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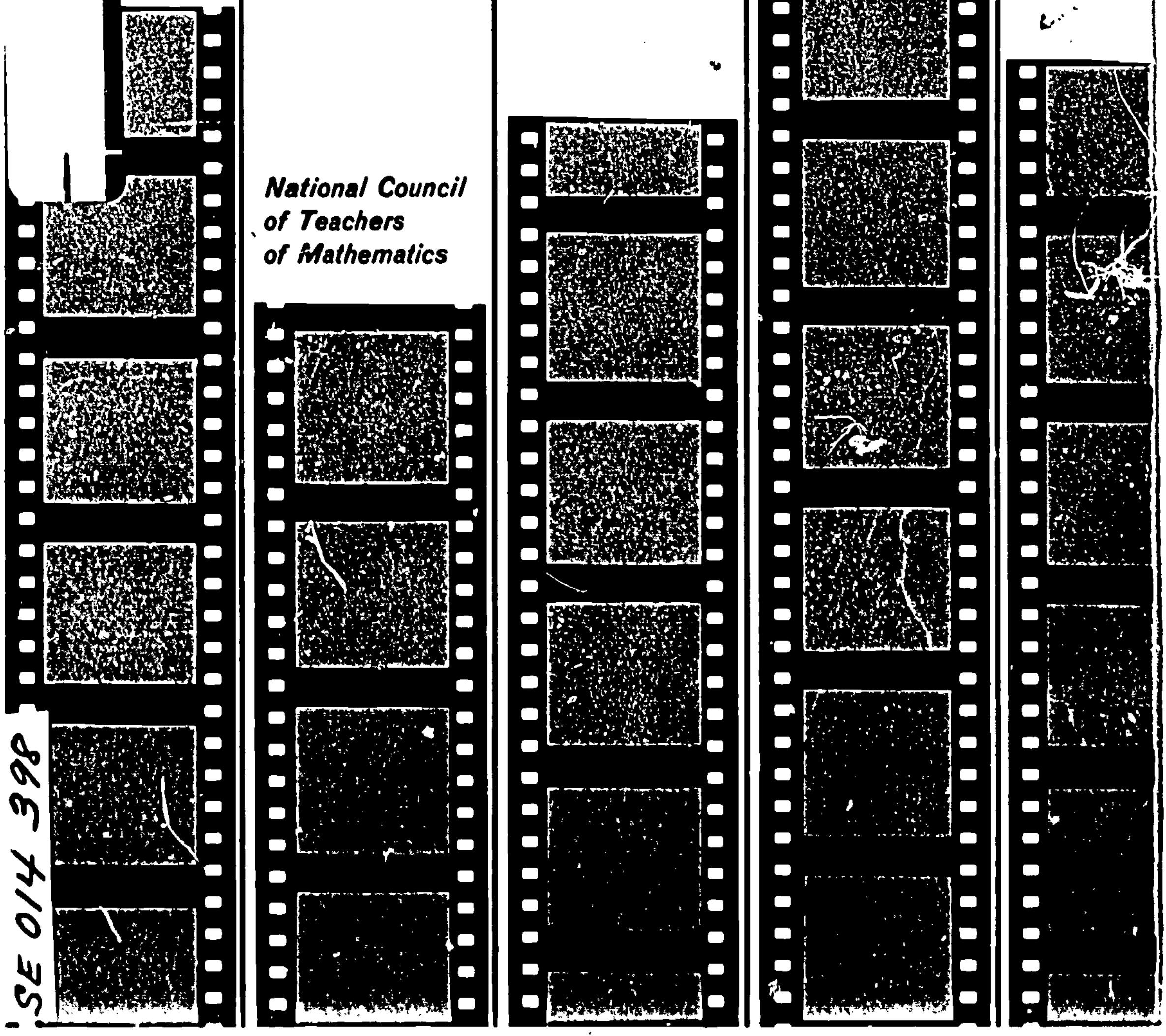
ABSTRACT

This book is an extension of the 1966 film/text series (ED 018 276) from the National Council of Teachers of Mathematics and was written to accompany 12 new teacher-education films. It is strong enough, however, to also serve alone as a text for elementary school teachers for the study of rational numbers. The 12 chapters corresponsing to the films were written separately by committee members with various methods of presentation. Aspects of rational numbers covered include a rationale for their introduction; the four operations with positive, decimal, and negative rational numbers; measurement; and graphing. (JM)

Mathematics for Elementary School Teachers

THE RATIONAL NUMBERS



National Council
of Teachers
of Mathematics

Mathematics for Elementary School Teachers

THE
RATIONAL
NUMBERS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth St., NW, Washington, D.C. 20036

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HISTORY AND ACKNOWLEDGMENTS

In 1966 the NCTM released for general distribution by Universal Education and Visual Arts a series of films, produced by Davidson Films under a National Science Foundation Grant, entitled "Mathematics for Elementary School Teachers." The films and an accompanying text were supervised and produced by a committee headed by Harry D. Ruderman. The topics covered by the films and text were:

1. Beginning Number Concepts
2. Development of Our Decimal Numeration System
3. Addition and Its Properties
4. Multiplication and Its Properties
5. Subtraction
6. Division
7. Addition and Subtraction Algorithms
8. Multiplication Algorithms and the Distributive Property
9. Division Algorithms
10. The Whole-Number System—Key Ideas

The success of the films and text was immediate. Soon the NCTM was besieged by teachers, teachers of teachers, and school systems to carry the series forward to include, at least, the rational numbers.

President Donovan A. Johnson named a committee consisting of Julius H. Hlavaty, *chairman*; Robert B. Davis, Abraham M. Glicksman, Leon A. Henkin, Donovan R. Lichtenberg, Joseph Moray, Harry D. Ruderman, David W. Wells, and Lauren G. Woodby to prepare a proposal for a new series of films.

In 1967 the NCTM, on the suggestion of this committee, signed a precedent-setting contract with General Learning Corporation, Davidson Films, and Harry D. Ruderman as director for the production of twelve teacher-training films (for the elementary level) and thirty short, single-concept films for students. These two series of films, under the title "Elementary Mathematics for Teachers and Students," were produced and released for distribution in 1970.

This book was written to accompany the twelve teacher-training films.

While it is complete and can stand on its own as a text for elementary school teachers for the study of rational numbers, it is intended to be a strong aid in the use of the films.

The committee named above, with Harry D. Ruderman as executive director, was given the responsibility for developing scripts for the films, following through with production of the films, and preparing the text materials. Lauren G. Woodby was the film teacher, and Joseph Moray was the classroom teacher. Many of the film scripts were prepared by members of the committee. However, the entire committee had an opportunity to react to all the scripts and all the films.

Each chapter in this text was written by a member of the committee. No editorial attempt has been made to impose on all the chapters any rigidly uniform method of presentation.

To expedite completion of the text, the Board of Directors designated Jack E. Forbes, Harry D. Ruderman, and Julius H. Hlavaty (*chairman*) as an editorial committee.

Thanks are expressed to the persons named above, to whom the NCTM and the profession owe a debt of gratitude for the films and the present text. With grateful acknowledgment of their special contributions, the following should be signalized:

Harry D. Ruderman, who has now directed with great success and special devotion two major projects for the NCTM

Francis Keppell, president of General Learning Corporation, whose insightful interest kept the project afloat and helped it to a conclusion

Jack W. Davidson and his staff, who contributed not only high technical skill but personal and creative input toward the high quality of the films

Joseph Moray, who was classroom teacher for both film projects mentioned above and, in addition, wrote all the scripts for the first film series, in which he was also the studio teacher

Lauren G. Woodby, who was the studio teacher for the second film series

James D. Gates, executive secretary of the NCTM, who was a large factor in the negotiations of the original contract and the production of this text

Charles R. Hucka, associate executive secretary of the NCTM, and his editorial staff, who made large contributions toward the actual production of this book

JULIUS H. HLAVATY

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Note to the Reader

You will find that sets of exercises appear in every chapter, and that answers to all of them appear in the back of the book. Before consulting the answers you will want to do your own figuring. For your convenience, working space has been provided beneath most of the exercises.

David W. Wells
Stuart A. Choate

BEYOND THE WHOLE NUMBERS

1

1. **Why do we need numbers other than whole numbers?**
2. **What is a nonnegative rational number?**
3. **What are some physical interpretations of the number $\frac{a}{b}$?**
4. **What are some strategies for introducing the meaning of a nonnegative rational number to elementary school pupils?**

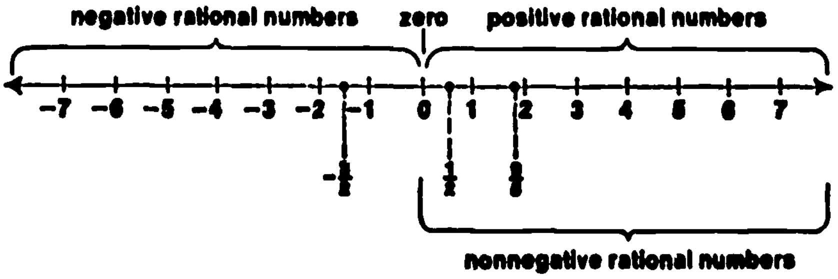
As long as early man needed only to determine the number of objects in a given collection, the set of numbers $\{0, 1, 2, 3, \ldots\}$, which is the set of whole numbers, served him well. Using only the whole numbers, he could determine the number of animals in his possession that could be used to provide food and clothing and the number of weapons needed by his family or tribe to hunt food or defend their home. The whole numbers were adequate for these and other situations that required only counting.

THE NEED FOR NUMBERS BEYOND THE WHOLE NUMBERS

There is also little doubt that the whole numbers were used to make crude measurements to the nearest whole unit. But when man began to construct permanent homes, engage in commercial trade, navigate the waters of his world, and assess taxes on his land and other possessions, the whole numbers were no longer adequate. When he attempted to determine, with greater precision than he had in the past, the measure of such properties of an object as length, area, volume, weight, capacity, and temperature, the subdividing of units of measure became necessary. To express measures in terms of these subunits, numbers beyond the whole numbers were needed.

The Rational Numbers

The set of numbers zero and greater that can be named by fractions was invented to satisfy the need. This set of numbers is called the set of nonnegative rational numbers. One-half, two-thirds, nine-fifths, eleven, and zero are all examples of nonnegative rational numbers. In general, any number that can be named by a fraction $\frac{a}{b}$, where a names a whole number and b names a whole number different from zero, is a nonnegative rational number.



Although the nonnegative rational numbers were invented to satisfy a practical need of man, there is also a mathematical need for this set of numbers—the need to be able to divide any and every whole number by a whole number greater than zero.

Students beginning the study of rational numbers will know from their previous work in mathematics that the sum of any two whole numbers is always a whole number. Also, the product of any two whole numbers is a whole number. Expressing these facts with frames as shown below, we say that if you use for the frames any pair of whole numbers, the result in each case will always be a whole number.

$$\square + \triangle = \_\_\_ \qquad \square \times \triangle = \_\_\_.$$

This is to say that the set of whole numbers is *closed* under the operations of addition and multiplication. In general terms,

> *For any pair of whole numbers a and b, a + b and a × b are whole numbers.*

However, the set of whole numbers is not closed under the operation of division. That is, not every pair of whole numbers has a quotient that is a whole number. For example, in the sentence $3 \div 4 = \square$ there is no whole number for $\square$ that will make the sentence true. Consequently, if we wish to always be able to divide any whole number by a whole number except zero, numbers beyond the whole numbers are needed. It will be shown later that the set of nonnegative rational numbers satisfies this mathematical need.

Man has continued to invent sets of numbers to satisfy new needs. To solve each of the four equations shown below, the set of numbers indicated at the right was invented.

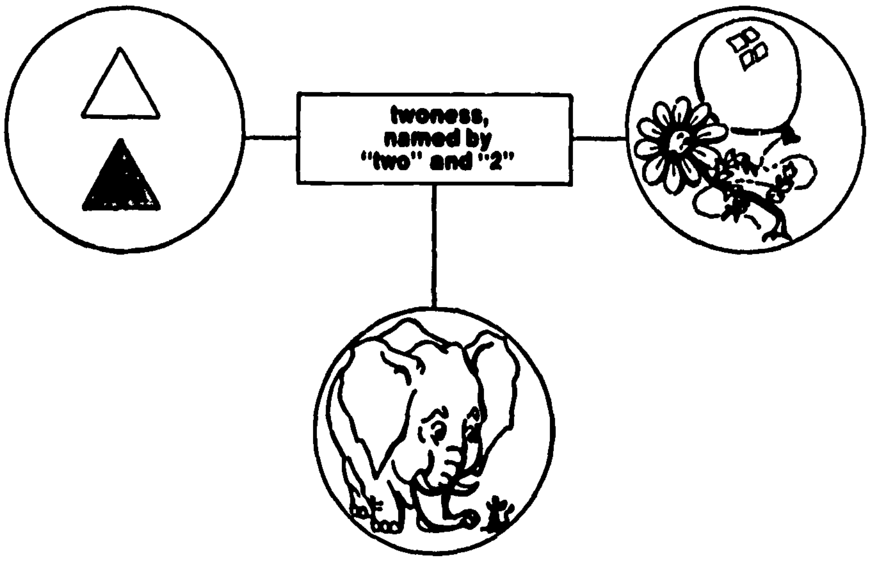
Equation	*Set of Numbers Invented*
$n + 9 = 5.$	Integers
$3n = 2.$	Rationals
$n \cdot n = 2.$	Real numbers
$n \cdot n = -4.$	Complex numbers

The invention of these sets of numbers and others has made possible the solution of some very important mathematical and practical problems. The sets of numbers named above are the ones usually encountered in elementary or secondary schools.

In this book the nonnegative and negative rational numbers, along with the integers, will be discussed. During the planning of the book the authors recognized that two strategies were available to them in developing the important ideas about the rational-number system. The first strategy was to extend the set of whole numbers to include the integers and then extend the set of integers to include the rational numbers. The second strategy was to extend the set of whole numbers to include the nonnegative rational numbers and then extend this set to include the negative rational numbers. Each of the strategies has some advantages to recommend it. However, the second strategy was chosen because it most nearly follows the development usually presented to elementary school pupils. Since the early chapters of this book focus on the nonnegative rational numbers and that is an awkward phrase to continue to write and to read, we use the term *rational number* to mean *nonnegative rational number* until the complete set of rational numbers is treated in a later chapter.

PHYSICAL MODELS FOR RATIONAL NUMBERS

When young children are first introduced to the ideas of rational numbers in school, they have usually had experience with the whole numbers and some of the numerals used as names for these numbers. For example, their past experience helps them understand that the sets shown in the diagram here, and all other sets equivalent to them, have only one property in common—namely, the number idea of twoness—and that the word "two" and the numeral "2" are names for the number two. It is also likely that children recognize that the number two can also be named by "1 + 1," "2 + 0," "3 − 1," "II," "2 × 1," "4 ÷ 2," and so forth. Some children may even recognize that the set of numerals for the number two is an infinite set. As teachers we can use these previous experiences with whole numbers as a foundation for developing the meaning of rational numbers and fractions with children in a way that is consistent with the pattern used for developing the meaning of whole numbers and their names.



Physical models can be drawn or constructed that are useful in developing some of the important ideas about rational numbers and a method for naming them.

Plane-Region Models

If we think of the square region shown here as a one-unit region, we can associate a number with the amount that is shaded. Notice that what

we know about whole numbers can be used to describe the physical situation. The unit is separated into two congruent regions, and *one* of the *two* congruent regions is shaded. The ordered number pair (1,2) can be used to describe the situation. A numeral for the rational number associated with the amount that is shaded includes the names of the numbers in the ordered number pair (1,2) and is written as $\frac{1}{2}$, read "one-half." The fraction $\frac{1}{2}$ is a reminder that 1 of 2 congruent parts is shaded. We say that $\frac{1}{2}$ of the unit region is shaded. To answer the question "How much of the unit region is shaded?" we reply, "One-half." It is interesting to note that a whole number cannot be used to answer this question.

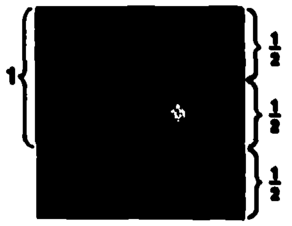
It is important to notice that at the outset a basic unit or unit region was established. Also, the example above illustrates three other important ideas concerning rational numbers. These ideas are listed below:

1. The rational number one-half is associated with the physical situation—the amount of the unit region that is shaded.
2. The number pair (1,2) may be used to indicate that 1 of the 2 congruent regions in the unit region is shaded.
3. The fraction $\frac{1}{2}$ is a name for the number one-half and is a reminder that 1 of 2 congruent regions is shaded.

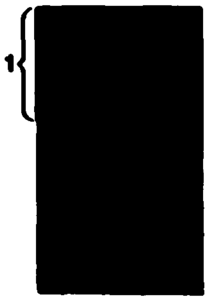
The one-unit rectangular region shown next is separated into four congruent regions, with three of the four shaded. The number associated with the amount that is shaded is three-fourths. The ordered number pair (3,4) indicates that 3 of the 4 congruent regions are shaded. The fraction $\frac{3}{4}$ is a name for the rational number three-fourths and is a reminder that 3 of the 4 congruent regions in the unit are shaded.

By again referring to this physical model we can see that the second number of the ordered pair (3,4), the denominator, designates the number of congruent regions into which the unit region is divided and the first number of the ordered pair, the numerator, indicates the number of congruent regions that are shaded. The ordered number pair (3,4) can be called the numerator-denominator pair.

A plane-region model can also be constructed for rational numbers such as $\frac{3}{2}$. Think of the one-unit region shown by the heavy black lines in the next figure. The unit region is separated into two congruent regions, and three such regions are shaded. The rational number $\frac{3}{2}$ is associated with the amount that is shaded. The numerator-denominator pair (3,2) indicates that the unit region is divided into two congruent regions and three such regions are shaded. The fraction $\frac{3}{2}$ names the rational number three-halves.



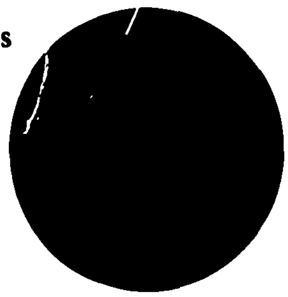
Study the plane-region models shown below and verify the correctness for each model of (1) the rational number, (2) the numerator-denominator pair, and (3) the fraction for the rational number. The unit regions are shown by the heavy black lines.



five-halves

(5,2)

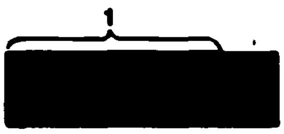
$\frac{5}{2}$



four-fourths

(4,4)

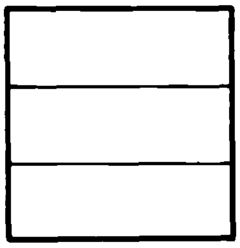
$\frac{4}{4}$



five-fourths

(5,4)

$\frac{5}{4}$

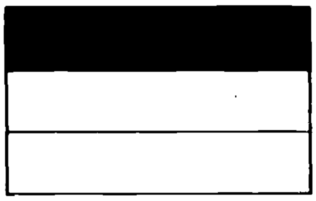


zero-thirds

(0,3)

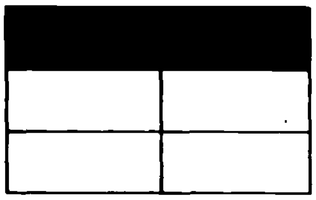
$\frac{0}{3}$

Every rational number can be named by each fraction in an infinite set of fractions. In the plane-region models shown below, each unit region is congruent to each of the other unit regions and the same amount is shaded. Notice that a different numerator-denominator pair describes each model and a different fraction names the same rational number associated with the shaded amount that is common to all the models.



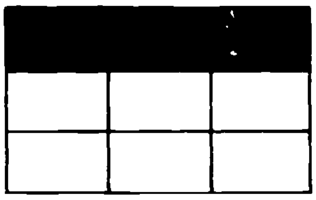
numerator-denominator pair (1,3)

fraction $\frac{1}{3}$



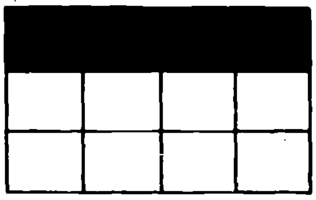
numerator-denominator pair (2,6)

fraction $\frac{2}{6}$



numerator-denominator pair (3,9)

fraction $\frac{3}{9}$



numerator-denominator pair (4,12)

fraction $\frac{4}{12}$

From these models it can be seen that the numerator-denominator pairs (1,3), (2,6), (3,9), and (4,12) all describe a model with the same amount shaded. Also, the fractions $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}$, and $\frac{4}{12}$ are all names for the number associated with the same shaded amount and therefore are equivalent fractions. To assert that $\frac{1}{3}$ and $\frac{3}{9}$ are equivalent fractions we write $\frac{1}{3} = \frac{3}{9}$. This assertion means only that the fractions $\frac{1}{3}$ and $\frac{3}{9}$ name the same rational number; it does not mean that the fractions are the same or identical.

If we use our imagination, additional models like these can be constructed in our minds to show that the rational number associated with the shaded amount has each numerator-denominator pair in the infinite set

$$\{(1,3), (2,6), (3,9), (4,12), \ldots\}.$$

Furthermore, each fraction in the infinite set of fractions

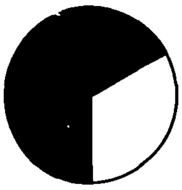
$$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots\right\}$$

is a name for the rational number one-third. This set of fractions is called an equivalence class of fractions for the rational number one-third. The set of all fractions that name the same rational number is an equivalence class of fractions for that rational number.

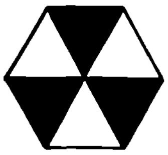
Exercise Set 1

1. Shown below are some models for rational numbers. For each unit region name the rational number, the numerator-denominator pair that describes the physical situation shown in the model, and the fraction that names the rational number. Assume that each figure represents one unit.

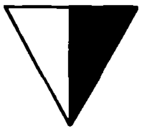
a.



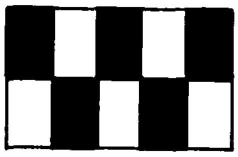
b.



c.



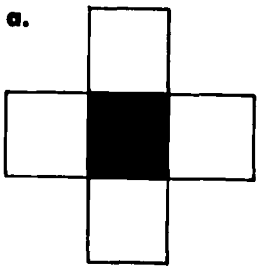
d.



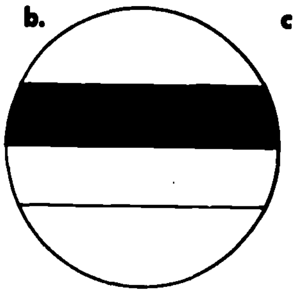
2. Draw a plane-unit-region model for each of the rational numbers $\frac{4}{5}$, $\frac{7}{8}$, and $\frac{4}{15}$.

3. Choose from the pictures below those that are good models for a rational number. Then tell why the others are not good models.

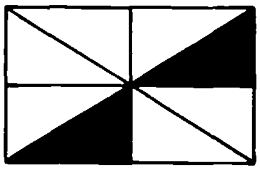
a.



b.

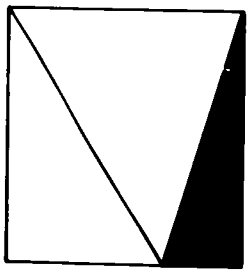


c.

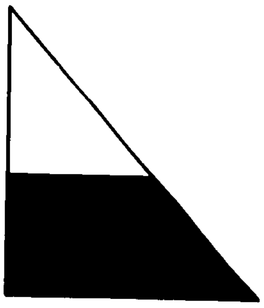


d.

e.



f.



4. Draw a plane–unit-region model for each of the rational numbers $\frac{4}{3}$, $\frac{6}{5}$, $\frac{6}{6}$, and $\frac{0}{4}$.

5. Draw plane–unit-region models to show that the rational number $\frac{3}{5}$ has the numerator-denominator pairs (3,5), (6,10), and (12,20).

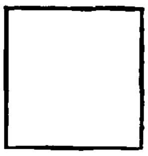
6. Draw plane–unit-region models to show that the fractions of each pair are equivalent.

a. $\frac{3}{4}, \frac{6}{8}$ **b.** $\frac{6}{10}, \frac{3}{5}$ **c.** $\frac{0}{2}, \frac{0}{3}$ **d.** $\frac{5}{3}, \frac{10}{6}$ **e.** $\frac{5}{5}, \frac{4}{4}$

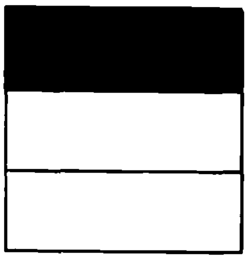
7. In drawing the models for exercise **6**, why is it necessary that the same unit region or two congruent unit regions be used to show that the fractions of each pair are equivalent?

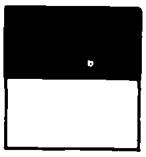
8. Draw a set of plane-unit-region models to show that $\frac{1}{4}$, $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, and $\frac{5}{20}$ are members of the same equivalence class of fractions.

9. If the plane region shown below is $\frac{1}{5}$ of the unit region, how can the unit region be constructed? If it is $\frac{3}{5}$ of the unit region?



10. During a class discussion, a pupil put the following diagrams on the board in an effort to show that $\frac{1}{3}$ is greater than $\frac{1}{2}$. After completing the diagram, he wrote $\frac{1}{3} > \frac{1}{2}$. Most of the other pupils did not agree (and correctly so). What is wrong with the pupil's argument that $\frac{1}{3} > \frac{1}{2}$?

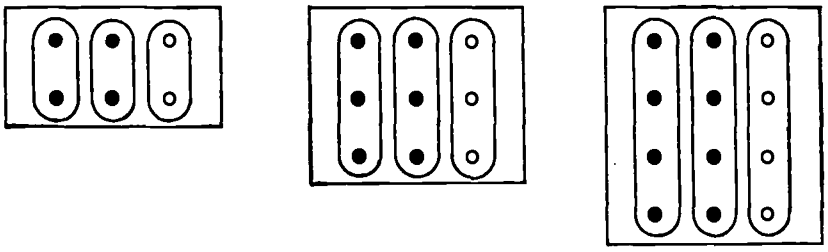




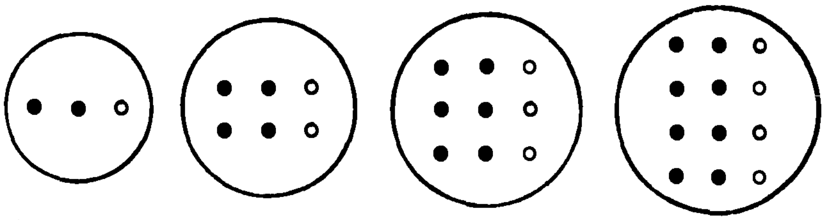
Sets as Models

Sets of objects can also be used as physical models for rational numbers. In the set of dots shown below the numerator-denominator pair (2, 3) can be used to indicate that 2 out of 3 of the dots are black, and the fraction $\frac{2}{3}$ compares the number of black dots to the total number of dots in the set. To express this comparison we say that $\frac{2}{3}$ of the dots are black.

Each of the sets in the next illustration has been separated into three equivalent subsets. The dots in two out of the three subsets in each set are black. Two-thirds compares the total number of black dots to the total number of dots. Now we can think about the infinite number of sets that might be constructed in the same pattern except that with each construction the number of dots in the equivalent subsets would be increased.



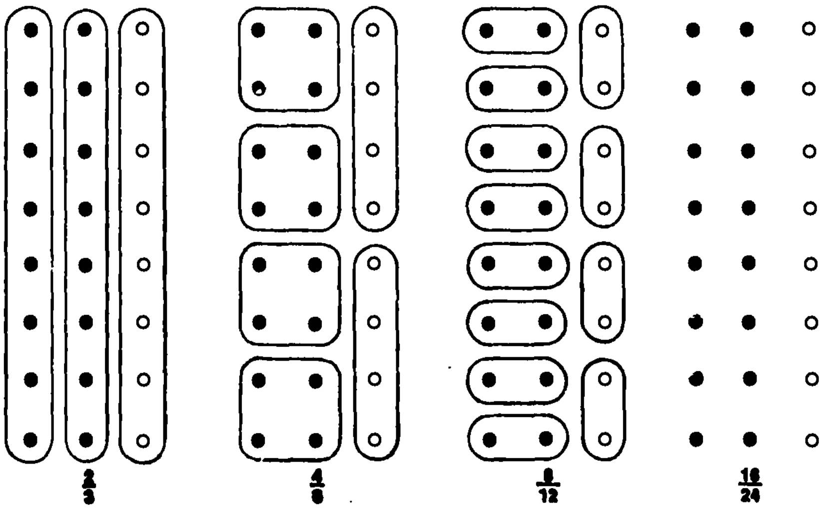
The following models for the rational number two-thirds are similar, except that the dots are not grouped in subsets. The arrangement of the dots within each set makes clear that the numerator-denominator pairs (2,3), (4,6), (6,9), and (8,12) can be used to describe the models for the



rational number two-thirds. Also, the fractions $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{8}{12}$ are all names for the rational number two-thirds. If one could continue without end to construct models for the number two-thirds in the same pattern and write the fraction for each model, he would have the equivalence class of fractions for the number two-thirds, which is

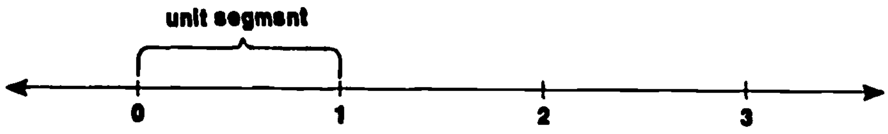
$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\right\}.$$

We now illustrate how the dots in a set of twenty-four can be regrouped into equivalent sets to show that $\frac{2}{3}$, $\frac{4}{6}$, $\frac{8}{12}$, and $\frac{16}{24}$ are equivalent fractions. The fractions $\frac{6}{9}$, $\frac{10}{15}$, and $\frac{12}{18}$ are also equivalent to $\frac{2}{3}$. However, since the denominators 9, 15, and 18 are not factors of 24, the equivalence cannot be shown by using a set containing twenty-four dots.

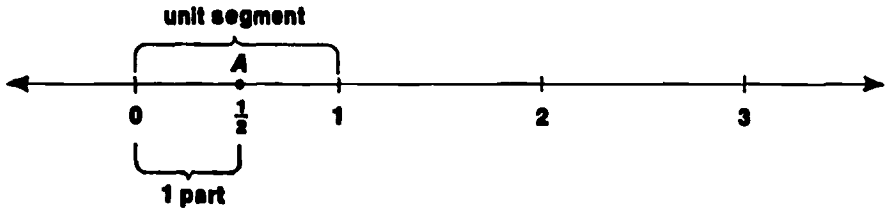


Number-Line Models

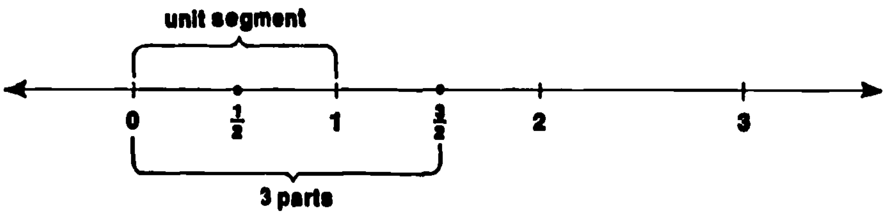
The last model for rational numbers to be considered in this chapter is that of the number line. Many children in the early elementary grades have had experience with the number line in their work with operations on whole numbers. Consequently, these children will recognize a number line such as the one shown below, where the whole numbers are placed in correspondence with points on a line. Furthermore, they will recognize that to construct a number line such as this, a unit segment is marked off and the numbers 0 and 1 are assigned to the endpoints of the unit segment. Then, beginning with the point corresponding to 1, segments congruent to the unit segment are marked off to the right. The whole numbers are then consecutively placed in correspondence with the endpoints of the segments. The whole number that corresponds to any point is the length of the segment beginning at the point for 0 and ending at the point that corresponds to that number. For example, 2 is the length of the segment beginning at 0 and ending at the point that corresponds to 2.



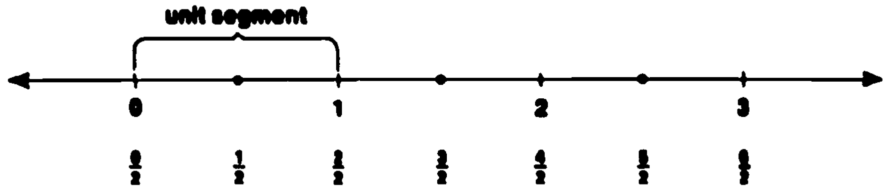
Children whose background includes experiences such as those just described are ready to use a number line as a model for rational numbers. In the next figure, a number line was first constructed as described above and then the unit segment was separated into two congruent segments by point *A*. The number associated with point *A* is one-half, and the length of the segment from 0 to *A* is $\frac{1}{2}$ unit. Notice that the unit is separated into two congruent parts and that one of the two parts is being used. The numerator-denominator pair (1,2) describes this physical situation, and the names of the numbers in the pair are included in the fraction $\frac{1}{2}$, a name for the number one-half.



To locate the point that corresponds to $\frac{3}{2}$, mark off a unit segment, separate it into two congruent segments, and then add another segment congruent to these, as shown below.

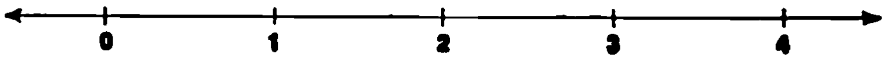


The points that correspond to $\frac{0}{2}$, $\frac{2}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$ can be located in like manner. The point that corresponds to $\frac{0}{2}$ is the same point that corresponds

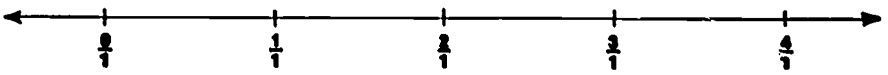


to 0. This seems reasonable because the fraction $\frac{0}{2}$ is a reminder that the unit segment is separated into two congruent segments and 0 of these segments have been counted off. On number line (*a*) shown below the points have been located that correspond to the whole numbers 0, 1, 2, 3, and 4; on number line (*b*) the points have been located that correspond to the rational numbers represented by fractions with denominator 1; on number line (*c*) the points have been located that correspond to the rational numbers represented by fraction. with denominator 2; and on number line (*d*) the points have been located that correspond to the rational numbers represented by fractions with denominator 4.

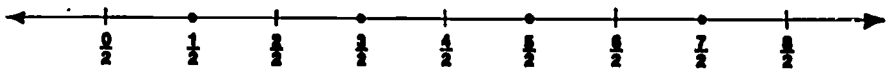
a



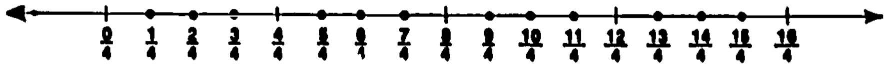
b



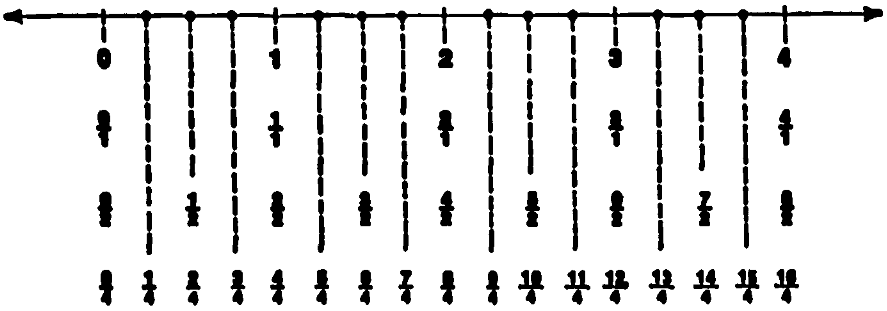
c



d



The next figure shows on a single number line the location of points that correspond to the whole numbers 1, 2, 3, and 4 and rational numbers represented by fractions with denominators of 1, 2, and 4.



Since $\frac{5}{2}$ and $\frac{10}{4}$ correspond to the same point, they are fractions for the same rational number. They are equivalent fractions. Also, $\frac{2}{1}$, $\frac{4}{2}$, and $\frac{8}{4}$ correspond to the same point as the whole number 2. Since the whole number 2 can be named by a fraction, 2 is a rational number. In general, any number that can be named by a fraction $\frac{a}{b}$, where a names a whole number and b names a whole number other than zero, is a rational number. From the number line shown above it can be seen that every whole number can be named by a fraction in which the numerator is a whole number and the denominator is a whole number different from zero.

By referring to the same number line it can be seen that if a rational number $\frac{a}{b}$ corresponds to a point to the right of the point for the rational number $\frac{c}{d}$, then $\frac{a}{b} > \frac{c}{d}$ and $\frac{c}{d} < \frac{a}{b}$. Children can use this idea to determine that of the three statements

$$\frac{3}{4} > \frac{13}{16}, \quad \frac{3}{4} < \frac{13}{16}, \quad \frac{3}{4} = \frac{13}{16},$$

only $\frac{3}{4} < \frac{13}{16}$ is true, because on an appropriate number-line model the point for $\frac{13}{16}$ is to the right of the point for $\frac{3}{4}$. Notice that if two fractions have the same denominator, the one that has the greater numerator corresponds to a point to the right of the point for the other number. That is, $\frac{9}{4}$ and $\frac{13}{4}$ have the same denominator, and the point for $\frac{13}{4}$ is to the right of the point for $\frac{9}{4}$; so $\frac{13}{4} > \frac{9}{4}$. Therefore, of two rational numbers named by fractions that have the same denominator, the one that has the greater numerator names the greater rational number.

It is also important to note that two rational numbers whose fractions have the same numerator can be compared by comparing the denominators. For example, in the figure, the fractions $\frac{3}{4}$ and $\frac{3}{2}$ have the same numerator. Since the denominator 4 is *greater* than the denominator 2, then $\frac{3}{4}$ is *less* than $\frac{3}{2}$.

As we think about the way the number-line model was constructed, challenging questions emerge.

1. For each rational number is there a corresponding point on the number line?
2. For each point to the right of the point for zero is there a corresponding rational number?
3. For each whole number there is a unique next whole number. For each rational number is there a unique next rational number?

Exercise Set 2

1. For each set of objects name a rational number suggested by the set, the numerator-denominator pair that describes the physical situation shown in the model, and the fraction that names the rational number.

a. 

b. 

c. 

2. Draw three different sets of dots so that $\frac{3}{4}$ of the dots in each se are black. Why is it that you can draw three different sets of dots, $\frac{3}{4}$ of which are black, without first knowing how many dots must be in each set?

3. Draw a set of 24 dots so that $\frac{5}{6}$ of them are black.

4. Draw a set of 30 dots with 20 of them black. Show how the 30 dots can be separated into subsets to illustrate that $\frac{2}{3}$ of the dots are black.

5. Draw a set of 36 objects and use it to show that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{6}{12}$, $\frac{9}{18}$, and $\frac{18}{36}$ are all names for the same rational number and therefore members of the equivalence class of fractions for one-half.

6. Draw a number-line model for each of the rational numbers shown below. Use a separate number line for each number.

a. $\frac{4}{5}$

b. $\frac{5}{8}$

c. $\frac{7}{5}$

d. $\frac{6}{6}$

e. $\frac{0}{3}$

f. $\frac{16}{8}$

7. What rational number corresponds to each of the points shown on this number-line diagram?

8. Draw number-line models to show that the fractions in each set are equivalent.

a. $\frac{2}{3}, \frac{4}{6}$ **b.** $\frac{8}{6}, \frac{4}{3}$ **c.** $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{18}{6}$

9. Draw a set of objects to show that the same rational number $\frac{1}{4}$ is represented by fractions with numerator-denominator pairs (1,4), (2,8), (3,12), and (4,16).

10. Use the number line shown on page 15 to help you tell why each statement is true.

a. $\frac{3}{4} > \frac{2}{4}$.

b. $\frac{1}{4} < \frac{1}{2}$.

c. $\frac{11}{4} > \frac{5}{2}$.

d. 1 is a rational number.

e. $\frac{0}{2} = \frac{0}{4} = 0$.

f. $\frac{4}{4} = \frac{2}{2}$.

g. The set of whole numbers is a subset of the set of rational numbers.

h. Some rational numbers are not whole numbers.

STRATEGIES FOR INTRODUCING RATIONAL NUMBERS

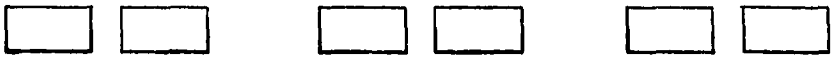
In the first section of this chapter it was stated that here the term *rational numbers* would refer to only the nonnegative rational numbers and that the complete set of rational numbers would be treated in a later chapter. Also, as stated, the desire to substitute the term *rational numbers* for *nonnegative rational numbers* was motivated by convenience. It is awkward to continue to write and to read such sentences as "Three-fourths is the nonnegative rational number associated with the shaded amount." Furthermore, the continued use of such cumbersome language can interfere with the reader's concentration on the main points in a discussion. Authors of current textbooks have recognized that there is no convenient name for the nonnegative rational numbers. In an effort to help both teachers and pupils focus their attention on the important ideas about these numbers, such names as "fractional numbers," "numbers of arithmetic," and "rational numbers" have been used.

Regardless of the name used for this set of numbers throughout a textbook series, the important ideas about them are introduced in essentially the same way as in the preceding sections here. The fact that there is no uniform agreement on the language to be used, different authors assigning different names to this set of numbers, points up the related fact that the ideas about rational numbers are more important for children to grasp than the names. The choice of a name for an idea such as "a nonnegative rational number" is arbitrary; it is the *idea* that must be understood by children. This attitude toward language is important in developing strategies for teaching important ideas about rational numbers.

Showing the Need for Numbers beyond the Whole Numbers

Long before beginning formal study of rational numbers it is important for pupils to engage in many exploratory activities to increase their awareness of the need for numbers beyond the whole numbers. Two examples of such activities are given below.

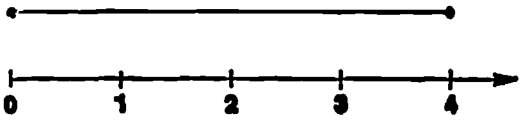
1. Place three candy bars of the same kind and size on a table and break each of them into two pieces of the same size, as shown below.



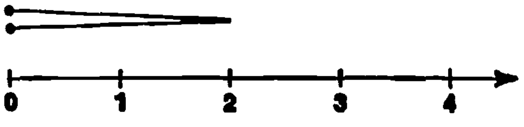
Ask these questions: (1) How many candy bars are on the table? (2) If I pick up one piece of the first bar [actually pick up half of the first bar], how much candy is left on the table? (3) If I pick up all of the second bar, how much is left?

The discussion generated by the pupils' responses to these and similar questions should focus on the need for numbers beyond the whole numbers to answer questions beginning with "How much." It is of secondary importance that during the activity children come up with the correct rational-number answers to the questions.

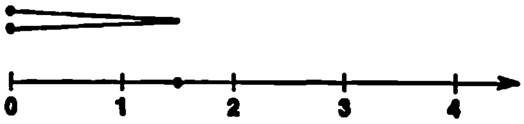
2. If the children have had some previous experience with measurement and the number line in their work with whole numbers, draw a number line on the chalkboard and place a piece of string along it as shown below. Then ask: How many units long is the piece of string? After they count to obtain four units,



bring the ends of the string together to make a double strand and ask: If we make this string into two pieces of the same length, how long will each piece be?



Then use a piece of string three units long and place it along the same number line. Ask: How long is the piece of string? After the children recognize that it is three units long, ask: If we make *this* string into two pieces of the same length, how long will each piece be? Some pupils may recognize that each piece is one and one-half units long. However, most pupils should be helped to discover that it is more than one and less than two units long and that there is no whole number for the length of the string. If we are going to have a number for that length, numbers beyond the whole numbers are needed.



Using Ordered Number Pairs

After reading the section "Physical Models for Rational Numbers," you may have felt that the emphasis on using ordered number pairs to describe the models was greater than necessary. This strategy was chosen deliberately. The ordered number pairs (numerator-denominator pairs) make a strong connection between the idea of a rational number and the fraction names for the number and direct the attention of pupils to the idea that a rational number involves a relation between two whole numbers. Furthermore, for pupils who continue the study of mathematics, recurring experiences with the use of ordered number pairs can serve as a solid building block for thinking of a rational number as an equivalence class of ordered pairs.

Sometime prior to using a textbook for a formal introduction of rational numbers we can help pupils become accustomed to using pairs of whole numbers. As we talk with pupils about some of the occurrences in the classroom or on the playground we can use ordered-number-pair language and encourage pupils to do the same. Some examples are listed below:

(a) 2 out of the 5 people who usually sit at this table are absent.
(b) 7 out of the 10 pencils are red.
(c) 2 out of the 7 cars are black.
(d) 1 out of 2 of Tom's tosses scored a point.
(e) 5 out of the 5 people at this table are boys.
(f) 17 out of the 20 plants grew.

Most textbook series use the same physical models for rational numbers as those used in this chapter. However, before a textbook is employed to introduce rational numbers, children can be given some experience in using number pairs to describe several situations of the types shown below. To focus attention on the properties of the models, some specific questions are suggested beneath each model.

The region is separated into four smaller regions of the same size. How many are shaded?

What does the number pair (3,4) tell about the region? What does (1,4) tell about the region?

Into how many segments of the same length is $\overline{AB}$ separated?

How many are shown in the shaded portion of the number line?

What number pair can be used to tell about the segments shown in the shaded portion? In the unshaded portion?

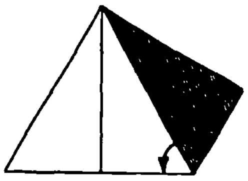
● ● ○ ○ ○

How many dots in the set?

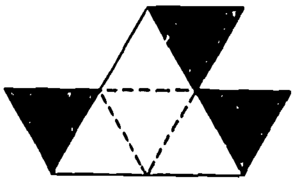
How many dots are black?

What does the number pair (2,5) tell about the set? The number pair (3,5)?

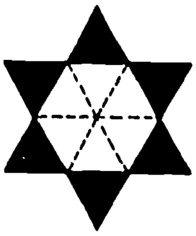
At the risk of seeming to dwell too long on the idea of using ordered pairs of numbers to describe physical models, we present one other technique. As shown in the following illustration, models cut from poster board have hinged flaps that can be manipulated to depict various situations to be described by ordered pairs. The flaps are shaded. Each white region represents one unit. The figure at the left has two congruent unit regions shown in white; the one in the middle, four; and the one at right, six.



(1, 2)



(1, 4)
(2, 4)
(3, 4)

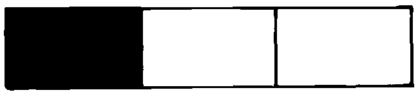


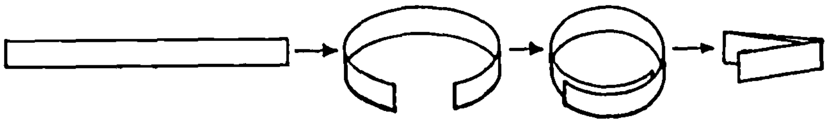
(1, 6) (4, 6)
(2, 6) (5, 6)
(3, 6) (6, 6)

Children's experiences using ordered number pairs to describe physical situations provide a good base on which to build an understanding of the idea of rational numbers.

Using Physical Models to Supplement the Textbook

Successful teachers realize that children gain by drawing or constructing for themselves some of the models shown on pages of their textbook. Suppose a picture of a model for the rational number one-third appears in the text as shown here. Children can use strips of adding-machine tape to make their own models to use for the discussion suggested by the text.



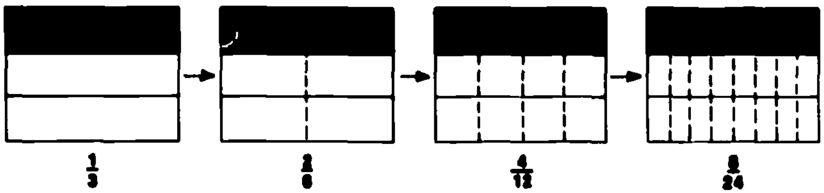


Providing pupils with an opportunity to construct their own models for rational numbers to be used along with the discussion in the textbook can increase the level of individual involvement in classroom activities. Pupils involved in a variety of well-thought-out activities are more likely to learn than those whose only individual activity is using a textbook. However, when pupils are using plane-region models, whether constructed models or pictures in a textbook, the following questions should be under continual consideration.

(a) What is the unit region?
(b) Into how many congruent regions is the unit separated?
(c) How many of the congruent regions are shaded?
(d) What number tells how much is shaded?
(e) What numerator-denominator pair describes the model?
(f) What is a fraction for the number?

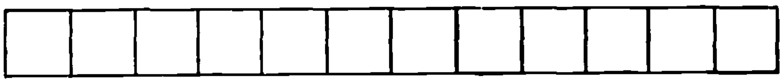
Although the questions listed above are suggested for use with plane-region models, a similar set of questions can be generated to serve the same purpose for use with sets of objects or a number line. Questions (a), (b), and (c) can be rephrased to stress that when a set of objects is the model, the unit is the set, and when the number line is used, the unit is a segment.

Paper folding can also be used to construct models to illustrate that a rational number can be named by many fractions. For example, suppose that pupils have constructed a plane-region model for the number one-third as shown in the first of the illustrations below. Then a series of consecutive folds is made to obtain the other three models. Dotted lines indicate the creases made by the folds.

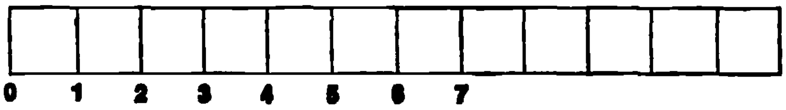


In using models such as those shown here to illustrate the meaning of equivalent fractions, questions should be used that emphasize that the shaded amount remains constant and that with each fold the number of congruent regions in the unit region is doubled, or multiplied by two. Sample questions: After the second fold, how many times as many congruent regions are in the unit region? How many times as many congruent regions are shaded? Why does folding seem to multiply by the same number both the number of congruent regions in the unit and the number of shaded congruent regions?

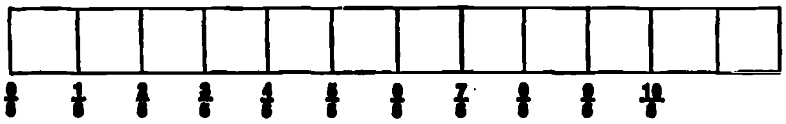
As pupils move through mathematics programs their use of the number-line model for rational numbers will continue to increase. So that a number line can always be available, some teachers have used the cork center of the tack strip over the chalkboard to tack small congruent strips of construction paper as shown below.



If each strip is considered a unit segment, then chalk numerals can be placed along the line as shown.



However, if a number line is needed with the unit segment separated into five congruent segments so that fifths can be counted along the number line, then five of the strips taken together can be considered the unit.



SUMMARY

Rational numbers were invented to answer questions that often could not be answered with whole numbers—questions beginning with such phrases as *how much*, *how long*, *how heavy*, *how far*, and so forth.

The idea of a rational number is developed by using physical models—plane regions, sets of objects, and number lines. These models provide the vehicle for developing the following ideas about rational numbers.

1. When thinking of a rational number, a unit is always specified or implied. The unit may be a segment, a set, or a plane region.

2. The physical model for a rational number can be described by an ordered pair of whole numbers. This ordered pair of numbers can be called a numerator-denominator pair.

3. A rational number can be named by a fraction that includes the names of the numbers in an ordered pair that describes a model.

4. Two fractions that name the same rational number are equivalent fractions.

5. There is an infinite set of ordered number pairs, just as there is an infinite set of equivalent fractions, for each rational number. The infinite set of equivalent fractions for a rational number is called an equivalence class.

Many of the difficulties encountered by children in working with rational numbers can be traced to a lack of good experiences with physical models. Experiences with physical models will be frequently recalled by children in their efforts to think through the algorithms for computing with fractions. The time spent on a careful development of the meaning of rational numbers and methods for naming them by using physical models will be time well spent for both teacher and pupil.

Leon A. Henkin

FRACTIONS AND RATIONAL NUMBERS

2

1. What is the difference between a fraction and a rational number?
2. What do we mean by equivalent fractions?
3. Of what use are equivalent fractions?
4. What are the difficulties in teaching the concept of equivalent fractions and how do we motivate students to overcome these difficulties?

As we have seen, *rational numbers* are numbers that are needed to describe certain situations and solve certain problems where the whole numbers by themselves are inadequate. The whole numbers are among the rational numbers, but there are many rational numbers that are not whole numbers.

Fractions are special symbols that are used as *names* for rational numbers. Each fraction is composed of the names of two whole numbers separated by a horizontal bar. The numeral above the bar denotes a whole number called the *numerator* of the fraction; the numeral below the bar denotes a whole number called the *denominator*. An important task of the teacher is to bring the students to a full understanding of the relation between the rational number named by a given fraction, and the numerator and denominator named by the upper and lower parts of the fraction.

Each fraction names exactly one rational number—no more and no less. However, every rational number—whether it is a whole number or not—has *many different* fractions among its names. Whenever two fractions name the same rational number they are called *equivalent fractions*. For example, "$\frac{4}{6}$" and "$\frac{10}{15}$" are equivalent fractions.

In developing the decimal system for naming the *whole* numbers, we provide each whole number with *just one* decimal numeral as its name.

Why, then, do we develop the system of fractions for naming *rational* numbers in such a way that each rational number gets *many different* fractions as its names? If the student gains a clear understanding of the answer to this question, he will be able to overcome the principal difficulties ordinarily encountered in passing from the study of whole numbers to the study of rational numbers.

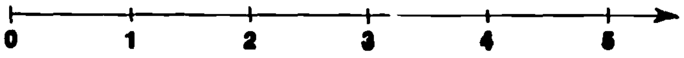
Actually, the question raised in the preceding paragraph has two different answers, and it is important for the student to understand each of them.

First, each rational number arises as a solution to problems of a certain kind. The formulation of any such problem, as we shall see below, involves two whole numbers. It is natural, therefore, to describe the rational number that *solves* the problem by means of a name (the fraction) made up from the names of these two whole numbers. However, it can easily happen that the same rational number solves two different problems of such a kind; hence this rational number will be described by the two different fractions corresponding to these two problems.

The second reason we have devised the system of fractional numerals so that it provides many different names for the same rational number is connected with the algorithms for *computing* sums, differences, products, and quotients of rational numbers. We shall see that these computations are greatly facilitated by an ability to pass back and forth among several different fractional names for the same rational number.

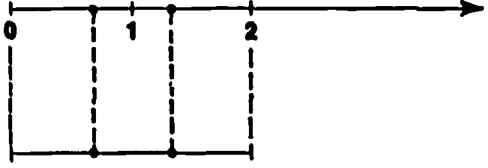
PROBLEMS GIVING RISE TO RATIONAL NUMBERS

Perhaps the simplest and most vivid way to see the need for other numbers besides the whole numbers is to work with the number line. The whole numbers serve as "addresses" for *certain* points on the line, but one glance shows that *most* points on the line lie between these.

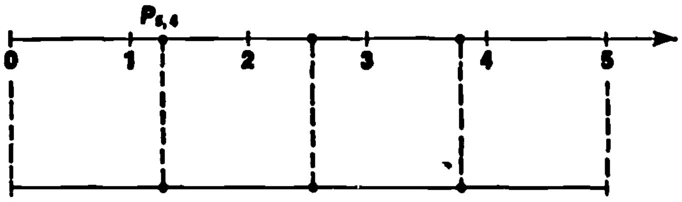


We want new numbers to serve as the addresses for points lying between points having whole-number addresses. Actually, many of these in-between points can be located with reference to two whole-number points.

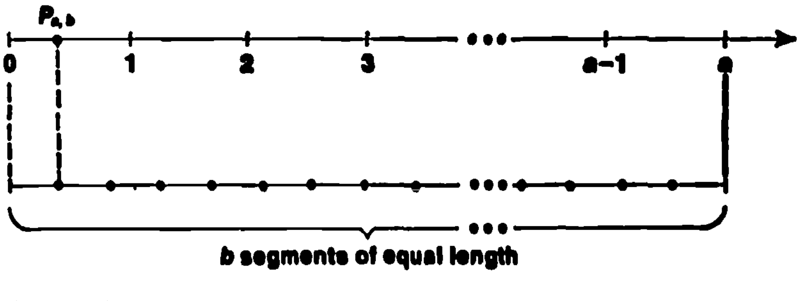
For example, consider the whole numbers 2 and 3. Let us take the segment of the number line from the point for 0 to the point for 2 and divide it, somehow, into 3 parts of equal length. We must insert two points of division into the segment, and we shall call the first of these the point $P_{2,3}$ (read aloud as "P two three").



Similarly, given the whole numbers 5 and 4, we locate the point $P_{5,4}$ on the number line by dividing the segment from the point for 0 to the point for 5 into 4 parts of equal length, and labeling the first of these division points $P_{5,4}$ (read aloud as "*P* five four").

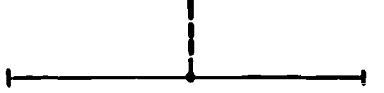


In general, if a and b are any whole numbers, the point $P_{a,b}$ is obtained by dividing the segment from point 0 to point a into b equal parts and taking $P_{a,b}$ to be the first division point.



Observe that we cannot divide a segment into "0 parts of equal length," so the notations $P_{2,0}$, $P_{5,0}$, and in general $P_{a,0}$ for any whole number a are undefined. In other words, we agree to use the notation $P_{a,b}$ only in case a and b are whole numbers with $b \neq 0$.

Another thing to observe is that in general the point $P_{a,b}$ is not the same as the point $P_{b,a}$. For example, the point $P_{3,2}$ is obtained by dividing the

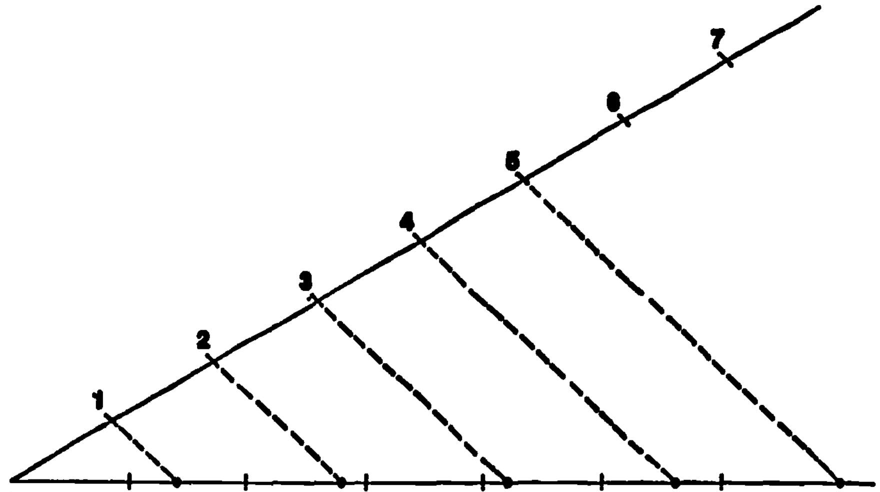


segment from point 0 to point 3 into 2 parts of equal length. Notice that $P_{3,2}$ is to the *right* of point 1, while $P_{2,3}$ is to the *left* of point 1 (as we see from the first part of the diagram). Hence $P_{3,2} \neq P_{2,3}$.

In the above discussion we have spoken of dividing a segment into b parts of equal length without looking into the question of how we can actually carry this out. To search for a practical way to divide a given line segment into b parts of equal length can be a valuable experience for elementary school students. Of course the easiest case is when $b = 2$. Someone is sure to have the idea of using a piece of string the same length as the given segment, then finding the midpoint by holding the endpoints together and finding the point at which the string folds over. After that the case $b = 4$, and then $b = 8$, should be tried, so that the students learn that it sometimes pays to repeat a good idea!

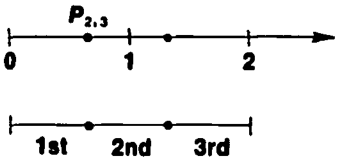
But the cases $b = 3$, 5, or 6 will not be so simple. Most students will find themselves led to experiment with various methods of approximating points such as $P_{1,3}$ and then of improving the approximation. This is, of course, an excellent form of mathematical activity—especially if the teacher urges the desirability of seeking a *systematic* way to improve any approximation.

By using Euclidean geometry one can find the point $P_{a,b}$ for any value of b ($b \neq 0$, of course) by a uniform method involving two number lines placed at an angle to one another. For example, $P_{7,5}$ may be found on the lower line by constructing a line connecting the point 5 on the upper line with the point 7 on the lower line. It will then be found that the lines parallel to this connecting line that pass through the points 1, 2, 3, and 4 on the upper line will divide the segment from 0 to 7 on the lower line into



5 parts of equal length. While elementary school children cannot be expected to follow a proof of this fact, the teacher can lead them by experiment to appreciate that a sequence of equally spaced parallel lines produces segments of the same length on *any* line that lies across them.

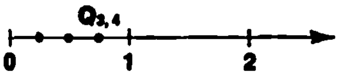
By looking back at the definition of $P_{2,3}$ we see that it can be described as the point on the number line such that if the segment from point 0 to $P_{2,3}$ is laid off 3 times (starting from 0), we arrive at the point 2. Similarly,



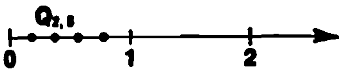
for any whole numbers a, b with $b \neq 0$, the point $P_{a,b}$ is such that if the segment from point 0 to $P_{a,b}$ is laid off b times, we arrive at the point a.

Now we want to describe a quite different way of arriving at a point on the number line, being given two whole numbers a, b such that $b \neq 0$. The new *point* thus constructed will be called $Q_{a,b}$ to distinguish it from $P_{a,b}$, described above.

Let us first illustrate the new process by an example. To construct the point $Q_{3,4}$ we divide the unit interval (i.e., the segment from point 0 to point 1) into 4 parts of equal length, we count off 3 of the little intervals produced by this division process (starting from point 0), and we take $Q_{3,4}$ to be the right endpoint of the last of the little intervals thus counted off.



In general, to obtain $Q_{a,b}$ we divide the unit interval into b parts of equal length, we count off a of the little intervals produced by this division process, and we take the right endpoint of the last of the little intervals thus counted off.

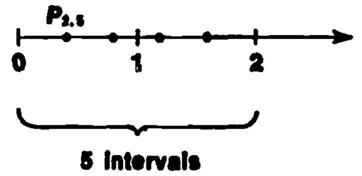


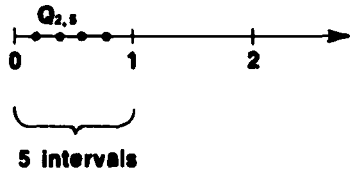
Although the description of the point $Q_{a,b}$ resembles that of $P_{a,b}$ in certain respects, the methods of finding these points are obviously different in various details. Children will probably be surprised, therefore, if they compare two points such as $P_{2,3}$ and $Q_{2,3}$, to find that these are at the same place on the number line. A few more such tries will probably convince them that $P_{a,b}$ and $Q_{a,b}$ are *always* the same. But why?

If children can be led to experiment and to seek an explanation of why $P_{a,b} = Q_{a,b}$ in general, they will gain a valuable mathematical experience.

Of course for the case where $a = 1$ it is easy enough to see that the definition of $Q_{1,b}$ reduces to that of $P_{1,b}$, for in both cases we must divide the interval from point 0 to point 1 into b parts of equal length, and then take the right endpoint of the first of the little intervals thus produced. The next case to consider is the case $a = 2$.

In order to understand why $P_{2,5} = Q_{2,5}$, for example, we have to consider the following ways of subdividing intervals. It is pretty clear that



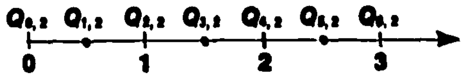


in order to understand the connection between the two processes it will be desirable to divide the interval from 1 to 2 into 5 segments of equal length in the second diagram, even though this is not needed for the construction of either $P_{2,5}$ or $Q_{2,5}$. It will then become apparent that *each* of the five intervals into which the interval from 0 to 2 is divided in finding $P_{2,5}$ (first diagram) coincides with *two* of the intervals of the length used in finding $Q_{2,5}$ (second diagram). Appreciation of this fact will provide a useful background of experience for later introduction of the concepts of ratio of lengths and scale factors in drawings, maps, or geometric figures.

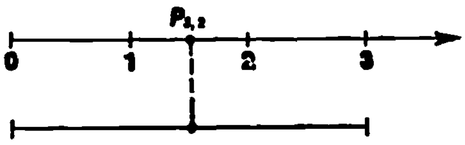
Traditionally, it will be recognized, the description of the point $Q_{2,5}$ is the one that is most often used to define the number $\frac{2}{5}$ at the elementary school level. That is, we take $\frac{2}{5}$ to be the address of the point we have called $Q_{2,5}$. Since we have seen that actually $P_{2,5}$ and $Q_{2,5}$ are the same point, it is clear that we can equally well use the description of the point $P_{2,5}$ as a way of introducing the fraction $\frac{2}{5}$ to school children. Mathematically, the two descriptions are equivalent. From the pedagogical viewpoint, however, each method has advantages and disadvantages when compared with the other. It will be instructive to look at some of these.

In describing the point $Q_{a,b}$ the instructions were to divide the interval from 0 to 1 into b parts of equal length and then, by counting from the left, to locate the division point numbered a. Obviously, however, this is possible only in case we have $a < b$. For example, $Q_{5,7}$ makes sense but $Q_{7,5}$ does not. On the other hand, the description of $P_{a,b}$ is equally applicable whether $a < b$, $a = b$, or $a > b$. For example, we have considered both of the points $P_{2,3}$ and $P_{3,2}$. From this point of view, the use of $P_{a,b}$ would appear to be advantageous relative to the use of $Q_{a,b}$, for it is certainly desirable for students to realize that $\frac{7}{5}$ and $\frac{5}{7}$ are equally good rational numbers, susceptible of definition by a common method.

Of course the definition of $Q_{a,b}$ can easily be modified to deal with $a = b$ or $a > b$. We simply provide that, instead of dividing just the interval from 0 to 1 into b parts of equal length, we divide *each* of the intervals from 0 to 1, from 1 to 2, from 2 to 3, and so forth, into b parts of equal length. Then we can count off these smaller intervals, starting from the point 0, and we can arrive at the one whose number is a, whether $a < b$, $a = b$, or $a > b$; the right endpoint of the last small interval thus counted is our point $Q_{a,b}$.



Each original interval divided into 2 parts of equal length; 3 small intervals counted off to get $Q_{3,2}$



The whole interval from point 0 to point 3 divided into 2 parts of equal length

In particular we note that $P_{3,1}$ and $Q_{3,1}$ are the same point, corresponding to the point for 3. It follows that $\frac{3}{1} = 3$. In general, for every whole number a, we have

$$\frac{a}{1} = a.$$

Another special case occurs when $a = b$ and neither is 0. Thus, $P_{3,3}$ and $Q_{3,3}$ are the same point, corresponding to the point for 1. It follows that $\frac{3}{3} = 1$. In general, for every whole number a that is not 0, we have

$$\frac{a}{a} = 1.$$

What can we say for the point $P_{0,6} = Q_{0,6}$? It isn't too hard to see that it makes sense to take this point for the 0 point. The interval from 0 to 0 divided into b equal parts will simply give the point for 0. It follows then that for every $b \neq 0$,

$$\frac{0}{b} = 0.$$

We could now ask, "Under what circumstances is

$$\frac{a}{b} = 0?"$$

In other words, when will $P_{a,b}$ be the point for 0? The only interval from the 0 point to the point for a that gives 0 as the right endpoint for the first interval, when divided into b equal parts, is the interval when $a = 0$. It follows then that if $\frac{a}{b} = 0$ we must have $a = 0$.

When the original description of $Q_{a,b}$ is modified in the manner just described, it has the advantage of suggesting that the original number line is simply being refined by using a smaller unit of length. Obviously, this enables the teacher to bring in the whole idea of measurement and the need for continually using smaller units of length in order to obtain more accurate estimates. Since this is one of the important uses of rational numbers, we may consider the use of $Q_{a,b}$ as having an advantage over using $P_{a,b}$ as a means of introducing the fraction $\frac{a}{b}$.

In another direction we can see an advantage to using the description of the point $P_{a,b}$, rather than that of $Q_{a,b}$, to introduce the rational number $\frac{a}{b}$ to an elementary school pupil. This has to do with the arithmetical relation connecting the rational number $\frac{a}{b}$ with the whole numbers a and b. This relation is simply the fact that

$$b \cdot \frac{a}{b} = a.$$

From the definition of $P_{a,b}$ it is immediately clear that when the segment from 0 to $P_{a,b}$ is added to itself b times we arrive at the point a. Hence if we define $\frac{a}{b}$ as the number representing the length of the segment from 0 to $P_{a,b}$, we shall have the desired equation $b \cdot \frac{a}{b} = a$. The corresponding fact is not as clear if we take the more traditional definition of $\frac{a}{b}$ as the address of the point $Q_{a,b}$.

The equation $b \cdot \frac{a}{b} = a$ is so basic to the significance of the rational number $\frac{a}{b}$ in all of its applications that some teachers prefer simply to define $\frac{a}{b}$ as the missing factor in the equation $b \cdot \square = a$. While this has some advantages, it sometimes leaves a pupil in doubt about whether there *is* such a number! Definitions on the number line make existence intuitively evident.

These few comparisons among several possible ways of introducing rational numbers suggest that there is no one method that can be considered superior in all ways to all other methods. Some teachers may prefer to use one method all the time; others prefer to try different approaches with different classes. Ways to develop these and other approaches in the classroom are described in chapter 3.

If time permits, a teacher might even consider mentioning more than one approach to the same class, for certainly the deepest understanding of a mathematical concept is brought about by considering it from different viewpoints. As we have indicated above, the student who discovers that the points $P_{a,b}$ and $Q_{a,b}$ are always the same, and who is led to wonder why, will find much that will help deepen his understanding of rational numbers and their uses.

Exercise Set 1

1. One basic application of rational numbers is their use to measure portions of a given pie, say, or to indicate the relative size of some subset of a given set. (Indeed, rational numbers are sometimes defined for pupils in this way.) From the point of view of relating to these applications, which is better: a definition of $\frac{a}{b}$ by means of the description of $P_{a,b}$ or of $Q_{a,b}$? Why?

2. For certain whole numbers a and b the rational number $\frac{a}{b}$ may turn out to be a whole number. From the point of view of explaining this phenomenon to pupils, which seems better: a definition of $\frac{a}{b}$ in terms of the point $P_{a,b}$ or of $Q_{a,b}$? Why?

3. Suppose we have a paper rectangle that we wish to divide into three vertical strips of equal width without using any instruments. We estimate (guess) where a line should be constructed if it is to be about one-third of the way from the right edge to the left, we fold the right edge over and make a crease at the estimated place, and then we open out the paper again. We label the crease *m*. Now we divide the space to the left of *m* in half by folding the left edge over so it lies just along *m* and making another crease. We open the paper again and label the new crease *q*.

[redacted]

[redacted]

a. How can we tell now, without measuring, whether the area to the right of *m* is *exactly* one-third of the area of the rectangle or a little less than one-third or a little more than one-third?

b. Suppose we find that the area to the right of *m* is *less* than one-third the area of the rectangle. How can we make another crease to get a line *n* that is *closer to the one-third position* than the line *m*? Describe exactly how to fold the paper so as to obtain the new line *n*. Will the area to the right of *n* be exactly one-third the area of the rectangle or a little more or a little less?

c. Can you describe a similar process for improving an estimate if the object is to divide a given rectangle into *five* strips of equal width?

FRACTIONS AS NUMERALS; EQUIVALENT NUMERALS

By a *numeral* we mean a symbol or a sequence of symbols that names or describes some particular number. Schoolchildren are often left confused by a discussion about numerals and their relation to numbers, and one of the chief reasons is that at a given point in the discussion they may be unsure whether we are talking of the number 3, say, or of the corresponding numeral "3."

Of course we use the numeral "3" when we talk about the number 3, since the numeral "3" is a name for the number 3. This is just like using the name "John" to talk about our friend John. But what do we use when we talk about the numeral "3" or about the name "John"? Obviously we need names for these symbols. And since the symbol "3" is different from the number 3, just as the name "John" is different from our friend John, we need a name for the symbol "3" or for the symbol "John" that is different from our name for the number 3 or for our friend John. In other words, the name for the symbol "3" should be different from the symbol "3" itself.

As the reader will see by going over the preceding paragraphs, we form a name for the symbol "3," that is, for the Arabic numeral denoting the number 3, by placing that symbol within quotation marks. Similarly, we use quotation marks around the name of our friend John to form a name for the symbol "John." In short, the symbol "3" is a name for the number 3, and the symbol "John" is a name for our friend John.

We shall follow the above convention carefully in the present section, although we do not advocate introducing this distinction systematically in a classroom; it is too subtle for most beginning students. Besides, most verbal communication in a classroom is oral rather than written, and it would be unbearably awkward for the teacher or student to say "The symbol quotation mark three quotation mark is a name for the number three, whereas a name for the symbol quotation mark three quotation mark is obtained by putting quotation marks around that symbol"! Instead, in oral communication about symbols and their use as names or descriptions of numbers, we may use the words "the numeral" or "the number" consistently before a name of one or another of these objects in order to signify whether we are talking about the concrete symbol or the abstract number. In this section we shall use these words systematically *in addition* to our use of quotation marks, even though in other parts of the book there will be no need for such extreme care.

Now then, what in fact are the numerals we use for denoting numbers? Let us first consider the whole numbers: 0, 1, 2, 3, $\cdots$.

Most commonly we use the *decimal numeration system*, which is a systematic method of forming a standard name for each whole number out of ten basic symbols called *digits*. Under this system *each whole number is denoted by one, and only one, standard decimal numeral.*

However, any given whole number possesses many names, or descriptions, other than its standard decimal numeral. For example, the numeral "11_{two}" (read aloud as "one one base two") is a name for the number 3 in the binary, or base-two, numeration system.

Even if we stick to the decimal numeration system, however, a given whole number such as 3 will have many names because we can form such names by the use of operation symbols. So, for example, the symbolic expression "$2 + 1$" denotes the number obtained when the numbers 2 and 1 are added; but since the number thus obtained is, in fact, 3, we see that the symbolic expression "$2 + 1$" denotes, or names, the number 3. Still another name for the number 3 is the symbolic expression "3×1."

When we have several names for the same number we call them *equivalent numerals.* Thus, the numerals "3," "11_{two}," "$2 + 1$," and "3×1" are all equivalent.

One cannot always tell at a glance whether two given numerals *are* equivalent. For example, the question whether the numeral

$$\text{"}((4 + 2) \div 6) \times (11 - 5)\text{"}$$

is equivalent to

$$\text{"}((6 + 2) \div 4) + ((11 + 5) - 15)\text{"}$$

would require a good bit of work from some of our students before they could answer it.

When a teacher gives a problem such as "Compute

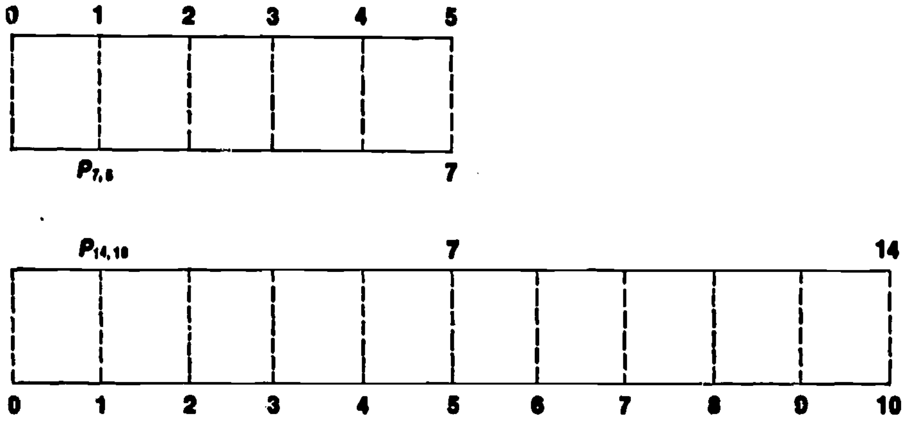
$$((4 + 2) \div 6) \times (11 - 5),\text{"}$$

the student is expected to find the *standard decimal name* for the number denoted by the given numeral.

Now let us look at the numeration system by which we provide a name for every *rational* number. As we have seen, each rational number is first introduced as the number that solves a certain problem formulated in terms of two *whole* numbers. We use the standard decimal numerals for these two whole numbers and combine them, by means of a third symbol called a "fraction bar," to form a name for the rational number that solves the problem.

For example, the problem may be to divide the interval from the point 0 to the point 7 on the number line into 5 intervals of equal length. We have called the first of these division points $P_{7,5}$. The number that is the address of this point (represents the length of the interval from the point 0 to the point $P_{7,5}$) is a rational number. We take the symbol "$\frac{7}{5}$" as a numeral denoting this number. The symbol "$\frac{7}{5}$" itself is called a *fraction*. The number 7 is called the *numerator* of the fraction; the number 5 is called the *denominator* of the fraction; the fraction itself is not a number, but it is a numeral denoting the rational number $\frac{7}{5}$ that is the address of the point $P_{7,5}$ on the number line.

While every *fraction* has a unique numerator and a unique denominator, a *rational number* does not. To understand why, let us consider the number-line problem of dividing the interval from the point 0 to the point 14 into 10 intervals of equal length. The first of these division points we have agreed to call $P_{14,10}$, and the rational number that is its address we denote by the fraction "$\frac{14}{10}$."

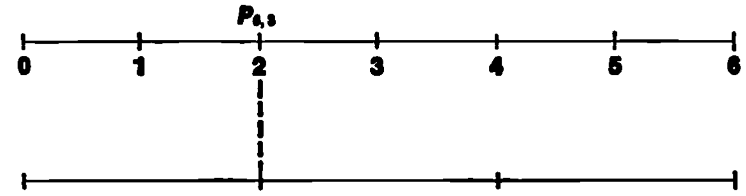


Now it happens that the point $P_{14,10}$ is the very same point on the number line as the point $P_{7,5}$, as shown above. Thus these points $P_{14,10}$ and $P_{7,5}$ have the same address, that is, the rational number $\frac{7}{5}$ is the same as the rational number $\frac{14}{10}$. We express this fact by writing "$\frac{7}{5} = \frac{14}{10}$."

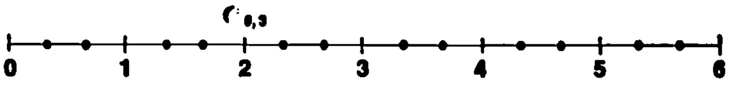
The fractions (i.e., symbolic expressions) "$\frac{7}{5}$" and "$\frac{14}{10}$" are equivalent but different; the former has the numerator 7, while the latter has the numerator 14. But the numbers $\frac{7}{5}$ and $\frac{14}{10}$ are *one and the same;* hence if we allowed ourselves to say that the number $\frac{7}{5}$ has a numerator 7, we could conclude by the logic of identity that the number $\frac{14}{10}$ has the numerator 7. Of course this would be absurd. Thus, *numbers have no numerators; only fractions do.*

We have seen that the fractions "$\frac{14}{10}$" and "$\frac{7}{5}$" are equivalent numerals. There are many other numerals equivalent to these, some of which are *not* fractions. For example, the symbolic expressions “$\frac{3}{5} + \frac{12}{15}$” and “$\frac{7}{3} \times \frac{6}{10}$” are also numerals that denote the rational number $\frac{7}{5}$; hence they are equivalent to the numerals “$\frac{7}{5}$” and “$\frac{14}{10}$.”

Some fractions (but not “$\frac{7}{5}$”) are names for whole numbers; hence such a fraction is equivalent to the standard decimal numeral denoting the whole number. For example, the fraction “$\frac{6}{3}$” is equivalent to the numeral “2,” since the rational number $\frac{6}{3}$ is the same as the whole number 2. This last fact, namely, that $\frac{6}{3} = 2$, can be seen by locating the first division point $P_{6,3}$ on the number line when the interval from the point 0 to the point 6 is divided into 3 equal intervals. That first division point is the point 2, so that the address of $P_{6,3}$ is 2. Similarly, we can see that the address of $Q_{6,3}$ must be 2.



Three segments of equal length



Each interval divided into 3 segments of equal length. Count off 6 small segments.

Just as we singled out the standard decimal numeral among all the many names of a whole number, so we may single out as standard numerals for rational numbers the fractions whose numerators and denominators are standard numerals for whole numbers. The difference is that while each whole number has *only one* standard decimal numeral, each rational number has many.

Among all the fractions for a given rational number, is there not some systematic way to select just one of them so that we could have a unique “selected numeral” corresponding to each rational number? Indeed there is—in fact, there are various systematic ways in which one can select a single fraction from each set of equivalent fractions. The most common way is to select the fraction that is in "lowest terms."

It turns out that among all the fractions "$\frac{x}{y}$" that are equivalent to a given fraction "$\frac{a}{b}$" there is always exactly one of them in which the whole number x, which is the numerator, and the whole number y, which is the denominator, have no common factor other than 1 (which, of course, is a factor of *every* whole number). This unique one of the equivalent fractions is said to be in *lowest terms.* For example, "$\frac{2}{3}$" is the unique fraction in lowest terms equivalent to "$\frac{14}{21}$."

Many traditional texts in elementary mathematics put a great deal of stress on getting pupils to represent a rational number by its unique standard fraction in lowest terms. However, nowadays we recognize that this is a somewhat misplaced effort, for the important things we do with rational numbers—such as computing with them or finding which ones solve certain kinds of problems—require us constantly to deal with fractions that are *not* in lowest terms. Thus, any two equivalent fractions are often regarded as equally acceptable names for the rational number they denote.

In the next section we shall examine some of the reasons why we replace one equivalent fraction by another, in the course of computations, and hence why a great emphasis on fractions in lowest terms is misplaced. Only the basic theoretical ideas will be presented; a fuller development of computation with fractions will be found in later chapters.

Exercise Set 2

1. Just as two numerals that denote the same number are called equivalent, so any two names or descriptions of the same object or person are called equivalent. Find equivalent expressions for each of the following:

a. "The President of the United States"

b. "The Pacific Ocean"

c. "The set {2,3}"

d. "The maternal grandfather of Abraham Lincoln"

e. "The National Council of Teachers of Mathematics"

2. Formulate a problem, using the numbers 3 and 5, whose solution is $\frac{5}{3}$. Formulate another problem, using the numbers 6 and 10, whose solution is $\frac{10}{6}$. Show directly from these problems, without computing their solutions, that the same number solves both problems.

WHY WE USE EQUIVALENT FRACTIONS IN COMPUTATION

Perhaps the simplest problem involving two given rational numbers is that of *comparing* them. Which of them is larger? Or are they the same?

Of any two distinct points on a number line, one will be to the left of the other. If p is the rational-number address of the point to the left and q is the rational-number address of the point to the right, we say that p is *less* than (or *smaller* than) q, and we write $p < q$. Or we may express the same situation by saying that q is *greater* than (or *larger* than) p and writing $q > p$.

But suppose we are given fractions "$\frac{a}{b}$" denoting the number p and "$\frac{c}{d}$" denoting the number q. If we have not located the points p and q on the number line, can we use the fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" to determine whether $p < q, p = q$, or $p > q$?

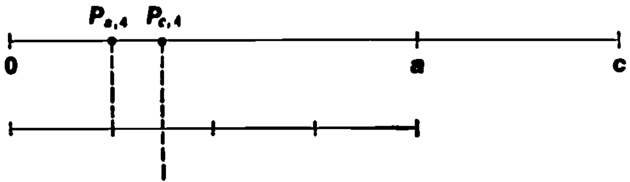
Under a certain special condition—namely, if the given fractions have *the same denominator*—there is a very simple answer to this question. Suppose, for example, that we wish to compare the rational numbers $\frac{8}{7}$ and $\frac{5}{7}$; clearly the former is larger than the latter $\left(\frac{8}{7} > \frac{5}{7}\right)$, and, equivalently, the latter is smaller than the former $\left(\frac{5}{7} < \frac{8}{7}\right)$. In general, if we are asked to compare two rational numbers given by fractions with the same whole-number denominator, say "$\frac{a}{b}$" and "$\frac{c}{b}$," the rule is:

If $a < c$, *then* $\frac{a}{b} < \frac{c}{b}$.

If $a = c$, *then* $\frac{a}{b} = \frac{c}{b}$.

If $a > c$, *then* $\frac{a}{b} > \frac{c}{b}$.

Of course these facts are intuitively obvious for those who have some familiarity in dealing with fractions, but they can easily be explained to those who have just been introduced to fractions. We have only to think of the points $P_{a,b}$ and $P_{c,b}$ on the number line. $P_{a,b}$ is the first division point when the interval from 0 to a is divided into b equal portions. $P_{c,b}$ is the first division point when the interval from 0 to c is divided into b equal portions. Clearly if the point a is to the left of the point c, so that the interval from 0 to a is smaller than the interval from 0 to b, then the division point $P_{a,b}$ will be to the left of the division point $P_{c,b}$, that is, we shall have $\frac{a}{b} < \frac{c}{b}$. If the point a is to the right of the point c, then $P_{a,b}$ will be to the right of $P_{c,b}$, that is, we shall have $\frac{a}{b} > \frac{c}{b}$. If the point a coincides with the point c, then $P_{a,b}$ will coincide with $P_{c,b}$, that is, we shall have $\frac{a}{b} = \frac{c}{b}$.

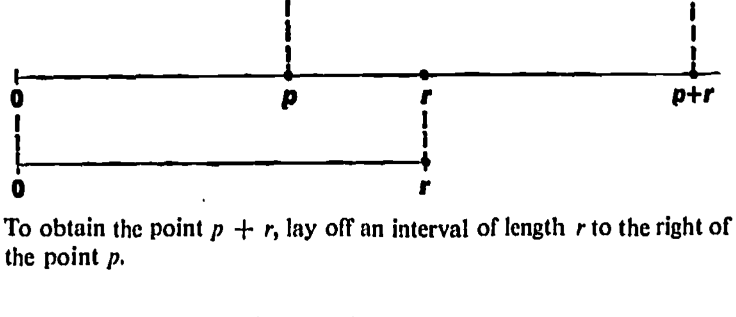


The intervals from 0 to a, and from 0 to c, each divided into 4 parts of equal length. Since $a < c$, we obtain $\frac{a}{4} < \frac{c}{4}$.

Suppose, however, we are asked to compare two rational numbers and these are named by fractions whose denominators do *not* happen to be the same. What then? We have no good method for dealing with this problem *directly*, so we reduce it to the case we have seen how to solve so easily. That is, we take one of the given fractions, say "$\frac{a}{b}$", and find an equivalent fraction "$\frac{r}{t}$"; similarly, we take the other given fraction, say "$\frac{c}{d}$", and find an equivalent fraction "$\frac{s}{t}$" *having the same denominator* as the fraction "$\frac{r}{t}$". Because these fractions have the same denominator, we can apply the previous method to tell whether the rational number $\frac{r}{t}$ is greater than, equal to, or less than the rational number $\frac{s}{t}$. But because the fractions "$\frac{r}{t}$" and "$\frac{a}{b}$" are equivalent, they name the same rational number, that is, $\frac{r}{t} = \frac{a}{b}$. Similarly we shall know that $\frac{s}{t} = \frac{c}{d}$. Hence when we have determined which of the relations $>$, $=$, or $<$ holds between $\frac{r}{t}$ and $\frac{s}{t}$, we shall know that the same relation holds between $\frac{a}{b}$ and $\frac{c}{d}$. In this way the original question will be answered.

Most elementary students will have no difficulty in comparing two rational numbers given by fractions having the same denominator. They will perceive the difficulty when the given fractions have unlike denominators; and they will be able to follow the idea sketched above, of carrying out the comparison by replacing each of the given fractions by an equivalent one in such a way that the two new fractions have the same denominator. "But," they will want to know, "how do we *find* such equivalent fractions having the same denominator?" When they have asked this question, they have been motivated to study the algorithm for "finding a common denominator."

Not only comparing, but also computing sums of rational numbers, is very easy *when the numbers are named by fractions having the same denominator.* If rational numbers have been introduced as addresses of points on a number line, the operation of adding them can be introduced by simply extending procedures developed earlier for interpreting the addition of whole numbers on the number line. That is, being given any rational numbers p and r, we first find the point p on the number line; we lay off to the right of it an interval having length r (i.e., having the same length as the distance between point 0 and point r); and the address of the new point at which we arrive, at the right end of the laid-off interval, is the rational number that we call $p + r$.

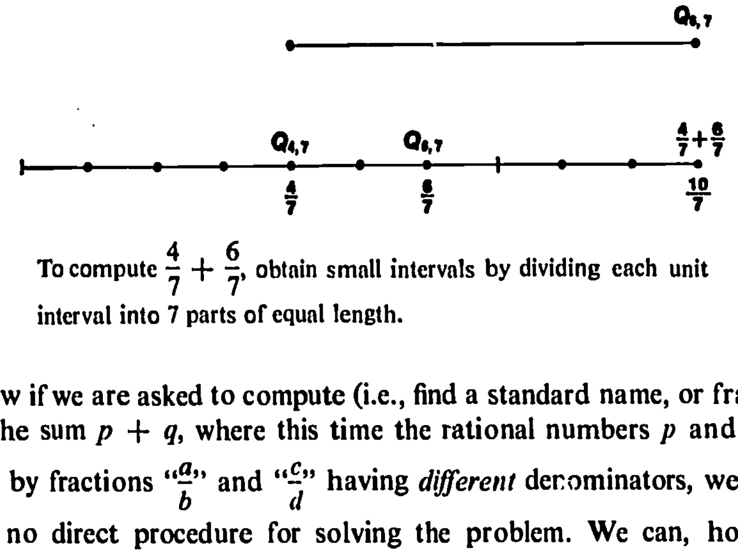


To obtain the point $p + r$, lay off an interval of length r to the right of the point p.

Now suppose that the given rational numbers p and r are named by fractions that happen to have the same denominator, say $p = \frac{a}{b}$ and $r = \frac{c}{b}$. How do we obtain a fraction representing the sum $p + r$?

We know that these numbers, $\frac{a}{b}$ and $\frac{c}{b}$, will be the respective addresses of certain points $Q_{a,b}$ and $Q_{c,b}$ obtained as follows: We divide each interval of the number line lying between two successive whole numbers into b parts of equal length; we then count off these smaller intervals, starting from point 0 and proceeding to the right; the point $Q_{a,b}$ will be the right endpoint of small interval number a, and the point $Q_{c,b}$ will be the right endpoint of small interval number c. Hence if we lay off an interval of the same length as the one from point 0 to point $Q_{c,b}$, starting from the point $Q_{a,b}$ and proceeding to the right, we shall arrive at the right endpoint of small interval number $a + c$, which is simply the point $Q_{a+c,b}$. But the address of this point is $\frac{a+c}{b}$. Thus we find that

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$



To compute $\frac{4}{7} + \frac{6}{7}$, obtain small intervals by dividing each unit interval into 7 parts of equal length.

Now if we are asked to compute (i.e., find a standard name, or fraction, for) the sum $p + q$, where this time the rational numbers p and q are given by fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" having *different* denominators, we again have no direct procedure for solving the problem. We can, however, proceed as we did with the problem of comparing two given rational numbers. That is, we pass from the fraction "$\frac{a}{b}$" to an equivalent fraction "$\frac{r}{t}$" and from "$\frac{c}{d}$" to an equivalent "$\frac{s}{t}$" *having the same denominator* as "$\frac{r}{t}$". Because of the equivalences, the fraction "$\frac{r}{t}$" is a name for the same rational number p, as the fraction "$\frac{a}{b}$" denotes; and, similarly, "$\frac{s}{t}$" denotes q, just as "$\frac{c}{d}$" does. Hence $p + q$ can be computed by adding with the fractions "$\frac{r}{t}$" and "$\frac{s}{t}$," which have the same denominator. This is a problem we have already seen how to solve.

As we have observed above, both the problems of comparing and of adding two rational numbers can be handled for *arbitrary* pairs of rational numbers—as long as we can pass from any given fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" to equivalent fractions "$\frac{r}{t}$" and "$\frac{s}{t}$" having the same denominator. There are, in fact, several algorithms for doing this, and we shall discuss some of these in the next section.

It is worthwhile observing that the question whether either or both of the fractions naming the given rational numbers are in lowest terms plays no role when it comes to applying our methods for comparing or adding the given numbers. Only the question whether the two fractions have the same denominator is important. This is one of the reasons we no longer stress the reduction of fractions to lowest terms.

NOTE.—In mathematics there is almost always more than one way to solve any given problem. Here we mention two other methods for comparing rational numbers to see which, if either, is the greater one.

Instead of considering the special case where the two given fractions have the same denominator, suppose they have the same *numerator*. How can we tell whether the rational number $\frac{a}{b}$ is less than, equal to, or greater than the rational number $\frac{a}{c}$? The rule is very simple:

$$\textit{If } b > c, \textit{ then } \frac{a}{b} < \frac{a}{c}; \textit{ if } b = c, \textit{ then } \frac{a}{b} = \frac{a}{c};$$

$$\textit{if } b < c, \textit{ then } \frac{a}{b} > \frac{a}{c}.$$

We leave the reader to verify this rule by going back to the definition of the points $P_{a,b}$ and $P_{a,c}$ and finding when $P_{a,b}$ is to the left of, coincides with, or is to the right of the point $P_{a,c}$.

The general case of comparing two given rational numbers can be reduced to this special case because we can always take two given fractions and convert each to an equivalent fraction so that the two new fractions have the same numerator. The reader can see for himself how much easier this is in an example such as comparing $\frac{2}{39}$ with $\frac{4}{77}$ than the method of finding fractions with a common denominator.

A second method for comparing rational numbers, of more limited applicability, consists in reducing each of the given fractions to an equivalent one in lowest terms. The two given rational numbers are equal if the *same* fraction is obtained when each of the given fractions is reduced to lowest terms. The two given fractions are unequal if the given fractions reduce to two *different* fractions in lowest terms. In the latter case, however, we must return to one of the earlier methods to find which of the given numbers is the greater one.

Exercise Set 3

1. The rule for comparing two rational numbers represented by fractions with the same denominator is as follows: If $a > c$, then $\frac{a}{b} > \frac{c}{b}$. In the text, this rule was explained using the fact that a rational number such as $\frac{a}{b}$ is the address of the point $P_{a,b}$. Use the fact that each rational number $\frac{a}{b}$ is the address of the point $Q_{a,b}$ to give an explanation of the same rule, dealing with the following particular case: Since $6 > 4$, we have $\frac{6}{5} > \frac{4}{5}$.

2. Suppose that a certain teacher has introduced rational numbers by the missing-factor approach, defining $\frac{a}{b}$ to be the number satisfying the open sentence $b \cdot \square = a$, where a and b are any whole numbers with $b \neq 0$. Indicate how this teacher could illustrate the rule for adding with fractions with the same denominator by using her definition to show that $\frac{7}{3} + \frac{10}{3} = \frac{17}{3}$.

3. Illustrate the rule for comparing the values of fractions having the same numerator in the case of the relation $\frac{7}{4} < \frac{7}{3}$ by showing that the point $P_{7,4}$ is to the left of the point $P_{7,3}$ on the number line.

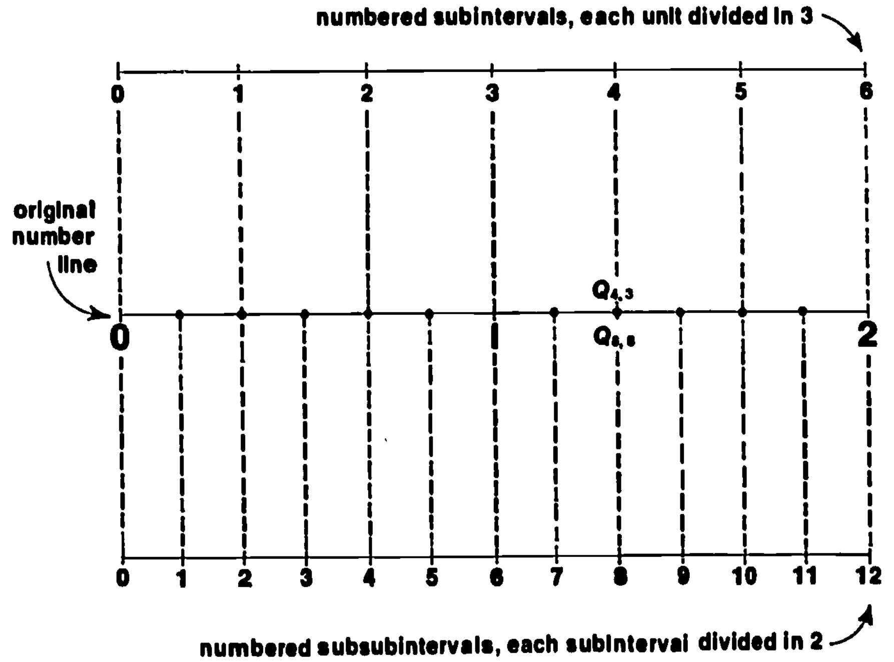
COMPUTING EQUIVALENT FRACTIONS

We have seen above how the problems of comparing or adding two given rational numbers lead us to the problem of passing from given fractions to equivalent fractions with different denominators. All problems of the latter type reduce to the following:

> Fundamental Principle for Fractions: *If* $\frac{a}{b}$ *is a rational number and c is a whole number other than* 0, *then* $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$.

One of the easiest ways to see the correctness of this principle is to compare the points $Q_{a,b}$ and $Q_{a \cdot c, b \cdot c}$ on the number line. We know that the rational number $\frac{a}{b}$ is the address of the point $Q_{a,b}$, while $\frac{a \cdot c}{b \cdot c}$ is the address of $Q_{a \cdot c, b \cdot c}$. Hence if we can show that the point $Q_{a,b}$ is the same point of the number line as $Q_{a \cdot c, b \cdot c}$, we shall know that these points have the same address, that is, that $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ as claimed in the Fundamental Principle just given.

