in mathematics, but because it suggests further exploration and research directions for this area as well.

Jon L. Higgins
Editor

This publication was prepared pursuant to a contract with the Office of Education, U.S. Department of Health, Education and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their judgment in professional and technical matters. Points of view or opinions do not, therefore, necessarily represent official Office of Education position or policy.

Foreword

Within the past two decades we have experienced a so-called revolution in school mathematics. Begle (1968) traces the origins of this revolution to the pre-1850 work of the mathematicians Abel and Galois, although their influence was not felt until the present century.

The chain of influence began with mathematical research, moving first to mathematics programs of study at the graduate-school level and then to the undergraduate level. During the 1950's the revolution spread to mathematics programs at the secondary-school level, finally moving downward to the elementary-school level during the 1960's.

It is Begle's (1968) belief that

> . . . this revolution has been successfully concluded. [And] no new revolution is clearly in sight. Even if the first stirrings of a new revolution might be taking place in mathematical research, its effects could not appear in the pre-college program for generations [p. 45].

But there has been another revolution of import--a revolution which influenced elementary-school mathematics programs and instruction in particular. It is a revolution which all too often has been lost in the shadow of the "modern mathematics" movement. We refer to "The Revolution in Arithmetic" characterized by Brownell (1954) in the first article of the first issue of The Arithmetic Teacher. This "other revolution" was rooted more in educational psychology than in mathematics per se. But it had an appreciable impact upon the goals and content of elementary-school mathematics programs as well as upon the process of instruction.

The focus of the present monograph is upon this "other revolution" which was associated closely with the so-called meaning theory of arithmetic (Brownell, 1935). Our consideration of this all-important era in the development of school mathematics programs in the United States is presented in three sections:

I. Points of View

II. Relevant Research

III. Concluding Observations

Three things should be noted at the outset:

1. This "other revolution" was associated almost exclusively with the elementary-school level. Rarely was the dominant theme of the meaning theory articulated at the secondary-school level. Rarely was there related research conducted within the context of secondary-school mathematics.

2. This "other revolution" emerged during a time when it was commonplace to view the content of a mathematics program for grades 1-6 or 1-8 more narrowly than we do today. Hence, most position papers and research reports referred to arithmetic, not elementary-school mathematics.

3. This "other revolution" predated our current penchant for preciseness in discourse pertaining to mathematical ideas (which at times impedes rather than facilitates communication!). In fact, by present standards there were many instances of outright fuzziness pertaining to a variety of mathematical notions.

It is to be hoped that this last thing, in particular, will not (when evident in quoted material) distract the reader from that which we believe to be the worth of the essence of the meaning theory--which is as timely today as when it was formulated.

I. POINTS OF VIEW

In planning this monograph there was a great temptation to include background from the psychological literature pertaining to meaning, and from related sources pertaining to the meaning of meaning. Although educational psychologists' concern for meaning ranges from little or no interest (e.g., Bugelski, 1964) to a central theoretical and applied interest (e.g., Ausubel, 1968), there is a large body of research which relates meaning and meaningfulness to rate of learning; to forgetting, retention, and retroactive inhibition; and to transfer and proactive facilitation. (See McGeoch and Irion, 1952, for an excellent earlier consideration of these relationships.) It is not uncommon, either, to find more than a few contemporary discussions of concept formation linked to meaning (e.g., Jenkins, 1966; Carroll, 1964).

It soon became evident that if we were to yield to the temptation to include some psychological and other relevant background pertaining to meaning, the "background" could become a complex, voluminous monograph in its own right! (And who are we to concoct that multi-course bill of fare?)

Hence, with few exceptions, we have considered the literature on meaning as it pertains only--and specifically--to mathematical learning and instruction, especially in association with the meaning theory of arithmetic in particular (and elementary-school mathematics in its broader sense).

Even with this delimitation, it is important for the reader to keep in mind that discussions of meaningful mathematical learning and instruction often have been confounded by other factors as suggested by

Figure 1. The rote-meaningful dimension (A), the reception-discovery dimension (B), and the concrete-symbolic dimension (C) are in a sense distinct--yet they may interact with each other and with a product-process dimension (D) which relates to the goals of learning and instruction.

For instance: Garry and Kingsley (1970) would classify the rote-meaningful dimension as a task variable and the reception-discovery dimension as a method variable. Confusion between these two things has led many persons to the following unfounded belief identified by Cronbach (1965):

> Discovered knowledge is meaningful, knowledge presented verbally is not.
>
> This is true only for stupid verbal presentation. Didactic teaching can be highly meaningful. That which is taught by discovery, moreover, is at best meaningful only to the student who discovers it, not to the many who fail to make the discovery [p. 115].

Similarly, confusion also has existed many times between the (A) rote-meaningful and (C) concrete-symbolic dimensions of Figure 1. As Weaver (1950) indicated some time ago:

> There is no doubt that the rise of the "meaning theory" has accelerated the use of representative materials in instructional activity. These materials are intended to make arithmetic more meaningful to children. It is unfortunate that many persons have misinterpreted the implications of this fact. Consequently they have harbored one or more of several misconceptions: (1) that the mere use of any concrete or semiconcrete materials guarantees meaningful instruction, [or] (2) that instruction cannot be meaningful without the use of these representative materials, . . . [p. 38].

The focus of this monograph is on the rote-meaningful dimension of mathematical learning and instruction. Other dimensions of Figure 1 will be of no interest unless they have some direct bearing upon considerations of the rote-meaningful dimension.

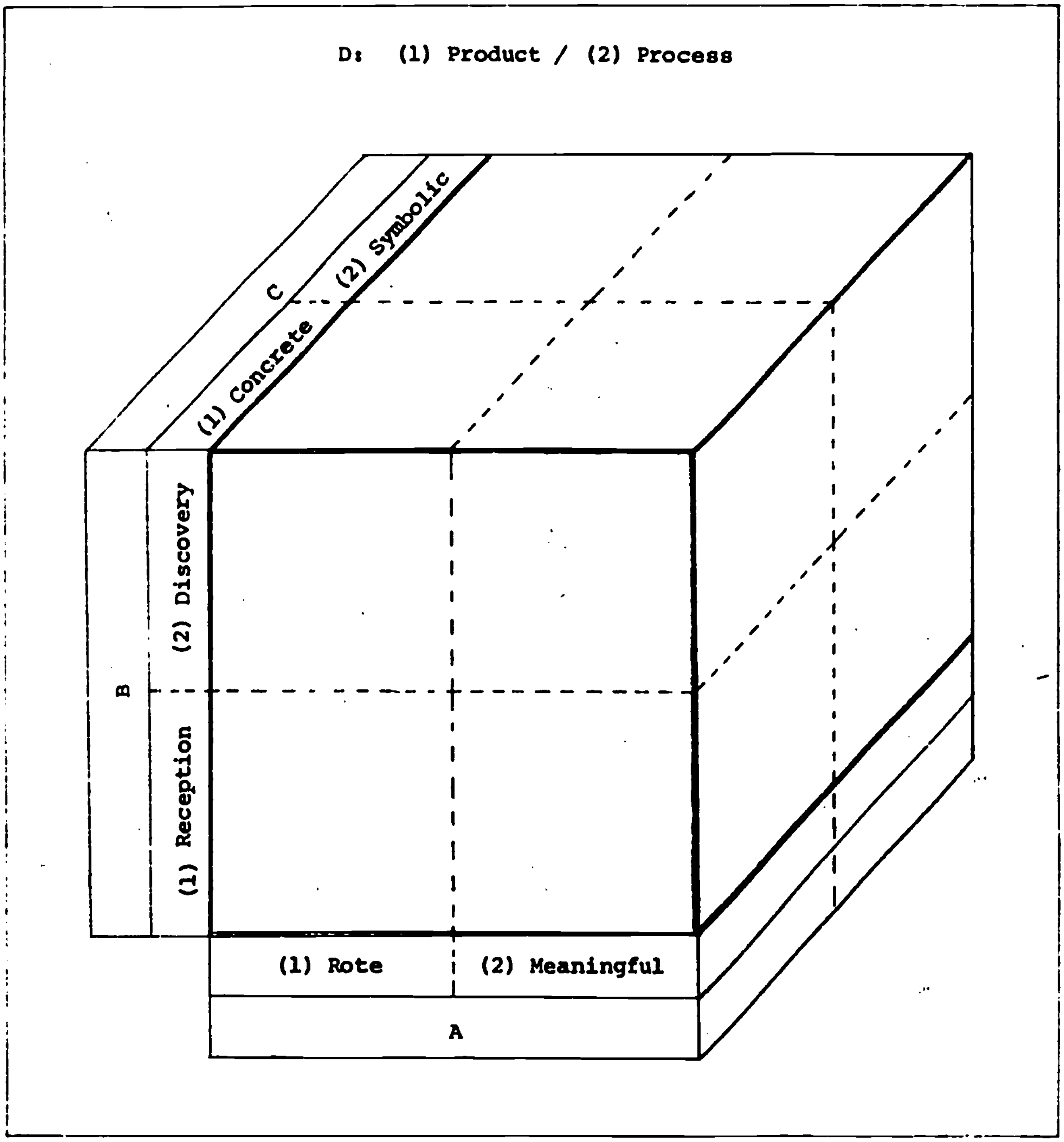


Figure 1. Some dimensions which confound considerations of meaningful learning and instruction

Establishing a Base Line

Mueller (1967) has indicated that "revolutions are essentially transformations, movements away from things as they were toward new ground, new objectives [p. 698]."

Away from what? Toward what?

One answer to the first of those questions, as it pertains to "The Revolution in Arithmetic," is exemplified by Knight's (1930) Introduction to the Twenty-Ninth Yearbook of the National Society for the Study of Education (Report of the Society's Committee on Arithmetic):

> The philosophy of this Yearbook . . . finds aims in the future as well as in the present. It suggests the desirability of preparation for adult living and holds it to be evident that a prediction of the demands of the future is feasible to a reasonable and useful degree of certainty. We should teach, then, those skills, informations, judgments, attitudes, habits, ideals, and ambitions which the child will find adequate and satisfying to the most important part of his whole self; that is, to his future adulthood as well as to his present childhood [p. 4].

This point of view clearly rejected a "progressive education" philosophy (popular at the time) which permitted children's emerging present needs within the context of "activity" or "experience" curricula to determine the scope and sequence of arithmetic content to be taught and learned. At the same time the Yearbook philosophy in effect supported a then-current trend to base curricular content principally upon surveys of arithmetic use within the everyday lives of adults.

Knight (1930) went on to assert that:

> Much may be said for the following point of view: What to teach should be decided by as wise adults as are available for the task, who will base their decisions as far as possible upon the available body of objective scientific data [p. 6].
>
> How to teach the child can be separated, in discussion, from what to teach--and how to teach is fundamentally more a matter

of psychology based on research and investigation than a matter of philosophy. . . . Theoretically, the main psychological basis [assumed in the Yearbook] is a behavioristic one, viewing skills and habits as fabrics of connections [pp. 4-5].

We must continue to seek . . . increased skill in the use of such aspects of learning as are suggested by the phrases: drill, continued effort, ability to withstand distraction, persistence though momentarily bored, effort sustained not by rewards at hand but by confidence in values forthcoming in the future, and the intent to master the matter in hand as a permanent possession rather than as a temporary accomplishment [p. 7].

Prelude to a Revolution

Although the points of view identified by Knight (1930) dominated the 29th NSSE Yearbook, there was evidence of demurrer within the Yearbook itself. Consider, for instance, Buckingham's (1930) comments:

. . . the curriculum, both as to content and method, should . . . devote more time to the growth of number concepts. . . . A reduction of arithmetic to the mere units and manipulations found to be employed in common affairs impoverishes the child's thinking [p. 18].

This whole idea of trimming the course of study down to the things which are going to be used directly defeats its own purpose. It means the learning of facts in isolation rather than in rich association; and it means, as usual, that facts so learned are less than half learned [p. 19].

Even the most narrowly utilitarian training recognizes that you must go beyond the mere treatment of the thing to be learned. . . . It is believed that a more rounded treatment should be given to the various topics of arithmetic in order that the true relations and inner connections of the subject may be appreciated. It is further believed that a fuller treatment of topics is actually necessary in order that the parts of the topics which are most frequently used may be successfully learned [pp. 19-20].

Just a few years earlier, Judd (1928) deplored the philosophy which viewed arithmetic as a "tool subject":

The figure of speech which one ought to use, if figure of speech is necessary, is not the figure of a tool, that is, of something that is now taken in the hand, now laid aside: the figure of speech which is appropriate is one which conveys the idea that number is an ever-guiding principle of life. . . . The number system . . . has changed the life of men. It has become a mode of thinking. It pervades every observation which man makes. It can no more be laid aside than can the right hand. It is not a tool [p. 5].

Judd (1928) also brought to the foreground another point of view which is critical in connection with this monograph:

What I am trying to say cannot be reconciled with that false educational psychology which is current in certain quarters where arithmetic is described as a mastery of so and so many different number combinations and a collection of so and so many distinct associations. Arithmetic is a general mode of thinking [p. 6]. . . .

I venture the prophecy that we are just at the point where we are about to leave behind the inadequate psychology which has in recent years taught that mental life is a bundle of particular ideas. We shall hear more and more in the years to come about general ideas [p. 7].

We are indebted to Brownell--one of Judd's doctoral students in educational psychology at the University of Chicago--for translating the preceding prophecy into a definitive base for the "other revolution" in school mathematics. In connection with his Ph.D. dissertation Brownell (1928) suggested that:

Many of the reasons why children find arithmetic difficult seem to be natural consequences of certain debatable theories regarding the nature of learning and teaching arithmetic.

There is, first of all, the conception of learning in arithmetic as the acquisition of a host of discrete isolated statements of fact. Children have learned arithmetic, according to this view, when they have committed to memory one hundred addition facts, one hundred subtraction facts, etc. . . . Learning in the case of each of these facts is held to consist in the immediate establishment of a specific, direct connection between given stimuli, such as 8 + 4, and a given response, 12. Each combination is to be learned as a separate item without relation to the facts previously learned. No encouragement is given the pupil to fit the facts together into a general coherent intellectual system; rather, he is taught the number facts in such a way as to prevent, if possible, his making use of facts which he has learned to assist him in acquiring new facts. . . . one criterion for the order in which facts are to be taught being the principle that each fact must be learned by itself, independent of all others [p. 195].

When the process of learning in arithmetic is conceived to be the mere acquisition of isolated, independent facts, the process of teaching becomes that of administering drill. This is the second debatable theory which now dominates instruction in primary arithmetic [p. 197].

Such a theory of teaching . . . fails to give adequate consideration, first, to the nature of the material which is to be learned and, second, to the behavior of children under conditions of drill [p. 198].

Brownell's dissertation, a classic in itself, was the forerunner of another classic in the literature on mathematics education, to which we now turn.

The "Meaning Theory"

In the opening chapter of the Tenth Yearbook of the National Council of Teachers of Mathematics, Brownell (1935) critically examined two then-current philosophies of arithmetic instruction--the "Drill Theory" and the "Incidental Learning Theory"--and formulated the first comprehensive statement of a third philosophy: the "Meaning Theory" of arithmetic instruction.

1. The "Drill Theory." According to Brownell (1935),

The drill conception of arithmetic may be outlined as follows: Arithmetic consists of a vast host of unrelated facts and relatively independent skills. The pupil acquires the facts by repeating them over and over again until he is able to recall them immediately and correctly. He develops the skills by going through the processes in question until he can perform the required operations automatically and accurately. The teacher need give little time to instructing the pupil in the meaning of what he is learning. . . . The main points in the theory are: (1) arithmetic, for the purposes of learning and teaching, may be analyzed into a great many units or elements of knowledge and skill which are comparatively separate and unconnected; (2) the

> pupil is to master these almost innumerable elements whether he understands them or not; (3) the pupil is to learn these elements in the form in which he will subsequently use them; and (4) the pupil will attain these ends most economically and most completely through formal repetition [p. 2].

Brownell (1935) then went on to say:

> Three major objections may be raised to drill as the sole, or even the principal, method of arithmetic instruction. The first objection is that the drill theory sets for the child a learning task the magnitude of which predetermines him to failure. The second objection is that drill does not generally produce in children the kinds of reaction it is supposed to produce. The third objection is that, even if under conditions of drill the proposed kinds of reactions were implanted, these reactions would constitute an inadequate basis for later arithmetical learning [p. 6].

In connection with his third objection, Brownell (1935) emphasized that:

> Arithmetic is best viewed as a system of quantitative thinking. . . .
>
> If one is to be successful in quantitative thinking one needs a fund of meanings, not a myriad of "automatic responses." . . . Drill does not develop meanings. Repetition does not lead to understandings [p. 10].

2. The "Incidental Learning Theory." Brownell (1935) characterized this theory, or group of theories--arising at least in part as a reaction against the "Drill Theory"--in the following manner:

> According to these theories, which differ chiefly in detail, children will learn as much arithmetic as they need, and will learn it better, if they are not systematically taught arithmetic. The assumption is that children will themselves, through "natural" behavior in situations which are only in part arithmetical, develop adequate number concepts, achieve respectable skill in the fundamental operations, discover vital uses of the arithmetic they learn, and attain real proficiency in adjusting to quantitative situations. The learning is through incidental experience [p. 12].

Although certain values of incidental learning were recognized, Brownell (1935) offered cogent criticisms of the "Incidental Learning Theory" and emphasized its impracticability. In particular, Brownell

(1935) contended that:

> However successful an occasional teacher may be in teaching arithmetic through incidental experience, general attainment of this success is not possible. The discriminating selection and orderly arrangement of vital and helpful learning situations involving number is no simple task. On the contrary, it calls for unusual insight into the mathematical and psychological nature of arithmetic on the one hand and into the psychology of childhood and of the learning process on the other hand. In a word, it calls for a degree and kind of insight which is, without aspersion of teachers as individuals, quite outside the equipment of the average teacher. Until teachers are differently selected and differently trained, it is fruitless to expect them adequately to teach children arithmetic through incidental experience [p. 18].

3. The "Meaning Theory." Brownell (1935) indicated that "this theory makes _meaning_, the fact that children shall see sense in what they learn, the central issue in the arithmetic instruction [p. 19, italics added]." At the outset of his discussion Brownell (1935) indicated ways in which this theory is related to the other two theories of arithmetic instruction:

> Within the "meaning" theory the virtues of drill are frankly recognized. There is no hesitation to recommend drill when those virtues are the ones needed in instruction. Thus, drill is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. But within the "meaning" theory there is absolutely no place for the view of arithmetic as a heterogeneous mass of unrelated elements to be trained through repetition. _The "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in "figuring." The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance_ [p. 19, italics added].
>
> [Also] The "meaning" theory allows full recognition of the value of children's experiences as means of enriching number ideas, of motivating the learning of new arithmetical abilities, and especially of extending the application of number knowledge and skill beyond the confines of the textbook. But the efficacy of incidental learning for developing all the types of ability

> which should be developed in arithmetic is held to be highly doubtful. . . [p. 20].

Brownell (1935) contended further that the "Meaning Theory" facilitates pupils' understanding of arithmetic in at least three ways: "First of all, it takes full account of the complexity of arithmetical learning [p. 20]." "In the second place, understanding of arithmetic is encouraged . . . through adapting the pace of instruction to the difficulty of the learning [p. 23]." And "The third way in which arithmetic instruction according to the "meaning" theory helps to make number sensible is by emphasizing relationships within the subject [p. 25]."

Brownell (1935) held strongly to the position that

> . . . the ultimate purpose of arithmetic instruction is the development of the ability to <u>think</u> in quantitative situations. . . . the ability merely to perform certain operations mechanically and automatically is not enough. . . . true arithmetical learning is seen to be a matter of growth [in quantitative thinking]. . . .
>
> . . . the teacher is unwise who measures progress purely in terms of the rate and accuracy with which the child secures his answers. These are measures of efficiency alone, not of growth. . . . The true measure of status and of development is . . . to be found in the level of the thought processes employed [pp. 28-29].

In concluding his discussion, Brownell (1935) stated:

> The record of arithmetic in the school is an unenviable one. The position taken in this chapter is that the fault lies in the type of instruction generally given. Arithmetic instruction has for a number of years inclined much too far in the direction of the drill theory of teaching. The trend now seems to be in the direction of the incidental learning theory of instruction. While this change in instructional theory represents distinct improvement, it does not . . . promise the kind and amount of reform needed. An attempt has been made in this chapter to outline a general shift in instructional emphasis and an altered view of the nature and purpose of arithmetical learning which may bring about the desired consequences. <u>The basic tenet in the proposed instructional reorganization is to make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence</u> [p. 31, italics added].

And so the "other revolution" was launched. In this same year several journal articles reflecting the "Meaning Theory" point of view emphasized explicit suggestions for the classroom teacher [e.g., Deans (1935), DeMay (1935)]. But it was Brownell's (1935) yearbook chapter which served as the launching pad for this "other revolution" which we now consider in greater detail.