Below we shall give the argument showing that, in general, $Q_{a,b}$ and $Q_{a \cdot c, b \cdot c}$ are the same point of the number line; the reader can follow this argument more easily by studying the following diagram, which illustrates the case $a = 4$, $b = 3$, $c = 2$, in which $Q_{4,3} = Q_{8,6}$ (showing that $\frac{4}{3} = \frac{8}{6}$).



Case for $a = 4$, $b = 3$, $c = 2$

Recall that to construct $Q_{a,b}$ we begin by dividing each unit interval into b equal parts, which we shall call subintervals. Then count off these small subintervals, starting from 0 and proceeding to the right. When we reach the subinterval whose number is a, its right endpoint is the point $Q_{a,b}$.

Now take *each* of the subintervals obtained above, and *divide each one into c equal parts;* let us call these new, very small intervals *subsubintervals.* Since each unit interval on the number line was divided into b subintervals and each subinterval is now divided into c subsubintervals, we clearly have $b \cdot c$ subsubintervals in each of the original unit intervals. Since all of the subsubintervals have the same length, we can therefore use them to locate the points $Q_{1,b\cdot c}, Q_{2,b\cdot c}, Q_{3,b\cdot c}, \cdots$ by counting off the subsubintervals, starting from 0 and proceeding to the right.

Now then, let us return to our point $Q_{a,b}$. We know that it is the right endpoint of one of the subintervals and that there are exactly a of these subintervals between the point 0 and the point $Q_{a,b}$. Since each of those a subintervals has now been divided into c subsubintervals, there will be $a \cdot c$ of these subsubintervals between the point 0 and the point $Q_{a,b}$. Hence when we start counting off the subsubintervals from the point 0 to get the points $Q_{1,b\cdot c}, Q_{2,b\cdot c}, Q_{3,b\cdot c}, \cdots$, we see that the point $Q_{a\cdot c,b\cdot c}$ will be the point $Q_{a,b}$. In short, the points $Q_{a,b}$ and $Q_{a\cdot c,b\cdot c}$ coincide. Hence the rational number $\frac{a}{b}$, which serves as the address for the point $Q_{a,b}$, is the same as the rational number $\frac{a\cdot c}{b\cdot c}$, which serves as the address for the point $Q_{a\cdot c,b\cdot c}$. In other words, $\frac{a}{b} = \frac{a\cdot c}{b\cdot c}$, as claimed in the *Fundamental Principle for Fractions*, and so "$\frac{a}{b}$" and "$\frac{a\cdot c}{b\cdot c}$" are equivalent fractions.

Let us now take up the problem whether, being given two fractions "$\frac{a}{b}$" and "$\frac{c}{d}$," we can find equivalent fractions for each of them so that these new fractions have the same denominator. It is easy to see that in the *special case where one of the given denominators is a factor of the other*, the Fundamental Principle for Fractions leads directly to a solution in which the given fraction with greater denominator does not have to be changed at all.

For example, if we are given the fractions "$\frac{2}{17}$" and "$\frac{13}{51}$," and if we notice that 17 is a factor of 51 (because $17 \times 3 = 51$), then we apply the Fundamental Principle to obtain $\frac{2}{17} = \frac{2 \times 3}{17 \times 3}$, thus passing from the given fraction "$\frac{2}{17}$" to the equivalent fraction "$\frac{6}{51}$" having the same denominator as the second given fraction, "$\frac{13}{51}$."

In the general case, suppose we are given fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" such that b is a factor of d. This means that we can find some whole number e such that $b \cdot e = d$. (Of course $e \neq 0$, for otherwise we would have $b \cdot e = 0$ and hence $d = 0$, contrary to the fact that d is the denominator of a fraction.)

By the Fundamental Principle we know that $\frac{a}{b} = \frac{a \cdot e}{b \cdot e}$; since $b \cdot e = d$, we get $\frac{a}{b} = \frac{a \cdot e}{d}$. Hence the fraction "$\frac{a \cdot e}{d}$" is equivalent to the given fraction "$\frac{a}{b}$" and has the same denominator as the other given fraction, "$\frac{c}{d}$."

Now suppose, however, that we study the given fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" and find that *neither* of the denominators is a factor of the other. What then? We see that we can always replace one of the given fractions, say "$\frac{c}{d}$," by an equivalent fraction whose denominator has the other given denominator, "b," as a factor. In fact we have $\frac{c}{d} = \frac{c \cdot b}{d \cdot b}$ by the Fundamental Principle, so that the fraction "$\frac{c \cdot b}{d \cdot b}$" is equivalent to "$\frac{c}{d}$" and obviously its denominator $d \cdot b$ has b as a factor.

We can, therefore, find a fraction equivalent to "$\frac{a}{b}$" having the same denominator as the fraction "$\frac{c \cdot b}{d \cdot b}$"; indeed, applying our previous method we obtain the fraction "$\frac{a \cdot d}{b \cdot d}$," which is equivalent to "$\frac{a}{b}$" and has the same denominator as "$\frac{c \cdot b}{d \cdot b}$."

Thus we have obtained the fraction "$\frac{a \cdot d}{b \cdot d}$," equivalent to the given fraction "$\frac{a}{b}$," and "$\frac{c \cdot b}{d \cdot b}$," equivalent to the given fraction "$\frac{c}{d}$"; and, comparing the denominators of the new fractions, we see that $b \cdot d = d \cdot b$.

For example, suppose the given fractions are "$\frac{2}{4}$" and "$\frac{5}{6}$," so that neither denominator is a factor of the other. We first compute

$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24},$$

getting the fraction "$\frac{20}{24}$," equivalent to the given fraction "$\frac{5}{6}$." The denominator of the other given fraction, "$\frac{2}{4}$," is a factor of the denominator of "$\frac{20}{24}$," since $4 \cdot 6 = 24$. Hence we compute $\frac{2}{4} = \frac{2 \times 6}{4 \times 6} = \frac{12}{24}$, getting the fraction "$\frac{12}{24}$," equivalent to the given fraction "$\frac{2}{4}$" and having the same denominator as "$\frac{20}{24}$." Finally, then, we have found the fractions "$\frac{12}{24}$" and "$\frac{20}{24}$" with the same denominator, equivalent respectively to the given fractions "$\frac{2}{4}$" and "$\frac{5}{6}$." And the fractions thus formed were obtained by two applications of the Fundamental Principle.

The teacher whose ideas about fractions have been dominated by an urge always to reduce them to lowest terms may find this example disturbing, since neither of the two new fractions—nor one of the given fractions—is in lowest terms. The important thing to remember is that for the purposes of comparing or of adding given rational numbers we must express the given numbers by fractions having the same denominator, and this we have done by a process that will work in the most general case. Whether fractions are in lowest terms or not is unimportant as far as most problems involving rational numbers are concerned.

Nevertheless, as a matter of convenience, we are sometimes interested in finding a pair of fractions with a common denominator that is *fairly small*, simply to reduce the fatigue involved in employing the multiplication algorithm. Consider, for example, the problem of adding the rational numbers $\frac{3}{26}$ and $\frac{2}{39}$.

If we seek to convert each of the given fractions "$\frac{3}{26}$" and "$\frac{2}{39}$" to an equivalent fraction in such a way that the two new fractions have the same denominator, the method developed above (involving two applications of the Fundamental Principle) will lead to the fractions "$\frac{3 \times 39}{26 \times 39}$" and "$\frac{2 \times 26}{39 \times 26}$," respectively. To express these, in turn, as standard fractions whose numerator and denominator are represented by whole-number decimal numerals, we must employ the multiplication algorithm to compute the products 3×39, 2×26, and 26×39—a matter not intrinsically difficult, but rather tiresome.

We may, therefore, seek a second way of expressing two given rational numbers by fractions having the same denominator, just as correct as our earlier method and less tiresome when it comes to applying algorithms. To arrive at such a method, let us look more closely at the results of applying the first method.

Being given the fractions "$\frac{a}{b}$" and "$\frac{c}{d}$," we convert the former to the equivalent fraction "$\frac{a \cdot d}{b \cdot d}$" and the latter to the equivalent fraction "$\frac{c \cdot b}{d \cdot b}$." Thus the two new fractions have, as their common denominator, the product of the two given denominators. Each of the given denominators is thus a factor of the new denominator. This is what enables us to use our Fundamental Principle for Fractions.

Any whole number, say m, that has each of the given denominators b and d among its factors is called a common multiple of b and d. *Whenever we find such a common multiple m*, we can use the Fundamental Principle to express each of the given fractions as an equivalent fraction having m as denominator.

For example, we may happen to notice that 156 is a common multiple of 26 and 39, because $26 \times 6 = 156$ and $39 \times 4 = 156$. Hence, in the example considered above, we can apply the Fundamental Principle to obtain $\frac{3}{26} = \frac{3 \times 6}{26 \times 6}$ and $\frac{2}{39} = \frac{2 \times 4}{39 \times 4}$ to obtain new fractions "$\frac{18}{156}$" and "$\frac{8}{156}$" having a common denominator, equivalent respectively to the given fractions "$\frac{3}{26}$" and "$\frac{2}{39}$." Clearly the multiplication algorithms employed here were less tiresome than those needed to find a pair of fractions having the common denominator 26×39.

In general, suppose the given fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" have denominators with a common multiple m, so that both b and d are factors of m.

To say that b is a factor of m means that we can find a whole number e such that $b \cdot e = m$. Hence the fraction "$\frac{a}{b}$," which is equivalent to "$\frac{a \cdot e}{b \cdot e}$," will be equivalent to "$\frac{a \cdot e}{m}$." Similarly, since d is a factor of m, we can find a whole number f such that $d \cdot f = m$, and then "$\frac{c}{d}$" will be equivalent to "$\frac{c \cdot f}{m}$."

The product of the given denominators is the easiest of their common multiples to find. However, if we happen to come upon a smaller common multiple, then we can use it to find the equivalent fractions we are seeking, and we shall expend less effort when it comes to applying the multiplication algorithm. How, then, can we look for smaller common multiples of the given denominators? Is there a systematic way in which we can find the *least* common multiple?

If we take either of our given denominators, say b, we can begin to form a list of *all* numbers having b as a factor by computing the products $1 \cdot b$,

$2\cdot b$, $3\cdot b$, $\cdots$. As we form each of these multiples of b, we can test it to see whether it has the other given denominator, d, as a factor. (We do this by the division algorithm, dividing our multiple of b by d and seeing whether the remainder is 0.) The first multiple of b that we find to have d as a factor will be the least common multiple (LCM) of b and d.

For example, let us return to our problem of adding $\frac{2}{39}$ to $\frac{3}{26}$. The first few multiples of 26 are 26, 52, 78, and 104, and if we test these for divisibility by 39, we find that 78 is the first in the list having 39 as a factor. Thus, 78 is the least common multiple of 26 and 39. Since $78 = 3 \times 26$, we replace the given fraction "$\frac{3}{26}$" by "$\frac{3 \times 3}{26 \times 3}$," so that $\frac{3}{26} = \frac{9}{78}$. Similarly, since $78 = 2 \times 39$, we replace "$\frac{2}{39}$" by "$\frac{2 \times 2}{39 \times 2}$," so that $\frac{2}{39} = \frac{4}{78}$. Hence $\frac{3}{26} + \frac{2}{39} = \frac{9}{78} + \frac{4}{78}$. If we had used the greater denominator, 39, we would have arrived at the LCM sooner.

Although using the LCM of the given denominators generally requires a less tedious application of the multiplication algorithm than using the product of the given denominators, the saving in time and trouble may be offset by the extra computation needed to *determine* the LCM. There are various ways in which this labor can be reduced. For example, suppose we notice that the given denominators, b and d, have a common factor, say g. This means we have $b = i\cdot g$ and $d = j\cdot g$ for certain whole numbers i and j. These numbers i and j are smaller, respectively, than b and d, so it is easier to find *their* LCM. If we find that the LCM of i and j is some number n, say, then the LCM of b and d will be $n\cdot g$.

Thus, in the example above, the given denominators 26 and 39 are seen to have 13 as a common factor. Indeed, $26 = 2 \times 13$, and $39 = 3 \times 13$. Since the LCM of 2 and 3 is their product, 6, the LCM of 26 and 39 must be 6×13, or 78.

Given a fraction "$\frac{a}{b}$," we have seen how to obtain equivalent fractions bearing a special relation to a second given fraction, "$\frac{c}{d}$." But suppose we are interested in *all* the fractions equivalent to "$\frac{a}{b}$"—is there some systematic way to find them?

Of course by using our Fundamental Principle for Fractions we can generate an infinite string of equivalent fractions:

$$\text{“}\frac{a}{b}\text{”},\ \text{“}\frac{a\cdot 2}{b\cdot 2}\text{”},\ \text{“}\frac{a\cdot 3}{b\cdot 3}\text{”},\ \ldots .$$

Are these all? We shall see that in certain special cases, the answer is yes—for instance, if $a = 3$ and $b = 4$. But in other cases, say if $a = 6$ and $b = 9$, the answer is no.

The point is that the fractions

$$\text{“}\frac{a\cdot 2}{b\cdot 2}\text{”}, \quad \text{“}\frac{a\cdot 3}{b\cdot 3}\text{”}, \quad \text{“}\frac{a\cdot 4}{b\cdot 4}\text{”}, \ldots$$

all have numerators and denominators greater than the numerator and denominator, respectively, of the given fraction “$\frac{a}{b}$.” And we know that sometimes a given fraction may be equivalent to another having a smaller numerator and denominator. For example, for the fraction “$\frac{21}{14}$,” we have the equivalent fraction “$\frac{3}{2}$.”

In general, suppose that we are given a fraction “$\frac{a}{b}$” and that the given numerator a and the denominator b have some common factor, f. Say $a = c\cdot f$ and $b = d\cdot f$. Then we know that the fraction “$\frac{a}{b}$” will be equivalent to the fraction “$\frac{c}{d}$,” since the Fundamental Principle assures us that the fractions “$\frac{c}{d}$” and “$\frac{c\cdot f}{d\cdot f}$” are equivalent. Of course the numerator and denominator of the fraction “$\frac{c}{d}$” will be smaller than those of the given fraction “$\frac{a}{b}$” in this case.

If the numerator and denominator of the new fraction “$\frac{c}{d}$” again have a common factor, we can, as before, determine still another equivalent fraction by removing this common factor from both numerator and denominator. Continuing in this way to remove common factors as far as possible, we eventually arrive—as is well known—at a fraction “$\frac{g}{h}$,” equivalent to the original fraction “$\frac{a}{b}$,” where the numerator g and denominator h have *no* common factor (other than 1). This fraction “$\frac{g}{h}$” is in lowest terms.

Once we have obtained this unique fraction in lowest terms, “$\frac{g}{h}$,” which is equivalent to the given fraction “$\frac{a}{b}$,” we get a complete list of *all* fractions equivalent to “$\frac{a}{b}$” by applying the Fundamental Principle to “$\frac{g}{h}$”, getting the list “$\frac{g\cdot 1}{h\cdot 1}$”, “$\frac{g\cdot 2}{h\cdot 2}$”, “$\frac{g\cdot 3}{h\cdot 3}$”, . . . , Thus, for example, to obtain all fractions equivalent to "$\frac{715}{572}$" we first reduce by a common factor 11 to get "$\frac{65}{52}$"; then reduce again by the common factor 13 to get "$\frac{5}{4}$", which is in lowest terms; and finally we get the list of fractions "$\frac{5}{4}$," "$\frac{10}{8}$," "$\frac{15}{12}$," $\cdots$, which consists of *all* fractions equivalent to "$\frac{715}{572}$." The fraction "$\frac{715}{572}$" will be the 143d term in the list.

Exercise Set 4

1. Find two fractions having the same *numerator*, equivalent respectively to "$\frac{3}{17}$" and "$\frac{6}{29}$." Use the result to decide which of the rational numbers $\frac{3}{17}$ and $\frac{6}{29}$ is the greater.

2. Find three fractions having the same denominator, equivalent respectively to "$\frac{3}{2}$," "$\frac{5}{3}$," and "$\frac{7}{5}$." Use the result to arrange the rational numbers $\frac{3}{2}$, $\frac{5}{3}$, and $\frac{7}{5}$ in order of magnitude (least number first).

3. A beginning pupil, when asked to compute a sum $\frac{a}{b} + \frac{c}{d}$, used a "simple-minded" rule and gave the answer $\frac{a+c}{b+d}$. Of course we know that this does not, in general, lead to the correct answer.

- **a.** Can you describe all those rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ for which this pupil's rule *does* give the right answer?

- **b.** In those cases where the pupil's rule gives the wrong answer, is the answer always too large or always too small, or does it depend on the particular fractions $\frac{a}{b}$ and $\frac{c}{d}$ that are being added? Justify your answer.

4. What are *all* the fractions equivalent to "$\frac{408}{255}$"? How would you convince someone that no fractions other than those you have listed can be equivalent to "$\frac{408}{255}$"?

5. How many rational numbers can be expressed by fractions whose numerators and denominators are selected from the numbers 1, 2, 3, 4? List all of these numbers in order of magnitude, least one first.

SUMMARY

1. There are many problems involving two whole numbers whose solution is not a whole number. Rational numbers provide solutions to many of these. Fractions are names for rational numbers, made up from the names of two whole numbers.

Among problems leading to rational numbers is the location of points, on a number line, lying between the whole-number points. Given whole numbers a and b with $b \neq 0$, we saw two different ways to use them to locate points on the number line that are called "$P_{a,b}$" and "$Q_{a,b}$." These turn out always to be at the same place, and the number giving their "address"—that is, their distance from the 0 point—is the rational number $\frac{a}{b}$.

2. A given rational number may be represented by many different fractions. Two fractions representing the same number are called "equivalent fractions." It is useful to be able to replace one fraction by an equivalent one for various computational purposes, especially for comparing two given rational numbers (to determine whether they differ and, if so, which is the larger) and for adding two given rational numbers.

3. Both comparing and adding are very simple in case the given rational numbers are represented by fractions having the same denominator. This motivates the search for methods of replacing any two given fractions by a new pair of fractions having a common denominator, equivalent respectively to the given fractions. Several methods to accomplish this are discussed.

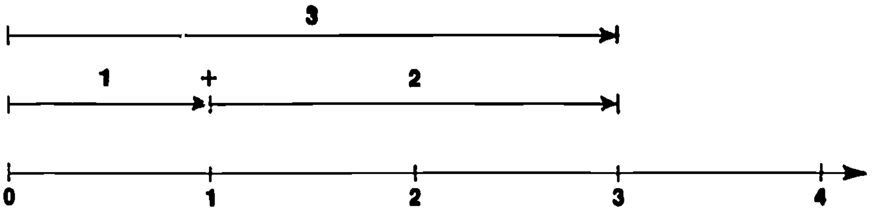
Joseph Moray

ADDITION OF RATIONAL NUMBERS

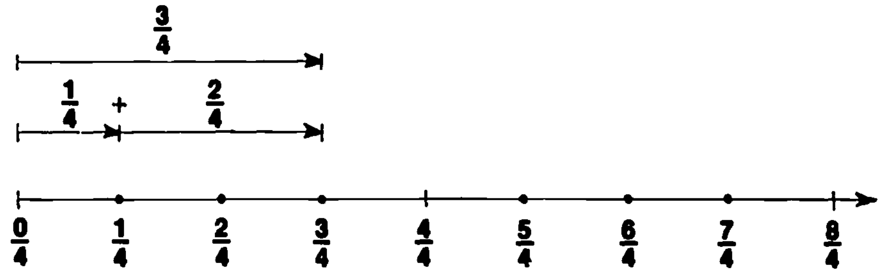
+3

1. **How can addition of rational numbers be related to addition of whole numbers?**
2. **What are some effective methods for finding a common denominator when computing sums of rational numbers?**
3. **Which properties of addition of whole numbers hold for addition of rational numbers?**
4. **What are the key ideas for understanding algorithms for computing sums of rational numbers?**

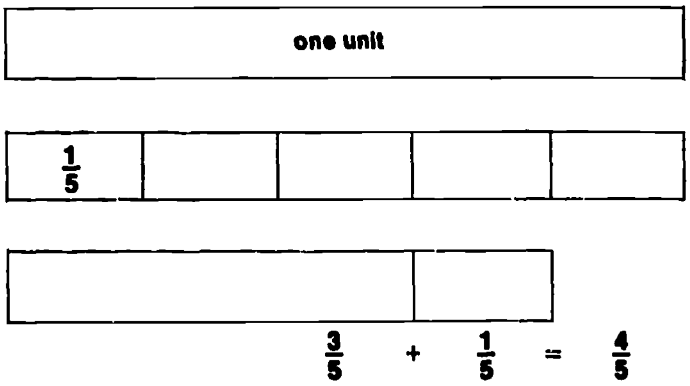
The number line offers an easy way of making a transition from addition of whole numbers to addition of rational numbers. When working with whole numbers students learned, for example, that 1 + 2 can be associated with a move on the number line from 0 of 1 unit to the right, followed by a move of 2 units to the right. This combination of moves was seen to be equivalent to a single move from 0 of 3 units to the right, resulting in the statement $1 + 2 = 3$. An arrow (or *vector*) diagram illustrates the result.



A similar procedure can be followed to compute the sum of two rational numbers, For example, a move from $\frac{0}{4}$ to $\frac{1}{4}$ followed by a move of $\frac{2}{4}$ to the right is seen to be equivalent to a single move from $\frac{0}{4}$ to $\frac{3}{4}$, resulting in the statement $\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.



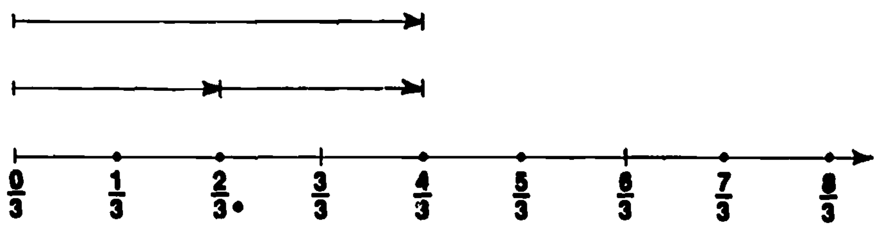
Strips of cardboard or rods made of wood or plastic (such as Cuisenaire Rods or Unifix Blocks) can be used in a similar manner to build meaning for addition of rational numbers.



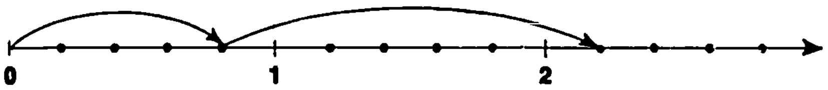
Exercise Set 1

1. Write an addition sentence (using fractions only—no mixed numerals—to name rational numbers) to go with each diagram:

a.



b.



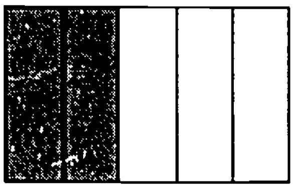
2. Draw a vector diagram (similar to **1a** above) for each of these sentences:

a. $\frac{2}{5}+\frac{4}{5}=\frac{6}{5}$.

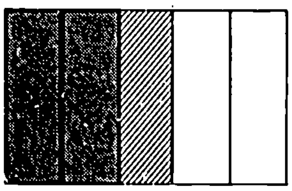
b. $\frac{5}{8}+\frac{4}{8}=\frac{9}{8}$.

REGIONS AND ADDITION

Another approach to addition of rational numbers is through the joining of parts of regions. Given a unit region partitioned into 5 parts of the same size, the student might be asked to shade in $\frac{2}{5}$ of the unit region and to write the fraction "2 over 5" to represent the amount shaded in. He is then directed to shade in $\frac{1}{5}$ more of the unit region. The amount now shaded in is $\frac{2}{5}+\frac{1}{5}$ of the unit region.



$\frac{2}{5}$



$\frac{2}{5}+\frac{1}{5}$

The size of each of the five parts into which the unit is partitioned is designated by the term *fifth*. The portion of the unit that is shaded in may therefore be considered as 2 + 1 fifths, or $\frac{2+1}{5}$. It is apparent then that the expressions "$\frac{2}{5}+\frac{1}{5}$" and "$\frac{2+1}{5}$" are equivalent. Since 2 + 1 = 3, we have

$$\frac{2}{5}+\frac{1}{5}=\frac{2+1}{5}=\frac{3}{5}.$$

Several similar examples, such as the following, can be worked out with the aid of partitioned unit regions, so that students can arrive at a generalization.

$$\frac{5}{8} + \frac{1}{8} = \frac{5 + 1}{8} = \frac{6}{8}.$$

$$\frac{4}{10} + \frac{3}{10} = \frac{4 + 3}{10} = \frac{7}{10}.$$

$$\frac{3}{5} + \frac{3}{5} = \frac{3 + 3}{5} = \frac{6}{5}.$$

$$\frac{28}{100} + \frac{77}{100} = ?$$

Students will recognize a pattern that can be applied to addition of any two rational numbers represented by fractions that have a common denominator: the sum of the numerators is the numerator of a fraction for the computed sum, and the common denominator is its denominator. Students are likely to put it more succinctly: "Add the numerators, and keep the same denominator." This generalization conforms to a definition of the sum of two rational numbers:

> *If* $\frac{a}{b}$ *and* $\frac{c}{b}$ *are any two rational numbers, then* $\frac{a}{b} + \frac{c}{b}$, *or* $\frac{a + c}{b}$, *is their sum, and we have*
>
> $$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

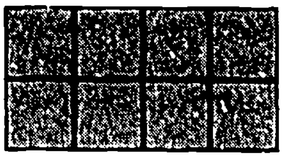
The operation of addition assigns a number called a "sum" to a pair of numbers called "addends." This statement holds for rational numbers as well as for whole numbers. For practical applications, however, it is often necessary to *compute* a sum, that is, to rename the sum with a standard numeral.

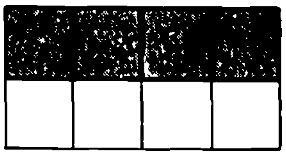
Renaming the sum of two whole numbers in "simplest form" offers no choice. For example, the sum of the pair of whole numbers 5 and 7 is 5 + 7, which is renamed in simplest form as "12." But how should the sum $\frac{5}{8} + \frac{7}{8}$ be renamed in simplest form? Before deciding, let us use regions to picture this sum.

$\frac{5}{8}$

$\frac{7}{8}$

Two unit regions are shown. Each unit is partitioned into 8 congruent parts. The shaded parts of the two units picture the sum $\frac{5}{8} + \frac{7}{8}$ as $\frac{5+7}{8}$, or $\frac{12}{8}$. The shaded parts can be rearranged as shown:





Here, too, we see that $\frac{5}{8} + \frac{7}{8} = \frac{12}{8}$. But in this case it might be desirable to go further with the computation and express the result with a mixed numeral:

$$\frac{5}{8} + \frac{7}{8} = \frac{12}{8} = \frac{8+4}{8} = \frac{8}{8} + \frac{4}{8} = 1 + \frac{4}{8} = 1 + \frac{1}{2} = 1\frac{1}{2}.$$

Of course, it is not necessary that all the steps shown above be stated explicitly. Each step, however, should be understood by the student in the process of learning an abbreviated algorithm, such as:

$$\frac{5}{8} + \frac{7}{8} = \frac{12}{8} = 1\frac{4}{8} = 1\frac{1}{2}.$$

But which is the simplest form, "$\frac{12}{8}$" or "$1\frac{4}{8}$" or "$1\frac{1}{2}$"? It depends on the application. While "$1\frac{1}{2}$" may be the easiest expression to use for some purposes, "$\frac{12}{8}$" or "$1\frac{4}{8}$" is appropriate when it is necessary to show the greater precision. A teacher who assigns computation with fractions should make clear to students which forms will be considered acceptable for their answers.

Exercise Set 2

1. Fill in the blanks to complete the steps for computing each sum.

a. $\frac{7}{10} + \frac{9}{10} = \frac{16}{10} = \frac{10}{10} + \frac{\quad}{10} = 1 + \frac{\quad}{10} = 1 + \frac{\quad}{5} = 1\frac{\quad}{5}.$

b. $\frac{3}{4} + \frac{3}{4} = \frac{\quad}{4} = \frac{\quad}{4} + \frac{2}{4} = 1 + \frac{2}{\quad} = 1 + \frac{\quad}{2} = 1\frac{\quad}{2}.$

2. Use the steps shown in the procedure above to compute the following sums:

a. $\frac{5}{6} + \frac{4}{6}$ b. $\frac{57}{100} + \frac{88}{100}$

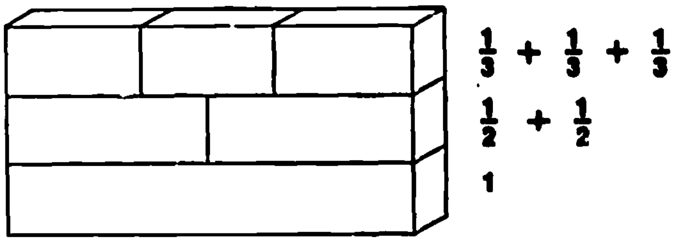
FINDING A COMMON DENOMINATOR

Adding with fractions that already have a common denominator is relatively simple for intermediate grade students. Computing sums with fractions that have *different* denominators, however, will involve some new addition computation skills.

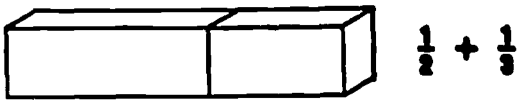
Computing a sum like $\frac{1}{2} + \frac{1}{4}$ may offer little difficulty, since many students can think of $\frac{1}{2}$ as $\frac{2}{4}$ and then compute the sum of $\frac{2}{4}$ and $\frac{1}{4}$, but computing a sum like $\frac{1}{2} + \frac{1}{3}$ can present a real problem. How might students (with minimum help from the teacher) figure out a solution to such a problem?

First, the students might be encouraged to estimate. Is $\frac{1}{2} + \frac{1}{3}$ greater than 1 or less than 1? How could the estimates be checked? The problem could be assigned to small groups in the classroom, with scissors, paper, number blocks, fraction kits, and so forth, as aids. Here are several ways in which students might arrive at a solution to $\frac{1}{2} + \frac{1}{3} = \square$.

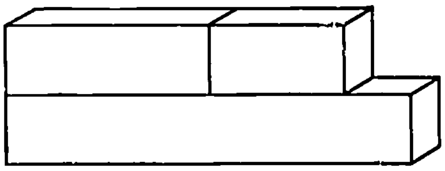
1. *Number blocks.* To compute $\frac{1}{2} + \frac{1}{3}$, the student first selects a unit length that can be "split" into halves and into thirds. Cuisenaire Rods, Stern Blocks, or similar materials will serve this purpose.



Next, $\frac{1}{2} + \frac{1}{3}$ is shown with the blocks:

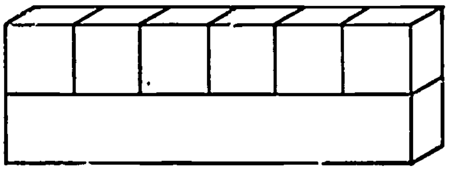


By comparison with the unit, $\frac{1}{2} + \frac{1}{3}$ is seen to be less than 1.



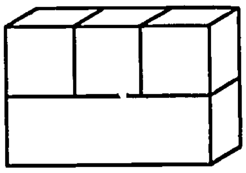
$\frac{1}{2} + \frac{1}{3} < 1.$

Can the unit be "split" into smaller pieces to aid in computing $\frac{1}{2} + \frac{1}{3}$?

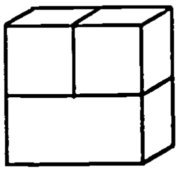


$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1.$

By matching, it can be seen that $\frac{1}{2} = \frac{3}{6}$, that $\frac{1}{3} = \frac{2}{6}$, and that $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

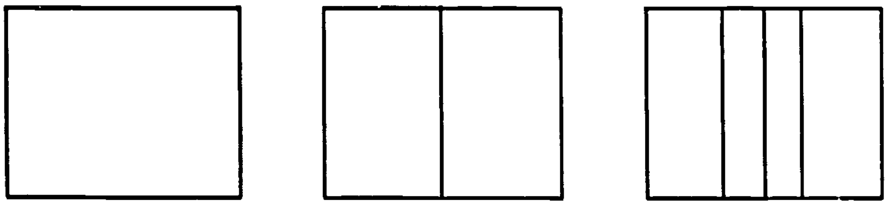


$\frac{1}{2} = \frac{3}{6}.$

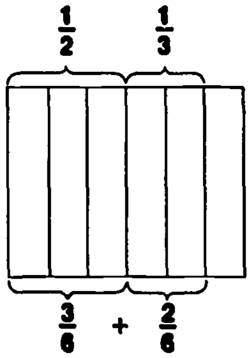


$\frac{1}{3} = \frac{2}{6}.$

2. *Regions.* To compute $\frac{1}{2} + \frac{1}{3}$, a unit region may be partitioned into halves, and also into thirds:

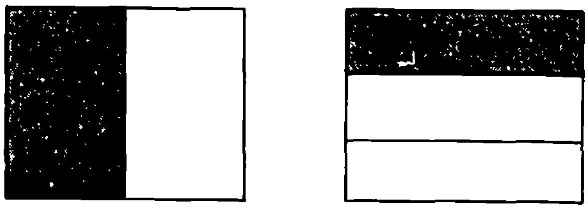


It might occur to a student that splitting the unit in half also splits the middle third in half and that by splitting each third in half he would have a way of computing $\frac{1}{2} + \frac{1}{3}$.

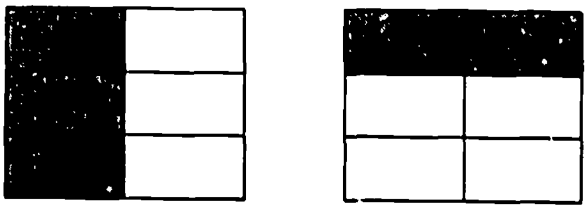


The unit is now split into 6 congruent parts. Each of these parts is $\frac{1}{6}$ of the unit. It can be seen that $\frac{1}{2}$ of the unit is also $\frac{3}{6}$ of the unit and that $\frac{1}{3}$ of the unit is also $\frac{2}{6}$ of the unit. It follows that $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

Another way to use regions to compute $\frac{1}{2} + \frac{1}{3}$ is to partition two unit regions, one into halves and the other into thirds, with one region split horizontally and the other vertically:

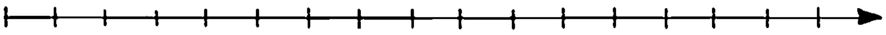


By superimposing the vertical and horizontal splits, the unit regions now appear as shown:

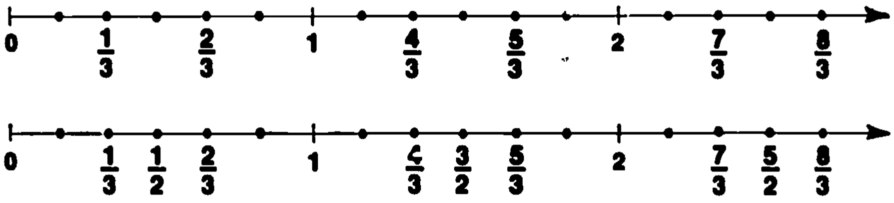


Each unit region is now partitioned into 6 congruent parts, and again it can be seen that $\frac{1}{2} = \frac{3}{6}$, that $\frac{1}{3} = \frac{2}{6}$, and that $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

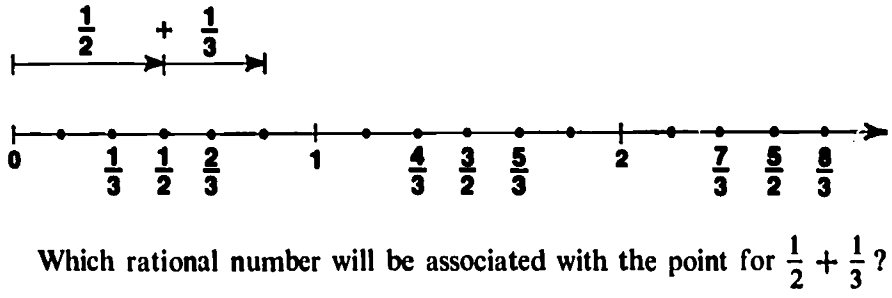
3. *Number line.* Since practical applications of addition of rational numbers often involve linear measurement, students may benefit from construction of number-line models for computing sums. To compute $\frac{1}{2} + \frac{1}{3}$, for example, a line can be marked off in units, halves, and thirds. If units are marked off first, students may have difficulty in splitting the units into thirds with sufficient accuracy. One way to avoid this difficulty is to begin with an unlabeled scale.



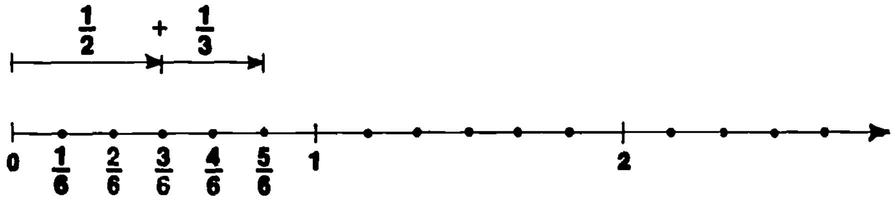
In order to show thirds, as well as halves of each unit, alternate marks are used for thirds. There are now six congruent segments marked off in each unit. Each two such segments make a length of $\frac{1}{3}$ unit, each three such segments make a length of $\frac{1}{2}$ unit.



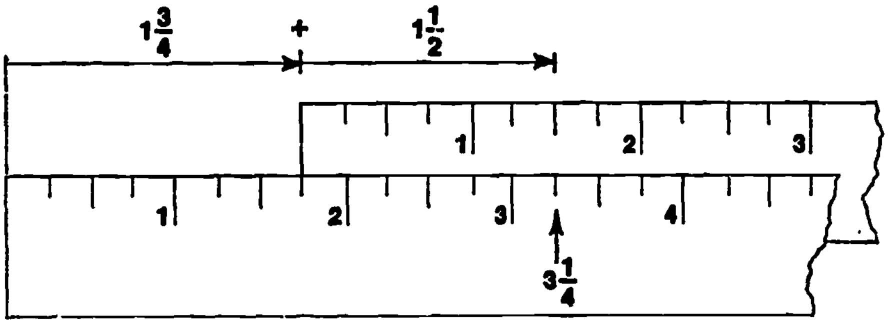
Now $\frac{1}{2} + \frac{1}{3}$ is pictured on the number line as a move of $\frac{1}{2}$ unit followed by a move of $\frac{1}{3}$ unit.



Which rational number will be associated with the point for $\frac{1}{2} + \frac{1}{3}$? Students can see that since each unit is split into six congruent segments, one such segment is $\frac{1}{6}$ unit. Then they can see that $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$ and that $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.



4. *Slide rule.* Two number-line strips can be used as a "slide rule" to compute a sum. If the denominators of the addends are 2, 4, 8, or 16, as occurs frequently in linear measurement, a slide rule for computing a sum can be made from two ordinary foot rulers. For example, to compute $1\frac{3}{4} + 1\frac{1}{2}$:



1. Slide the lower ruler to the left until the point for the first addend $\left(1\frac{3}{4}\right)$ is directly below 0 on the upper ruler.
2. On the upper ruler, find the point for the second addend $\left(1\frac{1}{2}\right)$. Directly below this point, the sum $\left(3\frac{1}{4}\right)$ is automatically registered on the lower ruler. (Of course, the roles of upper ruler and lower ruler may be interchanged.)

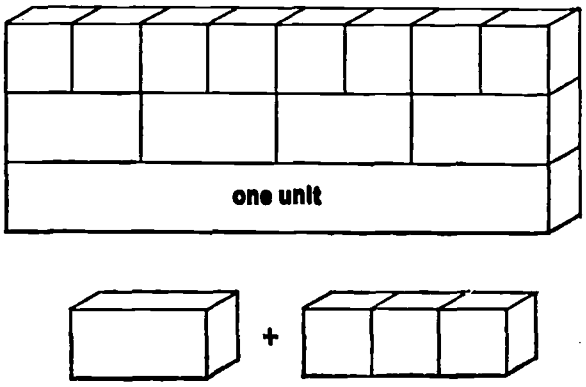
Rulers marked in sixteenths of an inch can be used in the same way when students are capable of working with more precise measurements. Slide rules can be constructed with other scales to compute sums with fractions whose denominators are not powers of 2: 2, 4, 8, 16, 32, 64.

The reason for the use of aids such as rods, regions (diagrams, folded paper, cutouts, etc.), and the number line in introducing addition of rational numbers is to give meaning to the abstract symbols and to the addition computation process. Otherwise, a student might add $\frac{1}{2}$ and $\frac{1}{3}$, get $\frac{2}{5}$, and not see that such an answer is inconsistent with the way rational numbers are usually applied in the real world.

Exercise Set 3

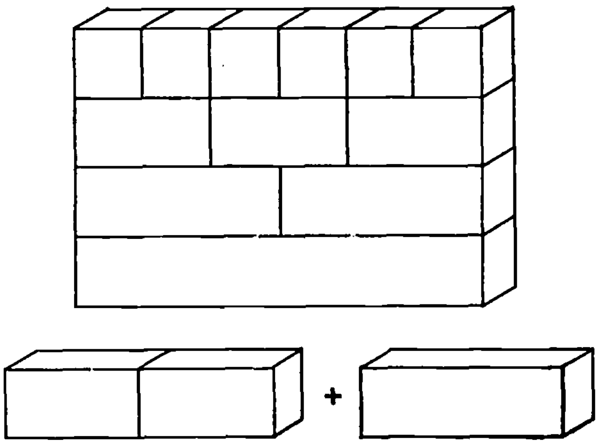
1. To compute each sum, refer to the illustration and supply the missing numerators and denominators.

a.



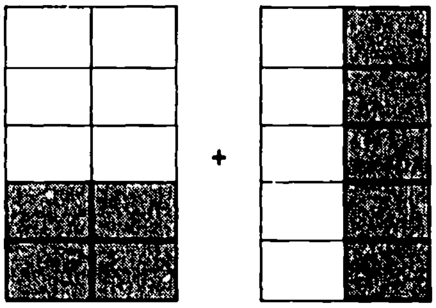
$$\frac{1}{\ } + \frac{3}{\ } = \frac{2}{\ } + \frac{3}{\ } = \frac{\ }{\ }$$

b.



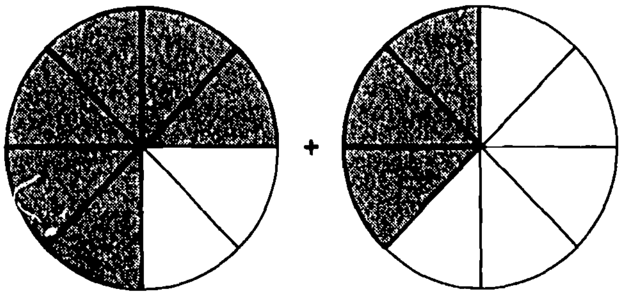
$$\frac{2}{\ } + \frac{\ }{2} = \frac{4}{\ } + \frac{\ }{6} = \frac{7}{\ }$$

c.



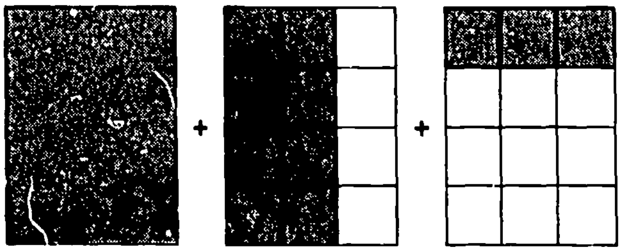
$$\frac{}{5} + \frac{}{2} = \frac{}{10} + \frac{}{} = \frac{}{10}.$$

d.



$$\frac{3}{} + \frac{}{8} = \frac{}{8} + \frac{}{8} = \frac{}{8}.$$

e.



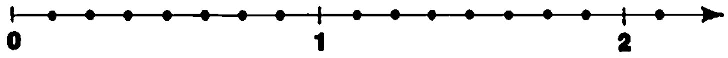
$$1\frac{}{3} + \frac{1}{} = 1\frac{8}{} + \frac{3}{} = 1\frac{}{}.$$

2. Starting at 0, draw two successive arrows to compute each sum. Express each computed sum in mixed numeral form.

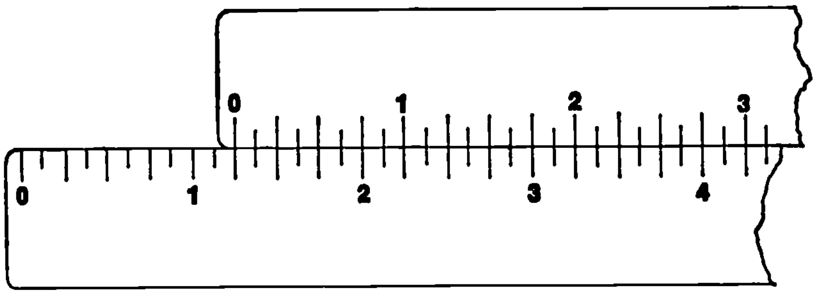
a. $\frac{2}{3} + \frac{1}{2} =$

b. $\frac{3}{4} + \frac{1}{3} =$

c. $1\frac{1}{4} + \frac{7}{8} =$



3. Read the slide rule illustrated below to supply the missing numerators and denominators for each addition sentence.



a. $1\frac{1}{} + 1 = 2\frac{}{}.$

c. $1\frac{1}{} + \frac{7}{} = 2\frac{}{8}.$

b. $1\frac{1}{} + \frac{0}{4} = 1\frac{1}{}.$

d. $1\frac{2}{} + 1\frac{}{8} = 3.$

FINDING COMMON DENOMINATORS TO COMPUTE SUMS

Two important prerequisites for learning efficient algorithms for computing sums with fractions that have different denominators are (1) the ability to rename rational numbers with equivalent fractions and (2) the ability to add with fractions that have the same denominator.

Exercises such as the following may be helpful in determining whether students are ready to learn efficient algorithms to compute sums of rational numbers.

Exercise Set 4

1. Supply the missing numerators and denominators.

a. $\frac{1}{3} = \frac{}{6} = \frac{3}{} = \frac{}{12}.$

c. $\frac{5}{4} = \frac{10}{} = \frac{}{20} = \frac{}{100}.$

b. $\frac{7}{8} = \frac{14}{} = \frac{}{24} = \frac{}{32}.$

d. $6\frac{3}{10} = 6\frac{}{20} = 6\frac{}{50}.$

2. Write *T* for True or *F* for False. If *False,* change the numerator or the denominator of the second fraction to make it an equivalent fraction.

a. $\frac{1}{2} = \frac{4}{10}$.

b. $\frac{3}{5} = \frac{60}{100}$.

c. $\frac{9}{8} = \frac{54}{40}$.

d. $2\frac{3}{4} = 2\frac{11}{12}$.

3. Supply the missing numerators to compute the sums.

a.

$$\begin{array}{r} \frac{3}{4} = \frac{3}{4} \\ + \frac{1}{2} = \frac{}{4} \\ \hline \frac{}{4} \end{array}$$

b.

$$\begin{array}{r} 1\frac{1}{4} = 1\frac{}{8} \\ + 1\frac{7}{8} = 1\frac{7}{8} \\ \hline 2\frac{}{8} = 3\frac{}{8} \end{array}$$

c.

$$\begin{array}{r} 4\frac{3}{10} = 4\frac{}{100} \\ + 3\frac{70}{100} = 3\frac{}{100} \\ \hline 7\frac{}{100} \end{array}$$

d.

$$\begin{array}{r} \frac{2}{5} = \frac{}{20} \\ + \frac{3}{4} = \frac{}{20} \\ \hline \frac{}{20} \end{array}$$

Let us assume that a student is now ready to "manipulate symbols" to compute sums like $\frac{2}{5} + \frac{3}{10}$, $\frac{2}{3} + \frac{1}{4}$, or any sums of the type $\frac{a}{b} + \frac{c}{d}$ where b and d are different denominators. Is there a particular sequence that is most effective? Probably not; the choice will depend on the teacher, the children, and the available materials. Some suggestions are given here.

Students who know how to compute $\frac{1}{4} + \frac{1}{4}$ usually can figure out for themselves how to compute $\frac{1}{2} + \frac{1}{4}$: "It's $\frac{3}{4}$ because $\frac{1}{2}$ is the same as $\frac{2}{4}$, and $\frac{2}{4} + \frac{1}{4}$ is $\frac{3}{4}$." Following such a response, students may be asked to compute other sums of the same type, such as $\frac{2}{5} + \frac{3}{10}$, or $\frac{3}{4} + \frac{7}{16}$, where it is necessary to rename only one of the addends. To compute the sum $\frac{2}{5} + \frac{3}{10}$, for example, the student may notice that one denominator is a multiple of the other; that is, 10 is a multiple of 5. This suggests that $\frac{2}{5}$ can be renamed with a fraction with 10 as its denominator, and using the generalization that multiplying the numerator and denominator by the same counting number produces an equivalent fraction, we have

$$\frac{2 \times n}{5 \times n} = \frac{?}{10}.$$

Since $5 \times 2 = 10$, $n = 2$ and $\frac{2}{5} = \frac{4}{10}$. The student may then compute as follows:

$$\frac{2}{5} + \frac{3}{10} = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}.$$

Some teachers may prefer to begin with an example like $\frac{2}{3} + \frac{1}{4}$, where neither denominator is a multiple of the other. One way to get students to focus on a pattern in the algorithm to be developed is described in this nonverbal sequence.[1]

$\begin{array}{r} \frac{2}{3} \\ + \frac{1}{4} \\ \hline \end{array}$	1. Teacher writes example on board.
$\begin{array}{r} \frac{2}{3} \\ + \frac{1}{4 \times 3} \\ \hline \end{array}$	2. Points to "3" (denominator of $\frac{2}{3}$), writes "× 3" to right of "4" (denominator of $\frac{1}{4}$).
$\begin{array}{r} \frac{2}{3} \\ + \frac{1 \times 3}{4 \times 3} \\ \hline \end{array}$	3. Points to "× 3." Holds up chalk, points to the right of "1" (numerator of $\frac{1}{4}$). If no student responds, teacher writes "× 3" to right of "1."
$\begin{array}{r} \frac{2}{3 \times 4} \\ + \frac{1 \times 3}{4 \times 3} \\ \hline \end{array}$	4. Points to "4" (denominator of $\frac{1}{4}$), writes "× 4" to right of "3" (denominator of $\frac{2}{3}$).
$\begin{array}{r} \frac{2 \times 4}{3 \times 4} \\ + \frac{1 \times 3}{4 \times 3} \\ \hline \end{array}$	5. Holds up chalk, nods to student to come up and write "× 4" to right of "2".

---

[1] In a nonverbal lesson, the teacher may work out a development on the chalkboard, leaving blanks at key spots. After setting the pattern, the teacher holds up the chalk and motions for someone to come to the board to fill in each blank. An incorrect response is erased. Occasionally the teacher may have to fill in a blank. As students "catch on," the number of blanks is increased until students can work out a complete example independently.

$$\begin{array}{r} \dfrac{2 \times 4}{3 \times 4} = \text{—} \\ + \dfrac{1 \times 3}{4 \times 3} = \dfrac{\quad}{12} \\ \hline \end{array}$$

6. Writes "=" and "—". Points to "4 × 3", points to space below "—", holds up chalk. If no response, writes "12".

$$\begin{array}{r} \dfrac{2 \times 4}{3 \times 4} = \dfrac{8}{12} \\ + \dfrac{1 \times 3}{4 \times 3} = \dfrac{3}{12} \\ \hline \dfrac{11}{12} \end{array}$$

7. Motions to students who are ready to fill in missing numerators and denominators of equivalent fractions for addends and for the computed sum.

Several other sums of the same type, such as $\frac{1}{2} + \frac{2}{5}$ and $\frac{2}{3} + \frac{3}{10}$, are computed in the same manner, but with students contributing more of the algorithm with each example. As soon as students "catch on" to the pattern, the multiplication notation may be omitted. For example, to compute the sum $\frac{2}{3} + \frac{3}{10}$, the teacher may begin with

$$\frac{2}{3} = \text{—}$$

$$\frac{3}{10} = \text{—},$$

and the next step would be to write "30" as the common denominator.

A mathematical description of the algorithm (which might appropriately be worked out by students in some classes) may be stated as follows:

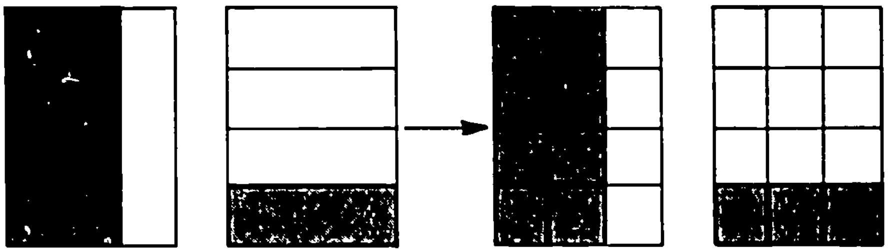
$$\begin{array}{r} \dfrac{a}{b} = \dfrac{a \times d}{b \times d} \\ + \dfrac{c}{d} = \dfrac{c \times b}{d \times b} \\ \hline \dfrac{(a \times d) + (c \times b)}{b \times d} \end{array}$$

Note that since multiplication is commutative, $d \times b = b \times d$, and the denominator of the sum appears as "$b \times d$". We can also change "$c \times b$" to "$b \times c$", and we have the following elegant generalization about addition of rational numbers:

For all rational numbers $\frac{a}{b}$ *and* $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{b \times d}.$$

To reinforce an understanding of this generalization for computing sums, a diagram similar to one shown previously can be used to illustrate the algorithm. For $\frac{2}{3} + \frac{1}{4}$, for example, one unit region is split vertically, another horizontally, to show $\frac{2}{3}$ and $\frac{1}{4}$, respectively. By superimposing the "splits" of each unit region on the other, each of the two unit regions becomes partitioned into the same number of congruent parts.

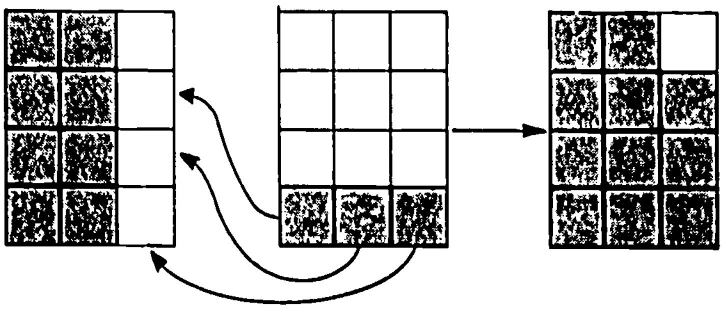


$$\frac{2}{3} \quad + \quad \frac{1}{4} \quad = \quad \frac{2 \times 4}{3 \times 4} \quad + \quad \frac{1 \times 3}{4 \times 3}.$$

Splitting the region showing thirds into fourths, horizontally, creates 4 × 3, or 12, congruent parts. Splitting the region showing fourths into thirds, vertically, creates 4 × 3, or 12, congruent parts.

The unit region with 2 thirds shaded now has 4 times as many parts, and 4 times as many of these parts are shaded; that is, 2×4 of the 3×4, or $\frac{2 \times 4}{3 \times 4}$, parts are shaded. The unit region with 1 fourth shaded now has 3 times as many parts, and 3 times as many of these parts are shaded; that is, $\frac{1 \times 3}{4 \times 3}$ of the unit region is shaded.

We now have 8 of 12 congruent parts shaded in one unit region and 3 of 12 congruent parts shaded in the other. Combining the shaded parts would show 11 of 12 congruent parts of a unit shaded.



The steps in developing the diagram for computing $\frac{2}{3} + \frac{1}{4}$ may be recorded in the following algorithm:

$$\frac{2}{3} + \frac{1}{4} = \frac{2 \times 4}{3 \times 4} + \frac{1 \times 3}{4 \times 3}$$

$$\frac{2}{3} + \frac{1}{4} = \frac{(2 \times 4) + (1 \times 3)}{3 \times 4}$$

$$\frac{2}{3} + \frac{1}{4} = \frac{8 + 3}{12}$$

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

The algorithm may be abbreviated in the form that appears in many textbooks.

$$\begin{array}{r} \frac{2}{3} = \frac{8}{12} \\ + \frac{1}{4} = \frac{3}{12} \\ \hline \frac{11}{12} \end{array}$$

Exercise Set 5

1. Draw diagrams of partitioned rectangular regions to illustrate how the product of the given denominators is used in computing these sums:

a. $\frac{2}{3} + \frac{1}{2}$ **b.** $\frac{2}{5} + \frac{1}{3}$

2. Use the abbreviated algorithm for the product-of-denominators method to compute these sums:

a. $\frac{3}{8} + \frac{5}{12}$ **b.** $\frac{13}{30} + \frac{7}{18}$

LEAST COMMON MULTIPLE

While it is always *possible* to use the product of two different denominators as a common denominator, other methods for determining a common denominator may be more convenient in some cases. We have already noted, for example, the case where one denominator is a multiple of the other, as in $\frac{2}{5} + \frac{3}{10}$. Here 10 is an easier common denominator to work with than 50.

To compute example **2a** in exercise set 5 the product 8 × 12, or 96, was used as a common denominator. Is there a common denominator for $\frac{3}{8} + \frac{5}{12}$ that is less than 96? To find out, we may consider the set of multiples for each of the denominators.

Multiples of 8: 8, 16, ㉔, 32, 40, ㊽, 56, 64, (72), · · ·
Multiples of 12: 12, ㉔, 36, ㊽, 60, (72), 84, 96, · · ·

The common multiples (which may serve as common denominators) are 24, 48, 72, and so on. The least common multiple (LCM) is 24. A simple technique for finding the least common multiple is to test each successive multiple of the larger denominator to see whether it is also a multiple of the smaller denominator. In the example above 12 is not a multiple of 8, but 24, the next multiple of 12 in the sequence, is also a multiple of 8; therefore, 24 is the least common multiple (LCM) of 8 and 12.

PRIME FACTORIZATION

To compute $\frac{13}{30} + \frac{7}{18}$ (exercise set 5, example **2b**) one may be tempted to use the product-of-denominators method to find a common denominator, rather than search for a multiple of 30 that is also a multiple of 18. There is another method, however, that is particularly appropriate in this case, a method that can be taught to upper-grade students who are familiar with prime numbers and prime factorizations.[2]

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[2] A prime number has exactly two divisors or *factors*. The number 1 is not prime, since it has only one factor, 1. The number 7 is prime; its factors are 1 and 7. The number 30 is not prime; it has more than two factors: 1, 2, 3, 5, 6, 10, 15, 30. Numbers that have more than two factors are *composite* numbers. Every composite number is a product of prime numbers, and as such, it has a unique *prime factorization*. For example, the prime factorization of 30 is 2 × 3 × 5, and it may be obtained in any one of the following ways (order doesn't matter, since multiplication is commutative and associative):

$$30 = 2 \times 15 = 2 \times 3 \times 5.$$
$$30 = 3 \times 10 = 3 \times 2 \times 5.$$
$$30 = 5 \times 6 \quad = 5 \times 2 \times 3.$$

$$\frac{13}{30} \to \frac{13}{2 \times 3 \times 5}$$
$$\frac{7}{18} \to \frac{7}{2 \times 3 \times 3}$$

1. Each denominator is expressed as a product of prime factors.

$$\frac{13}{30} \to \frac{13}{2 \times 3 \times 5} \to \frac{}{2 \times 3 \times 5 \times \mathit{3}}$$
$$\frac{7}{18} \to \frac{7}{2 \times 3 \times 3} \to \frac{}{2 \times 3 \times 3 \times \mathit{5}}$$

2. In order to obtain the least common denominator, all the prime factors that appear for either denominator will also have to appear in the common denominator. Why is

$$2 \times 3 \times 5$$

multiplied by 3? Why is

$$2 \times 3 \times 3$$

multiplied by 5? (Note that if 3 appears as a factor twice for one given denominator it must also appear as a factor at least twice for the common denominator.) The common denominator must have all the prime factors of *both* original denominators.

$$\frac{13}{30} \to \frac{13}{2 \times 3 \times 5} \to \frac{13 \times \mathit{3}}{2 \times 3 \times 5 \times \mathit{3}}$$
$$\frac{7}{18} \to \frac{7}{2 \times 3 \times 3} \to \frac{7 \times \mathit{5}}{2 \times 3 \times 3 \times \mathit{5}}$$

3. In each case, in order to obtain an equivalent fraction, the number multiplied by the denominator must also be multiplied by the numerator.

$$\frac{13}{30} = \frac{13 \times 3}{2 \times 3 \times 5 \times 3} = \frac{39}{90}$$
$$\frac{7}{18} = \frac{7 \times 5}{2 \times 3 \times 3 \times 5} = \frac{35}{90}$$
$$\frac{74}{90}$$

4. The steps for finding a common denominator by the prime-factorization method are incorporated in the algorithm shown here.

Exercise Set 6

1. Write a prime factorization for each number:

a. 12

b. 28

c. 70

d. 100

2. Use the prime-factorization method to find the least common multiple of each pair of numbers:

a. 6, 15

b. 15, 35

c. 16, 24

d. 28, 42

3. Use the product-of-denominators method to compute these sums:

a. $\frac{5}{6} + \frac{7}{10}$

b. $\frac{11}{42} + \frac{37}{48}$

c. $\frac{1}{6} + \frac{3}{8} + \frac{3}{10}$

4. Use the successive-multiples method to compute the sums in exercise **3** above.

5. Use the prime-factorization method to compute the sums in exercise **3.**

6. Which of the above three methods would you use to compute $\frac{5}{28} + \frac{7}{30} + \frac{9}{35}$? Use your choice to compute the sum.

PROPERTIES OF ADDITION OF RATIONAL NUMBERS

If students are aware of properties of addition of whole numbers, it would be reasonable for them to assume that these same properties should also apply to addition of rational numbers, since the set of whole numbers is included in the set of rational numbers. Familiar properties of addition extended to apply to rational numbers are listed below:

WHOLE NUMBERS	RATIONAL NUMBERS
Closure	
If a *and* b *are whole numbers, there is a unique whole number* $a + b$ *which is their sum.*	*If* $\frac{a}{b}$ *and* $\frac{c}{b}$ *are rational numbers, there is a unique rational number* $\frac{a+c}{b}$ *which is their sum.*[3]
Commutativity	
If a *and* b *are whole numbers, then* $a + b = b + a$.	*If* $\frac{a}{b}$ *and* $\frac{c}{b}$ *are rational numbers, then* $\frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}$.
Associativity	
If a, b, *and* c *are whole numbers, then* $(a + b) + c = a + (b + c)$.	*If* $\frac{a}{b}$, $\frac{c}{b}$, *and* $\frac{d}{b}$ *are rational numbers, then* $\left(\frac{a}{b} + \frac{c}{b}\right) + \frac{d}{b} = \frac{a}{b} + \left(\frac{c}{b} + \frac{d}{b}\right)$.
Additive Identity Element	
If a *is a whole number, then* $a + 0 = 0 + a = a$.	*If* $\frac{a}{b}$ *is a rational number, then* $\frac{a}{b} + 0 = 0 + \frac{a}{b} = \frac{a}{b}$.

Experiences can be provided that will tend to confirm the assumption that properties of addition of whole numbers will hold for addition of rational numbers.

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[3] Since it is always possible to represent two or more rational numbers with fractions that have a common denominator, we shall assume this has been done.

CLOSURE PROPERTY FOR ADDITION

The sum of any pair of whole numbers is always a whole number. (In contrast, the difference of any pair of whole numbers is not always a whole number; e.g., $2 - 3$ is not a whole number.) The sum of two whole numbers can always be associated with two successive moves to the right, starting from 0, on the number line. The result is a whole number corresponding to a particular point on the number line. Students will find that the sum of any two nonnegative rational numbers can be represented on the number line in the same manner.

We can also draw on two basic generalizations about rational numbers to show that the set of rational numbers is closed under addition:

1. *Any number that can be represented by a fraction* $\frac{a}{b}$ *where* a *and* b *are whole numbers,* $b \neq 0$, *is a rational number.*

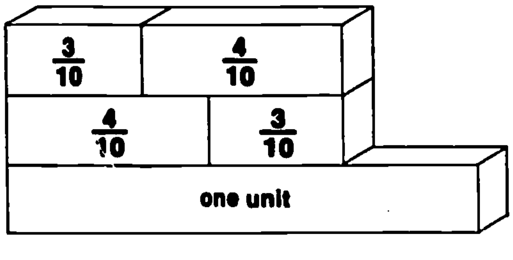
2. *For any rational numbers* $\frac{a}{b}$ *and* $\frac{c}{b}$,

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$

Since a and c are whole numbers, then $a + c$ is a whole number, by the closure property of addition for whole numbers. Since b is also a whole number (not 0), it follows that $\frac{a+c}{b}$ is a rational number, by statement 1 above, and the set of nonnegative rational numbers is closed under addition.

ADDITION IS COMMUTATIVE

Manipulative aids can be used to show that changing the order of two addends does not affect the sum. For example, number blocks make it



clear that $\frac{3}{10} + \frac{4}{10} = \frac{4}{10} + \frac{3}{10}$. After working several examples, students may conclude that changing the order of any two number blocks will not change their combined length, even when the number blocks represent rational numbers instead of whole numbers. Experience with partitioned regions, parts of sets, and number-line diagrams will serve to reinforce the principle that addition of rational numbers is commutative.

With more advanced students, a formal proof that $\frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}$ can be developed as follows:

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$ by addition of rational numbers;

$$= \frac{c + a}{b}$$ because addition of whole numbers is commutative;

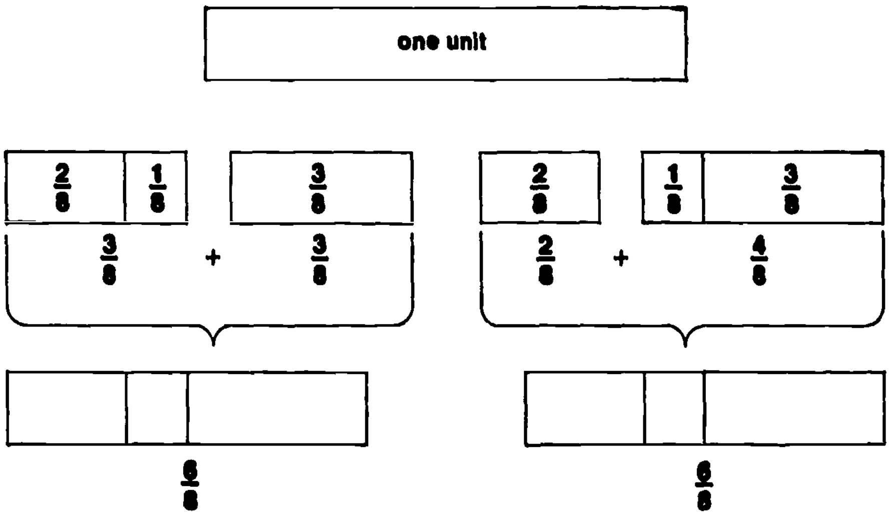
$$= \frac{c}{b} + \frac{a}{b}$$ by addition of rational numbers.

ADDITION IS ASSOCIATIVE

The number-block illustration below shows that

$$\left(\frac{2}{8} + \frac{1}{8}\right) + \frac{3}{8} = \frac{2}{8} + \left(\frac{1}{8} + \frac{3}{8}\right);$$

the way of grouping the addends does not affect the sum. Other concrete materials and diagrams can be used with a variety of examples to help develop the generalization that the grouping of addends does not affect the sum.



At a later stage, the generalization can be established deductively as follows:

$$\left(\frac{a}{b} + \frac{c}{b}\right) + \frac{d}{b} = \frac{a + c}{b} + \frac{d}{b} \qquad \text{by addition of rational numbers;}$$

$$= \frac{(a + c) + d}{b} \qquad \text{by addition of rational numbers;}$$

$$= \frac{a + (c + d)}{b} \qquad \text{because addition of whole numbers is associative;}$$

$$= \frac{a}{b} + \frac{c + d}{b} \qquad \text{by addition of rational numbers;}$$

$$= \frac{a}{b} + \left(\frac{c}{b} + \frac{d}{b}\right) \qquad \text{by addition of rational numbers.}$$

The commutative and associative properties used together make possible the rearrangement of addends in any combination without affecting the sum. Because of this "rearrangement principle" the grouping symbols (parentheses and brackets) may be omitted in an expression of a sum of more than two addends without creating any ambiguity. For example,

$$\frac{3}{16} + \left[\frac{1}{16} + \left(\frac{4}{16} + \frac{7}{16}\right)\right]$$

may be expressed as

$$\frac{3}{16} + \frac{1}{16} + \frac{4}{16} + \frac{7}{16},$$

which in turn may be rearranged as

$$\left(\frac{3}{16} + \frac{7}{16}\right) + \left(\frac{4}{16} + \frac{1}{16}\right).$$

No matter which way the addends are regrouped or reordered, the correctly computed sum will be $\frac{15}{16}$.

The rearrangement principle can be illustrated convincingly with materials such as Unifix or Stern Blocks or Cuisenaire Rods. Elementary students (and teachers) who sometimes find it difficult to make a distinction between associativity and commutativity will find it easier to deal with the more general rearrangement principle instead.

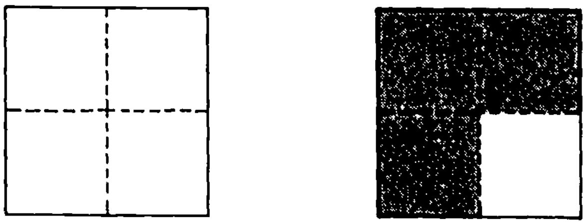
ADDITION PROPERTY OF 0

The number line (where a move of 0 is equivalent to "jumping in place"), as well as number blocks and diagrams, can be used to develop the generalization:

If 0 *is one of two addends, the sum is equal to the other addend.*

Because it has this "neutral" effect, 0 is called the *identity element for addition*, or the *additive identity*.

A diagram will illustrate an example of the addition property of 0 as applied to addition of rational numbers. How many fourths are shaded in both unit regions?



$$\frac{0}{4} + \frac{3}{4} = \frac{3}{4}.$$

Since 0 is a rational number, it can be represented by a fraction. The set of fractions for 0 is $\left\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \cdots\right\}$, so we have $0 = \frac{0}{b}$, where b is any whole number except 0. To establish that 0 is the identity element for addition of rational numbers, we draw upon the addition property of 0 for whole numbers, as follows:

$\frac{a}{b} + \frac{0}{b} = \frac{a + 0}{b}$	by addition of rational numbers;
$= \frac{a}{b}$	because $a + 0 = a$, by the addition property of 0 for whole numbers.
$\frac{0}{b} + \frac{a}{b} = \frac{0 + a}{b}$	by addition of rational numbers;
$= \frac{a}{b}$	because $0 + a = a$, by the addition property of 0 for whole numbers.

Since $\frac{a}{b} + \frac{0}{b} = \frac{0}{b} + \frac{a}{b}$ (because addition is commutative) and since $\frac{0}{b} = 0$, we have the general statement of the addition property of 0:

$$\frac{a}{b} + 0 = 0 + \frac{a}{b} = \frac{a}{b}.$$

OTHER PROPERTIES OF ADDITION

Two other properties of addition are noted briefly here.

1. The *well-defined*, or *uniqueness*, property of addition of rational numbers:

> *If* $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, *and* $\frac{g}{h}$ *are rational numbers,* $\frac{a}{b} = \frac{c}{d}$, *and* $\frac{e}{f} = \frac{g}{h}$, *then* $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{g}{h}$; *that is, sums do not depend on the particular fractions used to name the numbers—only on the numbers themselves.*

The uniqueness property of addition of rational numbers is often useful in solving equations, especially when negative numbers are involved.

2. The *cancellation* property of addition of rational numbers:

> *If* $\frac{a}{b}, \frac{c}{d}$, *and* $\frac{e}{f}$ *are rational numbers and if* $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$, *then* $\frac{a}{b} = \frac{c}{d}$.

This property is also useful in solving equations. For example: If $\frac{a}{b} + \frac{2}{3} = \frac{7}{8} + \frac{2}{3}$, then $\frac{a}{b} = \frac{7}{8}$.

At the elementary school level, an awareness of the uniqueness and cancellation properties can be developed informally through experiences with a balance scale.

Exercise Set 7

1. Which addition property applies in each of the following?

a. $\left(\frac{15}{16} + \frac{7}{16}\right) + \frac{3}{16} = \frac{15}{16} + \left(\frac{7}{16} + \frac{3}{16}\right).$

b. $\frac{3}{4} + \left(\frac{3}{10} + \frac{1}{4}\right) = \left(\frac{3}{10} + \frac{1}{4}\right) + \frac{3}{4}.$

c. $\frac{7}{8} + \left(\frac{0}{8} + \frac{1}{8}\right) = \frac{7}{8} + \left(\frac{1}{8} + \frac{0}{8}\right).$

d. $\frac{2}{3} + \frac{3}{4} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3} + \frac{1}{3} + \frac{3}{4} + \frac{1}{2}.$

2. Rearrange the addends to compute each sum quickly.

a. $\left(\frac{2}{3} + \frac{93}{100}\right) + \frac{7}{100}$

b. $\frac{5}{100} + \left(\frac{247}{100} + \frac{95}{100}\right)$

c. $\left(\frac{9}{10} + \frac{7}{8}\right) + \left(\frac{1}{10} + \frac{1}{8}\right)$

d. $\frac{13}{8} + \frac{13}{5} + \frac{11}{8} + \frac{12}{5}$

3. Which addition property is used in each step?

a. $\frac{3}{16} + \left[\frac{1}{16} + \left(\frac{4}{16} + \frac{7}{16}\right)\right] = \frac{3}{16} + \left[\frac{1}{16} + \left(\frac{7}{16} + \frac{4}{16}\right)\right]$

b. $= \frac{3}{16} + \left[\left(\frac{7}{16} + \frac{4}{16}\right) + \frac{1}{16}\right]$

c. $= \frac{3}{16} + \left[\frac{7}{16} + \left(\frac{4}{16} + \frac{1}{16}\right)\right]$

d. $= \left(\frac{3}{16} + \frac{7}{16}\right) + \left(\frac{4}{16} + \frac{1}{16}\right).$

MIXED NUMERALS AND SUMS

In a mixed numeral (sometimes referred to less precisely as a "mixed number") one part of the numeral names a whole number and the other part is a fraction for a rational number, as in the mixed numeral "$2\frac{3}{8}$." The symbol "$2\frac{3}{8}$" is by agreement an abbreviation of an expression for the sum $2 + \frac{3}{8}$. Since the two expressions are equivalent, we have

$$2\frac{3}{8} = 2 + \frac{3}{8}, \qquad \text{and} \qquad 2 + \frac{3}{8} = 2\frac{3}{8}.$$

An application in addition of rational numbers occurs as follows:

$2\frac{3}{8} + 1\frac{6}{8} = 2 + \frac{3}{8} + 1 + \frac{6}{8}$ by agreement;

$= (2 + 1) + \left(\frac{3}{8} + \frac{6}{8}\right)$ because addition is associative and commutative (rearrangement principle);

$= 3 + \frac{9}{8}$ by addition;

$= 3\frac{9}{8}$ by agreement.

If desired, $3\frac{9}{8}$ may be renamed as follows:

$3\frac{9}{8} = 3 + \frac{9}{8}$ by agreement;

$= 3 + \left(\frac{8}{8} + \frac{1}{8}\right)$ renaming $\frac{9}{8}$ as the sum of $\frac{8}{8}$ and $\frac{1}{8}$;

$= 3 + \left(1 + \frac{1}{8}\right)$ because $\frac{a}{a} = 1$ for $a \neq 0$;

$= (3 + 1) + \frac{1}{8}$ because addition is associative;

$= 4 + \frac{1}{8}$ by addition;

$= 4\frac{1}{8}$ by agreement.

Awareness of a mixed numeral as an abbreviation of an expression for a sum will be especially useful when computing differences with mixed numerals, as will be shown in the chapter "Subtraction of Rational Numbers."

Exercise Set 8

1. Fill in the blanks in each renaming sequence:

a. $1\frac{7}{5} = 1 + \frac{7}{\quad} = \frac{\quad}{5} + \frac{7}{\quad} = \frac{\quad + 7}{5} = \frac{\quad}{5}.$

b. $\frac{49}{4} = \frac{\quad + 1}{4} = \frac{\quad}{4} + \frac{1}{4} = 12 + \frac{1}{\quad} = 12\frac{1}{\quad}.$

2. Fill in the blanks, and justify each step in computing the sum:

a. $4\frac{2}{3} + 5\frac{2}{3} = 4 + \frac{\quad}{3} + 5 + \frac{\quad}{3}$

b. $= 4 + 5 + \frac{\quad}{3} + \frac{\quad}{3}$

c. $= 9 + \frac{\quad}{3}$

d. $= 9 + \frac{\quad}{3} + \frac{1}{\quad}$

e. $= 9 + 1 + \frac{1}{\quad}$

f. $= 10 + \frac{1}{\quad}$

g. $= 10\frac{1}{\quad}.$

SUMMARY

1. Important prerequisites for an understanding of addition of rational numbers are an understanding of addition and its properties for the system of whole numbers, an understanding of the association of a rational number with a set of equivalent fractions, and the ability to find equivalent fractions for given fractions. With this prerequisite knowledge and with the guided use of various manipulative and pictorial aids, students can participate in developing basic generalizations about addition of rational numbers.

2. Most of the difficulties that students have with addition of rational numbers arise when computing sums with fractions that have different denominators. In such cases it is always possible to rename the rational

numbers with fractions that have a common denominator, in order to follow a generalization that provides a basic pattern for computing a sum:

For any rational numbers $\frac{a}{b}$ *and* $\frac{c}{b}$,

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

A common denominator is a common multiple of two or more denominators. A simple method for finding a common denominator is to multiply the given denominators. This leads to another basic generalization about addition of rational numbers:

$$\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{b \times d}.$$

When different denominators have a common factor, it may be more efficient to use the least common multiple (LCM) as a common denominator. Two ways to find the LCM are (1) comparing successive multiples of the different denominators and (2) using the prime-factorization method.

3. Addition of rational numbers has the same properties as does addition of whole numbers: It has closure; it is commutative and associative; it has the same identity element, 0; it has the uniqueness and cancellation properties. Students can develop an understanding of these properties and of the algorithms for computing sums of rational numbers by working with concrete aids, and diagrams, to help them interpret the mathematical symbols they use in working with addition of rational numbers.

Joseph Moray

4

SUBTRACTION OF RATIONAL NUMBERS

1. **How is subtraction of rational numbers related to addition of rational numbers?**
2. **How does an understanding of subtraction of whole numbers help students to understand subtraction of rational numbers?**
3. **What are some difficulties students may encounter when computing differences of rational numbers?**
4. **What are some efficient algorithms for computing the difference of two rational numbers?**

A child who has learned to add rational numbers isn't likely to need any formal introduction to subtraction of rational numbers. If he knows that $\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$, for example, he will probably assume that $\frac{4}{5} - \frac{1}{5} = \frac{3}{5}$. Such an assumption makes sense, since it draws on the relationship of subtraction to addition in the whole-number system. If we draw on this relationship, we can see how meaning can be developed for the difference of two rational numbers.

In the system of whole numbers, addition assigns to the pair of numbers a and b the sum $a + b$. In the sentence $a + b = c$, a and b are called "addends" of the sum $a + b$. To take a specific example, addition assigns the sum 4 + 5, or 9, to the pair of numbers 4 and 5. The numbers 4 and 5 are addends of the sum.

$$\underbrace{\overset{\text{addend}}{4} \quad + \quad \overset{\text{addend}}{5}}_{\text{sum}} = \underset{\substack{\uparrow\\ \text{sum}\\ \text{(computed)}}}{9.}$$

Subtraction assigns to the pair of whole numbers c and b the missing addend in the sentence $\square + b = c$. The unknown, or missing, addend is the difference of the sum, c, and the given addend, b.

$$\underset{\substack{\nearrow \\ \text{missing addend}}}{\square} \quad + \quad \underset{\substack{\uparrow \\ \text{addend}}}{b} \quad = \quad \underset{\substack{\uparrow \\ \text{sum}}}{c.}$$

The difference of c and b is expressed as $c - b$, and we have

$$\square \qquad = \quad c \quad - \quad b.$$

(Until negative numbers are introduced, the expression "$c - b$" has no meaning if $c < b$. For example, in the sentence $\square = 3 - 7$, the missing addend is the difference of the sum, 3, and the given addend, 7. Since the missing addend plus the given addend equals the sum, we have $\square + 7 = 3$, but there is no whole-number solution for this sentence.)

Two subtraction sentences can be formed from the addition sentence $a + b = c$:

$$c - b = a, \qquad \text{and} \qquad c - a = b.$$

Thus, if $4 + 5 = 9$, then

$$9 - 5 = 4, \qquad \text{and} \qquad 9 - 4 = 5.$$

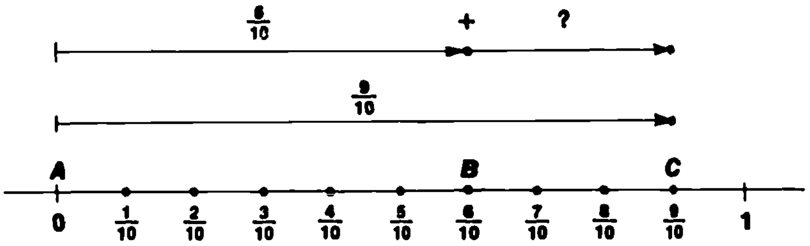
These sentences illustrate the relationship of subtraction to addition: the sum minus one addend is equal to the other addend. Further, subtracting a number is the inverse of adding that number. If we start with 4 and add 5, and then subtract 5 from that sum, we get back to 4:

$$(4 + 5) - 5 = 4.$$

EXTENSION OF SUBTRACTION TO RATIONAL NUMBERS

Working some problems on the number line will show that basic ideas about subtraction of whole numbers can be extended to the system of rational numbers. Two examples follow, each picturing a different interpretation of subtraction.

1. Along a highway, traveling in the same direction, the distance from A to B is $\frac{6}{10}$ of a mile, and the distance from A to C is $\frac{9}{10}$ of a mile. What is the distance from B to C?

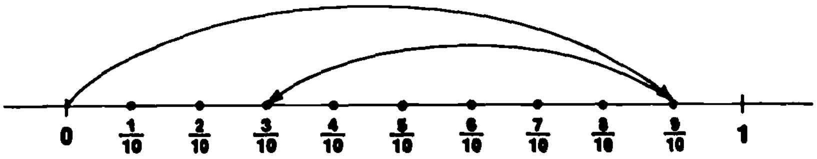


The above diagram shows one of the most useful interpretations of a difference of two numbers. The "arrows," or *vectors*, describe motions which in this case represent a given addend and a sum. The vector from the point for the given addend to the point for the sum (also given) determines the missing addend. The missing addend in this problem can be found by counting the number of tenths from $\frac{6}{10}$, the address for point *B*, to $\frac{9}{10}$, the address for point *C*.

Some students may count points instead of segments and begin counting at the wrong place. For example, a student may say "1, 2, 3, 4 tenths," as he points to the marks for $\frac{6}{10}$, $\frac{7}{10}$, $\frac{8}{10}$, and $\frac{9}{10}$, and conclude, "that's $\frac{4}{10}$, so $\frac{6}{10} + \frac{4}{10} = \frac{9}{10}$." If this occurs, it may help to have the student move a finger along the path from the point for $\frac{6}{10}$ toward the point for $\frac{7}{10}$, and say "$\frac{1}{10}$" as he arrives at the point for $\frac{7}{10}$. In this manner, the move from $\frac{6}{10}$ to $\frac{9}{10}$ will be shown as a move of $\frac{3}{10}$.

A move on the number line from a point for the given addend to a point for the sum, to determine the missing addend, can be associated with the difference of any two rational numbers. This technique will be especially helpful when students learn to subtract negative numbers, as will be pointed out later.

2. Ferdinand the Frog made two jumps. He first made a jump of $\frac{9}{10}$ of a meter, then he jumped back along the same path $\frac{6}{10}$ of a meter. How far is he from where he started? We can look at this problem as starting with the interval from 0 to $\frac{9}{10}$ and then retracing, or "taking away," the interval from $\frac{3}{10}$ to $\frac{9}{10}$ (an interval of $\frac{6}{10}$). This ties in with the take-away approach to subtraction of whole numbers, where the difference is associated with what is left after some objects are removed from a set.



In this problem the given numbers are a sum and one of two addends. The other addend is missing. If we let r represent the missing addend, then $r = \frac{9}{10} - \frac{6}{10}$. The number-line illustration shows that the point for the difference of $\frac{9}{10}$ and $\frac{6}{10}$ is also the point for $\frac{3}{10}$, so $r = \frac{3}{10}$.

The technique of moving to the right when adding and to the left when subtracting will work with positive numbers, but, as you will see later, it will not work with negative numbers. For this reason, generalizations like "Always move to the left when subtracting a number on the number line" should be avoided. It is better to be specific: "To subtract $\frac{6}{10}$, move $\frac{6}{10}$ to the left." (Students will learn later that subtracting $-\frac{6}{10}$ will mean a move of $\frac{6}{10}$ to the *right*.)

Exercise Set 1

1. Rewrite each sentence in the following form: missing addend equals sum minus given addend.

a. $\frac{2}{3} + n = \frac{10}{3}$.

b. $n + \frac{7}{8} = \frac{12}{8}$.

c. $\frac{4}{5} = \frac{1}{5} + n$.

d. $\frac{9}{10} - n = \frac{4}{10}$.

2. Make a vector diagram, with one vector for the given addend and another vector for the sum and then a vector for the missing addend, to compute each difference.

a. $\frac{7}{8} - \frac{3}{8}$

b. $1\frac{1}{4} - \frac{3}{4}$

3. Make a diagram showing two successive jumps on the number line to compute each difference.

a. $\frac{7}{8} - \frac{3}{8}$

b. $\frac{9}{5} - \frac{2}{5}$

4. For each sentence, determine whether n is a *sum* or an *addend.*

a. $n + \frac{1}{2} = \frac{3}{2}.$

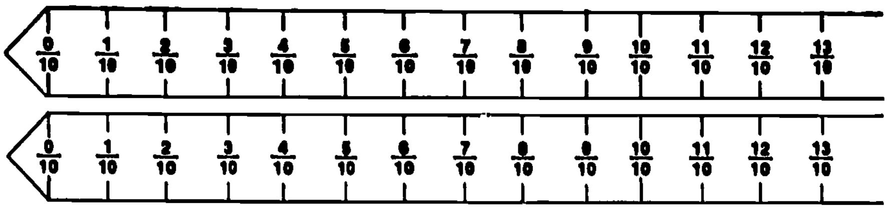
b. $n = \frac{3}{4} - \frac{1}{4}.$

c. $n - \frac{3}{8} = \frac{5}{8}.$

d. $\frac{7}{8} = n - \frac{3}{8}.$

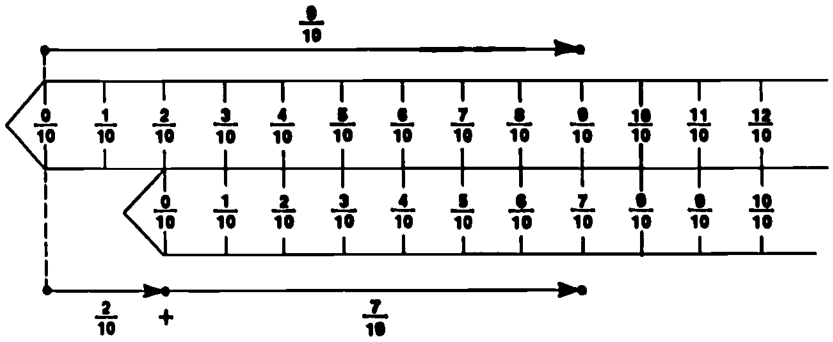
The use of a vector diagram to compute a difference has one slight disadvantage: the length of the move from the point for the given addend to the point for the sum is not revealed automatically. If an automatic reading is desired, it can be obtained by the use of parallel number lines, using a slide-rule technique. A slide rule for computing sums and differences can be constructed by marking two strips with the same scale. The slide rule works on the principle of the missing-addend approach to subtraction.

For example, to compute the difference $\frac{9}{10} - \frac{2}{10}$, we can use two number lines, marked off in tenths. The missing-addend approach to



subtraction tells us that the given addend plus the missing addend is equal to the sum. That is, $\frac{2}{10} + \square = \frac{9}{10}$. Actually the difference $\left(\frac{9}{10} - \frac{2}{10}\right)$ *is* the missing addend, since $\frac{2}{10} + \left(\frac{9}{10} - \frac{2}{10}\right) = \frac{9}{10}$. What we are really trying to obtain, then, is a simpler name for the number $\frac{9}{10} - \frac{2}{10}$.

The slide rule provides a model for the missing-addend approach; at the same time, it automatically produces a simpler name for the difference. Picture an "arrow," or vector, from the point for $\frac{0}{10}$ to the point for $\frac{9}{10}$, the sum. The difference, or missing addend, would then be associated with a vector from the point for $\frac{2}{10}$ to the point for $\frac{9}{10}$. We could count the number of tenths from $\frac{2}{10}$ to $\frac{9}{10}$, but by sliding the lower number line until the point for $\frac{0}{10}$ is directly under the point for $\frac{2}{10}$ on the upper number line, the computed difference, $\frac{7}{10}$, is indicated directly below the given sum, $\frac{9}{10}$. (Of course the roles of the upper number line and the lower number line may be interchanged.)



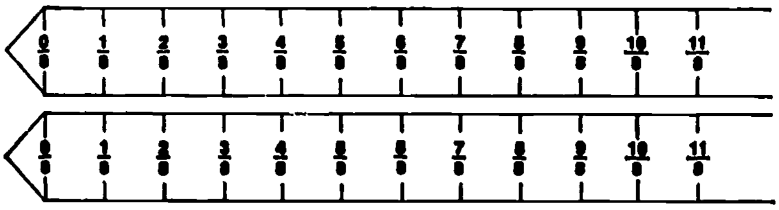
Nothing we have done so far is different in procedure or in principle from what would be done if the number lines were marked off with points for whole numbers instead of with points for rational numbers between the whole numbers. Essentially, what we have done is to develop or reinforce the notion that the basic ideas about the subtraction of whole numbers apply to the subtraction of rational numbers as well. In particular, we have extended the missing-addend interpretation of subtraction to include the subtraction of nonnegative rational numbers; that is, subtraction assigns to the pair of rational numbers $\frac{a}{b}$ and $\frac{c}{b}$ the missing addend in the sentence

$$\frac{c}{b} + \square = \frac{a}{b}.$$

In the above sentence $\frac{a}{b}$ is the sum and $\frac{c}{b}$ is the given addend. The missing addend is the *difference* of $\frac{a}{b}$ and $\frac{c}{b}$, or $\frac{a}{b} - \frac{c}{b}$. The missing addend, or difference, is a nonnegative rational number only if $\frac{a}{b}$ is not less than $\frac{c}{b}$. For example, $\frac{2}{3} - \frac{1}{3}$ is a *nonnegative* rational number because $\frac{2}{3}$ is not less than $\frac{1}{3}$; and, as students will learn later, $\frac{2}{3} - \frac{4}{3}$ is a *negative* rational number because the sum, $\frac{2}{3}$, is less than the given addend, $\frac{4}{3}$.

Exercise Set 2

1. Make a slide rule out of two strips of paper or cardboard. Mark off in eighths, as shown:



Use your slide rule to compute the following differences.

a. $\frac{7}{8} - \frac{2}{8}$ **c.** $\frac{5}{8} - \frac{1}{8}$

b. $\frac{3}{8} - \frac{0}{8}$ **d.** $\frac{9}{8} - \frac{5}{8}$

2. In which of the following is the number for the frame a nonnegative number?

a. $\frac{7}{3} + \square = \frac{9}{3}.$ **c.** $\frac{2}{8} = \frac{3}{8} + \square.$

b. $\frac{1}{4} = \square - \frac{3}{4}.$ **d.** $\square + \frac{1}{5} = \frac{1}{5}.$

3. Rewrite each sentence to conform to the following pattern:

missing addend = sum − given addend.

a. $\frac{1}{3} + \square = \frac{5}{3}$.

b. $\square + \frac{2}{10} = \frac{11}{10}$.

c. $\frac{5}{8} - \square = \frac{3}{8}$.

d. $\frac{14}{100} = \frac{100}{100} - \square$.

4. Rewrite each sentence to conform to the following pattern:

given addend + missing addend = sum.

a. $\frac{8}{5} - \frac{5}{5} = \square$.

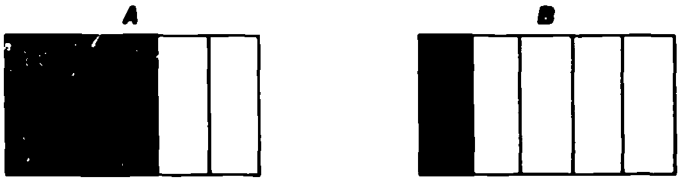
b. $\frac{11}{16} - \square = \frac{5}{16}$.

c. $\square = \frac{8}{10} - \frac{5}{10}$.

d. $\frac{67}{100} = \frac{99}{100} - \square$.

Folded paper, sets of rods or blocks, units cut up into parts of the same size, and drawings of partitioned regions are among the materials that can provide experiences related to the subtraction of rational numbers.

Just as we did for addition, we can use drawings of regions to develop generalizations that will be useful in subtraction computation. Consider the shaded parts of the two unit regions pictured below. How much more of *B* would have to be shaded in order to match the shaded part of *A*? This question can be expressed by either of these equivalent mathematical sentences:



$$\frac{1}{5} + \square = \frac{3}{5} \quad \text{or} \quad \square + \frac{1}{5} = \frac{3}{5}.$$

Experience with subtraction so far tells students that the missing addend is the difference of the sum and the given addend:

$$\overset{\substack{\textbf{missing addend}\\ \searrow}}{\square} + \overset{\substack{\textbf{given addend}\\ \downarrow}}{\frac{1}{5}} = \overset{\substack{\textbf{sum}\\ \downarrow}}{\frac{3}{5}}.$$

$$\underset{\substack{\nearrow\\ \textbf{missing addend}}}{\square} = \underset{\substack{\uparrow\\ \textbf{sum}}}{\frac{3}{5}} - \underset{\substack{\uparrow\\ \textbf{given addend}}}{\frac{1}{5}}.$$

Some students will compute the missing addend by recalling an addition, rather than a subtraction, combination: "One-fifth plus how many fifths equals three-fifths?" rather than "Three-fifths minus one-fifth equals how many fifths?"

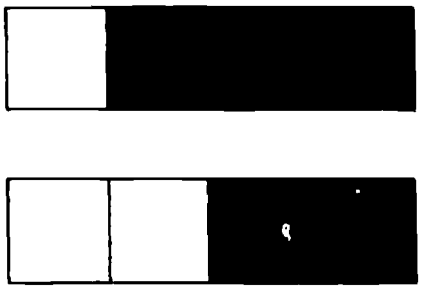
Regions can also be used to illustrate the take-away interpretation of subtraction. "How much more of *A* is shaded than *B*?" can be answered by removing or covering the amount of shading in *A* that is in *B*. That is, the difference $\frac{3}{5} - \frac{1}{5}$ can be computed by folding back or covering a shaded fifth. Then 3 — 1, or 2, fifths of the unit appear shaded, and the student finds that

$$\frac{3}{5} - \frac{1}{5} = \frac{3-1}{5} = \frac{2}{5}.$$

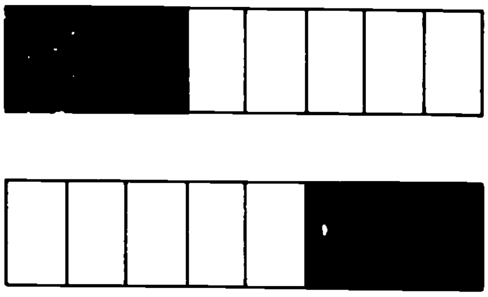
Exercise Set 3

1. Match the shaded parts in each pair of unit regions, and write a related subtraction sentence (in the form $\frac{a}{b} - \frac{c}{b} = \frac{d}{b}$).

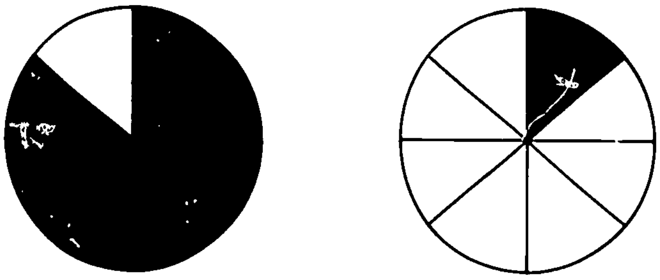
a.



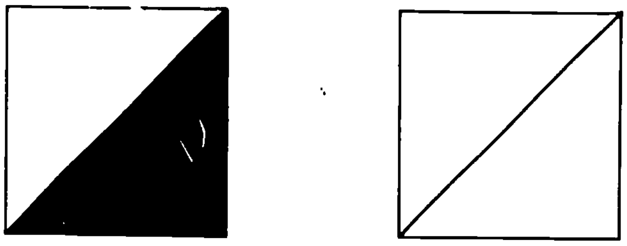
b.



c.



d.



2. Fill in the blanks to complete the steps in computing each difference:

a. $\frac{13}{10} - \frac{7}{10} = \frac{13 - \underline{\quad}}{10} = \frac{\underline{\quad}}{10}.$

b. $\frac{15}{16} - \frac{\underline{\quad}}{16} = \frac{\underline{\quad} - 4}{16} = \frac{\underline{\quad}}{16}.$

c. $\frac{74}{100} - \frac{\underline{\quad}}{100} = \frac{74 - \underline{\quad}}{100} = \frac{59}{100}.$

d. $1\frac{1}{4} - \frac{2}{4} = \frac{\underline{\quad}}{4} - \frac{2}{4} = \frac{\underline{\quad} - 2}{4} = \frac{\underline{\quad}}{4}.$

So far, we have been computing differences with pairs of fractions that have the same denominator. It is suggested that this practice be followed with students until basic ideas about subtraction of rational numbers have been established. Then computing with fractions with different denominators should be no more difficult for subtraction than for addition (except for a special case, which we shall deal with later in this chapter).

Recall that in order to compute the sum of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, if $b \neq d$, it is necessary to represent the numbers by fractions that have a common denominator. A sequence of steps for computing the difference of two such rational numbers is shown in the example below:

$$\begin{array}{l}
\frac{2}{3} \rightarrow \frac{2}{3 \times 4} \rightarrow \frac{2 \times 4}{3 \times 4} \rightarrow \frac{8}{12} \\
\frac{1}{4} \rightarrow \frac{1}{4 \times 3} \rightarrow \frac{1 \times 3}{4 \times 3} \rightarrow \frac{3}{12} \\
\hline
\hfill \frac{5}{12}
\end{array}$$

In the first step the product of the two given denominators (3×4, or 4×3) is the common denominator. In the next step the procedure for finding equivalent fractions is followed: since the denominator of $\frac{2}{3}$ is multiplied by 4, then the numerator must also be multiplied by 4 to get the equivalent fraction $\frac{8}{12}$. An equivalent fraction, also with the denominator 12, is obtained from $\frac{1}{4}$ by multiplying numerator and denominator by 3.

A general statement of this algorithm follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{a \times d}{b \times d} - \frac{b \times c}{b \times d} = \frac{(a \times d) - (b \times c)}{b \times d}.$$

Exercise Set 4

1. Use the algorithm form shown above to compute each difference. (The first example has been partially completed.)

a. $\frac{2}{3} - \frac{1}{2} = \frac{2 \times \quad}{3 \times 2} - \frac{3 \times 1}{\quad \times 2} = \frac{(2 \times \quad) - (3 \times \quad)}{3 \times 2} = \frac{4 - \quad}{6} = \frac{\quad}{6}.$

b. $\frac{3}{4} - \frac{3}{5} =$

c. $\frac{9}{10} - \frac{2}{3} =$

2. Compute the difference $\frac{7}{30} - \frac{5}{24}$, using two different methods for finding the common denominator (see chap. 3). Express result in lowest terms.

a. The product-of-denominators method

b. The least-common-multiple method

3. Compute the following differences. (Choose any method.)

a. $\frac{1}{2} - \frac{3}{10}$

b. $\frac{17}{20} - \frac{17}{50}$

c. $\frac{15}{24} - \frac{7}{18}$

d. $\frac{15}{32} - \frac{7}{40}$

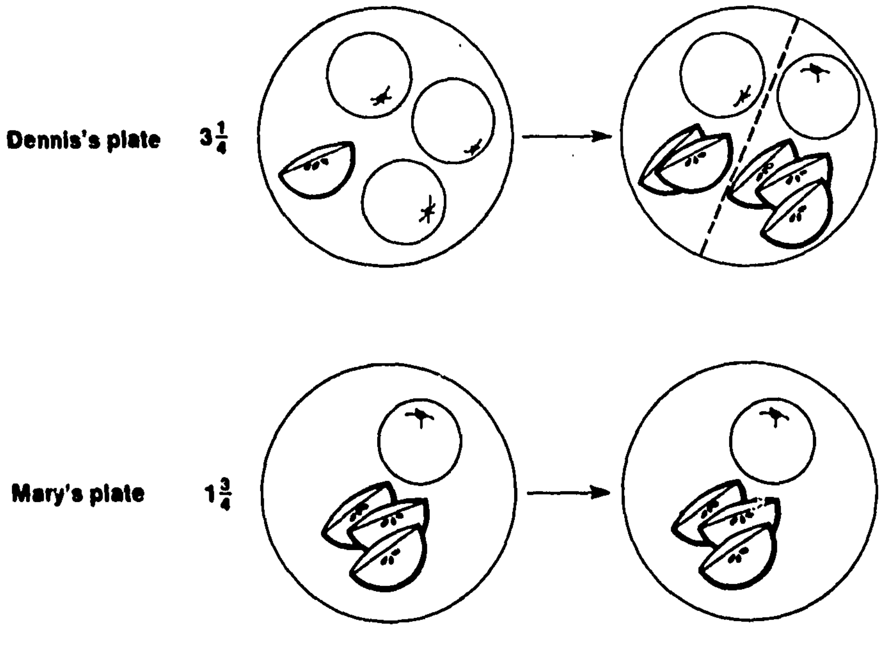
Students who are proficient in adding with fractions also tend to do well when subtracting with fractions. The processes for computing sums and computing differences are enough alike that some of the skills for both can be developed simultaneously. Certain prerequisites are necessary: facility in adding and subtracting whole numbers, an understanding of how rational numbers can be associated with partitioned units, the ability to obtain and to identify equivalent fractions, facility in renaming mixed numerals. Any computation process, or algorithm, that demands so many different skills can easily break down if there is weakness in just one of the component skills.

Complex algorithms such as those for subtracting or dividing with mixed numerals can cause frustration and produce fear of mathematics, a failure complex, and even trauma in some students. For this reason, we suggest that some evaluation be made of a student's skills and attitudes before he is required to cope with problems that deal with computing differences such as $27\frac{13}{16} - 9\frac{17}{20}$.

For our purposes at the moment, however, let us assume that we are working with students who are ready for an introduction to more difficult subtraction computation. We might begin by setting up a problem involving a familiar or interesting life experience, rather than by manipulating abstract symbols. For example, we place $3\frac{1}{4}$ oranges on one plate and $1\frac{3}{4}$ oranges on another plate, and we address the group we are working with: "Dennis, this plate is for you, and, Mary, this is yours. Who has more oranges? How much more?"

And we wait for answers—for discussion, estimates, arguments. We say, "How do you know?" or "You have to convince us," and we wait for explanations. We try to lead a little, and listen a lot.

And the explanations come. "If you cut 1 of the 3 oranges into quarters then you can take 3 of the quarters and 1 whole orange and match them with the $1\frac{3}{4}$ oranges on Mary's plate, and then Dennis would still have $1\frac{2}{4}$ oranges left."



But there are times when we may need to figure out problems without "cutting up the oranges." So we work with the students to develop a computation procedure, or algorithm, that we can use whenever we need to solve similar problems. Perhaps it will look like this,

$$\begin{array}{rcccl}
3\frac{1}{4} = & 1 + 1 + 1 + \frac{1}{4} = & 1 + 1 + \frac{3}{4} + \frac{1}{4} + \frac{1}{4} \\
-\,1\frac{3}{4} = & 1 + \frac{3}{4} = & 1 + \frac{3}{4} \\
\hline
 & & 1 \quad + \quad \frac{1}{4} + \frac{1}{4} = 1\frac{2}{4}
\end{array}$$

where the matching interpretation of subtraction is used to compute the difference, or like this:

$$\begin{array}{lr}
3\frac{1}{4} \rightarrow 2 + 1 + \frac{1}{4} \rightarrow 2 + \frac{4}{4} + \frac{1}{4} & \rightarrow 2\frac{5}{4} \\
-\,1\frac{3}{4} & \rightarrow -1\frac{3}{4} \\
\hline
 & 1\frac{2}{4}
\end{array}$$

Here, students will envision the problem as a take-away rather than as a matching process. Others might think of the problem this way,

$$1\frac{3}{4} + \square = 3\frac{1}{4},$$

and reason, "How much do I have to add to $1\frac{3}{4}$ to get $3\frac{1}{4}$? $1\frac{3}{4} + \frac{1}{4}$ makes 2, and $1\frac{1}{4}$ more is $3\frac{1}{4}$, so the missing addend is $\frac{1}{4} + 1\frac{1}{4}$, or $1\frac{2}{4}$, and $1\frac{3}{4} + 1\frac{2}{4} = 3\frac{1}{4}$."

$$1\frac{3}{4} + \boxed{\frac{1}{4} + 1 + \frac{1}{4}} = 3\frac{1}{4}.$$

$$1\frac{3}{4} + \quad \boxed{1\frac{2}{4}} \quad = 3\frac{1}{4}.$$

SUBTRACTION ALGORITHMS

Which subtraction algorithm should we teach? At first, the algorithm to use may be one that the students "make up." At least it should be one that they can easily follow—one that makes sense. Then a transition can be made to the algorithm that appears in the textbook they are using.

Actually, there is nothing sacred about a particular algorithm. There are many to choose from and others to invent. Some should be avoided.

Some students try to cut down on the amount of writing, and they may use one of these forms:

$$
\begin{array}{rl}
4\frac{2}{3} = & \frac{8}{12} \\[4pt]
-\ 1\frac{1}{4} = & \frac{3}{12} \\ \hline
 & 3\,\frac{5}{12}
\end{array}
\qquad\qquad
\begin{array}{r|l}
4\frac{2}{3} & \frac{8}{12} \\[4pt]
-\ 1\frac{1}{4} & \frac{3}{12} \\ \hline
3 & \frac{5}{12}
\end{array}
\qquad\qquad
\begin{array}{r}
4\frac{2}{3} \\[4pt]
-\ 1\frac{1}{4} \\ \hline
3\frac{5}{12}
\end{array}
$$

The notation on the left violates the meaning of the symbol for equality, and it should be avoided on that count alone. There is another reason for discouraging its use; students may write the fraction and forget about the rest of the mixed numeral. In this case they might easily write $\frac{5}{12}$ and not remember to compute $4 - 1$.

The form in the middle is compact, and some students use it successfully. But here, too, the whole numbers may be neglected, and the feeling of equality may be lost.

The form that appears on the right is one preferred by many of the faster students. They write the problem, work out all the steps mentally, and then, if they are permitted to, simply write the result. If a student is consistently accurate with this method, why not encourage him to use it? However, if he makes mistakes, he should be required to "spell things out" so his work can be checked to find out what causes the errors.

Some of the forms commonly found in textbooks are shown below:

$$
A.\quad
\begin{array}{r}
3\frac{1}{3} = 3\frac{5}{15} = 2\frac{20}{15} \\[4pt]
-\ 1\frac{4}{5} = 1\frac{12}{15} = 1\frac{12}{15} \\ \hline
1\frac{8}{15}
\end{array}
\qquad
C.\quad
\begin{array}{r}
3\frac{1}{3} = \quad 3\frac{5}{15} = \quad 2\frac{20}{15} \\[4pt]
-\ 1\frac{4}{5} = -\ 1\frac{12}{15} = -\ 1\frac{12}{15} \\ \hline
1\frac{8}{15}
\end{array}
$$

$$
B.\quad
\begin{array}{r}
3\frac{1}{3} \rightarrow 3\frac{5}{15} \rightarrow 2\frac{20}{15} \\[4pt]
-\ 1\frac{4}{5} \rightarrow 1\frac{12}{15} \rightarrow 1\frac{12}{15} \\ \hline
1\frac{8}{15}
\end{array}
\qquad
D.\quad
\begin{array}{r}
2\frac{20}{15} \\[4pt]
3\frac{1}{3} = \quad 3\frac{\cancel{5}}{\cancel{15}} \\[4pt]
-\ 1\frac{4}{5} = -\ 1\frac{12}{15} \\ \hline
1\frac{8}{15}
\end{array}
$$

Some textbook authors object to notation that seems to imply that equations are being subtracted, so they use arrows, as shown in *B*, or repeat the symbol for minus each time the difference is renamed, as shown in *C*. The crossing out that appears in *D* is reminiscent of a common practice when subtracting whole numbers. It helps some students, and it confuses others.

The most difficult procedure in computing differences occurs when the sum is renamed twice, as shown in the above examples. $3\frac{1}{3}$ is renamed first as $3\frac{5}{15}$, to obtain a common denominator, and again as $2\frac{20}{15}$, to obtain a greater numerator. The second renaming step is the nemesis of many students. This is what often happens:

$$\begin{array}{r}
3\frac{1}{3} = \overset{2}{\cancel{3}}\frac{{}^{1}5}{15} = 2\frac{15}{15} \\[2ex]
1\frac{4}{5} = 1\frac{12}{15} = 1\frac{12}{15} \\
\hline
1\frac{3}{15}
\end{array}$$

(This, of course, is not the correct answer.)

The error results from learning a mechanical procedure when subtracting whole numbers, and transferring that procedure blindly to subtracting with mixed numerals. Students who make such an error need to work through a more expanded algorithm, the key part of which is shown here:

$$\begin{aligned}
3\frac{5}{15} &= (2 + 1) + \frac{5}{15} \\
&= \left(2 + \frac{15}{15}\right) + \frac{5}{15} = 2 + \left(\frac{15}{15} + \frac{5}{15}\right) = 2 + \frac{20}{15} = 2\frac{20}{15}.
\end{aligned}$$

There is another algorithm, not frequently used but sometimes useful, which eliminates the source of the error shown above:

$$\begin{array}{r}
3\frac{1}{3} = \frac{10}{3} = \frac{50}{15} \\[2ex]
1\frac{4}{5} = \frac{9}{5} = \frac{27}{15} \\
\hline
\frac{23}{15}
\end{array}$$

In this algorithm, mixed numerals are replaced by fractions, thus eliminating the need for borrowing or regrouping. (Can you think of a disadvantage of this method?)

Exercise Set 5

1. Replace the mixed numerals by fractions to compute the differences.

a. $3\frac{1}{2} - 1\frac{2}{3}$

b. $27\frac{3}{16} - 19\frac{3}{5}$

2. Compute each of the differences in exercise **1**, using the form shown as *A* in the preceding section.

3. Explain the computation error in each example.

a.

$$\begin{array}{r} 8 \\ -\ \frac{1}{10} \\ \hline \frac{7}{10} \end{array}$$

b.

$$\begin{array}{rcr} \overset{3}{\cancel{4}}\,{}^{1}\frac{1}{3} & = & 3\frac{11}{3} \\ -\ 1\frac{2}{3} & = & 1\frac{2}{3} \\ \hline & & 2\frac{9}{3} \end{array}$$

c.

$$\begin{array}{rcr} 5\frac{77}{100} & = & \frac{77}{100} \\ -\ 3\ \frac{7}{10} & = & \frac{70}{100} \\ \hline & & \frac{7}{100} \end{array}$$

d.
$$\begin{array}{r} 6\frac{4}{8} \\ -\ 4\frac{3}{8} \\ \hline 2\frac{7}{8} \end{array}$$

e.
$$\begin{array}{r} 12\frac{3}{8} \\ -\ 1\frac{7}{8} \\ \hline 11\frac{4}{8} = 11\frac{1}{2} \end{array}$$

f.
$$\begin{array}{rr} 10\frac{3}{16} = & \overset{9}{\cancel{10}}\frac{19}{16} \\ -\ 2\frac{5}{8} = & -\ 2\frac{5}{16} \\ \hline & 7\frac{14}{16} \end{array}$$

EQUAL ADDITIONS

Another algorithm worth learning uses the equal-additions, or equal-increment, principle. This algorithm eliminates "borrowing" by making use of the principle that adding the same number to both the sum and the given addend does not affect the difference. For example, the difference of 162 and 98 is easier to compute if each number is increased by 2:

$$\begin{aligned} 162 - 98 &= (162 + 2) - (98 + 2) \\ &= 164 \qquad - \quad 100 \\ &= 64. \end{aligned}$$

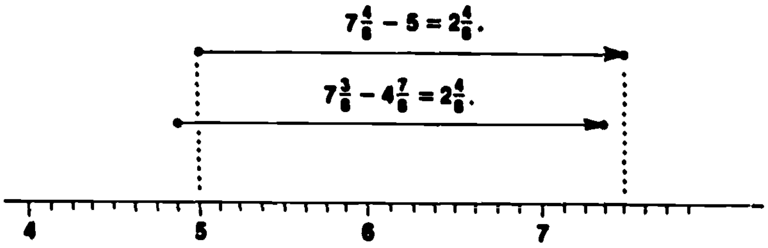
How can the principle of equal additions be applied to compute the difference $7\frac{3}{8} - 4\frac{7}{8}$? What number could you add to both the sum, $7\frac{3}{8}$, and the given addend, $4\frac{7}{8}$, to make computation easier?

$$\begin{array}{r}
7\frac{3}{8} \to 7\frac{3}{8}+\frac{1}{8} \to 7\frac{4}{8} \to 7\frac{4}{8}\\
\underline{4\frac{7}{8} \to 4\frac{7}{8}+\frac{1}{8} \to 4\frac{8}{8} \to 5}\\
2\frac{4}{8}
\end{array}$$

After students understand the steps shown above, the notation can be condensed. Two shorter algorithms are shown:

$$\begin{array}{r}
7\frac{3}{8}+\frac{1}{8}=7\frac{4}{8}\\
\underline{4\frac{7}{8}+\frac{1}{8}=5}\\
2\frac{4}{8}
\end{array}
\qquad\qquad
\begin{array}{r}
7\frac{3}{8}\to 7\frac{4}{8}\\
\underline{4\frac{7}{8}\to 5}\\
2\frac{4}{8}
\end{array}$$

Shifting a vector for the difference on the number line gives a clear picture of how the vector length remains constant when the principle of equal additions is applied.



Exercise Set 6

1. Use the principle of equal additions to rename each difference so that the given addend is changed to the nearest whole number, then complete the computation.

a. $4\frac{1}{3}-1\frac{2}{3}$

b. $9\frac{3}{10}-5\frac{6}{10}$

c. $6\frac{1}{4}-2\frac{3}{4}$

d. $88\frac{16}{100}-27\frac{97}{100}$

2. The principle of equal additions is applied in a different way in the following example. Explain how.

$$\begin{array}{rcr} 7\frac{17}{100} & \rightarrow & 7\frac{117}{100} \\ -\,4\frac{84}{100} & \rightarrow & 5\frac{84}{100} \\ \cline{3-3} & & 2\frac{33}{100} \end{array}$$

3. A method for computing the difference of two whole numbers is shown in the following example:

$$\begin{array}{rcrcrcr} 72 & \rightarrow & 60 + 1\overset{1}{\cancel{0}} + 2 & \rightarrow & 60 + 3 & \rightarrow & 63 \\ \underline{-\,49} & & \underline{-\,40 - \cancel{9}\quad} & & \underline{-\,40\quad} & & \underline{-\,40} \\ & & & & & & 23 \end{array}$$

In this example, use is made of what might be called an equal-subtractions principle: 9 is subtracted from 49 to get 40, and 9 is subtracted from one of the 7 tens in 72 to get (60 + 1) + 2, or 63. The difference 72 − 49 is changed to 63 − 40, for easier computation.

A similar algorithm is shown for computing the difference of two rational numbers in this example:

$$\begin{array}{rcrcrcrcr} 4\frac{2}{5} & \rightarrow & 3 + 1 + \frac{2}{5} & \rightarrow & 3 + \overset{\frac{2}{5}}{\frac{\cancel{5}}{\cancel{5}}} + \frac{2}{5} & \rightarrow & 3 + \frac{4}{5} & \rightarrow & 3\frac{4}{5} \\ \underline{-\,1\frac{3}{5}} & & \underline{-\,1 - \frac{3}{5}\quad} & & \underline{-\,1 - \frac{\cancel{3}}{\cancel{5}}\quad} & & \underline{-\,1 - 0} & & -\,1 \\ \cline{9-9} & & & & & & & & 2\frac{4}{5} \end{array}$$

The above algorithm may be abbreviated as follows:

$$\begin{array}{r} \overset{3}{\cancel{4}}\,\frac{\overset{4}{\cancel{2}}}{5} \\ -\,1\frac{\cancel{3}}{\cancel{5}} \\ \hline 2\frac{4}{5} \end{array}$$

[Note: $\frac{3}{5}$ is subtracted from 4, leaving $3\frac{2}{5}$; the $\frac{2}{5}$ is then added to the original $\frac{2}{5}$ to get $\frac{4}{5}$.] .

Use this abbreviated form to compute each of the following differences:

a. $6\frac{1}{4} - 1\frac{2}{4}$

b. $7\frac{3}{8} - 4\frac{7}{8}$

c. $1\frac{4}{16} - \frac{15}{16}$

d. $9 - 2\frac{5}{12}$

We have seen that there are many algorithms for computing the difference of two rational numbers. The traditional borrowing, or regrouping, method is featured in most textbooks published in the United States, and it is an effective algorithm for general use. However, there are good reasons for teaching one or more other algorithms. Some children may find the regrouping method confusing or difficult, while some other method might make sense to them. Another reason for learning other algorithms is that it often involves exploring and learning more mathematics.

PROPERTIES OF SUBTRACTION

After students learn that addition of rational numbers, just as of whole numbers, is both commutative and associative, they might investigate whether subtraction of rational numbers has these properties:

Is subtraction of rational numbers commutative? A student who has a good understanding of the missing-addend approach to subtraction can easily answer this question by applying it to a specific case. For example: Is it true that

$$\frac{3}{4} - \frac{1}{4} = \frac{1}{4} - \frac{3}{4}\ ?$$

The student will see that $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$, because $\frac{2}{4}$ is the missing addend in the sentence $\frac{1}{4} + \square = \frac{3}{4}$. He will also see that $\frac{1}{4} - \frac{3}{4} \neq \frac{2}{4}$ ($\frac{1}{4} - \frac{3}{4}$ is not equal to $\frac{2}{4}$), because $\frac{2}{4}$ is not the answer to the question $\frac{3}{4} + \square = \frac{1}{4}$. In fact, there is no positive-rational-number answer, and $\frac{1}{4} - \frac{3}{4}$ is undefined until negative rational numbers are introduced. At any rate, the student finds that

$$\frac{3}{4} - \frac{1}{4} \neq \frac{1}{4} - \frac{3}{4}.$$

One exception, or counterexample, is sufficient to show that *subtraction of rational numbers is not commutative.*

Is subtraction of rational numbers associative? Once again, a single example can serve to provide the answer. Is it true that

$$\left(\frac{7}{8} - \frac{3}{8}\right) - \frac{1}{8} = \frac{7}{8} - \left(\frac{3}{8} - \frac{1}{8}\right)?$$

Computing the differences as indicated by the grouping symbols reveals that

$$\left(\frac{7}{8} - \frac{3}{8}\right) - \frac{1}{8} = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$$

and

$$\frac{7}{8} - \left(\frac{3}{8} - \frac{1}{8}\right) = \frac{7}{8} - \frac{2}{8} = \frac{5}{8}.$$

Since $\frac{3}{8} \neq \frac{5}{8}$, then

$$\left(\frac{7}{8} - \frac{3}{8}\right) - \frac{1}{8} \neq \frac{7}{8} - \left(\frac{3}{8} - \frac{1}{8}\right).$$

This counterexample shows that *subtraction of rational numbers is not associative.*

THE ROLE OF 0 IN SUBTRACTION

Recall that the sum of zero and any rational number is equal to that rational number, that is,

$$\frac{a}{b} + 0 = \frac{a}{b}, \quad \text{and} \quad 0 + \frac{a}{b} = \frac{a}{b}.$$

Zero is the *identity element* or *additive identity* for addition of rational numbers. Does zero play the same role in subtraction?

From each addition sentence, two subtraction sentences can be formed. For example, from the sentence

$$\frac{2}{3} + 0 = \frac{2}{3},$$

the student should be able to obtain these two sentences:

$$\frac{2}{3} - \frac{2}{3} = 0 \quad \text{and} \quad \frac{2}{3} - 0 = \frac{2}{3};$$

and from the sentence

$$0 + \frac{7}{10} = \frac{7}{10},$$

these two sentences:

$$\frac{7}{10} - \frac{7}{10} = 0, \quad \text{and} \quad \frac{7}{10} - 0 = \frac{7}{10}.$$

After working a set of similar examples, students might be prompted to propose these generalizations about the role of zero in subtraction of rational numbers:

1. *Any rational number minus itself is equal to zero.*

$$\frac{a}{b} - \frac{a}{b} = 0.$$

2. *Any rational number minus zero is equal to that number.*

$$\frac{a}{b} - 0 = \frac{a}{b}.$$

These two generalizations are derived from the fact that 0 is the identity element for addition of rational numbers. We may not conclude from this, however, that 0 is the identity element for subtraction of rational numbers. While it is true that $\frac{a}{b} - 0 = \frac{a}{b}$, it is *not* true that $0 - \frac{a}{b} = \frac{a}{b}$. Both conditions would have to be met for 0 to be called the identity element for subtraction.

"SHIFTING OF TERMS" IN SUBTRACTION

To compute the difference $5\frac{2}{8} - 1\frac{3}{8}$, a student might proceed as follows:

$$\begin{array}{r}
5\frac{2}{8} = 4 + \frac{8}{8} + \frac{2}{8} = 4 + \frac{10}{8} \\
-\ 1\frac{3}{8} = 1 + \frac{3}{8} \qquad = 1 + \frac{3}{8} \\
\hline
\end{array}$$

He may then be deceived by the addition symbols and complete the computation incorrectly:

$$\begin{array}{l} 4 + \frac{10}{8} \\ \\ \underline{1 + \ \frac{3}{8}} \\ 3 + \frac{13}{8} = 3\,\frac{13}{8}. \end{array}$$

This error could be prevented by an understanding of a special property of subtraction of rational numbers:

For all ration numbers r, s, t, u,

$$(r + s) - (t + u) = (r - t) + (s - u).$$

Notice how the shifting of terms in this equality changes a difference of two sums into a sum of two differences. Applied to the example above, we have:

$$\left(4 + \frac{10}{8}\right) - \left(1 + \frac{3}{8}\right) = (4 - 1) + \left(\frac{10}{8} - \frac{3}{8}\right),$$

and the correct result is $3 + \frac{7}{8}$, or $3\,\frac{7}{8}$.

The shifting-of-terms principle can be confirmed by students by working out a subtraction problem with the help of concrete materials. A formal proof of the generalization

$$(r + s) - (t + u) = (r - t) + (s - u)$$

is presented here. Keep in mind the relationship of addition to subtraction and the fact that addition is commutative and associative.

1. Let $(r - t) = x$ and $(s - u) = y$. If $r - t = x$, then $r = (t + x)$, and if $s - u = y$, then $s = (u + y)$.
2. $(r + s) = (t + x) + (u + y)$

   $= (t + u) + (x + y)$.
3. $(r + s) - (t + u) = (x + y)$.
4. Recall that $x = r - t$ and $y = s - u$. (See step 1.)

Then $(r + s) - (t + u) = (r - t) + (s - u)$, which is the statement we set out to prove.

If we take a special case, where $s = u$, we arrive at another generalization that is useful in computing with mixed numerals.

$$\begin{aligned}(r + s) - (t + s) &= (r - t) + (s - s)\\ &= (r - t) + 0\\ &= (r - t).\end{aligned}$$

And we have

$$(r + s) - (t + s) = (r - t).$$

This was the generalization applied in an algorithm developed earlier, where adding the same number to both sum and given addend did not affect the missing addend. For example, if $r = 3\frac{2}{5}$ and $t = 1\frac{4}{5}$, then we may let $s = \frac{1}{5}$ to make it easier to compute the difference $3\frac{2}{5} - 1\frac{4}{5}$:

$$\begin{aligned}3\frac{2}{5} - 1\frac{4}{5} &= \left(3\frac{2}{5} + \frac{1}{5}\right) - \left(1\frac{4}{5} + \frac{1}{5}\right)\\ &= 3\frac{3}{5} - 2\\ &= 1\frac{3}{5}.\end{aligned}$$

SUMMARY

The principles of subtraction of whole numbers apply to subtraction of rational numbers as well. Subtraction is related to addition; subtracting a number is the inverse of adding that number. If $\frac{c}{b} + \frac{d}{b} = \frac{a}{b}$, then $\frac{c}{b} = \frac{a}{b} - \frac{d}{b}$ and $\frac{d}{b} = \frac{a}{b} - \frac{c}{b}$. Subtraction assigns to the sum, $\frac{a}{b}$, and an addend, $\frac{c}{b}$, the missing addend in the sentence $\square + \frac{c}{b} = \frac{a}{b}$, and since the missing addend is the difference of the sum and the given addend, we have $\square = \frac{a}{b} - \frac{c}{b}$.

The difference of two nonnegative rational numbers is a nonnegative rational number only if the sum is greater than or equal to the given addend. Subtraction of rational numbers is neither commutative nor associative. Zero plays a special role in subtraction: for any rational number $\frac{a}{b}$, $\frac{a}{b} - 0 = \frac{a}{b}$ and $\frac{a}{b} - \frac{a}{b} = 0$. Adding the same rational number to both the sum and the given addend does not affect the difference of two rational numbers.

There are many algorithms for computing the difference of two rational numbers. Learning several different algorithms can increase a student's efficiency in computing; it can also help to increase his understanding of arithmetic by providing him with more opportunities for applying the basic principles of subtraction of rational numbers.

Harry D. Ruderman

MULTIPLICATION OF RATIONAL NUMBERS

5

1. What meaning shall we give children for an "indicated" product of a pair of positive rational numbers?
2. What preparation will help children understand the traditional multiplication algorithm for fractions?
3. How can we explain to children the traditional multiplication algorithm for fractions?
4. What properties does the multiplication of rational numbers have? How are they different from those for the multiplication of whole numbers?

PREPARING THE STUDENT TO FIGURE OUT FOR HIMSELF: ASSOCIATING A RECTANGULAR REGION WITH AN INDICATED PRODUCT

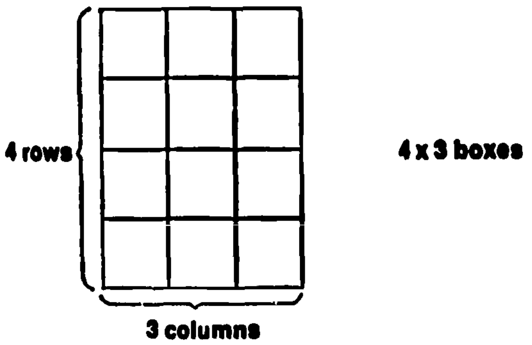
The ability to figure things out for oneself is evidence of understanding. How can we develop this ability in a student when introducing him to the multiplication of rational numbers? One way is to extend an interpretation of the multiplication of whole numbers to an interpretation of the multiplication of rational numbers—and to do so in a manner that will enable the student to reconstruct this extension himself.

We may begin to make this extension by recalling a meaning of the product of two whole numbers:

> *The product of whole numbers a and b is the number of elements in an array having a rows and b columns.*

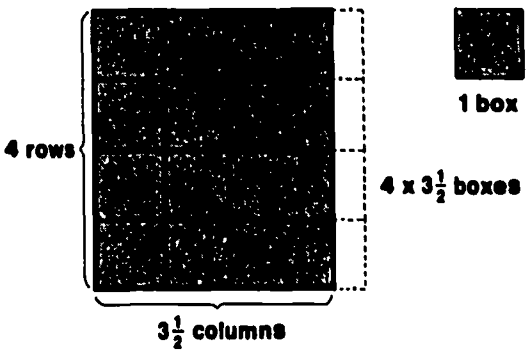
For example,

> 4×3 is the number of boxes in a rectangular array having 4 rows and 3 columns.



Counting tells us that there are 12 boxes in this array. Hence $4 \times 3 = 12$.

To consider products such as $4 \times 3\frac{1}{2}$ we can extend the rectangular array to 4 rows and $3\frac{1}{2}$ columns. The number of boxes in this new array is $4 \times 3\frac{1}{2}$. The picture shows that $4 \times 3\frac{1}{2}$ is between 4×3 and 4×4. If we view the rectangular region $4 \times 3\frac{1}{2}$ as a rug (measured, say, in feet), we see that it would cover a rug measuring 4×3 and fail to cover a rug measuring 4×4.



In mathematical symbols,

$$4 \times 3 < 4 \times 3\frac{1}{2} < 4 \times 4.$$

Hence $4 \times 3\frac{1}{2}$ is between $4 \times 3 = 12$ and $4 \times 4 = 16$. So we have

$$12 < 4 \times 3\frac{1}{2} < 16.$$

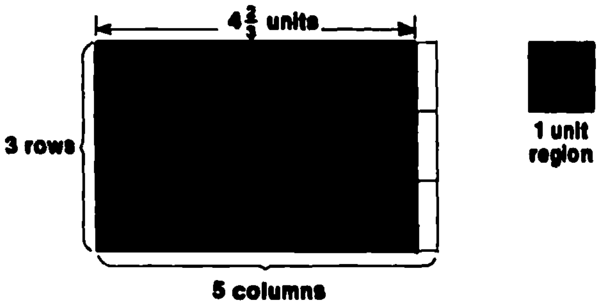
In many practical situations this information is all that is needed. We may refer to 12 as a *lower estimate* for $4 \times 3\frac{1}{2}$ and to 16 as an *upper estimate* for $4 \times 3\frac{1}{2}$.

There is much merit in postponing computation with rational numbers until after the student can draw on squared paper a picture that will go with indicated products such as

$$4 \times 3\frac{1}{2}$$

and can write the product expression for a given picture such as the one shown here.

Another important step before moving into computation is to obtain lower and upper estimates for products by replacing factors with lesser or greater whole numbers. Thus,

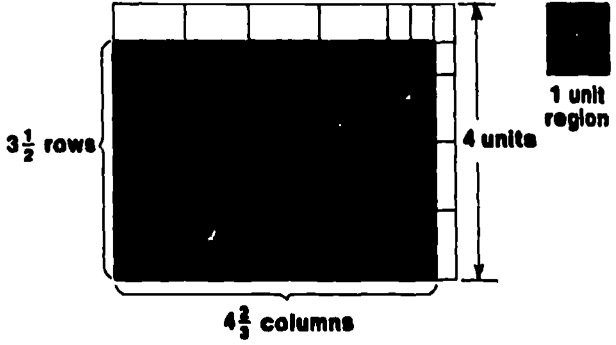


$$3 \times 4 < 3 \times 4\frac{2}{3} < 3 \times 5$$

or

$$12 < 3 \times 4\frac{2}{3} < 15.$$

If neither factor is a whole number, we can still proceed as before. Consider the product $3\frac{1}{2} \times 4\frac{2}{3}$.



The product for this rectangular region is

$$3\frac{1}{2} \times 4\frac{2}{3}.$$

A lower estimate is 3 × 4, and an upper estimate is 4 × 5. So we may write

$$3 \times 4 \; < \; 3\frac{1}{2} \times 4\frac{2}{3} \; < \; 4 \times 5$$

or

$$12 \; < \; 3\frac{1}{2} \times 4\frac{2}{3} \; < \; 20.$$

If time is given to

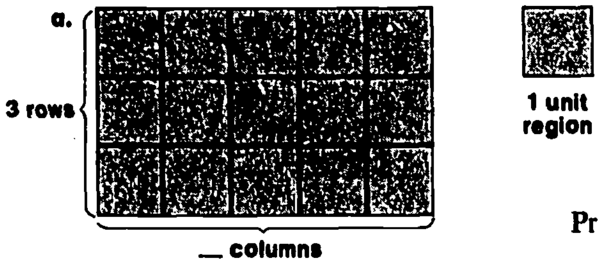
(1) obtaining rectangles for indicated products,
(2) obtaining product expressions for given rectangles, and
(3) obtaining lower and upper estimates for products,

students will be in a better position to figure out computed products that are likely to be within reason, even if not correct.

Exercise Set 1

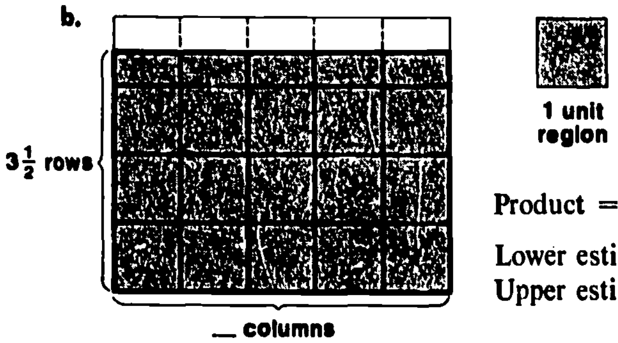
1. What product expression goes with each of the following rectangles? (Do not compute.) When asked to do so, show the lower and upper estimates for the product. (Do not compute.)

a.



Product = ___ × ___

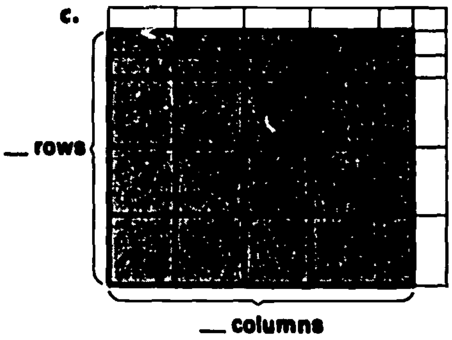
b.



Product = $3\frac{1}{2}$ × ___
Lower estimate = ___ × ___
Upper estimate = ___ × ___

Fill in: ___ × ___ < ___ × ___ < ___ × ___

c.



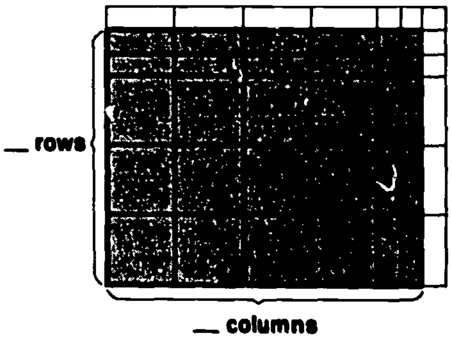
Product = ___ × ___

Lower estimate = ___ × ___

Upper estimate = ___ × ___

Fill in: ___ × ___ < ___ × ___ < ___ × ___.

d.



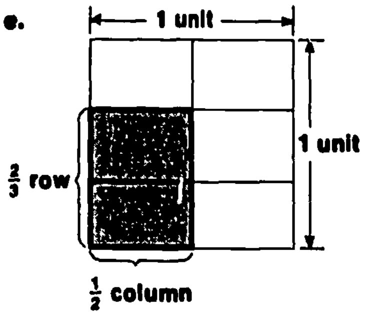
Product = ___ × ___

Lower estimate = ___ × ___

Upper estimate = ___ × ___

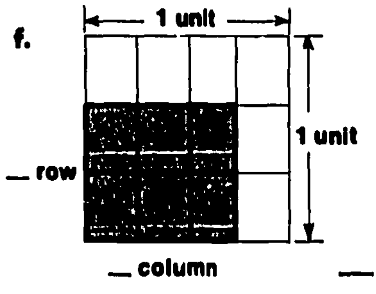
Fill in: ___ × ___ < ___ × ___ < ___ × ___

e.



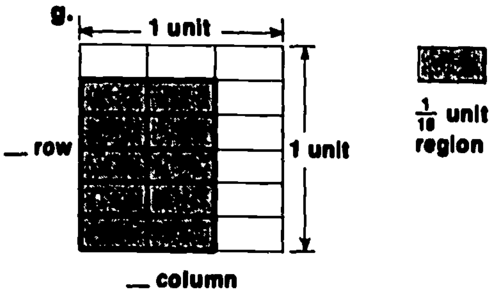
Product = ___ × ___

f.



Product = ___ × ___

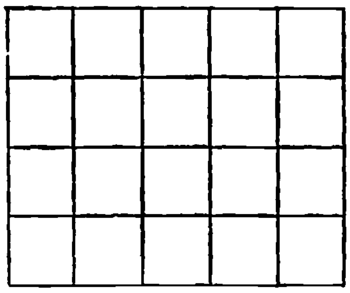
g.



Product = ___ × ___

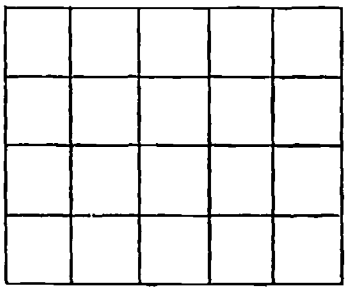
2. Shade a rectangular region that goes with each of the following products. When asked to so so, show the upper and lower estimates for each product. (Do not compute.)

a.



3×4

b.



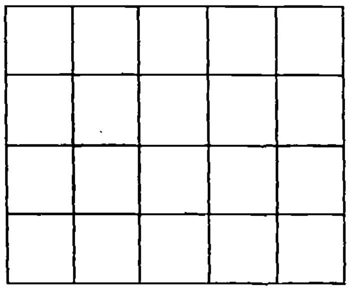
$3 \times 4\frac{2}{3}$

Lower estimate = ___ × ___

Upper estimate = ___ × ___

Fill in: ___ × ___ $< 3 \times 4\frac{2}{3} <$ ___ × ___

c.



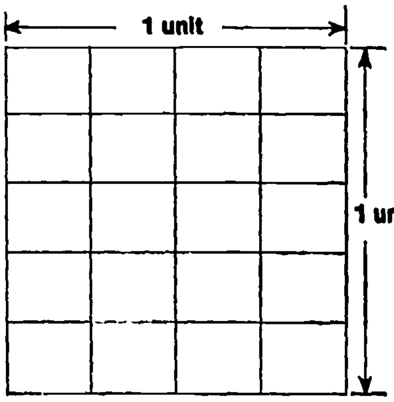
$3\frac{2}{3} \times 4\frac{1}{2}$

Lower estimate = ____ × ____

Upper estimate = ____ × ____

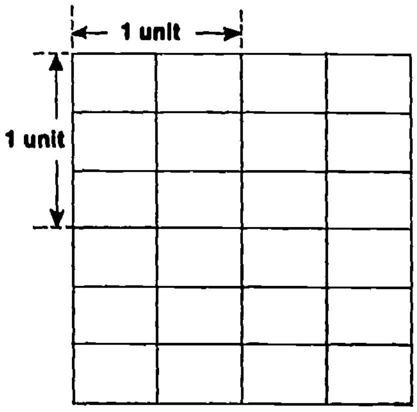
Fill in: ____ × ____ $< 3\frac{2}{3} \times 4\frac{1}{2} <$ ____ × ____

d.



$\frac{2}{5} \times \frac{3}{4}$

e.



$1\frac{2}{3} \times 1\frac{1}{2}$

f. Assign your own width and length for a unit region to obtain a rectangle for $1\frac{1}{5} \times 2\frac{1}{2}$. You may want to make the unit for width different from the unit for length.

3. Sketch a rectangular region that goes with each of the following products. Do not compute the products.

a. $3 \times 5\frac{1}{2}$

b. $3 \times 5\frac{2}{3}$

c. $3\frac{1}{2} \times 5\frac{2}{3}$

d. $\frac{2}{3} \times \frac{3}{2}$

e. $\frac{3}{4} \times \frac{4}{3}$

f. $\frac{3}{5} \times \frac{3}{5}$

g. $1\frac{1}{3} \times \frac{3}{5}$

h. $1\frac{1}{3} \times 1\frac{1}{2}$

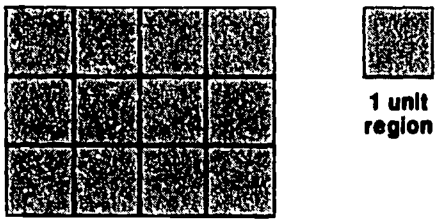
i. $1\frac{3}{4} \times 2\frac{1}{2}$

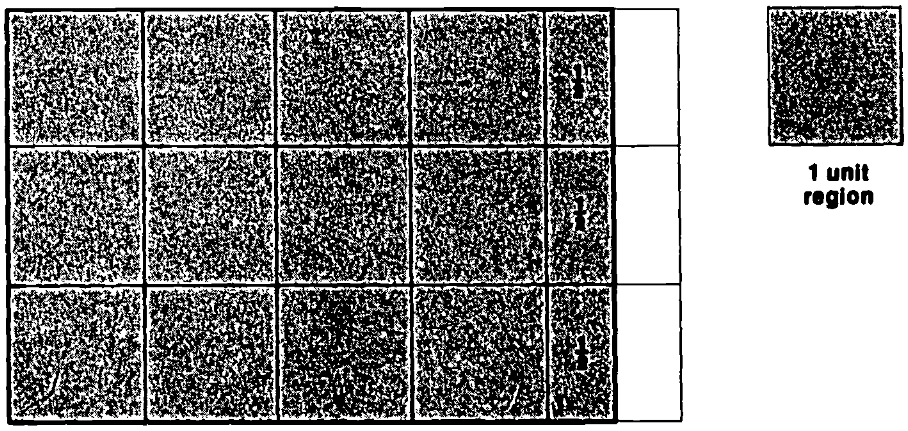
j. $\frac{1}{2} \times \frac{3}{3}$

k. $\frac{4}{5} \times \frac{2}{2}$

COMPUTING PRODUCTS OF RATIONAL NUMBERS WITHOUT A TRADITIONAL ALGORITHM

After a student has acquired the ability to obtain a rectangular region for an indicated product and to obtain an indicated product for a rectangular region, he is ready to compute products from their pictures. Just as a picture for 3 × 4 can be used to compute 3 × 4 by counting, a picture can be used to compute $3 \times 4\frac{1}{2}$. We count 3 × 4 full squares, giving 12. The remaining three $\frac{1}{2}$ squares give $1\frac{1}{2}$ squares. Adding, $12 + 1\frac{1}{2}$ gives $13\frac{1}{2}$ as the product for $3 \times 4\frac{1}{2}$. We could say that we "counted boxes" to obtain the computed product:



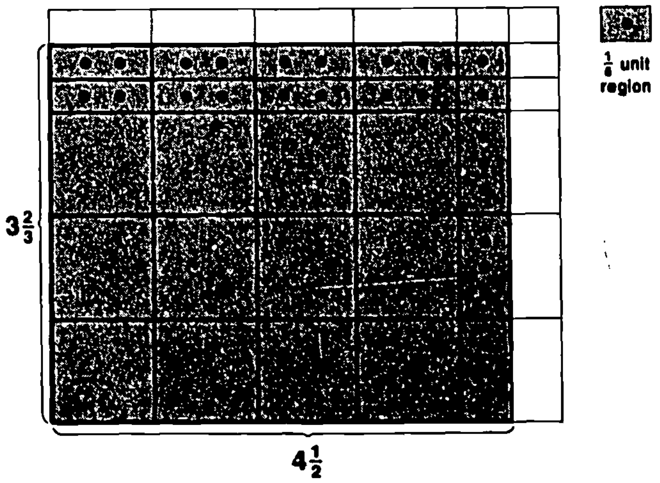


$$3 \times 4\frac{1}{2} = 13\frac{1}{2}.$$

This method of "counting boxes" can be used for all such problems. Let's try a slightly harder problem, say,

$$3\frac{2}{3} \times 4\frac{1}{2}.$$

First we see that there are 3 × 4, or 12, unit regions. The boxes marked



with three dots are each $\frac{1}{2}$ of a unit region. There are 3 of these boxes, giving $1\frac{1}{2}$ additional unit regions. The boxes marked with a single dot are each $\frac{1}{6}$ of a unit region because 6 of them make up a full unit region. Two of them make $\frac{2}{6}$, or $\frac{1}{3}$, of a full unit region. Finally, we see that there are 8 boxes marked by two dots; each of these is $\frac{1}{6} + \frac{1}{6}$, or $\frac{1}{3}$, of a unit region. These 8 boxes give a total of $\frac{8}{3}$, or $2\frac{2}{3}$, unit regions. We need only add:

Unit regions $3 \times 4 = 12$

One-half unit regions $3 \times \frac{1}{2} = 1\frac{1}{2}$

One-sixth unit regions $2 \times \frac{1}{6} = \frac{1}{3}$

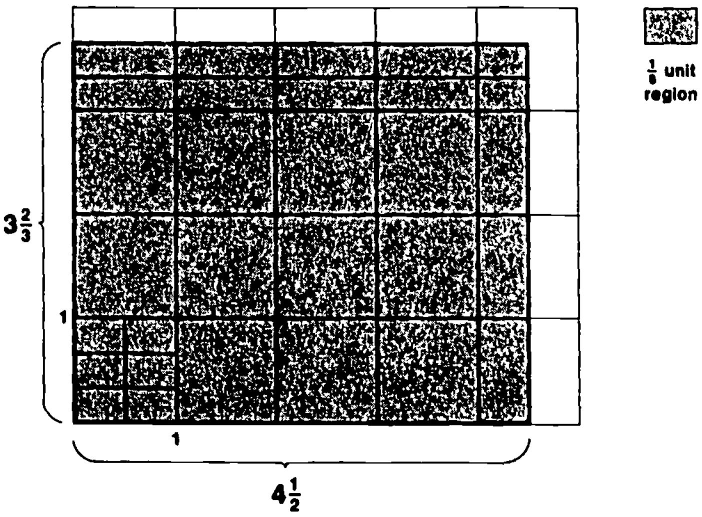
One-third unit regions $8 \times \frac{1}{3} = \underline{2\frac{2}{3}}$

$$15 + 1\frac{1}{2} = 16\frac{1}{2} \text{ unit regions.}$$

Hence the computed product for $3\frac{2}{3} \times 4\frac{1}{2}$ obtained by counting boxes is $16\frac{1}{2}$, so that

$$3\frac{2}{3} \times 4\frac{1}{2} = 16\frac{1}{2}.$$

Another approach by counting is to split all the boxes horizontally into thirds and vertically into halves. The shaded unit region contains 6 smaller boxes of the same size, so each small box is $\frac{1}{6}$ of a unit region. The rectangular region for



$$3\frac{2}{3} \times 4\frac{1}{2}$$

contains 11 × 9, or 99, such small boxes, each $\frac{1}{6}$ of a unit region. The number of unit regions for all 99 boxes is then $\frac{99}{6}$, so that

$$\begin{aligned} 3\frac{2}{3} \times 4\frac{1}{2} &= \frac{99}{6} \\ &= \frac{96}{6} + \frac{3}{6} \\ &= 16 + \frac{1}{2} \\ &= 16\frac{1}{2}, \end{aligned}$$

the same answer as before.

Students should be encouraged to find other ways of counting to obtain a computed product. The consistency of mathematics is revealed to students when they recognize that no matter which correct way is used, the same result is obtained.

Exercise Set 2

NOTE.—Exercises below marked "*" take us into the realm of algebra. They are *not* being suggested for use in elementary school except possibly for special work with advanced groups. They are offered as exercises in deductive reasoning for teachers who wish to explore, in greater depth, proofs for steps in algorithms.

1. For each of the following six exercises, find the computed product by *counting boxes*. If possible, try to find a second counting method for obtaining a computed product as a check. Where feasible, show a lower and upper estimate for your result as an additional check.

a. $2 \times 3\frac{1}{2}$ **c.** $2\frac{1}{3} \times 3\frac{1}{2}$ **e.** $2\frac{1}{2} \times 3\frac{2}{3}$

b. $2\frac{1}{2} \times 3$ **d.** $2\frac{1}{2} \times 3\frac{1}{3}$ **f.** $2\frac{2}{3} \times 3\frac{1}{2}$

* **2.** Exercises **1a–1f** suggest a property of computed products. Try to formulate a conjecture from your observations and test it. Try to show that your conjecture is correct.

3. Find the computed product by counting boxes. Try to formulate a conjecture from your observations on **a, b, c, d.** Try to show that your conjecture is correct.

a. $\frac{4}{6} \times \frac{3}{2}$ **d.** $1\frac{3}{4} \times \frac{4}{7}$ **f.** $3\frac{1}{2} \times 3\frac{1}{2}$

b. $\frac{9}{6} \times \frac{2}{3}$ **e.** $2\frac{1}{2} \times 2\frac{1}{2}$ **g.** $4\frac{1}{2} \times 4\frac{1}{2}$

c. $\frac{3}{5} \times \frac{10}{6}$

*** 4.** Exercises **3e–3g** could suggest a generalization. Try to find one and then test it on $5\frac{1}{2} \times 5\frac{1}{2}$ and $6\frac{1}{2} \times 6\frac{1}{2}$. Try to show that your generalization is correct.

5. Compute each product by counting boxes. Try to generalize from your observations of **a** and **b.**

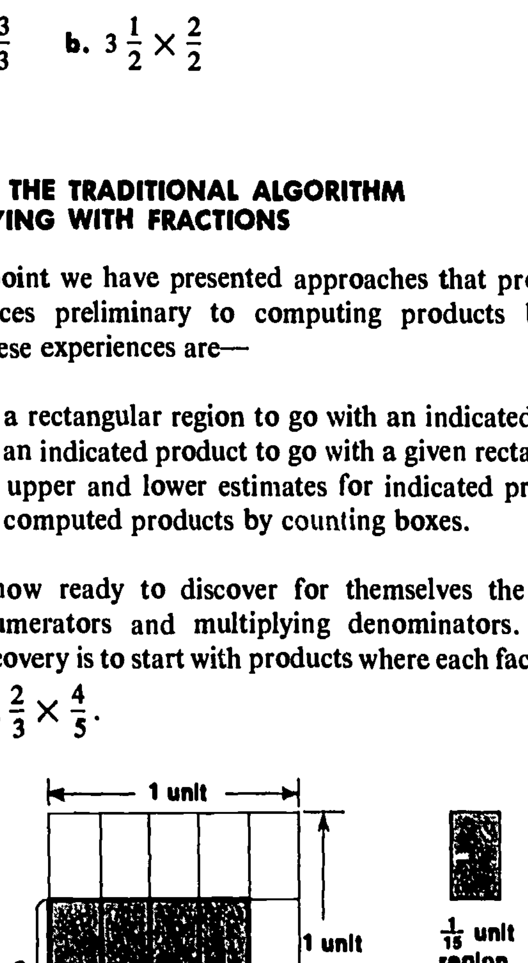
a. $2\frac{1}{2} \times \frac{3}{3}$ **b.** $3\frac{1}{2} \times \frac{2}{2}$

ARRIVING AT THE TRADITIONAL ALGORITHM FOR MULTIPLYING WITH FRACTIONS

Up to this point we have presented approaches that provide valuable pupil experiences preliminary to computing products by traditional algorithms. These experiences are—

1. obtaining a rectangular region to go with an indicated product;
2. obtaining an indicated product to go with a given rectangular region;
3. obtaining upper and lower estimates for indicated products;
4. obtaining computed products by counting boxes.

Students are now ready to discover for themselves the algorithm of multiplying numerators and multiplying denominators. One way of helping the discovery is to start with products where each factor is less than 1, for example, $\frac{2}{3} \times \frac{4}{5}$.



The first step is to assign a value to each of the small boxes. A count of the number of boxes making up the full unit region is 3 × 5, or 15. As 15 identical boxes make up a unit region, each of the 15 boxes is assigned a value of $\frac{1}{15}$. The shaded region for the product $\frac{2}{3} \times \frac{4}{5}$ has, by counting, 2 × 4, or 8, boxes, each $\frac{1}{15}$ of a unit region. It now follows that $\frac{2}{3} \times \frac{4}{5}$ and $\frac{8}{15}$ both give the value for the same rectangular region and so must be equivalent. We now have

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}.$$

It is not very hard for students to see that "15" tells into how many boxes of the same size the unit region is split. The unit region is split into 3 rows and 5 columns of boxes. The total number is therefore 3 × 5, or 15, which is the product of the denominators of the two fractions. The "8" tells how many boxes are in the rectangular region that goes with $\frac{2}{3} \times \frac{4}{5}$. The shaded array of small regions has 2 rows and 4 columns. The number of boxes for the shaded region is 2 × 4, or 8, which is the product of the numerators. We may now say that for the product $\frac{2}{3} \times \frac{4}{5}$ the product of the denominators, 3 × 5, tells into how many boxes of the same size the unit region is split, while the product of the numerators, 2 × 4, tells the number of these boxes in the shaded region for the indicated product. We have shown that

$$\begin{aligned}\frac{2}{3} \times \frac{4}{5} &= \frac{2 \times 4}{3 \times 5} \\ &= \frac{8}{15}.\end{aligned}$$

The last discussion strongly suggests that in multiplying with two fractions the product of the denominators gives the number of boxes of the same size into which a unit region is split, while the product of the numerators tells how many of these boxes fill up the rectangular region for the indicated product. The fraction obtained in this way gives the area of the corresponding region for the product. We now have two ways of expressing the value for the rectangular region—

1. as a product,

$$\frac{2}{3} \times \frac{4}{5};$$

2. as the quotient of a product of numerators and a product of denominators,

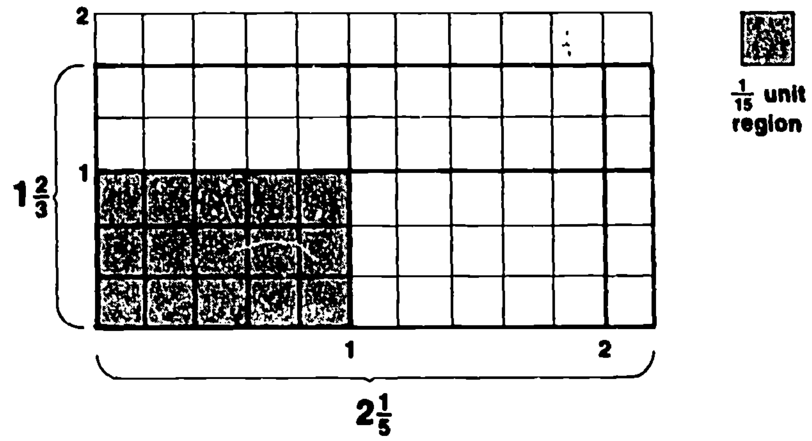
$$\frac{2 \times 4}{3 \times 5}.$$

As both must have the same value—the *area* of the same rectangular region—the traditional algorithm is justified. This algorithm is exemplified by

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}, \quad \text{or} \quad \frac{8}{15}.$$

Let us consider one more product before stating a generalization—say,

$$1\frac{2}{3} \times 2\frac{1}{5}, \quad \text{or} \quad \frac{5}{3} \times \frac{11}{5}.$$



Notice that our unit region is a rectangle that is *not* a square. For our purposes the unit region need not be a square.

The heavily outlined large rectangular region is for the indicated product

$$1\frac{2}{3} \times 2\frac{1}{5}, \quad \text{or} \quad \frac{5}{3} \times \frac{11}{5}.$$

The shaded region is our unit region and has 3×5, or 15, boxes, all of the same size. Each box, therefore, is worth $\frac{1}{15}$ of a unit region. The region for the product has 5×11, or 55, boxes, each worth $\frac{1}{15}$, so $\frac{55}{15}$ units is the area of the region for our product. Hence

$$\frac{5}{3} \times \frac{11}{5} = \frac{55}{15},$$

which is

$$\frac{5 \times 11}{3 \times 5}.$$

The two examples we worked,

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}, \quad \text{or} \quad \frac{8}{15},$$

and

$$\frac{5}{3} \times \frac{11}{5} = \frac{5 \times 11}{3 \times 5}, \quad \text{or} \quad \frac{55}{15},$$

strongly suggest the generalization, which is correct, that

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

This generalization is the traditional algorithm for multiplying with fractions. It was arrived at through many intermediate stages geared toward enabling the student to compute an indicated product by himself, even if he should forget the final algorithm.

Exercise Set 3

For each of the following indicated products, compute the product by a diagram and counting boxes. Check your result by the traditional algorithm.

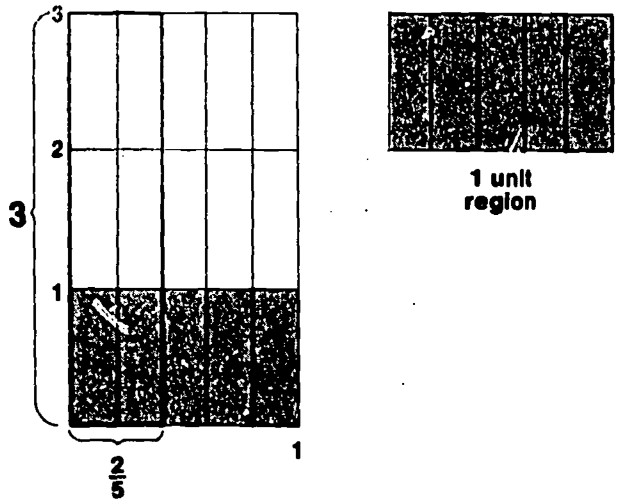
1. $\frac{3}{5} \times \frac{1}{2}$
2. $\frac{3}{4} \times \frac{2}{5}$
3. $\frac{1}{4} \times \frac{3}{5}$
4. $\frac{3}{4} \times \frac{5}{6}$
5. $3 \times 4\frac{1}{3}$
6. $3\frac{1}{3} \times 4$
7. $2\frac{2}{5} \times 3\frac{1}{2}$
8. $2\frac{1}{2} \times 3\frac{2}{5}$
9. $2\frac{1}{2} \times \frac{4}{4}$
10. $2\frac{2}{3} \times \frac{3}{3}$
11. $3\frac{1}{2} \times \frac{5}{5}$

AN IMPORTANT CONSEQUENCE OF THE ALGORITHM

As might be expected, multiplying by a whole number a (say, 3) is the same as multiplying by $\frac{a}{1}\left(\text{by } \frac{3}{1}\right)$. (See chap. 2.) For example,

$$3 \times \frac{2}{5}$$

may be computed by counting boxes.



As a unit region contains 5 boxes, all of the same size, each box is worth $\frac{1}{5}$. The region for $3 \times \frac{2}{5}$ contains 6 of these boxes, so $3 \times \frac{2}{5} = \frac{6}{5}$. Our algorithm for multiplying yields

$$\frac{3}{1} \times \frac{2}{5} = \frac{3 \times 2}{1 \times 5}$$

$$= \frac{6}{5}.$$

It seems that whenever a whole-number factor a appears, we may use $\frac{a}{1}$ in its place. Another way of saying this is that $a = \frac{a}{1}$ for all whole numbers a, and "$\frac{a}{1}$" may always be used for "a" in computations. In particular, $1 = \frac{1}{1}$. (See chap. 2.)

Our special problem may be expressed by

$$\begin{aligned} a \times \frac{b}{c} &= \frac{a}{1} \times \frac{b}{c} \\ &= \frac{a \times b}{1 \times c} \\ &= \frac{a \times b}{c}. \end{aligned}$$

In other words, to multiply by a whole number, merely multiply the numerator of the given fraction by this whole number.

Exercise Set 4

Compute each indicated product by counting squares and by the use of the traditional algorithm.

1. $3 \times \frac{2}{3}$

2. $2 \times \frac{3}{5}$

3. $3 \times \frac{1}{4}$

4. $2 \times \frac{3}{4}$

5. $3 \times \frac{2}{5}$

6. $3 \times \frac{3}{3}$

PROPERTIES OF THE MULTIPLICATION OF RATIONAL NUMBERS

Multiplication of rational numbers is always possible. In fact, our traditional algorithm tells us that

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

The product of a pair of whole numbers is always a whole number. Moreover, if neither factor is 0, the product cannot be 0. Hence the expression

$$\frac{a \times c}{b \times d}$$

always names a rational number, the product of the rational numbers

$$\frac{a}{b} \times \frac{c}{d}.$$

We say that

the set of rational numbers is closed under multiplication.

Strictly speaking, we have considered only the nonnegative rational numbers.

Because the multiplication of whole numbers is commutative, we can infer from our algorithm that

the multiplication of rational numbers is also commutative.

As a special case, note:

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{4 \times 2}{5 \times 3} = \frac{4}{5} \times \frac{2}{3};$$

thus

$$\frac{2}{3} \times \frac{4}{5} = \frac{4}{5} \times \frac{2}{3}.$$

In general:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d};$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{c \times a}{d \times b}$$

(as multiplication of whole numbers is commutative);

and

$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}.$$

This proves that multiplication of rational numbers is commutative.

Because the multiplication of whole numbers is associative, we can infer from our algorithm that

the multiplication of rational numbers is associative.

As a special case, note:

$$\left(\frac{2}{3} \times \frac{4}{5}\right) \times \frac{6}{7} = \left(\frac{2 \times 4}{3 \times 5}\right) \times \frac{6}{7}$$

$$= \frac{(2 \times 4) \times 6}{(3 \times 5) \times 7}$$

$$= \frac{2 \times (4 \times 6)}{3 \times (5 \times 7)}$$

(as multiplication of whole numbers is associative)

$$= \frac{2}{3} \times \frac{4 \times 6}{5 \times 7}$$

$$= \frac{2}{3} \times \left(\frac{4}{5} \times \frac{6}{7}\right).$$

The general case is handled in exactly the same way, giving

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right).$$

Because the multiplication of whole numbers distributes over addition,

multiplication of rational numbers also distributes over addition.

Again we show this for a special case in a way that suggests how the general proof can be given.

$$\begin{aligned}
\frac{2}{3} \times \left(\frac{4}{5} + \frac{7}{8}\right) &= \frac{2}{3} \times \left(\frac{4 \times 8}{5 \times 8} + \frac{5 \times 7}{5 \times 8}\right) \\
&= \frac{2}{3} \times \left(\frac{4 \times 8 + 5 \times 7}{5 \times 8}\right) \\
&= \frac{2 \times (4 \times 8 + 5 \times 7)}{3 \times (5 \times 8)} \\
&= \frac{2 \times (4 \times 8) + 2 \times (5 \times 7)}{3 \times (5 \times 8)} \\
&= \frac{(2 \times 4) \times 8}{(3 \times 5) \times 8} + \frac{2 \times (7 \times 5)}{3 \times (8 \times 5)} \\
&= \frac{2 \times 4}{3 \times 5} + \frac{(2 \times 7) \times 5}{(3 \times 5) \times 5} \\
&= \frac{2 \times 4}{3 \times 5} + \frac{2 \times 7}{3 \times 8}.
\end{aligned}$$

So

$$\frac{2}{3} \times \left(\frac{4}{5} + \frac{7}{8}\right) = \left(\frac{2}{3} \times \frac{4}{5}\right) + \left(\frac{2}{3} \times \frac{7}{8}\right).$$

In general, if r, s, t, are any rational numbers,

$$r(s + t) = rs + rt,$$

just as for whole numbers. The distributive property is especially useful in shortening computation and in solving equations.

Because 1 serves as an identity element for the multiplication of whole numbers,

1 also serves an an identity element for the multiplication of rational numbers.

This may be shown very easily from the fact that $\frac{1}{1} = 1$. It follows that

$$1 \times \frac{a}{b} = \frac{1}{1} \times \frac{a}{b}$$

$$= \frac{1 \times a}{1 \times b}$$

$$= \frac{a}{b}.$$

Similarly,

$$\frac{a}{b} \times 1 = \frac{a}{b} \times \frac{1}{1}$$

$$= \frac{a \times 1}{b \times 1}$$

$$= \frac{a}{b}.$$

We have previously noted that $\frac{c}{c} = 1$ whenever $c \neq 0$. It follows then that

$$\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times \frac{c}{c} = \frac{a \times c}{b \times c};$$

that is,

$$\frac{a}{b} = \frac{a \times c}{b \times c},$$

a result that we had previously obtained:

> *Multiplying both numerator and denominator by the same nonzero number does not change the value of a fraction, yielding an equivalent fraction.*

What conclusion can be drawn from the information that a and b are whole numbers and $a \times b = 1$? The only way this could happen is for both a and b to be 1. However, this need not be the case for rational numbers. Examples are

$$\frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1$$

and

$$\frac{4}{7} \times \frac{7}{4} = \frac{28}{28} = 1.$$

> *There are infinitely many different pairs of rational numbers with each pair having a product of 1.*

Whenever we have two rational numbers r and s for which

$$r \times s = 1,$$

we say that r is the reciprocal of s (or the multiplicative inverse of s) and that s is the reciprocal of r (or the multiplicative inverse of r). Thus, as

$$\frac{2}{3} \times \frac{3}{2} = 1,$$

$\frac{2}{3}$ is the reciprocal (or multiplicative inverse) of $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal (or multiplicative inverse) of $\frac{2}{3}$. It follows that the product of any number (not 0) and its reciprocal (or multiplicative inverse) is 1. In general, if $\frac{a}{b}$ is a rational number with $a \neq 0$, then $\frac{b}{a}$ is its reciprocal (or multiplicative inverse), as

$$\begin{aligned} \frac{a}{b} \times \frac{b}{a} &= \frac{a \times b}{b \times a} \\ &= 1. \end{aligned}$$

If $a = 0$, $\frac{a}{b}$ has no reciprocal. We shall have other opportunities to use reciprocals.

For whole numbers, the product of any number and 0 is 0. But since $0 = \frac{0}{1}$, and in fact $0 = \frac{0}{n}$ for any whole number n other than 0, we have:

$$\begin{aligned} 0 \times \frac{a}{b} &= \frac{0}{1} \times \frac{a}{b} \\ &= \frac{0 \times a}{1 \times b} \\ &= \frac{0}{b} \\ &= 0. \end{aligned}$$

This proves that

the product of 0 *and any rational number is* 0.

Suppose we try to find a rectangular region for an indicated product having one factor 0. We find that we cannot draw one, as it must have either length or width of size 0. This is another way of showing that when 0 is a factor of any indicated product, the product must be 0.

When a pair of whole numbers, each different from both 0 and 1, are multiplied, the product is always greater than either factor. This does not hold for rational numbers. The product may be greater or less than one or both factors. Thus

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \quad \text{and} \quad \frac{1}{6} < \frac{1}{2}, \quad \frac{1}{6} < \frac{1}{3};$$

$$\frac{1}{2} \times \frac{4}{3} = \frac{4}{6} \quad \text{and} \quad \frac{4}{6} > \frac{1}{2}, \quad \frac{4}{6} < \frac{4}{3};$$

$$\frac{3}{2} \times \frac{4}{3} = \frac{12}{6} \quad \text{and} \quad \frac{12}{6} > \frac{3}{2}, \quad \frac{12}{6} > \frac{4}{3}.$$

More generally, let $r > 0$, $s > 0$: If $r < 1$, then $rs < s$; if $r > 1$, then $rs > s$.

A very important property, especially useful in solving equations, is the following:

If r, s, t, are rational numbers, $s \neq 0$, and $rs = ts$, then $r = t$.

We may call this the restricted cancellation law for multiplication, the restriction being that the canceled factor must not be 0.

This law may be shown as follows: Suppose $s = \frac{a}{b}$. Then we have

$$r \times \frac{a}{b} = t \times \frac{a}{b},$$

then

$$\left(r \times \frac{a}{b}\right) \times \frac{b}{a} = \left(t \times \frac{a}{b}\right) \times \frac{b}{a}$$

and

$$r \times \left(\frac{a}{b} \times \frac{b}{a}\right) = t \times \left(\frac{a}{b} \times \frac{b}{a}\right)$$

because the multiplication of rational numbers is associative;

$$r \times 1 = t \times 1$$

because $\frac{a}{b} \times \frac{b}{a} = 1$, and

$$r = t$$

because 1 is a multiplicative identity.

In a similar way, we can show that if $sr = st$ and $s \neq 0$, then $r = t$.

The usefulness of these properties will be revealed in the exercises and in later work. But let us now do a problem where these properties help.

Consider the following computation problem:

$$\left(\frac{22}{7} \times 48\right) \times \frac{7}{22} = \frac{7}{22} \times \left(\frac{22}{7} \times 48\right) \text{---Multiplication is commutative.}$$

$$= \left(\frac{7}{22} \times \frac{22}{7}\right) \times 48 \text{ -- Multiplication is associative.}$$

$$= 1 \times 48 \text{ ------- The product of a number and its reciprocal is 1.}$$

$$= 48. \text{ ---------- 1 is the multiplicative identity.}$$

The product, 48, was obtained with very little computation and effort.

Exercise Set 5

1. Using the properties of multiplication, find easier ways of computing each of the following and state the properties used:

a. $\left(7 \times \frac{2}{3}\right) \times \frac{3}{2}$ **c.** $\frac{1}{2} \times \left(4 + \frac{2}{5}\right)$

b. $\left(\frac{22}{7} \times \frac{63}{63}\right) \times \frac{7}{22}$ **d.** $\left(6 + \frac{4}{5}\right) \times \frac{1}{2}$

2. Show that if $3a = 3b$, then $a = b$.

3. Show that if $\frac{a}{3} = \frac{b}{3}$, then $a = b$.

4. Show that if $\frac{a}{c} = \frac{b}{c}$, then $a = b$.

5. Show that if $\frac{3}{a} = \frac{3}{b}$, then $a = b$.

6. Show that if $\frac{c}{a} = \frac{c}{b}$, then $a = b$.

7. Show that if r, s, t, are rational numbers; the following statements are true:

a. $(rs)t = r(st)$.

b. $r(s + t) = rs + rt$.

c. $(s + t)r = sr + tr$.

d. If $sr = st$ and $s \neq 0$, then $r = t$.

8. What is the reciprocal of $\frac{5}{8}$? Show that your answer is correct.

9. What is the reciprocal of $\frac{4}{3}$? Show that your answer is correct.

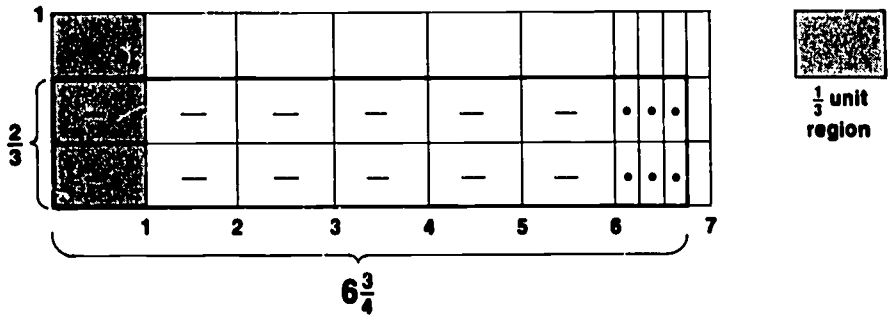
10. Show that the only multiplicative identity is 1.

11. Show that a rational number other than 0 has exactly one reciprocal. Hint: Use the property that if $r \times s = t \times s$ with $s \neq 0$, then $r = t$.

SOME SHORTCUTS BASED ON PROPERTIES OF MULTIPLICATION

By using the properties of multiplication, computation can often be greatly shortened. This is especially true for the distributive property of multiplication over addition. Let us return to our rectangular region for the problem

$$\frac{2}{3} \times 6\frac{3}{4}.$$



Each box marked with a dash is $\frac{1}{3}$ of a unit region, as 3 of these boxes fill up one unit region. Twelve of such boxes have an area of 4 unit regions. Each box marked with a dot is $\frac{1}{12}$ of a unit region, as 12 of them fill up one unit region. The rectangular region for our product contains 6 of such boxes, giving us $\frac{6}{12}$, or $\frac{1}{2}$, more. Hence the product $\frac{2}{3} \times 6\frac{3}{4}$ has the value $4 + \frac{1}{2}$, or $4\frac{1}{2}$. Note that the portion of the region made up of boxes with dashes has the value $\frac{2}{3} \times 6$, or $\frac{12}{3}$, or 4. The portion of the region marked with dots is $\frac{2}{3} \times \frac{3}{4}$, or $\frac{6}{12}$, or $\frac{1}{2}$. We summarize as follows:

$$
\begin{aligned}
\frac{2}{3} \times 6\frac{3}{4} &= \frac{2}{3} \times \left(6 + \frac{3}{4}\right) \\
&= \left(\frac{2}{3} \times 6\right) + \left(\frac{2}{3} \times \frac{3}{4}\right) \\
&= \frac{12}{3} + \frac{6}{12} \\
&= 4 + \frac{1}{2} \\
&= 4\frac{1}{2}.
\end{aligned}
$$

Computation can frequently be considerably shortened if reductions are introduced before completing computations. This reduction is based on the following property:

> *To divide a product by a number, we may divide any one of its factors by this number.*

Thus, instead of the computation

$$\frac{6 \times 8}{2} = \frac{48}{2} = 24,$$

we may compute

$$\frac{6 \times 8}{2} = \left(\frac{6}{2}\right) \times 8 = 3 \times 8 = 24$$

or, alternatively,

$$\frac{6 \times 8}{2} = 6 \times \left(\frac{8}{2}\right) = 6 \times 4 = 24.$$

Moreover, from the equality

$$\frac{a \times c}{b \times c} = \frac{a}{b}$$

we may say that

> *the value of a fraction is unchanged if both the numerator and the denominator are divided by the same number.*

This property enables us to "reduce" fractions to obtain equivalent ones with smaller numerators and denominators.

The last two properties mentioned are most useful in simplifying computations. In the following example, both the numerator and the denominator are divided by 9 and by 7:

$$\begin{aligned} \frac{27}{35} \times \frac{28}{45} &= \frac{\overset{3}{\cancel{27}} \times \overset{4}{\cancel{28}}}{\underset{5}{\cancel{35}} \times \underset{5}{\cancel{45}}} \\ &= \frac{3 \times 4}{5 \times 5} \\ &= \frac{12}{25}. \end{aligned}$$

Some of the steps could be eliminated to obtain the shorter form:

$$\frac{\overset{3}{\cancel{27}}}{\underset{5}{\cancel{35}}} \times \frac{\overset{4}{\cancel{28}}}{\underset{5}{\cancel{45}}} = \frac{12}{25}.$$

Exercise Set 6

NOTE.—Exercise **2**, marked *, is suggested for use by teachers, not pupils.

1. Using whatever shortcuts you can, compute in two different ways:

a. $\frac{48}{75} \times \frac{35}{64}$ **b.** $\frac{3}{4} \times 8\frac{4}{5}$ **c.** $3\frac{3}{4} \times 8\frac{4}{5}$

* **2.** For the numbers we have studied so far, if $r = s + p$ and $p > 0$, then $r > s$. Moreover, if $r > s$, then for some $p > 0$ we have $r = s + p$. Using these ideas, prove that—

a. if $r > s$ and $t > 0$, then $rt > st$;

b. if $\frac{a}{b} > \frac{c}{d}$ and $c > 0$, then $\frac{b}{a} < \frac{d}{c}$.

3. The following are common mistakes made by students. What would you do when they occur? What would you do to minimize their occurrence?

a. $5 \times (4 \times 6) = (5 \times 4) \times (5 \times 6)$.
Student's reason: distributivity

b. $5 \times (4 + 6) = 5 \times 4 + 6$.
Student's reason: distributivity

c. $\frac{\overset{2}{\cancel{4}} + 6}{\underset{1}{\cancel{2}}} = 2 + 6$, or 8.

Student's reason: dividing numerator and denominator by 2

d. $\frac{\cancel{6}4}{1\cancel{6}} = \frac{4}{1}$, or 4.

(Correct answer, but what about the work? What is wrong with canceling 6s?)

"OF" AND "TIMES"

In most quantitative sentences involving "of," the mathematical meaning is multiplication. However, there are occasions when that is not the case. For example:

> "Two *of* the three oranges were bad."

We do not mean here that 2 × 3, or 6, oranges were bad. We mean that two oranges were bad and one was good. The context should be sufficiently clear to tell whether or not multiplication is intended. Here is an example where multiplication is intended for "of":

> Three *of* the ten-dollar bills were counterfeit. How much collected money was counterfeit?

Students could profit from pointing out the meaning for "of" in a variety of quantitative statements.

Exercise Set 7

Decide on the meaning intended for "of" in each of the following quantitative sentences:

1. One-half *of* $5 was spent on food.

2. The rectangle has a width *of* 3 feet.

3. Two *of* the triplets were girls.

4. He ran $\frac{2}{3}$ *of* a mile.

5. He collected $\frac{2}{5}$ *of* the $10 owed.

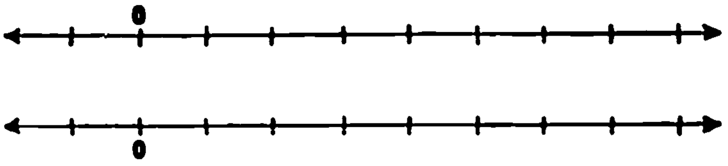
6. Two *of* the six apples were rotten. How many rotten apples were there?

7. One-half *of* the six apples were rotten. How many rotten apples were there?

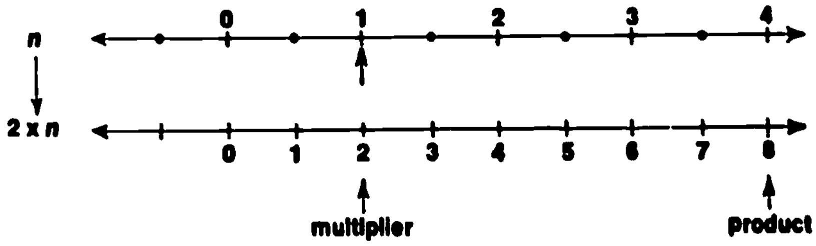
ALTERNATE APPROACHES TO MULTIPLICATION WITH FRACTIONS

Using the Number Line

In this approach one begins with two parallel lines having equally spaced division marks to be assigned numbers. Select two marks, one directly above the other, and assign the value 0 to each.

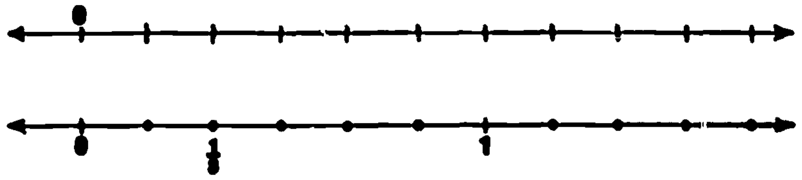


The next assignment depends on the numbers to be multiplied. For computing 2 × 4, we assign the number 1 to the mark that is one division to the right of the 0 mark of the lower scale. The assignment of 0 and 1 determines the values to be assigned to all the other scale marks. Above the scale mark for 2 (our multiplier) on the lower scale, assign 1 to the upper scale. Values for all the other division marks of the upper scale are now determined.



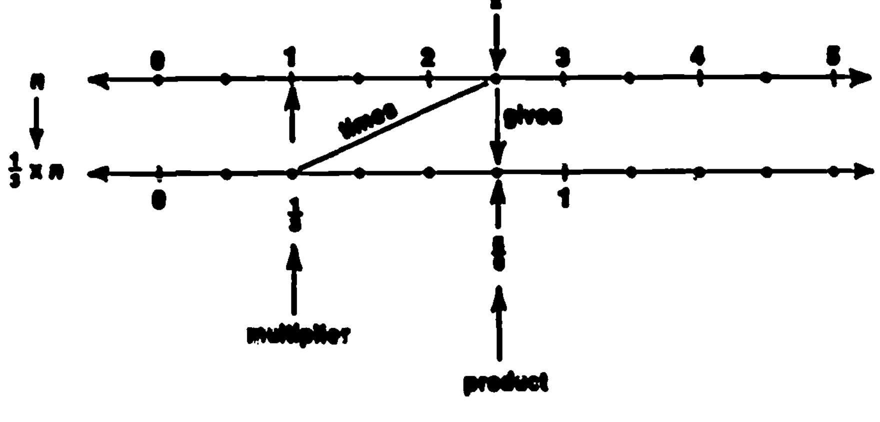
For the upper scale every two divisions count for 1, while for the lower scale single divisions count for 1. If we look carefully at the two scales, we notice that for each number of an upper-scale mark, the scale mark directly below is given a number that is its double. In particular, the mark directly under the mark for 4 has the value 8. (Of course, the roles of the upper scale and the lower scale may be interchanged.)

The same idea will now be used to compute $\frac{1}{3} \times \frac{5}{2}$. As before, we assign 0 to two division marks that are directly opposite each other. The product of the denominators—2 × 3, or 6—suggests that we use 6 divisions for 1 on the lower scale, so 1 is assigned to the mark 6 units to the right of "0" on the lower scale. The assignment of 0 and 1 determines what the assignments must be for the other scale marks. Label the scale mark for $\frac{1}{3}$ on the lower scale. Directly above $\frac{1}{3}$ on the lower scale (our multiplier), find the



mark of the upper scale and label it "1". Now that 0 and 1 are assigned on the upper scale, all the other scale marks have their values determined. We can now read on the lower scale the product of any number shown on the upper scale and the multiplier, $\frac{1}{3}$. For example, we show here that

$$\frac{1}{3} \times \frac{5}{2} = \frac{5}{6}.$$



We shall see in a later chapter that this method applies equally well to computing products involving negative factors. With very little change this method can also be used to compute quotients.

Stretchers and Shrinkers

The University of Illinois Committee on School Mathematics (UICSM) has been experimenting with the method of "stretchers and shrinkers" for use by slow learners and reports much success. Very briefly, one interprets a fraction such as $\frac{2}{3}$ to mean that when it is applied to something it stretches it by a factor of 2 and shrinks the result by a factor of 3. Moreover, the order of stretching and shrinking does not matter. Thus, in the indicated product $\frac{4}{3} \times \frac{2}{5}$ we have stretchers 4 and 2 giving a total stretch of 4 × 2, or 8. The shrinkers 3 and 5 produce a total shrink of 3 × 5, or 15, The net result is a stretch of 8 followed by a shrink of 15, which may be expressed by $\frac{8}{15}$. Hence, $\frac{4}{3} \times \frac{2}{5} = \frac{8}{15}$.

Exercise Set 8

1. Use a number-line approach to compute each of the following indicated products:

a. $3 \times 2\frac{1}{2}$ **c.** $\frac{2}{5} \times \frac{3}{4}$ **e.** $\frac{3}{2} \times \frac{5}{2}$

b. $\frac{2}{3} \times \frac{7}{5}$ **d.** $\frac{3}{5} \times \frac{1}{2}$ **f.** $\frac{2}{3} \times \frac{4}{5}$

2. Try to show that the number-line approach for computing products will always work. One way of doing this is to work with the general product $\frac{a}{b} \times \frac{c}{d}$. Count *bd* division marks to the right of "0" on the lower scale and label the final division mark "1." Find the division mark for $\frac{a}{b}$ on the lower scale and assign to the mark directly above it the value 1. Read the value for the product $\frac{a}{b} \times \frac{c}{d}$ directly under the scale mark for $\frac{c}{d}$.

SUMMARY OF KEY IDEAS

1. Have students find rectangular regions to go with indicated products. Have them obtain upper and lower estimates for the products but postpone computing them.

2. Have students find indicated products to go with given rectangular regions. Obtain upper and lower estimates for the products but postpone computing them.

3. Have students find rectangular regions to go with indicated products and compute the products by "counting boxes." Obtain upper and lower estimates for the products.

4. Have students find rectangular regions to go with indicated products and observe a pattern for the computed products. Try to help them observe that in each case the product of the numerators is the numerator of the computed product and the product of the denominators is the denominator of the computed product. Try to have them notice that the product of the denominators tells into how many equal parts the unit region is split and that the product of the numerators tells how many of these parts constitute the region that goes with the product.

5. Except for 0, all rational numbers have multiplicative inverses. "Inverting the fraction" is an easy way to obtain a fraction for the reciprocal.

6. Multiplication of rational numbers has the same properties a- multiplication of whole numbers, with some minor exceptions. Multiplication of rational numbers is closed, commutative, and associative; is distributes over addition; and it has an identity element.

Review Exercises

1. Find a rectangular region for each of the following indicated products. Obtain lower and upper estimates for each product. Compute each product by counting boxes. Check your result by using the traditional algorithm.

a. $\frac{2}{5} \times \frac{3}{4}$ **c.** $1\frac{2}{5} \times 2\frac{3}{4}$ **e.** $1\frac{4}{5} \times 2\frac{3}{4}$

b. $\frac{6}{5} \times \frac{3}{4}$ **d.** $1\frac{3}{4} \times 2\frac{2}{5}$ **f.** $1\frac{3}{4} \times 2\frac{3}{5}$

2. Try to explain each of the following errors frequently made by students. How can we make their occurrence less frequent?

a. $\frac{2}{7} \times \frac{3}{7} = \frac{5}{7}$. **c.** $2\frac{1}{2} \times 3\frac{1}{2} = 6\frac{1}{4}$.

b. $\frac{2}{7} \times \frac{3}{7} = \frac{6}{7}$. **d.** $3 \times \frac{2}{5} = \frac{6}{15}$.

3. List as many properties of multiplication of rational numbers as you can. Give one example to illustrate each property.

4. List as many properties as you can for multiplication of rational numbers that do not hold for multiplication of whole numbers. Illustrate each by an example.

5. Compute each of the following products. Conjecture a generalization these illustrate and try to prove it.

a. $1\frac{1}{2} \times 1\frac{1}{2}$ **c.** $3\frac{1}{2} \times 3\frac{1}{2}$ **e.** $5\frac{1}{2} \times 5\frac{1}{2}$

b. $2\frac{1}{2} \times 2\frac{1}{2}$ **d.** $4\frac{1}{2} \times 4\frac{1}{2}$ **f.** $6\frac{1}{2} \times 6\frac{1}{2}$

6. Compute each of the following products. Conjecture a generalization these illustrate and try to prove it.

a. $1\frac{1}{3} \times 1\frac{2}{3}$ **c.** $3\frac{1}{3} \times 3\frac{2}{3}$ **e.** $2\frac{2}{5} \times 2\frac{3}{5}$

b. $2\frac{1}{3} \times 2\frac{2}{3}$ **d.** $1\frac{2}{5} \times 1\frac{3}{5}$ **f.** $3\frac{2}{5} \times 3\frac{3}{5}$

7. Conjecture a generalization that includes both of the ones given in exercises **5** and **6** above.

8. What must be true about rational numbers r and s if the following conditions are true?

a. $rs = 0$. **b.** $rs = 1$.

9. Play the following game with someone.
Ten cards (library cards will do) are labeled with the ten numerals

0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

one numeral on each card. The cards are shuffled face down, and one card is drawn. The number for this card is then assigned by each player in turn to one of the letters A, B, C, or D in the following expression:

$$\frac{A}{B+1} \times \frac{B+C}{D+1}$$

Each player has his own copy of this expression for making assignments, After each assignment the card is replaced in the deck, the deck is again shuffled, a card is drawn, and both players make their individual assignments to their own identical expressions. The assignments are made until all the letters have been assigned numbers. Remember that an assignment for B must be made in two places. The player with the greater number wins. Here are assignments made by a player:

$$\begin{array}{llc} 7 \to A & 2 \to C & \dfrac{7}{6+1} \times \dfrac{6+2}{9+1} \\ 6 \to B & 9 \to D & \end{array}$$

What assignments would have given a greater value?

What assignment would have given the greatest value? Assume that the numbers are still 2, 6, 7, 9.

Harry D. Ruderman

DIVISION OF RATIONAL NUMBERS

6

1. **What meanings shall we give to a quotient of two rational numbers?**
2. **What is the missing-factor approach to computing quotients?**
3. **What is the reciprocal approach to computing quotients?**
4. **Why do we "invert and multiply" to compute quotients?**

A basic pedagogic strategy for enabling a student to figure something out for himself is to extend in a natural manner ideas already known to him to the new ideas being developed. This strategy was employed in moving from the multiplication of whole numbers to the multiplication of rational numbers. This strategy will also be used in moving from the division of whole numbers to the division of rational numbers.

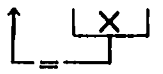
A STRATEGY FOR ENABLING A STUDENT TO FIGURE OUT FOR HIMSELF HOW TO COMPUTE QUOTIENTS

Most students would have little difficulty obtaining the number 4 for the frame in the open sentence

$$8 \div 2 = \square .$$

If a student is asked to check, he might say that 2 × 4 = 8. If he does, then he has learned the fundamental idea of division. In other words, the above division sentence asks that we find a number for the frame which, when multiplied by 2, gives 8. We may diagram this as shown below:

$$8 \div 2 = \square .$$



What number multiplied by 2 gives 8?

In this sentence 8 is the product, 2 is the given factor, and the number for the frame is the missing factor. This method for computing quotients will be referred to as the missing-factor method. It is the one we shall now use to compute indicated quotients of rational numbers. Notice how natural the extension is to rational numbers.

We begin by considering the problem of computing the indicated quotient

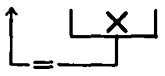
$$\frac{8}{9} \div \frac{2}{3}.$$

First we write an open sentence that goes with this problem and assume that the number for the frame may be represented by a fraction.

$$\frac{8}{9} \div \frac{2}{3} = \boxed{\frac{\ }{\ }}.$$

Just as with the whole numbers, this sentence asks for a number which when multiplied by $\frac{2}{3}$, gives the product $\frac{8}{9}$. We may diagram this problem just as before to emphasize its meaning:

$$\frac{8}{9} \div \frac{2}{3} = \boxed{\frac{\ }{\ }}.$$

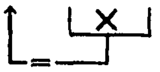


We are assuming that at this stage of his development, as is generally the case, the student knows the traditional algorithm for multiplying with fractions: multiply numerators to obtain the numerator of the product, and multiply denominators to obtain the denominator of the product. The missing factor for this problem is readily obtained as follows:

Its numerator is 4 because $2 \times 4 = 8$.
Its denominator is 3 because $3 \times 3 = 9$.

The missing factor for the frame is thus $\frac{4}{3}$. After "$\frac{4}{3}$" is entered into the frame, the diagram has this appearance:

$$\frac{8}{9} \div \frac{2}{3} = \boxed{\frac{4}{3}}.$$



A check of the result $\frac{4}{3}$ is immediate, for

$$\frac{8}{9} = \frac{2}{3} \times \frac{4}{3}.$$

Not only have we obtained a computed quotient for $\frac{8}{9} \div \frac{2}{3}$; we have also checked the result.

Brighter students might question this as a general method for computing quotients. They might remark that it is not very likely that the fraction for the product (or dividend) will have a numerator and a denominator that will be multiples of the numerator and the denominator of the fraction for the given factor (or divisor). This is a weakness that can now be corrected. We consider the problem of computing $\frac{7}{9} \div \frac{2}{3}$. An open sentence for this problem is:

$$\underbrace{\underset{\uparrow}{\frac{7}{9}} \div \underbrace{\frac{2}{3} = \boxed{\frac{}{}}}_{\times}}_{=}.$$

We observe that 9 is a multiple of 3 but that 7 is *not* a multiple of 2. In order to use the missing-factor method of computing quotients, the numerator of the product fraction must be a multiple of the numerator of the given factor fraction. We now have an opportunity to return to an earlier principle used when we added with fractions: "Multiplying both numerator and denominator by the same nonzero number gives an equivalent fraction." If we multiply both numerator and denominator of $\frac{7}{9}$ by 2, the numerator and the denominator of the resulting fraction, $\frac{14}{18}$, will have the desired property. We may now write the equivalent sentence

$$\frac{7 \times 2}{9 \times 2} \div \frac{2}{3} = \boxed{\frac{}{}},$$

or

$$\underbrace{\underset{\uparrow}{\frac{14}{18}} \div \underbrace{\frac{2}{3} = \boxed{\frac{}{}}}_{\times}}_{=}.$$

The missing factor for the frame is easily obtained as follows:

Its numerator is 14 ÷ 2, or 7.
Its denominator is 18 ÷ 3, or 6.

So the missing factor is $\frac{7}{6}$. Filling in the frame, we get

$$\underbrace{\underset{\uparrow}{\frac{14}{18}} \div \underbrace{\frac{2}{3} = \boxed{\frac{7}{6}}}_{\times}}_{=}.$$

We check by multiplying the given factor, $\frac{2}{3}$, and the missing factor, $\frac{7}{6}$:

$$\begin{aligned}\frac{2}{3} \times \frac{7}{6} &= \frac{14}{18}\\ &= \frac{7 \times 2}{9 \times 2}\\ &= \frac{7}{9},\end{aligned}$$

which is the given product.

It may happen that neither the numerator nor the denominator of the product fraction is an appropriate multiple. The following problem illustrates how this situation may be handled:

$$\frac{3}{4} \div \frac{2}{5} = \boxed{\frac{}{}}.$$

3 is *not* a multiple of 2.
4 is *not* a multiple of 5.

To use the missing-factor method, we must replace "$\frac{3}{4}$" by an equivalent fraction whose numerator and denominator are the right multiples. If we multiply both numerator and denominator by 2, giving

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8},$$

the new fraction, $\frac{6}{8}$, while "better," is still not suitable because 8 is not a multiple of 5. To remedy this defect, all that remains is to multiply both the numerator and the denominator of $\frac{6}{8}$ by 5 to obtain a fraction with the desired property:

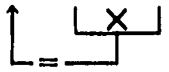
$$\frac{6}{8} = \frac{6 \times 5}{8 \times 5} = \frac{30}{40}.$$

The original open sentence

$$\frac{3}{4} \div \frac{2}{5} = \boxed{\frac{}{}}$$

may now be replaced by the equivalent open sentence

$$\frac{30}{40} \div \frac{2}{5} = \boxed{\frac{}{}}.$$



But

$$30 \div 2 = 15.$$

$$40 \div 5 = 8.$$

So the number for the frame is $\frac{15}{8}$. Filling in the frame, we get

$$\underbrace{\frac{30}{40} \div \underbrace{\frac{2}{5} = \frac{15}{8}}_{\times}}_{=}.$$

Of course at the very beginning we could have multiplied both numerator and denominator of $\frac{3}{4}$ by 10, which is 2×5, to obtain a fraction with the desired property, namely, $\frac{30}{40}$. Once again we see that our result checks by multiplying given factor and missing factor:

$$\begin{aligned} \frac{2}{5} \times \frac{15}{8} &= \frac{30}{40} \\ &= \frac{3 \times 10}{4 \times 10} \\ &= \frac{3}{4}. \end{aligned}$$

If we do not compute the products 30 and 40 but use instead indicated products, the traditional algorithm emerges:

$$\frac{3}{4} \div \frac{2}{5} = \boxed{\quad - \quad}.$$

$$\begin{aligned} \frac{3 \times 2 \times 5}{4 \times 2 \times 5} \div \frac{2}{5} &= \frac{3 \times 5}{4 \times 2} \\ &= \boxed{\frac{3}{4} \times \frac{5}{2}}. \end{aligned}$$

The original indicated quotient of the first line is equivalent to the indicated quotient just below it, so we obtain

$$\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2}.$$

This is recognized as the traditional invert-and-multiply method of computing quotients. We observe that the net effect of our procedure is to replace the given factor (the divisor) by its reciprocal and simultaneously to replace the division by multiplication. Another way of saying this is:

> *Dividing by a nonzero number gives the same result as multiplying by its reciprocal.*

The missing-factor method helps the student discover the traditional invert-and-multiply approach by himself. However, it is probably wiser to have him work many problems in division without using the traditional algorithm and then reveal it to him only if he doesn't discover it himself. Clearly, the missing-factor approach stresses important basic principles and provides a rationale that permits the student to figure out a computed quotient should he forget the traditional algorithm. Moreover, the traditional algorithm may be derived in all generality by the missing-factor approach. Before doing this, let us see one more division problem worked out, somewhat streamlined:

$$\frac{7}{5} \div \frac{2}{3} = \boxed{\frac{\;\;}{\;\;}}.$$

$$\frac{7 \times 2 \times 3}{5 \times 2 \times 3} \div \frac{2}{3} = \boxed{\frac{7 \times 3}{5 \times 2}}$$

$$= \frac{21}{10}.$$

Hence

$$\frac{7}{5} \div \frac{2}{3} = \frac{21}{10}.$$

Check:

$$\frac{2}{3} \times \frac{21}{10} = \frac{42}{30}$$

$$= \frac{7 \times 6}{5 \times 6}$$

$$= \frac{7}{5}.$$

The proof for the traditional algorithm proceeds in exactly the same way. We assume $c \neq 0$.

$$\frac{a}{b} \div \frac{c}{d} = \boxed{\frac{\;\;}{\;\;}}.$$

$$\frac{acd}{bcd} \div \frac{c}{d} = \frac{ad}{bc}$$

$$= \frac{a}{b} \times \frac{d}{c},$$

or

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

We have now justified the traditional algorithm in all generality and have shown that dividing by a nonzero rational number gives the same result as multiplying by its reciprocal.