The Next 25 Years

During the period 1935-1960 much was done to advance the centrality of meaning with respect to elementary-school arithmetic instruction. Serious attempts were made to base some (but not all) arithmetic textbook series for elementary-school pupils upon underlying tenets of the meaning theory. Professional books for teachers began to place much greater emphasis upon meaning--with respect to both content background [e.g., Buckingham's (1947) Elementary Arithmetic: Its Meaning and Practice] and classroom instruction [e.g., Brueckner and Grossnickle's (1947) How to Make Arithmetic Meaningful].

Admittedly (as we shall document in due time), some persons were not in complete sympathy with the emphasis being given to meaning. Other persons felt that many considerations of meaning lacked breadth and specificity of coverage. For instance, J. Murray Lee (1951), in his Editor's Introduction to Stokes's professional book for teachers stated that:

> The teaching of meanings of arithmetic has been one of the most discussed subjects in the profession. It has been a lot like the weather--much discussion but no one does much about it. This volume on "Teaching the Meanings of Arithmetic" is different! It actually shows how meanings can be taught [p. v].

As is so often the case, the essential nature and intent of a "movement" become distorted with the passage of time--and the meaning theory was no exception. As Swain (1960) indicated at the close of the 25-year period we are considering,

> Nearly all educational leaders and writers on arithmetic have climbed upon the bandwagon labeled "Meaning," but it is often no more than lip service that they render. Meaningful learning becomes for them a catch-all phrase useful for justifying whatever pedagogical procedure they may advocate [p. 272].

As we now consider in detail some of the points of view regarding meaning and arithmetic instruction which emerged during the period 1935-1960, you may find it helpful to refer to the time line of Figure 2--even completing it on a do-it-yourself basis as you follow our presentation.

Some important distinctions. Shortly after his characterization of the "Meaning Theory" of arithmetic, Brownell (1937) emphasized that elementary-school arithmetic programs should be concerned with both the significance and the meaning of number. Brownell had in mind the same distinction which Buckingham (1938) phrased in the following way:

> By the significance of number I mean its value, its importance, its necessity in the modern social order. . . . The idea of significance is therefore functional.
>
> On the other hand, the meaning of number, as I understand it, is mathematical. In pursuit of it we conceive of a closely knit, quantitative system.
>
> By making this distinction between significance and meaning I think we shall gain greatly. We shall perhaps begin to do two things; namely, to teach arithmetic as a social study and to teach it as mathematics [p. 26].

Buckingham (1938) then discussed another distinctive term:

> When I speak of insight I am talking no longer about arithmetic, no longer about the curriculum, but about the learner himself. Insight implies a person.

1935 — 10th NCTM Yearbook (The Teaching of Arithmetic)

1937 —

1938 —

1941 — 16th NCTM Yearbook (Arithmetic in General Education)

1944 —

1945 — 2nd Report of the NCTM Commission on Post-War Plans

1946 —

1947 —

1948 —

1949 —

1950 —

1951 — 50th NSSE Yearbook, Part II (The Teaching of Arithmetic)

1953 — 21st NCTM Yearbook (The Learning of Mathematics: Its Theory and Practice)

1955 —

1956 —

1957 —

1960 — 25th NCTM Yearbook (Instruction in Arithmetic)

Figure 2. Incomplete Time Line for the Period 1935-60

> When we confront children with a significant and meaningful experience, and when they make this experience theirs, they acquire insight, each to the degree that he is able [p. 27].

Confounding the distinctions. Despite these early attempts to distinguish between significance and meaning, it did not take long for the distinction to become muddled. As Brownell (1945) indicated:

> Nowadays it is fashionable to say that arithmetic should be taught meaningfully. . . . Every new or revised course of study gives prominence to meaningful outcomes in arithmetic. And the publication of each new or revised textbook series is accompanied by vigorous claims that here at last is a real program for making arithmetic meaningful. There is, then, general agreement at present that arithmetic should be taught meaningfully.
>
> This agreement in theory is not, however, matched by agreement in practice. In the first place, some advocates of what they call meaningful arithmetic disregard or minimize arithmetical meanings in favor of social applications, holding that experience in using arithmetical skills will make them meaningful. The fallacy in this thinking has been pointed out several times [Brownell, 1937; Buckingham, 1938]: experience in using skills may produce some awareness of the usefulness of number (that is, of its significance), but it cannot produce meanings. Meaning is to be sought in the structure, the organization, the inner relationships of the subject itself [p. 481].

In an attempt to clarify this last characterization, Brownell (1945) went on to identify three broad classes of "essential meanings of arithmetic":

1. Meanings associated with whole numbers, common fractions, etc., and with the decimal, place-value system of numeration.

2. Meanings associated with the nature of the operations of addition, subtraction, multiplication, and division.

3. Meanings associated with algorithms for computing.

A very few years later, Brownell (1947) rephrased the meaning-significance distinction by referring to the meanings of arithmetic and the meanings for arithmetic (i.e., its significance). He also reorganized and expanded his categorization of the meanings of arithmetic:

1. Basic number concepts (whole numbers, common fractions, etc.).

2. The nature of the operations of addition, subtraction, multiplication, and division.

3. Principles, relationships, and generalizations--which included those things which today we identify as "properties."

4. Meanings associated with our numeration system and its use in "rationalizing" computational procedures and algorithms.

Johnson (1948) preferred to identify three broad classes of meanings:

1. Structural meanings.

2. Functional meanings.

3. Operational meanings.

Somewhat more parsimoniously, Rosenquist (1949) identified scientific and functional meanings. Still other persons (e.g., Storm, 1948) were much more detailed in their classification and listing of arithmetic meanings.

It is no wonder that the original, helpful and important distinction between meaning and significance became lost along the way. And it is no wonder that several different persons could advocate meaningful arithmetic instruction--and could be advocating several quite different things.

Further considerations of meaning. It is not at all surprising that certain discussions of the nature of meaning as applied to arithmetic focused upon one aspect or another of language and symbolism: e.g., Riess (1944), Hickerson (1955), Morton (1955), and Hendrix (1950). In this connection, an excellent analysis that all too often has been overlooked in more recent times was presented in a series of two articles by Van Engen (1949), who first characterized a general theory of meaning:

> In any meaningful situation there are always three elements. (1) There is an event, an object, or an action. In general terms, there is a referent. (2) There is a symbol for the referent. (3) There is an individual to interpret the symbol as somehow referring to the referent. . . . It is important to remember that the symbol refers to something outside itself. This something may be anything whatsoever, even another symbol, subject only to the condition that in the end it leads to a meaningful act or a mental image [p. 323].
>
> . . . the whole object of arithmetic instruction clearly is to help the child devise a system of symbols which, in some sense, is representative of a realm of events--a series of symbolized operations with which the child has had direct experience. These operations, symbolized by the spoken word, the written word, or, in the case of mathematics, the mathematical symbol, are the primary instruments of knowledge. Awareness of the operation or of an event itself is not knowledge. Neither is awareness of the symbol knowledge. When, however, the two become associated in the mind of an individual, then there is knowledge [p. 325].
>
> A knowledge of arithmetic implies . . . that the individual becomes aware of a correspondence between a set of symbols and a set of operations. These operations are predominantly concerned, on elementary levels, with overt acts and images acquired as the result of experiences with the manipulation of objects. It is of utmost importance to note that the word "operation" is not used in the sense of "the fundamental operations of arithmetic" [addition, subtraction, multiplication, division]. "Operations," as used in this paper, designate the referent of a symbol--a written word, a mark on the board, a gesture, or a spoken word. This referent is an action considered with reference to the thing acted upon. Thus the term "operations" refers to overt acts. For example, the act of breaking a stick into halves is an operation [pp. 325-326].

After giving several illustrations of implications of this general theory for instructional practices in arithmetic, Van Engen (1949) discussed three particular theories of meaning in relation to arithmetic instruction:

1. The Social-Meaning Theory, in which it is believed that "the child will understand numbers provided he can observe and use these numbers in social situations [p. 395]." Van Engen (1949) makes explicit that a visit to the grocery store, for instance, may contribute to the desirable outcome of a child's "understanding of the usefulness of numbers. [but] Only in rare cases would it contribute to the child's understanding of the meaning of the symbolism used in the arithmetic class [p. 396]."

2. The Structural Theory of Meaning, in which it is believed that "arithmetic becomes meaningful when the child sees the structure of the subject. By 'structure' these educators mean the internal organization, the logic, of the subject [p. 396]." In order to appreciate Van Engen's critical reactions to this theory, we must distinguish between semantic and syntactic meanings--or between the semantic and syntactic dimensions of meaning [to which Van Engen (1953) alluded in a later source].

The semantic meaning of a symbol is derived from the referent(s) with which that symbol is associated. For Van Engen, operations involving things within the physical world are the principal, most important referents in the development of semantic meanings in arithmetic.

The syntactic meaning of a symbol, however, is derived from the way in which that symbol is used in association with other symbols: e.g., $2x$ and x^2 use the symbol "2" in a different sense, syntactically.

It was Van Engen's (1949) contention that:

> If those who espouse the structural theory have sinned, it is a sin of omission rather than a sin of commission. They have placed the emphasis on meanings which are of a higher order than those primary meanings which are the foundations of knowledge [p. 398].

The interrelationships which the structural-theory advocates have been espousing are syntactical meanings and are not the foundational meanings in arithmetic. "Syntactical meanings" are those very important meanings which are established after the child has acquired a semantic basis for these meanings of higher order [p. 398].

3. The Nihilistic Theory of Meaning, in which it is believed that "such symbols as 6 + 2 are meaningless symbols. However, such symbols as 6 apples + 2 apples are meaningful, so the argument goes, because they refer to apples, that is, something concrete [p. 399]." Van Engen (1949) dismisses this theory summarily, having considered earlier in his paper ways in which symbols such as "6 + 2" can be given meaning.