Exercise Set 1

1. Use the missing-factor method to compute each of the following indicated quotients, and check your answers.

a. $\frac{4}{9} \div \frac{1}{3}$

b. $\frac{9}{8} \div \frac{3}{2}$

c. $\frac{9}{10} \div \frac{3}{5}$

d. $\frac{4}{7} \div \frac{2}{3}$

e. $\frac{3}{10} \div \frac{2}{5}$

f. $\frac{1}{2} \div \frac{2}{3}$

g. $1\frac{1}{2} \div \frac{2}{3}$

h. $1\frac{1}{2} \div 1\frac{1}{3}$

2. Use the missing-factor method to compute each of the following indicated quotients, and check your answers.

a. $\frac{3}{2} \div \frac{5}{2}$

b. $\frac{3}{4} \div \frac{5}{4}$

c. $\frac{3}{7} \div \frac{5}{7}$

d. $\frac{3}{7} \div \frac{6}{7}$

e. $\frac{8}{5} \div \frac{2}{5}$

f. $\frac{8}{7} \div \frac{2}{7}$

g. $\frac{8}{9} \div \frac{2}{9}$

h. $\frac{8}{11} \div \frac{2}{11}$

3. What generalization is suggested by the problems in exercise 2? Try to prove your generalization.

4. Use the missing-factor method to compute each of the following pairs of indicated quotients, and check your answers.

a. $\frac{8}{9} + \frac{4}{3}$, $\frac{4}{3} + \frac{8}{9}$

b. $\frac{2}{9} + \frac{4}{3}$, $\frac{4}{3} + \frac{2}{9}$

c. $\frac{6}{5} + \frac{2}{3}$, $\frac{2}{3} + \frac{6}{5}$

d. $\frac{3}{4} + \frac{2}{5}$, $\frac{2}{5} + \frac{3}{4}$

e. $2\frac{1}{2} + \frac{4}{5}$, $\frac{4}{5} + 2\frac{1}{2}$

f. $2\frac{1}{2} + 1\frac{2}{3}$, $1\frac{2}{3} + 2\frac{1}{2}$

g. $\frac{2}{3} + 5$, $5 + \frac{2}{3}$

h. $7 + 5$, $5 + 7$

i. $8 + \frac{2}{3}$, $\frac{2}{3} + 8$

j. $\frac{2}{3} + \frac{4}{6}$, $\frac{4}{6} + \frac{2}{3}$

k. $1 + 1\frac{1}{2}$, $1\frac{1}{2} + 1$

l. $6 + 1\frac{1}{2}$, $1\frac{1}{2} + 6$

5. What generalization is suggested by the problems in exercise 4? Try to prove your generalization.

6. Use the missing-factor approach to compute each of the following indicated quotients, and check your result.

a. $2\frac{4}{5} \div 2$

b. $6\frac{4}{9} \div 2$

c. $6\frac{6}{7} \div 3$

d. $12\frac{3}{5} \div 3$

e. $12\frac{8}{9} \div 4$

f. $10\frac{5}{8} \div 5$

7. What generalization is suggested by the problems in exercise 6? Try to prove your generalization.

INTERPRETING A FRACTION AS A QUOTIENT

Recall that many fractions name whole numbers. For example:

$$\frac{6}{1} = 6, \quad \frac{6}{2} = 3, \quad \frac{6}{3} = 2, \quad \frac{6}{6} = 1.$$

Corresponding to each of the above fractions we have the following quotients:

$$6 \div 1 = 6, \quad 6 \div 2 = 3, \quad 6 \div 3 = 2, \quad 6 \div 6 = 1.$$

It follows, then, that

$$\frac{6}{1} = 6 \div 1, \quad \frac{6}{2} = 6 \div 2, \quad \frac{6}{3} = 6 \div 3, \quad \text{and} \quad \frac{6}{6} = 6 \div 6.$$

It would seem, therefore, that for any whole numbers a and b with $b \neq 0$, we have

$$a \div b = \frac{a}{b}$$

and, in particular,

$$2 \div 3 = \frac{2}{3}.$$

To show that this is so, we may argue as follows:

$$\begin{aligned} 2 \div 3 &= \frac{2}{1} \div \frac{3}{1} \\ &= \frac{2}{1} \times \frac{1}{3} \\ &= \frac{2 \times 1}{1 \times 3} \\ &= \frac{2}{3}. \end{aligned}$$

The general argument is an identical one. For $b \neq 0$,

$$\begin{aligned} a \div b &= \frac{a}{1} \div \frac{b}{1} \\ &= \frac{a}{1} \times \frac{1}{b} \\ &= \frac{a \times 1}{1 \times b} \\ &= \frac{a}{b}. \end{aligned}$$

In more advanced mathematics the symbol for division, "÷," is seldom seen. In its place the fraction bar is used almost exclusively. Frequently, instead of

$$\frac{2}{3} \div \frac{4}{5}$$

one sees the "complex" fraction

$$\frac{\frac{2}{3}}{\frac{4}{5}}$$

For typographical reasons this may sometimes be written as shown below:

$$\frac{2}{3}\Big/\frac{4}{5} \quad \text{or} \quad \frac{2/3}{4/5}$$

A complex fraction is a fraction with either the numerator, the denominator, or both named by a fraction or a mixed numeral. When complex fractions are used, we should remember their basic meaning.

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \div \frac{4}{5}$$

$$= \frac{2}{3} \times \frac{5}{4}$$

$$= \frac{10}{12}.$$

The lengths of the horizontal fraction bars eliminate possible ambiguities by showing which number is the divisor. For example,

$$\frac{\frac{1}{2}}{3} \text{ means } \frac{1}{2} \div 3 = \frac{1}{6},$$

while

$$\frac{1}{\frac{2}{3}} \text{ means } 1 \div \frac{2}{3} = \frac{3}{2}.$$

Exercise Set 2

1. Express each of the following quotients as a complex fraction:

a. $\frac{5}{2} \div \frac{4}{3}$ c. $\frac{5}{2} \div 4$ e. $\frac{2}{1} \div \frac{3}{1}$

b. $5 \div \frac{4}{3}$ d. $\frac{1}{2} \div \frac{4}{3}$ f. $1\frac{1}{2} \div 1\frac{1}{3}$

2. Compute the value of each of the following complex fractions:

a. $\dfrac{\frac{5}{2}}{\frac{3}{2}}$ **d.** $\dfrac{\frac{4}{5}}{2}$ **g.** $\dfrac{\frac{5}{2}}{\frac{5}{4}}$

b. $\dfrac{\frac{5}{4}}{\frac{3}{2}}$ **e.** $\dfrac{4}{\frac{5}{2}}$ **h.** $\dfrac{\frac{4}{5}}{\frac{5}{4}}$

c. $\dfrac{\frac{5}{3}}{\frac{5}{3}}$ **f.** $\dfrac{1\frac{1}{2}}{5\frac{1}{3}}$ **i.** $\dfrac{\frac{1}{2}}{\frac{1}{3}}$

INTERPRETING THE REMAINDER WHEN DIVIDING WHOLE NUMBERS

We are now in a position to give a meaning in terms of rational numbers to the *remainder* when dividing whole numbers. Consider the problem of dividing 43 by 8:

$$\begin{array}{r} 5 \\ 8\overline{)43} \\ \underline{40} \\ 3 \end{array}$$

(NOTE.—In the system of whole numbers, "43 ÷ 8" is actually a meaningless expression, as it names no whole number. However, the need for such quotients in practical problems requires that some meaning be given, even for whole numbers. Introducing remainders helps to fill the need.)

In this problem we obtain a quotient of 5 with a remainder of 3. But what meaning does 3 have other than just how many "left over"? The following steps suggest a meaning:

$$\begin{aligned} 43 \div 8 &= \frac{43}{8} \\ &= \frac{40 + 3}{8} \\ &= \frac{40}{8} + \frac{3}{8} \\ &= 5 + \frac{3}{8} \\ &= 5\,\frac{3}{8}. \leftarrow\text{remainder} \end{aligned}$$

It would appear that the remainder can be regarded as the numerator of the fraction in the mixed numeral for the computed quotient. With this problem as a background it is now appropriate to carry out the computation for 43 ÷ 8 as follows:

$$\begin{array}{r} 5\,\frac{3}{8} \\ 8\overline{)43} \\ \underline{40} \\ 3 \end{array}$$

We show the computed quotient as $5\,\frac{3}{8}$ rather than 5 with a remainder of 3. Notice that the remainder is divided by the divisor to obtain the fraction of the mixed-numeral quotient. To check the result, $5\,\frac{3}{8}$, we compute the product:

$$\begin{aligned} 8 \times 5\,\frac{3}{8} &= 8 \times \left(5 + \frac{3}{8}\right) \\ &= (8 \times 5) + \left(8 \times \frac{3}{8}\right) \\ &= 40 + 3 \\ &= 43. \end{aligned}$$

In practical problems the remainder may have other significances:

1. A limousine can take 8 passengers. How many limousines are needed

to take 43 passengers? Computing 43 ÷ 8 gives, as we have seen, $5\frac{3}{8}$. Clearly we cannot use $\frac{3}{8}$ of a limousine, so we must have 6 limousines.

2. A certain dress requires 8 yards of cloth. How many such dresses can be made from 43 yards? Dividing 43 by 8 gives the value $5\frac{3}{8}$. The question calls for the number of dresses that can be made. Interpreting this to mean complete dresses, the answer now is 5. Only 5 dresses can be made.

In the first problem we needed the smallest whole number greater than $5\frac{3}{8}$, which is 6. In the second problem we needed the greatest whole number less than $5\frac{3}{8}$, which is 5. In either case $5\frac{3}{8}$ is not the correct answer to the problem. However, if a cord 43 yards long is to be split into 8 pieces of equal length, each piece must be $5\frac{3}{8}$ yards long. The correct answer to a division problem depends, as we have just seen, on the nature of the problem as well as the computed quotient.

Exercise Set 3

1. Compute each of the following quotients, expressing each as a mixed numeral:

a. 37 ÷ 5

b. 49 ÷ 6

c. 100 ÷ 12

d. 100 ÷ 36

e. 1,000 ÷ 12

f. 1,000 ÷ 231

g. 1,728 ÷ 231

h. 1,728 ÷ 1,000

i. 1,000 ÷ 9

2. Find two practical situations for computing 37 ÷ 5, one requiring the next greater whole number, the other the next lesser whole number. Find a situation where $7\frac{2}{5}$ is the correct answer.

3. Show in two different ways that $100 \div 8 = 12\frac{1}{2}$.

PROPERTIES OF 0 AND 1 IN DIVISION

We shall now investigate expressions such as

$$0 \div \frac{2}{3}, \quad \frac{2}{3} \div 0, \quad 0 \div 0,$$

$$1 \div \frac{2}{3}, \quad \frac{2}{3} \div 1.$$

You might rightfully ask, What for? Rarely is there the need to compute such quotients. We could try to answer by saying that a bright student in your class might ask you to work such a problem. What kind of answer would you then give? An unsatisfactory answer could bother him considerably. Besides, having a correct answer to such questions rounds out the entire story of dividing with fractions, and it can be found with very little effort.

Let us consider each of the above quotient expressions, one at a time.

$$\begin{aligned} 0 \div \frac{2}{3} &= \frac{0}{1} \times \frac{3}{2} \\ &= \frac{0 \times 3}{1 \times 2} \\ &= \frac{0}{2} \\ &= 0. \end{aligned}$$

This result suggests that perhaps whenever $\frac{a}{b} \neq 0$ we have

$$0 \div \frac{a}{b} = 0.$$

Let us try to show this, using the very same method as for $\frac{2}{3}$. If $\frac{a}{b} \neq 0$,

$$\begin{aligned} 0 \div \frac{a}{b} &= \frac{0}{1} \times \frac{b}{a} \\ &= \frac{0 \times b}{1 \times a} \\ &= \frac{0}{a} \\ &= 0. \end{aligned}$$

(Note that if $\frac{a}{b} \neq 0$, then it follows that $a \neq 0$. If $a \neq 0$, then $\frac{0}{a} = 0$. See chap. 2.)

It follows then that for every rational number r different from 0,

$$0 \div r = 0 \quad \text{and} \quad \frac{0}{r} = 0.$$

Summarizing:

Whenever 0 *is divided by a nonzero number, the result is* 0.
For every number $r \neq 0$, $0 \div r = 0$ *and* $\frac{0}{r} = 0$.

Our next expression, "$\frac{2}{3} \div 0$," suggests the question: What can we say about expressions indicating division by 0? We use a division sentence and its equivalent multiplication sentence.

$$\frac{2}{3} \div 0 = \square$$

is equivalent to the multiplication sentence

$$\frac{2}{3} = \square \times 0.$$

This sentence asks for a number whose product with 0 gives $\frac{2}{3}$. But the product of every number and 0 is 0, so there is no number for the frame that makes the sentence true. The only conclusion we can come to is that "$\frac{2}{3} \div 0$" names *no* number. In place of $\frac{2}{3}$ we could just as well have used any nonzero number and have used the same reasoning to obtain a

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similar conclusion. In other words, if number $r \neq 0$, the expressions "$r \div 0$" and therefore also "$\frac{r}{0}$" name *no* number.

But what if $r = 0$? In other words, what number fits the frame here:

$$0 \div 0 = \square$$

or, equivalently,

$$0 = \square \times 0$$

(the corresponding multiplication sentence)? Suprisingly, every number fits! For example:

$$0 = \boxed{5} \times 0.$$

$$0 = \boxed{9} \times 0.$$

But in mathematics we like to have expressions assigned one and only one meaning. Since the above argument shows that "$0 \div 0$" could name any number with equal justification, the expression "$0 \div 0$" is ambiguous and we agree not to use it in meaningful mathematical sentences. In other words, we regard "$0 \div 0$" as naming *no* number.

Summarizing:

> *If an expression indicates division by* 0, *then the expression names no number. That is, for every number r the expressions* "$r \div 0$" *and* "$\frac{r}{0}$" *name no number.*

Let us now investigate indicated quotients involving 1. Let's begin with $1 \div \frac{2}{3}$.

$$\begin{aligned} 1 \div \frac{2}{3} &= \frac{1}{1} \times \frac{3}{2} \\ &= \frac{1 \times 3}{1 \times 2} \\ &= \frac{3}{2}, \end{aligned}$$

the reciprocal of the number $\frac{2}{3}$ with which we started. We could just as well have used $\frac{a}{b}$ in place of $\frac{2}{3}$ and obtained the generalization

$$1 \div \frac{a}{b} = \frac{b}{a},$$

the reciprocal of $\frac{a}{b}$ (provided $\frac{a}{b} \neq 0$), or, equivalently,

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

The next expression involves dividing by 1. From our information with whole numbers we might expect that dividing by 1 has no effect on the number being divided. Let us try this with $\frac{2}{3} \div 1$.

$$\frac{2}{3} \div 1 = \frac{2}{3} \div \frac{1}{1}$$

$$= \frac{2}{3} \times \frac{1}{1}$$

(from the division algorithm)

$$= \frac{2 \times 1}{3 \times 1}$$

$$= \frac{2}{3},$$

the very same as the number being divided. In exactly the same way we could have shown that

$$\frac{a}{b} \div 1 = \frac{a}{b}.$$

Summarizing:

> *If* 1 *is divided by a nonzero number, the reciprocal of the number is obtained. In general terms,*
>
> $$1 \div \frac{a}{b} = \frac{b}{a} \quad \textit{provided} \quad \frac{a}{b} \neq 0.$$
>
> *If a number is divided by* 1, *the result is that number. In general terms, for every number r,*
>
> $$r \div 1 = r \quad \textit{and} \quad \frac{r}{1} = r.$$

Exercise Set 4

1. Which of the following expressions name no number? Why?

a. $0 \div 5$

b. $5 \div 0$

c. $5 \div 5$

d. $5 \div 1$

e. $1 \div 5$

f. $0 \div 1$

g. $1 \div 0$

h. $1 \div \frac{0}{5}$

i. $1 \div \frac{5}{0}$

j. $\dfrac{0}{\frac{1}{0}}$

k. $\dfrac{0}{\frac{0}{1}}$

l. $\dfrac{1}{\frac{0}{1}}$

m. $\dfrac{\frac{1}{0}}{1}$

n. $\dfrac{0}{\frac{1}{0}}$

o. $\dfrac{0}{\frac{1}{1}}$

2. Compute each of the following indicated quotients when possible.

a. $1 \div \frac{3}{4}$

b. $\frac{3}{4} \div 1$

c. $0 \div \frac{3}{4}$

d. $0 \div 0$

e. $0 \div 1$

f. $0 \div 2$

g. $2 \div 0$

h. $\left(1 \div \frac{2}{3}\right) \div 1$

i. $\left(1 \div \frac{2}{3}\right) \div 0$

j. $1 \div \left(1 \div \frac{2}{3}\right)$

k. $\frac{7}{8} \div \left(\frac{89}{98} \div 0\right)$

l. $0 \div \left(\frac{89}{98} \div \frac{78}{87}\right)$

m. $\frac{89}{98} \div \left(\frac{78}{87} \div \frac{78}{87}\right)$

n. $\left(\frac{88}{66} \div \frac{22}{33}\right) \div 0$

o. $\frac{4}{4} \div 7$

p. $\frac{5}{5} \div \frac{3}{4}$

q. $\frac{3}{4} \div \frac{5}{5}$

r. $1\frac{1}{2} \div \frac{3}{3}$

s. $\frac{5}{5} \div 1\frac{1}{2}$

t. $\frac{1}{5} \div 2\frac{1}{2}$

u. $\frac{0}{5} \div 2\frac{1}{2}$

3. What can be said about numbers r and s if $r \div s = 0$?

4. What can be said about numbers r and s if $r \div s = 1$?

5. What can be said about numbers r and s if $r \times s = 0$?

6. What can be said about numbers r and s if $r \times s = 1$?

7. What can be said about numbers r and s if $r \div s = r$?

8. What can be said about numbers r and s if $r \div s = s$?

9. What can be said about numbers r and s if $r \times s = r$?

PROPERTIES OF DIVISION OF RATIONAL NUMBERS

One may question the usefulness or desirability of considering properties of division in the classroom. We may try to justify considering properties by appealing to the time that can be saved in computation if the properties yield shortcuts. For example, knowing that division of rational numbers has a restricted distributivity, one can compute the following almost at sight:

$$12\frac{3}{8} \div 3 = 4\frac{1}{8}.$$

$$\left(6\frac{3}{4} + 9\frac{3}{5}\right) \div 3 = 2\frac{1}{4} + 3\frac{1}{5}.$$

We can also argue that studying properties enables one to compare different operations with a view to recognizing differences and similarities. Also, a deeper insight into the nature of mathematics is obtained from such a study.

For whole numbers a and b the expression "$a \div b$" rarely names a whole number. However, in the realm of rational numbers, the expression

"$a \div b$" always names a rational number except when $b = 0$. In fact, for all rational numbers r and s with $s \neq 0$, the expression "$r \div s$" always names a rational number. However, in view of the fact that a divisor cannot be 0, we conclude that *the set of rational numbers is not closed with respect to division.*

Division is neither commutative nor associative for rational numbers because it is neither commutative nor associative for whole numbers, which are among the rational numbers. For example:

$$8 \div 4 \neq 4 \div 8.$$

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2).$$

We have already shown that $r \div 1 = r$ for every rational number r. However, it is not true that for every r, $1 \div r = r$. In particular, $1 \div 2 \neq 2$. It follows that 1 is not an identity element for division of rational numbers. In fact, *division has no identity element for rational numbers*. We may show this as follows. Suppose t were an identity element for division. Then for every number r

$$r \div t = r \qquad \text{and} \qquad t \div r = r$$

from the definition of identity element, which is really a "do nothing" element for the operation. In particular, the two equations must hold for, say, $r = 4$ so that

$$4 \div t = 4 \qquad \text{and} \qquad t \div 4 = 4.$$

The only number that fits the left equation is $t = 1$, while the only number that fits the right equation is $t = 16$. But the value of t must be the same for both equations if t is an identity element. It follows then that division has no identity element.

We know that multiplication distributes over addition; that is, for all rational numbers r, s, and t

$$r(s + t) = rs + rt.$$

Is there an analogous property for division? Let's try some examples. For example, is $8 \div (2 + 2) = (8 \div 2) + (8 \div 2)$? We have:

$$8 \div (2 + 2) = 8 \div 4 = 2.$$

$$(8 \div 2) + (8 \div 2) = 4 + 4 = 8.$$

So

$$8 \div (2 + 2) \neq (8 \div 2) + (8 \div 2)$$

because $2 \neq 8$. However,

$$(2 + 2) \div 8 = 4 \div 8 = \frac{1}{2},$$

and

$$(2 \div 8) + (2 \div 8) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

so that

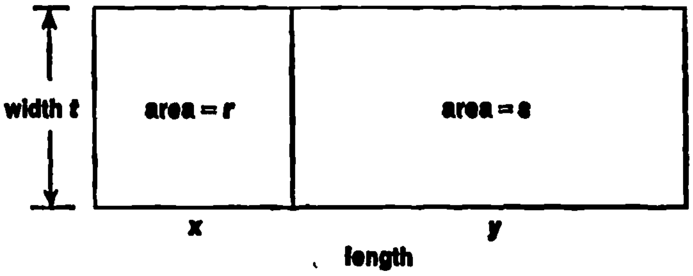
$$(2 + 2) \div 8 = (2 \div 8) + (2 \div 8).$$

This last example suggests that perhaps there is a partial distributivity:

for all rational numbers r, s, and $t \neq 0$

$$(r + s) \div t = (r \div t) + (s \div t).$$

Let us try to show that this is so by means of rectangles. The figure shows that



$$tx = r, \quad ty = s, \quad \text{and} \quad t(x + y) = r + s.$$

It follows then that

$$x = r \div t, \quad y = s \div t, \quad \text{and} \quad x + y = (r + s) \div t,$$

or

$$(r + s) \div t = x + y.$$

Hence

$$\begin{aligned}(r + s) \div t &= x + y\\ &= (r \div t) + (s \div t).\end{aligned}$$

Another way of expressing this result is to write

$$\frac{r + s}{t} = \frac{r}{t} + \frac{s}{t}.$$

We shall refer to this property by saying:

Division distributes over addition from the right.

Another very valuable property of division of rational numbers is revealed by the following example:

$$6 \div 4\frac{1}{2} = (6 \times 2) \div \left(4\frac{1}{2} \times 2\right)$$
$$= 12 \div 9$$

Let us check the property of multiplying both terms of a quotient by the same nonzero number. Computing the above quotient, we obtain

$$6 \div 4\frac{1}{2} = \frac{6}{1} \div \frac{9}{2}$$
$$= \frac{6}{1} \times \frac{2}{9}$$
$$= \frac{12}{9},$$

the same result.

It might appear from these two exercises that in general the following is true:

> *If both terms of a quotient are multiplied by the same nonzero number, the new quotient has the same value as the original one.*

To show that this is actually so, we may argue as follows:

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers with $c \neq 0$, $e \neq 0$. Then

$$\left(\frac{a}{b} \times \frac{e}{f}\right) \div \left(\frac{c}{d} \times \frac{e}{f}\right) = \frac{ae}{bf} \div \frac{ce}{df}.$$

Common multiplier is $\frac{e}{f}$.

$$= \frac{a\cancel{e}}{b\cancel{f}} \times \frac{d\cancel{f}}{c\cancel{e}}$$
$$= \frac{a}{b} \times \frac{d}{c}$$
$$= \frac{a}{b} \div \frac{c}{d}.$$

We shall refer to this property of quotients as the *equal-multiplication property*. Here is a division example using this property:

$$\frac{3}{4} \div \frac{2}{5} = \left(\frac{3}{4} \times \frac{5}{2}\right) \div \left(\frac{2}{5} \times \frac{5}{2}\right)$$

(multiply both terms by $\frac{5}{2}$)

$$= \frac{15}{8} \div \frac{10}{10}$$

$$= \frac{15}{8} \div 1$$

$$= \frac{15}{8}.$$

Exercise Set 5

1. Find a shortcut for computing each of the following and state the division property or properties used.

a. $15\frac{3}{8} \div 3$

b. $15\frac{3}{8} \div 5$

c. $15 \div 2\frac{1}{2}$

d. $15\frac{1}{2} \div 3$

e. $5\frac{1}{3} \div 1\frac{1}{3}$

f. $\left(6\frac{2}{5} + 8\frac{4}{5}\right) \div 2$

g. $\left(2\frac{3}{4} + 3\frac{5}{8}\right) \div \left(2\frac{1}{2} + 1\frac{1}{4}\right)$

h. $\left(8\frac{3}{4} \div 1\frac{1}{2}\right) \div 2\frac{1}{2}$

2. Compute each of the following, using two different methods.

a. $7\frac{3}{4} \div 2\frac{1}{4}$

b. $7\frac{1}{4} \div 2\frac{3}{4}$

c. $8\frac{1}{2} \div 2\frac{3}{4}$

d. $8\frac{3}{4} \div 2\frac{1}{4}$

e. $2\frac{1}{4} \div 7\frac{3}{4}$

f. $2\frac{3}{4} \div 7\frac{1}{4}$

g. $2\frac{3}{4} \div 8\frac{1}{2}$

h. $2\frac{1}{4} \div 8\frac{3}{4}$

3. Show that division of rational numbers has no identity element. Does it have a "partial" identity?

4. For what numbers r and s is $r \div s = s \div r$? We might call such number pairs commutative pairs for division. Why?

5. For what numbers r, s, and t is $r \div (s \div t) = (r \div s) \div t$? We may call such number triples associative triples for division. Why?

6. For what numbers r, s, and t is $r \div (s + t) = (r \div s) + (r \div t)$?

7. Give example to show that dividing by a rational number might result in a number greater than the original number. What can you say about all those rational numbers for which this is the case?

8. Suppose that the rational numbers r, s, and t are all greater than 0, and that $r < s$. Show that the following statements are true.

a. $rt < st$. b. $r \div t < s \div t$.

ALTERNATE APPROACHES TO DIVIDING WITH FRACTIONS

We have already considered in some detail the missing-factor and equal-multiplication methods for dividing with fractions. There are many other methods, some of which will now be given.

The Reciprocal Method

Many textbooks advocate this method in preference to the others. It will be illustrated by a particular example of computing

$$\frac{4}{5} \div \frac{2}{3},$$

which we use to obtain the division sentence

$$\frac{4}{5} \div \frac{2}{3} = \boxed{}.$$

An equivalent multiplication sentence is

$$\frac{4}{5} = \boxed{} \times \frac{2}{3}.$$

If, as a preliminary guess, we use for the frame $\frac{3}{2}$ (the reciprocal of $\frac{2}{3}$), we obtain

$$\frac{4}{5} = \boxed{\quad \frac{3}{2}} \times \frac{2}{3},$$

which is a false statement. The right member has the value 1 and not $\frac{4}{5}$, as we require. To meet this requirement and make the sentence true, all we need do is introduce the additional factor $\frac{4}{5}$ into the frame to give

$$\frac{4}{5} = \boxed{\frac{4}{5} \times \frac{3}{2}} \times \frac{2}{3},$$

which is a true statement. This shows that the number for the frame is $\frac{4}{5} \times \frac{3}{2}$. Returning to the first open sentence, it now follows that

$$\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2}.$$

The very same method can be used to obtain the general traditional algorithm.

The Same-Denominator Method

Many teachers prefer the same-denominator method because the initial steps closely resemble the algorithm for adding with fractions. We illustrate this method with the particular example $\frac{3}{4} \div \frac{1}{2}$. We begin by replacing, where necessary, the given fractions with equivalent fractions having the same denominator.

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4}.$$

The original problem is now regarded as equivalent to simply dividing the new numerators.

$$\begin{aligned} \frac{3}{4} \div \frac{1}{2} &= \frac{3}{4} \div \frac{2}{4} \\ &= 3 \div 2 \\ &= \frac{3}{2}. \end{aligned}$$

In other words, a quotient of rational numbers having the same denominator is equivalent to the quotient of their numerators. That this method will always give the correct result can be shown by either of the first two methods already considered. Let us show how the missing-factor method justifies this method.

$$\frac{a}{b} \div \frac{c}{b} = \boxed{};$$

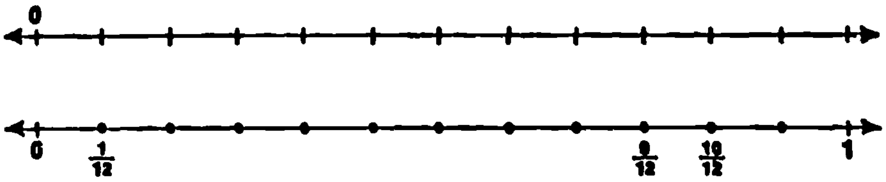
$$\frac{a \times c}{b \times c} \div \frac{c}{b} = \boxed{\frac{a}{c}}.$$

Using the Number Line for Computing Quotients

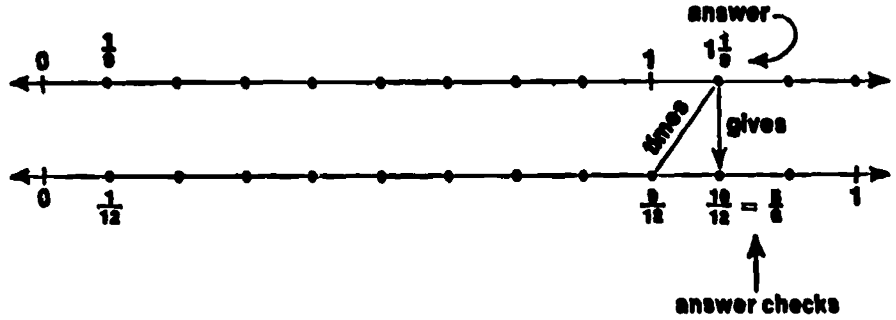
The number-line method used for computing quotients strongly resembles the number-line method for computing products, discussed in chapter 5. We illustrate this method by the problem to compute

$$\frac{5}{6} \div \frac{3}{4}.$$

First a common denominator for the two fractions is obtained, 12 in this case. We begin with two parallel scales marked uniformly with the marks for 0 on the two scales directly opposite each other. On the lower of the two scales, 12 divisions from the mark for 0, label the mark "1." This determines the values for all the other scale marks on the lower scale.



We show only 5 values for scale marks. Directly above the mark for the given factor, $\frac{3}{4}$, which is the same as $\frac{9}{12}$, goes the label "1." The assignments of 0 and 1 determine the assignments for all the other scale marks. Directly above the scale mark for the product $\frac{5}{6}$, which has the same value as $\frac{10}{12}$, will be found the scale mark for $\frac{5}{6} \div \frac{3}{4}$, whose value is seen to be $1\frac{1}{9}$.



Let's see if it checks.

$$\frac{3}{4} \times 1\frac{1}{9} = \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{4}}} \times \frac{\overset{5}{\cancel{10}}}{\underset{3}{\cancel{9}}}$$

$$= \frac{5}{6}.$$

Using Rectangles to Compute Quotients

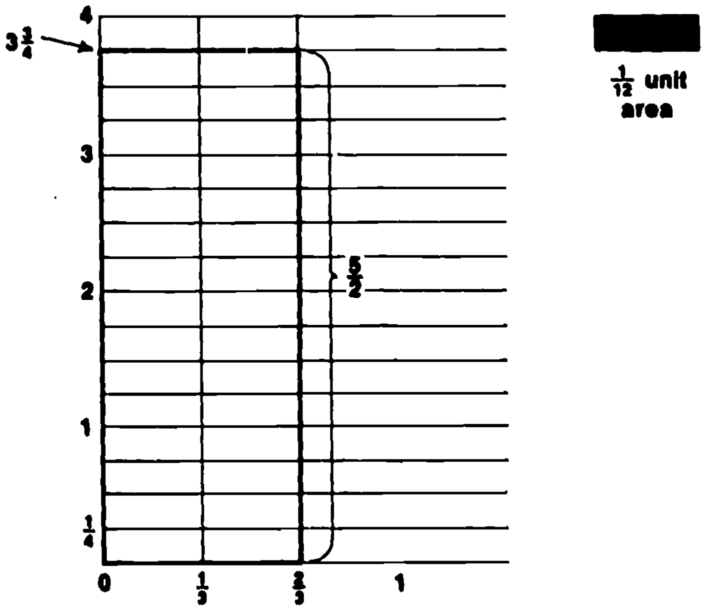
With this method a rectangle is constructed so that the given product is its area, while its length and width are the given factor and the missing factor. A segment for the given factor is marked off on a number line. On this segment we build a rectangle until its area has the value of the given product. The crux of this method is to obtain the right building blocks to build with. We illustrate this method by computing

$$\frac{5}{2} \div \frac{2}{3}.$$

If the product fraction has the "multiple" property needed in the missing-factor method, then its denominator gives the value of the building block to use. For our problem we have

$$\frac{5}{2} = \frac{5 \times 2 \times 3}{2 \times 2 \times 3} = \frac{30}{12}.$$

A building block that will work is $\frac{1}{12}$. Using these blocks, we build a rectangle on the segment for $\frac{2}{3}$ until the area has the value $\frac{30}{12}$. Its length will then be the value of the quotient $\frac{30}{12} \div \frac{2}{3}$, or of $\frac{5}{2} \div \frac{2}{3}$, which is just what we want. The block $\frac{1}{12}$ has a base of $\frac{1}{3}$, so its height must be $\frac{1}{4}$,



giving the scale value for the vertical scale. At a height of $3\frac{3}{4}$ we have obtained the required area of $\frac{30}{12}$. Hence, our answer is the number for the height, $3\frac{3}{4}$.

Check:

$$\begin{aligned}\frac{2}{3} \times 3\frac{3}{4} &= \frac{2}{3} \times \frac{15}{4} \\ &= \frac{30}{12} \\ &= \frac{5}{2}.\end{aligned}$$

Exercise Set 6

1. Compute each of the following, using all six methods:

Missing-factor	Same-denominator
Reciprocal	Number-line
Equal-multiplication	Rectangle

a. $\frac{3}{4} \div \frac{1}{2}$ **c.** $\frac{3}{2} \div \frac{1}{4}$

b. $\frac{1}{2} \div \frac{3}{4}$ **d.** $\frac{1}{4} \div \frac{3}{2}$

2. Justify the same-denominator method by a method other than the missing-factor one.

3. Use the rectangle method to compute the following:

a. $24 \div 4$ **b.** $24 \div 6$ **c.** $2 \div 1\frac{1}{2}$

SUMMARY OF KEY IDEAS

1. The missing-factor method of computing quotients makes for an easy transition from division of whole numbers to division of rational numbers expressed as fractions. This method rests strongly on the basic idea of interpreting a quotient as a missing factor.

2. Dividing by a number gives the same result as multiplying by its reciprocal (or multiplicative inverse).

3. When dividing whole numbers, the remainder is the numerator of the fraction (unreduced) in the computed mixed-numeral expression for the quotient.

$$5 \div 3 = 1\frac{2}{3}. \leftarrow \text{remainder}$$

$$14 \div 4 = 3\frac{2}{4}. \leftarrow \text{remainder}$$

4. Multiplying both terms of a quotient by the same nonzero number does not change its value. That is, $r \div s$ and $(rt) \div (st)$ are the same number for all rational numbers r, s, t, with $s \neq 0$, $t \neq 0$.

5. The properties of division of rational numbers are almost the same as for division of whole numbers. However, for rational numbers division is always possible except for division by zero.

Review Exercises

1. Compute each of the following, using all six methods:

   Missing-factor
   Reciprocal
   Equal-multiplication
   Same-denominator
   Number-line
   Rectangle

   a. $\frac{2}{3} \div \frac{1}{2}$
   b. $\frac{1}{2} \div \frac{2}{3}$
   c. $\frac{5}{2} \div \frac{1}{6}$
   d. $\frac{1}{6} \div \frac{5}{2}$
   e. $\frac{3}{4} \div \frac{5}{2}$
   f. $\frac{5}{2} \div \frac{3}{4}$
   g. $\frac{5}{3} \div \frac{1}{2}$
   h. $\frac{1}{2} \div \frac{5}{3}$

2. How does $a \div b$ compare with $b \div a$?

3. Which of all the methods for computing quotients can be justified with the least number of principles? Which method do you prefer to teach? Why?

4. In working the division problem $\frac{3}{4} \div \frac{1}{2}$ a student computes

$$\frac{4}{3} \times \frac{2}{1}$$

and argues "If I invert both fractions, then I surely will invert the right one." How would you respond to his remark?

5. Multiplying both terms of an indicated quotient by the same nonzero number does not change its value. What happens to its value if both terms are increased by the same number?

6. Prove that the following hold where $r, s, t, u,$ are any rational numbers with the provisos indicated:

a. $\frac{r}{s} = \frac{rt}{st}$ provided s and t are not zero.

b. $\frac{r}{s} + \frac{t}{s} = \frac{r+t}{s}$ provided s is not zero.

c. $\frac{r}{s} \times \frac{t}{u} = \frac{rt}{su}$ provided s and u are not zero.

d. $\frac{r}{s} \div \frac{t}{u} = \frac{r}{s} \times \frac{u}{t}$ provided the following are not zero: s, t, u.

David W. Wells
Philip L. Cox

DECIMALS:
ADDITION AND SUBTRACTION

±•7

1. **How is decimal numeration extended so that some rational numbers can be named by decimal numerals?**
2. **How are rational numbers such as $\frac{7}{10}$, $\frac{7}{100}$, $\frac{25}{100}$, and $11\frac{33}{100}$ named by decimals?**
3. **What rational numbers can be named by decimals?**
4. **What is the rationale for the traditional algorithms for computing sums and differences using decimals?**

It is not easy to determine when and where decimal numeration was first extended (in any systematic way) to numbers other than whole numbers. Although "decimal fractions" were used by the Chinese many centuries earlier, common fractions were used universally in Europe to express numerically parts of a whole until about 350 years ago.

Although not all historians agree, the invention of "decimal fractions" is most often credited to the Dutch mathematician Simon Stevin. His book of 1585, *La Disme* or *De Thiende* [The Tenth], was written to popularize the "decimal fraction" and its usage. It is interesting to note that Stevin also advocated use of the decimal system in all areas of weights and measures. Although the struggle is still on, this dream of Stevin's is as yet unfulfilled in the United States almost four hundred years later. In Stevin's decimal notation 15.912 would be written as

$$15⓪9①1②2③ \quad \text{or} \quad 1\overset{0}{5}\overset{1}{9}\overset{2}{1}\overset{3}{2}.$$

The "⓪" served as his "decimal point," indicating the location of the ones place. To date, the only significant improvement in Stevin's numeration has been in notation. The invention of logarithms gave impetus to the use of decimal fractions, and in a book published in 1619 Napier proposed the use of a decimal point as we in the United States use it today.

History records a wide variety of symbolism proposed for decimal numeration. Some of these suggestions are listed below. In each case, the symbolism is intended to represent the number that would presently be named in the United States by the decimal numeral 15.912.

$15\ 9'\ 1''\ 2'''$

15/912

15, 9 1 2

15,912

$15/\dot{9}\dot{1}\dot{2}$

$15/\underline{912}$

$15912^{\textcircled{3}}$

$15\ 9^{\mathrm{I}}\ 1^{\mathrm{II}}\ 2^{\mathrm{III}}$

15912^{III}

15.912

$15\overset{1\ 2\ 3}{\overline{)912}}$

Even today, despite the wide use of decimal notation, there is no universally accepted form for writing the "decimal point." For 3.14 in the notation of the United States, the English write 3·14, and the Germans and the French write 3,14. At times, we also resort to other forms of the "decimal point," as when we enter dollar amounts on balance sheets or write a dollar amount such as $\$3^{\underline{14}}$ on a personal check.

Regardless of the symbolism used, the development of decimal numeration for numbers other than whole numbers has to be regarded as one of man's greatest inventions. Those men who contributed to its development were responsible for making a very great contribution to our present civilization. Where would we be without it?

INTRODUCING THE EXTENSION OF DECIMAL NUMERATION

Prior to classroom study, students are usually aware of some decimal names for rational numbers (times in sports events, dollar amounts, · · ·), and some students can read some of the names. This awareness is a factor to be considered when planning for classroom instruction. Successful classroom teachers often use this awareness to motivate the study of decimals, the previous experiences of their students becoming the starting point for instruction. However, these previous experiences cannot be taken to mean that students understand the relationship of the names they have learned to the decimal system for representing numbers. More intensive study is needed before they will be able to use decimal notation effectively.

Exploratory activities can be planned to use the students' experiences with decimals and to make the extension of decimal numeration a natural one. Prior to beginning their formal study of decimal names for rational numbers, students could be asked to collect newspaper articles, advertisements, dials, gauges, and other objects that contain what they *think* are decimal names for rational numbers. This might include information such as amounts of precipitation in weather reports, dollar amounts, and times of winning racers, also objects such as odometers or FM radio dials. The material collected can be used as material for a student-constructed bulletin-board display.

Students' ideas are used by successful teachers as a guide to meeting the instructional needs of their students. When discussing the numerals and objects collected, student reactions and descriptions telling why they think these numerals name rational numbers may help form some initial judgments concerning the students' knowledge of decimal notation. In one sense, these reactions and your evaluation of them can be thought of as an informal pretest. (As the study of decimals progresses, these initial judgments may be refined through careful observation and evaluation of classroom performance.) You can use the outcomes of exploratory activities such as those described above as one basis for developing a successful unit of instruction—one that is closely geared to the instructional needs of your students.

One major concept to which the student was exposed in his study of whole numbers is the relationship between adjacent place values. The place values for each "4" in the numeral "4,444" are shown below.

Thousands	**Hundreds**	**Tens**	**Ones**
10 × 100	**10 × 10**	**10 × 1**	**1**
4	**4**	**4**	**4**

Moving to the left, the place value of each "4" is ten times the place value of the "4" at its right. For expressing large numbers, this place-value pattern can be continued to the left as far as we wish to go. In the following, place values are shown in italic:

$$4{,}444 = (4 \times \mathit{1{,}000}) + (4 \times \mathit{100}) + (4 \times \mathit{10}) + (4 \times \mathit{1})$$

$$\begin{aligned} 10 &= 10 \times 1. \\ 100 &= 10 \times 10. \\ 1{,}000 &= 10 \times 100. \\ \vdots \quad &\quad \vdots \qquad \vdots \end{aligned}$$

Students who have mastered this concept in dealing with whole numbers are ready to consider extending the decimal numeration system to include names of other rational numbers. Their previous experiences are the foundation block for building the ideas and skills necessary for introducing decimal place value to the right of the ones place.

As shown in the chart below, moving to the right in the numeral "4,444," each "4" has a place value $\frac{1}{10}$ that of the place value of the "4" to its left. Since students were able to extend the pattern of place values to the left as far as they wished, we can now add another 4 to the right of the ones place and ask what the value of the new place should be if the same left-to-right pattern between adjacent place values is preserved.

Ten Thousands	Thousands	Hundreds	Tens	Ones	?
	$\frac{1}{10} \times$ 10,000	$\frac{1}{10} \times$ 1,000	$\frac{1}{10} \times$ 100	$\frac{1}{10} \times$ 10	$\frac{1}{10} \times$?
	4	4	4	4	4

$$1{,}000 = \frac{1}{10} \times 10{,}000.$$

$$100 = \frac{1}{10} \times 1{,}000.$$

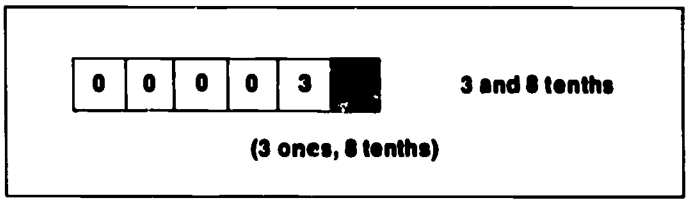
$$10 = \frac{1}{10} \times 100.$$

$$1 = \frac{1}{10} \times 10.$$

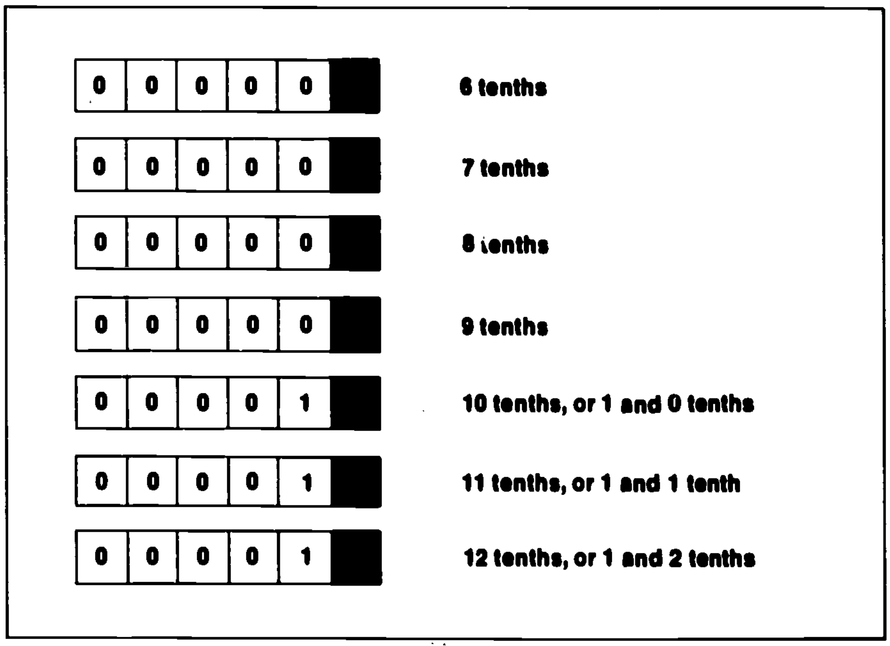
$$? = \frac{1}{10} \times ?$$

By examining this pattern, the student can be helped to see that the new place value would be $\frac{1}{10} \times 1$ or $\frac{1}{10}$, indicating tenths.

An automobile or bicycle odometer can be used as a classroom aid in reading decimal numerals after decimal numeration has been extended to include a tenths place. Some odometers have the tenths place indicated by a different color.



By turning the stem of an odometer, the number of tenths is recorded in sequence. Each group of ten tenths is recorded as one mile, illustrating the relationship between tenths and the ones place.



Suppose we wished to record this mileage on paper without drawing a picture of the odometer. If we wrote "23", this would be twenty-three miles, not two and three-tenths miles. When decimal numeration is extended to the right of the ones place, some method for indicating its location is needed.

It is at this time the decimal point can be introduced as a device for indicating the location of the ones place. By writing 2.3, the decimal point indicates that 2 is in the ones place, and the numeral 2.3 is read as "2 and 3 tenths." Notice that the decimal point is read as "and."

Rational numbers named by the numerals $\frac{7}{10}$ and $1\frac{3}{10}$ can now be named by decimals. Although the visual symbols for indicating tenths differ, the decimal names are the same.

$$\frac{7}{10} = 0.7 = \text{"seven tenths."}$$

$$1\frac{3}{10} = 1.3 = \text{"one and three tenths."}$$

After the student has explored the extension of decimal numeration to the tenths place, we extend the place-value system to include hundredths and then thousandths by helping him to see how the left-to-right pattern of place values continues. Adding places to the right of the tenths place as shown in the chart below, we can similarly ask, "If the place-value pattern continues, what is the place value of the place to the right of the tenths place?" After that question has been answered satisfactorily: "What is the place value of the place to the right of the hundredths place?"

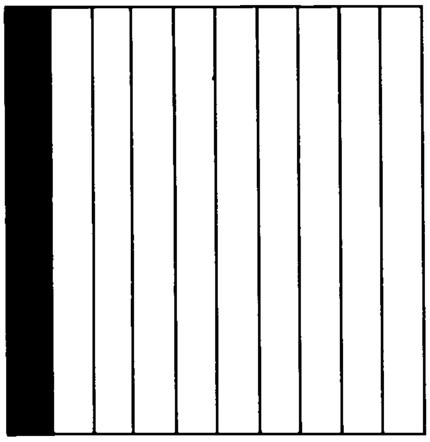
Ten Thousands	Thousands	Hundreds	Tens	Ones	Tenths	?	?
10,000	1,000	100	10	1	$\frac{1}{10}$	?	?

At the same time, the decimal notation is developed so that students can write decimal names for rational numbers involving hundredths and thousandths.

$$\frac{7}{100} = 0.07. \qquad \frac{23}{100} = 0.23.$$

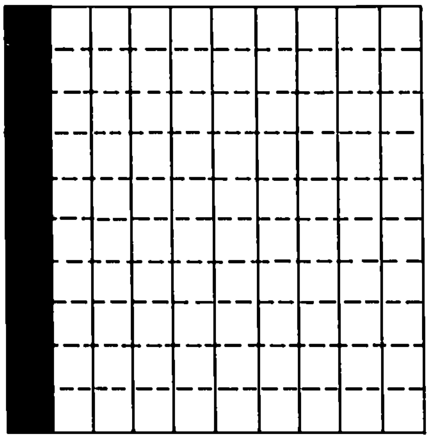
$$\frac{9}{1{,}000} = 0.009. \qquad 2\frac{3}{100} = 2.03.$$

When the place value of hundredths is introduced, square regions can be subdivided into 100 congruent regions and used to show the relationship between hundredths, tenths, and ones.



The shaded strip is $\frac{1}{10}$ of the square region.

$\frac{1}{10}$

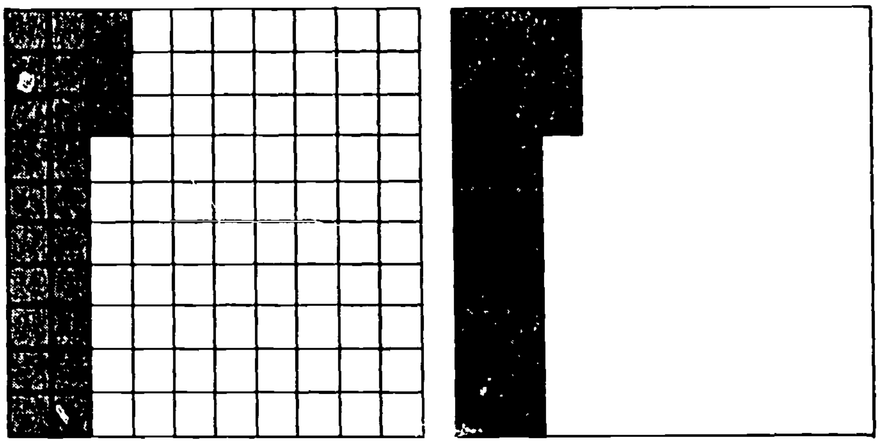


When the region is subdivided into 100 congruent regions, the shaded strip is $\frac{10}{100}$ of the square region.

Since the area of the shaded strip remains the same, we have shown that

$\frac{1}{10} = \frac{10}{100}$, or $0.1 = 0.10$.

$\frac{10}{100}$



$$\frac{23}{100} = \frac{20}{100} + \frac{3}{100} = \frac{2}{10} + \frac{3}{100} = 0.23.$$

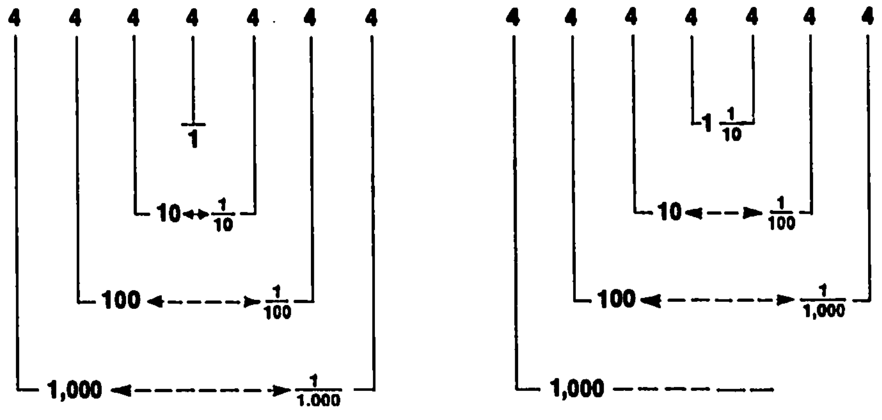
Having understood the extension of decimal numeration to tenths and hundredths, many students will not need to use visual materials in extending the pattern to include thousandths. If needed, the scale on a meterstick could be adapted as a model for visualizing the relationship between thousandths, hundredths, tenths, and ones.

Another model that is often used to reinforce understandings of place value (hundredths, tenths, and ones) is the analogy between the United States decimal and monetary notation. Since a dime is $\frac{1}{10}$ of a dollar and a penny is $\frac{1}{100}$ of a dollar, an amount such as $0.83 could be interpreted in several ways.

$0.83 = 8 dimes, 3 pennies.
$0.83 = 8 tenths of a dollar, 3 hundredths of a dollar.
$0.83 = 83 hundredths of a dollar.

Since 10 dimes (tenths of a dollar) or 100 pennies (hundredths of a dollar) are equivalent to a dollar, the analogy between money and decimal notation could also be used to further illustrate that $\frac{10}{10} = 1$ and $\frac{100}{100} = 1$.

During all of the types of activities suggested above it should be stressed that one important function of the decimal point is to locate the ones place in the numeral for the reader. Using the ones place as a point of reference, there is a symmetry between corresponding place value on either side of the ones place. For example, one place to the right is the tenths place and one place to the left is the tens place. The left-hand portion of the illustration exhibits the symmetry for the numeral 4,444,444. However, if the decimal point rather than the ones place is considered the center of the numeration system, the symmetry between corresponding place values no longer exists (see the right-hand portion of the illustration).



A secondary function of the decimal point is to separate the numeral into two component parts—one part representing a whole number and the other a number less than one.

This secondary function is reflected by the methods we use for reading decimal numerals. Two accepted methods for reading the decimal numeral 262.34 are "two, six, two, *point*, three, four" and "two hundred sixty-two *and* thirty-four hundredths."

The italicized words (*point* and *and*) indicate the location of the decimal point and separate the whole-number part of the numeral from the part representing a number less than one. The fact that both methods for reading decimal numerals (the student should be able to use either one) stress the location of the decimal point should not be allowed to overshadow the fact that the ones place, not the decimal point, is in the center of the decimal numeration system and that the primary function of the decimal point is to locate the ones place.

When reading decimal numerals, the word "and" should be reserved solely for indicating the location of the decimal point. For example, "420" should not be read as "four hundred and twenty"; this creates confusion. For a careful interpretation, compare the two statements shown below.

"Four hundred and twenty thousandths" = 400.020.
"Four hundred twenty thousandths" = 0.420.

Exercise Set 1

1. Write a decimal for each number.

a. $\frac{3}{10}$ **d.** $\frac{1}{100}$ **g.** $1\frac{56}{100}$

b. $1\frac{6}{10}$ **e.** $\frac{42}{100}$ **h.** $\frac{8}{1,000}$

c. $2\frac{0}{10}$ **f.** $3\frac{9}{100}$ **i.** $\frac{23}{1,000}$

2. Write a fraction or a mixed numeral for each number.

a. 0.1 **d.** 0.18 **g.** 0.004

b. 3.4 **e.** 3.47 **h.** 0.037

c. 0.07 **f.** 2.06

3. Write a decimal and a fraction (or mixed numeral) for:

a. six tenths;

b. two and three tenths;

c. six, point, seven;

d. four hundredths;

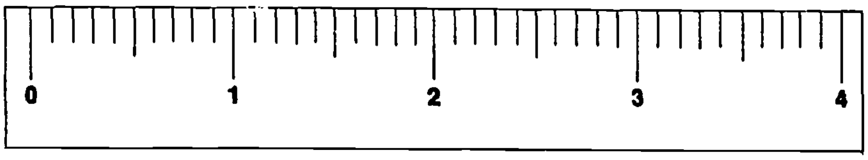
e. twelve hundredths;

f. twenty-three hundredths;

g. six and three hundredths;

h. seven thousandths;

i. fifty-four thousandths.



4. Using the ruler above, write a decimal for the number that is—

a. 0.3 greater than 0.5;

b. 0.4 less than 0.9;

c. 0.7 greater than 2.2;

d. 1.3 greater than 0.9;

e. 1.6 less than 3.0.

5. Complete each pattern below, and describe the pattern you used.

a. 0.06, 0.07, ___, ___, 0.10, ___, 0.12

b. 0.05, 0.10, ___, 0.20, ___, ___

c. 0.18, 0.15, 0.12, ___, ___, ___

d. 0.004, ___, 0.006, ___, ___, 0.009

e. 6.23, ___, 6.31, 6.35, ___, ___

6. Arrange in order beginning with the name for the smallest number (you may use the ruler above exercise **4**).

a. 3.4, 2.1, 0.4, 2.0, 1.8

b. 1.4, 3.2, 0.9, 1.0, 4.7

7. Answer the following questions. You may use the ruler again. Write your answer in decimal notation.

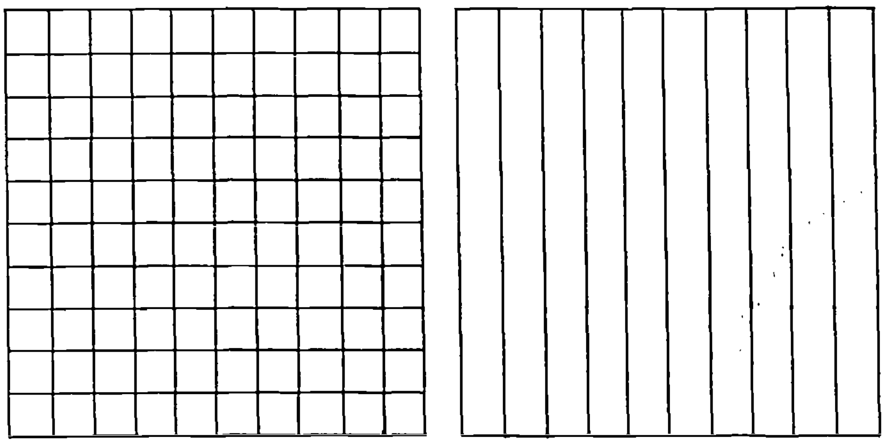
a. 0.8 is how much greater than 0.2?

b. 3.8 is how much greater than 2.5?

c. 0.9 is how much less than 1.0?

d. 1.3 is how much less than 4.7?

e. 3.3 is how much greater than 1.8?



For exercises **8** and **9**, refer to the square regions pictured above. Draw additional diagrams if needed.

8. Insert

>, <, or =

in the blank space to make each statement true.

a. 0.3 0.25. **d.** 0.8 0.9.
b. 0.10 0.1. **e.** 0.15 0.2.
c. 0.40 0.04. **f.** 0.5 0.50.

9. Arrange in order, beginning with the name for the smallest number.

a. 0.25, 0.1, 0.01, 0.15, 0.3 **c.** 0.09, 0.8, 0.25, 0.50, 0.4

b. 0.4, 0.33, 0.10, 0.2, 0.42

10. Place the decimal point in the numerals so that each statement is sensible.

- **a.** The classroom is 1 2 5 feet high.
- **b.** The length of my math book is 9 0 inches.
- **c.** A ticket to the baseball game costs $3 5 0.

11. Write a decimal for each number.

- **a.** 6 ones, 5 tenths
- **b.** 4 tenths, 2 hundredths
- **c.** 8 tens, 3 ones, 7 tenths, 9 hundredths
- **d.** 3 ones, 8 hundredths
- **e.** 9 thousandths
- **f.** 7 tenths, 2 hundredths, 3 thousandths
- **g.** $(3 \times 10) + (2 \times 1) + 4 \times \frac{1}{10} + \left(6 \times \frac{1}{100}\right)$
- **h.** $11 + \frac{2}{10}$
- **i.** $(8 \times 1) + \left(8 \times \frac{1}{100}\right)$
- **j.** $\frac{19}{10}$
- **k.** Three hundred twenty-five hundredths
- **l.** $(9 \times 100) + (7 \times 10) + (9 \times 1)$
- **m.** $(8 \times 100) + (3 \times 10) + (7 \times 1) + \left(9 \times \frac{1}{10}\right) + \left(6 \times \frac{1}{100}\right)$

12. Which distance in each pair is closer to 4 miles?

a. 3.8 miles, 4.3 miles	**c.** 3.8 miles, 3.6 miles
b. 4.4 miles, 4.2 miles	**d.** 3.9 miles, 3.89 miles

13. a. Draw number-line diagrams that could be used to illustrate the correctness of your responses in exercise **12.**

b. Draw pictures of square regions such as those used in exercises **8** and **9** to show that you are correct.

14. Which of the following is another name for—

a. $\frac{18}{10}$? 1.08, 1.8, 0.18

b. $2\frac{7}{100}$? 0.27, 2.7, 2.07

c. $32\frac{8}{10}$? 32.8, 3.28, 0.328

15. The manner in which we read decimal names for rational numbers sometimes creates confusion in translating the written or oral statements to numerical form. How would you interpret each of the following statements? What could be done to minimize confusion in translating each one?

a. Seven hundred thousandths (0.700 or 0.00007?)

b. Four hundred one thousandths (0.400 or 0.401?)

c. Six hundred and twenty-two thousandths (0.622 or 600.022?)

16. When introducing the naming of rational numbers by decimals in the classroom, what kinds of questions, exercises, and/or activities could be used that would help prepare students for computing sums and differences at a later date?

17. Draw a diagram (or diagrams) that could be used to visualize pictorically the truth of each of the following statements.

a. $\frac{13}{100} = \frac{10}{100} + \frac{3}{100} = \frac{1}{10} + \frac{3}{100} = 0.13.$

b. $\frac{234}{100} = \frac{200}{100} + \frac{30}{100} + \frac{4}{100} = 2 + \frac{3}{10} + \frac{4}{100} = 2.34.$

18. Place values for each "2" in the base-three numeral "222_{three}" are listed below under each "2".

2	2	2
three^2	three	one

Suppose the place-value system for the base-three numeration system is extended to the right of the ones place so that the pattern existing between adjacent place values is preserved. Write base-three names for the following:

a. $\frac{1}{3}$ **c.** $2\frac{2}{3}$

b. $1 + \left(2 \times \frac{1}{3}\right)$ **d.** $(2 \times 1) + \left(0 \times \frac{1}{3}\right) + \left(2 \times \left(\frac{1}{3}\right)^2\right)$

19. Write a base-ten fraction or mixed numeral equivalent to each of the following base-three numerals.

a. 0.2_{three} **c.** 21.2_{three}

b. 2.1_{three} **d.** 211.02_{three}

NAMING RATIONAL NUMBERS BY DECIMALS

Rational Numbers with Denominators That Are Powers of Ten

Writing and reading decimals for rational numbers such as $\frac{3}{10}$, $6\frac{7}{10}$, $\frac{9}{100}$, $8\frac{7}{100}$, and $\frac{5}{1,000}$ is a direct by-product of the extension of the decimal numeration system and the introduction of decimal-point notation. This was partially discussed in the previous section. When writing $\frac{15}{10}$ as $\frac{10}{10} + \frac{5}{10} = 1 + \frac{5}{10} = 1\frac{5}{10}$, or 1.5, the student also relies on previous work in naming fractions by mixed numerals.

The students' previous work with equivalent fractions in conjunction with the square regions or number lines mentioned in the previous section can also be used to help justify statements such as the following:

$$\frac{1}{10} = \frac{10}{100} = \frac{100}{1,000}. \qquad 4 = \frac{4}{1} = \frac{40}{10} = \frac{400}{100} = \frac{4,000}{1,000}.$$

$$\frac{3}{10} = \frac{30}{100} = \frac{300}{1,000}. \qquad \frac{23}{100} = \frac{230}{1,000}.$$

The equivalence of such fractions also can be used to justify in the following manner that $\frac{35}{100}$ and 0.35 are equivalent:

$$\frac{35}{100} = \frac{30}{100} + \frac{5}{100}$$

$$= \frac{3}{10} + \frac{5}{100} \qquad \left(\text{since } \frac{30}{100} = \frac{3}{10}\right)$$

$$= 0.35 \qquad \text{(3 tenths, 5 hundredths).}$$

The ability to generate equivalent decimals by using equivalent fractions is helpful in computing certain sums and differences (to be discussed in a later section) and in determining relative sizes of rational numbers named by decimals. Consider the following problem (see exercises **8** and **9** in exercise set 1 for an earlier informal approach to this kind of problem): Arrange in order beginning with the name of the smallest number.

0.23, 0.5, 0.067, 0.1

If each decimal is replaced by an equivalent decimal in thousandths, we have the following:

0.23 = 0.230. 0.067 = 0.067.

0.5 = 0.500. 0.1 = 0.100.

When each is written as a decimal numeral with the same number of decimal places to the right of the ones place, as above, the correct response is easier to see. What we might have actually done is to think of the decimals as $\frac{230}{1,000}$, $\frac{500}{1,000}$, $\frac{67}{1,000}$, and $\frac{100}{1,000}$. When this is done, all we must do is to compare the numerators (230, 500, 67, and 100) to determine the relative sizes of the numbers.

Rational Numbers with Denominators That Are Not Powers of Ten

Rational numbers such as $\frac{3}{10}$, $\frac{27}{100}$, and $\frac{18}{100}$ (with denominators that are powers of ten) can easily be named by decimals. What about rational numbers such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{3}{8}$? Do these rational numbers have decimal names?

Finding decimal names for these rationals can be reduced to a problem the students learned to solve earlier. Given any fraction such as $\frac{1}{2}$, several fractions equivalent to it can be generated by multiplying both numerator and denominator by the same number.

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}. \qquad \frac{1}{2} = \frac{1 \times 4}{2 \times 4} = \frac{4}{8}.$$

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}. \qquad \frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10}.$$

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} \dots .$$

Given a fraction such as $\frac{3}{10}$, $\frac{25}{100}$, or $\frac{125}{1,000}$, a decimal name for the number can be written.

$$\frac{3}{10} = 0.3. \qquad \frac{25}{100} = 0.25. \qquad \frac{125}{1,000} = 0.125.$$

Therefore, a rational number such as $\frac{1}{2}$ can be named by a decimal if an appropriate equivalent fraction with a denominator that is a power of 10 can be found.

$$\frac{1}{2} = \frac{?}{10} \qquad \frac{1}{2} = \frac{?}{100} \qquad \frac{1}{2} = \frac{?}{1,000}$$

Since

$$\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10},$$

$\frac{1}{2}$ can be named by 0.5

$$\left(\frac{1}{2} = \frac{5}{10} = 0.5\right).$$

When one decimal equivalent to $\frac{1}{2}$ is found, others can be generated by annexing zeros to the right. Since

$$\frac{1}{2} = \frac{5}{10} = \frac{50}{100} = \frac{500}{1,000},$$

then

$$\frac{1}{2} = 0.5 = 0.50 = 0.500.$$

Can $\frac{3}{4}$ be named by a decimal?

$$\frac{3}{4} = \frac{?}{10} \qquad \frac{3}{4} = \frac{?}{100} \qquad \frac{3}{4} = \frac{?}{1,000}$$

Since there is no whole number that 4 can be multiplied by to get 10 as a product (4 is not a factor of 10), $\frac{3}{4}$ does not have a decimal name in tenths. Is 4 a factor of 100? Yes, because $4 \times 25 = 100$. Therefore,

$$\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = 0.75.$$

As was the case with $\frac{1}{2}$, other decimal equivalents can now be generated.

Since

$$\frac{3}{4} = \frac{75}{100} = \frac{750}{1,000},$$

then

$$\frac{3}{4} = 0.75 = 0.750.$$

Since 8 is not a factor of either 10 or 100, $\frac{3}{8}$ has no decimal name in tenths or hundredths. However, 8 is a factor of 1,000, since $8 \times 125 = 1,000$. Therefore,

$$\frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1,000} = 0.375.$$

This procedure of obtaining decimal names for rational numbers such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{3}{8}$ can be summarized as follows:

1. Name the rational number by a fraction that shows a denominator that is a power of 10.

2. Write the decimal numeral for the numerator of the fraction and place the decimal point in the correct place in the decimal numeral, as indicated by the denominator.

It should be emphasized that this procedure involves no new mathematics for the student. We are merely again exposing the student to a potent technique for solving problems in mathematics and other areas: *To solve a new problem, reduce it to a problem that has already been solved.* The student is now able to find decimal names for many rational numbers.

Exercise Set 2

1. Complete the following chart:

Fraction	Equivalent Fraction	Equivalent Decimal
$\frac{1}{4}$	$\frac{1 \times 25}{4 \times 25} = \frac{25}{100}$.	0.25
$\frac{1}{2}$	$\frac{1 \times 5}{2 \times 5} = \frac{5}{10}$.	
$\frac{3}{20}$	$\frac{3 \times 5}{20 \times 5} =$	
$\frac{9}{25}$	$\frac{9 \times}{25 \times} =$	

2. For each fraction, find an equivalent decimal.

a. $\frac{3}{5}$ **c.** $\frac{5}{8}$ **e.** $\frac{9}{20}$

b. $\frac{3}{4}$ **d.** $\frac{3}{50}$ **f.** $\frac{4}{25}$

3. Find a decimal name equivalent to each mixed numeral.

a. $2\frac{1}{2}$ **b.** $3\frac{3}{4}$ **c.** $1\frac{1}{5}$

4. Complete the following:

a. $\frac{1}{2} = \frac{}{10} = \frac{}{100} = \frac{}{1,000}$. **c.** $\frac{100}{1,000} = \frac{}{100} = \frac{}{10}$.

b. $\frac{2}{5} = \frac{}{10} = \frac{}{100} = \frac{}{1,000}$. **d.** $\frac{750}{1,000} = \frac{}{100} = \frac{}{4}$.

5. Insert >, <, or = in the blank spaces to make each statement true.

a. $3\frac{2}{5}$ 3.4.

b. $\frac{1}{2}$ 0.4.

c. 0 .4 0.40.

d. 1.0 0.10.

e. $\frac{1}{4}$ 0.25.

f. 2.42 $2\frac{1}{2}$.

g. $\frac{4}{25}$ 0.04.

6. Complete the following:

a. If $\frac{1}{5} = 0.2$, then $\frac{2}{5} =$ ___ .

b. If $\frac{1}{8} = 0.125$, then $3\frac{1}{8} =$ ___ .

7. Write three decimals for each number given.

a. $\frac{1}{2}$

b. 0.10

c. $3\frac{1}{4}$

8. Discuss the meaning of the following statement: To name certain rational numbers by decimals we reduced the problem to a problem that had already been solved.

9. What skills and understandings of rational numbers named by fractions are needed by a student so that he can use the procedures outlined in this section for naming rational numbers by decimals?

CAN ALL RATIONAL NUMBERS BE NAMED AS DECIMALS?

In the previous section, you discovered that a decimal name for a rational number could be found if the number could be named by an appropriate fraction. Will this procedure result in a decimal name for *all* rational numbers?

Consider the rational number $\frac{1}{3}$. Can it be named by a decimal?

$$\frac{1}{3} = \frac{1 \times \square}{3 \times \square} = \frac{?}{10} \qquad \frac{1}{3} = \frac{1 \times \square}{3 \times \square} = \frac{?}{100} \qquad \frac{1}{3} = \frac{1 \times \square}{3 \times \square} = \frac{?}{1{,}000}$$

1. Is 3 a factor of 10? (Does there exist a whole number a such that $3 \times a = 10$?) Since $10 = (3 \times 3) + 1$, the remainder indicates that 3 is *not* a factor of 10.

2. Is 3 a factor of 100? Since $100 = (3 \times 33) + 1$, 3 is *not* a factor of 100.

3. Is 3 a factor of 1,000? Since $1{,}000 = (3 \times 333) + 1$, 3 is *not* a factor of 1,000.

4. The pattern thus far indicates that if we go on to any other power of 10, such as 10,000 or 100,000, we shall get a remainder of 1 when we divide the power of 10 by 3.

Therefore, there is no appropriate fraction equivalent to $\frac{1}{3}$. As a result, we cannot find a decimal name for $\frac{1}{3}$.

Such a series of questions used in the classroom will indicate the existence of at least one number, $\frac{1}{3}$, for which no decimal can be found. Are there others that have no decimal names? What technique can be used to determine whether or not a decimal exists for a given rational number?

To analyze these two questions we again use our procedure for finding decimals equivalent to given fractions. Our results thus far indicate that a decimal for a rational number can be found if it can be named by a fraction that shows a denominator that is a power of 10.

$\frac{1}{2} = \frac{5}{10} = 0.5.$ 2 is a factor of 10.

$\frac{3}{4} = \frac{75}{100} = 0.75.$ 4 is a factor of 100.

$\frac{3}{8} = \frac{375}{1{,}000} = 0.375.$ 8 is a factor of 1,000.

$\frac{1}{3} = ?$ 3 is not a factor of 10, 100, 1,000, $\cdots$.

Therefore, we cannot find a decimal for $\frac{1}{3}$.

A procedure similar to that followed in exercise set 3, below, could be used to indicate to the student a method for determining when a given rational number named by a fraction in lowest terms has a decimal name.

Exercise Set 3

1. Consider the following rational numbers named by fractions in lowest terms. Find, if possible, an equivalent decimal for each number. Indicate any numeral that cannot be replaced by a decimal.

a. $\frac{2}{5}$	**d.** $\frac{7}{50}$	**g.** $\frac{13}{20}$	**j.** $\frac{2}{11}$
b. $\frac{1}{3}$	**e.** $\frac{2}{9}$	**h.** $\frac{5}{8}$	**k.** $\frac{1}{2}$
c. $\frac{1}{4}$	**f.** $\frac{1}{7}$	**i.** $\frac{5}{6}$	**l.** $\frac{9}{25}$

2. a. List the denominators of those fractions that can be replaced by decimals.

b. Write the prime factorization for each of these denominators.

3. a. List the denominators of those fractions that could not be replaced by decimals.

b. Write the prime factors for each of these denominators.

4. Compare the prime factorization of the denominators in exercise **2b** with those in exercise **3b**. How do you think you could determine whether or not a rational number named by a fraction in lowest terms has a decimal name?

Although many students are not able to fully verbalize a conclusion after working problems in a sequence such as that shown in this exercise set, the pattern it exhibits does seem to indicate that rational numbers with decimal names have completely reduced fractions with denominators whose prime factorizations contain only 2s and 5s. The prime factorizations of the denominators of rational numbers that did not have decimal names may include 2s and/or 5s as factors but include other factors as well (not counting 1, of course).

The basis of this pattern is that all powers of 10 have prime factorizations consisting entirely of 2s and 5s ($10 = 2 \times 5$, $100 = 2 \times 2 \times 5 \times 5$, $1{,}000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$). Therefore, for a denominator to be a factor of some power of 10, its prime factorization must also consist entirely of 2s and/or 5s.

In summary:

1. A rational number named by a fraction in lowest terms has a decimal name if the prime factorization of the denominator consists entirely of 2s and/or 5s.

2. A rational number named by a fraction in lowest terms does not have a decimal name if the prime factorization of the denominator includes factors other than 2s and/or 5s.

Despite this, students should not be given the impression that decimal names are never used for rational numbers such as $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{7}$. It is true that decimal approximations for these rationals exist. For example, $\frac{1}{3}$ is often approximated by 0.3, 0.33, or 0.333. ($\frac{1}{3} \approx 0.3 \approx 0.33 \approx 0.333$.) However, these decimals are approximations—not names of the number $\frac{1}{3}$. To obtain decimals for *all* rational numbers the concept of decimals must be extended to include infinite decimals, to be discussed in chapter 8, or mixed decimals. Some mathematicians regard expressions such as $0.33\frac{1}{3}$, $1.2\frac{4}{7}$, and so on, as mixed decimals:

$$0.33\frac{1}{3} = \frac{33\frac{1}{3}}{100} = \frac{100}{300} = \frac{1}{3}.$$

$$1.2\frac{4}{7} = \frac{12\frac{4}{7}}{10} = \frac{88}{70} = 1\frac{9}{35}.$$

COMPUTING SUMS OF RATIONAL NUMBERS WITH DECIMALS

Competence in computing sums of whole numbers is a prerequisite for competence in using the addition algorithm with decimals. Except for the extension of the decimal numeration system to the right of the ones place, the addition algorithm is identical. As in computing sums of whole numbers, numbers named by digits in like place-value positions are added. Since our extension preserved the relationship between adjacent place values, similar procedures for regrouping in addition are used. Furthermore, since we did not introduce any new numbers, only a new method for naming certain rational numbers, the properties associated with rational and whole numbers (associative, commutative, distributive, $\cdots$) are also preserved and may be used as a basis for justifying the algorithm for computing sums with decimals.

Students have had prior experience in computing sums of numbers named by fractions. Before computing sums with decimals, a review of computation with rational numbers such as those below is helpful.

$$\frac{3}{10} + \frac{5}{10} = \frac{8}{10}.$$

$$\frac{1}{100} + \frac{5}{100} = \frac{6}{100}.$$

$$2\frac{3}{10} + 1\frac{2}{10} = 3\frac{5}{10}.$$

$$2\frac{2}{100} + 1\frac{5}{100} = 3\frac{7}{100}.$$

Ask questions such as these: In the first of the equations, what is the number of tenths in the first addend? The second addend? The sum? What is the decimal equivalent of $\frac{8}{10}$? In $2\frac{3}{10} + 1\frac{2}{10}$, what is the number of ones and tens in each addend? In the computed sum? What is the decimal equivalent of $3\frac{5}{10}$?

Also prior to developing the algorithm for computing with decimals it is desirable to discuss several questions such as the following: One of the equations shown below is true. Which one? How do you know?

$$0.3 + 0.2 = 50. \qquad 0.3 + 0.2 = 5. \qquad 0.3 + 0.2 = 0.5.$$

The discussion of questions such as these can help lay a good foundation for estimating sums of rational numbers named by decimals. This foundation can be useful in helping pupils develop the skills necessary to estimate sums and consequently make fewer errors in placing the decimal point in a numeral for a sum. As will be shown later, estimating sums is a powerful tool that can be used to place the decimal point correctly when computing with decimals.

To introduce the topic of computing sums with decimals we can again use the technique of translating a new problem into one we already know how to solve.

Consider this problem:

$$0.3 + 0.2 = ?$$

What are the fractions for 0.3 and 0.2?

$$\frac{3}{10} \quad \text{and} \quad \frac{2}{10}.$$

Rewrite the above problem using these equivalent fractions, and compute the sum.

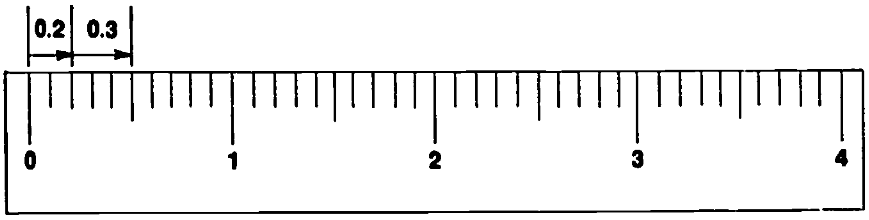
$$\frac{3}{10} + \frac{2}{10} = \frac{5}{10}.$$

Since $\frac{3}{10} + \frac{2}{10} = \frac{5}{10}$, and $\frac{5}{10} = 0.5$,

$$0.3 + 0.2 = 0.5.$$

On the basis of this and similar examples, it seems that sums with decimals can be computed in a similar manner to sums of whole numbers.

Sums such as 0.3 + 0.2 can also be visualized by using a number line or ruler graduated in tenths and interpreting the decimals as distances on the number line or ruler.



A similar procedure can be used to introduce the computation of sums such as those shown below.

(*a*) $0.03 + 0.04 = ?$

$$\frac{3}{100} + \frac{4}{100} = \frac{7}{100} = 0.07.$$

$0.03 + 0.04 = 0.07.$

(*b*) $0.6 + 0.7 = ?$

$$\frac{6}{10} + \frac{7}{10} = \frac{13}{10} = 1\frac{3}{10} = 1.3.$$

$0.6 + 0.7 = 1.3.$

(*c*) $0.12 + 0.37 = ?$

$$\frac{12}{100} + \frac{37}{100} = \frac{49}{100} = 0.49.$$

$0.12 + 0.37 = 0.49.$

(*d*) $1.3 + 2.5 = ?$

$$1\frac{3}{10} + 2\frac{5}{10} = 3\frac{8}{10} = 3.8.$$

$1.3 + 2.5 = 3.8.$

Again, most of these sums can be pictured by using a number line or ruler. Further analysis of examples such as 1.3 + 2.5 may be done by using the commutative and associative properties of addition to justify regrouping similar to that shown below.

$$\begin{aligned} 1.3 + 2.5 &= 1\frac{3}{10} + 2\frac{5}{10} \\ &= \left(1 + \frac{3}{10}\right) + \left(2 + \frac{5}{10}\right) \\ &= (1 + 2) + \left(\frac{3}{10} + \frac{5}{10}\right) \\ &= 3 + \frac{8}{10}. \end{aligned}$$

or

$$\begin{aligned} 1.3 + 2.5 &= (1 + 0.3) + (2 + 0.5) \\ &= (1 + 2) + (0.3 + 0.5) \\ &= 3 + 0.8 \\ &= 3.8. \end{aligned}$$

Since $3\frac{8}{10} = 3.8$, $1.3 + 2.5 = 3.8$.

Although the technique of restating each problem in fraction form, computing the sum, and converting back to decimal notation will always work, it can become quite laborious.

When adding two whole numbers such as 34 and 15 the digits are aligned vertically according to their respective place values ("like place values are added to each other").

$$\begin{array}{rl} 34 = & 3 \text{ tens } 4 \text{ ones} \\ +\ 15 = & \underline{1 \text{ ten } \ 5 \text{ ones}} \\ & 4 \text{ tens } 9 \text{ ones } = 49 \end{array}$$

Suppose we follow a similar pattern in findings the sum of 0.12 and 0.34.

$$\begin{array}{rl} 0.12 = & 1 \text{ tenth } \ 2 \text{ hundredths} \\ +\ 0.34 = & \underline{3 \text{ tenths } 4 \text{ hundredths}} \\ & 4 \text{ tenths } 6 \text{ hundredths } = 0.46 \end{array}$$

The above procedure can be justified by using the commutative and associative properties of addition.

$$\begin{aligned} 0.12 + 0.34 &= (0.1 + 0.02) + (0.3 + 0.04) \\ &= (0.1 + 0.3) + (0.02 + 0.04) \\ &= 0.4 + 0.06 \\ &= 0.46. \end{aligned}$$

Notice that when using either format, sums of numbers named by digits of the same place value are computed (tenths and tenths, hundredths and hundredths). The vertical format similar to the whole-number algorithm is more compact. The decimal point serves as a guide in aligning place values just as the ones place does when using the similar algorithm for computing sums of whole numbers.

For numbers greater than 1 the same algorithm can be used. Consider 3.5 + 1.2.

$$\begin{array}{rr} 3\frac{5}{10} & 3.5 \\ + 1\frac{2}{10} & + 1.2 \\ \hline 4\frac{7}{10} & 4.7 \end{array}$$

By aligning like place values, using the decimal point as our guide, we obtain the same result as when the computation is done with the equivalent mixed numerals.

By considering several problems using the algorithm and comparing the results with results obtained by other established methods, the student should be able to conclude—

1. that sums with decimals are computed as sums of whole numbers are computed;

2. that the decimal point can be used as a guide in vertically aligning place values so that sums of numbers named by digits in the same place value are computed.

This algorithm is also related to previous work in computing sums of rational numbers named by fractions. When we vertically align like place values on the right of the ones place we are computing the sum of numerators of like fractions.

Example:

$$\begin{array}{rl} 0.13 = & \frac{1}{10} + \frac{3}{100} \\ + 0.32 = & \frac{3}{10} + \frac{2}{100} \\ \cline{2-2} & \frac{4}{10} + \frac{5}{100} = 0.45. \end{array}$$

Once the vertical format has been established, examples that require regrouping can be introduced. Again, the procedure is analogous to regrouping when computing sums of whole numbers, since the relationship between adjacent place values is preserved.

Consider several examples such as the ones below.

$$\begin{array}{rl}
0.78 &= \text{7 tenths 8 hundredths}\\
\underline{+\ 0.47} &= \underline{\text{4 tenths 7 hundredths}}\\
&= \text{11 tenths 15 hundredths}\\
&\qquad\qquad\qquad (\text{15 hundredths} = \text{1 tenth, 5 hundredths})\\
&= \text{12 tenths 5 hundredths}\\
&= \text{1 one 2 tenths 5 hundredths}\\
&\qquad\qquad\qquad (\text{12 tenths} = \text{1 one, 2 tenths})\\
&= 1.25.
\end{array}$$

$$\begin{array}{rl}
3.23 &= \text{3 ones 2 tenths 3 hundredths}\\
\underline{+\ 8.39} &= \underline{\text{8 ones 3 tenths 9 hundredths}}\\
&= \text{11 ones 5 tenths 12 hundredths}\\
&= \text{11 ones 6 tenths 2 hundredths}\\
&= \text{1 ten 1 one 6 tenths 2 hundredths}\\
&= 11.62.
\end{array}$$

Once it is established that regrouping in addition with decimals is analogous to regrouping in addition of whole numbers, the student should be encouraged to regroup whenever it helps in computations. In this addition example, we show the regrouping.

$$\begin{array}{r}
{}^{1\,1}\\
.78\\
\underline{.47}\\
1.25
\end{array}$$

COMPUTING DIFFERENCES WITH DECIMALS

Subtraction can be thought of as finding a missing addend when one addend and the sum are known. Therefore

$$6.8 - 2.3 = \square$$

means

$$6.8 \qquad = \square + 2.3.$$

Since the addition algorithm for decimals is analogous to the algorithm for computing sums of whole numbers, it is natural to assume that differences with decimals can be computed as are differences of whole numbers.

Suppose they are. Then like place values are aligned vertically and $6.8 - 2.3 = \square$ is computed as follows.

$$\begin{array}{r} 6.8 \\ -\ 2.3 \\ \hline 4.5 \end{array}$$

If 4.5 is correct, it will make the statement $6.8 = \square + 2.3$ true.

$$\begin{array}{r} 4.5 \\ +\ 2.3 \\ \hline 6.8 \end{array}$$

It does. Therefore (in this case) the algorithm for computing differences with decimals is analogous to the whole-number algorithm for computing differences.

The above procedure can be justified (as was the procedure for computing sums with decimals) by replacing each decimal by an equivalent fraction, computing the difference, and then replacing the result by the equivalent decimal.

$$6.8 - 2.3 = \square.$$

$$6.8 = 6\frac{8}{10}.$$

$$2.3 = 2\frac{3}{10}.$$

$$\begin{array}{r} 6\frac{8}{10} \\ -\ 2\frac{3}{10} \\ \hline 4\frac{5}{10} \end{array}$$

Since $4\frac{5}{10} = 4.5$, $6.8 - 2.3 = 4.5$.

Computing differences with decimals can be introduced in the classroom in a manner exactly parallel to that for computing sums. As with addition, once vertical alignment is done correctly all goes exactly as with whole numbers.

"RAGGED DECIMALS"—PRO AND CON

Problems such as $1.3 + 5 + 2.02$ that have "ragged decimals"—decimals with varying numbers of digits to the right of the ones place—may not create problems for the student, but their value is debated by those who write textbooks. Some writers claim that such problems are obsolete and should not be included. They argue that rational numbers whose sums are to be computed (including those named by decimals) represent measurements and that it is not possible to have measurements to differing degrees of precision (like those above) in the same situation. Therefore, the argument goes, a problem such as $1.3 + 5 + 2.02$ will not occur in real life—only in some textbooks and on some exams!

We agree that in some situations the computation of such numbers is meaningless. However, such problems do have numerical results and will be considered in this section. What is open to question is not whether sums of such numbers can be computed but whether the sum has any meaning in a given problem situation.

How do we compute the sum of $1.3 + 5 + 2.02$?

Consider the same algorithm established earlier. If we align place values vertically, our problem and the result look like this:

$$\begin{array}{l} 1.3 \\ 5 \\ \underline{2.02} \\ 8.32 \end{array}$$

Is this answer reasonable? We know that $1 < 1.3 < 2$ and $2 < 2.02 < 3$. Therefore, the sum of $1.3 + 5 + 2.02$ should be between 8 and 10. It is, so our answer is reasonable.

Suppose we check the problem by rewriting the numbers in fraction form.

$$\begin{array}{lcl} 1\frac{3}{10} & = & 1\frac{30}{100} \\ 5 & = & 5 \\ 2\frac{2}{100} & = & \underline{2\frac{2}{100}} \\ & & 8\frac{32}{100} \end{array}$$

Since $8\frac{32}{100} = 8.32$, the result obtained earlier must be correct. Thus we have established that the same algorithm can be used even when the decimal names have varying numbers of digits to the right of ones place.

Recall that for any decimal an equivalent decimal can be generated by annexing zeros to the right of ones place. The equivalence of such names was discussed earlier and can be verified by using equivalent fractions or a manipulative aid such as a number line. For example,

$$0.3 = 0.30 = 0.300 = 0.3000 = \ldots .$$

$$1.7 = 1.70 = 1.700 = 1.7000 = \ldots .$$

$$3 = 3.0 = 3.00 = 3.000 = \ldots .$$

Using equivalent decimals, our original example can be rewritten as shown below:

Original Problem	Rewritten Problem
1.3	1.30
5	5.00
2.02	2.02
8.32	8.32

Since 1.3 = 1.30 and 5 = 5.00, each addend remains unchanged, and thus our result is the same. What we actually accomplished by rewriting the problem is equivalent to obtaining a common denominator as we did when working the same problem in fraction notation

$$\left(1.30 = 1\frac{30}{100},\ 5.00 = 5\frac{0}{100},\ 2.02 = 2\frac{2}{100}\right).$$

In the case of subtraction with ragged decimals, the annexing of zeros recommended for addition becomes essential for computing some differences.

Original Problem	Rewritten Problem
2.6	2.600
− .431	− .431
2.169	2.169

Exercise Set 4

1. Compare the three addition problems below. In what ways are they alike? In what ways are they different?

$$\begin{array}{r} 53 \\ +\ 24 \\ \hline 77 \end{array} \qquad \begin{array}{r} 5.3 \\ +\ 2.4 \\ \hline 7.7 \end{array} \qquad \begin{array}{r} 0.53 \\ +\ 0.24 \\ \hline 0.77 \end{array}$$

2. Compare the two methods of solution. In what ways are they alike? In what ways are they different?

$$\begin{array}{r} 3\frac{2}{10} \\ 2\frac{3}{10} \\ +\ 1\frac{4}{10} \\ \hline 6\frac{9}{10} \end{array} \qquad\qquad \begin{array}{r} 3.2 \\ 2.3 \\ +\ 1.4 \\ \hline 6.9 \end{array}$$

3. Complete the following.

- **a.** 12 hundredths = 1 tenth, ____ hundredths.
- **b.** 17 tenths = 1 one, ____ tenths.
- **c.** 6 tenths, 3 hundredths = 5 tenths, ____ hundredths.
- **d.** 8 tenths, 5 hundredths = 7 tenths, ____ hundredths.

4. Write a decimal for the computed sum.

- **a.** 5.8 + 3.2 + 0.7
- **b.** 0.23 + 0.45
- **c.** 0.69 + 0.52
- **d.** 1.82 + 3.17
- **e.** $1\frac{3}{10} + 3\frac{8}{10}$
- **f.** 0.23 + 4.2
- **g.** 0.249 + 0.843
- **h.** 14.21 + 23.83 + 8.05
- **i.** $6\frac{1}{2} + 11\frac{1}{4}$
- **j.** 20 + 4.5 + 0.67

5. Write a decimal for the computed difference.

- **a.** 8.5 − 2.3
- **b.** 5.79 − 4.3
- **c.** 0.94 − 0.48
- **d.** 6.0 − 1.7
- **e.** 3.850 − 1.647
- **f.** 0.48 − 0.3
- **g.** 0.400 − 0.278
- **h.** 9.3 − 6.59

6. In each pair subtract the smaller from the larger and write a decimal for the computed difference.

a. 0.24, 0.2 **b.** 0.9, 1.7 **c.** $\frac{3}{5}$, 0.5

7. In what ways does the computation of sums and differences with decimals differ from the computation of sums and differences of whole numbers? In what ways is it similar?

8. Discuss similarities between computing sums and differences with decimals and computing sums and differences of rational numbers named by fractions.

9. Consider each of the following statements. Do you agree with them? Why or why not?

- **a.** Students should not learn decimals as some new topic in the curriculum. The only phase of the work that is new is the notation.
- **b.** The distinguishing feature of decimals is the denominator and not the decimal point.

10. The following statements were made in the section on computing differences with decimals. What implications does each statement have for classroom instruction?

- **a.** Subtraction can be thought of as finding a missing addend when one addend and the sum are known.
- **b.** Computing differences with decimals can be introduced in the classroom in a manner parallel to that for computing sums with decimals.

SUMMARY

1. Having extended the decimal numeration system to the right of the ones place and learned to name rational numbers by decimals, the student can use previously learned algorithms for whole numbers to compute sums and differences of rational numbers.

2. Rational numbers with finite decimal names are those that have

fractional names with denominators whose prime factorizations contain only 2s and/or 5s.

3. It is not sufficient that the student gain *only* computational proficiency with these algorithms. It is hoped that students see the study of decimals for what it is—an "old" topic with new names for the numbers. This goal can be at least partially realized if instructional procedures such as those suggested in this chapter are used to integrate the study of decimals with previously learned concepts, operations, and properties of both rational and whole numbers.

Donovan R. Lichtenberg

DECIMALS: MULTIPLICATION AND DIVISION

×÷8

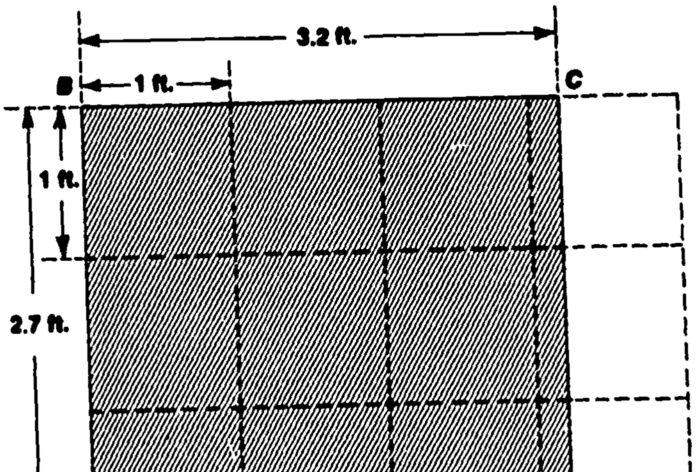
1. **Why are the multiplication and division algorithms used with decimals essentially the same as the ones used for whole numbers?**
2. **How can we help children learn to place the decimal point correctly in multiplication and division computation?**
3. **How can we extend the concept of decimal representation to include all rational numbers?**
4. **Are there any numbers that are not rational numbers?**

There are several reasons for the popularity of decimal notation, but the main one is probably the ease of computation which decimals afford. It seems that most people would rather compute with decimals than with fractions because the algorithms for decimals are essentially the same as for whole numbers, with the extra step of placing the decimal point. Simon Stevin, who first popularized decimal notation with his *La Disme* in 1585, wrote: "To speak briefly, *La Disme* teaches how all computations of the type of the four principles of arithmetic—addition, subtraction, multiplication and division [with decimals]—may be performed by whole numbers with as much ease as in counter-reckoning."

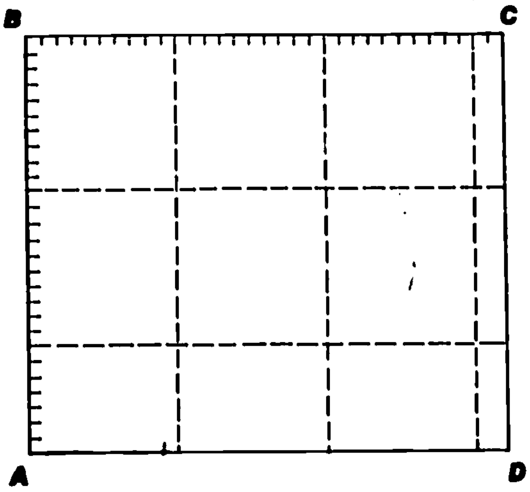
MULTIPLYING WITH DECIMALS

If a child can compute products of whole numbers, it should not be difficult for him to learn to compute products of rational numbers expressed in decimal form. But rather than simply learning rote rules for placing the decimal point, a child should learn the basis for these rules.

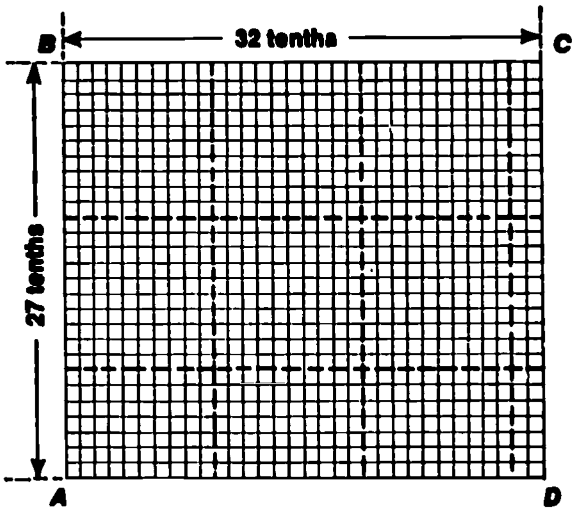
Consider the product 2.7 × 3.2. One of the first things that children should recognize is that this product is greater than 6 because $2.7 > 2$ and $3.2 > 3$. Similarly, they should see that it is less than 12 because $2.7 < 3$ and $3.2 < 4$. If children have learned to associate multiplication with areas of rectangular regions, the fact that 2.7 × 3.2 is between 6 and 12 can be illustrated vividly by means of a diagram. Suppose that in the figure below rectangle *ABCD* is 2.7 feet wide and 3.2 feet long. Then the area in square feet is 2.7 × 3.2. Since the squares in the figure represent square feet, it is clear that the area of the rectangular region is greater than 6 square feet and less than 12 square feet.



By refining this diagram, we can learn even more about the product 2.7 × 3.2. The figure below shows the same rectangular region with the edges marked off into tenths of a foot. Note that since the width is 2.7 feet, we can say that the width in feet is 27 tenths; the length is 32 tenths.



Now we can express the area in terms of some smaller square units. It can be seen in the next figure that each small square is one tenth of a foot on a side. Since it takes 100 small square regions to fill one unit square, each small square region has an area of one hundredth of a square foot. And it can be seen that rectangle *ABCD* encloses 27 × 32, or 864, of these smaller square regions. Hence the area is 864 hundredths, or 8.64, square feet. This illustrates why the digits in the computed product of 2.7 and 3.2 are the same as in the product of 27 and 32. This, therefore, is a way of providing the rationale for multiplying with decimals in the same way as with whole numbers.



It should be observed that once we have convinced a child that the digits in the product of 2.7 and 3.2 are the same as in the product of 27 and 32, we can teach him to place the decimal point by estimation. We know that $27 \times 32 = 864$, and we know that $6 < 2.7 \times 3.2 < 12$. Where can we place a point in "864" so that a number between 6 and 12 is named? Clearly, the point must be between the "8" and the "6."

The computation can also be explained by converting from decimals to fractions, as follows:

$$\begin{aligned} 2.7 \times 3.2 &= 2\frac{7}{10} \times 3\frac{2}{10} \\ &= \frac{27}{10} \times \frac{32}{10} \\ &= \frac{27 \times 32}{10 \times 10} \\ &= \frac{864}{100} \\ &= 8\frac{64}{100} \\ &= 8.64. \end{aligned}$$

We see again why we compute the product of 27 and 32 to obtain the product of 2.7 and 3.2.

Let's consider another example: 1.4×3.57.

$$\begin{aligned} 1.4 \times 3.57 &= 1\frac{4}{10} \times 3\frac{57}{100} \\ &= \frac{14}{10} \times \frac{357}{100} \\ &= \frac{14 \times 357}{10 \times 100} \\ &= \frac{4{,}998}{1{,}000} \\ &= 4\frac{998}{1{,}000} \\ &= 4.998. \end{aligned}$$

Here it was necessary to compute 14×357. By working through several examples in this manner, children begin to see a shortcut. They might think:

$$\boxed{\text{tenths}} \times \boxed{\text{hundredths}} = \boxed{\text{thousandths}}.$$

$$1.4 \quad \times \quad 3.57 \quad = \quad 4.998.$$

They will eventually discover the counting-off rule for placing the decimal point:

> *Count the total number of digits to the right of the decimal points in the numerals for the two factors, and place a point in the numeral for the product so that the number of digits to the right is equal to that total.*

Exercise Set 1

1. Draw rectangle diagrams that will show—

a. that 1.2 × 2.1 = 2.52;

b. that .7 × 3.5 = 2.45.

2. Compute each of the following products by first converting to fractions.

a. 3.7 × 6.2

b. 4.1 × .83

c. .8 × 17.5

d. .67 × 3.29

Many elementary textbook series introduce exponents. It is easy to see that the counting-off rule for placing the decimal point when multiplying is related to a property of exponents.

Children learn, for example, that $10^2 = 10 \times 10$. The exponent 2 indicates that 10 is used as a factor twice. Similarly, $10^3 = 10 \times 10 \times 10$. So $10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5$.

In general,

$$10^n = \overbrace{10 \times 10 \times \ldots \times 10}^{n \text{ factors}}$$

(10 is used as a factor n times)

with the understanding that $10^1 = 10$. Therefore,

$$10^m \times 10^n = \overbrace{(10 \times 10 \ldots \times 10)}^{m \text{ factors}} \times \overbrace{(10 \times 10 \times \ldots \times 10)}^{n \text{ factors}}$$

$$= \overbrace{10 \times 10 \times 10 \times \ldots \times 10}^{m + n \text{ factors}}.$$

Hence

$$10^m \times 10^n = 10^{m+n}.$$

Numbers such as 10, 100, 1,000, and so forth, which can be expressed in the form 10^n, are called "powers of 10". Any decimal is equivalent to a fraction in which the denominator is a power of 10. Now consider the product 1.23 × 4.785. We know that

$$1.23 \times 4.785 = 1\frac{23}{100} \times 4\frac{785}{1,000}$$

$$= \frac{123}{100} \times \frac{4,785}{1,000}$$

$$= \frac{123}{10^2} \times \frac{4,785}{10^3}.$$

We see that for each factor the number of places to the right of the decimal point is the same as the exponent in the denominator of the corresponding fraction. The product is

$$\frac{123 \times 4{,}785}{10^2 \times 10^3},$$

and since $10^2 \times 10^3 = 10^5$, we have

$$\frac{588{,}555}{10^5}.$$

Since the denominator is 10^5, we must count off 5 places from the right. So

$$1.23 \times 4.785 = 5.88555.$$

It should be noted again in this example that the position of the decimal point can easily be determined by observing that 1.23 is between 1 and 2 and 4.785 is between 4 and 5. The product, therefore, is between 4 and 10. It is clear that the decimal point must be placed after the "5" on the left, showing a product of 5.88555.

This method of placing the point by estimating should be stressed in the elementary school much more than it usually is. It forces the child to think about the reasonableness of his results, and it is less apt to become merely a mechanical process.

As a matter of fact, this is the method for placing the decimal point that people have necessarily used when computing with a slide rule. It is necessary because a slide rule usually does not give the exact digits of the numeral for a product. If one attempts to compute 1.23 × 4.785 on a slide rule, he will be able to read off approximately "589" as the product. He places the decimal point by reasoning as above. The slide-rule user knows that the product must be between 4 and 10, so it is approximately 5.89.

As another illustration, suppose one wishes to compute 2.4 × 35.7 on a slide rule. From the rule he will be able to read approximately "857". Now, where should the decimal point be placed? Since $2 \times 35 = 70$ and $3 \times 36 = 108$, the product of 2.4 and 35.7 must be approximately 85.7.

Exercise Set 2

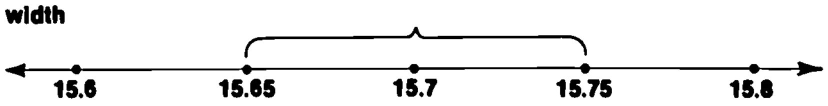
In each of the following a product is given, along with a slide-rule reading. Place the decimal point in the slide-rule reading by estimating the product.

1. 4.3×7.8	335	**5.** 12.3×13.5	166
2. 1.75×5.5	962	**6.** 8.7×9.7	844
3. 11.5×4.33	498	**7.** 0.51×8.07	412
4. $6.7 \times .83$	556	**8.** 0.23×20.5	472

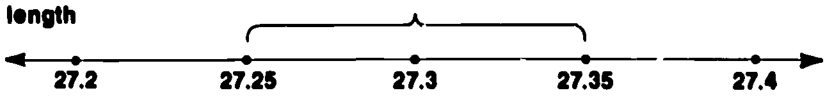
The above exercises illustrate that the slide rule usually gives a result that is "rounded off." This perhaps would be a good place to discuss the importance of rounding in applications of mathematics.

At the beginning of this chapter we assumed we were working with a rectangle whose width is 2.7 feet and whose length is 3.2 feet. From a pure-mathematics standpoint, the area of such a rectangle is exactly 8.64 square feet. But in the physical world, is it possible to find a rectangle that measures exactly 2.7 feet by exactly 3.2 feet?

All measurement is subject to error. This is a fact that surveyors, engineers, and others who apply mathematics learn to accept. If a surveyor measures a rectangle and says that it is 15.7 feet by 27.3 feet, he is indicating that he has measured to the nearest tenth of a foot. This means that the width is closer to 15.7 feet than it is to 15.6 feet or 15.8 feet. But this means that all that is known is that the width is between 15.65 and 15.75, as indicated on this number line:



Likewise, if the length is indicated as 27.3, it is between 27.25 and 27.35:



Now $15.7 \times 27.3 = 428.61$, but does it make sense to say that the area of the rectangle is 428.61? Do we really know what the area is, to the nearest hundredth? If we let w represent the exact width of the rectangle and l the length, we know that

$$15.65 < w < 15.75 \qquad \text{and} \qquad 27.25 < l < 27.35.$$

So all we really know about the area is that it is between 15.65×27.25 and 15.75×27.35. Carrying out the computations, we find that the area in square feet is between 426.4625 and 430.7625. You can see that it hardly makes sense to say that the area is 428.61 square feet. A surveyor would probably round this result to 429 square feet. That is, he would round off so that there are no more digits in the product than there are in either factor. It can be seen from the explanation above that we have no assurance even that 429 is correct to the nearest whole number!

Exercise Set 3

In the example above 426.4625 can be called a "lower bound" for the area of the rectangle, and 430.7625 can be called an "upper bound." Find upper and lower bounds for the area of each of the following rectangles, assuming that measurements have been made to the nearest tenth of a unit.

1. $w = 4.2,\ l = 7.3$

2. $w = 9.7,\ l = 14.1$

3. $w = 6.0,\ l = 9.1$

By the time children begin working with decimals, they have usually learned that it is easy to compute a product of whole numbers when one factor is a power of 10. They learn that if one factor is ten they can simply annex a zero to the numeral for the other factor. They learn to annex two zeros if the factor is 100, and so on. When exponents are introduced, they should see that the number of zeros annexed is the same as the exponent in the power of ten:

$27 \times 10 = 270, \quad 27 \times 10^2 = 2{,}700, \quad 27 \times 10^3 = 27{,}000, \quad$ etc.

Most people think in terms of annexing zeros, but it would be more correct to talk about digits moving to the left. That is, if a number is multiplied by 10, the original ones digit becomes the tens digit, the tens digit becomes the hundreds digit, and so forth. Children can construct a table such as the following:

Ten Thousands	Thousands	Hundreds	Tens	Ones	
			2	7	← 27
		2	7	0	← 27 × 10
	2	7	0	0	← 27 × 100
2	7	0	0	0	← 27 × 1,000

Each digit moves two places to the left when multiplying by 100, three for 1,000, and so forth. Assuming that the same thing would happen if one factor is, for example, 2.784, children can build the following table:

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths	
			2	7	8	4	← 2.784
		2	7	8	4		← 2.784 × 10
	2	7	8	4			← 2.784×10^2
2	7	8	4				← 2.784×10^3

This illustrates why it is possible to think of moving the decimal point to the right when multiplying by a power of 10. In reality the decimal point occupies a fixed position, namely, a position immediately to the right of the units place. But the digits move into new places when a number is multiplied by a power of ten.

By using the above idea, children can learn to express numbers in scientific notation. In this notation a number is expressed as a product in which one factor is a number between 1 and 10 and the other factor is a power of 10. Here are some examples:

$$43 = 4.3 \times 10$$

$$640 = 6.4 \times 10^2$$

$$5{,}837 = 5.837 \times 10^3$$

$$6{,}000{,}000 = 6 \times 10^6$$

$$52.3 = 5.23 \times 10$$

$$463.7 = 4.637 \times 10^2$$

It can be seen that to express a given number in scientific notation one can proceed as follows: (1) Write down the digits representing the number, omitting any decimal point. (2) Insert a decimal point in this numeral so that a number between 1 and 10 is named. (3) Write as the second factor whatever power of 10 is necessary to make the product equal to the given number.

Exercise Set 4

1. Express each of the following in standard decimal form.

a. 28.4×100

b. $6.7 \times 1{,}000$

c. 2.837×10^2

d. 6.1×10^5

2. Express each of the following in scientific notation.

a. 475

b. 28.9

c. 57,000

d. 3,287

e. 43.689

f. 35,000,000,000

DIVISION WITH DECIMALS

Once children have learned that multiplication with decimals is much like multiplication with whole numbers, they probably suspect that the same holds true for division. In view of the relationship between the two operations, it could hardly be otherwise. This can be brought out by asking children to complete a sequence of sentences such as the following:

Since 15 × 23 = 345, then 345 ÷ 23 = □.
Since 1.5 × 23 = 34.5, then 34.5 ÷ 23 = □.
Since 15 × 2.3 = 34.5, then 34.5 ÷ 2.3 = □.
Since 1.5 × 2.3 = 3.45, then 3.45 ÷ 2.3 = □.

If a child completes the sentences correctly, he sees that all of the numerals for the quotients have the same digits. It looks as if one should be able to use the algorithm for computing 345 ÷ 23 to compute each of the other quotients also. The only new thing to learn is how to place the decimal point.

It can be seen from the above examples that if the division process terminates, or "comes out even," the decimal point can be placed by using what we know about placing it in multiplication. To illustrate further, consider 3.75 ÷ 2.5. If we ignore the decimal points and compute as if we were computing 375 ÷ 25, the work would look like this:

$$\begin{array}{r} 15 \\ 2.5\overline{)3.75} \\ \underline{2\ 5} \\ 1\ 25 \\ \underline{1\ 25} \end{array}$$

Now since the quotient times the divisor equals the dividend, and since there is one place to the right of the point in the divisor and two in the dividend, we know that the point must be placed between the "1" and "5" in the quotient. That is 3.75 ÷ 2.5 = 1.5 because 1.5 × 2.5 = 3.75. If one wished to state a counting-off rule for division, it would be this:

> *The number of decimal places in the numeral for the quotient is the number of places in the numeral for the dividend minus the number of places in the numeral for the divisor.*

Exercise Set 5

Each of the following sentences can be made true by placing a decimal point in the numeral on the right-hand side. Place the decimal point by using the reasoning discussed above.

1. 4.06 ÷ 2.9 = 1 4. **4.** 7.955 ÷ 1.85 = 4 3.
2. 22.41 ÷ 2.7 = 8 3. **5.** .52416 ÷ 1.12 = 4 6 8.
3. 2.646 ÷ 4.2 = 6 3. **6.** 62.37 ÷ 231 = 2 7.

The last exercise above illustrates an important special case: If the divisor is a whole number, the number of decimal places in the numeral for the quotient is the same as in the numeral for the dividend. This leads to the well-known procedure of "placing the decimal point in the quotient immediately above that in the dividend," as in these examples:

$$\begin{array}{r} 7.8 \\ 9\overline{)70.2} \\ \underline{63} \\ 7\,2 \\ \underline{7\,2} \end{array} \qquad \begin{array}{r} .57 \\ 41\overline{)23.37} \\ \underline{20\,5} \\ 2\,87 \\ \underline{2\,87} \end{array}$$

Although this procedure works only in the case where the divisor is a whole number, it turns out to be a useful idea because every division problem can be transformed into one in which the divisor is a whole number. The general principle that allows us to make the transformation was discussed in chapter 6. The principle referred to is that if the divisor and dividend are both multiplied by the same nonzero number, the quotient is unchanged. That is,

$$a \div b = (a \times c) \div (b \times c) \qquad \text{if} \qquad c \neq 0.$$

We combine this principle with the fact that any rational number that can be expressed as a decimal can be multiplied by an appropriate power of ten to produce a whole number. To compute 43.4 ÷ 1.24, for example, we multiply both 43.4 and 1.24 by 100 and then divide the resulting numbers, that is, 4,340 by 124. This is the explanation for what most adults know as "moving the decimal points":

$$1\underset{\smile}{.24}\overline{)43\underset{\smile}{.40}}$$

After we have transformed the problem so that the divisor is a whole number, we can place the decimal point in the quotient above that in the dividend:

$$\begin{array}{r} 35. \\ 124.\overline{)4340.} \\ \underline{37\,2} \\ 6\,20 \\ \underline{6\,20} \end{array}$$

We have found that 43.4 ÷ 1.24 = 4,340 ÷ 124 = 35.

It is worth noting that it isn't always necessarily most convenient to multiply by a power of ten. If dividing 6.75 by .25, for example, we can take advantage of the fact that .25 × 4 = 1. When we multiply both 6.75 and .25 by 4, we have

$$\begin{aligned} 6.75 \div .25 &= (6.75 \times 4) \div (.25 \times 4) \\ &= (6.75 \times 4) \div 1 \\ &= 6.75 \times 4. \end{aligned}$$

We see that we have eliminated any division computation; we need only compute 6.75 × 4. Most children would find this easier than computing 675 ÷ 25, which is what they would have to do if 6.75 and .25 were both multiplied by 100.

The good student will recognize that the above explanation agrees with what he has already learned about division of rational numbers. After all, decimals are simply names for rational numbers, and the student has learned that dividing by a rational number (other than zero) is the same as multiplying by its reciprocal, that is, by its multiplicative inverse. Since .25 × 4 = 1, the numbers .25 and 4 are multiplicative inverses of one another. So 6.75 ÷ .25 = 6.75 × 4.

Exercise Set 6

Tell whether each of the following statements is true or false. If a statement is false, form a true statement by changing the position of a decimal point on the right-hand side or by inserting one.

1. 67.4 ÷ 1.3 = 674 ÷ 13.
2. 6.53 ÷ .7 = 653 ÷ 7.
3. 32 ÷ .16 = 3,200 ÷ 16.
4. .038 ÷ .02 = 3.8 ÷ 2.
5. .00064 ÷ .004 = 6.4 ÷ 4.
6. 3.9 ÷ .003 = 390 ÷ 3.
7. 8.49 ÷ .236 = 8,490 ÷ 236.
8. 454 ÷ 20 = 45.4 ÷ 2.

The explanation of division given thus far has ignored some troublesome complications, and it is now necessary to come to grips with them. One complication is illustrated by 48 ÷ 7.5. If we multiply both 48 and 7.5 by 10 and proceed with the computation, our work looks like this:

$$\begin{array}{r} 6. \\ 7.5_{\wedge}\overline{)48.0_{\wedge}} \\ \underline{45\ 0} \\ 3\ 0 \end{array}$$

Now, certainly we cannot say that 48 ÷ 7.5 = 6, because the division did not "come out even." However, if we recall that 480 = 480.0, we can take care of the problem:

$$\begin{array}{r} 6.4 \\ 7.5\overline{)48.0\,0} \\ \underline{45\,0} \\ 3\,0\,0 \\ \underline{3\,0\,0} \end{array}$$

So 48 ÷ 7.5 = 6.4. (It can be seen that for the counting-off rule mentioned earlier to be valid it is necessary to think of 48 as 48.00.)

Some students might conclude from examples such as the preceding that if division computation does not "come out even" we can always annex one or more zeros so that it will. But experience will convince them that it is really a rare case where this is so. In other words, the quotient of two numbers usually cannot be expressed exactly as a decimal. So what can we do? One answer is that we can round off to whatever degree of precision is desired. We may wish, for example, to compute 23.6 ÷ 7 to the nearest tenth. Our computation would be as follows:

$$\begin{array}{r} 3.37 \\ 7\overline{)23.60} \\ \underline{21} \\ 26 \\ \underline{21} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

We know there are more digits to the right of the "7", which we have not yet computed, so the quotient is between 3.37 and 3.38. This tells us that 23.6 ÷ 7 is closer to 3.4 than to 3.3, so we write

$$23.6 \div 7 \approx 3.4.$$

EXTENSION OF THE DECIMAL CONCEPT—INFINITE DECIMALS

Children should be taught how to round off, but they should also be encouraged to extend a computation such as the example in the preceding paragraph to see what happens. That is, they should annex some more zeros and continue the work with an alert eye for some sort of pattern.

This is what they will find if six more zeros are annexed:

$$\begin{array}{rr}
 & 3.37142857 \\
 & 7\overline{)23.60000000} \\
 & \underline{21} \\
 & 26 \\
 & \underline{21} \\
\rightarrow & 50 \\
 & \underline{49} \\
 & 10 \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{35} \\
\rightarrow & 50
\end{array}$$

When the alert student gets to the line indicated by the second arrow, he will realize that he is again at the stage indicated by the first arrow. He will see that the work from that point on is going to be repeated over and over. The block of digits "714285" will repeat in the quotient infinitely many times.

Now how should this quotient be interpreted? Can the infinite repeating decimal that results from applying the division process be thought of as representing a number? These are not trival questions. Complete answers lie beyond the scope of this book. You should see that if we are to think of the infinite decimal as representing a number, an extension of the decimal concept is required, because a decimal represents a sum. For example,

$$2.438 = 2 + \frac{4}{10} + \frac{3}{100} + \frac{8}{1,000}.$$

An infinite decimal would, then, seem to indicate an infinite sum. Using techniques of higher mathematics, it can be shown that certain types of infinite sums can make sense and that it is reasonable to talk about an infinite decimal as representing a number. There is a definite point on the number line that corresponds to any infinite decimal.

The result of the computation above can be shown like this:

$$23.6 \div 7 = 3.3\overline{714285}$$

The bar over the block of six digits means that this block repeats infinitely many times. Notice that in this statement "=" is used rather than "≈". This means that the infinite decimal on the right is considered a name for the quotient. Of course, the quotient is a rational number, because we could write:

$$23.6 \div 7 = 236 \div 70 = \frac{236}{70} = \frac{118}{35}.$$

Now that we have extended the notion of decimals to include infinite ones, we shall call the type of decimals discussed originally either "finite decimals" or "terminating decimals." Some rational numbers can be expressed as terminating decimals. (If you do not remember which rational numbers can be so expressed, refer back to chapter 7.) Is it perhaps true that any rational number that is not representable as a terminating decimal will have a repeating decimal representation? Let's once again look at the computation for 23.6 ÷ 7. Note the circled remainders:

$$
\begin{array}{r@{}l}
 & \underline{\;\;3.3714285} \\
7) & 23.6000000 \\
 & \underline{21} \\
 & \;\,26 \\
 & \;\,\underline{21} \\
 & \;\;\;\textcircled{5}0 \\
 & \;\;\;\underline{49} \\
 & \;\;\;\;\textcircled{1}0 \\
 & \;\;\;\;\;\;\underline{7} \\
 & \;\;\;\;\;\textcircled{3}0 \\
 & \;\;\;\;\;\underline{28} \\
 & \;\;\;\;\;\;\textcircled{2}0 \\
 & \;\;\;\;\;\;\underline{14} \\
 & \;\;\;\;\;\;\;\textcircled{6}0 \\
 & \;\;\;\;\;\;\;\underline{56} \\
 & \;\;\;\;\;\;\;\;\textcircled{4}0 \\
 & \;\;\;\;\;\;\;\;\underline{35} \\
 & \;\;\;\;\;\;\;\;\;\;5
\end{array}
$$

Each of the numbers 1, 2, 3, 4, 5, and 6 occurs as a remainder after the point where the annexed zeros are used. You should see that no other nonzero remainders are possible when dividing by 7. Consequently, the next remainder must be 0 or a remainder that has occurred previously. In the latter case a repeating decimal results.

The decimal representation for any rational number $\frac{a}{b}$ can be determined by dividing *a* by *b*. Suppose the division process does not terminate. Then 0 does not occur as a remainder, and there are only so many nonzero numbers that can occur. If all of these numbers have occurred as remainders after the point where the annexed zeros are used, the next remainder will have to duplicate a previous one. It can happen that a duplicate will occur before all of the possible remainders have been used. (Exercise **2** below will illustrate this.) In either case a repeating decimal results. Hence, any rational number can be expressed as either a terminating or a repeating decimal.

Exercise Set 7

1. Compute each of the following quotients and round to the nearest tenth.

a. $45 \div 2.3$

b. $321 \div 17$

c. $5.7 \div .13$

d. $13 \div 7$

2. Express each of the following rational numbers as a repeating decimal. Use the bar notation.

a. $\frac{2}{3}$ **d.** $\frac{3}{13}$

b. $\frac{4}{7}$ **e.** $\frac{7}{37}$

c. $\frac{5}{11}$

3. Suppose that a rational number $\frac{a}{b}$ cannot be expressed as a terminating decimal. What is the maximum number of digits that could be in the repeating block in the infinite-decimal representation?

If a number is rational, then it can be expressed as either a terminating or an infinite repeating decimal. Is the converse of this statement true? Is it true that if a number can be expressed as a terminating or repeating decimal, then it is a rational number? For terminating decimals, the answer to this question becomes apparent when one thinks of converting from decimals to fractions. Examples are

$$0.7 = \frac{7}{10}, \quad 2.53 = 2\frac{53}{100} = \frac{253}{100}, \quad \text{and} \quad 0.371 = \frac{371}{1{,}000}.$$

Any terminating decimal corresponds to a fraction whose denominator is a power of ten. So any number that can be expressed as a terminating decimal can also be expressed in the form $\frac{a}{b}$ where a and b are whole numbers, and hence any such number is a rational number.

It is not so easy to see whether or not a given repeating decimal represents a rational number. Consider, for example, the repeating decimal $3.24\bar{7}$. If this represents a rational number, we should be able to find whole numbers a and b such that $\frac{a}{b} = 3.24\bar{7}$. This can be done. To see how to find a and b, let's begin by letting w represent the number under consideration. That is, if $w = 3.247777 \cdots$,

$$100 \times w = 324.7777\ldots \quad \text{and} \quad 1{,}000 \times w = 3{,}247.7777\ldots.$$

(We are assuming that the rules for arithmetic with infinite decimals are the same as for finite decimals. It can be shown that these rules are still valid.)

Now if you have never seen this technique before, you are probably wondering why we multiplied w by 100 and then by 1,000. The reason should become clear in the next step. Notice that in the decimal expressions for $100 \times w$ and $1{,}000 \times w$ the part to the right of the decimal point is the same for each (the digit "7" repeats forever). Now notice what happens when we subtract $100 \times w$ from $1{,}000 \times w$:

$$\begin{array}{r} 1{,}000 \times w = 3{,}247.7777\ldots \\ \underline{100 \times w = 324.7777\ldots} \\ 900 \times w = 2{,}923 \end{array}$$

The difference is a whole number. Since $900 \times w = 2{,}923$, $w = \frac{2{,}923}{900}$. This shows that w is a rational number. The reader should divide 2,923 by 900 to verify that the repeating decimal $3.24\overline{7}$ is obtained.

A little thought will convince you that the procedure used above will work for any repeating decimal. Consequently, any repeating decimal represents a rational number. Let's look at one more example. We shall show that $.4\overline{239}$ represents a rational number by finding a fraction for it. Let's call the number y.

$$y = \quad .4239239239\ldots$$

Then $\quad 10 \times y = \quad 4.239239239\ldots$ (Why do we multiply by 10?)

and $\quad 10{,}000 \times y = 4{,}239.239239\ldots$ (Why do we multiply by 10,000?)

Now we subtract $10 \times y$ from $10{,}000 \times y$:

$$\begin{array}{r} 10{,}000 \times y = 4{,}239.239239\ldots \\ \underline{10 \times y = \quad\quad 4.239239\ldots} \\ 9{,}990 \times y = 4{,}235 \quad\quad\quad\quad \end{array}$$

So $y = \frac{4{,}235}{9{,}990}$. This fraction is not in lowest terms, but that is irrelevant here. We have shown that $.4\overline{239}$ represents a rational number.

With a slight adjustment in our thinking, our findings about decimal representations of rational numbers can be summarized in a very concise manner. Any terminating decimal can be thought of as a repeating decimal in which the digit zero repeats forever after a certain point. For instance, $0.5 = 0.50000\ldots$. With this in mind, we can say that a number is rational if and only if it can be expressed as an infinite repeating decimal.

Exercise Set 8

Show, by finding fractions for the following repeating decimals, that each of them represents a rational number.

1. $.3\overline{8}$	**3.** $2.\overline{7}$
2. $.\overline{43}$	**4.** $.4\overline{9}$

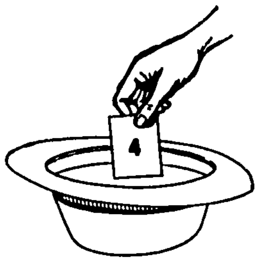
Were you perhaps surprised by the result in the last exercise above? The result indicates that the repeating decimal .49999. . . represents the rational number $\frac{1}{2}$. But we also know that $.50000\ldots = \frac{1}{2}$. This seems to mean that $\frac{1}{2}$ has two repeating-decimal representations. It happens that this is an inescapable fact. Not only is it true of $\frac{1}{2}$, it is true of every non-zero rational number that can be expressed as a terminating decimal. Every such rational number has two repeating-decimal representations, one in which 0 repeats and one in which 9 repeats. Here are some more examples:

$$\frac{1}{4} = .250000\ldots = .249999\ldots.$$

$$\frac{3}{8} = .3750000\ldots = .3749999\ldots.$$

$$\frac{2}{5} = .40000\ldots = .39999\ldots.$$

We have been talking about infinite repeating decimals. Are there other kinds of infinite decimals? You should see that it is at least possible to think of an infinite decimal that does not repeat. Suppose we have a hat in which we have placed ten cards, each of which has one of the digits 0 through 9 on it. Now suppose we reach into the hat and, without looking, pick a card. Perhaps we draw the 4. We then write "4" with a decimal point to its left. After replacing the card in the hat, we draw again. We write whatever digit is drawn to the right of the first one. If we continue like this, we shall generate a decimal numeral. After ten drawings we might have



.4018873971

Now think of continuing this forever. There is certainly no reason to suspect that this technique would generate a repeating decimal, although of course it might occasionally. It cannot represent a rational number if it does not repeat. Does the resulting infinite decimal, then, represent a number at all? It can be shown that there is a precise point on the number line for any infinite decimal. So it is reasonable to think of an infinite nonrepeating decimal as representing a number. We simply say that each such decimal represents an irrational number.

An example of an irrational number is the number whose square is 2, that is, $\sqrt{2}$. We sometimes read that $\sqrt{2} = 1.414$, but 1.414 is only an approximation for $\sqrt{2}$ because

$$(1.414)^2 = 1.999396.$$

It is because $\sqrt{2}$ is an irrational number and consequently cannot be expressed as a fraction or terminating decimal or repeating decimal that we must denote it by using the radical sign (or doing something equivalent). It can be shown that the square root of any whole number is either a whole number itself ($\sqrt{9} = 3$) or else it is an irrational number. Another well-known example of an irrational number is π, which is often approximated by 3.14. Again we can say that it is because of the fact that this is an irrational number that a special symbol must be used for it.

Most of the day-to-day affairs of the world are conducted using only rational numbers. But one should not conclude from this that irrational numbers are scarce. In an important sense there are more irrational numbers than there are rational numbers. However, a discussion of this matter is beyond the scope of this book.

The rational and irrational numbers should be thought of as intermingled on the number line. It was pointed out in chapter 2 that between any two rational numbers there is always another rational. It is also true that between any two rationals there is an irrational. Similarly, between any two irrationals there is always another irrational and also a rational.

Exercise Set 9

1. Give two different repeating decimal representations for each of the following rational numbers.

a. $\frac{3}{4}$

b. $\frac{5}{2}$

c. $\frac{7}{8}$

d. $\frac{6}{25}$

2. Think of the infinite decimal expression

$$.10100100010000100000 1\ldots,$$

where the ones are separated by zeros and in each block of zeros there is one more zero than in the preceding block. Does this expression represent a rational number or an irrational number?

3. Write a decimal expression for an irrational number between the rational numbers $.34\overline{2}$ and $.34\overline{25}$.

Some students are fascinated by discoveries they can make about repeating decimals. If asked to find the decimal representations of $\frac{1}{7}$ and $\frac{2}{7}$, their work would look as follows:

```
      .142857          .285714
    7)1.000000       7)2.000000
       7                1 4
       30                60
       28                56
  →     20                40
        14                35
         60                50
         56                49
          40                10
          35                 7
           50                30
           49                28
            1                 2
```

So $\frac{1}{7} = .\overline{142857}$ and $\frac{2}{7} = .\overline{285714}$. We note that the repetend (the repeating block of digits) for $\frac{2}{7}$ consists of the same digits as the repetend for $\frac{1}{7}$. Moreover, the digits are in the same cyclical order. An examination of the computation shows why this is so. The beginning stage of the computation for $\frac{2}{7}$ matches the stage indicated by the arrow in the computation for $\frac{1}{7}$. Thereafter, the steps are identical, so the same digits have to appear in the quotient. As a matter of fact, by just looking at the computation for $\frac{1}{7}$, we can also conclude the following:

$$\frac{3}{7} = .\overline{428571}$$

$$\frac{4}{7} = .\overline{571428}$$

$$\frac{5}{7} = .\overline{714285}$$

$$\frac{6}{7} = .\overline{857142}$$

If the denominator is 7 and the numerator is other than a multiple of 7, the repetend will consist of the same six digits. There are other denominators that exhibit a similar property. Examples are 17, 19, and 23.

There is another aspect of repeating decimals that will be of interest to some students. From the argument showing why the decimal for $\frac{a}{b}$ repeats if it does not terminate, it can be seen that the maximum number of digits in the repetend is $b - 1$ (see exercise **3** in set 7). But this maximum number will not always occur. If, for example, we wish to rename $\frac{3}{13}$ as a decimal we would get the following:

```
      .230769
 13)3.000
    2 6
      40
      39
       10
        0
       100
        91
         90
         78
         120
         117
           3
```

When the remainder 3 is reached we realize that we are back to the starting point. We see that $\frac{3}{13} = .\overline{230769}$, and the repetend contains only 6 digits rather than the maximum of 13 − 1, or 12. Upon examining the work, we see that the numbers 4, 1, 10, 9, 12, and 3 occur as remainders. The numbers 2, 5, 6, 7, 8, and 11 do not and of course will not, even though the process goes on forever. Is there a relationship between the denominator and the number of digits in the repetend? There is a very simple relationship in the case where the denominator is a prime number. Perhaps you have already guessed it. The exercises below will give you some more clues.

Exercise Set 10

1. Complete this table.

REPEATING DECIMALS

$\frac{a}{b}$	Repeating Decimal for $\frac{a}{b}$	Number of Digits in Repetend	$b - 1$
$\frac{1}{7}$	$0.\overline{142857}$	6	6
$\frac{3}{13}$	$0.\overline{230769}$	6	12
$\frac{1}{3}$			
$\frac{6}{11}$			
$\frac{4}{37}$			
$\frac{7}{41}$			
$\frac{15}{73}$			
$\frac{11}{101}$			
$\frac{30}{271}$			

2. How are the numbers in the third column in the above table related to the numbers in the fourth column?

3. We saw that in computing the decimal for $\frac{3}{13}$ the number 2 did not occur as a remainder. Compute the decimal for $\frac{2}{13}$ and compare the remainders with the ones obtained in the computation for $\frac{3}{13}$.

SUMMARY

1. If a student knows the multiplication and division algorithms for whole numbers, he needs to learn only how to place the decimal point in order to be able to compute with decimals. Procedures for placing the decimal point can and should be taught in meaningful ways.

2. The division process does not always terminate. If it doesn't, the quotient can be approximated as closely as we please by a finite decimal.

The decimal concept can be extended to include infinite decimal expressions. It can be shown that any infinite decimal corresponds to a point on the number line and hence represents a number.

3. A number is rational if and only if it can be expressed as an infinite repeating decimal.

An infinite nonrepeating decimal represents an irrational number.

Lauren G. Woodby

MEASUREMENT

9

1. **How are rational numbers related to measurement?**
2. **What is the role of arbitrary units?**
3. **What are some strategies for measurement of area?**

A *laboratory approach* was chosen for this chapter because of the nature of measurement. Manipulation and observation are involved in the measurement process, and the best way for anyone to learn about measurement is to measure things. You should do the selected measurement activities, just as children do them, in order to see more clearly how rational numbers and the fractions that name these numbers arise naturally from a concrete setting. Your experience with these measurement activities will suggest others that are appropriate for children in your classes.

Arbitrary units are emphasized because the author believes that an emphasis on standard units tends to obscure the basic notions children should learn about measurement. In the suggested measurement activities for length, area, and weight, a unit object is to be chosen arbitrarily.

Similar triangles are used to find length by indirect measurement. Two simple homemade instruments are described in this section.

The approximate nature of a measurement is made clear by the experiment with successive refinements in finding the inner and outer areas of a region having a curved boundary. This activity leads to an intuitive notion of the limit process.

DISCRETE AND CONTINUOUS QUANTITIES

The size of a crowd and the capacity of a parking lot are quantities that can be found by counting discrete objects. The numbers are whole numbers and are exact (i.e., without error) if the counting is done correctly. However, the height of a person, the amount of paint in a can, and the size of a playground are continuous quantities. The measures are usually not whole numbers, and there always exists some uncertainty in determining these numbers. Our main concern is with measurement of continuous quantities.

THE MEASUREMENT PROCESS

Measurement is a process of assigning a number to a physical object in order to tell how much of some particular property that object possesses. When we measure an object to find how much of some property it possesses, we usually compare that object with a unit object and assign a number. But how is this comparison done? If the property we are considering is weight, we really answer the question "How many unit objects are needed to balance the given object?" We compare *objects* with respect to some property. Our first step is to choose some object as the unit object for that property.

The choice of a unit object is arbitrary. The concern about standard units comes long after children learn to measure with arbitrary units. For example, our arbitrary unit of length might be the length of a pencil. We often use the term "unit of length" to mean the unit object itself as well as its length. The number assigned to the unit object is "1."

It is common practice to use the same term, for example, "weight," to denote the *property* and also the *measure* of that property. An object has weight (property), and for a particular unit the weight (measure) is a unique number, but we can never hope to find this number exactly; we can only get close to the number that we assume exists. In the process of getting close, we need rational numbers to obtain increasingly better approximations of the measure. For example, we might be satisfied to know the weight of an object to the nearest pound. For a finer measure we could use $\frac{1}{2}$-pound weights. For still finer measures we could use $\frac{1}{4}$-pound weights or $\frac{1}{8}$-pound weights. If a piece of brass labeled "1 gram" is the unit object and a decision to the nearest gram is not precise enough to suit us, we could use fractional parts of the unit, such as $\frac{1}{10}$ gram or $\frac{1}{100}$ gram or even $\frac{1}{1,000}$ gram. We can get closer and closer approximations, but there is always an error involved in the measurement of continuous quantities.

ORDER

Before measurement tasks make much sense to a child, he must be able to compare two objects with respect to the property being measured. At this premeasurement stage, numbers are not involved. The child needs only some physical means of comparing to decide which object has more of an attribute (property). One way to provide this experience and at the same time focus attention on the property to be measured is to have children work individually at four tables identified by signs "Length," "Area," "Weight," and "Volume." Suggested activities, and comments on them, follow.

Activity 1. Ordering according to Length

1. Select three objects. Arrange in order from shortest to longest.

2. Select another object and decide where it belongs in the ordered arrangement. Put it there.

3. Repeat with another object.

4. Continue until you have at least seven objects arranged in order of length.

5. Arrange three of your classmates in order of height.

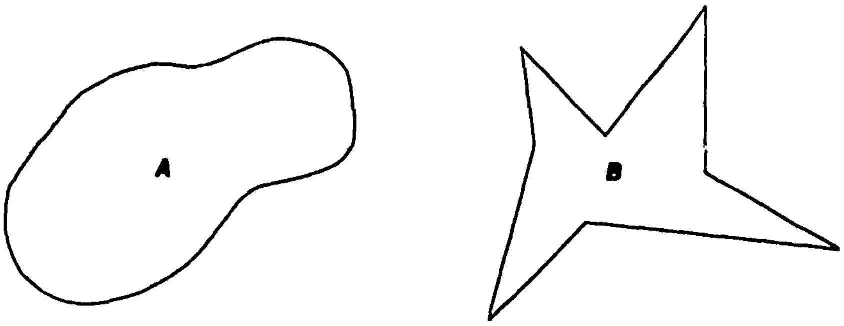
6.

Transfer each line segment to the working line with one end at the point marked "Begin." Decide the order of the segments from shortest to longest.

Activity 2. Ordering according to Area

1. Select six cardboard shapes. Arrange in order from least to most area. You may want to cut and fit pieces.

2. Which region covers more space, *A* or *B*?



3. Select a banana and a tangerine. Which do you think has more skin area? Test your guess.

Children develop interesting ways to compare the areas of two objects. When a child peels a tangerine and pushes the flattened pieces close together he achieves considerable insight into the problem. Some children will try to cover two objects with foil or paper and then compare the two covers in some way. The general problem of comparing surface areas of three-dimensional objects helps to clarify the question "What is area?" When comparing the areas of two plane regions, children can cut one region into pieces and rearrange to try to cover the other. Numbers are not needed, since the comparison is made directly.

Activity 3. Ordering according to Weight

Several balances should be available at this table. Children should first estimate which of a pair of objects is heavier by holding one in each hand and judging by feel. Then they should check their estimate with a balance.

1. Select six objects and arrange them in order from lightest to heaviest.

2. Where does your pencil belong in this ordering?

3. Where does a quart of water belong in this ordering?

Activity 4. Ordering according to Volume

A supply of rice or water will be needed at this table. A variety of containers of assorted shapes should be available for selection.

1. Select six containers and order them from least to most capacity. Make an estimate first and then check your estimate.

ARBITRARY UNITS

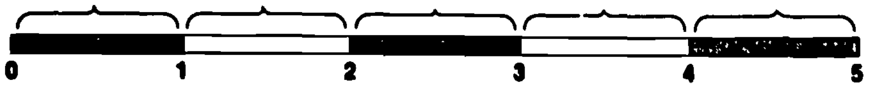
One source of confusion about measurement is premature emphasis on standard units. Arbitrary units are natural for children to use in their introductory work in measurement, and teachers find that children learn basic notions of measurement best if they are allowed to choose their own units. In particular, children focus their attention on the notion of an interval when they measure length using a soda straw or a popsicle stick (or any other object they select) for their unit object.

Activity 5. Measurement of Length (Arbitrary Units)

Choose any unit object having length and use it to measure the lengths of at least five other objects. Record your data in a table like this:

To Measure This Object	Use This Unit Object	Length Measure		
		Greater than	Less than	Closer to
Red Paper Strip	Soda Straw	3	4	3
Dowel Rod				
Etc.				

Children see the connection between their unit and distance on a number line if they tape several unit objects (e.g., soda straws) end to end on the chalkboard. The counting numbers assigned to points on their crude number line are measures of the distance from the starting point. Attention is focused on the intervals between the points.



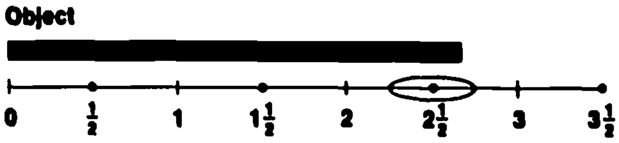
Children see that the intervals between the marked points are each one unit long. The length of the object pictured below can be given as "between 3 and 4." Notice that two numbers are used to designate this interval.

The next stage is to use one number for all lengths that lie within an interval. For example, the length of the object above can also be expressed as "3 units," meaning that the length is closer to 3 than to either 2 or 4. In turn, this means that when an object is placed with one endpoint at 0 on the number-line scale, the other endpoint lies in the one-unit interval whose center is at the mark for 3. The observer must make a judgment about the location of the halfway points (which are shown on this scale) between marks on the scale. This is an important skill, and plenty of practice should be given in reading to the nearest mark.

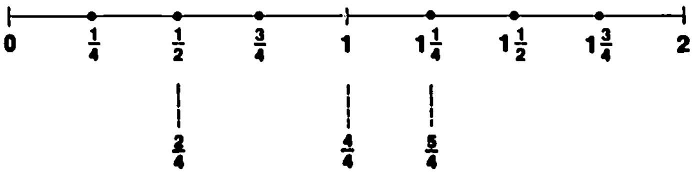
NEED FOR RATIONAL NUMBERS

At this stage a child naturally wants to use halves. He will not be satisfied with the precision of the measure "5 soda straws" for his height. He can fold the soda straw to get a half-unit, and he can fold it again to get a quarter-unit. He needs rational numbers, and he has a way to get them that he understands. It makes sense to divide the unit length into 2 same-size pieces and name the length of one of them by the symbol "$\frac{1}{2}$." He can also get the lengths $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$ of his unit.

Now that $\frac{1}{2}$-unit and $\frac{1}{4}$-unit lengths are available, the child can measure lengths of objects to the nearest $\frac{1}{2}$ unit or the nearest $\frac{1}{4}$ unit.



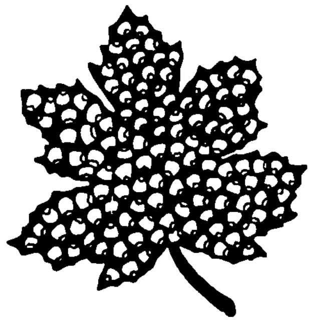
In the picture, we judge that the endpoint lies in the $\frac{1}{2}$-unit interval centered at the $2\frac{1}{2}$ mark. Thus, the length of the object (to the nearest $\frac{1}{2}$ unit) is $2\frac{1}{2}$ units. Any other object placed with the left endpoint at 0 and its right endpoint anywhere in the same interval also has a length (to the nearest $\frac{1}{2}$ unit) of $2\frac{1}{2}$ units. In order to measure lengths to the nearest $\frac{1}{4}$ unit, we need marks at $\frac{1}{4}$-unit intervals. It is natural to name these points with the fractions shown along the number line pictured below. Through measurement, rational numbers become associated with points on the number line.



Another outcome of this activity is that children observe that “$\frac{1}{2}$” and “$\frac{2}{4}$” name the same point on the scale. They also see that “$\frac{4}{4}$” is another name for “1.” They are discovering equivalent fractions in a meaningful physical setting.

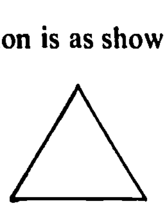
AREA MEASUREMENT—ARBITRARY UNITS

To measure the area of a leaf a child might count the kernels of corn that cover it.

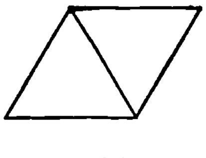


A regular polygon of arbitrary size can be chosen as the unit object to define a unit of area. Triangular shapes are very simple to use and can be arranged conveniently into natural groupings of 2, 3, 4, and 6 that have interesting shapes.

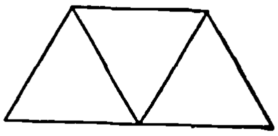
For example, if the unit region is as shown below,



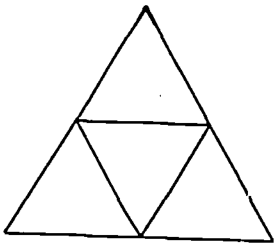
then the measure of

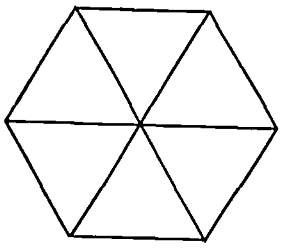


is 2, and

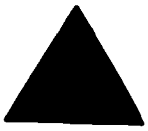


has measure 3. Groupings of 4 and 6 units are shown next.

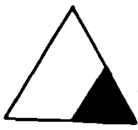


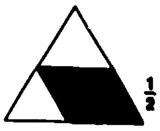


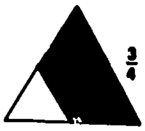
Moreover, again using a triangle for the unit region, certain fractions appear quite naturally. For examples, see below.

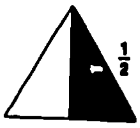


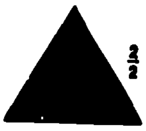
1 unit

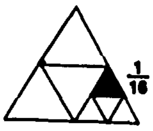




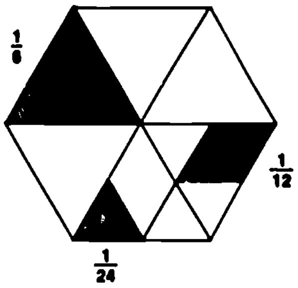




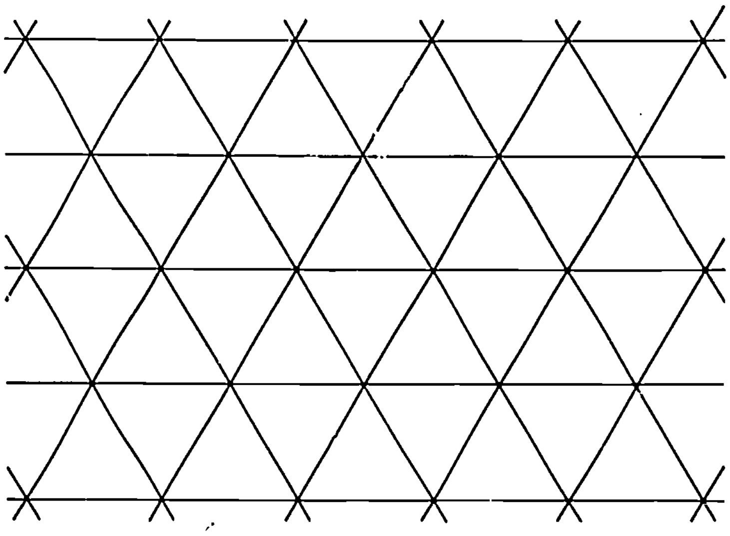




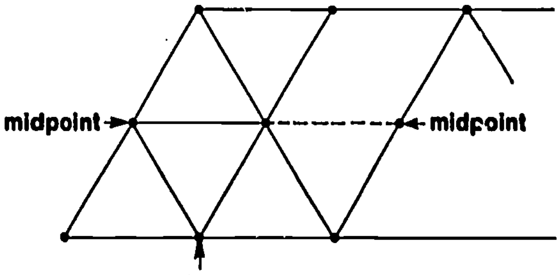
Hexagonal shapes offer rich opportunities for obtaining certain fractions. For example, if the unit region now is a hexagon, then the shaded regions have the measures indicated.



Specially ruled paper (called "isometric ruled") is convenient for drawing triangular shapes and hexagonal shapes. Here is a sample.



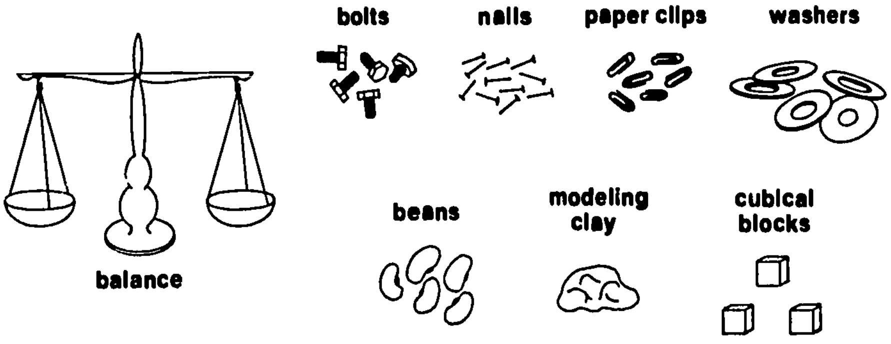
Such paper can be easily prepared and duplicated for use by children. Begin by constructing a (comparatively) large equilateral triangle. Make some copies, find midpoints of segments, and extend lines as indicated below.



WEIGHT MEASUREMENT—ARBITRARY UNITS

Arbitrary units of weight provide another concrete approach to fractions and rational numbers.

Activity 6. Weighing with Arbitrary Units



1. Choose any object you wish for your unit object and use it to weigh several objects of your choice, recording results in a table like this:

To Weigh This Object	Use This Unit Object	Weight Measure		
		More than	Less than	Closer to
Eraser Block of Wood Etc.	Washer	6	7	7

2. Put a washer (or whatever other object you chose for your unit object) on one pan. On the other pan put just the right amount of modeling clay to balance the washer. Now remove the washer and separate the clay into 2 parts so that the 2 pieces of clay have the same weight. Divide one of these pieces again into 2 equal-weight parts.

3. Use your clay weights (and washers) to weigh the same objects you weighed before and record your results.

4. How could you improve your results even more?

After much experience with an arbitrary unit of weight, such as the weight of a steel washer, the children usually want to get "closer" or "better" results, but the washers can't conveniently be cut into parts.

Give them a supply of beans or buttons or paper clips that are fairly uniform and ask them to figure out a way to get a finer result. Some child may find that 12 beans weigh the same as 1 washer, so he will have the fractions "$\frac{1}{12}$," "$\frac{2}{12}$," "$\frac{3}{12}$," etc. Another child may find and use fifths. This activity illustrates clearly the use of fractions with larger denominators for finer measurement. If 5 buttons weigh the same as 1 washer, each button weighs $\frac{1}{5}$ as much as a washer; if 12 paper clips weigh as much as 1 washer, a paper clip weighs $\frac{1}{12}$ as much as a washer. It is clear that weighing to the nearest $\frac{1}{12}$ unit is more precise than weighing to the nearest $\frac{1}{5}$ unit.

RATIO

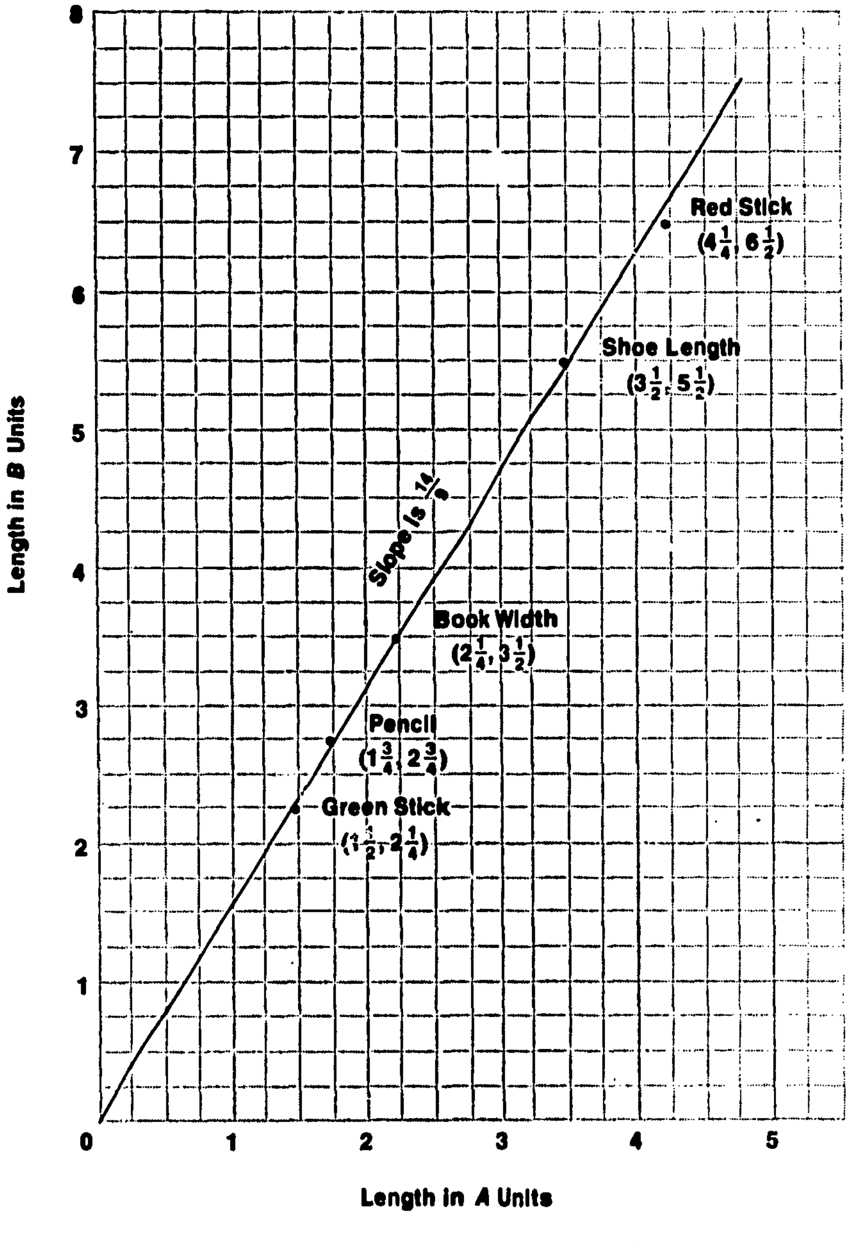
There is a natural approach to the concept of ratio, again using arbitrary units. Begin with this problem:

How can you compare the lengths of two objects, *A* and *B*, without measuring the objects? (*A* and *B* could be strips of paper).

1. Use *A* and *B* as arbitrary units to measure the lengths of several other objects. Record results in a table like this:

Object	Length in *A* Units	Length in *B* Units
Red Stick	$4\frac{1}{4}$	$6\frac{1}{2}$
Shoe Length	$3\frac{1}{2}$	$5\frac{1}{2}$
Etc.		

2. Now graph the number pairs for the length of each object. Try to draw a straight line that seems to fit the pattern best. What can you say about this line?



In the sample graph shown, the steepness of the line is about "3 up for 2 over." One statement of this result is "The length of 3 of the *B* strips is about the same length as 2 of the *A* strips." Another statement is "The length of *A* is about $\frac{3}{2}$ the length of *B*." Can you interpret the results in some other way? Suppose you measured an object with an *A* strip and

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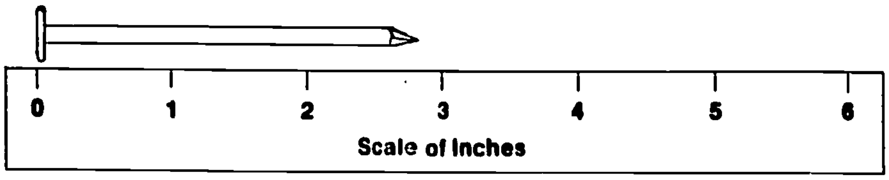
found its length to be about 6 *A* strips. What would be its length in *B* strips? The fraction $\frac{3}{2}$ can be used to compare the length of *A* to the length of *B*. When used in this way, "$\frac{3}{2}$" is called a "ratio."

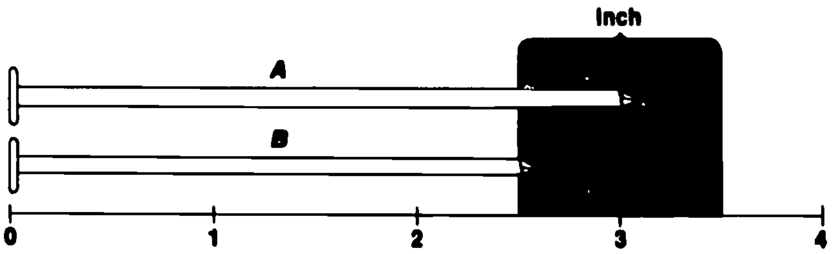
STANDARD UNITS

So far no standard units have been mentioned, and this is as it should be. Standard units are often introduced too soon and overemphasized in textbooks. Moreover, most standard units are really arbitrary. After children learn how to measure with units of their choice, it is easy for them to use standard units. The basic notions are the same. Children readily see that standard units are necessary for communication.

A foot ruler with one-inch units subdivided into sixteenths is commonly used in school. This instrument is too complicated—cluttered up with too many marks—for young children. Instead, a simple six-inch ruler marked only in one-inch intervals is suggested for young children. If simple rulers without any marks between the one-inch marks are not available, children can make them of cardboard. Another technique is to cover the markings on the complicated ruler so that only the one-inch marks are seen.

The nail in the illustration (which is not shown actual size) is 3 inches long *to the nearest inch* because its endpoint is closer to 3 than to 2 or to 4. The observer should imagine a 1-inch interval centered at the "3" mark. Any nail whose endpoint lies in that interval is 3 inches long, to the nearest inch. Nail *B*, though longer than nail *A*, is also 3 inches long to the nearest inch. The precision, or fineness, of measures obtained with this scale is 1 inch because the length of the interval is 1 inch. In order to measure more precisely we need a smaller interval, just as with the scale of arbitrary units.





THE GEOBOARD

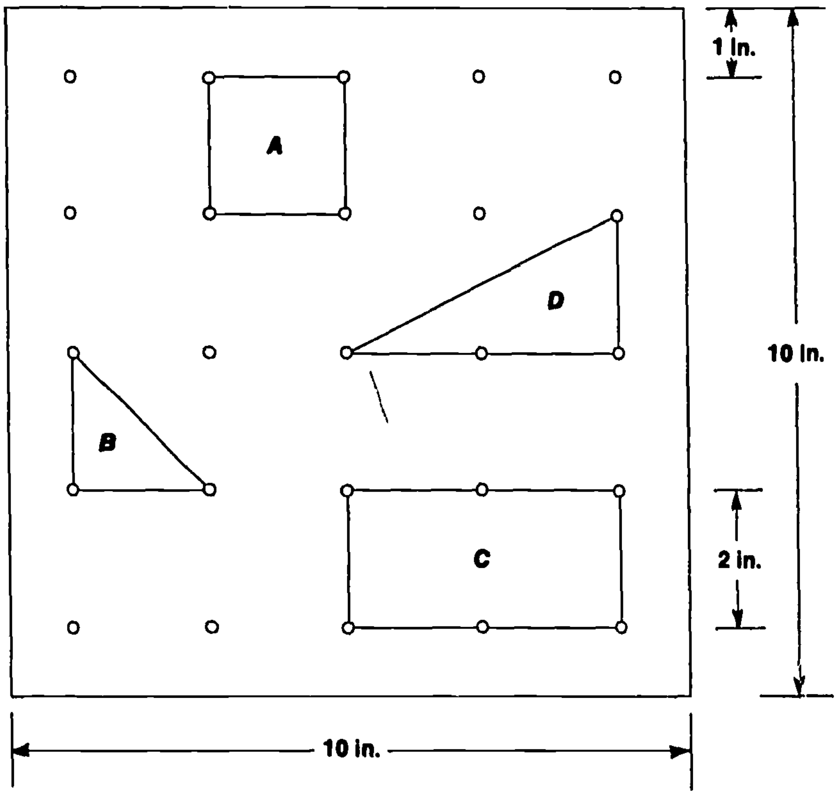
One of the most versatile teaching devices is a geoboard, or nailboard. Homemade geoboards are in some ways better than commercially made ones, and the cost of materials is less than 50 cents a board. You should actually make a geoboard to experience firsthand the simplicity and ease of construction; then have every child in your room make one.

Activity 7. Making a Geoboard

The materials for making the geoboard to be described here are a 10-by-10-inch piece of plywood, $\frac{1}{2}$ inch thick; sandpaper; a paper grid with 2-inch squares; 25 no. 15 brass nails (escutcheon pins) about $\frac{3}{4}$ inch long; and a hammer.

Sand the piece of plywood to remove sharp edges. Allowing a 1-inch border, use the paper grid as a template and pound in a pin at each lattice point so that there are 25 pins in a 5 × 5 array, as shown in the figure.

Having a 1-inch border makes it possible to use two or more of the boards, placed together, to give more lattice points.



Geoboard

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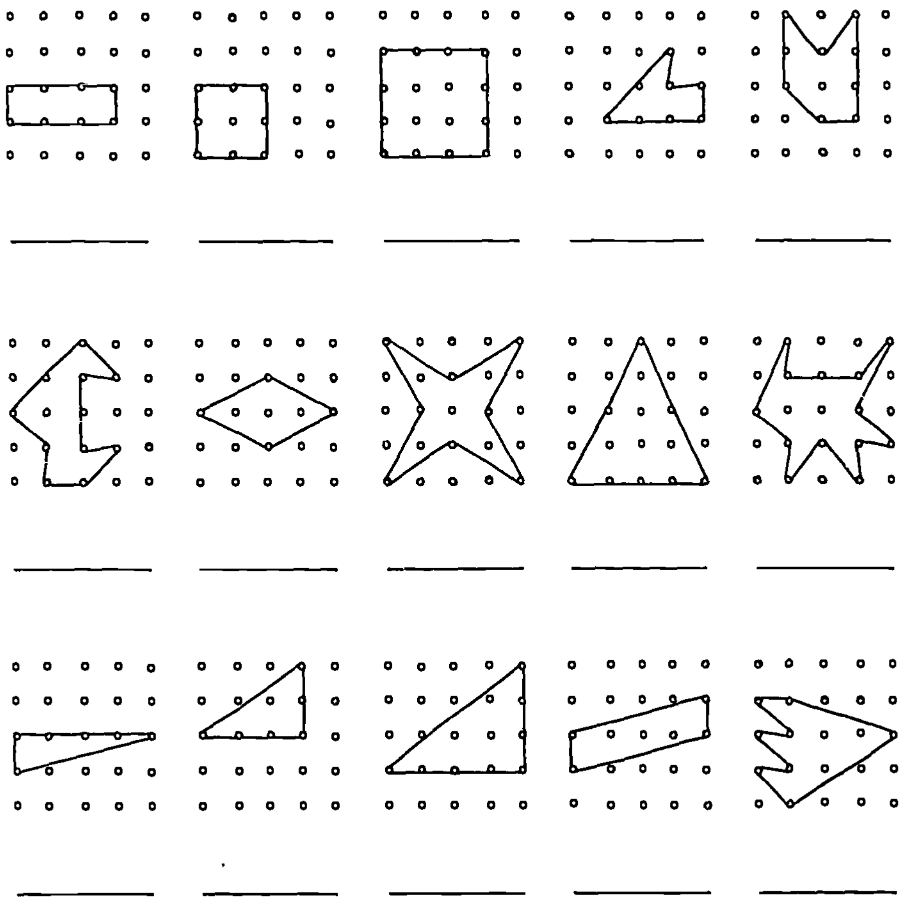
Activity 8. Using the Geoboard for Area

On the geoboard, rubber bands are stretched around pins to indicate idealized geometric figures. Some basic shapes are shown on the geoboard pictured above. Usually a square shape, *A*, is used for the unit of area. Shape *B* has half as much area as *A*. Children *see* this intuitively, but they really believe it when they tear a square piece of paper that just covers *A* and fold it along a diagonal to form two triangular pieces of paper that fit one on top of the other over *B*. In this figure if *A* is the unit region, then *B* is $\frac{1}{2}$ and *C* is 2. The area of *D* is seen intuitively to be half the area of *C*. Children should prove it by cutting or tearing a piece of paper that fits on *C* to get two same-size pieces that fit on *D*.

Now use rubber bands to make shapes like those pictured below and to create some original shapes. Find the area of each.

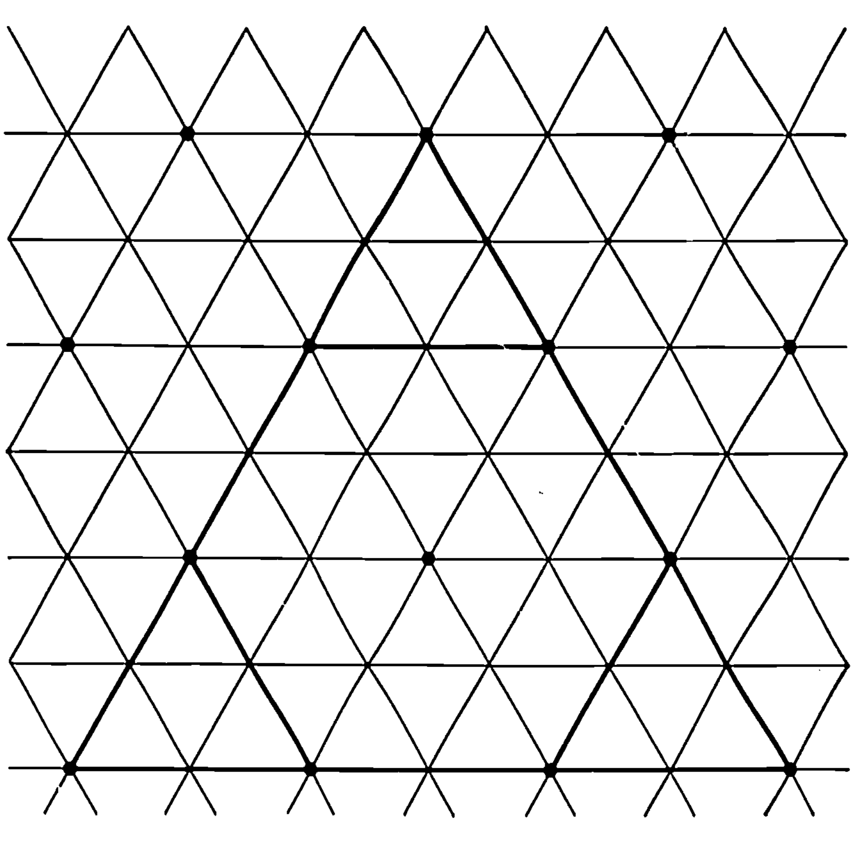


Activity 9. Additional Geoboard Activities

1. What other ways can you use the geoboard in working with fractions?

2. What could you choose for the unit region in order to show sixteenths conveniently?

3. Make a geoboard with triangles, rather than squares, as the basic shapes. Isometric paper can be used as a template. What might this geoboard be used for? (See the figure below.)



INDIRECT MEASUREMENT OF LENGTH

Many measurements are made indirectly. For example, to find how far it is between two houses in your town you could measure the distance on a map and then use the scale. The figure below shows a way to use triangles to find a distance, *BC*, that cannot be measured directly because of an obstacle.

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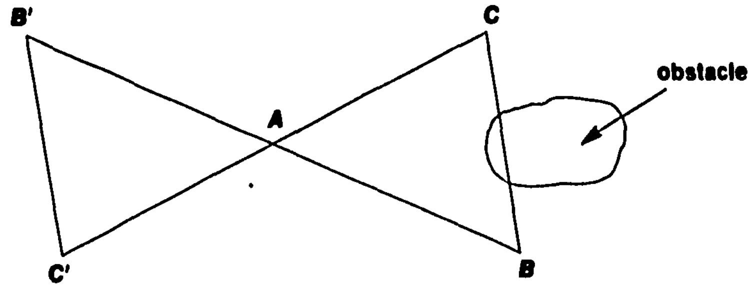
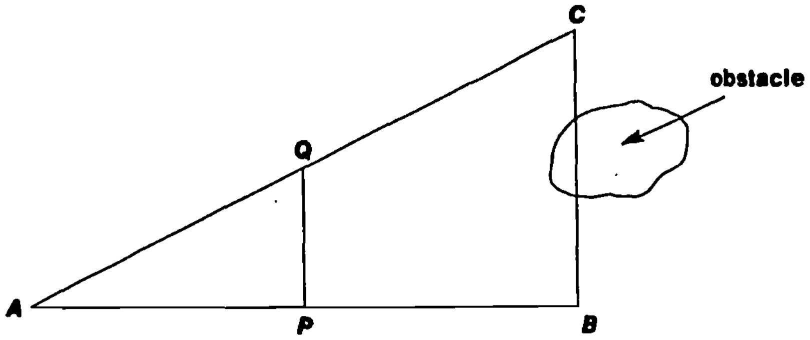


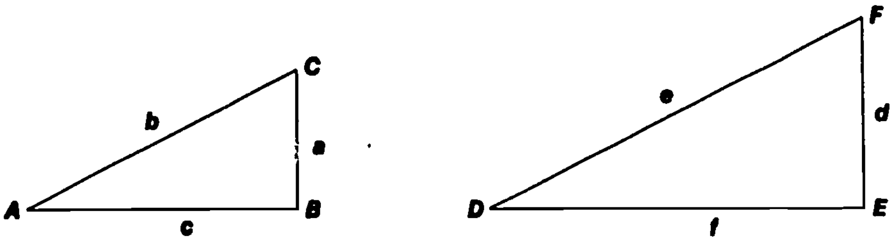
Choose any convenient point *A* and stretch a string from *B* to *B'* through *A* so that *AB'* is just as long as *AB*; stretch a string from *C* to *C'* through *A* so that *AC'* is just as long as *AC*. Now measure the distance from *B'* to *C'*. This length will also be the length of *BC* because the new triangle *AB'C'* is the same size and shape as triangle *ABC*.

An alternative solution to the problem above is to find *P*, the midpoint of *AB*, and *Q*, the midpoint of *AC*. Now measure the distance *PQ*. The desired distance, *BC*, is twice *PQ*.



This alternative solution is based on the notion of similar triangles—triangles that have the same shape. Each side of triangle *ABC* is twice as long as the corresponding side of triangle *APQ*. Or, to put it the other way, since *AP* is half as long as *AB* and *AQ* is half as long as *AC*, *PQ* is half as long as *BC*.

Similar figures have the same shape but not necessarily the same size. A good example is a photograph and its enlargement. Triangles *ABC* and *DEF*, below, have the same shape; that is, they look alike. More specifically, the pairs of corresponding angles *A* and *D*, *B* and *E*, and *C* and *F* are the same size. We call the triangles similar.



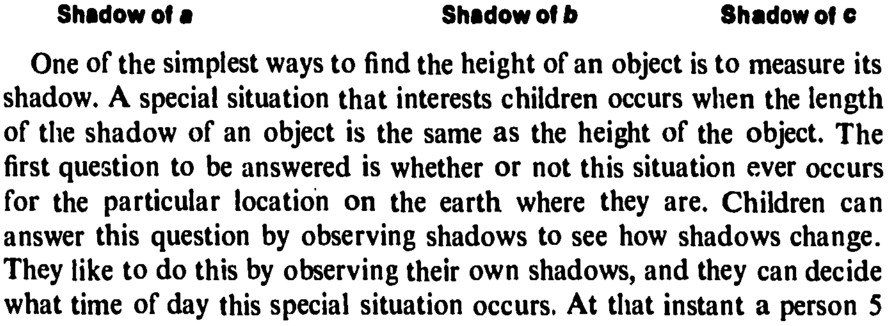
The interesting and useful thing about similar figures is that corresponding sides are proportional; that is,

$$\frac{a}{d} = \frac{c}{f}, \quad \frac{a}{d} = \frac{b}{e},$$

and also

$$\frac{a}{c} = \frac{d}{f}, \quad \frac{a}{b} = \frac{d}{e}.$$

Activity 10. Height Measurement Using Shadows



Shadow of a Shadow of b Shadow of c

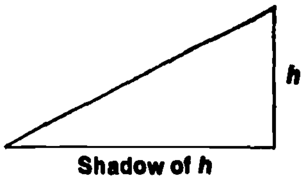
One of the simplest ways to find the height of an object is to measure its shadow. A special situation that interests children occurs when the length of the shadow of an object is the same as the height of the object. The first question to be answered is whether or not this situation ever occurs for the particular location on the earth where they are. Children can answer this question by observing shadows to see how shadows change. They like to do this by observing their own shadows, and they can decide what time of day this special situation occurs. At that instant a person 5 feet tall has a shadow 5 feet long; a flag pole 70 feet tall casts a shadow 70 feet long. So, at that instant, the height of a building can be found by measuring the length of its shadow. To illustrate with equations:

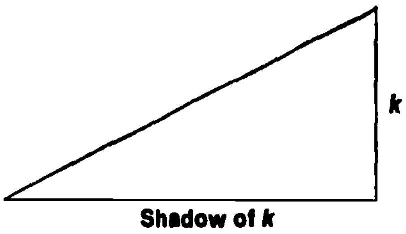
$$\frac{\text{height of } a}{\text{length of shadow of } a} = 1, \quad \frac{\text{height of } b}{\text{length of shadow of } b} = 1,$$

and

$$\frac{\text{height of } c}{\text{length of shadow of } c} = 1$$

A variation of this technique is to find another special situation—for example, when the length of a person's shadow is twice the person's height.





In that special situation,

$$\frac{\text{height of } k}{\text{length of shadow of } k} = \frac{1}{2}$$

A building 52 feet tall would cast a 104-foot shadow.

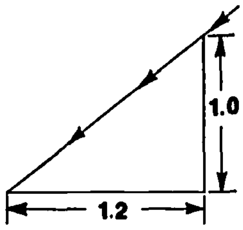
Another way to use similar triangles in finding heights is to measure the length of the shadow of an object of known height and, at about the same time, measure the length of the shadow of an object whose height is to be determined. For example, if the length of shadow of a meterstick is 1.2 meters, the length of the shadow of *any* object (at that same time) is 1.2 times as long as its height. We know that $\frac{h}{s} = \frac{1.0}{1.2}$. If s (the length of the shadow of h) is found to be 60 feet, we know that h is a little less than 60. If $\frac{h}{60} = \frac{1.0}{1.2}$, $h = 50$, as shown below.

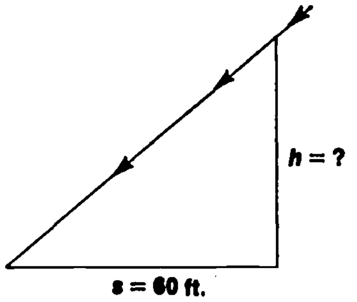
$$60 \times \frac{h}{60} = 60 \times \frac{10}{12},$$

so

$$h = \frac{600}{12}$$

$$= 50$$





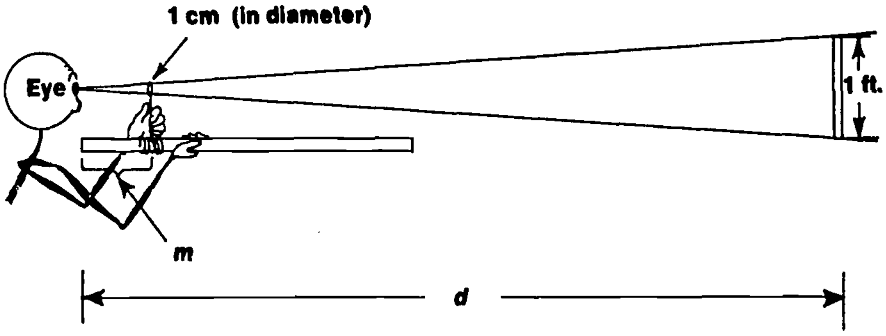
TWO INDIRECT-MEASUREMENT INSTRUMENTS

The two distance-measuring devices described here are selected because of their simplicity. They are inexpensive and very easy to make. Both are easy to use, and children can get surprisingly accurate results with both. By using these devices children learn about similar triangles and ratio.

Activity 11. Making and Using a Stadia Device

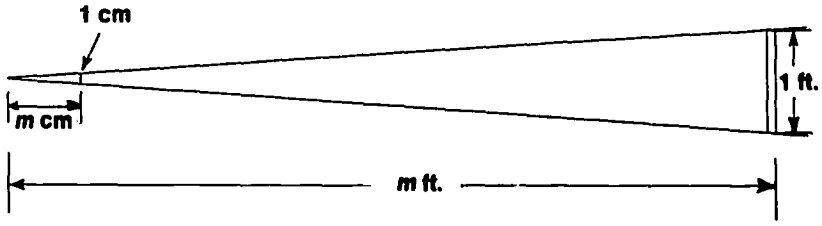
The materials needed are a meterstick, a piece of no. 27 copper wire about 8 inches long, and a 1-foot rule.

To make the instrument form a loop in the wire and twist to make a ring 1 centimeter in diameter, then wrap the ends around the meterstick to hold the loop in the upright position shown in the diagram below, which illustrates its use.



To use the instrument, follow these steps: Place the 1-foot rule in a vertical position. Sight along the meterstick through the ring. Slide the ring until the ends of the rule appear to be just inside the ring. Read the distance *m*, on the meterstick, in centimeters. The distance *d* to the rule is *m* feet.

To see why this is true, examine the diagram of the two triangles formed by the lines of sight.

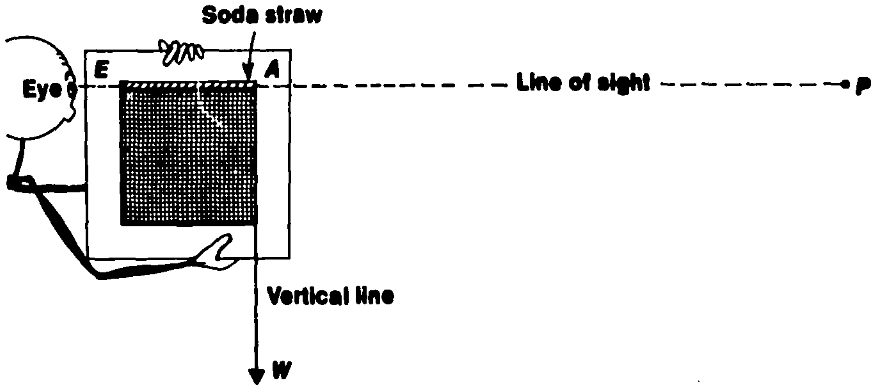


The small triangle is similar to the large one. The ratio we are interested in is $\frac{1 \text{ cm}}{m \text{ cm}}$ in the small triangle and $\frac{1 \text{ ft.}}{m \text{ ft.}}$ in the large triangle.

Activity 12. Making and Using a Hypsometer

The hypsometer is an ingenious arrangement of a plumb bob, a sighting device, and squared paper to get similar triangles and thus measure heights indirectly. Materials needed are a sheet of squared paper, stiff cardboard, a soda straw, string, a fishline sinker (or something similar), and masking tape or glue.

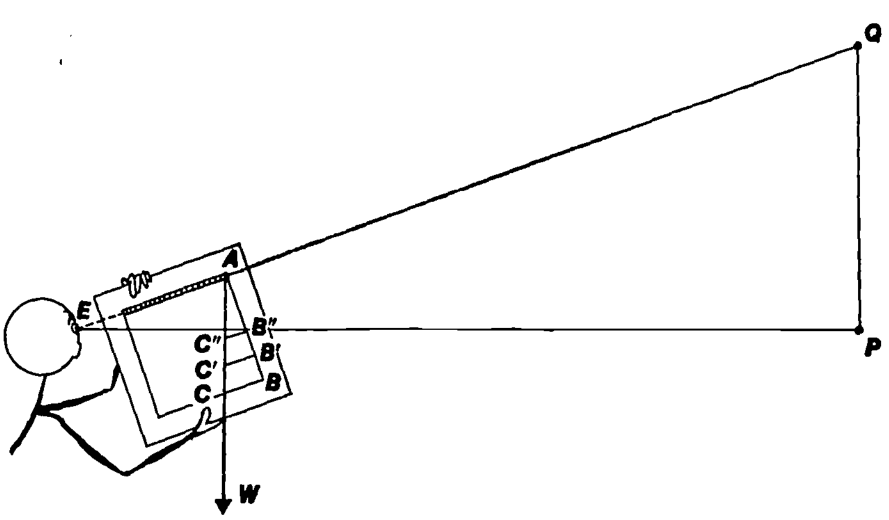
To make the instrument mount the squared paper on the cardboard and fasten the straw along the top of the paper, as shown in the sketch. Attach the weight on the piece of string from A, the corner of the ruled lines.



The first use that is suggested is to determine a level line. The string AW is vertical because of gravity. If the hypsometer is held so that AW falls along a ruled line on the paper, the soda straw will be level. The line of sight through the soda straw will be a level line, and you can locate a point P on an object level with your eye. Assume that this object is a building whose height you want to determine.

To find the height from P to the top of the building, sight through the straw to the top of the building. The string AW is vertical, so AC is perpendicular to *EP*. The two right triangles *ABC* and *EPQ* are similar, and the correspondence is

$$A \leftrightarrow E \qquad B \leftrightarrow P \qquad C \leftrightarrow Q.$$



Measure the horizontal distance *EP*. Choose a suitable scale and locate *B''* so that *AB''* represents *EP*. The line segment *B''C''* represents the desired distance *PQ*. Simply use the scale and read the distance directly from the squared paper.

Activity 13. Indirect Measurement

1. Estimate the height of the ceiling in a room. Use the hypsometer to measure the height of the ceiling. Check by direct measurement.

2. Estimate the height of a flagpole or a tall building and then measure its height by using the hypsometer.

3. Find out, by observing shadows and measuring their length, about what time of day your shadow is just as long as you are tall. Use this information to find the height of a building. Find the time when your shadow is twice your height.

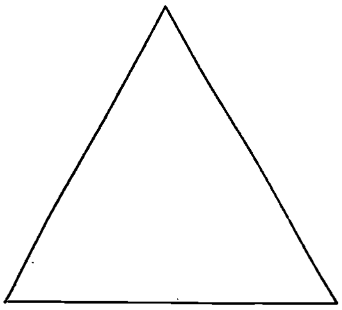
UNITS OF AREA

What is meant by a unit of area? In simple terms, it is any identifiable plane region whose area we agree to call "1." As with length, we are free to choose any region we wish for our unit region. Some units of area that people use are an acre, a square inch, a square mile, a hectare, a square kilometer.

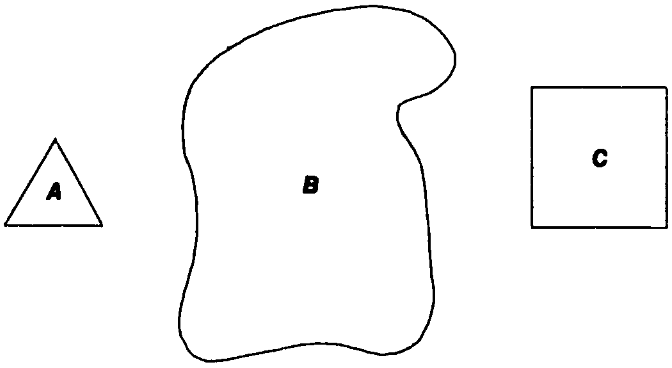
There is no need for the unit region to have a square shape. In fact, an acre was originally described as the land inside a rectangle 40 rods long and 4 rods wide. The space inside a circle of any given radius could be our unit of area; so could the space inside an equilateral triangle of specified length of sides.

Activity 14. Area Measurement

1. Consider the triangular shape pictured at the right as the unit region, and show $\frac{1}{4}$ of the unit. Show $\frac{1}{16}$ of the unit.



2. Consider *A* as the unit region, and estimate the area of *B*.

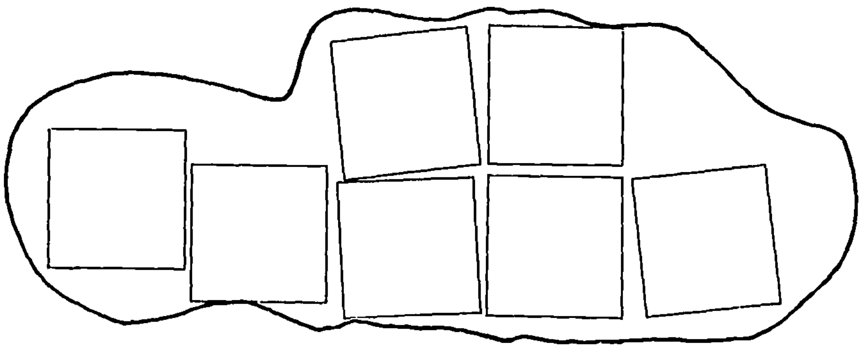


3. Consider *C* as the unit region, and estimate the area of *B*.

DIRECT MEASUREMENT OF AREA

Children should have experience in comparing regions, ordering according to area before attempting to measure area.

Direct measurement of area of a region by covering with unit regions gives children considerable insight into the nature of area as well as an insight into the concept of a limit. Given a certain region, a child is asked to estimate the area in square inches. A natural approach would be for him to find how many square pieces, one inch on a side, can be put inside the boundary without any overlapping. He should have a supply of these unit squares and fit them in—then count to find how many. The result might look like this:

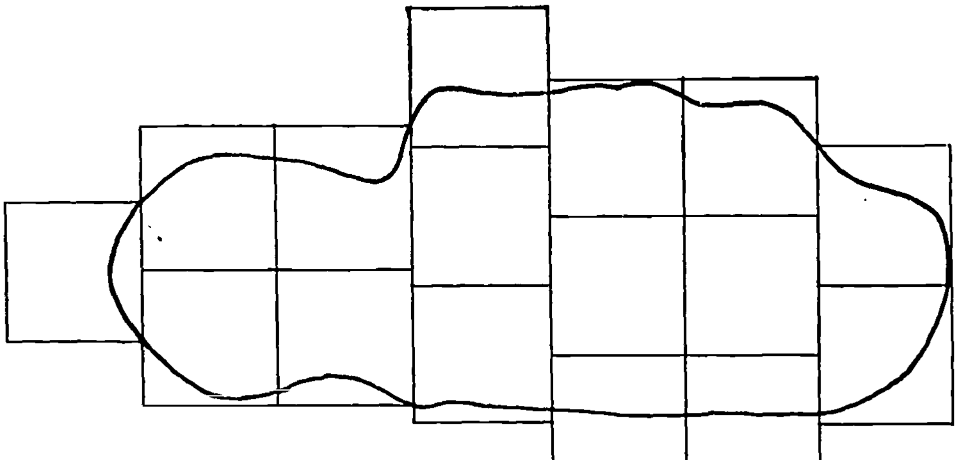


(Not shown actual size)

He knows his result is too small—that is, he knows 7 is less than the area—because there is some space not covered. Thus the question "What is the area?" has only a partial answer, and the child is not usually satisfied with the result.

How can he improve the result? One way is to cut some unit squares into fractional pieces and fit the pieces inside the region. He might work with eighths and find $\frac{73}{8} < A$. This is a little more satisfying; he is closer, but he doesn't know how much he is still "off." Although this approach is informative, a slightly different approach can give more information.

Suppose he covered the region completely, with some of the square pieces extending outside the boundary. His result might look something like this:



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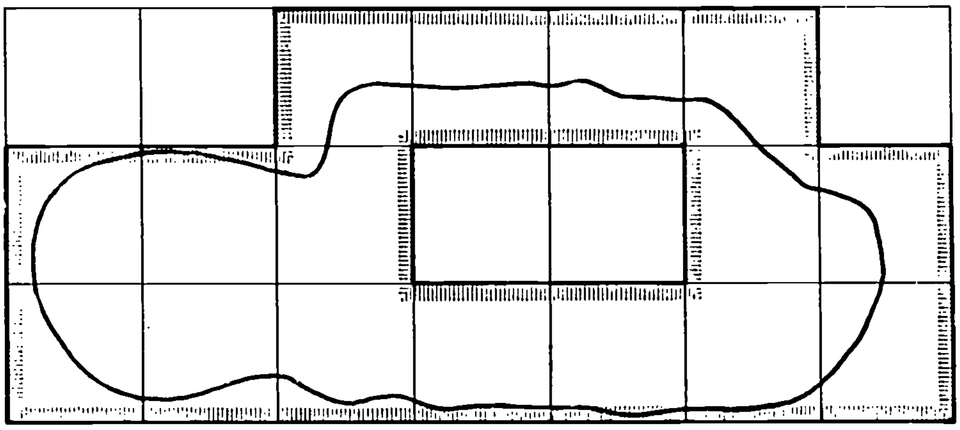
2.70

He knows that the area is less than 16 square inches. Notice that now the area is known to be between two numbers, 7 and 16. This result is considerably more satisfying because the child knows something about the uncertainty: He knows that the area measure is some number in the interval between 7 and 16.

Children will want to improve the precision by fitting in half-units and quarter-units. They should be encouraged and allowed to try to narrow down the interval by fitting in smaller pieces. Eventually they will want to use a more systematic procedure because the physical handling of the many small pieces is difficult.

One procedure that simplifies the work is to have the children trace the boundary of the region on graph paper. Another procedure is to place a transparent grid over the region. With either procedure the result might look like this for the one-inch paper. Children will get different results, depending on how the grid is located with respect to the boundary.



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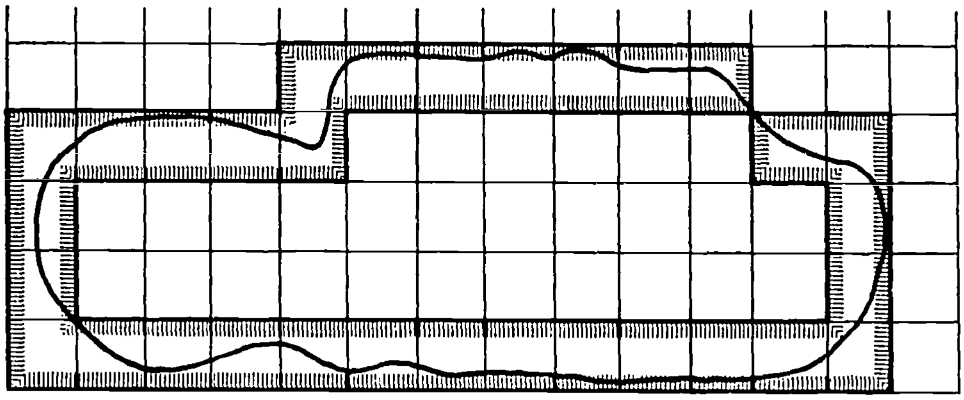
Here the number of squares entirely inside the boundary is 2. The number of squares that are inside or partly inside and partly outside the boundary is 18. In other words, the outlined inner area is 2 square inches, and the outer area (also outlined) is 18 square inches.

This result can be stated as follows:

$$2 \text{ sq. in.} < \text{Area} < 18 \text{ sq. in.}$$

The next step is to use smaller squares. Here is the result for $\frac{1}{2}$-inch squares. Each square has an area of $\frac{1}{4}$ square inch.



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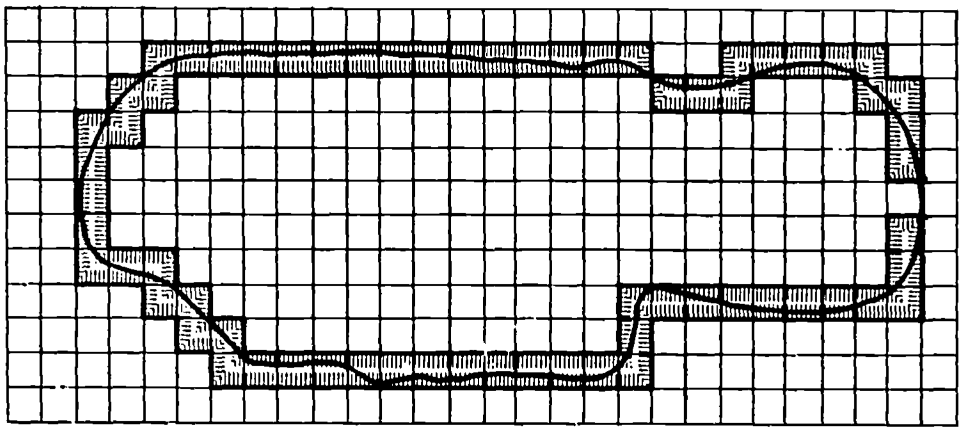
In this case, the number of squares entirely inside the boundary (arrived at by counting) is 28, and the number of squares that are inside or partly inside and partly outside the boundary is 59. So the inner area is $\frac{28}{4}$ square inches, and the outer area is $\frac{59}{4}$ square inches. This result can be stated as follows:

$$\frac{28}{4}\text{ sq. in.} < \text{Area} < \frac{59}{4}\text{ sq. in.}$$

In the next refinement, $\frac{1}{4}$-inch squares are counted. Each of these squares has an area of $\frac{1}{16}$ square inch. The result is shown below with the inner and outer areas outlined.



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In this case the number of squares completely inside the boundary of the region is 153. The number of squares inside or partly inside and partly outside the boundary is 222. Thus the inner area is $\frac{153}{16}$ square inches, and the outer area is $\frac{222}{16}$ square inches.

This result can be stated as follows:

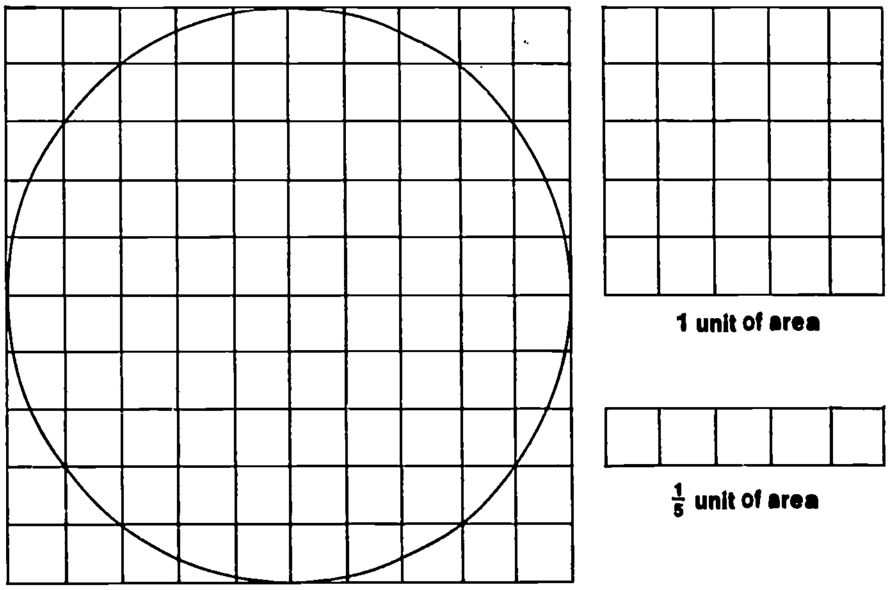
$$\frac{153}{16} \text{ sq. in.} < \text{Area} < \frac{222}{16} \text{ sq. in.},$$

or

$$9\frac{9}{16} \text{ sq. in.} < \text{Area} < 13\frac{14}{16} \text{ sq. in.}$$

Successive refinements could be made, each time reducing the size of the interval. But no matter how carefully the work is done, there will always be an interval in which the measure lies. We can never hope to find the "true" measure.

Activity 15. Measuring Area by Weighing



Use a compass and on graph paper draw a circle like the one shown above. Now mount it on cardboard and cut it out. Mount other graph paper on cardboard of the same weight and cut out several square pieces of paper with the side of the square just as long as the radius of your circle. The weight of one of these square pieces is to be your unit of weight. Also cut out some fractional units; you will want some $\frac{1}{5}$ units and possibly some smaller ones.

Weigh the disk with these squares and fractional parts of squares. You will need a pan balance that is sensitive enough for your needs.

How much does your disk weigh? Is there anything special about this result?

Are you really measuring area? What assumption are you making?