In another source Van Engen (1953), like Hendrix (1950), makes an important distinction between "meaning" and "understanding":

> Understanding refers to something that is in the possession of an individual. The individual who understands is aware of a satisfying feeling, a psychological closure, which results from having fitted everything into its proper place. Of course, this psychological closure must be tested because a child may think he understands when he does not understand. . . .
>
> The pupil who understands is in possession of the cause and effect relationships--the logical implications and the sequences of thought that unite two or more statements by means of the bonds of logic. The statement which is understood is seen to follow from statements accepted previously by the pupil [p. 75].
>
> Meaning is that which is "read into" a symbol by the pupil. The pupil realizes that the symbol is a substitute for an object. It is a triadic relationship between a pupil, a symbol, and the referent. Understanding is more nearly a process of integrating concepts--placing them in a certain sequence according to a set of criteria. Meaning, in its semantic sense, is a substitution process. It is a substitution of symbol for object, or symbol for symbol or symbol for concept. Understanding is an organizational process.
>
> From these considerations it would seem that the phrase, "I know what you mean but I do not understand it" is not a mere play of words. In a particular instance the pupil may know the referent: he may know what to do but he may not know why it

should be done: he cannot make the logical connection between the "situational need" and the response [p. 76].

The teacher of mathematics will teach for both understanding and meaning. . . . She should realize that certain methods are appropriate for the development of understanding and still other methods may be appropriate for the development of meaning. She should realize that the same methods are not necessarily appropriate for the development of both understanding and meaning. The teacher who makes these distinctions and adjusts her methods accordingly must of necessity be a better teacher than the one who blindly strives to "teach meaningfully" [pp. 76-77].

Goals of arithmetic instruction. The preceding considerations stemmed from a distinction between _meaning_ and _significance_ which is related to points of view regarding the principal goal(s) of arithmetic instruction. As might be expected, different emphases were in evidence during the period 1935-1960.

McSwain (1950), for instance, suggested that "The major purpose in teaching arithmetic is to help pupils to discover arithmetical meanings and to develop ability to do quantitative thinking [p. 267]." On the other hand, Wilson (1951)--a long-time proponent of a "social utility theory" of arithmetic--believed that "The basic and dominating aim of arithmetic in the schools is to equip the child with the useful skills for business [p. 12]."

The most common viewpoint held during this period, however, was expressed in the following way in the Second Report of the NCTM Commission on Post-War Plans (1945):

We must conceive of arithmetic as having both a mathematical and a social aim.

The fundamental reason for teaching arithmetic is represented in the social aim. No one can argue convincingly for an arithmetic which is sterile and functionless. If arithmetic does not contribute to more effective living, it has no place in the elementary curriculum. To achieve the social aim of arithmetic children must be led to see its worth and usefulness. . . .

We may grant the paramount importance of the social aim, and yet insist that it can be achieved only to a limited extent if the mathematical aim is neglected. The latter aim relates to the acquisition of the content of arithmetic, to the learning of arithmetical skills and ideas (concepts, principles, generalizations, and the like). Both skills and ideas should be made sensible to children through their mathematical relationships [p. 200].

The purposes of arithmetic cannot be fully attained unless children understand what they learn and know when and how to use it. We face here the problem of developing meanings, both mathematical and social meanings; and meanings have not customarily received their share of attention in classroom instruction [pp. 200-201].

It became commonplace to assert that "Arithmetic should be both mathematically meaningful and socially significant [Brueckner (1941), italics added]."

An interesting demurrer from this dual-aim point of view came from Wheat (1946) who objected to what he termed a "bifurcated design" in which "At least half of our arithmetic is mathematical (!), it is conceded, and the other half is social [p. 141]."

In an earlier source Wheat (1941) expressed this viewpoint quite strongly:

Arithmetic is a system of ideas. It is not a collection of objects. It is not a set of signs. It is not a series of physical activities. Arithmetic is a system of ideas. Being ideas, arithmetic exists and grows only in the mind. It does not flourish in the world of things. It does not arise out of sensory perceptions. It has nothing to do with the amount of chalk dust forty pupils can raise in a schoolroom in thirty minutes. Arithmetic exists and grows only in the mind. Being a system, arithmetic must be taught as a system. It is not an outgrowth of the individual's everyday experiences. It is not learned according as the interests or the whims of pupils may suggest. It is not anyone's personal discovery or invention. Arithmetic must be taught as a system [p. 80].

At a later time Wheat (1951) contended that:

We encounter no more arithmetic in the affairs of our lives than our knowledge of arithmetic permits us and leads us to meet [p. 22].

> The way to think that is arithmetic creates its own need. In a very real sense, we do not need it until we have it. This is the reason why our attempts to provide a need for arithmetic in advance of learning it are largely futile. We do not put in front what follows in rear. We do not make the trailer pull the car [p. 23].

The preceding comments illustrate well the treachery of quoting out of context. Without reading completely Wheat's (1941, 1951) two papers, one might grossly misjudge the point he was making and feel that he was depreciating the role of objects and actions. But as Van Engen (1949) emphasized when referring to one of Wheat's earlier works:

> Many of the modern practices in arithmetic must be classed as meaningful ways of teaching arithmetic from the point of view of an operational arithmetic. Many of the excellent suggestions found in Wheat's [1937] book on The Psychology and Teaching of Arithmetic fit into the operational theory of meaning. In particular, Wheat's "steps of progress in the study of groups through analysis and synthesis" constitute a beautiful example of the operational approach to foundational meanings in arithmetic [p. 400].

Other demurrers. Although proponents of a meaning theory of arithmetic instruction anticipated a number of objections that were bound to arise relative to an emphasis upon meanings [e.g., see Brownell (1935, 1945, 1947)], an interesting set of three journal articles centered upon certain of these objections.

Buell (1944) fired the opening salvo:

> "The keynote of the new arithmetic is that it should be meaningful rather than mechanical." I say lets [sic] make it increasingly mechanical and then go on to something more abstract. Let us continually make the difficult into the mechanical, and go on to the more difficult.
>
> Let us not look for "meanings" that are not there. Let us not become too mystical about this whole business. And please, let us not load up our own minds and the children's minds with round-about philosophies, methods, procedure, "meanings," et cetera, et cetera. Let us teach him that 8 and 6 are fourteen; that is all there is to it. He'll learn that as a definite fact without a lot of lacework built around it; and it has all the

> meaning it needs to have, the right kind of meaning, the speedy, definite meaning [p. 307].
>
> . . . mathematical "reasons" are the things we have less use for and forget most easily. Applications are remembered because used [p. 307].

Wheat (1945) was quick to respond with a variety of counter arguments, including the point of view that mathematical meaning is crucial in connection with applications:

> Meaning makes clear the practical application. The essence of the "social" situation to which number is "applied" is the number relation that is involved. . . . The practicality of the situation is in proportion to the meaning the pupil brings to it. Meaning makes the practical application practical [p. 102].

Bernard (1945) attempted to serve in the role of peacemaker regarding the Buell-Wheat controversy over meaning and application, but did seem to side more with Wheat than with Buell on at least one point:

> . . . one is limited in his application of mathematics if he is not fully appreciative of the meaning of numbers, their properties, and the principles of their combination [p. 260].
>
> Mathematics teachers can best serve those who will use mathematics in other fields by teaching them the theorems of mathematics and by making as clear to them as possible the logical structure of mathematics [p. 263].

Is this grossly different from Fehr's (1955) more recent position?

> . . . in his experience alone a child will never meet all number situations called for in later life. Nor can he, from a large number of isolated arithmetic situations, ever come to have a basic knowledge for use in later life. Accordingly, we must teach the pupils a structure of arithmetic and sufficient applications of the structure, so that in new situations, in life problems, he can use the structure for the necessary solution of problems [p. 27].

In a somewhat different vein Johnson (1948) argued for a <u>how-why</u> sequence of instruction pertaining to computational procedures and their rationalizations, contending that

> The why cannot be given before the "how-a-thing-is-done" is known because the step which we are trying to tell the reason for in a process must be known before the reason for it can be given [p. 364, hyphens and quotation marks added].

Weaver (1951) countered with a case for a why-how instructional sequence, emphasizing its advantages and suggesting disadvantages and dangers of an exclusive adherence to a how-why temporal sequence. In particular, Weaver indicated that relevant meanings can be used by pupils to develop algorithms--to develop the how of a variety of computational procedures. He pointed out that any advantage for such an approach akin to "guided discovery" is lost if how precedes why.

Meaning and skill. During the period 1935-1960 there were recurring discussions [Dickey (1938), McConnell (1941), Brownell (1944a, 1944b), Buswell (1951), Fehr (1953), Van Engen (1955)] which stressed:

(1) rejection of a connectionistic or association psychology, in which learning was viewed as a mechanical process, with principal concern for the products of learning, such as skill mastery; and

(2) the acceptance of a field or Gestalt psychology, in which learning was viewed as a growth process involving the reorganization of experience, moving from less mature to more mature patterns of thought and performance, with as much (or more) concern for the learning process itself as for the products of learning.

Meanings became highly important for arithmetic instruction within this latter psychological context, but the attainment of skill in computing was in no way ruled out as a desirable instructional objective. It was generally conceded, however, that the development of skill should follow the development of meaning and understanding.

But activities that are appropriate for the development of meaning, or the development of understanding, or the development of significance, are not the ones which develop skill. For that purpose, varied repetition is much more appropriate--but repetition does little or nothing in itself to contribute to the development of meaning, understanding, and significance.

Several writers such as Stretch (1935), Buckingham (1941), Sueltz (1953), and Brownell (1956) emphasized the unfortunate use of "drill" as <u>the</u> method of instruction under connectionistic approaches to arithmetic and the appropriate place of "drill" as <u>an</u> instructional technique for the attainment of skill <u>following</u> the development of meaning and understanding.

<u>In retrospect</u>. Much has been made at times of Bruner's (1960) four general claims which he feels can be made for teaching the "fundamental structure" of a subject. We cite the three that are especially relevant here:

> 1. . . . understanding fundamentals makes a subject more comprehensible.
>
> 2. Perhaps the most basic thing that can be said about human memory . . . is that unless detail is placed in a structured pattern, it is rapidly forgotten. . . . What learning general or fundamental principles does is to ensure that memory loss will not mean total loss, that what remains will permit us to reconstruct the details when needed.
>
> 3. . . . an understanding of fundamental principles and ideas . . . appears to be the main road to adequate "transfer of training" [pp. 23-26].

It is not without interest to observe that fifteen years earlier Brownell (1945) suggested that

> There are at least four good reasons why meanings should be taught in arithmetic:

(1) Arithmetic can function in intelligent living only when it is understood.

(2) Meanings facilitate learning.

(3) Meanings increase the chances of transfer.

(4) Meaningful arithmetic is better retained and is more easily rehabilitated than is mechanically learned arithmetic [p. 494].

In a later source Brownell (1947) expanded his listing in these ways:

From the standpoint of the teacher, meaningful arithmetic is interesting to teach. The need to develop understandings is much more stimulating than the task of listening to memorized facts and of administering mechanical drill.