Miscellaneous Problems

1. Suppose a yardstick is defective in that its length is really $\frac{1}{2}$ inch too short. By this yardstick, a room is found to be 43 $\frac{1}{2}$ feet long. What is your estimate of the length of the room?

2. A stopwatch is used to record the time of an event at 15 minutes 23.7 seconds. If the watch is known to lose time at the rate of 4 minutes a day, how will this error affect the time for the event? What should the time be?

3. Use a scale model of a car, a boat, an airplane, or any other object. Find the overall length of the real object by making a measurement on the model and adjusting for the scale.

4. If a globe that is a model of the earth is 1 foot in diameter, what (approximately) is the scale? With this same scale, what size is a model of the moon?

5. Make two balls from modeling clay, one with diameter twice the other. Estimate the ratio of their weights. Check by weighing.

Robert B. Davis

NEGATIVE RATIONALS

$^-10$

1. **Having started with the numbers 0, 1, 2, . . . and found some more numbers such as $\frac{1}{2}, \frac{3}{7}, \ldots$ (which are also quite important), can we carry such explorations further? Are there still more numbers that we can easily find? Would they be useful?**
2. **How can we help children to explore these new numbers informally and intuitively?**
3. **How do we compute the sum $^-2 + {}^+3$?**

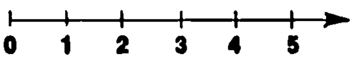
In the present chapter we shall explore a new mathematical terrain, the world of negative numbers.

Before we begin our explorations it may be well to get our bearings. What new numbers are we to explore? How is this exploration to be carried out?

We began our story in chapter 1 knowing the whole numbers,

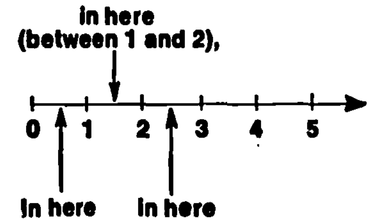
$$0, 1, 2, 3, \ldots.$$

We know that these whole numbers can be pictured on a number line, as shown:



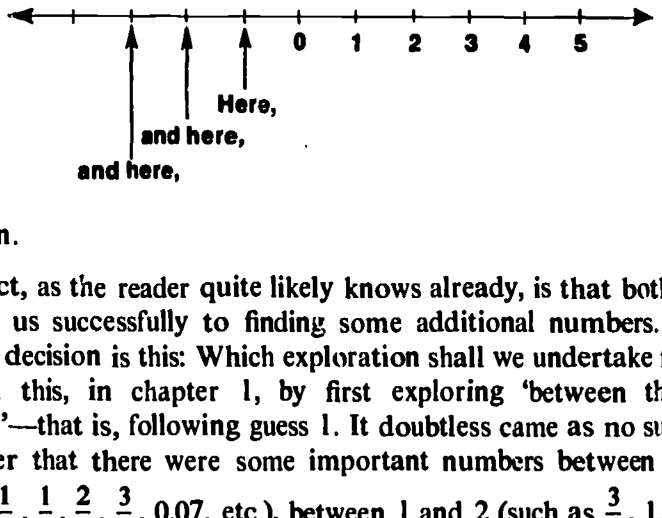
Just looking at this geometrical representation of the whole numbers suggests that, quite likely, there ought to be some other useful numbers if we go about finding them correctly. There seem to be two obvious guesses as to where to look.

Guess 1. Let's look between the whole numbers:



and so on.

Guess 2. Since the whole numbers lie to the right of zero,[1] let's ask if the line might not show some symmetry and offer us numbers to the left of zero that are, in effect, the "reflections" of the whole numbers, as seen in a mirror held at zero. This would mean looking for numbers as shown below.



and so on.

The fact, as the reader quite likely knows already, is that both guesses will lead us successfully to finding some additional numbers. Perhaps the main decision is this: Which exploration shall we undertake first? We answered this, in chapter 1, by first exploring 'between the whole numbers"—that is, following guess 1. It doubtless came as no surprise to the reader that there were some important numbers between 0 and 1 (such as $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{5}$, 0.07, etc.), between 1 and 2 (such as $\frac{3}{2}$, 1.37, etc.), and indeed between *any* two whole numbers.

By looking first for the nonnegative rational numbers, we followed the sequence most commonly used in elementary schools: that is to say, we considered numbers like $\frac{1}{2}$ and $3\frac{7}{8}$ *before* we considered numbers like

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[1] The statement that "the whole numbers lie to the right of zero" is an informal way of saying that "the points corresponding to the whole numbers lie to the right of the point corresponding to zero." It often saves words if we refer to "numbers" on the number line, although of course we always really mean "the points corresponding to" these numbers, since (when we are being very precise about our language), a "line" consists not of "numbers" but rather of "points."

$^{-}3$ or $^{-}2\frac{1}{2}$. One should not, however, consider this order to be the only possible one. From the point of view of the professional mathematician, either exploration can be undertaken first. It's entirely a matter of choice.

We stress this fact because there is reason to believe that the new problems we shall meet in chapters 10 and 11, such as

$$^{+}5 + {}^{-}3 \qquad \text{and} \qquad {}^{+}2 \times {}^{-}7,$$

are actually *easier* for children than problems such as

$$\frac{1}{2} \div \frac{3}{5}$$

—even though the latter problem usually appears earlier in the school curriculum. We are likely to witness the emergence soon of new curricula and new textbooks that attempt to follow more closely this "path of gradually increasing complexity" by allowing an earlier introduction of negative numbers and, possibly, deferring some work with fractions until later. Such a change, although a departure from tradition, would be consistent with our increasing knowledge of the intellectual growth of children and would not violate any principles of sound mathematics.

But the question of which area to explore first is not the only one that needs to be answered. In this chapter we shall be guided by two other principles: (1) *Mathematical activity involves sensible responses to sensible problems*; we want to present mathematics to children so that this fact will emerge clearly. (2) In most cases, *informal intuitive ideas develop first, and their more precise and abstract formalization comes later.*

THE GAME "GUESS MY RULE"

We are seeking some interesting mathematics that children can enjoy doing and that will set the stage for the natural emergence of negative numbers. One excellent choice is the game "Guess My Rule," which we now explain here.[2]

The point of the game is that someone, whom we shall call "the leader," has made up a "rule." He may, for example, have decided that whatever number we tell him, he will double it and tell us the answer. The other players' task is to guess what the leader's rule is. They tell him whatever number they like—say, perhaps, "2." He applies his rule to 2, and tells

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[2] The use of this mathematical game with elementary school children was suggested by Professor W. Warwick Sawyer.

them that the answer is "4." They can record their progress thus far by means of a table:

The number the other players tell the leader	2
The number the leader gives in response	4

If they tell him "10," then (using his rule) he responds "20"; if they tell him "7," he responds "14." This extends the table to look like this:

The number the other players tell the leader	2	10	7
The number the leader gives in response	4	20	14

Probably by now the other players can guess that what the leader is doing is taking their number and doubling it. They have "guessed his rule." Now it will be someone else's turn to make up a rule, and the rest of the class will try to guess this new rule.

Exercise Set 1

1. John made up a rule, which the class is trying to guess. When the class said "5," John answered "7." When the class said "12," John answered "14." When the class said "100," John answered "102." Here is a table of the game thus far:

The class asked John to use his rule on this number	5	12	100
John responded by answering	7	14	102

Can you guess John's rule?

2. In one classroom there is a wise old owl. When the children whisper "15" to the owl, he responds by hooting "10." When the children whisper

"50," the owl hoots "45." When the children whisper "5," the owl hoots "0." Here is a table of the results thus far:

The numbers we whisper to the owl	15	50	5
The numbers the owl hoots back	10	45	0

Suppose you whispered "7" to the owl; what number would he hoot in response? Can you add to the preceding table to show this? Do you know the rule that the owl is using?

3. Children playing Guess My Rule have recorded this table on the chalkboard:

We tell Billy to use his rule on this number	12	25	100	0
This is what Billy answers	19	32	107	7

Who made up the rule? What was the first number that the children told Billy? What did Billy reply? What was Billy's response when the children said "100"? If the children told Billy "40," what would Billy say in response? Do you know Billy's rule?

(Obviously, the game Guess My Rule can be used for many different purposes in the classroom; it is not at all limited to negative numbers. We present it here because it provides an excellent way to get children interested in number patterns, and such patterns, in turn, can naturally lead to negative numbers.[3])

A FIRST EXPERIENCE WITH NEGATIVE NUMBERS

The Elementary Mathematics for Teachers and Students film T10, *Negative Rationals*, begins with a classroom scene of children playing

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[3] The "Guess My Rule" game is illustrated by an actual classroom lesson shown on a 16-mm sound motion picture film entitled *Guessing Functions*, available from the Madison Project, 918 Irving Ave., Syracuse, N.Y. 13210. The game is also discussed at some length in chapter 25 of *Explorations in Mathematics, Teacher's Edition*, by Robert B. Davis (Reading, Mass.: Addison-Wesley Publishing Co., 1966).

Guess My Rule.[4] (In fact, the teacher uses a "wise old owl" as a prop, as in exercise **2** of set 1, above.) The children tell the owl "7," the owl uses his rule on 7, and answers "4." Then the children tell the owl "5," and using his rule on 5, the owl answers "2."

Can we guess the owl's rule? You might want to try a few more numbers to be sure, but in fact the owl's rule is "Whatever number you tell me, I'll subtract 3 from it, and tell you the answer."

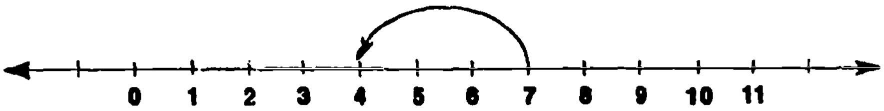
Thus, if we were to tell the owl "15," he would answer "12," and if we told him "11," he would answer "8."

We can show this by a table:

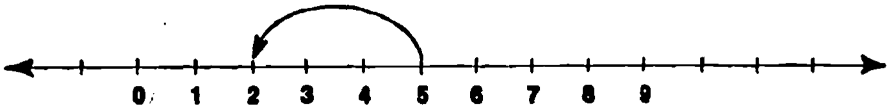
We tell him this	7	5	15	11
He answers this	4	2	12	8

However, for our present purposes it will be even more useful to show Owl's rule on a number line:

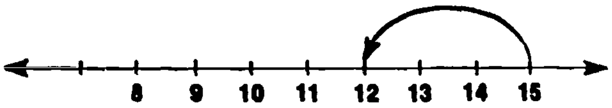
If we tell him "7," he answers "4":



If we tell him "5," he answers "2":



If we tell him "15," he answers "12":



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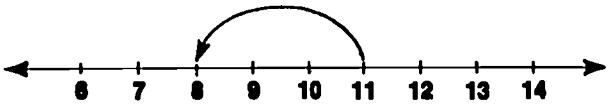
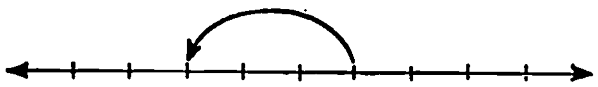
[4] The film series referred to was developed by the National Council of Teachers of Mathematics in cooperation with General Learning Corp. and is available from Silver Burdett Co., 250 James St., Morristown, N.J. 07960.

And, finally, if we tell him "11," he answers "8":

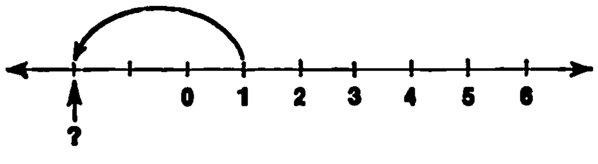


The advantage of the number line becomes apparent when we suppose we tell the owl "1." What will he answer?

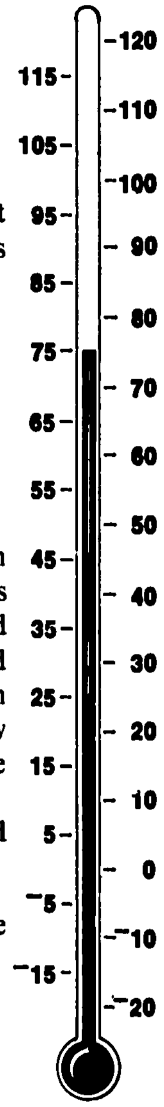
Assuming that the pattern of "subtracting 3" or "jumping three giant steps to the left" still holds,



then when we start at 1 we end our jump by coming down at the point marked "?" in the illustration. But what *number* corresponds to this point?



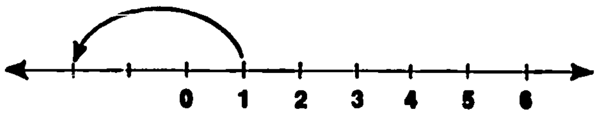
(The question might well be asked: What experiences may the children have had that can help them decide which number corresponds to this point? One answer that is likely to hold true is that they will have had experience with an outdoor thermometer, which shows both positive and negative numbers. It is often helpful to use a diagram or picture of an outdoor thermometer in the classroom. What other places do you know where children might encounter a "number line" that includes negative numbers?)



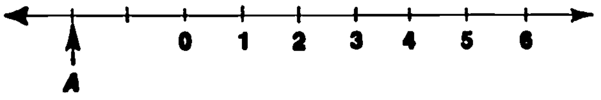
Let's summarize where we stand now in the Guess My Rule game played with Owl:

1. We have recognized the common pattern of "take three steps to the left on the number line":

2. We have asked Owl to use this same rule, starting at the point labeled "1":



3. Consequently, we know the *point* on the number line where our jump has ended; it is the point that we now label "*A*" for convenience:



4. But, of course, we suspect (or at least we hope) that point *A* may also have a *number* name, *and it is this number name that we'd like to find.*

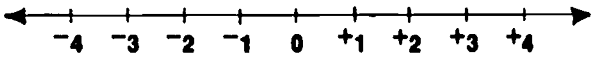
5. Technically, mathematics is nowadays commonly thought of as a rather elaborate structure that man has created. Thus far, in this book, we have not created any number names for points to the left of zero. Thus, technically, there *is no number name yet created* that applies to point *A*.

MATHEMATICS AND LEGISLATION

The creation of formal mathematics is in many ways similar to the creation of formal legislation in legal work. Our ancestors were unaware of most modern dangers of environmental pollution and hence made no laws to deal with them.

But we try to draw our formal laws from underlying notions of social needs, social justice, and so on. As we *recognize* environmental problems, we try to pass legislation to deal with them. Hence pollution legislation does not start in legislative chambers but rather with biological, chemical, and economic knowledge about our environment. From this knowledge, we move on to create legislation.

What is the parallel for our present mathematical problem? Well, we know that we have not yet created an "official" number name for point *A*. (*We have not yet passed appropriate legislation.*) But we do have some commonsense knowledge that suggests a possibility (*just as we can have scientific knowledge that indicates an ecological need*). This commonsense knowledge comes, for example, from our experience with thermometers (or perhaps, if we're lucky, from ammeters, etc.). What does the commonsense knowledge suggest? Specifically, a number line with *positive* *numbers* to the right of zero, and *negative numbers* to the left of zero. Using a common modern notation, we would label points like this:



Remark on status. We have *not* yet passed our formal legislation, so the picture above is a plausible guess, a reasonable conjecture—but perhaps *not yet a fact.*

Remark on notation. Some years ago the symbol "$-$" was commonly used for three different ideas:

1. The idea of subtraction, as in $8 - 2 = 6$.

2. The idea of additive inverse, as in $-x$ to denote a number that, when added to x, produced the answer 0, as in $x + (-x) = 0$.

3. As part of the symbol for a negative number, as in -5, which might (for example) mean "5 degrees below zero."

Not only was the same written symbol used for these three different ideas, but it was often read as "minus" in all three cases. This led to considerable confusion.

Consequently, an increasing number of modern books use three different symbols for these three different ideas:

1. For subtraction the traditional symbol is used, and it is read "minus" (or perhaps, with younger children, "take away").

Written	*Read*
$10 - 3 = 7.$	"Ten minus three equals seven."

2. For additive inverse a small circle (or letter o in roman type) is used, written as a raised prefix.

Written	*Read*
$^{\circ}(5)$	"The additive inverse of five" or, sometimes, "The opposite of five."

Hence we have

$$5 + {}^{\circ}(5) = 0,$$

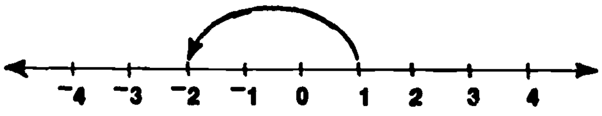
which we would read "Five plus the additive inverse of five equals zero."

3. For the names of negative numbers we use a symbol that is like

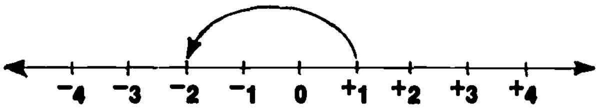
the "minus" sign but is a raised prefix. Sometimes, not always, it is shorter than the "minus" sign.

Written	*Read*
$^-8$	"Negative eight"
$^-1$	"Negative one"

We can now summarize our guess as to a suitable name for point *A* in the owl's game: If we tell Owl to start at "one," his jump will end at "negative two."



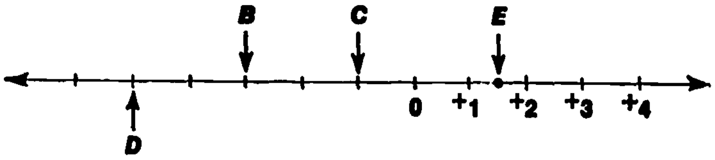
(Sometimes, largely for pedagogical emphasis, our positive numbers are also written with a raised prefix, as was done in an earlier number-line representation, and are read "positive one," etc. If we follow this convention, as below, we would say: "When we tell Owl to start his jump at positive one, he ends at negative two.")



Exercise Set 2

Remembering that we have not yet made these ideas "official," we can nonetheless try to recognize what seem to be sensible patterns, and we can use them as the basis for trying to answer these questions:

1. What number name would you assign to points *B*, *C*, *D*, and *E*, shown below?



2. Suppose you start at the point labeled "$^+2$" and take a jump of 5 units to the left. Where will you land?

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3. Suppose you start at the point for $^{+}2\frac{1}{2}$ and then take a jump of 5 units to the left. Where will you land?

4. Suppose that, at 8:00 P.M. Friday, the temperature is 3° Fahrenheit. During the night the temperature drops 10 degrees, reaching its coldest point at 5 A.M. Saturday morning. What was the temperature at 5 A.M. Saturday morning?

5. A man had a checking account that gave him automatic credit. Suppose he had a balance of $25. Then he wrote a check for $100. What was his new balance?

A CONCRETE MODEL FOR NEGATIVE INTEGERS

Experience appears to indicate that the most effective way to introduce young children to negative numbers is to use concrete examples.

First, let's be clear in our own minds as to why we need numbers beyond the original "natural" numbers. We need positive rational numbers such as $\frac{1}{2}$ or $\frac{2}{5}$ or 3.71 for processes of sharing things ("one-half for each of us"), for use in measurement problems (Next Exit $\frac{1}{4}$ Mile), and to allow us to solve various abstract mathematical problems such as finding a nonempty solution set for the equation $2 \times \square = 3$ or extending the natural numbers so that the new system will be closed under division by a nonzero number.

Why do we need negative numbers? In a practical situation, we need them when we must start counting or measuring from an arbitrary starting point. When does this happen? Primarily when there is no absolute starting point available—or none that is convenient. Thus, temperature is ordinarily measured from an arbitrary "zero" (on either the Fahrenheit or centigrade scale) and may be "above zero" (positive) or "below zero" (negative). An absolute starting point ("absolute zero") is available, but most everyday scales were firmly established before this absolute zero was known.

We cannot measure years from the year of the creation of the earth because we do not know when that occurred. Therefore we measure the year from the birth of Christ, with the result that the date can be either "A.D." (positive) or "B.C." (negative)—or from some other designated starting point.

We also use relative change from an arbitrary reference point in measuring altitude ("300 ft. above sea level" vs. "100 feet below sea level"), in computing common economic indices ("the Dow-Jones Industrial Average fell 9 points"), and in any other case where there is no convenient, absolute beginning point.

Thus, in using the number line in the Guess My Rule game played with Owl, the filmed lesson *Negative Rationals* gave the children a natural (and rather typical) reason for turning to negative numbers: If we want number names for points on a line, we cannot "start with the first point" (for there *is* no "first" point on a line), and when we have one point, we cannot "count the *next* point" (for there is no "next" point on a line).

Can we offer children a model of positive and negative numbers that will be even more concrete than the number line? The answer is yes. Extensive trials show a model called "Pebbles in the Bag" will work well, even with young children. It covers positive and negative integers and zero. Here is how it goes:

We have a bag, partly filled with pebbles. We also have a pile of extra pebbles on a table. Thus we can put pebbles into the bag, or we can remove pebbles from the bag. This will allow us to manipulate the pebbles so as to create concrete situations that will correspond to problems such as those shown below:

$$3 + 4 = ? \qquad 3 - 2 = ? \qquad 5 - 8 = ?$$

Obviously, if we ask, in the usual sense, how many pebbles are in the bag, we will get an ordinary "counting" answer, such as "Fifty-seven." We will never encounter negative numbers. Therefore, this is *not* the question we shall use. We assume there are enough pebbles in the bag so that we don't want to go back and count "from the beginning" (just as we don't usually measure altitude "from the center of the earth"). Instead, *we establish a reference point*. Remembering that we are dealing with children, we establish the reference point dramatically: We have some child (Anne, say) clap her hands loudly and say "Go!"

Now we have our reference point. We *now*, for example, put three pebbles into the bag and remove five, to dramatize

$$3 - 5,$$

and we then ask: "Are there more pebbles in the bag than there were when Anne said 'Go,' or are there fewer?" If children are ready for this topic, they will be able to say, with confidence, that there are fewer. Our next question is: "How many fewer?" The children should be able to answer "*Two fewer*," and we write this as shown:

$$3 - 5 = {}^{-}2$$

(read as "Three minus five is negative two" or "Three minus five equals negative two").[5] Notice that the "negative two" does not mean that the bag is "more empty than empty." It means: "There are now two pebbles fewer in the bag than there were *when Anne said 'Go!'* "

Similarly,

$$5 - 1 = {}^{+}4$$

would mean that after someone (Toby, say) said "Go!" we put five pebbles into the bag, then we removed one pebble from the bag, and now there are four more pebbles in the bag than there were *when Toby said "Go!"* We would read

$$5 - 1 = {}^{+}4$$

as "Five minus one equals positive four."

Exercise Set 3

Make up a Pebbles in the Bag story to match each of these problems:

1. $8 - 2$

2. $3 - 4$

3. $5 - 5$

THE ADDITION OF NEGATIVE RATIONALS

Operating "intuitively" or "informally," we have learned to add or subtract nonnegative integers, as in

$$x + y = ? \qquad \text{and} \qquad x - y = ?$$

and to express the answer as an integer that will be nonnegative in the case of $x + y$ and will also be nonnegative in the case of $x - y$ if it happens

---

[5] An actual classroom lesson, with second-grade children working with this "Pebbles in the Bag" model, can be seen in the film *A Lesson with Second Graders,* available from the Madison Project. The game is discussed further in chapter 4 of *Explorations in Mathematics.*

that x is larger than y or equal to y. For $x - y$, the answer will be a negative integer if y is larger than x.

Examples:

$$3 + 4 = {}^+7, \qquad 8 - 2 = {}^+6, \qquad 5 - 5 = 0, \qquad 3 - 9 = {}^-6$$

(the fourth example being the case of $x - y$ where y is larger than x, since $3 < 9$). This leaves a glaring omission. We can cope nicely with

$$x + y = z \qquad \text{and} \qquad x - y = z,$$

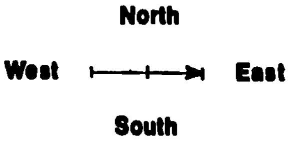
provided that x and y are nonnegative. Whether z is negative or not raises no difficulties.

But—*suppose x or y were allowed to be negative?*

To put it another way: We have introduced negative integers (at least intuitively), but we don't yet know how to add negative integers, or to subtract them, or to multiply them, or to divide them.

We now develop an informal method for adding negative integers by considering the important mathematical notion of a *vector*, as follows:

Imagine thirty children standing in a gymnasium, all facing the east end of the room. Suppose you ask them to "take two giant steps forward." We could represent this movement by an arrow two units long, pointing eastward:



We shall call this arrow a "vector." Unlike coordinates, vectors are not being used here to indicate *position*, but rather *motion*. Where each child ends depends on where he started. It will be different for different children. But each child took two giant steps to the east, so (assuming, for simplicity, that their steps are all the same length) each child carried out the same movement. Suppose we now ask the children to "take three (more) giant steps to the east." Assuming that no one has been so unfortunate as to collide with the gymnasium wall, each child will now have moved a total of five giant steps to the east:

We can use these "arrow" pictures of vectors even more easily if we draw them on a number line:

The question now is: Can we relate these arrows (or vectors) to integers?

Exercise Set 4

1. What integer would you match up with the vector shown below?

2. What integer would you match up with this vector?

3. What vector would you match up with the integer $^+2$?

4. What vector would you match up with the integer $^+6$?

5. Use vectors to compute the sum $^+2 + {}^+6$.

6. But, after all, we probably suspect that $^+2$ is not really different from our old friend 2, and $^+6$ is not really different from our old friend 6. Trust this suspicion, for the moment, and see if it suggests a way to compute the sum $^+2 + {}^+6$.

7. Did you get the same answer in exercise **6** that you got in exercise **5**?

8. Remember that the children are facing the east wall of the gymnasium. Suppose we now ask them to "take four giant steps backward." We can represent this by the following vector:

What integer would you associate with this vector?

9. What vector would you associate with the integer $^-7$?

10. What can you say about the vectors that correspond to negative integers?

11. What can you say about the vectors that correspond to positive integers?

12. Use arrow pictures of vectors to compute the following:

- **a.** $^{+}2 + {}^{+}7$
- **b.** $^{-}3 + {}^{-}2$
- **c.** $^{+}5 + {}^{-}3$
- **d.** $^{+}4 + {}^{-}7$
- **e.** $^{-}2 + {}^{+}5$

13. We spoke earlier of "additive inverses." We said "the additive inverse of $^{+}5$" would mean "the number we can add to $^{+}5$ to get a sum of 0." That is to say, the additive inverse of $^{+}5$ (written as $^{o}(^{+}5)$) is the unique element in the solution set of the open sentence

$$^{+}5 + \square = 0.$$

What is a simpler name for $^{o}(^{+}5)$?

14. We have seen (in chap. 5) that every positive rational number has a multiplicative inverse. Do you suppose that every integer has an additive inverse?

NEGATIVE RATIONALS

The idea of thinking of positive and negative integers as *vectors* along the number line turns out to be very useful. For one thing, we can now easily extend our system from integers, such as

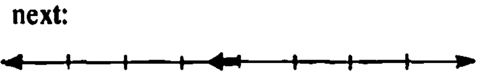
$$^{+}3 \quad \text{or} \quad ^{-}2 \quad \text{or} \quad ^{+}21,$$

to *rational numbers*, such as

$$^{+}3.5 \quad \text{or} \quad \frac{^{-}1}{2} \quad \text{or} \quad ^{+}1\frac{7}{8} \quad \text{or} \quad ^{-}2\frac{1}{3}.$$

We need only think of $^{+}3.5$ as "three and a half giant-steps forward," that is, as the vector

and to think of $\frac{^{-}1}{2}$ as "one-half a giant step backwards," that is to say, as the vector shown next:



Exercise Set 5

1. Compute the following sums:

a. ${}^{+}2\frac{1}{2} + {}^{-}3\frac{1}{2}$

b. ${}^{-}3\frac{1}{2} + {}^{-}3\frac{1}{2}$

c. ${}^{+}5 + {}^{-}10$

d. ${}^{+}5 + {}^{-}5\frac{1}{2}$

e. $\frac{^{-}1}{2} + {}^{+}1$

f. $\frac{^{-}1}{2} + {}^{-}2\frac{1}{2}$

g. $\frac{^{-}1}{2} + {}^{+}9$

We are now face to face with the entire set of rational numbers, examples of which are

$${}^{+}3, \quad {}^{-}5, \quad {}^{+}2.5, \quad {}^{-}1\frac{1}{2}, \quad 0, \quad \text{and} \quad \frac{^{+}11}{13}.$$

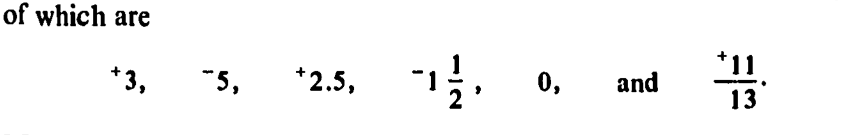
Moreover, we know—at least informally—how to add any two rational numbers, whether they are positive, negative, or zero.

Exercise Set 6

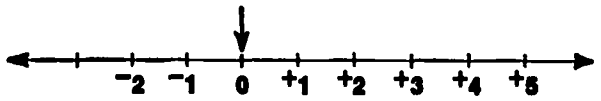
1. Do you think every rational number (whether positive, negative, or zero) has an additive inverse?

2. Can you give a reason to make your answer to exercise **1** seem plausible?

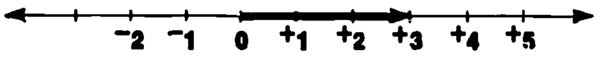
The Elementary Mathematics for Teachers and Students film T10, *Negative Rationals*, also uses a second way to let children have concrete experiences with the addition of integers. This second method can be used with a chalkboard drawing that has two parallel lines or with a manipulatable device made by marking the number lines on separate pieces of cardboard or wood. Here is a picture of the manipulatable version. It resembles a slide rule, and we shall call it that.



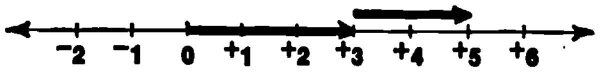
To relate this slide rule to our "vector" or "arrow" diagrams, one can consider that we start at zero,



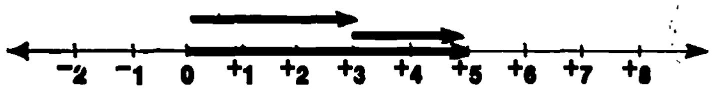
take three giant steps forward (an arrow 3 units long, pointing to the right),



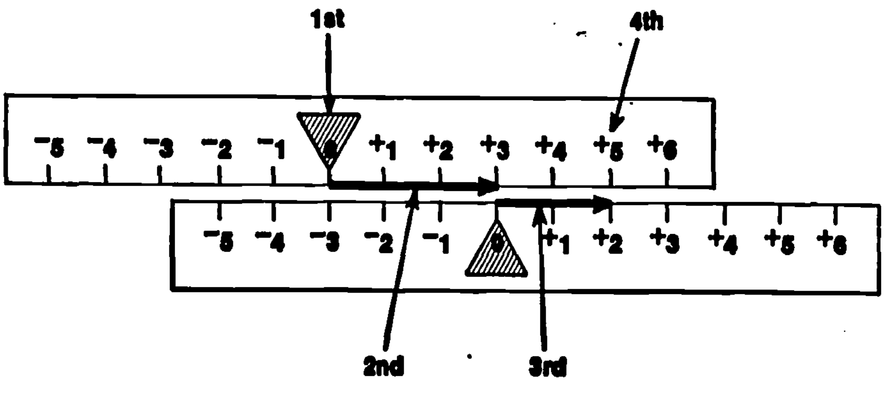
then take two giant steps forward,



which is equivalent (in terms of our final resting place) to taking five giant steps forward:



The slide rule, in the position that has just been pictured, allows us to visualize all of the arrows shown below and follow the steps they represent: first, start here; second, move 3 units to the right; third, then count over 2 *more* units to the right; fourth and finally, read the result, 5.



Slide-Rule Diagram

We have just used the slide rule to obtain

$$^+3 + {}^+2 = {}^+5.$$

Exercise Set 7

1. Slide the two pieces of "wood" shown in the slide-rule diagram into a new position to indicate the sum of $^{+}4 + {}^{+}2$.

2. Arrange the slide rule to correspond to $^{+}4 + {}^{+}3$.

3. Use the slide rule to show $^{+}4 + {}^{-}5$.

4. Compute each of the following sums:

a. $^{+}3 + {}^{-}4$

b. $^{+}3 + {}^{-}1$

c. $^{+}4 + {}^{-}4$

d. $^{-}3 + {}^{+}5$

e. $^{+}5 + {}^{-}3$

5. Mark off a slide rule so that you can easily deal with half-unit distances and thus with problems like $^{-}2\frac{1}{2} + {}^{-}2\frac{1}{2}$.

THE QUESTION OF "ORDER"

Properties such as $3 < 5$, the fact that $a < b$ and $b < c$ tell us automatically that $a < c$, and so on, are often referred to as the "order properties" of the natural numbers. It is reasonable to ask about the order properties of the set of all integers (positive, negative, and zero) or the set of all rational numbers (positive, negative, and zero).

An interest in this question arises naturally with children. They sooner or later come to a question such as "Which is larger, $^{-}100$ or $^{+}\frac{1}{2}$?"

This question has two "sensible" answers. The fact is that everyday lay usage does not coincide entirely with the language used by professional mathematicians. In the everyday world, we speak of "large indebtedness" or "large losses" in the sense that, say, losing a hundred dollars is ordinarily a more significant event than finding fifty cents. This everyday usage suggests that, at least in certain senses, $^{-}100$ is "larger" than $^{+}\frac{1}{2}$. This notion exists also within mathematics, but it is *not* the "order" notion that mathematicians most commonly use.

The clearest way to contrast the two notions is geometrically, by referring to the number line. When we say that $3 < 5$ or $\frac{1}{2} < 2$, we are (among other things) saying: "Three lies to the left of five on the number line or "One-half lies to the left of two on the number line." It is this meaning ($a < b$ says that a lies to the left of b on the number line) that is most fundamental for mathematicians. Consequently, we shall move into the new territory of negative numbers by extending this notion: For *all* rational numbers, whether positive or negative or zero,

$$a < b$$

shall be construed to mean that "a lies to the left of b on the number line."

Once we agree on this fundamental, we can easily answer the question: Which of the following statements is true?

$$^{-}100 < \frac{^{+}1}{2} \quad \text{or} \quad \frac{^{+}1}{2} < {}^{-}100$$

Exercise Set 8

1. Which of the following statements is true?

$$^{-}100 < \frac{^{+}1}{2} \quad \text{or} \quad \frac{^{+}1}{2} < {}^{-}100$$

2. Which statements are true, and which are false?

   a. $^{+}3 < {}^{+}5$

   b. $^{-}3 < {}^{-}5$

   c. $^{+}2 < {}^{+}2\frac{1}{2} < {}^{+}3$

   d. $^{-}3 < {}^{-}2\frac{1}{2} < {}^{-}2$

   e. $^{-}1{,}000{,}000 < {}^{+}0.003$

   f. Every negative number is less than zero.

   g. If a is a negative number and b is a positive number, then $a < b$.

   h. $\frac{1}{3} < \frac{1}{2}$

   i. $\frac{^{-}1}{3} < \frac{^{-}1}{2}$

3. Mark $^{-}2\frac{1}{2}$ on the number line. Do you see a possibility of students' making errors here?

This leads us to a discussion of the *geometrical meaning of "absolute value."* We said earlier that there were *two* possible meanings of "Which is larger, $^{-}100$ or $\frac{^{+}1}{2}$?" We have agreed that the fundamental meaning for mathematicians is "Which lies to the right on the number line, $^{-}100$ or $\frac{^{+}1}{2}$?" But mathematics *does* recognize also the other everyday meaning (which might be expressed as asking which is "more important"), and we can state that meaning geometrically on the number line by asking: "Which lies further away from 0 on the number line, $^{-}100$ or $\frac{^{+}1}{2}$?" This can be written in standard mathematical notation by using absolute value. The absolute value of $^{-}100$, which we write as

$$|^{-}100|,$$

is the distance between 0 and $^{-}100$ on the number line. Therefore

$$|^{-}100| = {}^{+}100$$

and

$$\left|\frac{^{+}1}{2}\right| = \frac{1}{2}.$$

Exercise Set 9

1. State whether each of the following is true or false.

a. $|^{-}5| = {}^{+}5.$

b. $|^{+}3| = {}^{+}3.$

c. $\left|\frac{^{+}1}{2}\right| = \frac{^{+}1}{2}.$

d. For any number x, if $x \neq 0$, then $0 < |x|$.

2. Which of the following is true?

$$|^{-}100| < \left|\frac{^{+}1}{2}\right| \quad \text{or} \quad \left|\frac{^{+}1}{2}\right| < |^{-}100|.$$

3. If $a < b$, which of the following is true?

$$|a| < |b| \quad \text{or} \quad |b| < |a|.$$

AN INFORMAL LOOK AT THE STRUCTURE OF THIS NEW NUMBER SYSTEM

We now have available to us all the whole numbers, all the nonnegative rationals, and—finally—all the negative rationals. For convenience, we shall call this new, enlarged number system "the set of rational numbers," sometimes denoted by Q (for *quotient* of integers).

> NOTE.—The words "rational numbers" have now changed their meaning. Prior to chapter 10, we did not *have* any numbers available to use except *positive numbers* and *zero*. Hence "integers" really meant "nonnegative integers", that is, 0, 1, 2, 3, 4, ... , and "rational numbers" really meant "nonnegative rational numbers." Now we *do* have negative numbers available to us. Hence, from now on, whenever we say "integers" we mean *all the integers* (positive, negative, and zero),
>
> $$\ldots {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots,$$
>
> and whenever we say "rational numbers" we shall mean *all the rational numbers*, whether positive, negative, or zero.

With all of these additional numbers available to us, it is reasonable to wonder how this new number system will be similar to those we have known in the past and how it will be different.

What equations can we solve? When we had only whole numbers 0, 1, 2, 3, ... we were unable to find a nonempty solution set for (say) the equation

$$2 \times \square = 5. \tag{1}$$

Clearly 2 is too small to be a root of this equation and 3 is too large, and there was nothing available in between. With the extension of our system to include the nonnegative rational numbers, we did have a solution for equation (1), namely, the number $2\frac{1}{2}$. Indeed, with the nonnegative rationals, we could solve any equation of the form

$$a \times \square = b,$$

provided merely that a was not zero.

But even with the nonnegative rationals, we could not solve the equation

$$\square + 5 = 3. \qquad (2)$$

Now, however, with negative numbers available to us, we can solve equation (2), since

$$^{-}2 + 5 = 3.$$

Indeed, we can now solve any equation of the form $\square + a = b$, where a and b are *any* rational numbers.

The Existence of Inverses

Closely related to the preceding situation with equations is the question of inverses. With only the nonnegative integers, we did not have "multiplicative inverses"—that is to say, we had no solutions for equations such as

$$2 \times \square = 1. \qquad (3)$$

When we augmented our number system by including the nonnegative rational numbers, then every number except zero had a unique multiplicative inverse. For equation (3), we had

$$2 \times \frac{1}{2} = 1;$$

for any positive integer N we had

$$N \times \frac{1}{N} = 1;$$

and for any positive rational number $\frac{a}{b}$ we had

$$\frac{a}{b} \times \frac{b}{a} = 1.$$

Just as the extension to positive rationals gave us multiplicative inverses, the extension to negative numbers gave us *additive inverses*. The parallel between multiplicative and additive inverses is very close and helpful; we can get all of our discussion of additive inverses from the preceding discussion of multiplicative inverses by merely replacing

"1" by "0", "$\times$" by "$+$", "$\frac{1}{N}$" by "^{o}N", and
"$\frac{b}{a}$" (as the multiplicative inverse of $\frac{a}{b}$) by "$^{o}\left(\frac{a}{b}\right)$".

If we make these changes, we would now say: Before we had negative numbers, we could not solve either

$$5 + \square = 0 \quad \text{or} \quad 3\frac{1}{2} + \square = 0.$$

Now that we have negative numbers available to us, we have

$$5 + {}^{-}5 = 0,$$

so we say that "${}^{-}5$ is the additive inverse of 5"; and we also have

$$3\frac{1}{2} + {}^{-}3\frac{1}{2} = 0,$$

so we say that "${}^{-}3\frac{1}{2}$ is the additive inverse of $3\frac{1}{2}$."

Since, in fact,

$${}^{-}5 + 5 = 0,$$

we can also say that "5 is the additive inverse of ${}^{-}5$." Using the symbol "^{o}" to denote additives inverses (a convention apparently introduced by the University of Illinois Committee on School Mathematics), we can write

$$^{o}({}^{+}5) = {}^{-}5, \quad {}^{o}\left({}^{+}3\frac{1}{2}\right) = {}^{-}3\frac{1}{2}, \quad \text{and} \quad {}^{o}({}^{-}5) = {}^{+}5.$$

Indeed, *every* rational number now has an additive inverse.

Some Basic Identities

Three familiar properties of our earlier number systems still hold (as you can probably convince yourself by examples using the vector model):

The commutative law for addition—

For all rational numbers x and y, we have

$$x + y = y + x.$$

The associative law for addition—

For all rational numbers x, y, and z, we have

$$x + (y + z) = (x + y) + z.$$

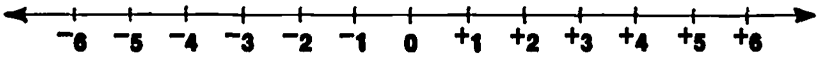
Existence of an additive identity element—

For every rational number x, it is always true that

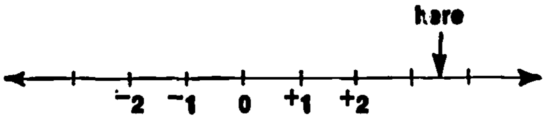
$$x + 0 = x \quad \textit{and} \quad 0 + x = x.$$

The Symmetry Property on the Number Line

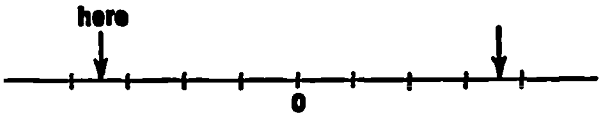
We really have answered (for the present) our question about what numbers lie to the left of zero on the number line. The line has a kind of symmetry that "balances" one side of zero against the other:



and, in fact, whenever you can find a number on one side—say, as shown below for the number "$3\frac{1}{2}$" —



you can always be sure that the symmetric point has a number name, specifically the additive inverse of the "balancing" point on the opposite side of zero:



(and, in this case, since

$$^{o}\left(^{+}3\frac{1}{2}\right) = {}^{-}3\frac{1}{2},$$

we know that the point on the left has the number name "$^{-}3\frac{1}{2}$").

Some Properties of Order

For any negative number x, we know that

$$x < 0,$$

and if y is any positive number, we know that

$$x < y.$$

Moreover, there are certain "rewriting rules" that allow us to rewrite an inequality without changing its meaning. For example, if c is positive, we know that

$$a < b \text{ is true} \quad \text{if and only if} \quad ac < bc;$$

and if c is negative, we know that

$$a < b \text{ is true} \qquad \text{if and only if} \qquad bc < ac.$$

A FORMAL DEVELOPMENT OF THE NEGATIVE INTEGERS

Thus far, we have looked at examples and thought about them. We may have—indeed, we should have—the feeling that we are beginning to understand negative numbers reasonably well.

However, all of this work has been "unofficial." To use our analogy with legislation about environmental pollution, we could say: "We've looked at lakes and rivers and biological data (and so on), and we're beginning to feel that we see what the situation is. However, we have not yet passed any legislation to deal with the problem." Consequently, at this stage our understanding of negative numbers is informal.

It could be made formal. One of the features that distinguish "modern mathematics" from "classical mathematics"—at least in areas like arithmetic—is the possibility nowadays of saying very precisely and explicitly what we mean by $^-4$, for example, or the addition of negative integers. This level of explicit precision was not available to the great mathematicians of the classical period— not available to Newton, nor to Euler, nor to Leibniz, nor to Descartes. (In some ways, if we wish to carry our "legislative" analogy further, the modern explicit mathematics might be likened to a democratic government that is based on explicitly stated laws passed by legislatures, whereas classical intuitive mathematics might be likened to earlier social systems based on an implicit community consensus or else on the judgment of a monarch. In both earlier systems, people may have *known* what they were doing, but they didn't ordinarily *say it* with any great explicit precision.)

For reasons of space, we do not present here a formal development of the negative integers. Many teachers will nonetheless want to know at least one way to give a precise, explicit development of the system of negative integers. (Several different methods exist.) Readers who are familiar with the notion of *equivalence classes* may be able to work out the development themselves by constructing equivalence classes of ordered pairs of counting numbers so that, for example, the equivalence class

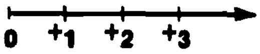
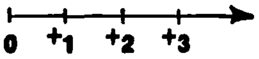
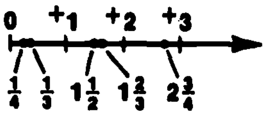
$$\{(1,3), (2,4), (3,5), \ldots\}$$

will be given the name "$^-2$." (Hint: You must next define the operation of *addition.*)

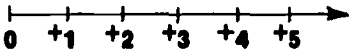
An alternative method for carrying out an explicit development of the system of negative integers is presented in chapter 11 of *Retracing Elementary Mathematics.*[6]

SUMMARY

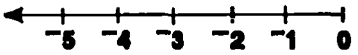
At the end of chapter 9 we had these numbers:

Name	*Examples*	*Corresponding Number-Line Points*
Nonnegative integers	0, 1, 2, 3, 4, . . .	
Nonnegative rationals	0, 1, 2, 3, 4, . . .	
	$\frac{1}{4}, \frac{1}{3}, 1\frac{1}{2}, 1\frac{2}{3}, 2\frac{3}{4}, \ldots$	

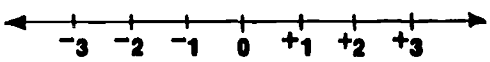
1. Our main theme in chapter 10 has been to ask whether there might not be some other numbers that would also be useful. To pursue this search, we started with the nonnegative integers,



and found the negative integers, $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, $^{-}5$, . . . , that correspond to points symmetric to the positive integers:



Combining negative integers and nonnegative integers, we obtained the numbers called *integers*, . . . $^{-}3$, $^{-}2$, $^{-}1$, 0, $^{+}1$, $^{+}2$, $^{+}3$, . . . , corresponding to these points on the number line



Notice that at this point the word "integer" changes its meaning: From

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[6] Leon Henkin, W. Norman Smith, Verne J. Varineau, and Michael J. Walsh (New York: Macmillan Co., 1962).

now on, the *integers* may be positive, zero, or negative.[7] (Prior to chapter 10, "integers" were either zero or positive. Negative integers hadn't been "legislated" yet.)

2. To give children a concrete experiential meaning for negative integers, we used the "Pebbles in the Bag" model.

3. Having obtained all the integers, our next task was to decide how to *add* them. We used a concrete model: we picked one direction (east) as "positive" and called the opposite direction (west) "negative." We interpreted $^{+}2$ as "two giant steps forward" (if we are facing east) and $^{-}3$ as "three giant steps backwards." Thus the mathematical sentence

$$^{+}2 + {}^{-}3 = {}^{-}1$$

has the following concrete meaning: "If we take two giant steps east, then three giant steps west, we end up one giant step west of where we started." This concrete interpretation allows us to carry out additions, such as those shown below.

$$\begin{array}{lll}
^{+}2 + {}^{+}3 = {}^{+}5. & ^{-}4 + {}^{+}5 = {}^{+}1. & ^{+}5 + 0 = {}^{+}5. \\
^{+}5 + {}^{-}4 = {}^{+}1. & ^{-}8 + {}^{+}3 = {}^{-}5. & ^{-}1 + 0 = {}^{-}1. \\
^{+}3 + {}^{-}8 = {}^{-}5. & ^{-}6 + {}^{+}6 = 0. & 0 + 0 = 0. \\
^{+}6 + {}^{-}6 = 0. & ^{-}5 + {}^{-}2 = {}^{-}7. &
\end{array}$$

Note that this list contains one of each possible case: both addends positive; one positive and one negative, where the positive number has a larger absolute value; one positive and one negative, where the negative number has the larger absolute value; where absolute values are equal; rearrangements of the last three using the commutative property of addition; the case where both addends are negative; and the three cases where one addend is zero. One could study this list and formulate some verbal rules describing the answers, but we recommend against it. Most elementary verbal rules at this stage are complicated and confusing; and they are unnecessary, since children can work directly with the "giant steps forward or backward" model whenever they want to get an answer.

To avoid confusion, the three different meanings that used to be

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[7] Analogously to this, "the United States" referred, a few years ago, to forty-eight states, which were contiguous. Now the phrase "the United States" refers to a nation of fifty states, not all contiguous.

assigned to the single symbol "−" and the single word "minus" have been assigned to three distinct symbols:

Symbol (in Use)	*Name*	*Meaning*	*Examples*
$5 - 3$	"*minus*" (In this example, "five minus three.")	Subtraction	$5 - 3 = 2.$
$^{o}(^{+}3)$	"*additive inverse*" (In this example, "the additive inverse of positive three.")	The additive inverse of positive three is the number you add to three to get an answer zero.	$^{+}3 + {}^{o}(^{+}3) = 0.$ $^{o}(^{+}3) = {}^{-}3.$ $^{o}(^{+}5) = {}^{-}5.$
$^{-}3$	"*negative*" (In this example, "negative three.")	Part of the name of any negative number	$^{-}3$

4. At this point in the story we went back to the nonnegative rationals,

$$0, 1, 2, 3, \ldots \quad \text{and} \quad \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots,$$

and again added a symmetric portion of the number line, corresponding this time to newly created *negative rationals,*

$$^{-}1, {}^{-}2, {}^{-}3, \ldots \quad \text{and} \quad \frac{^{-}1}{2}, \frac{^{-}1}{3}, \frac{^{-}2}{3}, \frac{^{-}1}{4}, \frac{^{-}3}{4}, \frac{^{-}1}{5}, \ldots,$$

so as to get one of the most important of all number systems, the system of *rational numbers:*

$$\ldots {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, \; {}^{+}2, {}^{+}3, \ldots$$

(and, not in order of size:)

$$\ldots, \frac{^-1}{5}, \frac{^-3}{4}, \frac{^-1}{4}, \frac{^-2}{3}, \frac{^-1}{3}, \frac{^-1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots$$

Notice that, at this point in our study of mathematical systems, the term "rational numbers" was given a new meaning: Henceforth it includes also the *negative* rational numbers.

We extended our "giant steps" model to include "one-half a giant step forward" and so forth and could thus use it to add rational numbers as well as integers.

5. At this point our exploration was incomplete; we needed to decide how to subtract, multiply, and divide these new numbers (which, in fact, we shall do in chapter 11). We also needed to formalize our work. Thus far it would be impossible to settle arguments or answer questions in the way mathematicians prefer—by reference to clear, explicit foundations. No such foundations had been created.. In fact, this formalization was not carried out here, but suggestions and references make it possible for the reader to carry this out if he wishes.

Abraham M. Glicksman

OPERATIONS EXTENDED TO NEGATIVE RATIONALS

11

1. **How can the extended number line be used to teach subtraction with negative rationals?**
2. **How can missing addends be used in computing differences with negative rationals?**
3. **How can missing factors be used in computing quotients with negative rationals?**
4. **How can a "pattern" approach be used to teach the "rules of signs" in multiplication?**
5. **How can parallel number lines be used to obtain products with negative rationals?**

SUBTRACTION INVOLVING NEGATIVE RATIONAL NUMBERS

In chapter 10 we used the number line with addition problems involving rational negative numbers. We shall now see that number lines can also be used with *subtraction* problems that involve negative rational numbers.

The key for our approach to subtraction is the missing-addend idea. In the subtraction problem

$$a - b = \square$$

the difference $a - b$ may be interpreted as the missing addend in a corresponding addition sentence,

$$a \quad = \square + b.$$

Instead of asking "What is $a - b$?" we ask: "To what missing addend must we add b in order to obtain a as the sum?"

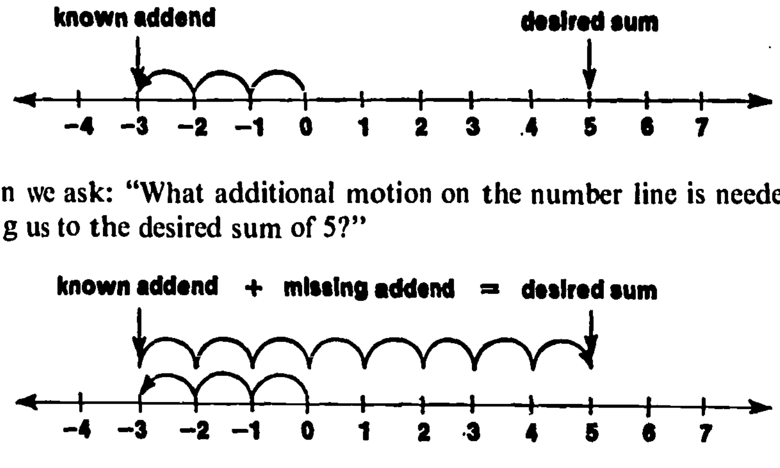
For example, suppose we want to compute

$$5 - (-3) = \square .$$

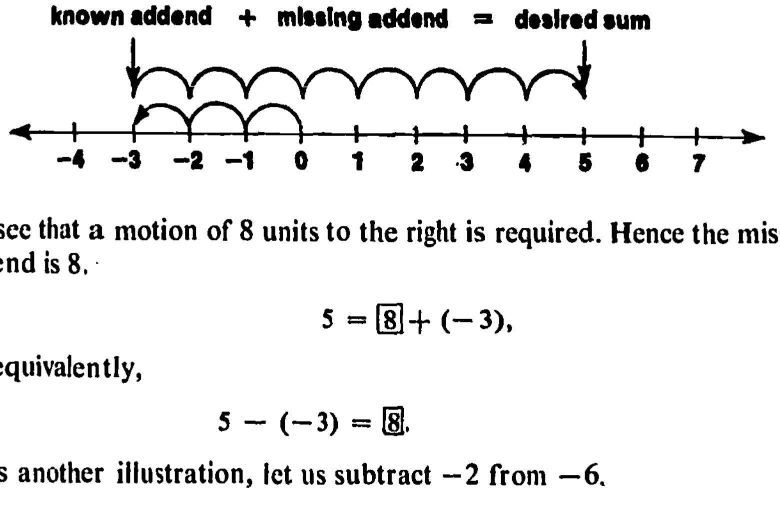
We convert this subtraction sentence into an equivalent addition sentence,

$$5 = \square + (-3).$$

Now we ask: "To what missing addend must we add -3 in order to obtain 5 as the sum?" By the commutative law for addition, this has the same answer as the question "What missing addend must we add to -3 to obtain 5 as the sum?" The answer can easily be found using a number line in the manner described in the preceding chapter. We start at 0 on the number line and interpret the known addend, -3, as a motion of 3 units to the left.



Then we ask: "What additional motion on the number line is needed to bring us to the desired sum of 5?"



We see that a motion of 8 units to the right is required. Hence the missing addend is 8,

$$5 = \boxed{8} + (-3),$$

or, equivalently,

$$5 - (-3) = \boxed{8}.$$

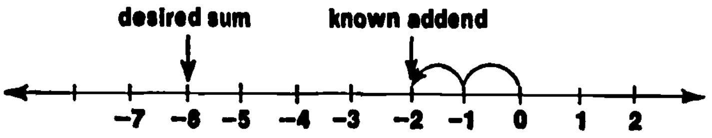
As another illustration, let us subtract -2 from -6.

$$(-6) - (-2) = \square .$$

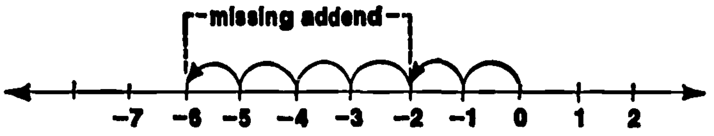
An equivalent addition sentence is

$$(-6) = \square + (-2).$$

We start by locating the known addend, -2, on the number line.



To obtain the desired sum, −6, we need an additional motion of 4 units toward the left:



Therefore, the missing addend is −4:

$$(-6) = \boxed{-4} + (-2).$$

Hence

$$(-6) - (-2) = \boxed{-4}.$$

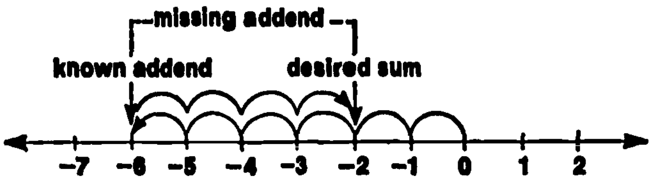
Suppose we want to subtract −6 from −2. This is equally easy to do on the number line. The subtraction sentence is

$$(-2) - (-6) = \square.$$

An equivalent addition sentence is

$$(-2) = \square + (-6).$$

This time the known addend is −6 and the desired sum is −2. We first locate the known addend. Then, to obtain the desired sum, −2, we must now move 4 units to the right:



Therefore the missing addend is 4:

$$(-2) = \boxed{4} + (-6).$$

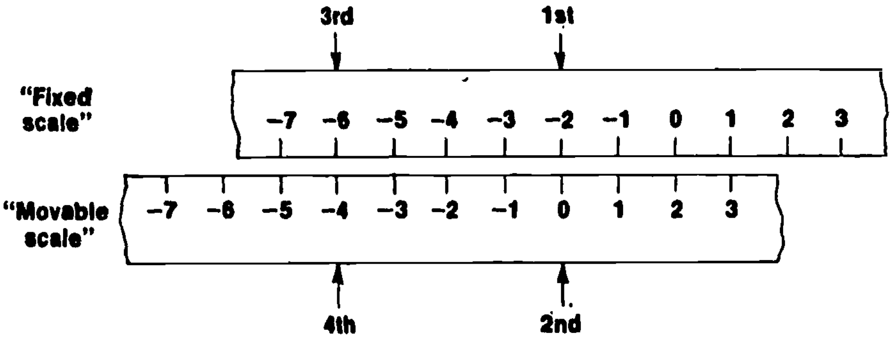
Thus, we conclude that

$$(-2) - (-6) = \boxed{4}.$$

A "slide rule" of the type used in the preceding chapter to illustrate addition can serve equally well as a device for teaching subtraction. For example, in order to compute

$$(-6) - (-2) = \square,$$

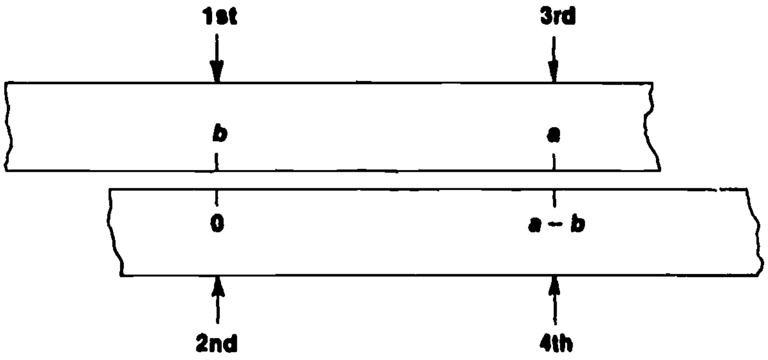
we first locate the given addend, -2, on the "fixed" scale. Second, we slide the "movable" scale until 0 is aligned with -2. Third, we locate the desired sum, -6, on the fixed scale. Fourth, we find the missing addend on the movable scale, opposite the desired sum, and learn that it is -4.



In general, to compute

$$a - b = \square$$

we first locate the given addend, b, on the fixed scale, then (second) slide the movable scale until 0 on this scale is aligned with b on the other. Third, we locate the desired sum, a, on the fixed scale. Fourth, opposite a we find the missing addend, $a - b$.



Further examples of this slide-rule technique will be found in exercise set 1.

Notice that by extending the number system to including negative rational numbers we have made subtraction an *unrestricted* operation.

Earlier in arithmetic, when we were limited to whole numbers or to positive rational numbers, $a - b$ was defined only for $a > b$. After introducing negative numbers, this restriction is no longer necessary. Using the number line, we can compute $3 - 7$ just as easily as $7 - 3$.

As children gain skill using the number line for computing sums and differences of rational numbers, they will undoubtedly begin to see shortcuts, and they may possibly formulate rules for the various situations that can arise. This is fine, but such rules should not be imposed by the teacher, nor should the pupils be required to memorize such rules. If a student discovers a correct rule and wants to use it—good! Do not discourage him. However, do not insist that other pupils use this rule unless they choose to do so on their own.

Exercise Set 1

1. Compute each of the following by using a missing-addend approach and by tracing the motions on a number line where that is convenient.

a. $2 - (-5)$

b. $5 - (-2)$

c. $(-2) - 5$

d. $(-5) - 2$

e. $(-2) - (-5)$

f. $\frac{3}{4} - \frac{1}{2}$

g. $\frac{1}{2} - \frac{3}{4}$

h. $\frac{3}{4} - \left(-\frac{1}{2}\right)$

i. $\frac{1}{2} - \left(-\frac{3}{4}\right)$

j. $\left(-\frac{3}{4}\right) - \frac{1}{2}$

k. $\frac{3}{2} - \frac{3}{4}$

l. $\frac{3}{4} - \frac{3}{2}$

m. $\left(-\frac{3}{4}\right) - \frac{3}{2}$

n. $\left(-\frac{3}{4}\right) - \left(-\frac{3}{2}\right)$

o. $\left(1\frac{3}{4}\right) - \frac{3}{2}$

p. $\frac{3}{2} - 1\frac{3}{4}$

q. $\left(-\frac{3}{2}\right) - \left(-1\frac{3}{4}\right)$

r. $2\frac{3}{4} - 1\frac{1}{2}$

s. $1\frac{1}{2} - 2\frac{3}{4}$

t. $\left(-\frac{3}{4}\right) - 1\frac{1}{2}$

u. $\left(-1\frac{1}{2}\right) - \left(-2\frac{3}{4}\right)$

v. $2.7 - 1.2$

w. $1.2 - 2.7$

x. $(-2.7) - 1.2$

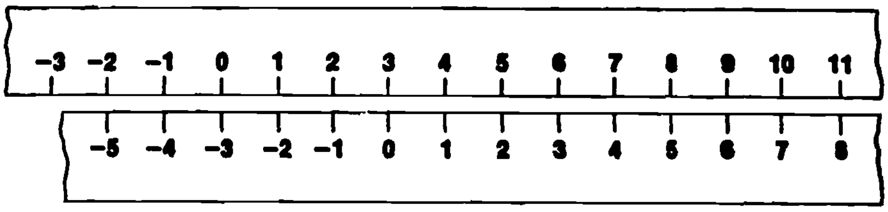
y. $2.7 - (-1.2)$

z. $(-1.2) - (-2.7)$

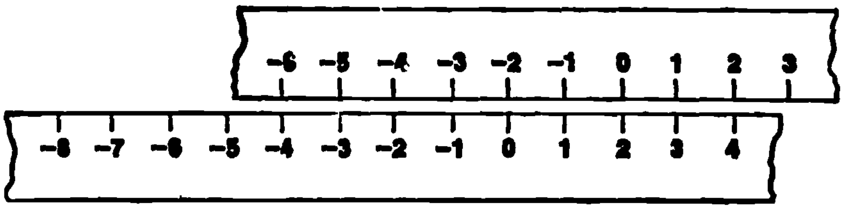
2. Construct a pair of sliding number-scales and use them as slide rules to verify your answers to problems **a–e** of exercise **1.**

3. For each of the following slide-rule settings specify at least three subtraction problems that can be solved with the slide in the position shown.

a.



b.



4. Verify that $a - (-b) = a + b$ by showing that both computations in each of the following exercises yield the same result.

a. $2 - (-3), \quad 2 + 3$

b. $(-2) - 3, \quad (-2) + (-3)$

c. $2 - 3, \quad 2 + (-3)$

d. $(-2) - (-3), \quad (-2) + 3$

5. In computing $a - b$, the result is not altered if the same number c is added to both a and b. This may be expressed as follows:

$$(a + c) - (b + c) = a - b \qquad \text{(for all } a, b, c\text{).}$$

This property can often be used to simplify a subtraction computation. For example, to compute

$$(-6) - (-5)$$

we might add 6 to both numbers to get

$$\begin{aligned}(-6) - (-5) &= (-6 + 6) - (-5 + 6)\\ &= 0 - 1\\ &= -1,\end{aligned}$$

or we might add 5 to both numbers:

$$\begin{aligned}(-6) - (-5) &= (-6 + 5) - (-5 + 5)\\ &= -1 - 0\\ &= -1.\end{aligned}$$

Use this idea to simplify each of the problems **a–e** of exercise **1**.

MULTIPLICATION INVOLVING NEGATIVE RATIONAL NUMBERS

After students have learned to compute sums and differences with positive and negative numbers, it is only natural that they consider products and quotients. In this section we shall discuss some strategies for teaching multiplication involving negative numbers. (We shall consider division in the next section.)

There are several good ways to introduce multiplication with negative rationals to a class. One way is to use a "pattern" approach. Starting with a series of products, each involving two positive factors, a pattern is developed that leads pupils to discover what they must do when one factor in a product is negative. This, in turn, leads to a pattern that enables them to discover what must be done when two factors in a product are negative.

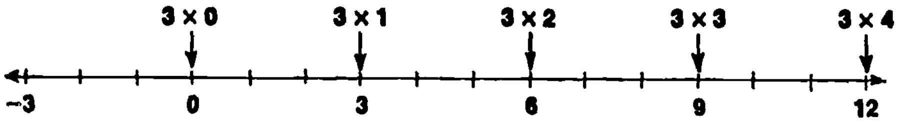
Consider, for example, the following sequence of indicated products:

$$\begin{aligned}&3 \times 4\\ &3 \times 3\\ &3 \times 2\\ &3 \times 1\\ &3 \times 0\\ &3 \times (-1)\\ &3 \times (-2)\\ &3 \times (-3)\\ &3 \times (-4)\end{aligned}$$

Students who have learned their elementary multiplication facts for whole numbers will readily compute the first five products:

$$
\begin{aligned}
3 \times \quad 4 &= 12\\
3 \times \quad 3 &= 9\\
3 \times \quad 2 &= 6\\
3 \times \quad 1 &= 3\\
3 \times \quad 0 &= 0\\
3 \times (-1) &= ?\\
3 \times (-2) &= ?\\
3 \times (-3) &= ?\\
3 \times (-4) &= ?
\end{aligned}
$$

If the pattern is not already obvious, it will be easily recognized if the students proceed to locate each of the known products on a number line in the following manner:

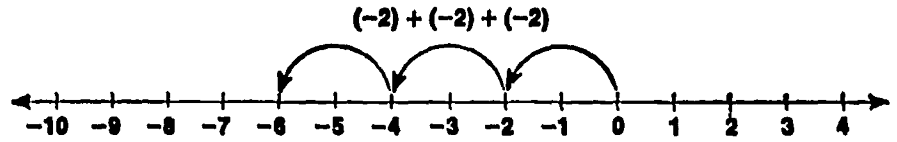


If the pattern of getting 3 less is to be preserved, then the remaining products are:

$$
\begin{aligned}
3 \times (-1) &= -3\\
3 \times (-2) &= -6\\
3 \times (-3) &= -9\\
3 \times (-4) &= -12
\end{aligned}
$$

Still another method that can be used to supplement and reinforce the results arrived at by the pattern approach above is to have pupils interpret each product as a repeated addition. For example, $3 \times (-2)$ is interpreted as

$$3 \times (-2) = (-2) + (-2) + (-2) = -6$$



Through activities of this type a class soon learns the general rule:

> *The product of a positive number and a negative number is a negative number.*

Observe that the rule emerges because of the very natural desire to preserve a multiplication pattern (or interpretation of multiplication) that applied originally to nonnegative numbers. The natural extension of the pattern (or interpretation) of the product to the case where one factor is negative leads us to the desired rule. Moreover, since multiplication of positive numbers is commutative, it is only natural to require that the operation remain commutative even when one of the factors is negative. For example, we agree that

$$(-2) \times 3 = 3 \times (-2) = -6$$

$$(-4) \times 3 = 3 \times (-4) = -12$$

and so forth. Therefore our extended multiplication rule may now be expressed as follows:

> *A product is negative whenever exactly one of the factors is negative and no factor is zero.*

With this principle clearly understood, the class can then tackle the problem of finding a product of *two negative* factors. Once again, a pattern that starts with known products can be useful:

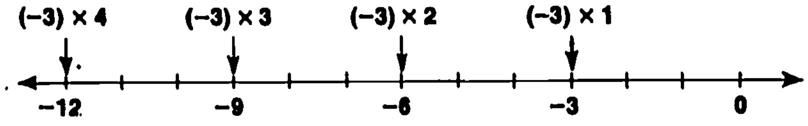
$$(-3) \times 4 = -12$$

$$(-3) \times 3 = -9$$

$$(-3) \times 2 = -6$$

$$(-3) \times 1 = -3$$

Once again, we locate the known products on a number line:



The students will quickly see that the unknown products must be as shown below.

$$(-3) \times 0 = 0$$

$$(-3) \times (-1) = 3$$

$$(-3) \times (-2) = 6$$

$$(-3) \times (-3) = 9$$

$$(-3) \times (-4) = 12$$

Through activities such as these there emerge the following rules:

The product of any number and zero is zero.
The product of any two negative numbers is a positive number.

A second possible approach to multiplication with negative numbers is based on preserving the distributivity law. This method is suitable for classes that have previously studied the distributivity law along with other properties of the whole numbers. For such classes, the teacher would first review the distributivity law:

For all whole numbers a, b, and c

$$a \times (b + c) = (a \times b) + (a \times c).$$

He can then introduce a specific simple addition problem such as

$$(-3) + 3 = 0$$

and proceed to investigate the consequences of multiplying each member of this equation by 2:

$$2 \times (-3 + 3) = 2 \times 0$$

Assuming that the distributivity law applies even when negative numbers are involved, we get

$$2 \times (-3) + 2 \times 3 = 2 \times 0$$

Then, applying the known number facts, we have

$$2 \times (-3) + 6 = 0$$

This shows that $2 \times (-3)$ added to 6 yields a sum of 0. But we already know that

$$(-6) + 6 = 0$$

and since -6 is the only additive inverse for 6, it follows that

$$2 \times (-3) = -6$$

A similar argument can be used to show that

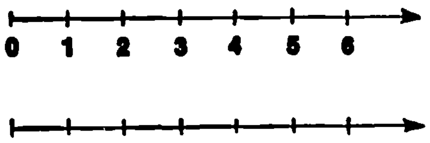
$$(-3) \times 2 = -6,$$

but classes that are well versed in number properties may prefer to view this result as a natural consequence of the previous result, by assuming that *commutativity* of multiplication is also preserved when this operation is extended to products involving negative numbers.

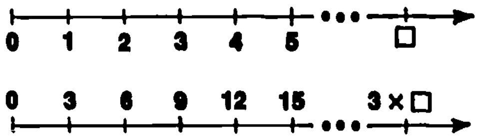
Arguments of this type are likely to appeal to the more "mathematically minded" students. For slower students the pattern approach is probably preferable.

In either case, students can see that the rules for multiplication with negative numbers are by no means arbitrary or haphazard. These rules are forced on us by our natural desire to preserve properties and patterns that hold true for nonnegative numbers. Extension of these properties to products that involve negative factors necessitates our adoption of the new rules.

There is yet another approach that teachers may find helpful when teaching multiplication of positive and negative numbers. This approach makes use of two number lines and has the advantage that the factors need not be integers. The method starts with parallel lines carrying equally spaced division marks. One of these lines is an ordinary whole-number line. The other is temporarily left unlabeled.



Let us construct a "multiply by three" rule. To do this we assign 0 to the initial mark on the new line and 3 to the next mark (i.e., the mark that is opposite and corresponds to 1 on the original number line). All other labels for the new line are now determined by these two initial assignments. In fact, each point of the new line is now assigned three times the number that was assigned to the point directly above it:

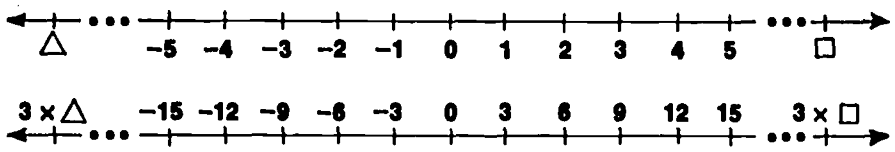


We can now quickly read off such products as

$$3 \times 2 = 6, \quad 3 \times 3 = 9, \quad \cdots\cdot$$

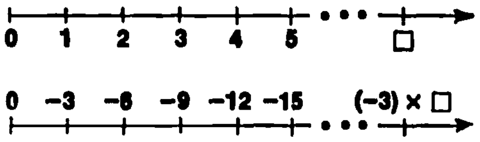
If we then extend each number line to the left, maintaining the obvious pattern on each of the lines, we can now read off the new products:

$$3 \times (-1) = -3, \qquad 3 \times (-2) = -6, \qquad \cdots\cdot$$

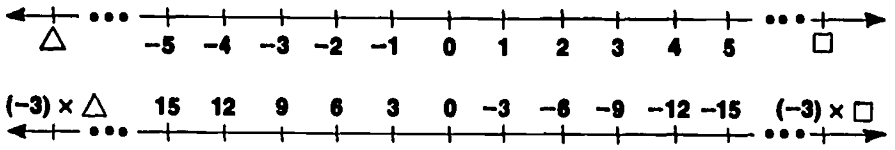


In this manner we readily establish the rule for a *product of a positive number and a negative number.*

To obtain the rule for finding a product of two negative numbers we can use a similar procedure. This time we construct a "multiply by minus three" rule.



Each point on the lower line is now assigned *minus three* times the number assigned to the point directly above it. Then, as before, we extend each number line to the left, maintaining the obvious pattern on each line.



This time we obtain the previously unknown products such as

$$(-3) \times (-2) = 6, \qquad (-3) \times (-4) = 12, \qquad \cdots,$$

and we verify the rule for a *product of two negative factors.*

Various other scale factors should be used as multipliers for the lower line, providing further practice and reinforcement for the students. The important objective here is *getting students to determine for themselves* when a product is positive and when it is negative.

Exercise Set 2

1. Compute each of the following products:

a. $2 \times (-5)$

b. $(-5) \times 2$

c. $(-2) \times (-5)$

d. $6 \times \left(-\frac{1}{2}\right)$

e. $\frac{1}{2} \times (-6)$

f. $(-6) \times \left(-\frac{1}{2}\right)$

g. $3 \times \left(-\frac{1}{2}\right)$

h. $\frac{1}{2} \times (-3)$

i. $\left(-\frac{1}{2}\right) \times (-3)$

j. $\frac{3}{4} \times (-4)$

k. $\frac{3}{4} \times (-2)$

l. $\frac{3}{4} \times (-5)$

m. $\left(-\frac{3}{4}\right) \times 4$

n. $\left(-\frac{3}{4}\right) \times 2$

o. $\left(-\frac{3}{4}\right) \times (-4)$

p. $\left(-\frac{3}{4}\right) \times (-2)$

q. $\left(-\frac{3}{4}\right) \times (-5)$

r. $(-2) \times \left(-\frac{3}{4}\right)$

s. $1\frac{1}{2} \times (-4)$

t. $4 \times \left(-1\frac{1}{2}\right)$

u. $(-4) \times \left(-1\frac{1}{2}\right)$

v. $\left(-1\frac{1}{2}\right) \times (-4)$

w. $\left(-1\frac{1}{2}\right) \times 6$

x. $(-6) \times \left(-1\frac{1}{2}\right)$

y. $2\frac{1}{4} \times \left(-1\frac{1}{2}\right)$

z. $\left(-2\frac{1}{4}\right) \times \left(-1\frac{1}{2}\right)$

aa. $2.5 \times (-1.2)$

bb. $(-2.5) \times (-1.2)$

cc. $(-2.5) \times 1.3$

dd. 1.36×2.5

ee. $(-2) \times \left(-\frac{3}{4}\right) \times (-6)$

ff. $1.5 \times (-4) \times \frac{1}{2} \times (-.6)$

2. The "Postman Game" is yet another way to teach multiplication with negative numbers. In this game we interpret the product 2 × 3 to mean that a postman delivers in our mail two checks, each for $3. Then we interpret 2 × (−3) to mean that the postman delivers two *bills* for $3. Similarly we interpret (−2) × 3 to mean that we *mail out* two checks, each for $3, and we interpret (−2) × (−3) to mean that we mail out two *bills* (due us), each for $3.

a. In each of these cases, do we expect to be richer or poorer, and by how much?

b. Which "rule of signs" is illustrated by each of the four situations?

c. Interpret problems **a**, **b**, and **c** of exercise **1** in terms of the Postman Game.

d. Can the game be applied to problems **d**, **e**, **f**, **g**, and **h** of exercise **1**? Explain your answer in each case.

3. Using a distributivity argument similar to that in the text, show that the following statements are true:

a. $2 \times (-5) = -10.$

b. $(-2) \times 5 = -10.$

c. $(-2) \times (-5) = 10.$

d. $a \times (-b) = -(a \times b).$

e. $(-a) \times (-b) = a \times b.$

DIVISION INVOLVING NEGATIVE RATIONAL NUMBERS

Division is related to multiplication in the same way that subtraction is related to addition. The key to subtraction is finding a missing addend. The key to division is finding a missing factor.

The division sentence

$$12 \div 3 = \square$$

is associated with the multiplication sentence

$$12 \quad = \square \times 3.$$

Then use of knowledge of multiplication with whole numbers gives the missing factor, 4.

This same missing-factor approach applies equally well to division problems involving any rational numbers, whether they are positive or negative, integral or fractional. For example, the division sentence

$$(-12) \div 3 = \square$$

is replaced by the equivalent multiplication sentence:

$$(-12) \qquad = \square \times 3.$$

The missing factor is then deduced to be -4, using previous knowledge of multiplication involving negative numbers.

Similarly, the division sentence

$$(-12) \div (-3) = \square$$

is replaced by the equivalent multiplication sentence

$$(-12) \qquad = \square \times (-3).$$

This time the missing factor is clearly seen to be 4.

After working many examples like

$$\begin{aligned} (-12) \div 3 \quad &= -4, \\ (-12) \div (-3) &= 4, \\ 12 \div (-3) &= -4, \\ 10 \div (-2) &= -5, \\ (-10) \div (-2) &= 5, \end{aligned}$$

and so forth, students will readily see that the "sign rules" for division are the same as the sign rules for multiplication. However, the formal rules should never be imposed by the teacher. The students should be allowed to compute without formal rules by referring to meanings and by stressing the interrelationships among the various operations.

Observe also that it is common practice to extend the concept of a fraction to allow any rational number as numerator and any nonzero rational number as denominator. Such fractions represent the *quotient* of numerator and denominator. With this definition we can also express the above results as

$$\frac{-12}{3} = -4 = -\left(\frac{12}{3}\right),$$

and so forth. It can be shown that all the rules we have developed for fractions whose numerators and denominators are whole numbers hold equally well for fractions whose numerators and denominators are rational numbers.

For all rational r and all nonzero rational s,

$$\frac{-r}{s} = -\frac{r}{s} = \frac{r}{-s}.$$

We shall feel free to use such properties whenever they are needed.

Exercise Set 3

1. Compute the following quotients:

a. $(-6) \div 3$

b. $6 \div (-3)$

c. $(-6) \div (-3)$

d. $(-15) \div 5$

e. $(-4) \div (-2)$

f. $(-2) \div (-4)$

g. $(-2) \div 8$

h. $5 \div (-15)$

i. $(-7) \div 2$

j. $-\frac{3}{4} \div (-3)$

k. $3 \div \left(-\frac{3}{4}\right)$

l. $\frac{1}{2} \div \frac{3}{2}$

m. $-\frac{1}{2} \div \left(-\frac{3}{2}\right)$

n. $-\frac{3}{2} \div \frac{1}{2}$

o. $2\frac{1}{2} \div \left(-\frac{3}{4}\right)$

p. $-2\frac{1}{2} \div \left(-\frac{3}{4}\right)$

q. $\frac{3}{4} \div \left(-2\frac{1}{2}\right)$

r. $-3\frac{1}{2} \div 1\frac{1}{2}$

s. $-1\frac{1}{2} \div \left(-3\frac{1}{2}\right)$

t. $-2.5 \div 5$

u. $-2.5 \div (-5)$

v. $-2.5 \div .5$

w. $-2.5 \div (-.5)$

x. $5 \div 2.5$

y. $-5 \div (-2.5)$

z. $-5 \div (-25)$

aa. $.5 \div (-.25)$

2. In earlier chapters we discussed the following principle:

In a division computation a ÷ b, the result is not altered if both numbers a and b are multiplied by the same nonzero factor c:

$$a \div b = (a \times c) \div (b \times c) \qquad \text{if} \qquad c \neq 0.$$

This property can often be used to simplify a division computation. For example, to compute

$$\left(-7\frac{1}{2}\right) \div 5$$

we might multiply both numbers by 4:

$$\left(-7\frac{1}{2}\right) \div 5 = \left(-7\frac{1}{2} \times 4\right) \div (5 \times 4)$$

$$= (-30) \div 20$$

$$= -\frac{3}{2}, \text{ or } -1\frac{1}{2}.$$

(Other multipliers are, of course, possible.) Use this idea to work out problems **k, l, o, p, q, v, z,** and **aa** of exercise **1**.

3. Another important property of division is expressed by

$$a \div b = a \times \frac{1}{b}.$$

Use this property to obtain rapidly the answers to problems **j, k, m, o,** and **q** of exercise **1**.

SUMMARY

1. By extending the number line to the left, we can work with negative as well as positive addends, interpreting positive addends by motion to the right and negative addends by motion to the left.

2. Since subtraction is equivalent to finding a missing addend, we can

subtract with negative rationals by looking for the appropriate missing addend on the extended number line.

3. Computations involving division with negative rationals make use of the same missing-factor approach that was used when dividing with whole numbers or with positive rationals.

4. Multiplication can be extended to negative rationals by first observing patterns involving previously known products. Preserving these patterns when one or more factors becomes negative leads naturally to the "rules of signs" in multiplication. These rules of signs also arise naturally when we attempt to preserve the pattern of distributivity of multiplication over addition.

5. Parallel number lines carrying appropriately chosen scales can be used effectively to compute products even when one or both factors are negative.

Review Exercises

1. Compute each of the following differences.

a. $(-5) - (-2)$

b. $\left(-\frac{3}{4}\right) - \left(-\frac{1}{2}\right)$

c. $\frac{3}{4} - \left(-\frac{3}{2}\right)$

d. $\left(-\frac{3}{4}\right) - \left(-1\frac{1}{2}\right)$

e. $(-2.7) - (-1.2)$

2. Compute each of the following products.

a. $5 \times (-2)$

b. $(-5) \times (-2)$

c. $\left(-\frac{1}{2}\right) \times (-6)$

d. $\frac{3}{4} \times (-1)$

e. $\left(-1\frac{1}{2}\right) \times 4$

f. $(-6) \times 1\frac{1}{2}$

g. $(-1.2) \times 2.5$

h. $(-1.36) \times (-2.5)$

i. $(-3) \times (-4) \times (-.5)$

j. $2 \times \left(-\frac{1}{4}\right) \times (-6) \times (-.1) \times (-5)$

3. Compute the following quotients.

a. $15 \div (-5)$

b. $(-15) \div (-5)$

c. $2 \div (-8)$

d. $(-5) \div (-15)$

e. $7 \div (-2)$

f. $\frac{3}{4} \div (-3)$

g. $(-3) \div \frac{3}{4}$

h. $\left(-\frac{1}{2}\right) \div \frac{3}{2}$

i. $\left(-\frac{3}{2}\right) \div \left(-\frac{1}{2}\right)$

j. $\left(-2\frac{1}{2}\right) \div \frac{3}{4}$

k. $3\frac{1}{2} \div \left(-1\frac{1}{2}\right)$

l. $2.5 \div (-5)$

m. $(-2.5) \div 5$

n. $2.5 \div (-.5)$

o. $(-.5) \div 2.5$

p. $(-.25) \div (-.05)$

4. Compute the following.

a. $(3 \times (-2)) - (4 \times (-5))$

b. $(6 \times (-3)) \div (-9)$

c. $\frac{(-1.2) \times (-.5)}{3 \times (-.1)}$

d. $(-1.2 \div (.5)) - \left(-\frac{1}{2} \times (-8.6)\right)$

e. $(-1 \times 2 \times (-3)) - (4 \times (-.5) \times 6) + \left((-8) \div \left(-1\frac{1}{3}\right)\right)$

Lauren G. Woodby

GRAPHING

12

1. **What kinds of graphs are used to communicate number information?**
2. **How can graphs help children understand equivalent fractions?**
3. **How can a graph picture the "less than" relation for an open sentence?**
4. **How can a graph picture reciprocals?**
5. **How can graphs help solve problems?**

One of the clearest ways to tell certain kinds of number stories is by means of a graph because a graph can picture number information in a condensed form that is easy to understand. A graph often helps to clarify a mathematical formula or to predict a result. Because of the usefulness of graphs, there has been increased attention to graphing at all levels in schools and especially in the elementary school.

There are many ways to introduce graphs. In the primary grades, very young children like to make "living graphs"; for example, the children could line up in front of pictures of pets to indicate the favorite pet. In this case each child is actively involved—he is a part of the picture. Later, when he sees a graph made by someone else, he realizes that this graph, too, tells some kind of story.

Children like to collect and organize information about objects or events. A child can picture the number of people in his family by putting paper cutouts in a column above his name, and in making this kind of graph he uses the notion of one-to-one correspondence. Children see the relations "less than" and "greater than" when the information is organized in concise form in a picture graph or bar graph.

An important teaching strategy for early work with graphs is to have each child write his own story to accompany his graph. This is usually a story of his data-collecting process and his interpretation of the results pictured by the graph.

Another teaching strategy is to allow children to select their own topics and collect their own data so that the graph is a help in telling a story. The graph is a means to an end, rather than an end in itself.

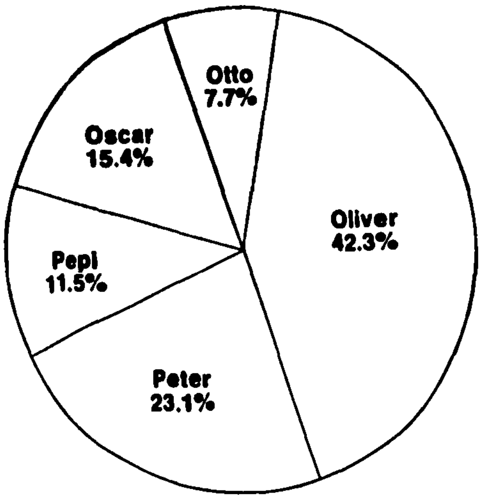
As children become more experienced in collecting and sorting data, they can use squared paper to make bar graphs and line graphs, but squared paper should not be introduced too early. Obtaining graph paper of the proper kind is a very real problem for teachers. One suggestion is to make several master sheets, $8\frac{1}{2}$ by 11 inches, with one-inch squares, half-inch squares, and quarter-inch squares, and then duplicate a supply. However, sheets 17 by 22 inches or larger are often needed, and these are generally hard to find. Stationery stores, art supply houses, and engineering supply houses are possible sources.

In gathering data, children get experience in counting and measuring. Rational numbers result naturally from measurement, so the children become acquainted with halves, thirds, and fourths in a natural setting. Circle graphs provide experiences with fractions, percent, and decimals, as well as measurement of angles. For example, the results of voting by the class to choose a name for a new pet could be presented in a circle graph, or pie chart.

DATA

CHOICE OF NAME FOR GERBIL

Name	Number of Votes	Percent (to nearest tenth)	Degrees (to nearest degree)
Oscar	4	15.4	55
Otto	2	7.7	28
Oliver	11	42.3	152
Peter	6	23.1	83
Pepi	3	11.5	42
Totals	26	100.0	360



Graph

Exercise Set 1

1. In the example of a pie chart for the choice of name for the gerbil, would you expect that the percent column will always add up to exactly 100 percent? Why?

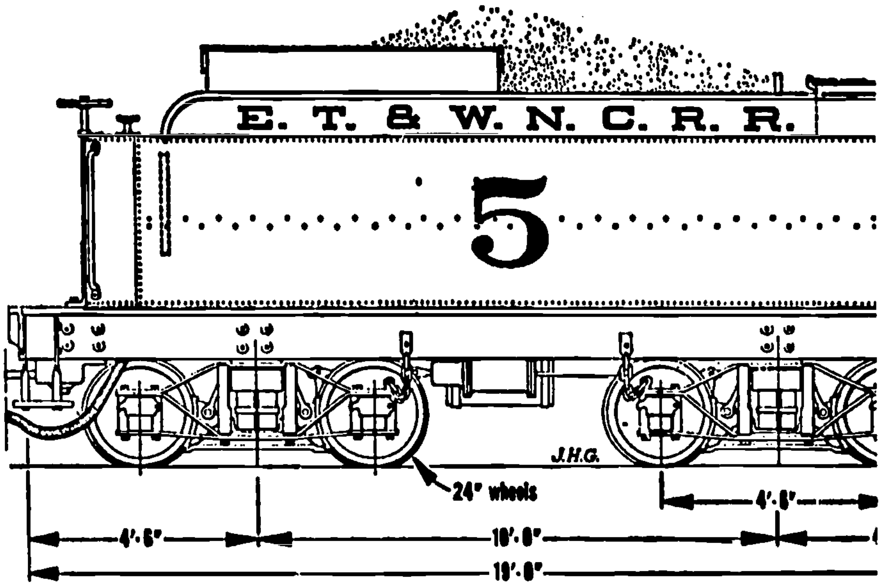
2. Collect or prepare samples of graphs that can be used to tell a story such as "Favorite Television Programs of People in My Class," "Pets in My Neighborhood," "Heights of My Classmates," "How Far Do Children Live from My School?"

3. For each graph, devise a question about the number information displayed—for example, "If you collected the information next week, what changes might you expect?"

4. Read some accounts of children's work in graphing in *Freedom to Learn: An Active Learning Approach to Mathematics*, by Edith E. Biggs and James R. MacLean (Addison-Wesley [Canada]: Don Mills, Ont., 1969).

SCALE DRAWING

Scale drawing is another form of graphing that can be introduced in primary grades. Children learn about shape, size, and relative position by making a scale drawing of their classroom or the street where they live. A map of the neighborhood provides a meaningful introduction to a coordinate system and serves as a source of many problems involving distance and direction. Scale models apply notions of rational numbers in a way that makes sense in many real problems in measurement. For example, a boy who lays out a track plan for his HO-scale-model train track knows what is meant by the statement "The scale is 1 to 87," and the model-ship builder knows that the expression "$\frac{1}{8}$-inch scale" means that $\frac{1}{8}$ inch represents 1 foot. In the drawing reproduced here in part, $\frac{1}{4}$ inch represents 1 foot. The ratio of lengths on the drawing to lengths on the tender is 1 to 48. Thus, the 24-inch wheels are drawn as $\frac{1}{2}$-inch circles.



Model Railroader, December 1968

The following table shows four popular railroad modeling scales:

Name	HO	N	S	O
Ratio	1:87.1	1:160	1:64	1:48
Scale (inches per foot)	0.138	0.075	0.188	0.25
Standard Gauge (inches)	0.649	0.353	0.883	1.177

Exercise Set 2

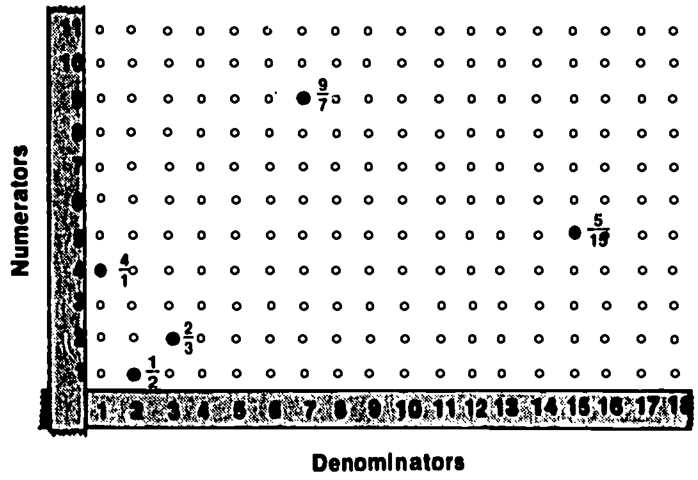
1. Have a child bring a scale model he has made and discuss it.

2. Obtain an N-scale railroad car and an HO-scale car. Measure the length of each and use the scale 1:160 or 1:87.1 to find the length of the real railroad car.

3. A standard-gauge railroad track is 4 feet $8\frac{1}{2}$ inches, inside width. What is the inside width of the track for an HO model? An N model? An S model? An O model? Check your results with the data given in the table above.

GRAPHS OF FRACTIONS ON PEGBOARD

A teaching strategy that works for many children who are confused by the idea of equivalent fractions is to graph fractions as ordered pairs on pegboard. The figure below shows a portion of a 2-by-2-foot pegboard that has masking tape marked with number lines. The horizontal number line is for denominators, and the vertical number line is for numerators. For the moment, we shall rule out 0 as either a denominator or numerator.



To picture the positive fraction $\frac{2}{3}$, first locate the denominator 3 on the horizontal scale, then locate the numerator, 2, on the vertical scale. There is a unique location (hole in the pegboard) for each fraction whose denominator and numerator are natural numbers. This one-to-one correspondance between fractions and holes in the pegboard permits us to picture any fraction; moreover, every hole in the pegboard is associated with exactly one fraction.

The first skill in pegboard graphing of fractions is to be able to locate a given fraction. Care must be taken to locate the denominator first, along with the horizontal number line. This may seem like a strange way to do it, since we usually *say* the numerator first; but there is a good reason why we want the denominator first (horizontal) and the numerator next (vertical). It will become clear when we examine the "slopes" of lines that are associated with the rational numbers represented by fractions. The fractions $\frac{2}{3}, \frac{1}{2}, \frac{4}{1}, \frac{9}{7}$, and $\frac{5}{15}$ are shown. Golf tees are put in the pegboard holes to mark the locations. The teacher should do the exercises below to sense the discovery experience that children will have.

Exercise Set 3

1. Mark, with a golf tee, each of these fractions:

$$\frac{1}{2}, \frac{3}{5}, \frac{4}{3}, \frac{7}{12}, \frac{1}{1}, \frac{3}{4}, \frac{15}{18}, \frac{8}{5}, \frac{3}{1}$$

2. Mark all these with golf tees of the same color:

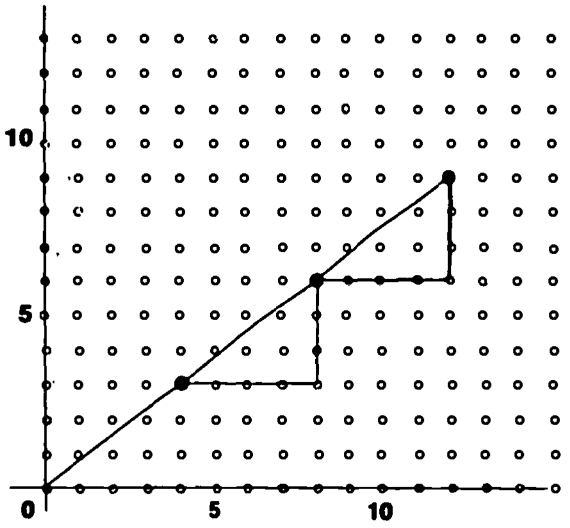
$$\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}$$

3. Mark all these with golf tees of another color:

$$\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}$$

Now look carefully at the fractions equivalent to $\frac{3}{4}$. Some of these are $\frac{15}{20}, \frac{6}{8}, \frac{9}{12}, \frac{18}{24}$. Graph these fractions on the pegboard with golf tees of one color. Notice the pattern. Are there other holes that fit the pattern which are not marked? Notice that from any golf tee you can go "over 4 and up 3" to find a hole for another equivalent fraction. This is an important idea. (The slope of the line on which all these tees lies is $\frac{3}{4}$.)

Look again at the colored golf tees for fractions equivalent to $\frac{2}{3}$ (exercise **3** of set 3). Add golf tees of the same color to the points for any other fractions equivalent to $\frac{2}{3}$ for which there is room on the pegboard. What is the slope of this line? $\left(\text{You should get } \frac{2}{3}.\right)$



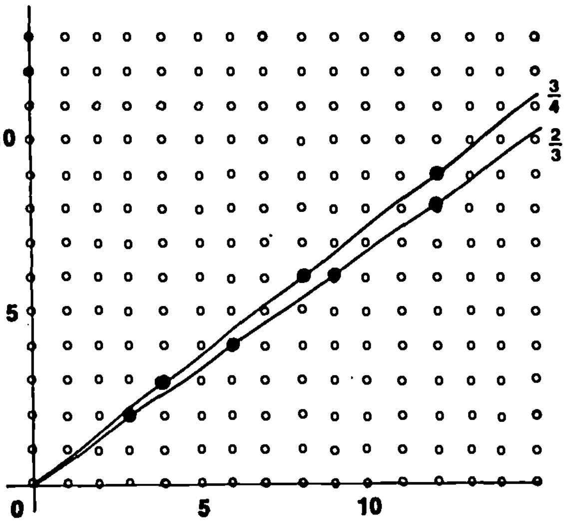
In the exercises you looked at some special fractions that are equivalent to $\frac{1}{1}$. What is the slope of the line of golf tees picturing those fractions? Now graph some fractions equivalent to $\frac{2}{1}$, and notice the steepness (slope) of the line of golf tees.

The striking result is that each collection of equivalent fractions can be pictured by golf tees that lie in a straight line. The slope of the line of golf tees for fractions equivalent to $\frac{2}{3}$ is $\frac{2}{3}$, the slope of the line of golf tees for fractions equivalent to $\frac{3}{4}$ is $\frac{3}{4}$, and the slope of the line of golf tees for fractions equivalent to $\frac{1}{1}$ is 1. In other words, the slope of the line is the feature that stands out for each collection of equivalent fractions. The slope of the line for any collection of equivalent fractions is the rational number named by the equivalent fractions.

Notice that all these lines pass through the (0, 0) point of the graph. We now come back and examine fractions having 0 in either the numerator or the denominator. The fraction $\frac{2}{0}$, for example, would be located at the point "over 0 and up 2", that is, on the vertical number line. Although $\frac{2}{0}$ is formed just like other fractions, it does not name a rational number, so we rule it out. We shall not consider those points on the vertical axis. What about the fraction $\frac{0}{3}$? This fraction would be located at the point "over

3 and up 0", that is, on the horizontal number line. It is a fraction that names the rational number "zero." In fact, the horizontal axis pictures the collection of all fractions with numerator 0 and has a slope of 0.

Two rational numbers can be compared readily by examining the slopes of the lines that picture the fractions for the numbers—simply observe which line has the greater slope. To see why this is so, consider the two rational numbers $\frac{2}{3}$ and $\frac{3}{4}$. Select equivalent fractions with a common denominator, say $\frac{8}{12}$ and $\frac{9}{12}$. The line from the origin to (12,9) is steeper than the line from the origin to (12,8). This example illustrates how the common denominator method of comparing two rational numbers is related to the graphic method of comparing slopes. (See the figure.) Notice that the slopes can also be compared by looking at the golf-tee locations for the fractions $\frac{6}{9}$ and $\frac{6}{8}$.



Exercise Set 4

1. Picture the fraction $\frac{5}{2}$ with a golf tee at (2,5). Now find several fractions equivalent to $\frac{5}{2}$ and picture each one with a golf tee of the same color. Describe the pattern. What is the slope of the line?

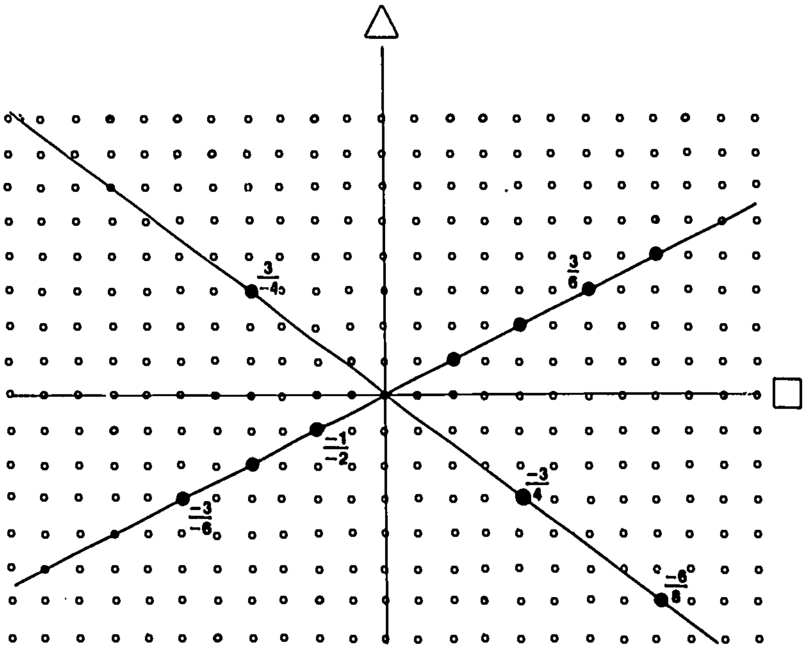
2. Select another fraction whose numerator is greater than the denominator (i.e., an "improper" fraction). Find several fractions equivalent to your selected fraction and picture them with golf tees. What is the slope of the line? Which fraction names the greater number, $\frac{5}{2}$ or the fraction you selected?

3. Graph the fractions $\frac{2}{5}$ and $\frac{3}{7}$ using golf tees of different colors. Find some fractions equivalent to $\frac{2}{5}$ and graph them. Find some fractions equivalent to $\frac{3}{7}$ and graph them. From your graph decide which fraction names the greater rational number. (Hint: Look especially at the fractions $\frac{6}{15}$ and $\frac{6}{14}$).

4. Use the pegboard to picture the fractions $\frac{4}{5}$, $\frac{6}{7}$, and $\frac{9}{10}$. From the graph decide which fraction names the greatest number; the smallest number.

After the study of rational numbers has been extended to include negative numbers, pegboard graphing of fractions becomes even more useful. We are no longer restricted to graphing in the first quadrant. We can graph any fraction, since fractions are now considered as ordered pairs of integers, including the negatives.

The pegboard now looks like this:



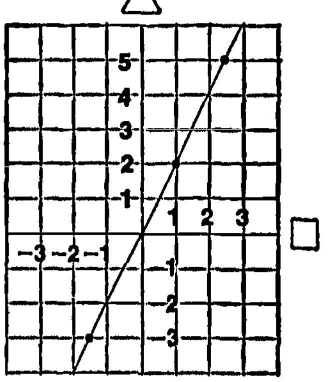
The graph of the fraction $\frac{-3}{4}$ is shown. Notice that golf tees that picture fractions equivalent to $\frac{-3}{4}$ (e.g., $\frac{-6}{8}$ and $\frac{3}{-4}$) lie in a line that has a negative slope. This line pictures the rational number negative $\frac{3}{4}$.

The golf-tee graph for the fraction $\frac{-1}{-2}$ lines up with the graph for $\frac{-2}{-4}$, $\frac{-3}{-6}$ as well as $\frac{1}{2}, \frac{3}{6}, \frac{4}{8}$. The slope of the line is $\frac{1}{2}$.

GRAPHS OF FUNCTIONS

After children have learned to locate on cross-section paper points that correspond to number pairs, they can graph the solution set for an open sentence such as $\triangle = \square - 3$ or $\triangle = 2 \times \square$. The fact that these graphs have a straight-line pattern gives children another view of the consistency of the rules for the operations of addition and multiplication. For example, the open sentence $\triangle = 2 \times \square$ is satisfied by the following number pairs:

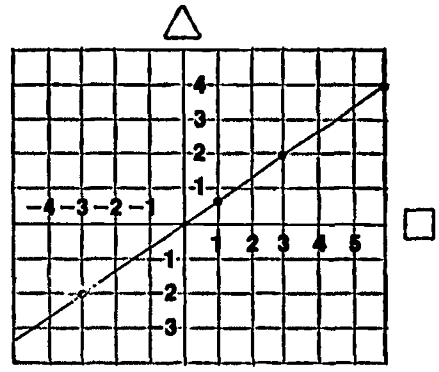
□	△
1	2
$2\frac{1}{2}$	5
0	0
$-\frac{1}{2}$	−1
$-1\frac{1}{2}$	−3



The points for these number pairs lie in a straight line that goes through the origin. The fact that (0,0) satisfies the open sentence reminds children of a property of 0; the product of any number and 0 is 0. The other feature of this line is that its slope is 2. From any point on the line "up 2 and over 1" locates another point on the line.

Now examine the graph for the open sentence $\triangle = \frac{2}{3} \times \square$. Some number pairs that satisfy the sentence are shown below:

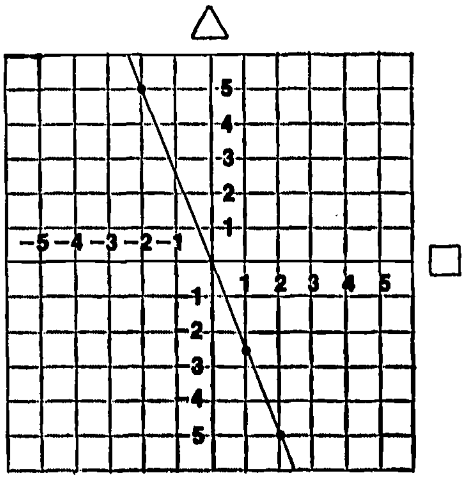
□	△
3	2
0	0
6	4
1	$\frac{2}{3}$
−3	−2



The points for these number pairs lie on a straight line. The point (0,0) lies on the line, and the slope of the line is $\frac{2}{3}$.

A suggested teaching strategy here is to have children graph enough open sentences of the type $\triangle = m \times \square$, where m is any rational number, to become convinced that the slope of the line is m. The open sentence $\triangle = -\frac{5}{2}\square$ is satisfied by these pairs of numbers:

□	△
0	0
1	$-2\frac{1}{2}$
2	−5
−2	5



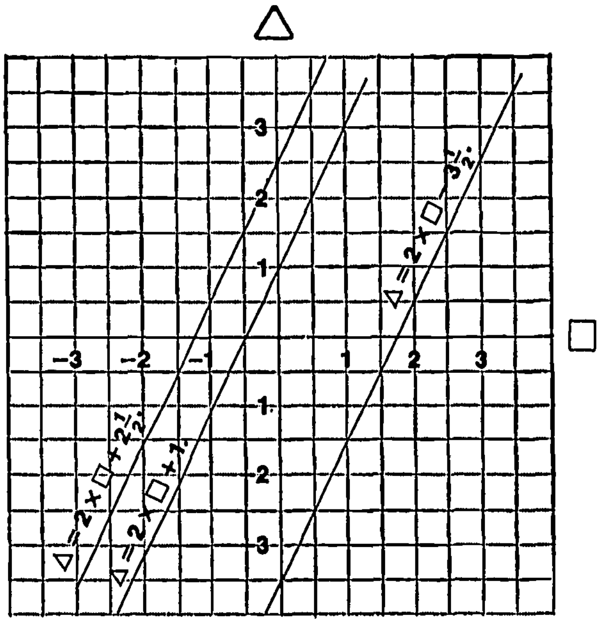
These points lie on a straight line through (0,0), but the slope of the line is negative $\frac{5}{2}$. The line slopes down and to the right (or up and to the left). From any point on the line "down 5 and to the right 2" locates another point on the line. The graph makes the rule for the product of two negative numbers plausible. If -2 is used for $\square$, the result, 5, fits the straight-line pattern observed for the graphs of similar open sentences.

Graphs of open sentences such as

$$\triangle = 2 \times \square + 1, \quad \triangle = 2 \times \square + 2\tfrac{1}{2},$$

$$\triangle = 2 \times \square - 3\tfrac{1}{2},$$

call attention to the idea of a family of lines all having the same slope, 2.

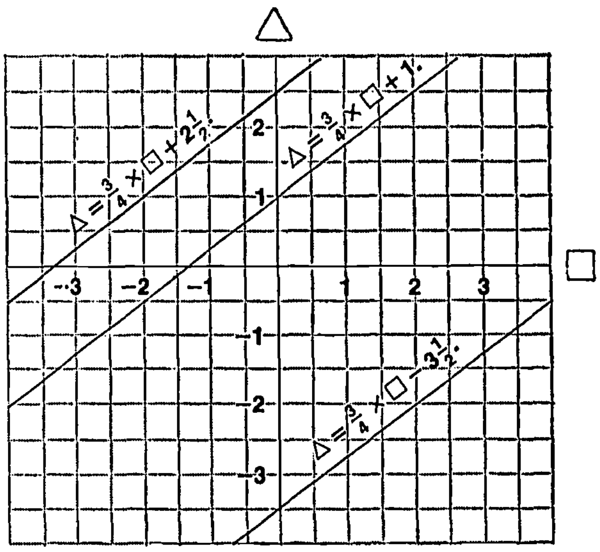


Graphs of open sentences such as

$$\triangle = \tfrac{3}{4} \times \square + 1, \quad \triangle = \tfrac{3}{4} \times \square + 2\tfrac{1}{2},$$

$$\triangle = \tfrac{3}{4} \times \square - 3\tfrac{1}{2},$$

call attention to another family of lines all having the same slope, $\frac{3}{4}$.



Graphs of the open sentences

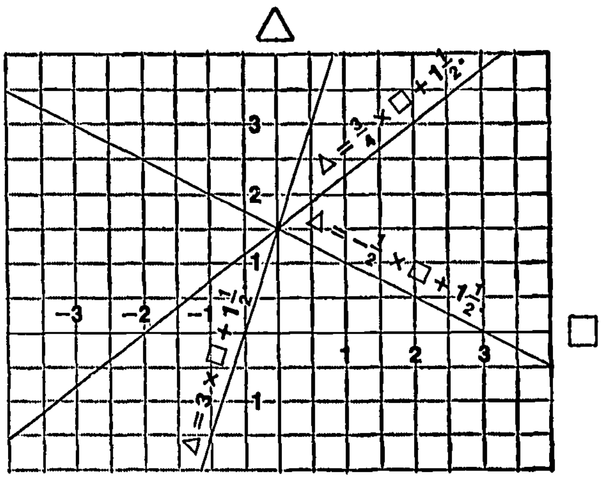
$$\triangle = 3 \times \square + 1\frac{1}{2}, \quad \triangle = -\frac{1}{2} \times \square + 1\frac{1}{2},$$

$$\triangle = \frac{3}{4} \times \square + 1\frac{1}{2},$$

call attention to a family of lines that pass through the point $\left(0,\ 1\frac{1}{2}\right)$. Here again it can be observed that open sentences of the type

$$\triangle = m \times \square + k$$

have straight lines as graphs.



Exercise Set 5

1. Find several number pairs that satisfy the open sentence

$$\triangle = \frac{3}{2} \times \square - \frac{1}{2}.$$

Graph those pairs. Find the slope of the line from your graph.

2. Graph, on the same set of axes, the open sentences

$$\triangle = -\frac{3}{4} \times \square - \frac{1}{2} \quad \text{and} \quad \triangle = \frac{1}{4} \times \square - \frac{1}{2}.$$

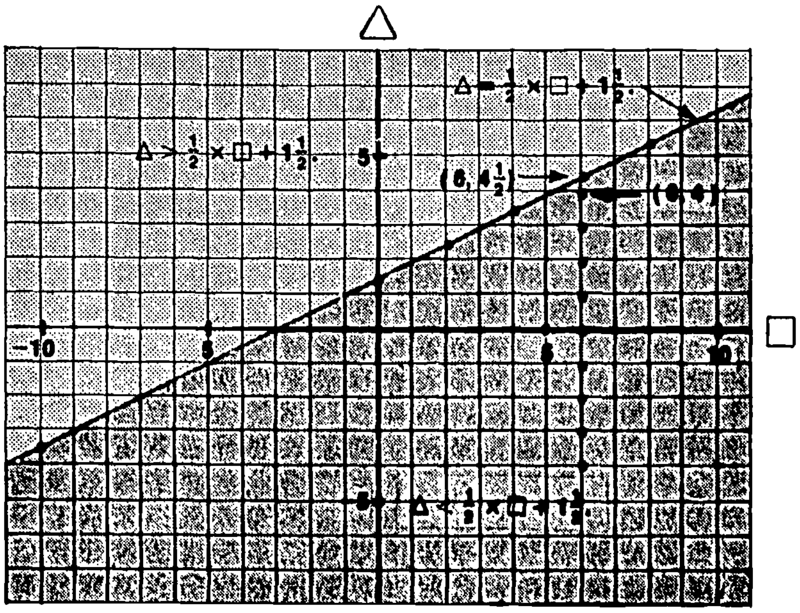
Describe your results.

GRAPHS OF OPEN SENTENCES FOR INEQUALITIES

Once children can graph open sentences with the "equals" relation, there is a natural and easy extension to graphing "less than" and "greater than" relations. The solution set for

$$\triangle = \frac{1}{2} \times \square + 1\frac{1}{2}$$

is pictured as points on a line in the graph shown here.



Points that lie on the line satisfy the open sentence

$$\triangle = \frac{1}{2} \times \square + 1\frac{1}{2};$$

points not on the line do not satisfy the open sentence. For example, $\left(6, 4\frac{1}{2}\right)$ is on the line, and $4\frac{1}{2} = \frac{1}{2} \times 6 + 1\frac{1}{2}$. But consider the point (6,4) just below $\left(6, 4\frac{1}{2}\right)$. $4 < \frac{1}{2} \times 6 + 1\frac{1}{2}\left(=4\frac{1}{2}\right)$. In fact all the points (6,a) directly below $\left(6, 4\frac{1}{2}\right)$ satisfy the relation $\triangle < \frac{1}{2} \times \square + 1\frac{1}{2}$, since for all these points $a < 4\frac{1}{2}$. Similarly, all the points directly above the point $\left(6, 4\frac{1}{2}\right)$ have number pairs that satisfy the relation

$$\triangle > \frac{1}{2} \times \square + 1\frac{1}{2}.$$

Exercise Set 6

1. Find some number pairs that satisfy the open sentence $y = \frac{-3x}{5} + 1$. Graph these points and, from the pattern, find other pairs that satisfy the open sentence.

2. Now find a number pair that does *not* satisfy the open sentence $y = \frac{-3x}{5} + 1$. Which of the following sentences does that number satisfy?

$$y > \frac{-3x}{5} + 1.$$

$$y < \frac{-3x}{5} + 1.$$

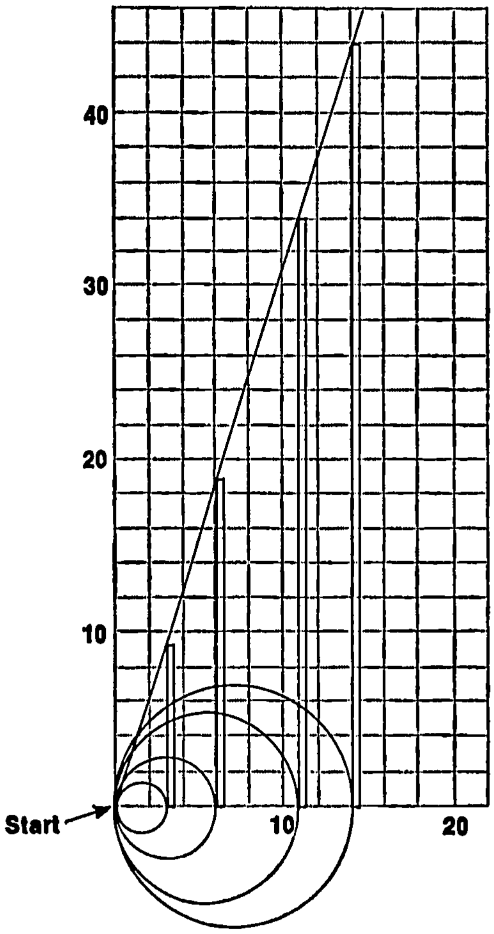
3. Select any other point on the same side of the line for $y = \frac{-3x}{5} + 1$. What number pair is associated with that point? Test to see which open sentence the number pair satisfies. Repeat for several other points.

4. Now select a point on the other side of the line from your points in exercise **3** to see which open sentence its number pair satisfies. Describe the set of points that satisfy the open sentence $y < \frac{-3x}{5} + 1$.

GRAPHING A SPECIAL FUNCTION

The following is a class exercise in graphing without numbers. It is a laboratory approach involving the whole class that has been found to be an unusually successful discovery-type lesson.

A large sheet of cross-section paper, about 2 feet by 4 feet, is taped flat on a table and one corner marked "start," as shown.



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Children, working in pairs, are asked to find a circular object and cut a strip of colored tape that just fits around the object. A child places the circular object on the graph sheet, as shown in the figure, so that the end of the diameter can be marked on the horizontal axis. His partner then unwinds the tape from the object and sticks it onto the sheet, vertically, from the marked end of the diameter. If no children select a very small circular object, suggest one, such as a coin. If no very large circular object was chosen, suggest one, such as a wastebasket. After the children have put on all the strips, ask them to describe what they see. When someone observes that the ends of the strips seem to be in line, stretch a string from the origin to check this observation and fasten the string with tape. Place a different circular object on the sheet and ask the children to predict the length of the circumference. Now ask the inverse question; hold up a length of tape and ask for the diameter of a circular object that the tape would just fit around. Notice that up to this point, no numbers have been used at all—just distances marked on the paper and strips of tape.

In order to measure these distances, label the inch markings on the horizontal and vertical lines through the origin. Ask the children to find, from the graph, the circumference of a circle whose diameter is 1 inch; 10 inches; 20 inches; 30 inches.

What is the slope of the line (the stretched string)?

This graph can be used to solve a practical problem in measurement. The problem, which must be solved by anyone who wants to make a trundle wheel that measures in yards, is "How can a piece of plywood be marked to make a circular disk whose circumference is one yard?" From the graph the required diameter is found by locating 36 inches on the vertical axis, moving to the right to the string, then down to the horizontal axis. This length can be transferred directly to the plywood. It is the diameter we seek. Compare this solution with the usual computational solution: $C = \pi D$, so $D \approx 36 \div 3.14$.

A teaching strategy that may be used is to ask children to make up other problems that can be solved directly by the graph and to solve them. Some samples are:

1. What would be the circumference of a wheel $8\frac{1}{2}$ inches in diameter?

2. If a bicycle tire is 75 centimeters in diameter, what is the circumference?

3. What is the diameter of a circle whose circumference is 19 inches? 19 feet? 19 meters?

The graph can be thought of as a function machine for the particular function $x \rightarrow \pi \cdot x$; the function can be called the "multiply by π" function. Notice also that we have automatically the "divide by π" function.

AREA AND PERIMETER

A graphic approach to the study of area and perimeter of rectangles seems to make sense to most children. The following sequence of activities is suggested. Children can work in groups of three or four at a table.

1. Cut out from $\frac{1}{2}$-inch cross-section paper about a dozen square pieces of various sizes. For each square piece find the distance around it (its perimeter). Make a table of number pairs. It might begin like this:

Length of Side of Square, in Inches	$3\frac{1}{2}$	
Perimeter of Square, in Inches	14	

Graph the number pairs. Now use the graph to find the perimeters of squares whose sides are $1\frac{3}{4}$ inches; $6\frac{1}{8}$ inches; 3.2 inches.

2. Using the same collection of square pieces, find the area of each in square inches and again make a table of number pairs:

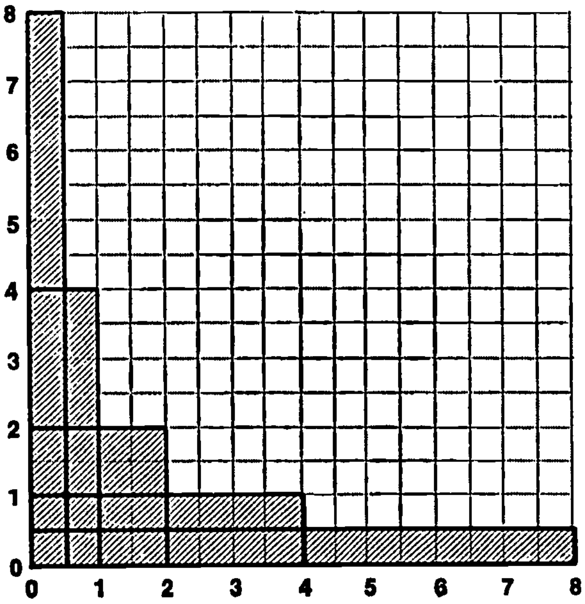
Length of Side of Squares, in Inches	$3\frac{1}{2}$	
Area of Square, in Square Inches	$12\frac{1}{4}$	

Graph the number pairs. Now use the graph to find the areas of squares whose sides are $1\frac{3}{4}$ inches; $6\frac{1}{8}$ inches; 3.2 inches.

The next two activities continue the strategy of using geometric models (this time, rectangles made by the children) for which the desired number information can be found by counting. Graphs can picture the data in a concise form.

3. Cut out from $\frac{1}{2}$-inch cross-section paper about a dozen rectangles, each with a perimeter of 12 inches. Children can check this condition with a 12-inch piece of string. Determine the area of each and graph the number pairs (length of one side, area). Describe any special cases.

4. Cut out some rectangles whose *areas* are all the same, say 4 square inches. Graph the number pairs (length of one side, length of the adjacent side). Describe any special cases.



Direct Graph of Rectangles of Area = 4

RECIPROCALS

The reciprocal relation is extremely useful in many applications, and a graph helps children understand this relation.

For what numbers n is it true that $n \times \frac{1}{n} = 1$? Almost every number you try works. For example, if n is 5, $\frac{1}{n}$ is $\frac{1}{5}$ and $5 \times \frac{1}{5} = 1$. How about $n = \frac{1}{7}$? In this case $\frac{1}{n}$ is $\frac{1}{\frac{1}{7}}$, which is 7, and $\frac{1}{7} \times 7 = 1$.

Try a negative number, say -15. If n is -15, $\frac{1}{n}$ is $\frac{1}{-15}$ and

$-15 \times \frac{1}{-15} = 1$. Are there any numbers that do not work? How about $n = 0$?

The relation can be described by the open sentence $\square \times \triangle = 1$ or $\triangle = \frac{1}{\square}$ (with $\square \neq 0$). Another way to describe the relation is

$$n \rightarrow \frac{1}{n},$$

which says to pair up n with $\frac{1}{n}$. Of course, we could use the following:

$$x \cdot y = 1.$$

$$y = \frac{1}{x}, \qquad x \neq 0.$$

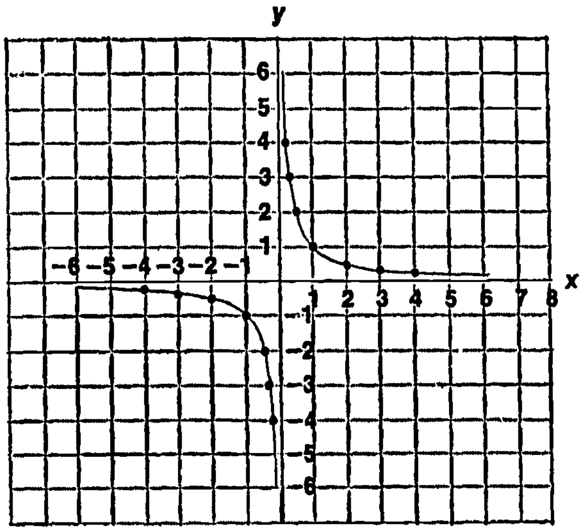
$$x \rightarrow \frac{1}{x}, \qquad x \neq 0.$$

A table of pairs of numbers that satisfy the open sentence $x \cdot y = 1$ is easy to make. Some pairs are shown in the table at the left, and some more in the table at the right.

x	y
1	1
2	$\frac{1}{2}$
$\frac{1}{2}$	2
3	$\frac{1}{3}$
$\frac{1}{3}$	3
4	$\frac{1}{4}$
$\frac{1}{4}$	4

x	y
-1	-1
-2	$-\frac{1}{2}$
$-\frac{1}{2}$	-2
-3	$-\frac{1}{3}$
$-\frac{1}{3}$	-3
-4	$-\frac{1}{4}$
$-\frac{1}{4}$	-4

The graph of these number pairs looks like this:



Exercise Set 7

1. Suppose you wanted to find the speed of automobiles by measuring with a stopwatch the time for each vehicle to travel a fixed distance, say 250 feet. A chart that can be used to translate time in seconds into speedometer reading would simplify computation. First complete the following table, using the formula $d = r \times t$.

Miles per Hour	*Feet per Second*	*Time in Seconds to Travel 250 Feet*
90		
60	88	2.8
50		
45		
40		
30	44	5.7
20		
15	22	11.4
10		
5		

2. For each speedometer reading in miles per hour, there is a time in seconds for the car to travel 250 feet. Make a graph of speedometer reading against time for traveling 250 feet. Could the time ever be as long as 40 seconds? Use the graph to find the time for traveling 250 feet if the car is going 27 miles per hour. What is the speed of the car if the time for the 250-foot distance is 15 seconds?

ANSWERS TO EXERCISES

BEYOND THE WHOLE NUMBERS

Exercise Set 1, pp. 8–10

1. **a.** Two-thirds, (2,3), $\frac{2}{3}$

b. Three-sixths, (3,6), $\frac{3}{6}$

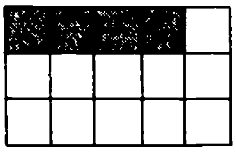
c. One-half, (1,2), $\frac{1}{2}$

d. Five-tenths, (5,10), $\frac{5}{10}$

2.

$\frac{4}{5}$

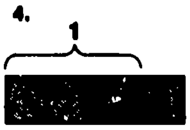
$\frac{7}{8}$



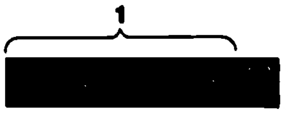
$\frac{4}{15}$

3. Models a, c, and d are good models. Models b, e, and f are not subdivided into congruent regions, and this is a requisite for representing rational numbers.

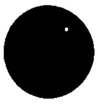
4.



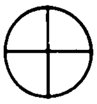
$\frac{4}{3}$



$\frac{8}{5}$



$\frac{6}{6}$



$\frac{0}{4}$

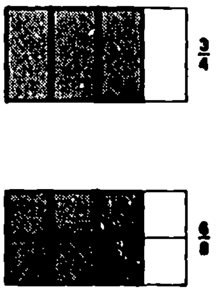
5.

(3, 5)

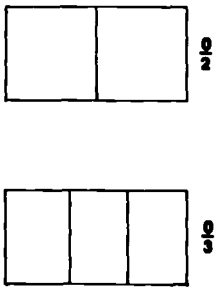
(6, 10)

(12, 20)

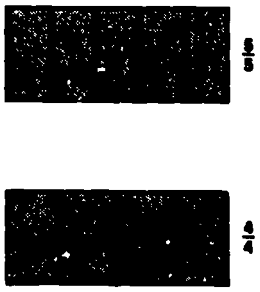
6. **a.**



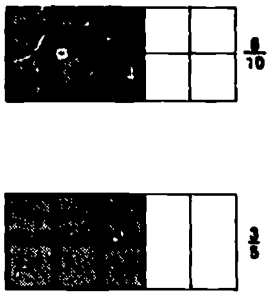
c.



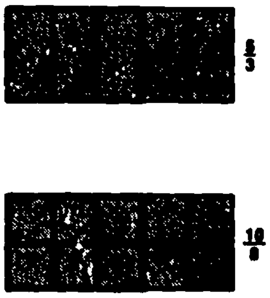
e.



b.

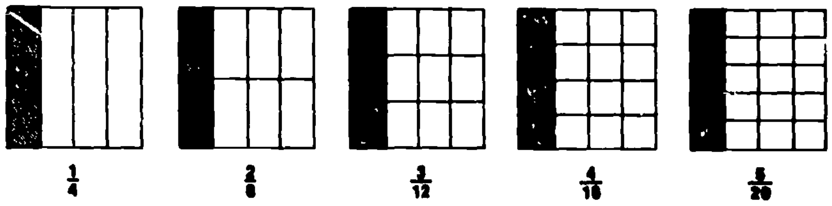


d.



7. Two fractions are compared on the basis of shaded regions. If the shaded regions of the two figures are equal, then the fractions are equivalent. Such a comparison can be made only if the two figures (the region models) have the same *unit area.*

8.



9. The unit region $\left(\frac{5}{5}\right)$ can be constructed by using five of the given regions (each of which represents $\frac{1}{5}$ of the unit).

If the given region represents $\frac{3}{5}$, divide it into 3 congruent parts (each will represent $\frac{1}{5}$ of the unit region) and build a region that is 5 times the smaller subdivision $(5 \cdot \left(\frac{1}{5}\right) = \frac{5}{5}$, or 1 unit region).

10. The comparison was made using noncongruent regions, and this does not provide a basis for comparing the shaded regions.

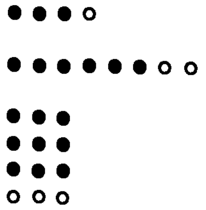
Exercise Set 2, pp. 16–18

1. **a.** Three-fourths, (3,4), $\frac{3}{4}$

b. Two-thirds, (2,3), $\frac{2}{3}$

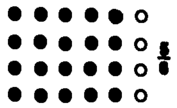
c. Five-sevenths, (5,7), $\frac{5}{7}$

2.

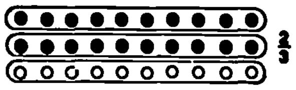


For every four dots drawn, there must be three dots shaded. Any multiple of (3,4) can be used to produce a representation of $\frac{3}{4}$.

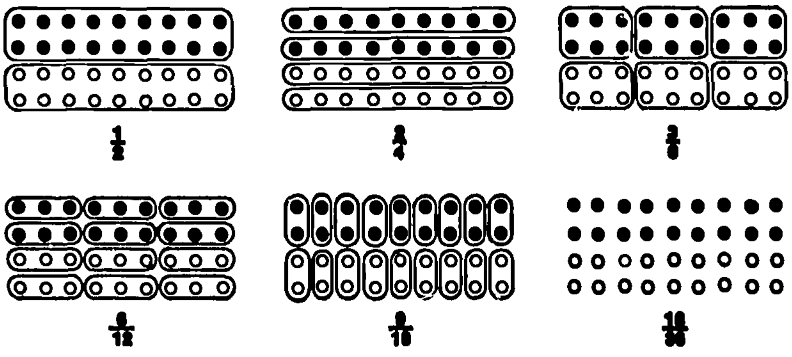
3.



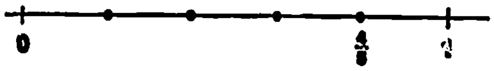
4.



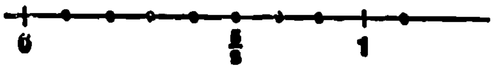
5.



6. a.



b.



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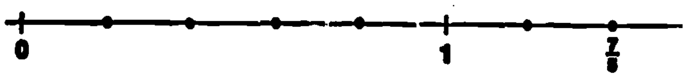

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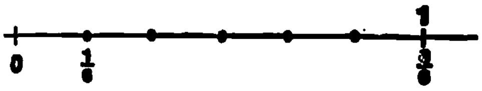
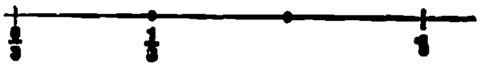
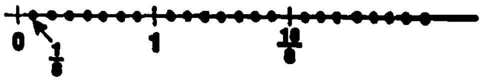


e.



f.



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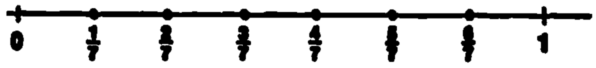
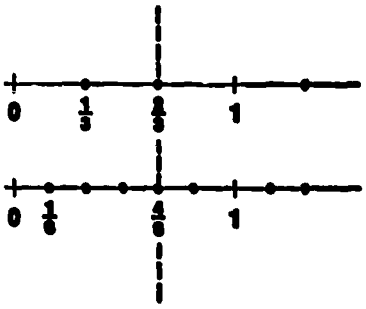
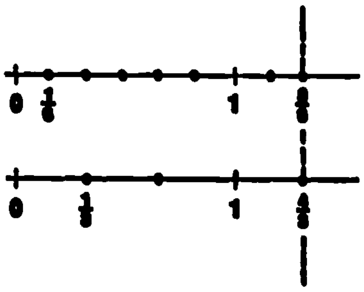


8. a.



b.



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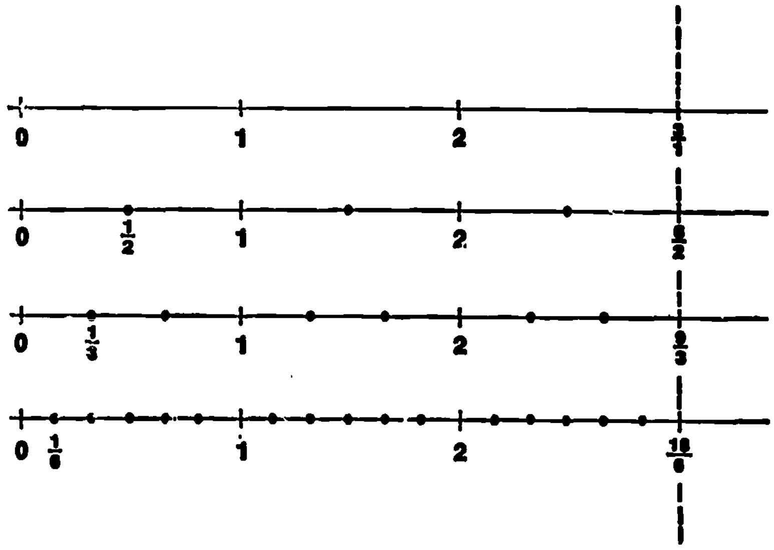
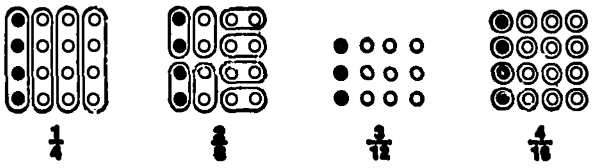


9.



Note that $\frac{3}{12}$ cannot be shown with a set of 16 objects, since 12 is not a factor of 16.

10. **a.** $\frac{3}{4}$ appears to the right of $\frac{2}{4}$ on the number line.

b. $\frac{1}{4}$ appears to the left of $\frac{1}{2}$ on the number line.

c. $\frac{11}{4}$ appears to the right of $\frac{5}{2}$ on the number line.

d. 1 can also be written as a fraction $\frac{a}{b}$ where a and b are whole numbers and $b \neq 0$.

e. All three are represented by the same point on the number line, 0.

f. $\frac{4}{4}$ and $\frac{2}{2}$ correspond to the same point on the number line.

g. Every whole number is a rational number, and therefore the set of whole numbers is a subset of the set of rational numbers.

h. As illustrated by the number line, there are many rational numbers between consecutive whole numbers.

FRACTIONS AND RATIONAL NUMBERS

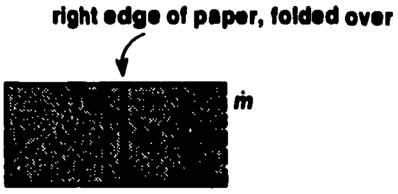
Exercise Set 1, pp. 33–34

1. The introduction of the rational number $\frac{a}{b}$ as an address for the point $Q_{a,b}$ lends itself more readily to the application of sharing a pie. The pie corresponds to the unit interval on the number line; dividing the pie fairly among b children is analogous to dividing the unit interval into b parts of equal length, such as is done in locating the point $Q_{a,b}$. Finally, if a of the b children are girls, then the part of the pie distributed to girls has measure $\frac{a}{b}$; we obtain this by counting off a of the b parts, again as is done in locating the point $Q_{a,b}$. The location of the point $P_{a,b}$ on the number line would be more analogous to the problem of dividing a pies among b children—a situation rarely encountered in practice, except for the special cases $a = 0$ and $a = 1$!

2. The use of the point $P_{a,b}$ to define the rational number $\frac{a}{b}$ lends itself naturally to an explanation of the fact that certain rational numbers $\frac{a}{b}$ coincide with whole numbers. For example, to locate the point $P_{6,3}$ we must divide the interval from point 0 to point 6 into 3 parts of equal length. Well, it is obvious that the division points come at point 2 and point 4. By definition, the first of these division points is the point $P_{6,3}$. Thus $P_{6,3}$ coincides with point 2. Since the address of $P_{6,3}$ is the rational number $\frac{6}{3}$, we thus get $\frac{6}{3} = 2$. Of course the same conclusion can be reached by considering point $Q_{6,3}$ instead of $P_{6,3}$, but the reasoning is a little less direct.

3. **a.** We refold the paper rectangle along the crease m. If the right edge of the paper then coincides with the crease q, the area to the right of m is exactly one-third of the area of the rectangle. If the right edge of the paper is to the right of the crease q, then the area to the right of m is less than one-third of the total area. Finally, if the right edge of the paper is to the left of q, the area to the right of m is more than one-third.

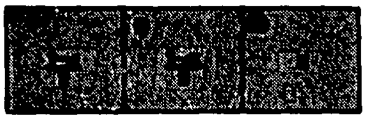
b. Suppose that when we apply the test in **a** above we find that the area to the right of m is less than one-third the total area. This means that when we refold the paper along crease m, the right edge of the paper lies to the right of the crease q.



What we do, then, is to pull the right edge over toward the left so that it coincides with the crease q, then make a new crease where the paper is folded at the right. If we now open the paper and label the new crease n, we shall find that n is to the left of m.

The area to the right of n will still be less than one-third the total area, but of course it is greater than the area to the right of m, so that we have a better approximation to one-third. Of course the whole process can be repeated to obtain a still better approximation.

The facts described above can be ascertained with a very little algebra. Suppose that the fraction of the total area that lies to the right of crease m is x. Then the fraction to the left of m is $1 - x$. Since the crease q divides the area to the left of m in half, the fraction of the total area to the left of q will be $\frac{1-x}{2}$.

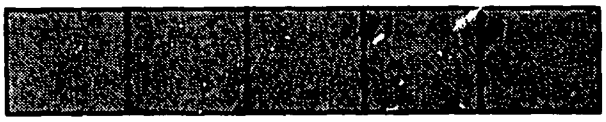


The fraction of the total area lying to the right of the crease q will be

$$\frac{1-x}{2} + x = \frac{1-x}{2} + \frac{2x}{2} = \frac{1+x}{2}.$$

This is the area we divide in half when we form the new crease, n. Hence the fraction of the total area that lies to the right of n will be one-half of $\frac{1+x}{2}$, or $\frac{1+x}{4}$. Since we assumed to begin with that $x < \frac{1}{3}$, we find $1 + x < \frac{4}{3}$ and hence $\frac{1+x}{4} < \frac{1}{3}$. Thus the area to the right of crease n is still less than one-third the total area, as claimed above.

c. Start by putting a crease, say p, at an estimated point so that the area to the right of p is approximately one-fifth the total area. Then divide the area to the left of p into four equal parts by folding the left edge over to coincide with p, then folding a second time (without opening the paper after the first fold). When the paper is opened out, there will now be 3 creases to the left of p; label the one closest to p with the letter r.



Now if we refold the paper along p, we can see whether the right edge coincides with, is to the left of, or is to the right of r. This determines whether the area to the right of p is exactly, a little more, or a little less than one-fifth the total area. Suppose we are a bit off. By pulling the right edge to coincide with r and making a new fold near p, we can improve our estimate.

Exercise Set 2, pp. 39–40

1. There are many possible correct answers to each part of this problem. Examples:

- **a.** Identifying the President by name or as "the husband of the wife of the President of the United States"
- **b.** "The largest ocean" or "the ocean at the west coast of the United States"
- **c.** "The set of whole numbers between 1 and 4" or "the set of those numbers x such that $x^2 - 5x + 6 = 0$"
- **d.** "The father of Abraham Lincoln's mother" or "the maternal grandfather of the sixteenth president of the United States"
- **e.** "The National Council of Teachers of Mathematics, Inc." or "the teachers' organization that published this book"

2. Problem 1. A 5-pound object is divided into 3 parts of equal weight. How much does each part weigh?

Problem 2. A 10-pound object is divided into 6 equal parts. How much does each part weigh?

Without computing the solution to problem (1), let us use the letter "x" to stand for the weight of any of the 3 parts of the 5-pound object. Obviously, then, we have

$$x + x + x = 5.$$

From this, by the logic of identity, we can infer that

$$(x + x + x) + (x + x + x) = 5 + 5,$$

so that

$$x + x + x + x + x + x = 10.$$

Since "x" occurs six times in the left side of the last equation, x represents the weight of each of the 6 parts mentioned in problem 2. Thus problems 1 and 2 have been shown to have the same solution.

Exercise Set 3, p. 45

1. To show that $\frac{6}{5} > \frac{4}{5}$ we must show that the point $Q_{6,5}$ is to the right of the point $Q_{4,5}$ on the number line. After dividing each unit interval on the number line into 5 parts of equal length, we count off 4 of these little intervals (starting from 0) to get $Q_{4,5}$, and we count off 6 of these little intervals (starting from 0) to get $Q_{6,5}$. Since $6 > 4$, this shows that $Q_{6,5}$ will lie to the right of $Q_{4,5}$; hence $\frac{6}{5} > \frac{4}{5}$.

2. According to the teacher's definition, $\frac{7}{3}$ is the number satisfying the open sentence $3 \cdot \square = 7$, that is, we have $3 \times \frac{7}{3} = 7$. Similarly, we have $3 \times \frac{10}{3} = 10$. It follows, by the logic of equality, that

(1) $$\left(3 \times \frac{7}{3}\right) + \left(3 \times \frac{10}{3}\right) = 7 + 10.$$

But by the distributive law, which we assume to hold for rational numbers here, we know that

(2) $$\left(3 \times \frac{7}{3}\right) + \left(3 \times \frac{10}{3}\right) = 3 \times \left(\frac{7}{3} + \frac{10}{3}\right).$$

Combining the last two equations, we get

(3) $$3 \times \left(\frac{7}{3} + \frac{10}{3}\right) = 17.$$

Now by the teacher's definition of rational numbers, $\frac{17}{3}$ is defined to be the number satisfying the open sentence $3 \cdot \square = 17$. Comparing this with equation (3) derived above, we see that $\frac{7}{3} + \frac{10}{3} = \frac{17}{3}$.

3. To get the point $P_{7,4}$ on the number line, we divide the interval from point 0 to point 7 into 4 parts of equal length; then $P_{7,4}$ is the right endpoint of the first of these parts. To get $P_{7,3}$, we take the same interval, from point 0 to point 7, and divide it into 3 parts of equal length; then $P_{7,3}$ is the right endpoint of the first of *these* parts. Obviously if we divide a given interval first into 4 equal parts and then into 3 equal parts, the former parts will be shorter than the latter parts. In particular the first interval of the division into 4 parts is shorter than the first interval of the division into 3 parts; but both these intervals begin at point 0, so the right endpoint $P_{7,4}$ of the former will be to the left of the right endpoint $P_{7,3}$ of the latter. Hence $\frac{7}{4} < \frac{7}{3}$.

Exercise Set 4, pp. 53–54

1. $\frac{3}{17} = \frac{6}{34}$. Since $29 < 34$, we get $\frac{6}{29} > \frac{6}{34}$. Hence $\frac{6}{29} > \frac{3}{17}$.

2. $\frac{3}{2} = \frac{45}{30}, \frac{5}{3} = \frac{50}{30}, \frac{7}{5} = \frac{42}{30}$. Hence $\frac{7}{5} < \frac{3}{2}$ and $\frac{3}{2} < \frac{5}{3}$.

3. **a.** We know that

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}.$$

Hence we want to find all rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ such that

$$\frac{a + c}{b + d} = \frac{a \cdot d + b \cdot c}{b \cdot d}.$$

By "cross-multiplying" we get

$$(a \cdot b \cdot d) + (c \cdot b \cdot d) = (a \cdot d \cdot b) + (b^2 \cdot c) + (a \cdot d^2) + (b \cdot c \cdot d).$$

Thus we must have

$$0 = (b^2 \cdot c) + (a \cdot d^2).$$

Since the denominators b and d are different from 0, we must have $c = 0$ and $a = 0$. Thus $\frac{a}{b} = 0$ and $\frac{c}{d} = 0$—these are the only rational numbers for which the "simple-minded" addition rule $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ is true.

b. Now suppose we do *not* have both $a = 0$ and $b = 0$, so that $\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$. Then which is larger, $\frac{a}{b} + \frac{c}{d}$ or $\frac{a+c}{b+d}$? Since $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + d \cdot c}{b \cdot b}$, we must *compare* the fractions

$$\frac{a \cdot d + b \cdot c}{b \cdot d} \quad \text{and} \quad \frac{a+c}{b+d}.$$

To do this, we express these by equivalent fractions having a common denominator:

$$(1) \quad \frac{a \cdot d + b \cdot c}{b \cdot d} = \frac{(a \cdot d + b \cdot c) \cdot (b + d)}{(b \cdot d) \cdot (b + d)}$$

$$= \frac{(a \cdot b \cdot d) + (b^2 \cdot c) + (a \cdot d^2) + (b \cdot c \cdot d)}{(b \cdot d) \cdot (b + d)}.$$

$$(2) \quad \frac{a}{b} + \frac{c}{d} = \frac{(a + c) \cdot (b \cdot d)}{(b + d) \cdot (b \cdot d)}$$

$$= \frac{(a \cdot b \cdot d) + (b \cdot c \cdot d)}{(b \cdot d) \cdot (b + d)}.$$

Now, comparing the numerators of the two right-hand fractions having a common denominator, we see that the numerator in equation (1) is greater than in equation (2). Hence

$$\frac{a}{b} + \frac{c}{d} > \frac{a+c}{b+d}$$

in this case.

4. We have $\frac{408}{255} = \frac{136}{85} = \frac{8}{5}$. Hence the list of all fractions equivalent to "$\frac{408}{255}$" is

$$\frac{8}{5}, \frac{16}{10}, \frac{24}{15}, \frac{32}{20}, \ldots, \frac{408}{255}, \ldots.$$

Suppose that "$\frac{a}{b}$" is *any* fraction equivalent to "$\frac{408}{255}$." Then, representing these by fractions with a common denominator, we see that

$$\text{“}\frac{408 \times b}{255 \times b}\text{”}$$

will be equivalent to

$$\text{“}\frac{a \times 255}{b \times 255}.\text{”}$$

But two fractions with the same denominator can be equivalent only if they have the same numerator (by our rule for comparing fractions with a common denominator). Hence

$$408 \times b = 255 \times a.$$

Since $408 = 51 \times 8$ and $255 = 51 \times 5$, we get

$$8 \times b = 5 \times a.$$

From this we can infer that 5 is a factor of b and 8 is a factor of a; so “$\frac{a}{b}$” is in the list

$\frac{8}{5}, \frac{16}{10}, \frac{24}{15}, \frac{32}{20}, \ldots.$

5. We get 16 fractions:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}; \quad \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}; \quad \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}; \quad \frac{4}{1}, \frac{4}{2}, \frac{4}{3}, \frac{4}{4}.$$

Using the common denominator 12, we express these as

$$\frac{12}{12}, \frac{6}{12}, \frac{4}{12}, \frac{3}{12}; \quad \frac{24}{12}, \frac{12}{12}, \frac{8}{12}, \frac{6}{12};$$

$$\frac{36}{12}, \frac{18}{12}, \frac{12}{12}, \frac{9}{12}; \quad \frac{48}{12}, \frac{24}{12}, \frac{16}{12}, \frac{12}{12}.$$

Counting the number of distinct numerators in this list, we see that we have altogether 11 distinct rational numbers. In order of magnitude these numbers are:

$$\frac{3}{12}, \frac{4}{12}, \frac{6}{12}, \frac{8}{12}, \frac{9}{12}, \frac{12}{12}, \frac{16}{12}, \frac{18}{12}, \frac{24}{12}, \frac{36}{12}, \frac{48}{12},$$

or

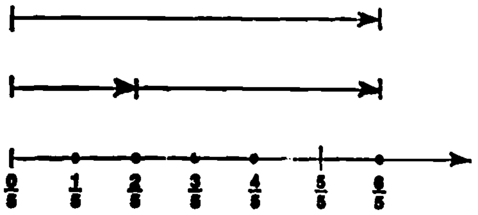
$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}.$$

ADDITION OF RATIONAL NUMBERS

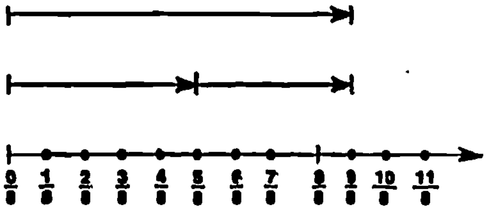
Exercise Set 1, pp. 56–57

1. a. $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$ **b.** $\frac{4}{5} + \frac{7}{5} = \frac{11}{5}.$

2. a.



b.



Exercise Set 2, pp. 59–60

1. a. $\frac{7}{10} + \frac{9}{10} = \frac{16}{10} = \frac{10}{10} + \frac{6}{10} = 1 + \frac{6}{10} = 1 + \frac{3}{5} = 1\frac{3}{5}.$

b. $\frac{3}{4} + \frac{3}{4} = \frac{6}{4} = \frac{4}{4} + \frac{2}{4} = 1 + \frac{2}{4} = 1 + \frac{1}{2} = 1\frac{1}{2}.$

2. a. $\frac{5}{6} + \frac{4}{6} = \frac{9}{6} = \frac{6}{6} + \frac{3}{6} = 1 + \frac{3}{6} = 1 + \frac{1}{2} = 1\frac{1}{2}.$

b. $\frac{57}{100} + \frac{88}{100} = \frac{145}{100} = \frac{100}{100} + \frac{45}{100} = 1 + \frac{45}{100} = 1 + \frac{9}{20} = 1\frac{9}{20}.$

Exercise Set 3, pp. 65–67

1. a. $\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}.$

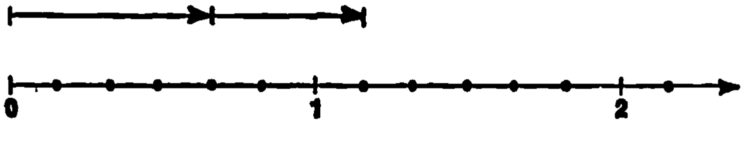
b. $\frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}.$

c. $\frac{2}{5} + \frac{1}{2} = \frac{4}{10} + \frac{5}{10} = \frac{9}{10}.$

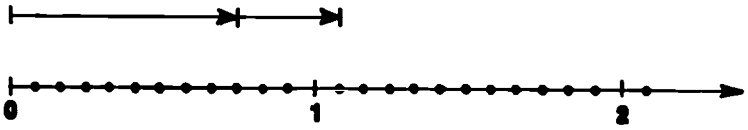
d. $\frac{3}{4} + \frac{3}{8} = \frac{6}{8} + \frac{3}{8} = \frac{9}{8}.$

e. $1\frac{2}{3} + \frac{1}{4} = 1\frac{8}{12} + \frac{3}{12} = 1\frac{11}{12}.$

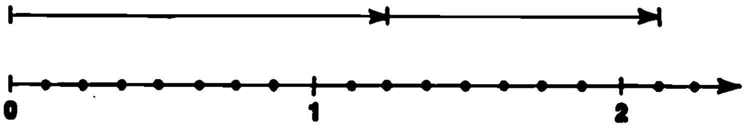
2. a. $\frac{2}{3} + \frac{1}{2} = 1\frac{1}{6}.$



b. $\frac{3}{4} + \frac{1}{3} = 1\frac{1}{12}$.



c. $1\frac{1}{4} + \frac{7}{8} = 2\frac{1}{8}$.



3. a. $1\frac{1}{4} + 1 = 2\frac{1}{4}$. **c.** $1\frac{1}{4} + \frac{7}{8} = 2\frac{1}{8}$.

b. $1\frac{1}{4} + \frac{0}{4} = 1\frac{1}{4}$. **d.** $1\frac{2}{8} + 1\frac{6}{8} = 3$.

Exercise Set 4, pp. 67–68

1. a. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$. **c.** $\frac{5}{4} = \frac{10}{8} = \frac{25}{20} = \frac{125}{100}$.

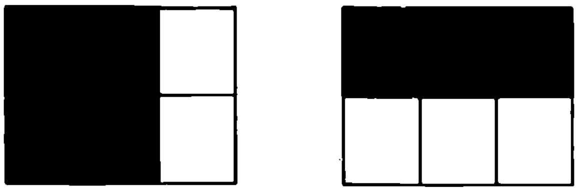
b. $\frac{7}{8} = \frac{14}{16} = \frac{21}{24} = \frac{28}{32}$. **d.** $6\frac{3}{10} = 6\frac{6}{20} = 6\frac{15}{50}$.

2. a. *F*, $\frac{5}{10}$ **b.** *T* **c.** *F*, $\frac{45}{40}$ or $\frac{54}{48}$ **d.** *F*, $2\frac{9}{12}$

3. a. $\begin{array}{r} \frac{3}{4} \\ +\frac{2}{4} \\ \hline \frac{5}{4} \end{array}$ **b.** $\begin{array}{r} 1\frac{2}{8} \\ 1\frac{7}{8} \\ \hline 2\frac{9}{8} = 3\frac{1}{8} \end{array}$ **c.** $\begin{array}{r} 4\frac{30}{100} \\ 3\frac{70}{100} \\ \hline 7\frac{100}{100} = 8 \end{array}$ **d.** $\begin{array}{r} \frac{8}{20} \\ \frac{15}{20} \\ \hline \frac{23}{20} \end{array}$

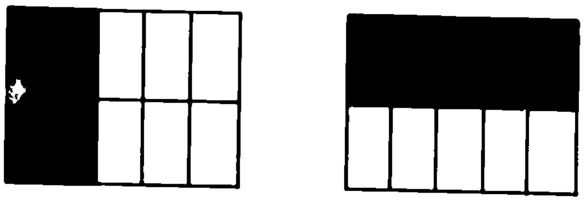
Exercise Set 5, p. 72

1. a.



$\frac{2 \times 2}{3 \times 2} + \frac{1 \times 3}{2 \times 3}$

b.



$\frac{2 \times 3}{8 \times 3}$ + $\frac{1 \times 8}{3 \times 8}$

(Other diagrams are possible.)

2. a.
$$\begin{array}{r} \frac{3}{8} = \frac{36}{96} \\ \underline{\frac{5}{12} = \frac{40}{96}} \\ \frac{76}{96} \end{array}$$

b.
$$\begin{array}{r} \frac{13}{30} = \frac{234}{540} \\ \underline{\frac{7}{18} = \frac{210}{540}} \\ \frac{444}{540} \end{array}$$

Exercise Set 6, p. 75

1. a. $2 \times 2 \times 3$ **b.** $2 \times 2 \times 7$ **c.** $2 \times 5 \times 7$ **d.** $2 \times 2 \times 5 \times 5$

2. a.
$$\left.\begin{array}{r} 6 \rightarrow 2 \times 3 \\ 15 \rightarrow 3 \times 5 \end{array}\right\} 2 \times 3 \times 5 \rightarrow 30$$

b.
$$\left.\begin{array}{r} 15 \rightarrow 3 \times 5 \\ 35 \rightarrow 5 \times 7 \end{array}\right\} 3 \times 5 \times 7 \rightarrow 105$$

c.
$$\left.\begin{array}{r} 16 \rightarrow 2 \times 2 \times 2 \times 2 \\ 24 \rightarrow 2 \times 2 \times 2 \times 3 \end{array}\right\} 2 \times 2 \times 2 \times 2 \times 3 \rightarrow 48$$

d.
$$\left.\begin{array}{r} 28 \rightarrow 2 \times 2 \times 7 \\ 42 \rightarrow 2 \times 3 \times 7 \end{array}\right\} 2 \times 2 \times 3 \times 7 \rightarrow 84$$

3. a.
$$\begin{array}{r} \frac{5}{6} = \frac{50}{60} \\ \underline{\frac{7}{10} = \frac{42}{60}} \\ \frac{92}{60} \end{array}$$

b.
$$\begin{array}{r} \frac{11}{42} = \frac{528}{2,016} \\ \underline{\frac{37}{48} = \frac{1,554}{2,016}} \\ \frac{2,082}{2,016} \end{array}$$

c.
$$\begin{array}{r} \frac{1}{6} = \frac{80}{480} \\ \frac{3}{8} = \frac{180}{480} \\ \underline{\frac{3}{10} = \frac{144}{480}} \\ \frac{404}{480} \end{array}$$

4\. a.
$$\begin{array}{r} \frac{5}{6} = \frac{25}{30} \\ \frac{7}{10} = \frac{21}{30} \\ \hline \frac{46}{30} \end{array}$$

b.
$$\begin{array}{r} \frac{11}{42} = \frac{88}{336} \\ \frac{37}{48} = \frac{259}{336} \\ \hline \frac{347}{336} \end{array}$$

c.
$$\begin{array}{r} \frac{1}{6} = \frac{20}{120} \\ \frac{3}{8} = \frac{45}{120} \\ \frac{3}{10} = \frac{36}{120} \\ \hline \frac{101}{120} \end{array}$$

5\. Same as 4, except that the LCM is arrived at as shown.

a.
$$\frac{5}{6} = \frac{5 \times 5}{2 \times 3 \times 5}$$
$$\frac{7}{10} = \frac{7 \times 3}{2 \times 5 \times 3}$$

b.
$$\frac{11}{42} = \frac{11 \times 2 \times 2 \times 2}{2 \times 3 \times 7 \times 2 \times 2 \times 2}$$
$$\frac{37}{48} = \frac{37 \times 7}{2 \times 2 \times 2 \times 2 \times 3 \times 7}$$

c.
$$\frac{1}{6} = \frac{1 \times 2 \times 2 \times 5}{2 \times 3 \times 2 \times 2 \times 5}$$
$$\frac{3}{8} = \frac{3 \times 3 \times 5}{2 \times 2 \times 2 \times 3 \times 5}$$
$$\frac{3}{10} = \frac{3 \times 2 \times 2 \times 3}{2 \times 5 \times 2 \times 2 \times 3}$$

6\. Personal choice. The prime-factorization method is shown.

$$\begin{array}{r} \frac{5}{28} = \frac{5 \times 3 \times 5}{2 \times 2 \times 7 \times 3 \times 5} = \frac{75}{420} \\ \frac{7}{30} = \frac{7 \times 2 \times 7}{2 \times 3 \times 5 \times 2 \times 7} = \frac{98}{420} \\ \frac{9}{35} = \frac{9 \times 2 \times 2 \times 3}{5 \times 7 \times 2 \times 2 \times 3} = \frac{108}{420} \\ \hline \frac{281}{420} \end{array}$$

Exercise Set 7, pp. 81–82

1. a. Addition is associative.
   b. Addition is commutative.
   c. Addition is commutative.
   d. Addition is commutative and associative (rearrangement).

2\. a. $\frac{2}{3} + \left(\frac{93}{100} + \frac{7}{100}\right) = 1\frac{2}{3}$.

b. $\left(\frac{5}{100} + \frac{95}{100}\right) + \frac{247}{100} = 3\frac{47}{100}$.

c. $\left(\frac{9}{10} + \frac{1}{10}\right) + \left(\frac{7}{8} + \frac{1}{8}\right) = 2$.

d. $\left(\frac{13}{8} + \frac{11}{8}\right) + \left(\frac{13}{5} + \frac{12}{5}\right) = 3 + 5 = 8$.

3\. a, b. Addition is commutative.
c, d. Addition is associative.

Exercise Set 8, p. 84

1\. a. $1\frac{7}{5} = 1 + \frac{7}{5} = \frac{5}{5} + \frac{7}{5} = \frac{5+7}{5} = \frac{12}{5}$.

b. $\frac{49}{4} = \frac{48+1}{4} = \frac{48}{4} + \frac{1}{4} = 12 + \frac{1}{4} = 12\frac{1}{4}$.

2\. a. $4 + \frac{2}{3} + 5 + \frac{2}{3}$ by agreement on mixed numerals;

b. $4 + 5 + \frac{2}{3} + \frac{2}{3}$ by rearrangement principle for addition;

c. $9 + \frac{4}{3}$ by addition $(4 + 5 = 9, \frac{2}{3} + \frac{2}{3} = \frac{4}{3})$;

d. $9 + \frac{3}{3} + \frac{1}{3}$ by addition $\left(\frac{4}{3} = 3 + \frac{1}{3} = \frac{3}{3} + \frac{1}{3}\right)$;

e. $9 + 1 + \frac{1}{3}$ because $\frac{3}{3} = 1$;

f. $10 + \frac{1}{3}$ because addition is associative and $9 + 1 = 10$;

g. $10\frac{1}{3}$ by agreement on mixed numerals.

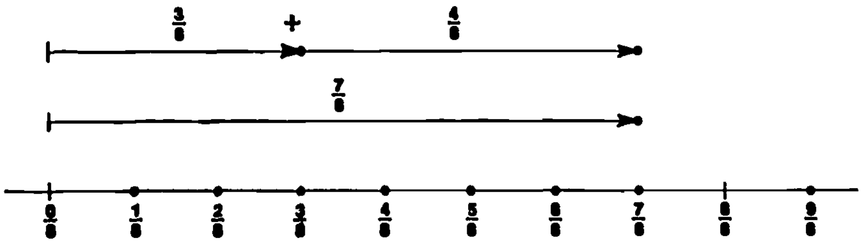
SUBTRACTION OF RATIONAL NUMBERS

Exercise Set 1, pp. 89–90

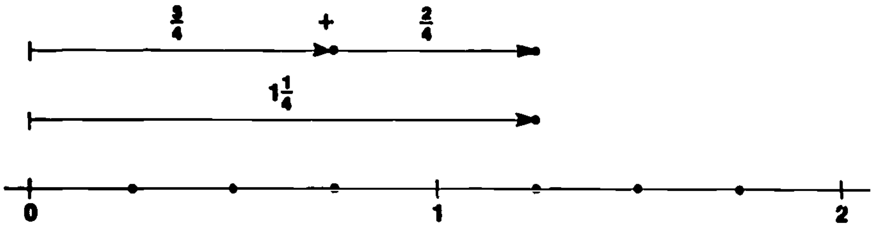
1\. a. $n = \frac{10}{3} - \frac{2}{3}$. c. $n = \frac{4}{5} - \frac{1}{5}$.

b. $n = \frac{12}{8} - \frac{7}{8}$. d. $n = \frac{9}{10} - \frac{4}{5}$.

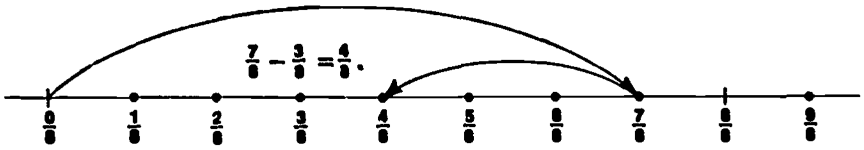
2\. a.



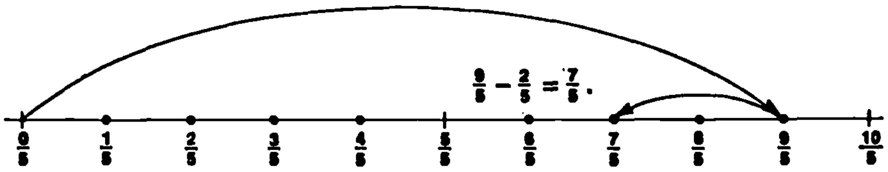
b.



3\. a.



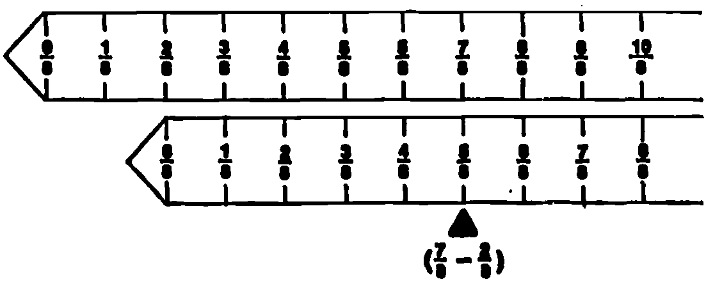
b.



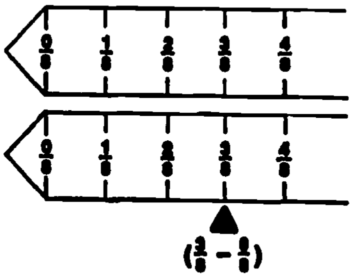
4\. a. Addend b. Addend c. Sum d. Sum

Exercise Set 2, pp. 92–93

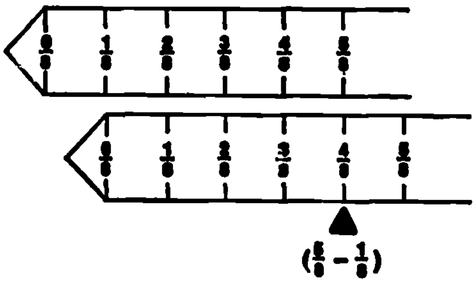
1\. a.



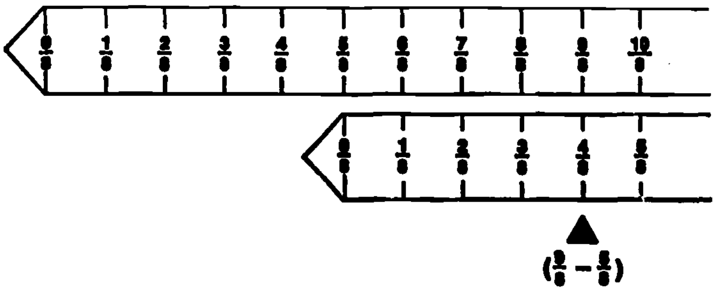
b.



c.



d.



2. a, b, d

3. a. $\square = \frac{5}{3} - \frac{1}{3}.$ c. $\square = \frac{5}{8} - \frac{3}{8}.$

b. $\square = \frac{11}{10} - \frac{2}{10}.$ d. $\square = \frac{100}{100} - \frac{14}{100}.$

4. a. $\frac{5}{5} + \square = \frac{8}{5}.$ c. $\frac{5}{10} + \square = \frac{8}{10}.$

b. $\frac{5}{16} + \square = \frac{11}{16}.$ d. $\frac{67}{100} + \square = \frac{99}{100}.$

Exercise Set 3, pp. 94–95

1. a. $\frac{3}{4} - \frac{2}{4} = \frac{1}{4}.$ c. $\frac{5}{6} - \frac{1}{6} = \frac{4}{6}$, or $\frac{2}{3}.$

   b. $\frac{3}{8} - \frac{3}{8} = \frac{0}{8}$, or 0. d. $\frac{1}{2} - \frac{0}{2} = \frac{1}{2}\left(\text{or } \frac{1}{2} - 0 = \frac{1}{2}\right).$

2. a. $\frac{13}{10} - \frac{7}{10} = \frac{13 - 7}{10} = \frac{6}{10}.$

   b. $\frac{15}{16} - \frac{4}{16} = \frac{15 - 4}{16} = \frac{11}{16}.$

   c. $\frac{74}{100} - \frac{15}{100} = \frac{74 - 15}{100} = \frac{59}{100}.$

   d. $1\frac{1}{4} - \frac{2}{4} = \frac{5}{4} - \frac{2}{4} = \frac{5 - 2}{4} = \frac{3}{4}.$

Exercise Set 4, pp. 96–97

1. a. $\frac{2}{3} - \frac{1}{2} = \frac{2 \times 2}{3 \times 2} - \frac{3 \times 1}{3 \times 2}$

   $$= \frac{(2 \times 2) - (3 \times 1)}{3 \times 2} = \frac{4 - 3}{6} = \frac{1}{6}.$$

   b. $\frac{3}{4} - \frac{3}{5} = \frac{3 \times 5}{4 \times 5} - \frac{4 \times 3}{4 \times 5}$

   $$= \frac{(3 \times 5) - (4 \times 3)}{4 \times 5} = \frac{15 - 12}{20} = \frac{3}{20}.$$

   c. $\frac{9}{10} - \frac{2}{3} = \frac{9 \times 3}{10 \times 3} - \frac{10 \times 2}{10 \times 3}$

   $$= \frac{(9 \times 3) - (10 \times 2)}{10 \times 3} = \frac{27 - 20}{30} = \frac{7}{30}.$$

2. a.
$$\begin{array}{rcl} \frac{7}{30} & = & \frac{168}{720} \\ -\frac{5}{24} & = & \frac{150}{720} \\ \hline & & \frac{18}{720} = \frac{1}{40} \end{array}$$

   b.
$$\begin{array}{rcl} \frac{7}{30} & = & \frac{28}{120} \\ -\frac{5}{24} & = & \frac{25}{120} \\ \hline & & \frac{3}{120} = \frac{1}{40} \end{array}$$

3. a. $\frac{2}{10}$, or $\frac{1}{5}$ c. $\frac{17}{72}$

b. $\frac{51}{100}$ d. $\frac{47}{160}$

Exercise Set 5, pp. 102–3

1. a.
$$\begin{array}{r} 3\frac{1}{2} = \frac{7}{2} = \frac{21}{6} \\ -\ 1\frac{2}{3} = \frac{5}{3} = \frac{10}{6} \\ \hline \frac{11}{6}, \text{ or } 1\frac{5}{6} \end{array}$$

b.
$$\begin{array}{r} 27\frac{3}{16} = \frac{435}{16} = \frac{435}{16} \\ -\ 19\ \frac{3}{8} = \frac{155}{8} = \frac{310}{16} \\ \hline \frac{125}{16}, \text{ or } 7\frac{13}{16} \end{array}$$

2. a.
$$\begin{array}{r} 3\frac{1}{2} = 3\frac{3}{6} = 2\frac{9}{6} \\ -\ 1\frac{2}{3} = 1\frac{4}{6} = 1\frac{4}{6} \\ \hline 1\frac{5}{6} \end{array}$$

b.
$$\begin{array}{r} 27\frac{3}{16} = 27\frac{3}{16} = 26\frac{19}{16} \\ -\ 19\ \frac{3}{8} = 19\frac{6}{16} = 19\frac{6}{16} \\ \hline 7\frac{13}{16} \end{array}$$

3. a. Probably subtracted numerator 1 from whole number 8. Correct result is $7\frac{9}{10}$.

b. Changed 1 to $\frac{10}{3}$ instead of $\frac{3}{3}$. Correct result: $2\frac{2}{3}$.

c. Forgot to compute 5 − 3. Correct result: $2\frac{7}{100}$.

d. Computed $\frac{4}{8} + \frac{3}{8}$ instead of $\frac{4}{8} - \frac{3}{8}$. Correct result: $2\frac{1}{8}$.

e. Computed $\frac{7}{8} - \frac{3}{8}$ instead of changing $12\frac{3}{8}$ to $11 + \frac{8}{8} + \frac{3}{8}$. Correct result: $10\frac{4}{8}$, or $10\frac{1}{2}$.

f. $2\frac{5}{8} = 2\frac{10}{16}$, not $2\frac{5}{16}$. Correct result: $7\frac{9}{16}$.

Exercise Set 6, pp. 104–6

1. a. $4\frac{1}{3} - 1\frac{2}{3} = \left(4\frac{1}{3} + \frac{1}{3}\right) - \left(1\frac{2}{3} + \frac{1}{3}\right) = 4\frac{2}{3} - 2 = 2\frac{2}{3}$.

b. $9\frac{3}{10} - 5\frac{6}{10} = \left(9\frac{3}{10} + \frac{4}{10}\right) - \left(5\frac{6}{10} + \frac{4}{10}\right)$

$$= 9\frac{7}{10} - 6 = 3\frac{7}{10}.$$

c. $6\frac{1}{4} - 2\frac{3}{4} = \left(6\frac{1}{4} + \frac{1}{4}\right) - \left(2\frac{3}{4} + \frac{1}{4}\right)$

$= 6\frac{2}{4} - 3 = 3\frac{2}{4}$, or $3\frac{1}{2}$.

d. $88\frac{16}{100} - 27\frac{97}{100} = \left(88\frac{16}{100} + \frac{3}{100}\right) + \left(27\frac{97}{100} + \frac{3}{100}\right)$

$= 88\frac{19}{100} - 28 = 60\frac{19}{100}$.

2. $7\frac{17}{100} - 4\frac{84}{100} = \left(7\frac{17}{100} + 1\right) - \left(4\frac{84}{100} + 1\right)$

$= \left(7\frac{17}{100} + \frac{100}{100}\right) - \left(4 + \frac{84}{100} + 1\right)$

$= 7\frac{117}{100} - 5\frac{84}{100}$.

3. a. $$\begin{array}{r} \overset{5}{\cancel{6}}\,\frac{\overset{3}{\cancel{1}}}{4} \\ -1\,\frac{\cancel{2}}{\cancel{4}} \\ \hline 4\,\frac{3}{4} \end{array}$$

c. $$\begin{array}{r} \cancel{1}\,\frac{\overset{8}{\cancel{4}}}{16} \\ -\ \frac{\cancel{15}}{\cancel{16}} \\ \hline \frac{5}{16} \end{array}$$

b. $$\begin{array}{r} \overset{6}{\cancel{7}}\,\frac{\overset{4}{\cancel{3}}}{8} \\ -4\,\frac{\cancel{7}}{\cancel{8}} \\ \hline 2\,\frac{4}{8} \end{array}$$, or $2\frac{1}{2}$

d. $$\begin{array}{r} \overset{8}{\cancel{9}}\,\frac{7}{12} \\ -2\,\frac{\cancel{5}}{\cancel{12}} \\ \hline 6\,\frac{7}{12} \end{array}$$

MULTIPLICATION OF RATIONAL NUMBERS

Exercise Set 1, pp. 115–19

1. a. 5 columns. Product = 3×5.

   b. 5 columns. Product = $3\frac{1}{2} \times 5$. Lower estimate = 3×5.

   Upper estimate = 4×5. $3 \times 5 < 3\frac{1}{2} \times 5 < 4 \times 5$.

   c. $3\frac{2}{3}$ rows, $4\frac{1}{2}$ columns. Product = $3\frac{2}{3} \times 4\frac{1}{2}$. Lower estimate = 3×4.

Upper estimate = 4×5. $3 \times 4 < 3\frac{2}{3} \times 4\frac{1}{2} < 4 \times 5$.

d. $3\frac{2}{3}$ rows, $4\frac{2}{3}$ columns. Product = $3\frac{2}{3} \times 4\frac{2}{3}$. Lower estimate = 3×4.

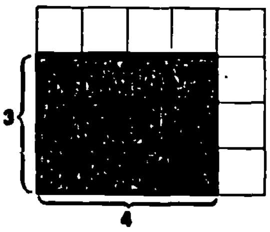
Upper estimate = 4×5. $3 \times 4 < 3\frac{2}{3} \times 4\frac{2}{3} < 4 \times 5$.

e. $\frac{2}{3}$ row, $\frac{1}{2}$ column. Product = $\frac{2}{3} \times \frac{1}{2}$.

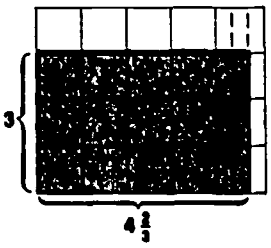
f. $\frac{2}{3}$ row, $\frac{3}{4}$ column. Product = $\frac{2}{3} \times \frac{3}{4}$.

g. $\frac{5}{6}$ row, $\frac{2}{3}$ column. Product = $\frac{5}{6} \times \frac{2}{3}$.

2. a.



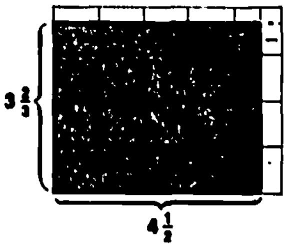
b.



Lower estimate = 3×4.
Upper estimate = 3×5.

$3 \times 4 < 3 \times 4\frac{2}{3} < 3 \times 5$.

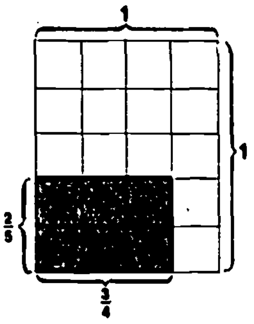
c.



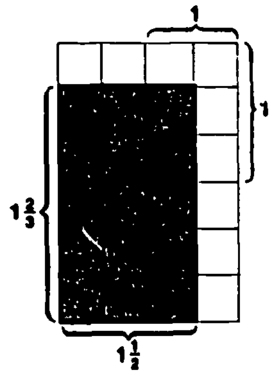
Lower estimate = 3×4.
Upper estimate = 4×5.

$3 \times 4 < 3\frac{2}{3} \times 4\frac{1}{2} < 4 \times 5$.

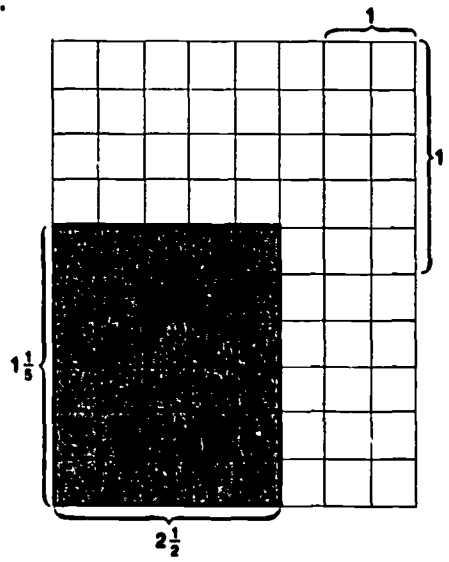
d.



e.

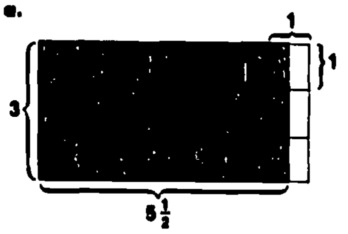


f.



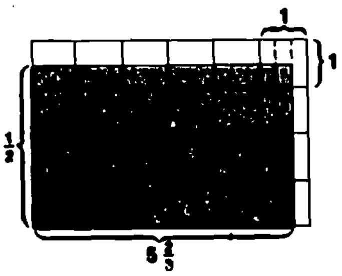
$\frac{1}{10}$ for each small box because 10 boxes make up the unit rectangular region.

3. a.

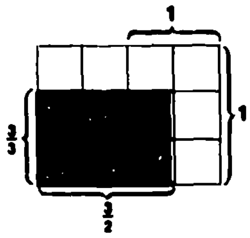


b.

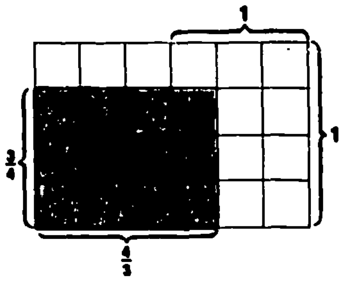
c.


1
1
$3\frac{1}{2}$
$5\frac{2}{3}$

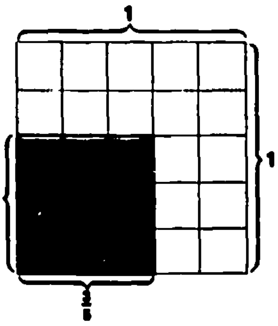
d.


1
1
$\frac{2}{3}$
$\frac{3}{2}$

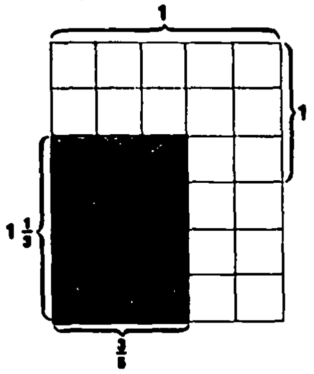
e.


1
1
$\frac{3}{4}$
$\frac{4}{3}$

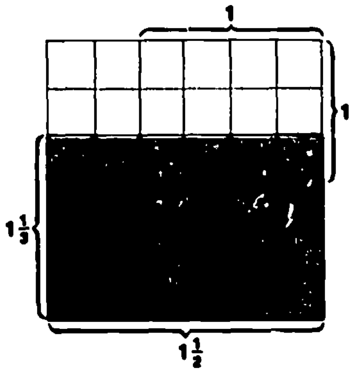
f.


1
1
$\frac{3}{5}$
$\frac{3}{5}$

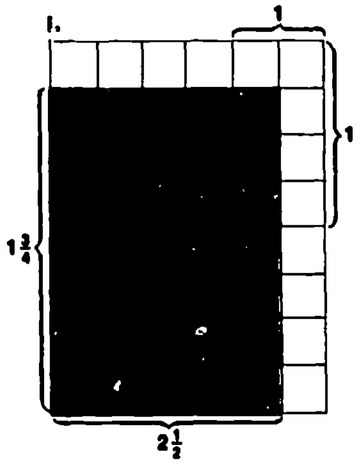
g.


1
1
$1\frac{1}{3}$
$\frac{3}{5}$

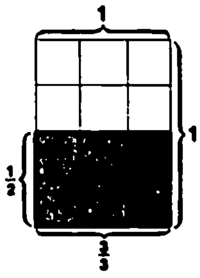
h.


1
1
$1\frac{1}{3}$
$1\frac{1}{2}$

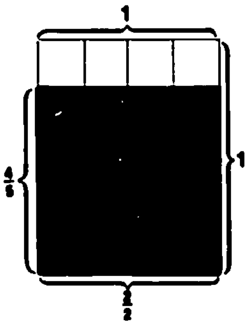
i.


1
1
$1\frac{3}{4}$
$2\frac{1}{2}$

j.



k.

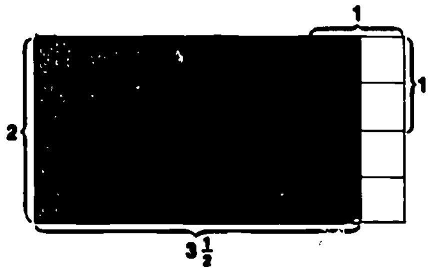


Exercise Set 2, pp. 123–24

1. a. There are 6 unit regions and 2 half-unit regions. $6 + \frac{1}{2} + \frac{1}{2} = 7$, so $2 \times 3\frac{1}{2} = 7$. If split into halves, the count is 14, and 14 halves, or $\frac{14}{2}$, is 7.

$$2 \times 3 < 2 \times 3\frac{1}{2} < 2 \times 4.$$

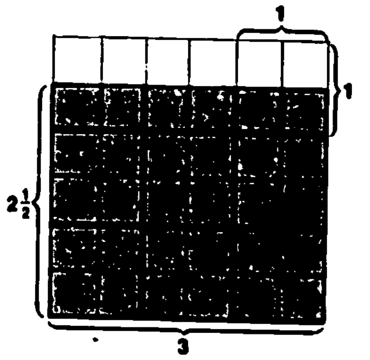
$$6 < \quad 7 \quad < 8.$$



b. There are 6 unit regions and 3 half-unit regions. $6 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 7\frac{1}{2}$, so $2\frac{1}{2} \times 3 = 7\frac{1}{2}$. If split into halves, the count is 15, and 15 halves is 14 + 1 halves, which is $7\frac{1}{2}$.

$$2 \times 3 < 2\frac{1}{2} \times 3 < 3 \times 3.$$

$$6 < 7\frac{1}{2} \quad < 9.$$

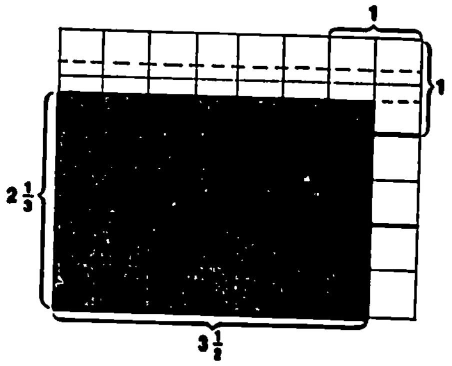


c. There are 6 unit regions, 2 half-unit regions, 3 third-unit regions, and 1 sixth-unit region. $6 + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + \frac{1}{6} = 6 + 1 + 1 + \frac{1}{6} = 8\frac{1}{6}$. If split into sixths, the count is 7 × 7, or 49.

$$\frac{49}{6} = \frac{48}{6} + \frac{1}{6} = 8\frac{1}{6}.$$

$$2 \times 3 < 2\frac{1}{3} \times 3\frac{1}{2} < 3 \times 4.$$

$$6 < \quad 8\frac{1}{6} \quad < 12.$$

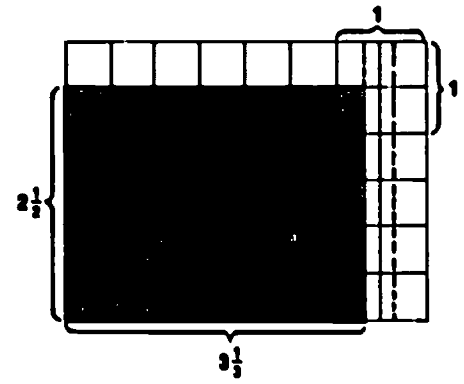


d. There are 6 unit regions, 3 half-unit regions, 2 third-unit regions, and 1 sixth-unit region. $6 + (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{3}) + \frac{1}{6} = 6 + 1 + \frac{1}{2} + \frac{2}{3} + \frac{1}{6} = 7 + \frac{7}{6} + \frac{1}{6} = 7 + \frac{8}{6} = 8\frac{1}{3}$. If split into sixths, the count is 5 × 10 = 50.

$$\frac{50}{6} = \frac{48}{6} + \frac{2}{6} = 8\frac{1}{3}. \text{ So } 2\frac{1}{2} \times 3\frac{1}{3} = 8\frac{1}{3}.$$

$$2 \times 3 < 2\frac{1}{2} \times 3\frac{1}{3} < 3 \times 4.$$

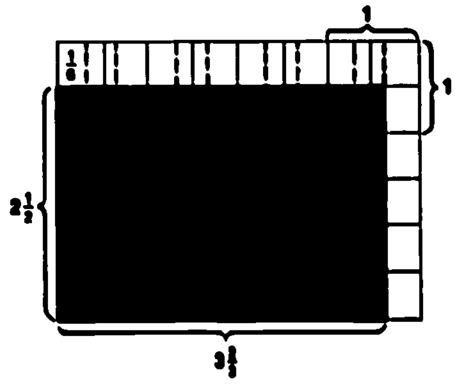
$$6 < \quad 8\frac{1}{3} \quad < 12.$$



e. The small boxes are each $\frac{1}{6}$ of a unit region, as 6 of them make up a full unit region. There are $5 \times 11 = 55$ of these small boxes. $\frac{55}{6} = \frac{54}{6} + \frac{1}{6} = 9\frac{1}{6}$. So $2\frac{1}{2} \times 3\frac{2}{3} = 9\frac{1}{6}$. We could slide 9 of the 10 boxes in the last column beyond 3 to fill in the boxes above $2\frac{1}{2}$, to make a 3×3 array. $3 \times 3 +$ the 1 box $= 9 + \frac{1}{6} = 9\frac{1}{6}$.

$$2 \times 3 < 2\frac{1}{2} \times 3\frac{2}{3} < 3 \times 4.$$

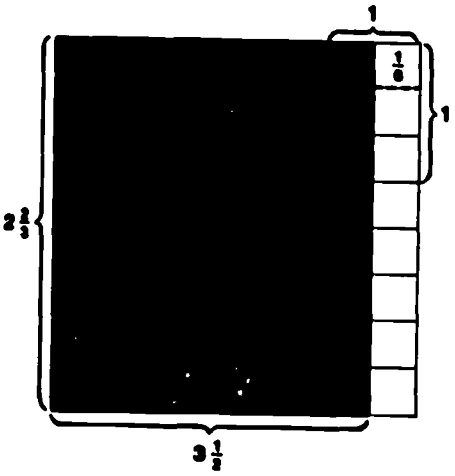
$$6 < \quad 9\frac{1}{6} \quad < 12.$$



1. There are 6 small boxes making up 1 unit region, so each small box is $\frac{1}{6}$ of a unit region. The shaded rectangle has $8 \times 7 = 56$ of these boxes. $\frac{56}{6} = \frac{54}{6} + \frac{2}{6} = 9\frac{1}{3}$. So $2\frac{2}{3} \times 3\frac{1}{2} = 9\frac{1}{3}$. Moving boxes beyond 3 could fill in a square 3×3 with 2 boxes left over. $3 \times 3 + \frac{2}{6} = 9\frac{1}{3}$.

$$2 \times 3 < 2\tfrac{2}{3} \times 3\tfrac{1}{2} < 3 \times 4.$$

$$6 < \quad 9\tfrac{1}{3} \quad < 12.$$



* 2. We use a table to organize some information with the hope that a pattern will be revealed.

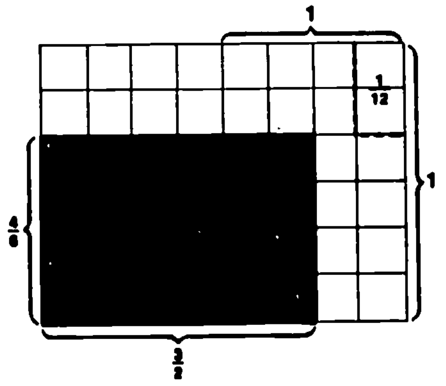
Ex.	*Factors*		*Product*	*Sum*	*Difference*
a.	2	$3\frac{1}{2}$	$2 \times 3\frac{1}{2} = 7.$	$2 + 3\frac{1}{2} = 5\frac{1}{2}.$	$3\frac{1}{2} - 2 = 1\frac{1}{2}.$
b.	$2\frac{1}{2}$	3	$2\frac{1}{2} \times 3 = 7\frac{1}{2}.$	$2\frac{1}{2} + 3 = 5\frac{1}{2}.$	$3 - 2\frac{1}{2} = \frac{1}{2}.$
c.	$2\frac{1}{3}$	$3\frac{1}{2}$	$2\frac{1}{3} \times 3\frac{1}{2} = 8\frac{1}{6}.$	$2\frac{1}{3} + 3\frac{1}{2} = 5\frac{5}{6}.$	$3\frac{1}{2} - 2\frac{1}{3} = 1\frac{1}{6}.$
d.	$2\frac{1}{2}$	$3\frac{1}{3}$	$2\frac{1}{2} \times 3\frac{1}{3} = 8\frac{1}{3}.$	$2\frac{1}{2} + 3\frac{1}{3} = 5\frac{5}{6}.$	$3\frac{1}{3} - 2\frac{1}{2} = \frac{5}{6}.$

Ex.	*Factors*		*Product*	*Sum*	*Difference*
e.	$2\frac{1}{2}$	$3\frac{2}{3}$	$2\frac{1}{2} \times 3\frac{2}{3} = 9\frac{1}{6}.$	$2\frac{1}{2} + 3\frac{2}{3} = 6\frac{1}{6}.$	$3\frac{2}{3} - 2\frac{1}{2} = 1\frac{1}{6}.$
f.	$2\frac{2}{3}$	$3\frac{1}{2}$	$2\frac{2}{3} \times 3\frac{1}{2} = 9\frac{1}{3}.$	$2\frac{2}{3} + 3\frac{1}{2} = 6\frac{1}{6}.$	$3\frac{1}{2} - 2\frac{2}{3} = \frac{5}{6}.$
	$a - r$	$a + r$	$(a - r)(a + r)$ $= a^2 - r^2.$	$(a - r) + (a + r)$ $= 2a.$	$(a + r) - (a - r)$ $= 2r.$
	$a - s$	$a + s$	$(a - s)(a + s)$ $= a^2 - s^2.$	$(a - s) + (a + s)$ $= 2a.$	$(a + s) - (a - s)$ $= 2s.$

The sums are the same for exercises **1a** and **1b**, for exercises **1c** and **1d**, and for exercises **1e** and **1f**. The products are different. The greater product goes with the smaller difference in every case. The last two lines of the table have two pairs of numbers having the same sums, each $2a$. If $s < r$ it then follows that the product $a^2 - s^2 > a^2 - r^2$, so the generalization holds, namely, if the sums of two pairs of factors are the same, then the pair having the smaller difference has the greater product.

3. a. There are 12 small boxes in 1 unit region, so each box is $\frac{1}{12}$ of a unit region.

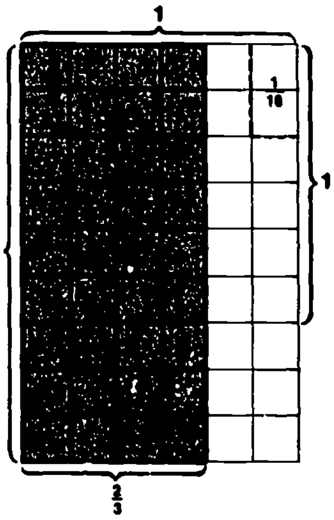
We have $4 \times 3 = 12$ boxes for the shaded rectangular region. $\frac{12}{12} = 1$, so $\frac{4}{6} \times \frac{3}{2} = 1$. The last row of 4 shaded boxes may be moved to fill in a unit region in the first two columns.



b. There are 18 small boxes in 1 unit region, so each box is $\frac{1}{18}$ of a unit region.

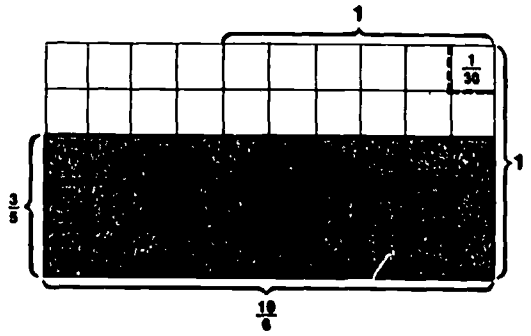
We have for the shaded region $9 \times 2 = 18$ small boxes. $\frac{18}{18} = 1$, so $\frac{9}{6} \times \frac{2}{3} = 1$.

The bottom 3 rows of shaded boxes may be moved to fill in the last empty column to complete a unit region.



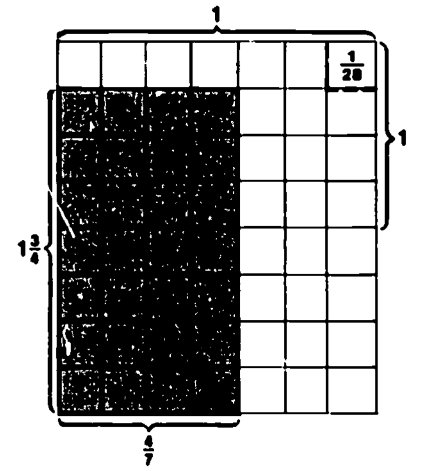
c. There are 30 small boxes in 1 unit region, so each box is $\frac{1}{30}$ of a unit region.

The shaded region has 3 × 10 = 30 small boxes, so $\frac{3}{5} \times \frac{10}{6} = \frac{30}{30} = 1$. If we move 3 × 4 boxes on the lower left corner to the upper right corner a unit region is completely filled.



d. There are 28 small boxes in 1 unit region, so each box is $\frac{1}{28}$ of a unit region.

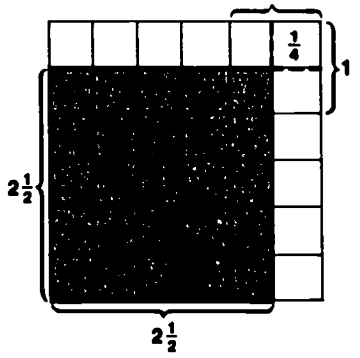
The shaded region has 7 × 4 = 28 small boxes. So $1\frac{3}{4} \times \frac{4}{7} = \frac{28}{28} = 1$. If we move the 4 × 4 boxes in the lower left corner, they may be used to fill in a full unit region.



e. There are 4 small boxes in 1 unit region, so each box is $\frac{1}{4}$ of a unit region.