From the standpoint of the pupil meaningful arithmetic--

1. Gives assurance of retention.

2. Equips him with the means to rehabilitate quickly skills that are temporarily weak.

3. Increases the likelihood that arithmetical ideas and skills will be used.

4. Contributes to ease of learning by providing a sound foundation and transferable understandings.

5. Reduces the amount of repetitive practice necessary to complete learning.

6. Safeguards him from answers that are mathematically absurd.

7. Encourages learning by problem-solving in place of unintelligent memorization and practice.

8. Provides him with a versatility of attack which enables him to substitute equally effective procedures for procedures normally used but not available at the time.

9. Makes him relatively independent so that he faces new quantitative situations with confidence.

10. Presents the subject in a way which makes it worthy of respect.

Some of these claims find support in the research to be summarized in Section II of this monograph. Other claims find support from less formal and less systematic evidence. In any event, these claims prompted no particular dispute in the literature on mathematics education during either the remainder of the 1935-1960 period or the subsequent period to which we now turn our attention.

The Recent Past: Since 1960

In our consideration of points of view expressed during the preceding period (1935-1960), we were concerned principally with discussions (pro and con) pertaining to the case for emphasizing mathematical <u>meaning</u> (as distinguished from social significance) in programs of arithmetic instruction. This included the position that an association-like psychology of learning--with its focus on connections established and reinforced through rote drill as <u>the</u> instructional method--was an inappropriate approach to the development of meaning. A different, field- or Gestalt-like psychology of learning--with an emphasis upon "structure" and "relationship"--was suggested as a more suitable guide for instruction.

In terms of position papers and the like which were aimed at "building a case for meaning" in connection with elementary-school mathematics instruction, the period of The Recent Past is a barren one. And for understandable reasons: the case had been built during the preceding 25 years.

Shulman (1971) has indicated that

> By the end of the 1960's, it appears that the rote-meaningful argument has been mercifully put to rest. In contrast to

> the 1920's and 1930's, general advocates of drill without understanding either have retired or are in hiding. This is not to imply that rote learning has ceased to occur in our classrooms. Far from it. It now occurs, however, through inadvertence rather than through careful planning.

Shulman's statement would be equally true if the first sentence were changed to read, "By the end of the 1950's, it appeared that the rote-meaningful argument had been mercifully put to rest."

Furthermore, with respect to program philosophy and serious attempts on the part of leaders in mathematics education to implement that philosophy, the Cambridge Conference on School Mathematics (1963) misrepresented the state of affairs by overly generalized assertions such as, "The traditional curriculum has stressed arithmetic drill throughout the elementary school [p. 16]."

In other instances the Cambridge Conference (1963) report simply reaffirmed points of view which had been stressed recurringly since about 1935; e.g., "The conference felt strongly that the understanding of the algorithms justifying the manipulations will in the end lead to better skills. . . [p. 16]."

As the "modern mathematics movement" progressed downward to the elementary-school level during this period of The Recent Past, it indirectly sharpened and extended a focus on mathematical meaning; but it did not generate that focus which, in fact, emerged and brightened during the 25 years prior to 1960. Throughout the present period since 1960 there have been many position papers, etc. related to the philosophy and content of "modern" mathematics programs for the elementary school, but any concern for "meaning" generally has been implicit rather than explicit.

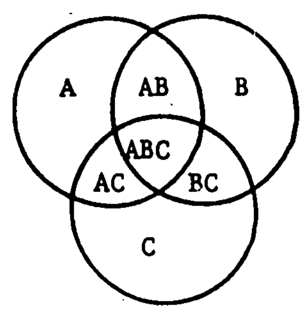
It is not inconceivable that in this present period, with its emphasis upon "activity learning," "real-life situations," etc., etc., there may be arising a "new confusion" regarding the meaning of meaningful (Olson, 1969). We may be coming about full circle, once again failing to distinguish clearly between content, activities, and experiences which contribute to mathematical meaning and those which contribute to social significance. Hopefully, however, we may profit from the past rather than disregard it.

II. RELEVANT RESEARCH

It is evident from the preceding section that meaning has been recognized as a vital component in the teaching/learning process. Both psychologists and mathematics educators have espoused it, and presented arguments to indicate and support its role. What does the research from mathematics education contribute to our understanding of the role of meaning and its importance?

Somewhat surprisingly, the evidence is sparse: there have been a relatively limited number of studies in which meaning is identified or isolated as a factor. These studies are of two types. There are some reports on research which attempted to ascertain the importance of meaning in learning mathematics, and others which, taking the need for meaning as a fact, explored the effect of teaching various procedures with meaning. With the evidence that meaningful instruction facilitates learning, the task shifted quickly from one of expounding it, to one of explaining it and elaborating on it.

As we searched for a logical model to aid in summarizing and describing the findings, it became evident that three factors are involved in all studies of both types. These factors are (A) the mathematical context, (B) the instructional context, and (C) the degree or extent of meaningfulness. A Venn diagram provides a way of viewing them:



These factors assume varying degrees of importance: in some studies, one or another is of most concern; in a few studies, there is a balance or overlapping of emphasis.

The mathematical context (A) includes the specific content which is to be learned, how the content was organized, and the nature of the meanings which are inherent in that content. Both methods and materials are aspects of the instructional context (B): how the teacher structures the learning environment, what he does as he interacts with the children, how materials were used, and how the outcomes were measured provide the way of inserting and inferring meaning. And the degree or extent of meaningfulness (C) must be considered as a factor beyond its involvement in the mathematical and instructional contexts.

Studies to Establish and Affirm the Importance of and Need for Meaning

Before the publication of Brownell's statement on the meaning theory in 1935, some research in mathematics education had already been published which provided support for it. In two studies of transfer, Overman (1930) and Olander (1931) reported that methods which emphasized generalization resulted in a greater amount of transfer from taught to untaught addition and subtraction combinations. McConnell (1934) presented evidence that teaching with meaning might not result in greater achievement when the criterion is prompt and automatic recall of content similar to that learned, but was better when the response required transfer.

Judd (1927) concluded that learning to work effectively with number is a developmental process, and that understanding of one aspect must be gained before the child can learn higher-level material. A similar conclusion was reached by Brownell (1928), from an investigation of the ability of children at each grade level to work with concrete materials and abstract ideas. He stated the need for developing understanding and meaning as the child moved through several stages to attain mature methods.

Brownell and Chazel (1935) demonstrated the ineffectiveness of premature drill. They summarized their study with third graders by stating that drill must be preceded by meaningful instruction. The type of thinking which is developed and the child's facility with the process of thinking is of greater importance than mere recall. Drill in itself makes little contribution to growth in quantitative thinking, since it fails to supply more mature ways of dealing with numbers.

Findings from a comprehensive investigation with children in grades 3 to 5 by Brownell and Carper (1943) suggested that activities and experiences which contribute to pupils' understanding of the mathematical nature of multiplication should precede work which focuses on memorization of facts.

Brownell, Kuehner and Rein (1939) studied 16 third-grade classes to determine the usefulness of a "crutch," as a meaningful procedure, in learning subtraction with the decomposition procedure. The crutch was found to facilitate accuracy at the early stages of learning for both bright and slow children and to aid understanding. They inferred an advantage for a method that makes a process meaningful rather than rote.

In MacLatchy and Hummel's (1942) study, a socially meaningful orientation was combined with mathematically meaningful teaching:

> Number experiences are isolated; the scheme of organization used in number is explained at first by markers and later theoretically; the social uses of number are given a second place. The purpose is to give children a meaningful skill which they can manipulate correctly and which they will recognize when they see it used in social situations or will themselves be able to use effectively in social situations in which it is pertinent. This scheme does not exclude functional arithmetic, but might most advantageously be used in conjunction with it [p. 227].

The teacher in the study began by assessing what the third and fourth graders knew about counting and adding. She asked simple questions which would show a child's understanding of the meaning of and his working familiarity with number. She then planned many varied experiences, grouping pupils to meet their indicated needs. Markers, social situations having number implications, encouraging various ways of solving problems, emphasis on accuracy, checking, and searching for uses of number in everyday living were all used, with careful preparation of all materials and logical analysis of steps involved in a process. High scores on a computation test were realized, and retention was good.

Thiele (1939), in his study of second grade arithmetic, reported highly significant superiority for the generalization method. Both he and McConnell (1934) found that programs stressing relationships and generalizations were preferable for developing understanding and the ability to transfer. But, as in many other studies, the teacher factor was not controlled by Thiele--rather, groups who had merely been exposed to a procedure were given tests designed to bring out differences in learning, and such differences were then ascribed to differences in teaching.

The study by Brownell and Moser (1949) was one in which the teaching method was specified and controlled. In this monumental study, the mathematical context was subtraction--or, more specifically, two subtraction algorithms--and the instructional context provided for a comparison of meaningful versus mechanical procedures. This was one of the few definitive experimental tests of the value of meaningfulness. The extent of meaningfulness can be rather well identified by the procedures which are described in detail in the study; meaning as it was involved in both mathematical and instructional contexts is evident.

In the study, the relative merits of teaching subtraction by the equal additions (EA) algorithm or by the decomposition (D) algorithm were investigated. Half of the classes were taught to borrow using decomposition; the other half, by equal additions. Each half was divided again, so that one part learned the subtraction procedure meaningfully, or rationally (R); the other part, mechanically (M). The investigation was conducted with approximately 1400 third-grade children who were having their first experience with "borrowing" in subtraction. Since the 41 schools involved varied in the extent to which arithmetic had been taught meaningfully in the first two grades, it was possible to note the effect of different backgrounds on the teaching of "borrowing" meaningfully and mechanically.

The classes were subjectively assigned to treatment so as to be as nearly as possible equal in ability. Teachers volunteered to participate and indicated which of the four experimental types of class they preferred to teach. Each day's work was prescribed for the fifteen days of the experimental period; teachers were told both what to do and what not to do each day. In both meaningfully-taught groups, the teachers led pupils to understand the procedures by:

(1) using actual objects (e.g., bundles of sticks) and drawings, as necessary

(2) writing the example in an expanded notation

(3) writing the "crutch" digit

(4) delaying the learning of the verbal pattern until they understood what it means.

The rationale to be developed for DR was the idea that 65 = 6 tens + 5 ones, or 5 tens + 15 ones. The rationale for EAR was that the difference between two numbers is not changed if the same amount is added to both numbers; thus one ten is added to both minuend and subtrahend.

In both mechanically-taught groups, pupils were taught to borrow in a purely mechanical, rote fashion; they were "directly" given the verbal pattern at the outset of learning, without explanation but with much drill to establish mastery. No hints were offered concerning the rationale of the procedure.

The verbal patterns or thought processes to be developed initially are indicated in Figure 3.

At the end of the experimental period, all pupils were given computation tests in subtraction, and all were interviewed individually. Six weeks later, during which no additional instruction was given on borrowing, a test to measure retention of skills was administered, and children in two centers were interviewed a second time. Evaluative criteria included measures of (1) rate, (2) accuracy, (3) smoothness of thought processes or work habits (included to avoid confusion of this factor with understanding), (4) understanding of procedures, (5) transfer to untaught types of subtraction, and (6) retention of efficiency and understanding.

$$\begin{array}{r} 65 \\ \underline{-28} \end{array}$$

DR	EAR
I can't take 8 from 5, so I borrow a ten from the 6 tens. I cross out the 6 and write a little '5,' to show that I borrowed a ten. I write a little '1' in front of the 5 to show that I now have 15 instead of 5. Then I subtract.	I can't take 8 from 5, so I add a ten to 5 and get 15. Now I subtract the ones and write 7. Since I added ten to the ones in the top number, I must add a ten to the tens in the bottom number, and 2 becomes 3. Now I subtract the tens, and write 3 in the tens' place.
DM	**EAM**
I can't take 8 from 5, so I think of 5 as 15. 8 from 15 is 7, and I write 7. Since I thought of 5 as 15, I must think of 6 as 5. 2 from 5 is 3, and write 3.	I can't take 8 from 5, so I think of 5 as 15. 8 from 15 is 7, and I write 7. Since I thought of 5 as 15, I must think of 2 as 3. 3 from 6 is 3, and I write 3.

Figure 3. Verbal Patterns To Be Developed

In summarizing the data from the three centers, the authors noted:

> Advocates of meaningful arithmetic (the writers included) argue that learning is more economical, that ideas and skill are better retained, and that these products of learning are more available for useful transfer when children see sense in the arithmetic they study [p. 147].
>
> As a whole, the findings show that meaningful instruction, especially in the case of decomposition, produced results superior to those produced by mechanical instruction [p. 149].
>
> [But], . . . the findings do not in themselves make a complete case against mechanical and for meaningful instruction in arithmetic [footnote, p. 149].

Among the conclusions are:

(1) The equal additions algorithm appears satisfactory for children who have a background of meaningful arithmetic, but for children with limited background the decomposition algorithm, taught with meaning, is better regardless of the criteria employed.

(2) The equal additions algorithm is difficult to rationalize.

(3) Some proficiency can be produced by mechanical instruction with either the decomposition or equal additions algorithm.

(4) Crutches were needed, but were more helpful for the decomposition algorithm than for the equal additions algorithm.

(5) Understanding appears to be cumulative.

(6) It is not simple to build meaningful learning upon a foundation of mechanical learning.

(7) "Readiness" is determined, not by the children's age or grade, but by the kind of arithmetic they have had.

The findings may be summarized in another way:

• Meaningful instruction, especially for decomposition, produced results superior to those produced by mechanical instruction.

• On tests immediately following instruction, mechanical instruction produced higher achievement than meaningful instruction.

• On measures of retention and transfer, the meaningfully taught groups scored significantly higher.

Swenson (1949) stated that the purpose of her investigation was:

> . . . to study learning, transfer of training, and retroactive inhibition as they appear in the learning of the one hundred addition facts by second grade children taught by three different methods of instruction, the chief variable among methods being the degree of emphasis upon organization and generalization in the learning process [p. 9].

Involved were 332 pupils from 14 second grades, who were stratified by MA and randomly assigned to treatment. The 100 facts were divided into three sets, each taught during a prescribed period of the study.

Teaching methods were of particular importance. The "generalization method" was based on the meaning theory of teaching and learning arithmetic. Teachers encouraged children to build up interrelationships among addition facts. Swenson noted:

> Because the meaning theory assumes that children are able to grasp number relationships, and that it is worthwhile to allow them to "figure out" these relationships, the generalizations were not dictated to the children as rules to be learned. Rather, the facts which centered around a generalization were presented together, in such a way that the teacher could, by skillful instruction, lead the pupils to their own formulation of the generalization. . . . The meaning theory holds that children should be allowed to continue relatively immature methods of arriving at answers so long as they need them as aids to understanding [pp. 12-13].

Therefore, counting, partial counting, dependence upon easier, known combinations, and concrete objects were allowed. The time spent on drill was necessarily limited by the large amount of time allowed for "figuring out" answers and for developing generalizations. Practice exercises were shorter, and were introduced after generalizations had been developed, and <u>during</u> rather than <u>before</u> the development of understanding. It was also pointed out that

> The emphasis placed by meaning theorists upon the organization of the number system was represented in the generalization method by grouping facts around generalizations, by encouraging children to derive one fact from another related one, by building up the decimal idea in teaching the adding of 10, and by allowing much original manipulation of number relations in connection with a miscellaneous set of facts in the last period of instruction [p. 13].

The "<u>drill</u> method" was based on the assumption that the learning of each addition fact is a discrete bond or connection to be formed; thus the facts were presented as abstract "facts to be learned" in miscellaneous order, with no organizational pattern. "According to the repeated warnings of drill theorists that children should not be allowed to reason out the answers [p. 14]," counting and other devious ways of deriving answers were discouraged. If a pupil hesitated or gave a wrong answer, the teacher told him the correct answer. Speed of response was emphasized. Little time was spent on the developmental part of the lesson; most of the time was spent in drill, with much repetition--interesting, varied, avoiding practice in error, with the most difficult facts repeated most.

In the "<u>drill-plus</u> method" each fact was introduced concretely or semi-concretely, and children verified each <u>new</u> fact by counting and manipulating concrete objects. From then on, drill procedures were followed. The facts were presented in groups which depended on organization by size of sum, though no attention was directed toward generalizations.

Possible intervening variables as well as both teaching and testing organization and procedures were carefully controlled; teachers attended seven pre-experiment sessions, were given explicit manuals, and were visited during the 20-week experimental period. The placement of facts was carefully planned.

Five addition tests, a test on the 100 untaught subtraction facts, tests of the 100 addition "decade facts," and one test with more difficult, complex addition examples were administered.

The generalization group made the highest net achievement record (on all tests) for original facts, interpolated facts, and final facts (but the differences between groups were not significant). The advantage of the generalization group on the test of original facts was highly significant in comparison with the drill-plus group and significant in comparison with the drill group. The differences in terms of interpolated facts (for retention) were nonsignificant. The advantage in terms of final facts (for transfer) was highly significant over the drill group and nearly significant over the drill-plus group, while the drill-plus group had a highly significant advantage over the drill group.

Thus, the generalization method resulted in greater initial learning, more transfer, more retention and less loss than drill-plus or drill methods, with the drill group appearing to be better than the drill-plus group.

Anderson (1949) viewed learning, under the meaning theory, as a process of reorganization of behavior that results from meaningful experience. He stated that the meaning theory has the following characteristics:

> (a) Arithmetic is a closely-knit system of ideas, principles, and processes. (b) This system has an organized, logical structure which can be understood by the pupil. (c) Learning in arithmetic consists of understanding number relations and the

mathematical significance of quantitative situations. Understandings of the number system may be called generalizations. (d) Drill or repetition is recognized as valuable for increasing proficiency in arithmetical situations after understanding has been developed or generalizations have been made [p. 41].

His experiment, less well-controlled than Swenson's, was conducted in 18 fourth-grade classes in 18 schools; data were obtained for 208 pupils in the drill group and 181 pupils in the meaning group. The experiment lasted from November until May, including all phases of arithmetic instruction during this time.

Teachers (with their classes) were selected because they seemed to have followed one of the two methods most closely in their previous teaching, and held an educational point of view consistent with the method. Day-by-day procedures were not prescribed, nor were teachers held to parallel courses concerning objectives, content, time, repetitions, or materials; instead, broad objectives were set and the teachers were given relatively free rein. Visits to the classrooms ascertained that instruction conformed to the theories. Teachers met for four sessions with the experimenter before the experiment, and twice during the experiment, at which manuals on the teaching method were studied. Logs kept by teachers indicated that in drill classes, about 11 minutes per day were spent on instruction and 24 minutes on drill; in the meaning classes, about 27 minutes a day were spent on instruction and 18 minutes on drill.

On tests measuring direct learning, the differences between groups taught arithmetic by drill or by meaning methods could, for the most part, have been accounted for by chance. (There was no significant difference in achievement of the two groups.) On tests of the ability "to think mathematically in quantitative situations," requiring ability "to adapt learning to new situations," a significant difference favoring the meaning method was obtained on the test for transfer. Those instructional procedures

> . . . which emphasized relational learning, discovery, and generalization were more productive, especially for children of high ability but of inferior achievement. There was evidence that drill procedures might be better for children of low ability and high achievement [p. 69].

Support for the meaning theory is thus implied, and there is also the implication that instruction must be individualized. What is best for pupils of one pattern of ability and achievement may not be best for pupils of other patterns. Also, Anderson noted,

> . . . it may be that attention to meanings, relations, or understanding of materials being learned is more important during early stages of learning, and that practice procedures are more important in late stages [p. 71].

This supports Brownell's conclusion that practice follows <u>after</u> understanding had been established.

Howard (1950) attempted to measure whether the development of meaning, through the extensive use of audio-visual aids, was worth the time in the teaching of the addition of common fractions. Fifteen classes in grades 5 and 6 were assigned to one of three 16-week treatments: (1) the pupils were shown how to do the computation or solve the problem, and then given practice in computation; (2) pupils used many manipulative objects and charts designed to bring out the meaning of each new step, after which they practiced by solving verbal problems; (3) pupils used objects and charts, and then were given practice on both computation and verbal problems. No significant differences in achievement were obtained, but the third group was found to be significantly better on a retention test.

Miller (1957) studied 360 seventh graders; he observed normal classroom techniques and classified the teachers as using a rule or meaning method. The meaning method was found to be more effective for computation of fractions and for decimals and percentage during the semester, while the rule method was superior for measurement. The meaning method was better for retention in computation and understanding, and for comprehension of complex analysis. It was more effective for average and high IQ groups, while the rule method seemed to be more effective for low IQ groups.

In a later study which compared the effects of using a meaningful and a mechanical or rote method, Krich (1964) worked with the mathematical content of division of fractions by fractions. He noted,

> While considerable evidence is available to show the value of teaching arithmetic with understanding, there is still the need to clarify the place of meaning and drill in learning specific processes in arithmetic [p. 697].

Eight sixth grades in eight schools were selected, with experimental and control groups matched for socioeconomic area, IQ, and arithmetic ability. In the meaningfully-taught group, the method was based on explaining rationally the meanings of the number symbols and arithmetic processes involved in multiplication and division of fractions, with the relationship of the two operations stressed. Children "were permitted to discover the inversion technique, but were not told to invert as a rule of procedure [p. 699]." In the mechanically-taught group, the rules were explained and children were told how to apply the rule. Drill on the material presented concluded each lesson.

Each group was taught through a series of five programmed lessons, to assure control of the teacher variable. A two-part test, measuring both understanding and computational skills, was used as pre-, post-, and retention test (administered two months after the posttest).

No significant differences were found between low IQ groups taught meaningfully or mechanically. The normal IQ group taught meaningfully scored significantly higher than the comparable mechanically-taught group on the retention test for understanding. Normal and high IQ groups achieved from both methods, but the meaningfully-taught group scored higher on retention while the mechanically-taught group scored significantly lower.

Tredway and Hollister (1963) were also interested in comparing meaningful and mechanical methods. They attempted to ascertain if teaching the basic concepts of percentage might be improved if taught meaningfully rather than by "rule, rote, and repetition." The meaningful procedure involved introducing the three cases of percentage at one time, and developing the interrelationships among them. They reported that such meaningful teaching of percentage provided significantly better results at all levels of intelligence than rote, textbook-oriented procedures. It also provided for better retention for those pupils of average intelligence.

Dawson and Ruddell (1955) reviewed five of these studies which supported the meaning theory approach. They concluded that "meaningful teaching leads to (1) greater retention over periods of time, (2) greater transfer potential, and (3) increased ability to solve new processes independently [p. 399]." The implications for teachers are clear: increased use of representative materials; more class time devoted to discussion and explanations; and short, specific practice periods following thorough development of topics.

Howard (1950), Shuster and Pigge (1965), Sebold (1946), and Feinstein (1952) also support the importance of using meaningful methods for work with fractions.

Studies Exploring the Effect of Teaching Various Procedures with Meaning

There are many more studies of this second type than of the first type. Those of this second type are ones in which the need for meaning is taken as a fact, so two or more procedures, <u>each</u> taught meaningfully, are compared.

Probably a majority of the studies of teaching techniques conducted during the past 15 years have involved the use of meaningful methods to teach the techniques. Some studies in actuality deal with meaning, even though the word is never used. We have included a sampling of studies in which the incorporation of meaningful methods was indicated by the researchers, and a few of those in which the method is clear by the use of associated words, or by the procedures which are described.

Pincus (1956) began with the premise that drill <u>should follow the development of understanding</u>. But he noted that there is disagreement on whether drill should consist of practice through number relationships, or merely repetitive "procedures and devices common before, as well as since, the development of teaching arithmetic meaningfully [p. 1415]." So four classes of third graders were taught second-decade addition and subtraction facts, and then divided into two matched groups, each of which used one of the two drill procedures. The addition and subtraction drill periods were each six weeks in length, followed by a seven-week period in which there were no addition or subtraction activities. At the end of each six-week period, both groups were tested for automatic recall of second decade addition or subtraction facts as well as for ability to transfer this recall to higher decades. After the seven-week lapse of instruction, both groups were tested for retention.

Considerable gains were made by both groups, regardless of the type of drill used. There was no significant difference (analyzed by t-ratios on the difference between means) in effectiveness between the two types of drill either in immediate learning, transfer, or retention, though there appears to be a "trend in favor of the drill-through-relationships method [p. 1415]."

Pincus added, "The considerable gains made by children taught by either method of instruction [for drill] may be taken as an indication of the value of teaching arithmetic meaningfully when followed by drill [p. 1415]."

Stokes (1958) experimented with a "cone of experience" meaningful method which had been developed and sequenced on the basis of surveys of the social problems met and arithmetic needed by 72,000 children. Children achieved successfully when ~~allowed to~~ operate at their own levels of meaning.

A status study, with no control of teaching method, was conducted by Rappaport (1958) to determine the relationship between computational skill and the understanding of the meanings in arithmetic. He tested students in grades 7 and 8 on the four operations and, three to four weeks later, on their understanding. Scores on the two tests were analyzed and correlated. Among the conclusions were:

(1) The pupils did not have an adequate understanding of meanings in arithmetic, assuming a score below 50 per cent is inadequate.

(2) Computational skill was not an indication of the understanding of meanings of processes used in computation.

(3) Correlations between computation and meanings tests were .63 for each total grade, lower for sub-groups.

Miller (1965) investigated the degree of relationship between "meaningful understanding" and computational skill of the arithmetic operations. He tested fifth and sixth graders, using (1) the computation section of a standardized test, (2) an experimenter-developed computation test, (3) an experimenter-developed instrument to diagnose understanding, and (4) standardized reading and intelligence tests. He found that children who demonstrated proficiency of computation of arithmetic operations did not necessarily also demonstrate understanding of computation. However, children who demonstrated understanding of operation revealed high computational skill of operation.

To determine how the use of class time affects achievement, Shipp and Deer (1960) compared four groups, in which 75 per cent, 60 per cent, 40 per cent or 25 per cent of class time was spent on group developmental work while the remainder was spent on individual practice. The developmental activities

> . . . were intended to increase understanding of the number system, the fundamental processes, and the general usefulness of number and quantity in everyday experience . . . [and included] explanations, discussions, and demonstrations by teacher and class; handling, inspecting, and arranging visual and manipulative materials; and group reading, drawing, construction work, and committee projects [pp. 117-118].

Higher achievement in computation, problem solving, and mathematical concepts was obtained when more than half of the time was spent on developmental activities.

In replications of this experiment, Shuster and Pigge (1965) and Zahn (1966) used other time allocations. They confirmed the finding that when the greater proportion of time is spent on developmental activities, achievement is higher.

In Hopkins' (1965) study with fifth graders, the control group continued the regular program that included about 50 per cent of the arithmetic time to be spent on meaningful activities and about 50 per cent on drill. The experimental group continued to use 50 per cent of their time for meaningful activities, but replaced the drill time with informal investigation of problems involving mathematical concepts usually met in the secondary school or college.

There was no significant difference between class means determined by a test for proficiency in computation; while there was significant difference between ability group means, there was no significance of treatment effects of different ability levels. There were significant differences for both class and ability group means on a test of understanding of arithmetic principles, but interaction showed no significance of treatment effects for different ability levels.

Hopkins concluded that the amount of time spent on drill can be reduced substantially and still retain equivalent computational proficiency. Utilizing time once spent for drill as a time to explore large mathematical concepts results in a better understanding of basic principles than is derived from using drill as a cognitive process.

When Tilton (1947) provided only 20 minutes per day for one day in each of four weeks (80 minutes in all) of individual remedial instruction in addition, subtraction and multiplication for two small groups of fourth graders, he found that emphasis on meanings resulted in significant achievement gain.

Binter (1962) compared the effectiveness of "the conventional method utilizing teacher telling, demonstrating, explaining, and the textbook as the means of presenting arithmetical understandings and computational skills" with

> . . . a prescribed curriculum which stressed pupil self-discovery in which pupils were encouraged to explore, experiment, and discover arithmetic facts and generalizations and various computational techniques to solve problems [p. 3942].

Teachers of fourteen sixth-grade classes were randomly assigned to teach the first case of per cent using one of the two methods. After a seven-day teaching period, a posttest was administered; a retention test was given five weeks later, with no teaching of per cent during the interim.

There was no significant difference in mean performance on the posttest, but results of the retention test showed a significant difference favoring the group taught by "self-discovery." However, on a transfer test, there was a significant difference favoring "teacher-telling."

The short length of time was noted to be a limitation, especially for the "self-discovery" group, and the fact that the textbook used by the "teacher-telling" group illustrated the second case of per cent as well as the first gave them an advantage on the transfer test. Binter's conclusion that

> . . . a teaching procedure which encouraged pupil self-discovery would be more defensible in terms of classroom procedure than one in which the mode of pupil intellectual operation was

> limited by teacher or textbook explaining, telling, and demonstrating [p. 3943]

does not appear to be supported by his data.

Lyda and Morse (1963) concluded that "When meaningful methods of teaching arithmetic are used, changes in attitudes toward arithmetic take place [p. 138]." Negative attitudes became positive, and the intensity of positive attitudes was increased when the experimenter stressed mathematical meanings in teaching fourth graders. These meanings were incorporated into lessons on the decimal numeration system, the process of counting, and each of the operations. Significant gains in computation and reasoning achievement also resulted. However, a halo effect emanating from the experimenter might have affected results in this study with only <u>one</u> class.

Greathouse (1966) investigated three methods of teaching in grades 5 and 6: group-oriented meaningful, individual-oriented meaningful, and drill-computation; analysis of results was on the basis of computational ability, quantitative reasoning, and mathematical understanding. Possible predictor variables were measured (reading comprehension, reading vocabulary, intelligence quotient, and pupil perception), and strongly biasing variables were controlled in the subsequent analysis of covariance. Both reading comprehension and intelligence quotient were found to be significant predictors of criterion achievement gain, but since the two were highly correlated, only IQ was used directly as a covariate.

There were no significant differences between methods. Yet, in most cases, the samples taught by the individual-oriented meaningful method achieved more than those taught by either other method.

Greathouse concluded that

> . . . a meaningful approach to arithmetic instruction with its emphasis on mathematical understanding as well as computational skills is recommended strongly as a supplant of a teaching approach which has the sole objective of developing computational competency [p. 5913].

In a study with fifth and sixth graders, Worthen (1968) compared two methods that differed only in terms of sequence characteristics. In the expository method, the verbalization of the required concept or generalization was the initial step in the sequence. Mathematical principles were explained verbally and symbolically to the pupil, who then worked with examples. In the discovery method, the pupil was presented with an ordered, structured series of examples of a generalization. No explanation was given, nor any hint that there was an underlying principle to be discovered. The pupil was expected to acquire the mathematical concept or generalization through an inference of his own. In both methods, the meaning inherent in the mathematics was an underlying basis for the lessons.

The content selected for the study included: (a) notation, addition, and multiplication of integers; (b) the distributive principle of multiplication over addition; and (c) exponential notation and multiplication and division of numbers expressed in exponential notation.

When the data were first analyzed, using the number of <u>pupils</u> in the statistical tests, the two sequences of presentation (with carefully described teaching behaviors) resulted in significantly different pupil performance on several types of tests. The expository method was found to be better than the discovery method on the initial test of learning, but discovery was better on retention tests administered after five and eleven weeks. The discovery group also transferred concepts more readily and used discovery problem solving approaches to new situations better. No differences were found in pupil attitude toward the two approaches. However, when the data were reanalyzed with the unit of analysis changed from pupil scores to class means, no significant differences were found (Worthen and Collins, 1971).

Christ (1969) reported that third graders learned time-telling concepts equally well in an active game situation as in a "traditional classroom" developmental-meaningful approach.

Fullerton (1955) compared two methods of teaching the "easy" multiplication facts to third-graders: (1) an inductive method by which pupils developed multiplication facts from word problems, using a variety of procedures; and (2) a "conventional" method which presented multiplication facts to pupils without involving them in the development of such facts. In this instance a significant difference in favor of the inductive method was found on a measure of immediate recall of taught facts as well as on measures of transfer and retention.

On the basis of multiple criteria, Schrankler (1967) evaluated the relative effectiveness of two algorithms for teaching multiplication with whole numbers to fourth grade pupils. As interacting factors, he considered (1) three intelligence levels and (2) two readiness backgrounds. From a variety of findings Schrankler concluded that methods using general ideas based on the structure of the number system are more successful than other methods investigated in achieving the objectives of increased computational skills, understanding of processes, and problem solving abilities associated with the multiplication of whole numbers between 9 and 100.

From a rather comprehensive investigation with third-grade pupils and their beginning work with multiplication, Gray (1965) found that an emphasis upon distributivity led to "superior" results when compared with an approach that did not include work with this property. The superiority was statistically significant on three of four measures: posttest of transfer ability, retention test of multiplication achievement, and retention test of transfer. On the remaining measure--posttest of multiplication achievement--children who had worked with distributivity scored higher than those who had not, but the difference was not statistically significant.

Van Engen and Gibb (1956) compared the effectiveness of the subtractive algorithm and the conventional distributive algorithm for division of whole numbers. The distributive algorithm was taught almost exclusively during the 1940's and 1950's:

```
      2
  23)552     First think, '2's in 5?'
     46
      92
     etc.
```

The subtractive approach has come back into use in recent years:

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  23)552
     230     10 x 23
     322
     230     10 x 23
      92
     etc.
```

Achievement, understanding, transfer, and retention were assessed. In defining the treatments, Van Engen and Gibb stated:

> All [eleven] teachers, regardless of the method taught, were urged to focus attention on arithmetic principles and generalizations, and all teachers were encouraged to use objects and socially-significant quantitative situations for problem-solving experiences. Thus, the primary difference between the two

methods was in the conceptual orientation of the process. The conventional method continued to place emphasis on the relation of division to the multiplication process. The subtractive method emphasized the concept of division as repeated subtraction of multiples of the divisor. The problem experiences for the subtractive involved one-digit and two-digit divisors but controlled the size of the dividend so that quotients would not exceed a two-digit number [p. 82].

Following analysis of both experimenter-developed tests and interviews, Van Engen and Gibb concluded:

(1) Children taught the conventional algorithm for division achieved more in solving the kinds of problems taught than did children taught the subtractive algorithm. However, the subtractive group worked with problems where both the number of digits in the divisor and dividend increased; only the dividend increased in number of digits in problems for the conventional group.

(2) The subtractive algorithm for division was more effective in enabling children to transfer to unfamiliar situations where the general context remains the same.

(3) Children taught the subtractive algorithm had a better understanding of the idea of division. For them the processing does not appear to be a matter of using a series of steps (such as "divide, multiply, subtract, compare, etc."), as was true of the children using the conventional algorithm.

(4) Retention appeared to be more a function of teaching procedure than of the algorithmic form used.

(5) Division for a measurement situation was easier for those taught the subtractive algorithm, while division for a partition situation was easier for those using the conventional algorithm.

(6) Children with low intellectual ability had less difficulty

understanding the process of division when they used the subtractive method.

In a study of the division algorithm with twelve fourth grade classes, Dawson and Ruddell (1955) compared the effectiveness of (1) "common textbook practices" and (2) a procedure in which division was presented as "a special case of subtraction." The second procedure also stressed "meaningful" instruction through much use of discussion and manipulative materials. The investigators concluded that this latter approach resulted in significantly higher achievement (immediately following instruction as well as after a retention period of seven weeks), and increased ability to solve examples in a new situation. It also helped pupils to develop greater understanding of division and its interrelationships with subtraction, multiplication and addition than did the "common textbook practices" approach. Whether these findings were related primarily to the emphasis on (1) subtractive concepts or (2) method of instruction or (3) use of materials cannot, however, be ascertained from the design of the study.

Bidwell (1971) compared three methods for teaching division of fractional numbers meaningfully: the common denominator method, the complex fraction method, and the inverse operation method. He found that the inverse operation method was most effective in terms of both learning structure, immediate computational skill, and retention.

O'Brien (1968) reported that pupils taught decimals with an emphasis on the principles of numeration, with no mention of fractions, scored lower on tests of computation with decimals than those taught either (a) the relation between decimals and fractions, with secondary emphasis on principles of numeration, or (b) rules, with no mention of fractions or principles of numeration. On later retention measures, the numeration approach was significantly lower than use of the rules approach, but not significantly different from the fraction-numeration approach.

In a study involving eleventh graders, Kepner (1971) examined the effect of "interpolated meaningful mathematical material" on the retention of previously learned similar material. A six-day unit on vectors was used as the original learning material, and a six-day unit on complex numbers was used as the related interpolated material. He found no significant differences in the retention of either vector addition or multiplication between classes in which the related material was interpolated and those in which a unit on probability was interpolated. He concluded that this suggested that an interference theory is inappropriate in describing "retroactive effects on the retention of meaningful learning by similar but conflicting material [p. 4551]."

The relative effects of two forms of spiral organization (area or topical) and two instructional modes (inductive or deductive) were studied by Armstrong (1968). Sixth graders were assessed at each of six cognitive levels, within three areas (set theory, number theory, and geometry) and on four topics (terminology, relations, operations and properties). She reported that the form of spiral organization of the curriculum did not affect the mathematical learning of the four topics. The inductive mode of presentation fostered the learning of operations, while the deductive mode resulted in greater learning of mathematical properties. The interaction of curriculum organization and instructional presentation variables was not found to significantly affect mathematical learning.

Ekman (1967) concluded that use of manipulative materials with third grade pupils learning addition and subtraction ideas resulted in increased understanding and transfer ability over the use of pictures or algorithms only.

No significant differences in overall learning of a mathematical principle between second grade groups who used a meaningful symbolic model and those who used a meaningful concrete model were reported by Fennema (1970). They were able to learn a principle by using either model when the model was related to knowledge the children had, providing evidence that "making the teaching of mathematical principles meaningful is as important as are the materials used to demonstrate that principle [p. 5339]." Children who had learned with a symbolic model could transfer this learning to solving untaught symbolic instances significantly better than could children who had learned with a concrete model. Thus learning facilitated by a symbolic model was more easily generalized than learning facilitated by a concrete model.

Some recent research has been related to that of Ausubel and the use of advance organizers; e.g., Gubrud, 1971; Ratzlaff, 1971. However, support for the use of such organizers appears tenuous; it may be that, for mathematical material, they are of more use to those with relatively high abstract thinking ability.

All in all, from research conducted during the past several decades we have come to be reasonably certain that particular advantages will accrue from meaningful mathematics instruction as opposed to rote instruction; but we are much less certain about advantages that may accrue from one meaningful approach, method, etc. vis-à-vis another meaningful one.

III. CONCLUDING OBSERVATIONS

Although a case for "meaning" in connection with mathematics instruction has been established through position papers and supporting research findings which have been summarized in this monograph, it would be in error to suggest that all mathematics instruction now is meaningful. We still need to be vitally concerned about implementation of mathematically meaningful instruction within the context of the school and the classroom. Specific suggestions for such implementation are beyond the bounds of this monograph, but there is at least one broad point we wish to make in relation to the contemporary scene.

Shulman (1970) has indicated that "mathematics educators have shown themselves especially adept at taking hold of conveniently available psychological theories to buttress previously held instructional proclivities [p. 23]." Some of this kind of thing is being done today in various attempts that are made to invoke Piaget or Bruner or Gagné or whomever in support of sundry suggested practices pertaining to mathematics instruction. And if we were looking for contemporary psychological support for meaningful instruction, it would not be at all unexpected if we turned to Ausubel.

In fact, we shall turn to Ausubel--but for a different reason. We are not contending that all facets of Ausubel's theory of meaningful verbal learning are exemplified by the original "Meaning Theory" of arithmetic instruction and its subsequent refinements and extensions. But we do wish to emphasize the relevance of certain distinctions which Ausubel (1968) has made:

> Meaningful learning involves the acquisition of new meanings, and new meanings, conversely, are the products of ...

meaningful learning. That is, the emergence of new meanings in the learner reflects the completion of a meaningful learning process [p. 37].

> The essence of the meaningful learning process . . . is that symbolically expressed ideas are related in a nonarbitrary and substantive (nonverbatim) fashion to what the learner already knows, namely, to some existing relevant aspect of his structure of knowledge (for example, an image, an already meaningful symbol, a concept, or a proposition). Meaningful learning presupposes both [1] that the learner manifest a meaningful learning set, that is, a disposition to relate the new material nonarbitrarily and substantively to his cognitive structure, and [2] that the material he learns be potentially meaningful to him, namely, relatable to his structure of knowledge on a nonarbitrary and nonverbatim basis [pp. 37-38].

Furthermore, according to Ausubel (1935):

> Logical meaning . . . refers to the meaning that is inherent in certain kinds of symbolic material by virtue of its very nature. . . . In short, logical meaning depends only on the material [pp. 44-45].

> Psychological (actual or phenomenological) meaning, on the other hand, is a wholly idiosyncratic cognitive experience. . . . Subject-matter can, at best, have logical meaning. It is the nonarbitrary and substantive relatability of logically meaningful propositions to a particular learner's cognitive structure that makes them potentially meaningful to him, and thereby creates the possibility of transforming logical into psychological meaning in the course of meaningful learning. Thus the emergence of psychological meaning depends not only on presenting the learner with material manifesting logical meaning, but also on the latter's actual possession of the necessary ideational background [p. 45].

Past and present emphases upon aspects of mathematical structure contribute to confronting children with material which has logical meaning. This is a necessary but insufficient condition for that material to have psychological meaning for a pupil. Meaningful mathematics instruction facilitates meaningful mathematical learning, and each of these demands that we look beyond the logical meaning inherent in mathematical content *per se*.

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