The shaded region has 5 × 5 = 25 small boxes. So $2\frac{1}{2} \times 2\frac{1}{2} = \frac{25}{4} = 6\frac{1}{4}$. We could move the last column to fill in the top row to make a 3 × 2 array with 1 small box left over. That tells us that

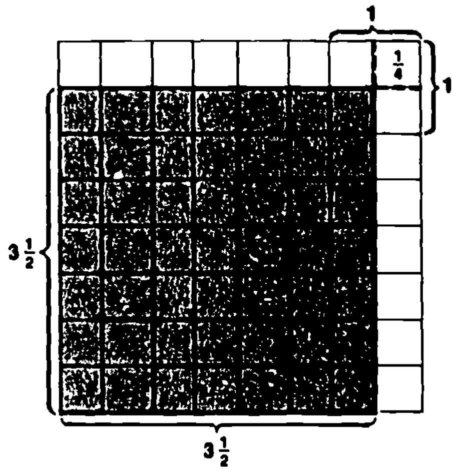
$$2\frac{1}{2} \times 2\frac{1}{2} = 2 \times 3 + \frac{1}{4} = 6\frac{1}{4}.$$



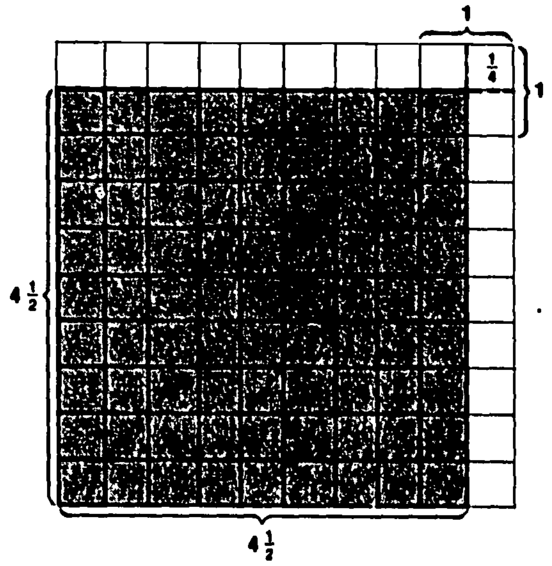
f. There are 4 small boxes in 1 unit region, so each box is $\frac{1}{4}$ of a unit region.

The shaded region has 7 × 7 = 49 small boxes. So $3\frac{1}{2} \times 3\frac{1}{2} = \frac{49}{4} = 12\frac{1}{4}$. We could move the last column of shaded boxes to fill in the top row and make a 4 × 3 array with 1 small box left over. Hence

$$3\frac{1}{2} \times 3\frac{1}{2} = 4 \times 3 + \frac{1}{4} = 12\frac{1}{4}.$$



3. There are 4 small boxes in 1 unit region, so each box is $\frac{1}{4}$ of a unit region. The shaded region is $9 \times 9 = 81$ small boxes. So $4\frac{1}{2} \times 4\frac{1}{2} = \frac{81}{4} = 20\frac{1}{4}$. We could move the last column of shaded boxes to the top row and make a 5×4 array with 1 small box left over. Hence $4\frac{1}{2} \times 4\frac{1}{2} = 5 \times 4 + \frac{1}{4} = 20\frac{1}{4}$.



* 4. The last three products may be written as follows:

$$2\frac{1}{2} \times 2\frac{1}{2} = 2 \times 3 + \frac{1}{4} = 6\frac{1}{4}.$$

$$3\frac{1}{2} \times 3\frac{1}{2} = 3 \times 4 + \frac{1}{4} = 12\frac{1}{4}.$$

$$4\frac{1}{2} \times 4\frac{1}{2} = 4 \times 5 + \frac{1}{4} = 20\frac{1}{4}.$$

This suggests the generalization that for any number n,

$$\left(n + \frac{1}{2}\right) \times \left(n + \frac{1}{2}\right) = n(n + 1) + \frac{1}{4}.$$

But

$$\left(n + \frac{1}{2}\right)\left(n + \frac{1}{2}\right) = n^2 + n + \frac{1}{4},$$

and

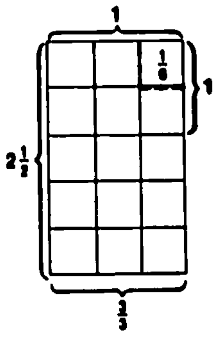
$$n(n + 1) + \frac{1}{4} = n^2 + n + \frac{1}{4},$$

establishing the generalization.

$$5\frac{1}{2} \times 5\frac{1}{2} = 5 \times 6 + \frac{1}{4} = 30\frac{1}{4}.$$

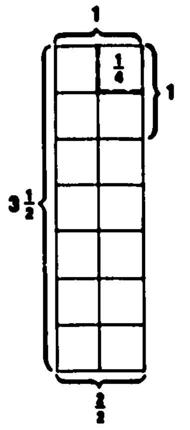
$$6\frac{1}{2} \times 6\frac{1}{2} = 6 \times 7 + \frac{1}{4} = 42\frac{1}{4}.$$

5. a. The region for $2\frac{1}{2} \times \frac{3}{3}$ has $5 \times 3 = 15$ small boxes, each worth $\frac{1}{6}$ of a unit region. So $2\frac{1}{2} \times \frac{3}{3} = \frac{15}{6} = 2\frac{1}{2}$. We could easily split the rectangle into $1 + 1 + \frac{1}{2}$.



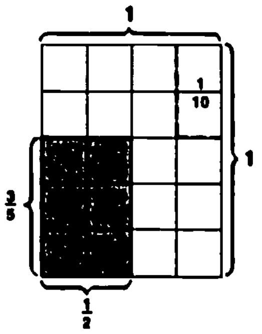
b. The region for $3\frac{1}{2} \times \frac{2}{2}$ has $7 \times 2 = 14$ small boxes, each $\frac{1}{4}$ of a unit region.

So $3\frac{1}{2} \times \frac{2}{2} = \frac{14}{4} = 3\frac{1}{2}$. We could easily split the entire rectangle into regions of $1 + 1 + 1 + \frac{1}{2} = 3\frac{1}{2}$.



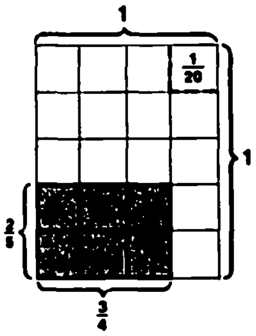
Exercise Set 3, p. 127

1. Each small box is $\frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$. The shaded area is $\frac{3}{5} \times \frac{1}{2}$, consisting of 3 such boxes. Hence $\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$, which is the same value obtained by the traditional algorithm.

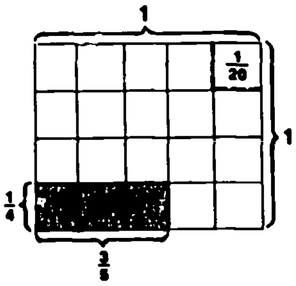


2. Each box is $\frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$. The shaded area is $\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$, the same value obtained by using the traditional algorithm.

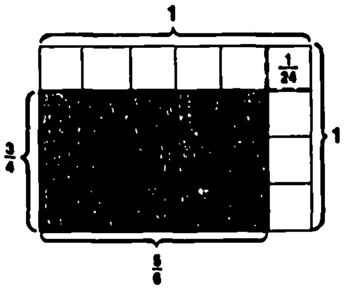
$$\frac{3}{4} \times \frac{2}{5} = \frac{2}{5} \times \frac{3}{4} = \frac{6}{20}.$$



3. Each box is $\frac{1}{20}$ of a unit region. The shaded area has 3 boxes. So $\frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$, which is the same answer one obtains from the traditional algorithm.



4. Each box is $\frac{1}{24}$ of a unit region, as 24 of them make 1 unit region. The shaded region has 3 × 5, or 15, boxes, each $\frac{1}{24}$ of a unit region. So $\frac{3}{4} \times \frac{5}{6} = \frac{15}{24}$, which is the same value obtained by using the traditional algorithm.

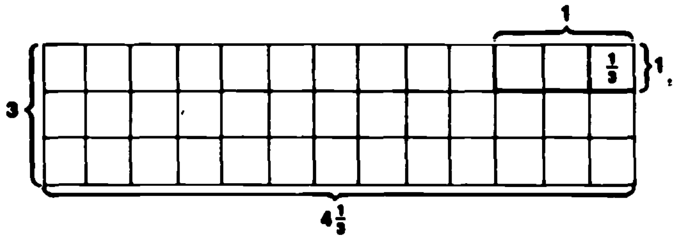


5. Each small box is $\frac{1}{3}$ of a unit region, as 3 make up a full unit region. The rectangle for the product has $3 \times 13 = 39$ small boxes. So $3 \times 4\frac{1}{3} = \frac{39}{3} = 13$.

The traditional algorithm yields

$$3 \times \frac{13}{3} = \frac{3}{1} \times \frac{13}{3} = \frac{39}{3} = 13,$$

the same result.

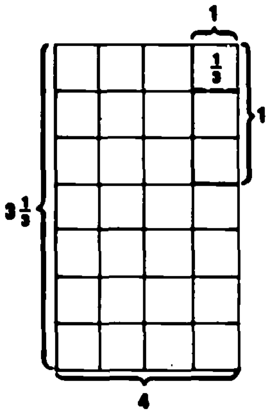


6. Each small box is $\frac{1}{3}$ of a unit region. The rectangle for the product $3\frac{1}{3} \times 4$ has $10 \times 4 = 40$ small boxes. It follows then that $3\frac{1}{3} \times 4 = \frac{40}{3} = 13\frac{1}{3}$.

The traditional algorithm yields

$$3\frac{1}{3} \times 4 = \frac{10}{3} \times \frac{4}{1} = \frac{40}{3} = 13\frac{1}{3},$$

the same result.

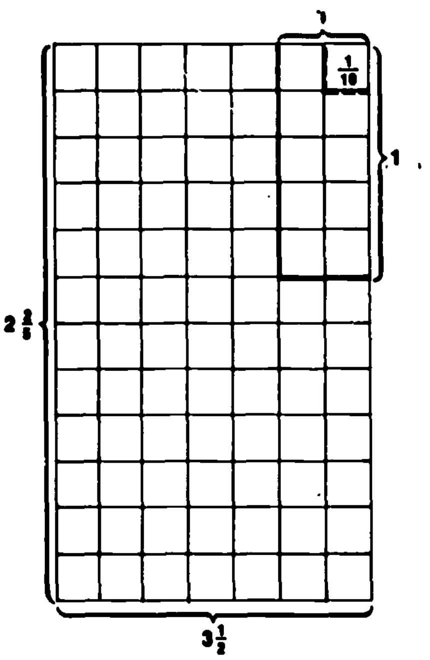


7. Each small box is $\frac{1}{10}$ of a unit region. The rectangle for the product $2\frac{2}{5} \times 3\frac{1}{2}$ contains $12 \times 7 = 84$ small boxes. Hence $2\frac{2}{5} \times 3\frac{1}{2} = \frac{84}{10} = 8\frac{2}{5}$.

The traditional algorithm yields

$$2\frac{2}{5} \times 3\frac{1}{2} = \frac{12}{5} \times \frac{7}{2} = \frac{84}{10} = 8\frac{2}{5},$$

the same result as the previous one.

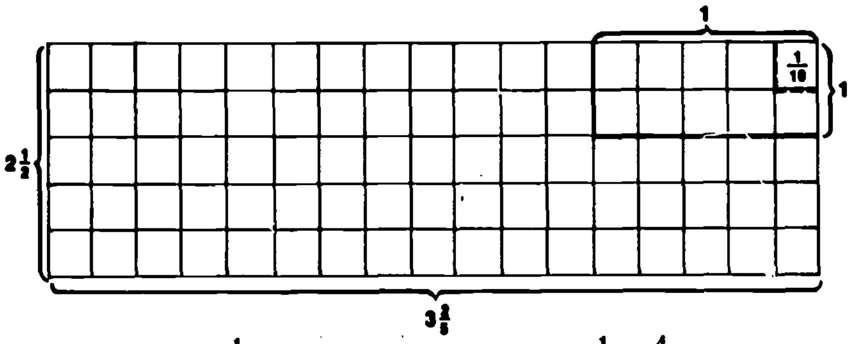


8. Each small box is $\frac{1}{10}$. The rectangle for the product $2\frac{1}{2} \times 3\frac{2}{5}$ contains $5 \times 17 =$ 85 small boxes. Hence $2\frac{1}{2} \times 3\frac{2}{5} = \frac{85}{10} = 8\frac{1}{2}$.

The traditional algorithm yields

$$2\frac{1}{2} \times 3\frac{2}{5} = \frac{5}{2} \times \frac{17}{5} = \frac{85}{10} = 8\frac{1}{2},$$

the same result.

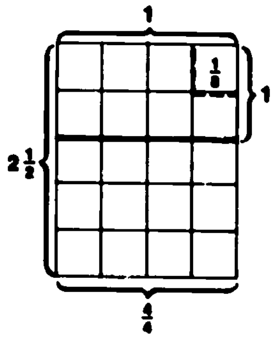


9. Each small box is $\frac{1}{8}$. The rectangle for the product $2\frac{1}{2} \times \frac{4}{4}$ has $5 \times 4 = 20$ boxes. Hence $2\frac{1}{2} \times \frac{4}{4} = \frac{20}{8} = 2\frac{1}{2}$.

The traditional algorithm yields

$$2\frac{1}{2} \times \frac{4}{4} = \frac{5}{2} \times \frac{4}{4} = \frac{20}{8} = 2\frac{1}{2},$$

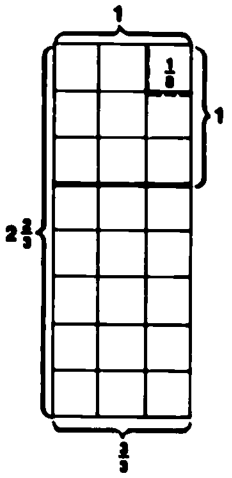
the same result.



10. Each small box is $\frac{1}{9}$. The rectangle for the product $2\frac{2}{3} \times \frac{3}{3}$ has $8 \times 3 = 24$ boxes. The traditional algorithm yields

$$2\frac{2}{3} \times \frac{3}{3} = \frac{8}{3} \times \frac{3}{3} = \frac{24}{9} = 2\frac{6}{9},$$

the same result.



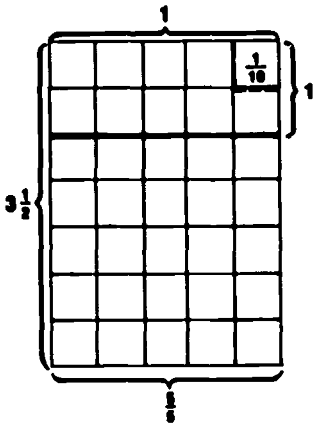
11. Each box is $\frac{1}{10}$. The rectangle for the product $3\frac{1}{2} \times \frac{5}{5}$ contains $7 \times 5 = 35$ boxes. Hence

$$3\frac{1}{2} \times \frac{5}{5} = \frac{35}{10} = 3\frac{1}{2}.$$

The traditional algorithm yields

$$3\frac{1}{2} \times \frac{5}{5} = \frac{7}{2} \times \frac{5}{5} = \frac{35}{10} = 3\frac{1}{2},$$

the same result.



Exercise Set 4, p. 129

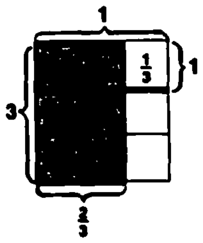
1. Each box is $\frac{1}{3}$. The shaded rectangle has 3 × 2 boxes, or 6 boxes. Hence

$$3 \times \frac{2}{3} = \frac{6}{3} = 2.$$

The traditional algorithm yields

$$3 \times \frac{2}{3} = \frac{3}{1} \times \frac{2}{3} = \frac{6}{3} = 2,$$

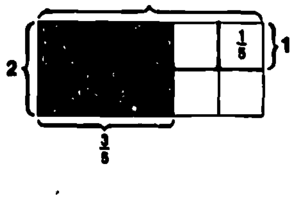
the same result.



2. Each box is $\frac{1}{5}$. The shaded rectangle has 2 × 3 = 6 boxes. Hence

$$2 \times \frac{3}{5} = \frac{6}{5}.$$

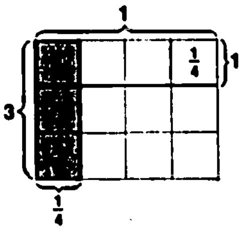
The same result is obtained from the traditional algorithm.



3. Each box is $\frac{1}{4}$. The shaded rectangle has $3 \times 1 = 3$ boxes. So

$$3 \times \frac{1}{4} = \frac{3}{4}.$$

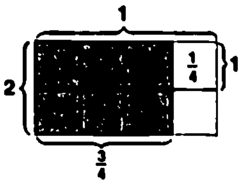
The same result is obtained from the traditional algorithm.



4. Each box is $\frac{1}{4}$. The shaded rectangle has $2 \times 3 = 6$ boxes. So

$$2 \times \frac{3}{4} = \frac{6}{4}.$$

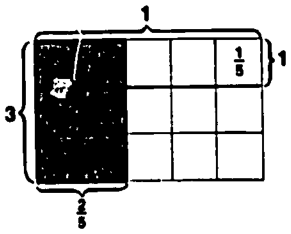
The same result is obtained from the traditional algorithm.



5. Each box is $\frac{1}{5}$. The shaded rectangle has $3 \times 2 = 6$ boxes. So

$$3 \times \frac{2}{5} = \frac{6}{5}.$$

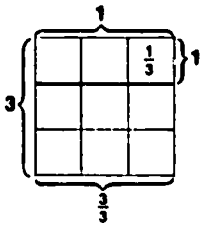
The same result is obtained from the traditional algorithm.



6. Each box is $\frac{1}{3}$. The rectangle for the product $3 \times \frac{3}{3}$ has $3 \times 3 = 9$ boxes. So

$$3 \times \frac{3}{3} = \frac{9}{3} = 3.$$

The traditional algorithm yields the very same computation and the same result.



Exercise Set 5, pp. 135–36

1. a. $\left(7 \times \frac{2}{3}\right) \times \frac{3}{2} = 7 \times \left(\frac{2}{3} \times \frac{3}{2}\right)$ . . Multiplication is associative.

$= 7 \times 1$ . . . . The product of a number and its reciprocal is 1.

$= 7$. . . . . 1 is the multiplicative identity.

b. $\left(\frac{22}{7} \times \frac{63}{63}\right) \times \frac{7}{22}$

$= \left(\frac{22}{7} \times 1\right) \times \frac{7}{22}$ . . If $a \neq 0, \frac{a}{a} = 1$.

$= \frac{22}{7} \times \frac{7}{22}$ . . . . . . 1 is the multiplicative identity.

$= 1$ . . . . The product of a number and its reciprocal is 1.

c. $\frac{1}{2} \times \left(4 + \frac{2}{5}\right)$

$= \left(\frac{1}{2} \times 4\right) + \left(\frac{1}{2} \times \frac{2}{5}\right)$ . . Manipulation distributes over addition.

$= 2 + \frac{1}{5}$

$= 2\frac{1}{5}$.

d. $\left(6 + \frac{4}{5}\right) \times \frac{1}{2} = \frac{1}{2} \times \left(6 + \frac{4}{5}\right)$ . . Multiplication is commutative.

$= \left(\frac{1}{2} \times 6\right) + \left(\frac{1}{2} \times \frac{4}{5}\right)$ Multiplication distributes over addition.

$= 3 + \frac{2}{5}$

$= 3\frac{2}{5}$.

2. If $3a = 3b$, then

$$\frac{1}{3} \times (3a) = \frac{1}{3} \times (3b)$$

$\left(\frac{1}{3} \times 3\right) \times a = \left(\frac{1}{3} \times 3\right) \times b$ . . . . Multiplication is associative.

$1 \times a = 1 \times b$ . . . . The product of a number and its reciprocal is 1.

$a = b$. . . . . 1 is the multiplicative identity.

3. If $\frac{a}{3} = \frac{b}{3}$, then

$$3 \times \frac{a}{3} = 3 \times \frac{b}{3}$$

$3 \times \left(\frac{1}{3} \times \frac{a}{1}\right) = 3 \times \left(\frac{1}{3} \times \frac{b}{1}\right)$. . . . . $\frac{a}{3} = \frac{1}{3} \times \frac{a}{1}$; $\frac{b}{3} = \frac{1}{3} \times \frac{b}{1}$.

$\left(3 \times \frac{1}{3}\right) \times a = \left(3 \times \frac{1}{3}\right) \times b$. . Multiplication is associative; $\frac{a}{1} = a$, $\frac{b}{1} = b$.

$1 \times a = 1 \times b$. . . . . $3 \times \frac{1}{3} = 1$.

$a = b$. . . . . $1 \times a = a$, $1 \times b = b$, as 1 is the multiplicative identity.

4. If $\frac{a}{c} = \frac{b}{c}$, then

$$\frac{c}{1} \times \frac{a}{c} = \frac{c}{1} \times \frac{b}{c}$$

$$\frac{c \times a}{1 \times c} = \frac{c \times b}{1 \times c}$$

$$\frac{a \times c}{1 \times c} = \frac{b \times c}{1 \times c}$$

$$\frac{a}{1} = \frac{b}{1}$$

$$a = b.$$

5. If $\frac{3}{a} = \frac{3}{b}$, then

$$(a \times b) \times \frac{3}{a} = (a \times b) \times \frac{3}{b}$$

$$\frac{(a \times b) \times 3}{a} = \frac{(a \times b) \times 3}{b}$$

$$\frac{a \times (b \times 3)}{a} = \frac{(b \times a) \times 3}{b}$$

$$\frac{b \times 3}{1} = \frac{b \times (a \times 3)}{b}$$

$$3 \times b = 3 \times a$$

$$b = a.$$

See exercise 2.

6. Replace "3" by "c" in exercise 5.

7. Let $r = \frac{a}{b}$, $s = \frac{c}{d}$, $t = \frac{e}{f}$.

a.

$$\begin{aligned}(rs)t &= \left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} \\ &= \left(\frac{a \times c}{b \times d}\right) \times \frac{e}{f} \\ &= \frac{(a \times c) \times e}{(b \times d) \times f} \\ &= \frac{a \times (c \times e)}{b \times (d \times f)} \\ &= \frac{a}{b} \times \frac{c \times e}{d \times f} \\ &= \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) \\ &= r \times (s \times t) \\ &= r(st).\end{aligned}$$

b.

$$\begin{aligned}r(s + t) &= r \times (s + t) \\ &= \frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right)\end{aligned}$$

$$= \frac{a}{b} \times \left(\frac{c \times f}{d \times f} + \frac{d \times e}{d \times f}\right)$$

$$= \frac{a}{b} \times \frac{c \times f + d \times e}{d \times f}$$

$$= \frac{a \times (c \times f + d \times e)}{b \times (d \times f)}$$

$$= \frac{a \times (c \times f) + a \times (d \times e)}{b \times (d \times f)}$$

$$= \frac{a \times (c \times f)}{b \times (d \times f)} + \frac{a \times (d \times e)}{b \times (d \times f)}$$

$$= \frac{(a \times c) \times f}{(b \times d) \times f} + \frac{a \times (e \times d)}{(b \times d) \times f}$$

$$= \frac{a \times c}{b \times d} + \frac{(a \times e) \times d}{b \times (d \times f)}$$

$$= \left(\frac{a}{b} \times \frac{c}{d}\right) + \frac{(a \times e) \times d}{b \times (f \times d)}$$

$$= (r \times s) + \frac{(a \times e) \times d}{(b \times f) \times d}$$

$$= (r \times s) + \frac{a \times e}{b \times f}$$

$$= (r \times s) + \left(\frac{a}{b} \times \frac{e}{f}\right)$$

$$= (r \times s) + (r \times t)$$

$$= rs + rt.$$

c. $(s + t)r = (s + t) \times r$

$$= r \times (s + t)$$

$$= (r \times s) + (r \times t)$$

$$= (s \times r) + (t \times r)$$

$$= sr + tr.$$

d. If $sr = st$, then

$$s \times r = s \times t$$

$$\frac{c}{d} \times \frac{a}{b} = \frac{c}{d} \times \frac{e}{f}$$

$$\frac{d}{c} \times \left(\frac{c}{d} \times \frac{a}{b}\right) = \frac{d}{c} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$$

$$\left(\frac{d}{c} \times \frac{c}{d}\right) \times \frac{a}{b} = \left(\frac{d}{c} \times \frac{c}{d}\right) \times \frac{e}{f}$$

$$1 \times \frac{a}{b} = 1 \times \frac{e}{f}$$

$$\frac{a}{b} = \frac{e}{f}$$

$$r = s.$$

8. The reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$ because $\frac{5}{8} \times \frac{8}{5} = \frac{40}{40} = 1$.

9. The reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$ because $\frac{4}{3} \times \frac{3}{4} = \frac{12}{12} = 1$.

10. Suppose that d were another multiplicative identity. Then for every rational number r we must have $d \times r = r$ and $r \times d = r$. In particular, $d \times 1 = 1$ and $1 \times d = 1$. But 1 is a multiplicative identity, so $d \times 1 = d$ and $1 \times d = d$. Then $d = d \times 1 = 1$, so $d = 1$ and d cannot be an identity different from 1.

11. Suppose that the rational number s had two reciprocals, say, r and t. Then $r \times s = 1$, and $r \times t = 1$. But then $r \times s = t \times s$, from which it follows that $r = t$ and s has but one multiplicative inverse, that is, one reciprocal. We made use of the property of restricted cancellation.

Exercise Set 6, pp. 138–39

1. a. $\frac{7}{20}$ b. $6\frac{3}{5}$ c. 33

2. a. If $r > s$, then $r = s + p$ for some number $p > 0$ and

$$r \times t = (s + p) \times t$$
$$= s \times t + p \times t,$$

so $r \times t > s \times t$ because $p \times t > 0$, or $rt > st$.

b. We sketch a proof. If $\frac{a}{b} > \frac{c}{d}$, then for some $p > 0$

$$\frac{a}{b} = \frac{c}{d} + p.$$

Now, multiplying both members of the equation by bd, we obtain

$$ad = bc + bdp.$$

Dividing both members by ac,

$$\frac{d}{c} = \frac{b}{a} + \frac{bdp}{ac}.$$

Therefore

$$\frac{d}{c} > \frac{b}{a} \quad \text{or} \quad \frac{b}{a} < \frac{d}{c}.$$

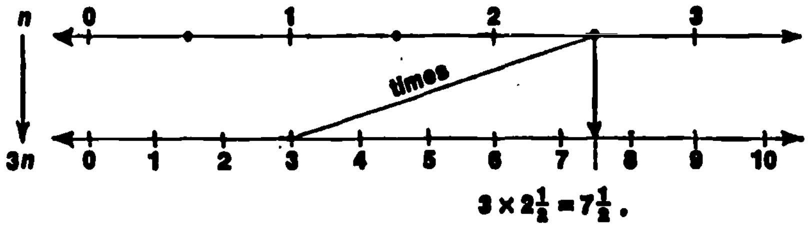
3. a. Multiplication does not distribute over multiplication.
   b. Distributivity over addition was not applied properly.
   c. If the numerator had been divided by 2, the result would have been $\frac{2 + 3}{1}$, or 5, which is correct.
   d. It just does not work in almost all cases. If canceling 6s worked, then $\frac{46}{16}$, $\frac{46}{61}$, and $\frac{64}{61}$ ought to all equal 4.

Exercise Set 7, p. 140

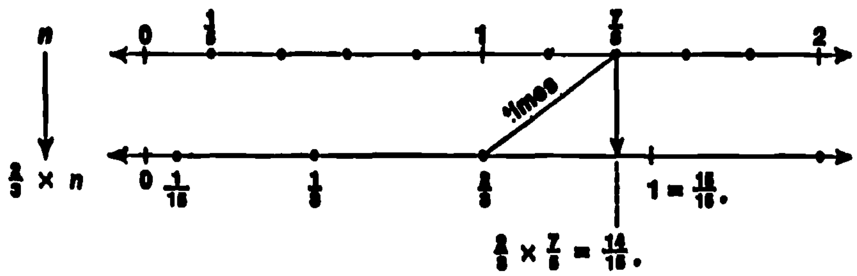
1–7. The *of* has the meaning of multiplication in exercises 1, 4, 5, and 7.

Exercise Set 8, p. 143

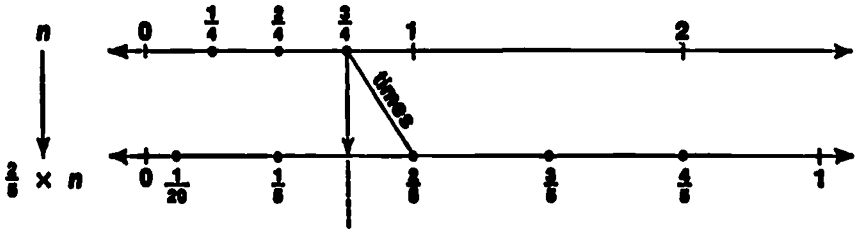
1. a.



b.

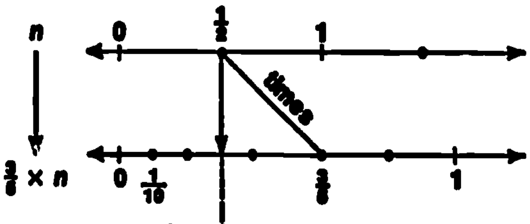


c.



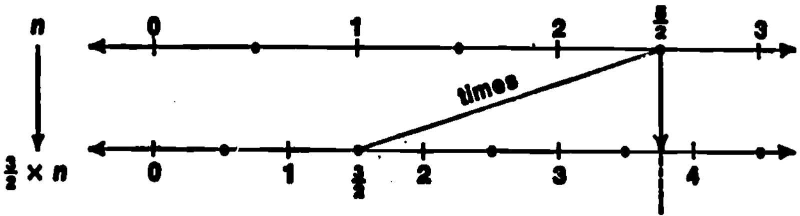
$\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$.

d.



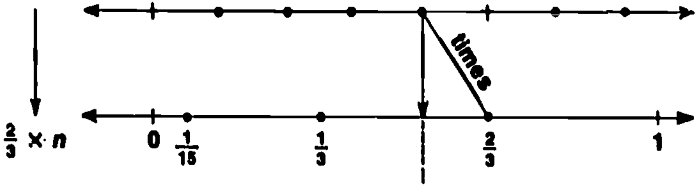
$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

e.



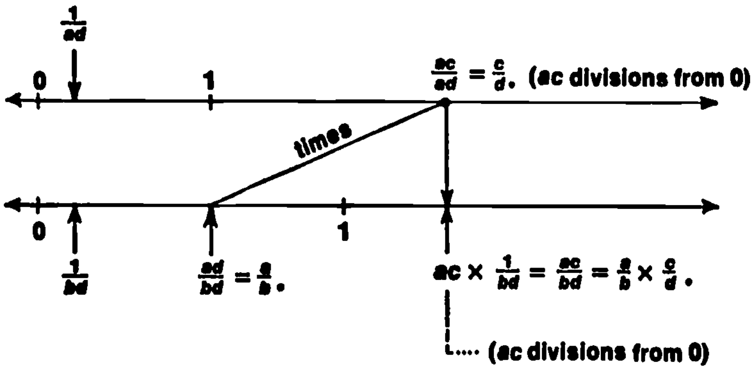
$\frac{3}{2} \times \frac{5}{2} = 3\frac{3}{4}$.

f.



$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

2.



Review Exercises, pp. 141–46

1. a. $\frac{6}{20}$, or $\frac{3}{10}$

   b. $\frac{18}{20}$, or $\frac{9}{20}$

   c. $\frac{77}{20}$, or $3\frac{17}{20}$

   d. $4\frac{1}{5}$

   e. $4\frac{19}{20}$

   f. $4\frac{11}{20}$

2. a. The student thought that this was an addition problem. Have problems in which the question calls for the operation with no computation.

   b. The student probably got addition and multiplication algorithms mixed up. Preparing rectangles might help him.

   c. The student took the easy way out, which is not correct. Two partial products were omitted here: $2 \times \frac{1}{2}$ and $\frac{1}{2} \times 3$. Preparing rectangles for the product might help.

   d. This is a frequent error, multiplying both numerator and denominator when multiplying by a whole number. Among the ways of preventing such an error are the following: (1) Express the whole number as a fraction: $3 = \frac{3}{1}$. (2) Recall that the value of a fraction is not changed if both numerator and denominator are multiplied by the same nonzero number.

3. The answer to this question will be found in the section of this chapter titled "Properties of the Multiplication of Rational Numbers."

4. These are some properties that hold for multiplication of rational numbers but not for multiplication of whole numbers:

If a product of two numbers is 1, it is possible for neither factor to be 1.

$$\text{Example: } \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1.$$

Every rational number except 0 has a multiplicative inverse or reciprocal.

Example: $\frac{4}{7}$ and $\frac{7}{4}$ are multiplicative inverses.

Products may be smaller than either factor or both factors without being 0.

$$\text{Example: } \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}. \quad \left(\frac{1}{6} < \frac{1}{2}, \frac{1}{6} < \frac{1}{3}.\right)$$

5. **a.** $2\frac{1}{4}$ **c.** $12\frac{1}{4}$ **e.** $30\frac{1}{4}$

b. $6\frac{1}{4}$ **d.** $20\frac{1}{4}$ **f.** $42\frac{1}{4}$

See exercise set 2, exercise **15**.

6. **a.** $2\frac{2}{9}$ **c.** $12\frac{2}{9}$. **e.** $6\frac{6}{25}$

b. $6\frac{2}{9}$ **d.** $2\frac{6}{25}$ **f.** $12\frac{6}{25}$

Let $a + b = 1$. Then

$$(n + a) \times (n + b) = n \times (n + 1) + a \times b.$$

Proof:

$$n \times (n + 1) + a \times b = n^2 + n + ab.$$

$$\begin{aligned}(n + a) \times (n + b) &= (n + a) \times n + (n + a) \times b \\ &= n \times n + a \times n + n \times b + 1 \times b \\ &= n^2 + n(a + b) + ab \\ &= n^2 + n \times 1 + ab \\ &= n^2 + n + ab,\end{aligned}$$

the same result as above.

7. Let $a + b = c$, then

$$(n + a) \times (n + b) = n \times (n + c) + a \times b.$$

Proof:

$$\begin{aligned}(n + a) \times (n + b) &= n^2 + (n \times a + n \times b) + a \times b\\ &= n^2 + n(a + b) + ab\\ &= n^2 + n(c) + ab\\ &= n \times (n + c) + ab.\end{aligned}$$

8. a. $r = 0$, or $s = 0$, or both r and s are 0.
   b. $r \neq 0$ and $s \neq 0$, and r and s are reciprocals of each other.

9. A better assignment and its value are shown below.

$7 \rightarrow A$ $\quad$ $6 \rightarrow B$ $\quad$ $9 \rightarrow C$ $\quad$ $2 \rightarrow D$

$$\begin{aligned}\frac{A}{B+1} \times \frac{B+C}{D+1} &= \frac{7}{6+1} \times \frac{6+9}{2+1}\\ &= \frac{7}{7} \times \frac{15}{3}\\ &= 1 \times 5\\ &= 5.\end{aligned}$$

The best assignment and its value are shown below.

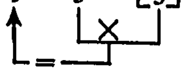
$9 \rightarrow A$ $\quad$ $6 \rightarrow B$ $\quad$ $7 \rightarrow C$ $\quad$ $2 \rightarrow D$

$$\begin{aligned}\frac{9}{6+1} \times \frac{6+7}{2+1} &= \frac{9}{7} \times \frac{13}{3}\\ &= \frac{117}{21}\\ &= 5\frac{4}{7}.\end{aligned}$$

DIVISION OF RATIONAL NUMBERS

Exercise Set 1, pp. 153–55

1. a. $\frac{4}{9} \div \frac{1}{3} = \left[\frac{4}{3}\right].$



Check: $\frac{1}{3} \times \frac{4}{3} = \frac{4}{9}.$

b. $\frac{9}{8} \div \frac{3}{2} = \left[\frac{3}{4}\right].$

Check: $\frac{3}{2} \times \frac{3}{4} = \frac{9}{8}.$

c. $\frac{9}{10} \div \frac{3}{5} = \left[\frac{3}{2}\right].$

Check: $\frac{3}{5} \times \frac{3}{2} = \frac{9}{10}.$

d. $\frac{4}{7} \div \frac{2}{3} = [\ \ ].$

$\frac{4 \times 3}{7 \times 3} \div \frac{2}{3} = [\ \ ].$

$\frac{12}{21} \div \frac{2}{3} = \left[\frac{6}{7}\right].$

Check: $\frac{2}{3} \times \frac{6}{7} = \frac{12}{21} = \frac{4}{7}.$

e. $\frac{3}{10} \div \frac{2}{5} = [\ \ ].$

$\frac{3 \times 2}{10 \times 2} \div \frac{2}{5} = [\ \ ].$

$\frac{6}{20} \div \frac{2}{5} = \left[\frac{3}{4}\right].$

Check: $\frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}.$

f. $\frac{1}{2} \div \frac{2}{3} = [\ \ ].$

$\frac{1 \times 2 \times 3}{2 \times 2 \times 3} \div \frac{2}{3} = [\ \ ].$

$\frac{6}{12} \div \frac{2}{3} = \left[\frac{3}{4}\right].$

Check: $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}.$

g. $1\frac{1}{2} \div \frac{2}{3} = [\ \ ].$

$\frac{3}{2} \div \frac{2}{3} = [\ \ ].$

$\frac{3 \times 2 \times 3}{2 \times 2 \times 3} \div \frac{2}{3} = [\ \ ].$

$\frac{18}{12} \div \frac{2}{3} = \left[\frac{9}{4}\right].$

Check: $\frac{2}{3} \times \frac{9}{4} = \frac{18}{12} = \frac{3}{4}.$

h. $1\frac{1}{2} \div 1\frac{1}{3} = [\ \ ].$

$\frac{3}{2} \div \frac{4}{3} = [\ \ ].$

$\frac{3 \times 4 \times 3}{2 \times 4 \times 3} \div \frac{4}{3} = \left[\frac{3 \times 3}{2 \times 4}\right]$

$= \left[\frac{9}{8}\right]$

Check: $\frac{4}{3} \times \frac{9}{8} = \frac{36}{24} = \frac{3}{2}.$

2\. a. $\frac{3}{5}$

b. $\frac{3}{5}$

c. $\frac{3}{5}$

d. $\frac{3}{6}$, or $\frac{1}{2}$

e. $\frac{8}{2}$, or 4

f. $\frac{8}{2}$, or 4

g. $\frac{8}{2}$, or 4

h. $\frac{8}{2}$, or 4

3\. $\frac{a}{d} \div \frac{c}{d} = \frac{a}{c}.$

Proof: $$\underset{\uparrow}{\frac{a \times c}{d \times c}} \div \underbrace{\frac{c}{d} = \frac{a}{c}}_{\times}.$$

$$\overset{=}{\longleftarrow}$$

4\. a. $\frac{2}{3}, \frac{3}{2}$

b. $\frac{1}{6}$, 6

c. $\frac{9}{5}, \frac{5}{9}$

d. $\frac{15}{8}, \frac{8}{15}$

e. $\frac{25}{8}, \frac{8}{25}$

f. $\frac{3}{2}, \frac{2}{3}$

g. $\frac{2}{15}, \frac{15}{2}$

h. $\frac{7}{5}, \frac{5}{7}$

i. 12, $\frac{1}{12}$

j. 1, 1

k. $\frac{2}{3}, \frac{3}{2}$

l. 4, $\frac{1}{4}$

5\. $\frac{a}{b} \div \frac{c}{d}$ and $\frac{c}{d} \div \frac{a}{b}$ are reciprocals.

Proof: $$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}.$$

$$\frac{c}{d} \div \frac{a}{b} = \frac{b \times c}{a \times d}.$$

6\. a. $1\frac{2}{5}$

b. $3\frac{2}{9}$

c. $2\frac{2}{7}$

d. $4\frac{1}{5}$

e. $3\frac{2}{9}$

f. $2\frac{1}{8}$

7\. $$\left(\frac{a}{b} + \frac{c}{d}\right) \div \frac{e}{f} = \frac{a}{b} \div \frac{e}{f} + \frac{c}{d} \div \frac{e}{f}.$$

Proof: $$\left(\frac{a}{b} + \frac{c}{d}\right) \div \frac{e}{f} = \left(\frac{a \times d + b \times c}{b \times d}\right) \div \frac{e}{f}$$

$$= \frac{(a \times d + b \times c) \times f}{(b \times d) \times e}$$

$$= \frac{(a \times d) \times f + (b \times c) \times f}{(b \times d) \times e}$$

$$= \frac{(a \times d) \times f}{(b \times d) \times e} + \frac{(b \times c) \times f}{(b \times d) \times e}$$

$$= \frac{a \times d}{b \times d} \times \frac{f}{e} + \frac{b \times c}{b \times d} \times \frac{f}{e}$$

$$= \frac{a}{b} \times \frac{f}{e} + \frac{c}{d} \times \frac{f}{e}$$

$$= \frac{a}{b} \div \frac{e}{f} + \frac{c}{d} \div \frac{e}{f}.$$

Exercise Set 2, pp. 157–58

1. a. $\frac{\frac{5}{2}}{\frac{4}{3}}$
   b. $\frac{5}{\frac{4}{3}}$
   c. $\frac{\frac{5}{2}}{4}$
   d. $\frac{\frac{1}{2}}{\frac{4}{3}}$
   e. $\frac{\frac{2}{1}}{\frac{3}{1}}$
   f. $\frac{1\frac{1}{2}}{1\frac{1}{3}}$

2. a. $\frac{5}{3}$
   b. $\frac{5}{6}$
   c. 1
   d. $\frac{2}{5}$
   e. $\frac{8}{5}$
   f. $\frac{9}{32}$
   g. 2
   h. $\frac{16}{25}$
   i. $\frac{3}{2}$

Exercise Set 3, pp. 160–61

1. a. $7\frac{2}{5}$
   b. $8\frac{1}{6}$
   c. $8\frac{1}{3}$
   d. $2\frac{7}{9}$
   e. $83\frac{1}{3}$
   f. $4\frac{76}{231}$
   g. $3\frac{37}{77}$
   h. $1\frac{91}{125}$
   i. $111\frac{1}{9}$

2. An outfit requires 5 yards of cloth. How many outfits can be made from 37 yards of cloth?

A tree is expected to yield 5 bushels of fruit. How many trees are needed to obtain a yield of 37 bushels?

Five boys worked on a job for which they received collectively \$37. If the boys want to divide the amount equally, how much should each boy get?

3. $\frac{100}{8} = \frac{96}{8} + \frac{4}{8} = 12\frac{1}{2}.$

$$\begin{array}{r} 12\frac{4}{8} = 12\frac{1}{2}. \\ 8\overline{)100} \\ \underline{8} \\ 20 \\ \underline{16} \\ 4 \end{array}$$

Exercise Set 4, pp. 164–66

1. The expressions in **b, g, i, j, k, l, m,** and **n** name no number because 0 appears as a divisor.

2. **a.** $\frac{4}{3}$
b. $\frac{3}{4}$
c. 0
d. Meaningless
e. 0
f. 0
g. Meaningless
h. $\frac{3}{2}$
i. Meaningless
j. $\frac{2}{3}$
k. Meaningless
l. 0
m. $\frac{89}{98}$
n. Meaningless
o. $\frac{1}{7}$
p. $\frac{4}{3}$
q. $\frac{3}{4}$
r. $1\frac{1}{2}$
s. $\frac{2}{3}$
t. $\frac{2}{25}$
u. 0

3. $r = 0$.

4. $r = s, r \neq 0, s \neq 0$.

5. $r = 0$, or $s = 0$, or both r and s are 0.

6. $r \neq 0, s \neq 0$, r and s are reciprocals.

7. $s = 1$, or $r = 0$, or both.

8. $s \neq 0$, and $r = s \times s$.

9. $s = 1$, or $r = 0$, or both.

Exercise Set 5, pp. 170–72

1. **a.** $5\frac{1}{8}$

b. $3\frac{3}{40}$

c. 6

d. $5\frac{1}{6}$

e. 4

f. $7\frac{3}{5}$

g. $1\frac{7}{10}$

h. $2\frac{1}{3}$

2. **a.** $\frac{31}{9}$, or $3\frac{4}{9}$

b. $\frac{29}{11}$, or $2\frac{7}{11}$

c. $\frac{34}{11}$, or $3\frac{1}{11}$

d. $\frac{35}{9}$, or $3\frac{8}{9}$

e. $\frac{9}{31}$

f. $\frac{11}{29}$

g. $\frac{11}{34}$

h. $\frac{9}{35}$

3. See the section preceding this exercise set for the answer to the first part. We sometimes say that 1 is a partial identity or right identity for division because for every number r we have $r \div 1 = r$.

4. For $r^2 = s^2$ and $r \neq 0$. Then r and s commute under division.

5. For $r = 0$ or $t^2 = 1$. In all cases $s \neq 0$ and $t \neq 0$. The associative property holds for r, s, and t under division.

6. $r = 0, s \neq 0$, and $t \neq 0$.

7. $3 \div \frac{1}{2} = 6$. They are less than 1 and greater than 0.

8. **a.**

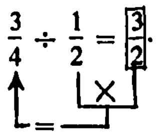
$$\begin{aligned} r + p &= s. \\ (r + p)t &= st. \\ rt + pt &= st. \\ rt &< st. \end{aligned}$$

b.

$$\begin{aligned} r + p &= s. \\ \frac{r + p}{t} &= \frac{s}{t}. \\ \frac{r}{t} + \frac{p}{t} &= \frac{s}{t}. \\ \frac{r}{t} &< \frac{s}{t}. \end{aligned}$$

Exercise Set 6, p. 176

1. **a.** *Missing-factor method*



Reciprocal method

$$\frac{3}{4} = \boxed{\phantom{\frac{3}{4}\times\frac{2}{1}}} \times \frac{1}{2}$$

$$\frac{3}{4} = \boxed{\phantom{\frac{3}{4}\times}\frac{2}{1}} \times \frac{1}{2}$$

$$\frac{3}{4} = \boxed{\frac{3}{4} \times \frac{2}{1}} \times \frac{1}{2}$$

$$\frac{3}{4} \div \frac{1}{2} = \boxed{\frac{3}{4} \times \frac{2}{1}}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}.$$

Equal-multiplication method

$$\frac{3}{4} \div \frac{1}{2} = \boxed{\frac{3}{4} \times \frac{2}{1}} \div \boxed{\frac{1}{2} \times \frac{2}{1}}$$

$$= \frac{6}{4} \div \frac{2}{2}$$

$$= \frac{3}{2} \div 1$$

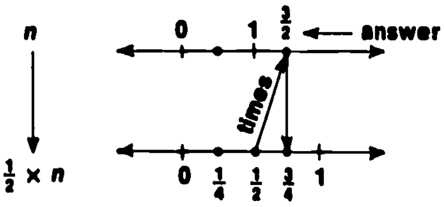
$$= \frac{3}{2}.$$

Same-denominator method

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4}$$

$$= \frac{3}{2}.$$

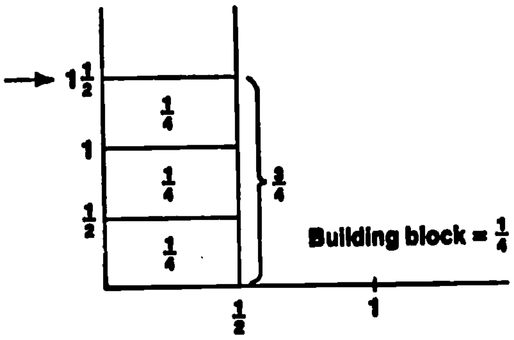
Number-line method



Check: $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}.$

Rectangle method

$$\frac{3}{4} \div \frac{1}{2}$$



b. *Missing-factor method*

$$\frac{1}{2} \div \frac{3}{4} = \boxed{}$$

$$\frac{12}{24} \div \frac{3}{4} = \boxed{\frac{4}{6}} = \frac{2}{3}$$

$$\times \qquad =$$

Reciprocal method

$$\frac{1}{2} \div \frac{3}{4} = \boxed{}$$

$$\frac{1}{2} \qquad = \boxed{\frac{4}{3}} \times \frac{3}{4}$$

$$\frac{1}{2} \qquad = \boxed{\frac{1}{2} \times \frac{4}{3}} \times \frac{3}{4}$$

$$\frac{1}{2} \div \frac{3}{4} = \boxed{\frac{1}{2} \times \frac{4}{3}}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

Equal-multiplication method

$$\frac{1}{2} \div \frac{3}{4} = \boxed{\frac{1}{2} \times \frac{4}{3}} \div \boxed{\frac{3}{4} \times \frac{4}{3}}$$

$$= \frac{4}{6} \div 1$$

$$= \frac{4}{6}$$

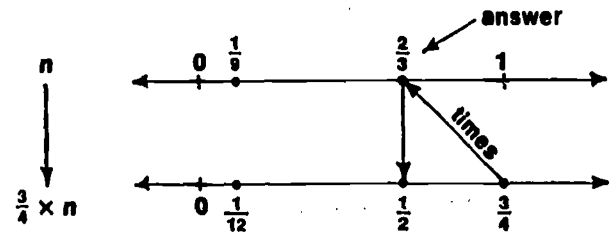
$$= \frac{2}{3}.$$

Same-denominator method

$$\frac{1}{2} \div \frac{3}{4} = \frac{2}{4} \div \frac{3}{4}$$

$$= \frac{2}{3}.$$

Number-line method

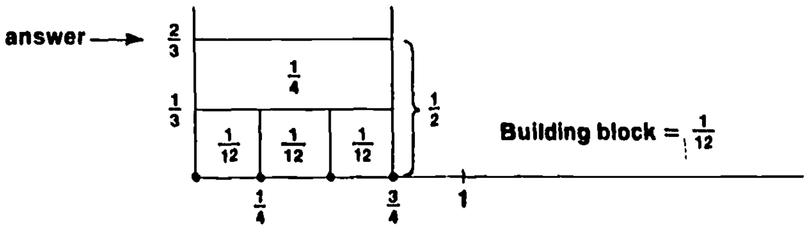


Check: $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}.$

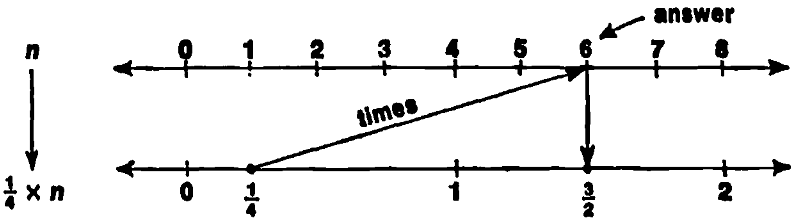
Rectangle method

$$\frac{1}{2} \div \frac{3}{4} = \frac{1 \times 6}{2 \times 6} \div \frac{3}{4}$$

$$= \frac{6}{12} \div \frac{3}{4}.$$



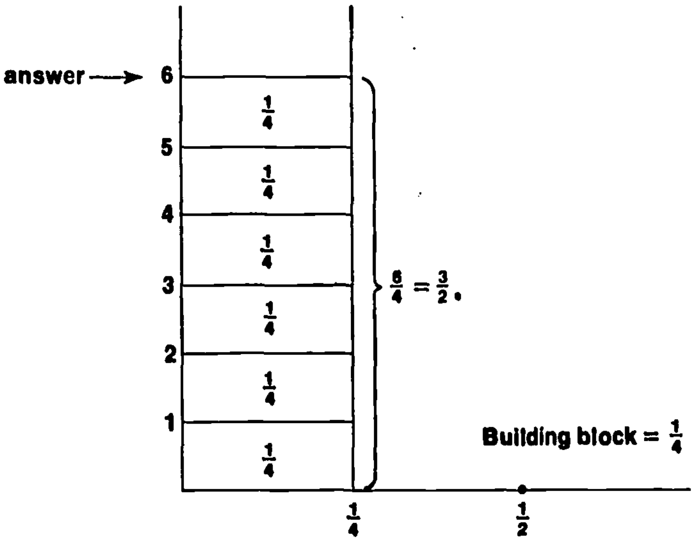
c. *Number-line method*



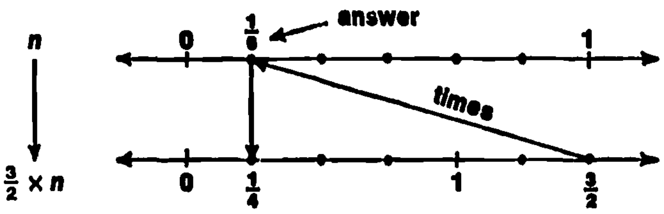
Check: $\frac{1}{4} \times 6 = \frac{6}{4} = \frac{3}{2}$.

Rectangle method

$$\frac{3}{2} \div \frac{1}{4} = \frac{6}{4} \div \frac{1}{4}.$$



d. *Number-line method*

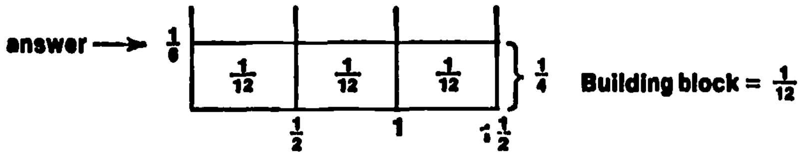


Check: $\frac{3}{2} \times \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$.

Rectangle method

$$\frac{1}{4} \div \frac{3}{2} = \frac{1 \times 3}{4 \times 3} \div \frac{3}{2}$$

$$= \frac{3}{12} \div \frac{3}{2}.$$



2. Justifying by the reciprocal method:

$$\frac{a}{d} \div \frac{c}{d} = \boxed{}$$

$$\frac{a}{d} \quad = \boxed{} \times \frac{c}{d}.$$

$$\frac{a}{d} \quad = \boxed{\frac{d}{c}} \times \frac{c}{d}.$$

$$\frac{a}{d} \quad = \boxed{\frac{a}{d} \times \frac{d}{c}} \times \frac{c}{d}.$$

$$\frac{a}{d} \div \frac{c}{d} = \boxed{\frac{a}{d} \times \frac{d}{c}}$$

$$= \frac{a \times d}{d \times c}$$

$$= \frac{a}{c}.$$

Justifying by the equal-multiplication method:

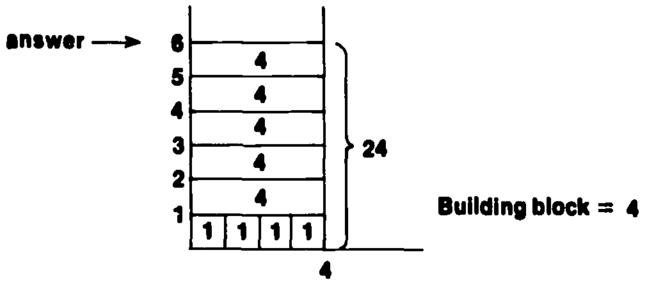
$$\frac{a}{d} \div \frac{c}{d} = \boxed{\frac{a}{d} \times \frac{d}{c}} \div \boxed{\frac{c}{d} \times \frac{d}{c}}$$

$$= \frac{a \times d}{d \times c} \div 1$$

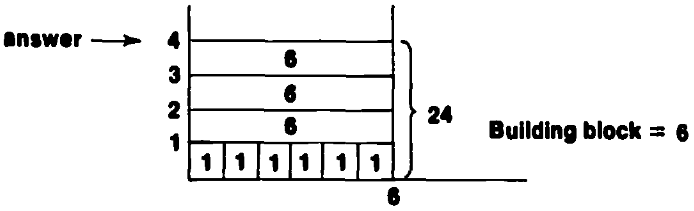
$$= \frac{a \times d}{c \times d}$$

$$= \frac{a}{c}.$$

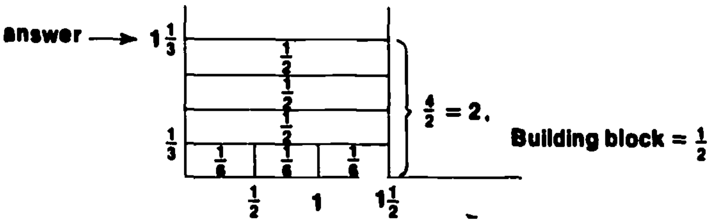
3. a.



b.



c. $$2 \div 1\frac{1}{2} = \frac{2}{1} \div \frac{3}{2} = \frac{2 \times 6}{1 \times 6} \div \frac{3}{2}.$$



$$\text{Check: } 1\frac{1}{3} \times 1\frac{1}{2} = \frac{4}{3} \times \frac{3}{2}$$
$$= \frac{12}{6}$$
$$= 2.$$

Review Exercises, pp. 177–78

1. Answers are the same, regardless of the method used. They are shown below.

a. $\frac{4}{3}$

b. $\frac{3}{4}$

c. 15

d. $\frac{1}{15}$

e. $\frac{3}{10}$

f. $\frac{10}{3}$

g. $\frac{10}{3}$

h. $\frac{3}{10}$

2. If $a \neq 0$ and $b \neq 0$, then $a \div b$ and $b \div a$ are reciprocals.

3. Probably the missing-factor method.

4. "The right one will be inverted, but so will the wrong one." Go back to basic principles when such a situation arises.

5. If the numerator and the denominator are the same, then adding the same number to both does not change the value of the fraction.

If the numerator is less than the denominator, then increasing both by the same number increases the value of the fraction: for example,

$$\frac{1}{2} < \frac{1+1}{2+1} = \frac{2}{3}.$$

A general proof that this is so may go like the following. Let $a < b$, and let r be any number greater than 0. To show that

$$\frac{a+r}{b+r} > \frac{a}{b},$$

it will suffice to show that

$$\frac{a+r}{b+r} - \frac{a}{b} > 0$$

or that

$$\frac{b(a+r) - a(b+r)}{b(b+r)} > 0$$

or that

$$ba + br - ab - ar > 0$$

or that

$$br - ar > 0$$

or that

$$r(b-a) > 0,$$

which is the case as $b > a$.

If the numerator is greater than the denominator, then increasing both by the same number decreases the value of the fraction: for example,

$$\frac{5}{2} > \frac{5+1}{2+1} = \frac{6}{3} = 2\cdot$$

A general proof that this is so can follow along the same lines as the one given above.

6. Let $r = \frac{a}{b}, \quad s = \frac{c}{d}, \quad t = \frac{e}{f}, \quad u = \frac{g}{h}.$

a.

$$\frac{rt}{st} = (rt) \div (st)$$

$$= \frac{ae}{bf} \div \frac{ce}{df}$$

$$= \frac{ae \times df}{bf \times ce}$$

$$= \frac{a \times d}{b \times c}$$ (dividing both numerator and denominator by *ef*)

$$= \frac{a}{b} \div \frac{c}{d}$$

$$= r \div s$$

$$= \frac{r}{s}.$$

b.

$$\frac{r}{s} + \frac{t}{s} = (r \div s) + (t \div s)$$

$$= \frac{ad}{bc} + \frac{ed}{fc}$$

$$= \left(\frac{a}{b} + \frac{e}{f}\right) \times \frac{d}{c}$$

$$= (r + t) \div \frac{c}{d}$$

$$= (r + t) \div s$$

$$= \frac{r + t}{s}.$$

c.

$$\frac{r}{s} \times \frac{t}{u} = (r \div s) \times (t \div u)$$

$$= \left(\frac{ad}{bc}\right) \times \left(\frac{eh}{fg}\right)$$

$$= \frac{a}{b} \times \frac{d}{c} \times \frac{e}{f} \times \frac{h}{g}$$

$$= \frac{a}{b} \times \frac{e}{f} \times \frac{d \times h}{c \times g}$$

$$= (r \times t) \div \frac{c \times g}{d \times h}$$

$$= (r \times t) \div (s \times u)$$

$$= \frac{r \times t}{s \times u}$$

$$= \frac{rt}{su}.$$

d. $$\frac{r}{s} \div \frac{t}{u} = (r \div s) \div (t \div u)$$

$$= \left(\frac{ad}{bc}\right) \div \left(\frac{eh}{fg}\right)$$

$$= \frac{ad}{bc} \times \frac{fg}{eh}$$

$$= \frac{ad}{bc} \times \left(\frac{g}{h} \times \frac{f}{e}\right)$$

$$= \frac{ad}{bc} \times \left(\frac{g}{h} \div \frac{e}{f}\right)$$

$$= (r \div s) \times (u \div t)$$

$$= \frac{r}{s} \times \frac{u}{t}.$$

DECIMALS: ADDITION AND SUBTRACTION

Exercise Set 1, pp. 188–93

1. a. 0.3
   b. 1.6
   c. 2.0
   d. 0.01
   e. 0.42
   f. 3.09
   g. 1.56
   h. 0.008
   i. 0.023

2. a. $\frac{1}{10}$
   b. $3\frac{4}{10}$
   c. $\frac{7}{100}$
   d. $\frac{18}{100}$
   e. $3\frac{47}{100}$
   f. $2\frac{6}{100}$
   g. $\frac{4}{1,000}$
   h. $\frac{37}{1,000}$

3. **a.** 0.6, $\frac{6}{10}$

b. 2.3, $2\frac{3}{10}$

c. 6.7, $6\frac{7}{10}$

d. 0.04, $\frac{4}{100}$

e. 0.12, $\frac{12}{100}$

f. 0.23, $\frac{23}{100}$

g. 6.03, $6\frac{3}{100}$

h. 0.007, $\frac{7}{1,000}$

i. 0.054, $\frac{54}{1,000}$

4. **a.** 0.8 **b.** 0.5 **c.** 2.9 **d.** 2.2 **e.** 1.4

5. Answers are in italic type.

a. 0.06, 0.07, *0.08*, *0.09*, 0.10, *0.11*, 0.12. The numbers increase by 0.01.
b. 0.05, 0.10, *0.15*, 0.20, *0.25*, *0.30*. The numbers increase by 0.05.
c. 0.18, 0.15, 0.12, *0.09*, *0.06*, *0.03*. The numbers decrease by 0.03.
d. 0.004, *0.005*, 0.006, *0.007*, *0.008*, 0.009. The numbers increase by 0.001.
e. 6.23, *6.27*, 6.31, 6.35, *6.39*, *6.43*. The numbers increase by 0.04.

6. **a.** 0.4, 1.8, 2.0, 2.1, 3.4 **b.** 0.9, 1.0, 1.4, 3.2, 4.7

7. **a.** 0.6 **b.** 1.3 **c.** 0.1 **d.** 3.4 **e.** 1.5

8. **a.** > **b.** = **c.** > **d.** < **e.** < **f.** =

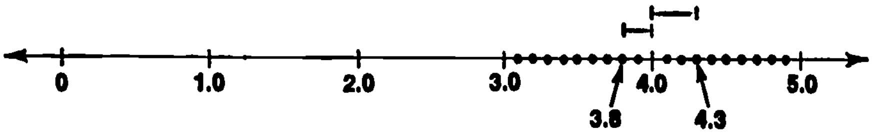
9. **a.** 0.01, 0.1, 0.15, 0.25, 0.3 **c.** 0.09, 0.25, 0.4, 0.50, 0.8
b. 0.10, 0.2, 0.33, 0.4, 0.42

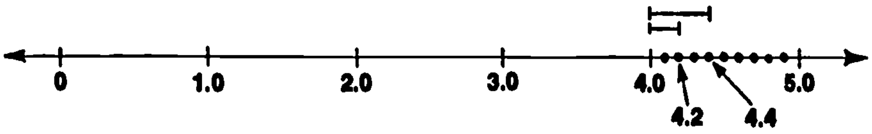
10. **a.** 12.5 **b.** 9.0 **c.** $3.50

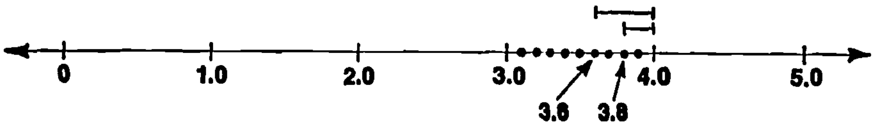
11. **a.** 6.5
b. 0.42
c. 83.79
d. 3.08
e. 0.009
f. 0.723
g. 32.46
h. 11.2
i. 8.08
j. 1.9
k. 3.25
l. 979
m. 837.96

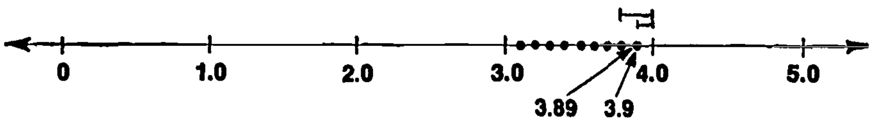
12. **a.** 3.8 miles **b.** 4.2 miles **c.** 3.8 miles **d.** 3.9 miles

13. **a.**

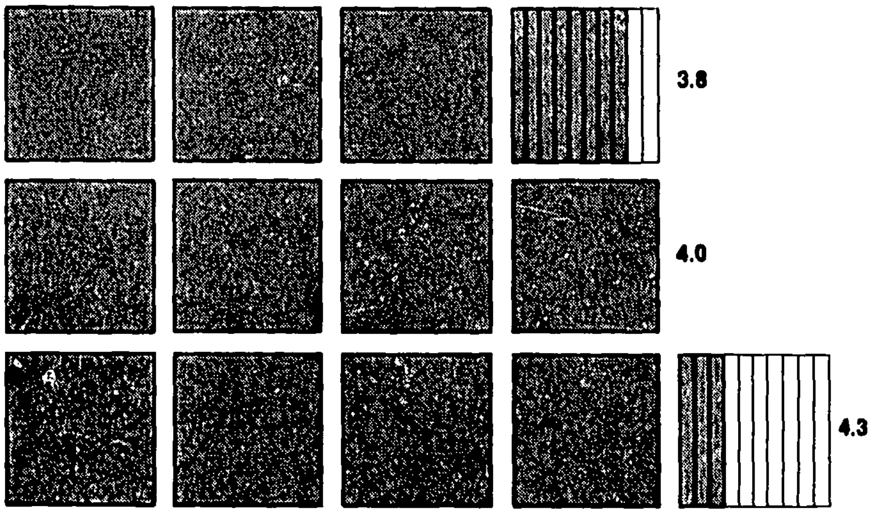








b.



These pictures are for the responses in **12a**. Similar diagrams can be used for the responses in parts **b**, **c**, and **d**.

14. **a.** 1.8 **b.** 2.07 **c.** 32.8

15. Preferable answers are the following:

a. 0.700 **b.** 0.401 **c.** 600.022

One method used to minimize confusion in situations like parts **a** and **b** is to omit or use hyphens to make the meaning clear:

Seven hundred thousandths = 0.700.

Seven hundred-thousandths = 0.00007.

Four hundred one thousandths = 0.401.

Four hundred one-thousandths = 0.400.

The statement in part c should present no difficulty if "and" is used solely to indicate the location of the decimal point.

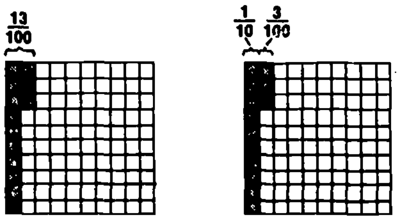
600.022 = six hundred and twenty-two thousandths.

0.622 = six hundred twenty-two thousandths.

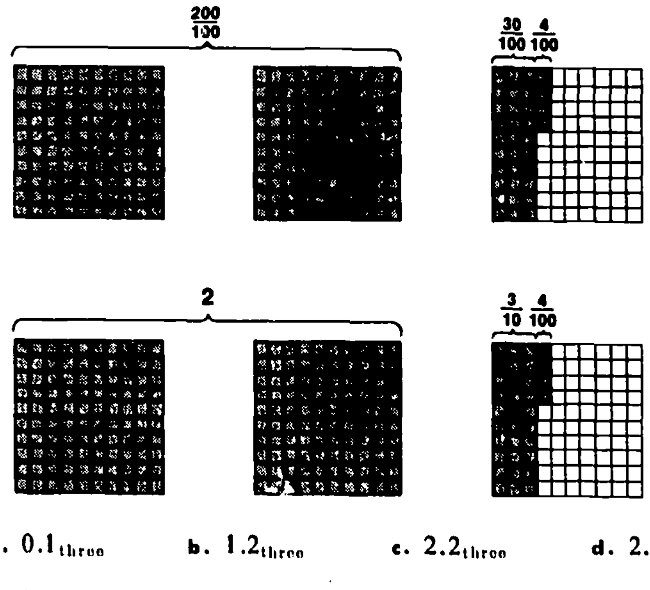
These answers do not remove oral ambiguities.

16. Answers will vary. Some activities suggested by this chapter include exercises similar to those in numbers **4** and **7**. Any exercises, activities, or questions that focus on the meaning of place value should be of help.

17. a.



b.



18. **a.** 0.1_{three} **b.** 1.2_{three} **c.** 2.2_{three} **d.** 2.02_{three}

19. **a.** $\frac{2}{3}$ **b.** $2\frac{1}{3}$ **c.** $7\frac{2}{3}$ **d.** $22\frac{2}{9}$

Exercise Set 2, pp. 197–98

1.

Fraction	*Equivalent Fraction*	*Equivalent Decimal*
$\frac{1}{4}$	$\frac{1 \times 25}{4 \times 25} = \frac{25}{100}.$	0.25
$\frac{1}{2}$	$\frac{1 \times 5}{2 \times 5} = \frac{5}{10}.$	0.5
$\frac{3}{20}$	$\frac{3 \times 50}{20 \times 5} = \frac{15}{100}.$	0.15
$\frac{9}{25}$	$\frac{9 \times 4}{25 \times 4} = \frac{36}{100}.$	0.36

2. a. 0.6 b. 0.75 c. 0.625 d. 0.06 e. 0.45 f. 0.16

3. a. 2.5 b. 3.75 c. 1.2

4. a. $\frac{1}{2} = \frac{5}{10} = \frac{50}{100} = \frac{500}{1,000}.$ c. $\frac{100}{1,000} = \frac{10}{100} = \frac{1}{10}.$

b. $\frac{2}{5} = \frac{4}{10} = \frac{40}{100} = \frac{400}{1,000}.$ d. $\frac{750}{1,000} = \frac{75}{100} = \frac{3}{4}.$

5. a. $=$ b. $>$ c. $=$ d. $>$ e. $=$ f. $<$ g. $>$

6. a. 0.4 b. 3.125

7. a. .5, .50, .500 b. .1, .100, .1000 c. 3.25, 3.250, 3.2500

8. We knew how to name as a decimal a rational number with a denominator that was a power of ten. Therefore, when given a rational number without such a denominator, we found an equivalent fraction that showed a denominator that was a power of ten.

9. Answers will vary. The prerequisite skills and understandings required include the ability to name rational numbers with a denominator that is a power of ten as a decimal, the ability to generate equivalent fractions for any given rational number, and an understanding of place value.

Exercise Set 3, p. 200

1. a. 0.40
   b. Cannot
   c. 0.25
   d. 0.14
   e. Cannot
   f. Cannot
   g. 0.26
   h. 0.625
   i. Cannot
   j. Cannot
   k. 0.50
   l. 0.36

2. a. 5, 4, 50, 20, 8, 2, 25
   b. $5 = 5.$
   $4 = 2 \times 2.$
   $50 = 2 \times 5 \times 5.$
   $20 = 2 \times 2 \times 5.$
   $8 = 2 \times 2 \times 2.$
   $2 = 2.$
   $25 = 5 \times 5.$

3. **a.** 3, 9, 7, 6, 11
b. 3 = 3. 7 = 7. 11 = 11.
9 = 3 × 3. 6 = 2 × 3.

4. Answers will vary.

Exercise Set 4, pp. 209–11

1. Answers will vary. One likeness is that the same digits are used. The major difference is that the place values represented by the digits differ.

2. Answers will vary. Although both examples represent the sum of the same numbers, they differ in the way the numbers are named.

3. **a.** 2 **b.** 7 **c.** 13 **d.** 15

4. **a.** 9.7
b. 0.68
c. 1.21
d. 4.99
e. 5.1
f. 4.43
g. 1.092
h. 46.09
i. 17.75
j. 25.17

5. **a.** 6.2
b. 1.49
c. 0.46
d. 4.3
e. 2.203
f. 0.18
g. 0.122
h. 2.71

6. **a.** $0.24 - 0.2 = 0.24 - 0.20 = 0.04.$
b. $1.7 - 0.9 = 0.8.$
c. $\frac{3}{5} - 0.5 = 0.6 - 0.5 = 0.1.$

7. The major difference is in the place values represented by the digits. Since the pattern between adjacent place values was an extension of whole-number place values, similarities include the regrouping procedure and the adding of digits in the same place-value position.

8. Both algorithms involve the computing of sums (or differences) of numbers that can be shown by like fractions.

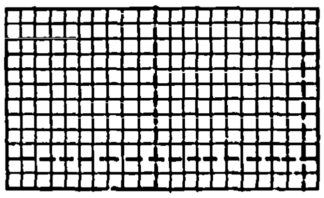
9. Answers will vary.

10. Answers will vary.

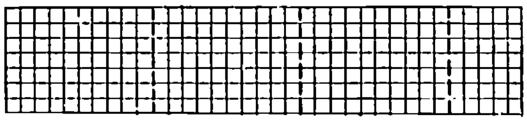
DECIMALS: MULTIPLICATION AND DIVISION

Exercise Set 1, pp. 217–18

1. **a.**



b.



2. **a.**

$$\begin{aligned} 3.7 \times 6.2 &= 3\frac{7}{10} \times 6\frac{2}{10} \\ &= \frac{37}{10} \times \frac{62}{10} \\ &= \frac{37 \times 62}{10 \times 10} \\ &= \frac{2,294}{100} \\ &= 22\frac{94}{100} \\ &= 22.94. \end{aligned}$$

b.

$$\begin{aligned} 4.1 \times .83 &= 4\frac{1}{10} \times \frac{83}{100} \\ &= \frac{41}{10} \times \frac{83}{100} \\ &= \frac{41 \times 83}{10 \times 100} \\ &= \frac{3,403}{1,000} \\ &= 3\frac{403}{1,000} \\ &= 3.403. \end{aligned}$$

c.

$$\begin{aligned} .8 \times 17.5 &= \frac{8}{10} \times 17\frac{5}{10} \\ &= \frac{8}{10} \times \frac{175}{10} \\ &= \frac{8 \times 175}{10 \times 10} \\ &= \frac{1,400}{100} \\ &= 14. \end{aligned}$$

d.

$$\begin{aligned} .67 \times 3.29 &= \frac{67}{100} \times 3\frac{29}{100} \\ &= \frac{67}{100} \times \frac{329}{100} \\ &= \frac{67 \times 329}{100 \times 100} \\ &= \frac{22,043}{10,000} \\ &= 2\frac{2,043}{10,000} \\ &= 2.2043. \end{aligned}$$

Exercise Set 2, pp. 219–20

1. 33.5	**3.** 49.8	**5.** 166	**7.** 4.12
2. 9.62	**4.** 5.56	**6.** 84.4	**8.** 4.72

Exercise Set 3, p. 221

1. Upper bound = 4.25 × 7.35 = 31.2375.
Lower bound = 4.15 × 7.25 = 30.0875.

2. Upper bound = 9.75 × 14.15 = 137.9625.
Lower bound = 9.65 × 14.05 = 135.5825.

3. Upper bound = $6.05 \times 9.15 = 55.3575$.
Lower bound = $5.95 \times 9.05 = 53.8475$.

Exercise Set 4, p. 223

1. **a.** 2,840 **b.** 6,700 **c.** 283.7 **d.** 610,000

2. **a.** 4.75×10^2 **c.** 5.7×10^4 **e.** 4.3689×10
b. 2.89×10 **d.** 3.287×10^3 **f.** 3.5×10^{10}

Exercise Set 5, p. 224

1. 1.4 **2.** 8.3 **3.** .63 **4.** 4.3 **5.** .468 **6.** .27

Exercise Set 6, p. 226

1. True.
2. False. $6.53 \div .7 = 65.3 \div 7$.
3. True.
4. True.
5. False. $.00064 \div .004 = .64 \div 4$.
6. False. $3.9 \div .003 = 3{,}900 \div 3$.
7. True.
8. True.

Exercise Set 7, pp. 230–31

1. **a.** 19.6 **b.** 18.9 **c.** 43.8 **d.** 1.9

2. **a.** $.\overline{6}$ **b.** $.\overline{571428}$ **c.** $.\overline{45}$ **d.** $.\overline{230769}$. **e.** $.\overline{189}$

3. If the decimal does not terminate, the possible remainders are 1, 2, 3, . . . , $b - 1$, so the maximum number of digits in the repeating block is $b - 1$.

Exercise Set 8, p. 232

1. Let $w = .38888\ldots.$
Then $100 \times w = 38.8888\ldots,$
and $10 \times w = 3.8888\ldots.$
So $90 \times w = 35,$
and $w = \frac{35}{90} = \frac{7}{18}.$

2. Let $w = .43434343\ldots.$
Then $100 \times w = 43.434343\ldots.$
So $99 \times w = 43,$
and $w = \frac{43}{99}.$

3. Let $w = 2.77777\ldots.$
Then $10 \times w = 27.77777\ldots.$
So $9 \times w = 25,$
and $w = \frac{25}{9}.$

4. Let $w = .499999\ldots.$
Then $100 \times w = 49.99999\ldots,$
and $10 \times w = 4.99999\ldots.$
So $90 \times w = 45,$
and $w = \frac{45}{90} = \frac{1}{2}.$

Exercise Set 9, pp. 234–35

1. **a.** $\frac{3}{4} = .75\overline{0} = .74\overline{9}$ **c.** $\frac{7}{8} = .875\overline{0} = .874\overline{9}$

b. $\frac{5}{2} = 2.5\overline{0} = 2.4\overline{9}$ **d.** $\frac{6}{25} = .24\overline{0} = .23\overline{9}$

2. Since it is nonrepeating, the expression represents an irrational number.

3. One such number is .34231010010001 . . . (where the decimal continues in the manner described in exercise **2**).

Exercise Set 10, pp. 237–38

1.

REPEATING DECIMALS

$\frac{a}{b}$	Repeating Decimal for $\frac{a}{b}$	Number of Digits in Repetend	$b - 1$
$\frac{1}{7}$	$.\overline{142857}$	6	6
$\frac{3}{13}$	$.\overline{230769}$	6	12
$\frac{1}{3}$	$.\overline{3}$	1	2
$\frac{6}{11}$	$.\overline{54}$	2	10
$\frac{4}{37}$	$.\overline{108}$	3	36
$\frac{7}{41}$	$.\overline{17073}$	5	40
$\frac{15}{73}$	$.\overline{20547945}$	8	72
$\frac{11}{101}$	$.\overline{1089}$	4	100
$\frac{30}{271}$	$.\overline{11070}$	5	270

2. The number of digits in the repetend is a divisor of $b - 1$.

3. In the computation for $\frac{2}{13}$ the numbers 7, 5, 11, 6, 8, and 2 occur as remainders. These are the numbers less than 13 that did not occur as remainders in the computation for $\frac{3}{13}$.

MEASUREMENT

A laboratory approach was chosen for this chapter, and it suggests the use of activities in place of giving exercise sets, as other chapters do. Answers will, in most cases, vary a great deal; only a few are given below.

Activity 1, p. 241

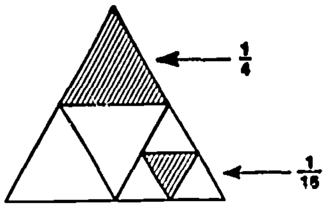
The answer to 6 is *b*, *c*, *e*, *a*, *d*.

Activity 8, p. 253

Reading across, the answers are as follows: first row, 3, 4, 9, 3, $4\frac{1}{2}$; second, $5\frac{1}{2}$, 4, 8, 8, 8; third, 2 , 3, 6, 4, 6.

Activity 14, p. 261

The answer to question 1 is shown below.



Activity 15, pp. 265–66

I am not really measuring area. My assumption is that the ratio of the areas is the same as the ratio of the weights.

Miscellaneous Problems, p. 266

1. 43 feet (or 42 ft. 11 in.)

2. The watch reading is too small. The number of seconds in a day is 86,440, and the number of seconds in 15 minutes 23.7 seconds is 923.7. Therefore the correct reading should be increased by $\frac{4 \times 60}{86,400} \times 923.7$, about 2.6 seconds. The correct

reading will then be

$$15 \text{ min. } 23.7 \text{ sec.} + 2.6 \text{ sec.} = 15 \text{ min. } 26.3 \text{ sec.}$$

3. Answers will vary.

4. 1 foot represents 8,000 miles or 1:42,000,000. The model of the moon would be about 3 inches in diameter.

5. The ratio is 8:1.

NEGATIVE RATIONALS

Exercise Set 1, pp. 270–71

1. John's reply number = 2 + class's number, or $y = 2 + x$.

2. Owl's reply number = class's number − 5.

3. Billy made up the rule. 12 is the first number given Billy, who replied, "19." Billy's reply for 100 is "107." Billy's rule is to add 7 to the number given him by the children.

Exercise Set 2, pp. 276–77

1. $B, {}^-3;\ C, {}^-1;\ D, {}^-5;\ E, {}^+1\frac{1}{2}$.

2. At B.

3. At the point for ${}^-2\frac{1}{2}$.

4. The temperature was −7°, or 7° below zero.

5. His new balance was ⁻75 dollars, or "$75 in the red."

Exercise Set 3, p. 279

1. We put 8 pebbles in the bag and take out 2, so there are now 6 more pebbles in the bag than there were before. $8 - 2 = {}^+6$.

2. We put 3 pebbles in the bag and take 4 out, so now there is 1 pebble less in the bag than before. $3 - 4 = {}^-1$.

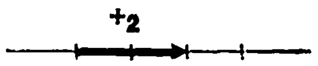
3. We put 5 pebbles in the bag and take 5 pebbles out, so there is still the same number of pebbles in the bag as when we started. $5 - 5 = 0$.

Exercise Set 4, pp. 281–82

1. ⁺2

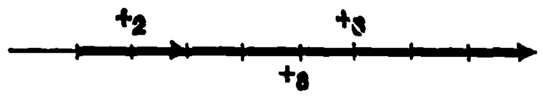
2. ⁺3

3.



4.

5.



+2 + +6 = +8.

6. +2 + +6 = 2 + 6 = 8.

7. Yes.

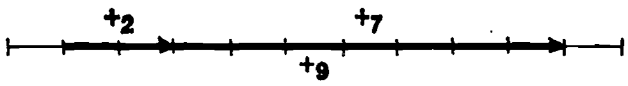
8. −4

9.

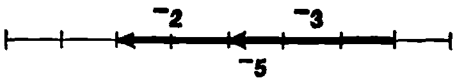
10. They point to the left.

11. They point to the right.

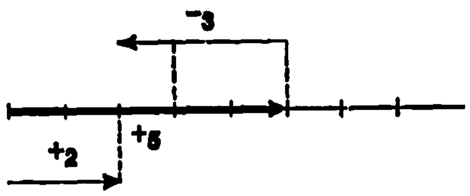
12. **a.** +2 + +7 = +9.



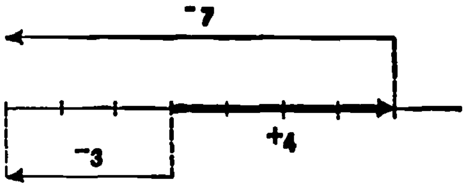
b. −3 + −2 = −5.



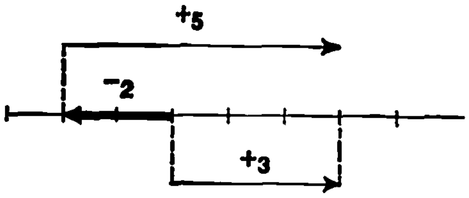
c. +5 + −3 = +2.



d. $^{+}4 + ^{-}7 = ^{-}3.$



e. $^{-}2 + ^{+}5 = ^{+}3.$



13. $^{-}5$

14. Yes.

Exercise Set 5, p. 283

1. a. $^{-}1$

b. $^{-}7$

c. $^{-}5$

d. $\frac{^{-}1}{2}$

e. $\frac{^{+}1}{2}$

f. $^{-}3$

g. $^{+}8\frac{1}{2}$

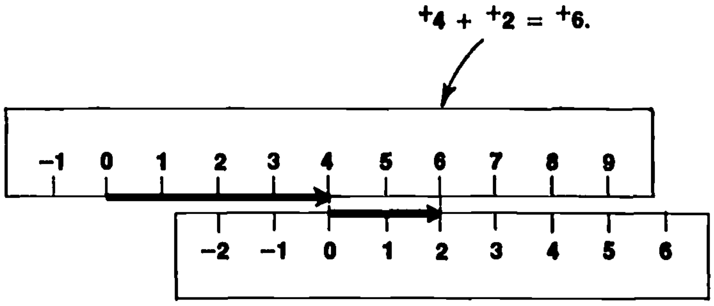
Exercise Set 6, p. 283

1. Yes.

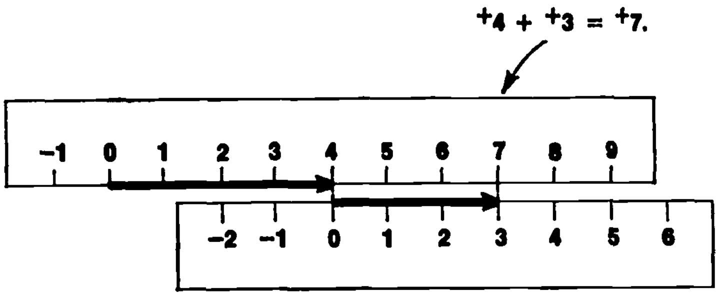
2. Use the number that indicates a motion in the opposite direction but the same distance.

Exercise Set 7, p. 285

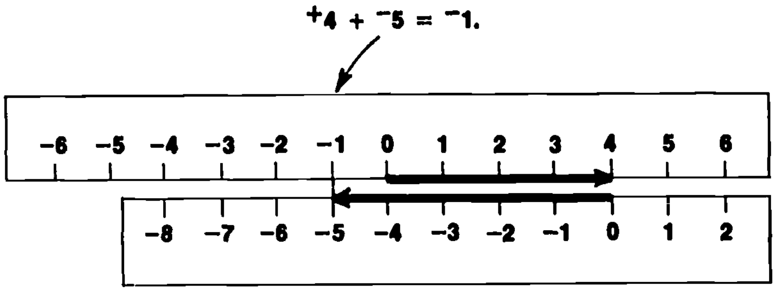
1.



2.

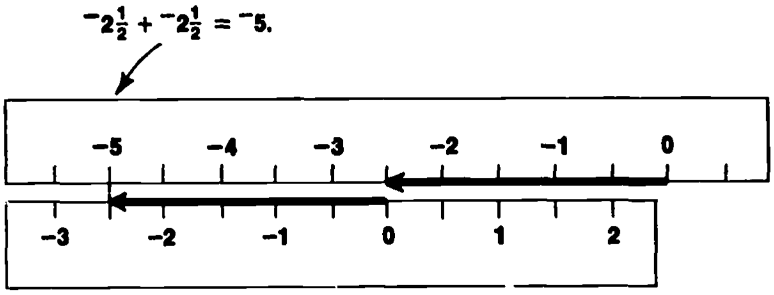

$^+4 + {}^+3 = {}^+7.$

3.


$^+4 + {}^-5 = {}^-1.$

4. a. $^-1$ b. $^+2$ c. 0 d. $^+2$ e. $^+2$

5.

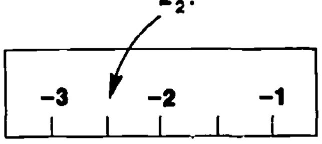

$^-2\frac{1}{2} + {}^-2\frac{1}{2} = {}^-5.$

Exercise Set 8, pp. 286–87

1. $^-100 < {}^+\frac{1}{2}.$

2. True: a, c, d, e, f, g, h. Statements b and i are false.

3.


$-2\frac{1}{2}.$

Yes. They frequently use the point midway between $^-1$ and $^-2$ for $^-2\frac{1}{2}$.

Exercise Set 9, pp. 287–88

1. All of the statements are true.
2. $\left|{}^+\frac{1}{2}\right| < |^-100|$.
3. Neither statement is true for all numbers a and b for which $a < b$.

OPERATIONS EXTENDED TO NEGATIVE RATIONALS

Exercise Set 1, pp. 301–3

1. a. 7
   - b. 7
   - c. -7
   - d. -7
   - e. 3
   - f. $\frac{1}{4}$
   - g. $-\frac{1}{4}$
   - h. $1\frac{1}{4}$, or $\frac{5}{4}$
   - i. $1\frac{1}{4}$, or $\frac{5}{4}$
   - j. $-1\frac{1}{4}$, or $-\frac{5}{4}$
   - k. $\frac{3}{4}$
   - l. $-\frac{3}{4}$
   - m. $-2\frac{1}{4}$, or $-\frac{9}{4}$
   - n. $\frac{3}{4}$
   - o. $\frac{1}{4}$
   - p. $-\frac{1}{4}$
   - q. $\frac{1}{4}$
   - r. $1\frac{1}{4}$, or $\frac{5}{4}$
   - s. $-1\frac{1}{4}$, or $\frac{5}{4}$
   - t. $-2\frac{1}{4}$, or $-\frac{9}{4}$
   - u. $1\frac{1}{4}$, or $\frac{5}{4}$
   - v. 1.5
   - w. -1.5
   - x. -3.9
   - y. 3.9
   - z. 1.5

3. a. $7 - 3 = 4$, $(-1) - 3 = -4$, $0 - 3 = -3$. Other variations are possible.
   b. $1 - (-2) = 3$, $0 - (-2) = 2$, $(-6) - (-2) = -4$. Other variations are possible.

4. a. Both yield 5.
   b. Both yield -5.
   c. Both yield -1.
   d. Both yield 1.

5. a. $2 - (-5) = (2 + 5) - (-5 + 5) = 7 - 0 = 7.$
   b. $5 - (-2) = (5 + 2) - (-2 + 2) = 7 - 0 = 7.$
   c. $(-2) - (5) = (-2 + 2) - (5 + 2) = 0 - 7 = -7.$
   d. $(-5) - (2) = (-5 + 5) - (2 + 5) = 0 - 7 = -7.$
   e. $(-2) - (-5) = (-2 + 5) - (-5 + 5) = 3 - 0 = 3.$

Exercise Set 2, pp. 309–10

1. a. -10
   b. -10
   c. 10
   d. -3
   e. -3
   f. 3
   g. $-\frac{3}{2}$, or $-1\frac{1}{2}$
   h. $-\frac{3}{2}$, or $-1\frac{1}{2}$
   i. $\frac{3}{2}$, or $1\frac{1}{2}$
   j. -3
   k. $-\frac{6}{4}$, or $\frac{3}{2}$, or $1\frac{1}{2}$
   l. $-\frac{15}{4}$, or $-3\frac{3}{4}$
   m. -4
   n. $-\frac{6}{4}$, or $-\frac{3}{2}$, or $-1\frac{1}{2}$
   o. 3
   p. $\frac{6}{4}$, or $\frac{3}{2}$, or $1\frac{1}{2}$
   q. $\frac{15}{4}$, or $3\frac{3}{4}$
   r. $\frac{6}{4}$, or $\frac{3}{2}$, or $1\frac{1}{2}$
   s. -6
   t. -6
   u. 6
   v. 6
   w. -9
   x. 9
   y. $-\frac{27}{8}$, or $-3\frac{3}{8}$
   z. $\frac{27}{8}$, or $3\frac{3}{8}$
   aa. -3
   bb. 3
   cc. -3.25
   dd. 3.4
   ee. -9
   ff. 1.8

2. a. Richer, poorer, poorer, richer (by \$6 each time)
   b. Product of two positive numbers
      Product of a positive and a negative number
      Product of a negative and a positive number
      Product of two negative numbers
   c. Problem a of exercise 1 means that the postman delivers two bills for \$5 each. Problem b means that we mail out five checks for \$2 each. Problem c means that we mail out two bills for \$5 each.
   d. Yes in problems d, f, and g of exercise 1, provided we interpret "$-\frac{1}{2}$" as a

bill for 50 cents. No in problems g and h because there cannot be such a thing as one-half of a check or one-half of a bill.

3. a. $2 \times (-5) + 2 \times 5 = 2 \times (-5 + 5) = 2 \times 0 = 0$. Therefore $2 \times (-5)$ is the additive inverse of 2×5, that is, it is -10.
   b. $(-2) \times 5 + 2 \times 5 = (-2 + 2) \times 5 = 0 \times 5 = 0$. Therefore $(-2) \times 5$ is the additive inverse of 2×5, that is, it is -10.
   c. $(-2) \times (-5) + (-2) \times 5 = (-2) \times (-5 + 5) = (-2) \times 0 = 0$. Therefore $(-2) \times (-5)$ is the additive inverse of $(-2) \times 5$, that is, the additive inverse of -10, which is 10.
   d. $a \times (-b) + a \times b = a \times (-b + b) = a \times 0 = 0$. Therefore $a \times (-b)$ is the additive inverse of $a \times b$, that is, $-(a \times b)$.
   e. $(-a) \times (-b) + a \times (-b) = (-a + a) \times (-b) = 0 \times (-b) = 0$. Therefore $(-a) \times (-b)$ is the additive inverse of $a \times (-b)$, that is, the additive inverse of $-(a \times b)$, which is $a \times b$.

Exercise Set 3, pp. 312–13

1. a. -2
   b. -2
   c. 2
   d. -3
   e. 2
   f. $\frac{1}{2}$
   g. $-\frac{1}{4}$
   h. $-\frac{1}{3}$
   i. $-3\frac{1}{2}$, or $-\frac{7}{3}$
   j. $\frac{1}{4}$
   k. -4
   l. $\frac{1}{3}$
   m. $\frac{1}{3}$
   n. -3
   o. $-3\frac{1}{3}\left(\text{or } -\frac{10}{3}\right)$
   p. $3\frac{1}{3}\left(\text{or } \frac{10}{3}\right)$
   q. $-\frac{3}{10}$
   r. $-2\frac{1}{3}\left(\text{or } -\frac{7}{3}\right)$
   s. $\frac{3}{7}$
   t. $-.5$
   u. $.5$
   v. -5
   w. 5
   x. 2
   y. 2
   z. $\frac{1}{5}$ (or .2)
   aa. -2

2. k. $3 \div \left(-\frac{3}{4}\right) = (3 \times 4) \div \left(-\frac{3}{4} \times 4\right) = 12 \div (-3) = -4.$
   l. $\frac{1}{2} \div \frac{3}{2} = \left(\frac{1}{2} \times 2\right) \div \left(\frac{3}{2} \times 2\right) = 1 \div 3 = \frac{1}{3}.$

o. $2\frac{1}{2} \div \left(-\frac{3}{4}\right) = \left(2\frac{1}{2} \times 4\right) \div \left(-\frac{3}{4} \times 4\right) = 10 \div (-3) = -3\frac{1}{3}.$

p. Similarly gives $(-10) \div (-3) = 3\frac{1}{3}.$

q. Similarly gives $3 \div (-10) = \frac{3}{-10} = -\frac{3}{10}.$

v. $(-2.5) \div .5 = (-2.5 \times 2) \div (.5 \times 2) = (-5) \div 1 = -5.$

z. Similarly gives $(-1) \div (-5) = \frac{-1}{-5} = \frac{1}{5}$, or .2.

aa. Similarly gives $2 \div (-1) = \frac{2}{-1} = -2.$

3. **j.** $\left(-\frac{3}{4}\right) \div (-3) = \left(-\frac{3}{4}\right) \times \left(-\frac{1}{3}\right) = \frac{1}{4}.$

k. $3 \div \left(-\frac{3}{4}\right) = 3 \times \left(-\frac{4}{3}\right) = -4.$

m. $\left(-\frac{1}{2}\right) \div \left(-\frac{3}{2}\right) = \left(-\frac{1}{2}\right) \times \left(-\frac{2}{3}\right) = \frac{1}{3}.$

o. $2\frac{1}{2} \div \left(-\frac{3}{4}\right) = 2\frac{1}{2} \times \left(-\frac{4}{3}\right) = \frac{5}{2} \times \left(-\frac{4}{3}\right) = -\frac{20}{6} = -\frac{10}{3}.$

q. $\frac{3}{4} \div \left(-2\frac{1}{2}\right) = \frac{3}{4} \div \left(-\frac{5}{2}\right) = \frac{3}{4} \times \left(-\frac{2}{5}\right) = -\frac{6}{20} = -\frac{3}{10}.$

Review Exercises, pp. 314–15

1. **a.** -3 **b.** $-\frac{1}{4}$ **c.** $\frac{9}{4}$, or $2\frac{1}{4}$ **d.** $\frac{3}{4}$ **e.** -1.5

2. **a.** -10

b. 10

c. 3

d. $-\frac{3}{4}$

e. -6

f. -9

g. -3

h. 3.4

i. -6

j. 1.5, or $1\frac{1}{2}$, or $\frac{3}{2}$

The Rational Numbers

3. a. -3
   b. 3
   c. $-\frac{1}{4}$
   d. $\frac{1}{3}$
   e. $-\frac{7}{2}$, or $-3\frac{1}{2}$
   f. $-\frac{1}{4}$
   g. -4
   h. $-\frac{1}{3}$
   i. 3
   j. $-\frac{10}{3}$, or $-3\frac{1}{3}$
   k. $-\frac{7}{3}$, or $-2\frac{1}{3}$
   l. $-.5$, or $-\frac{1}{2}$
   m. $-.5$, or $-\frac{1}{2}$
   n. -5
   o. $-\frac{1}{5}$, or $-.2$
   p. 5

4. a. 14 b. 2 c. -2 d. -6.7 e. 24

GRAPHING

Exercise Set 1, p. 318

1. No, because of rounding off to the nearest tenth.

Exercise Set 2, p. 320

2. Answers vary.

3. The track width is 56.5 inches.

For an HO model, the track width is

$$56.5 \div 87.1 = .649 \text{ inches.}$$

For an N model, the track width is

$$56.5 \div 160 = .353 \text{ inches.}$$

For an S model, the track width is

$$56.5 \div 6.4 = .883 \text{ inches.}$$

For an O model, the track width is

$$56.5 \div 48 = 1.177 \text{ inches.}$$

Exercise Set 4, pp. 323–24

1. The golf tees lie on a line. The slope is $\frac{5}{2}$, or $2\frac{1}{2}$.

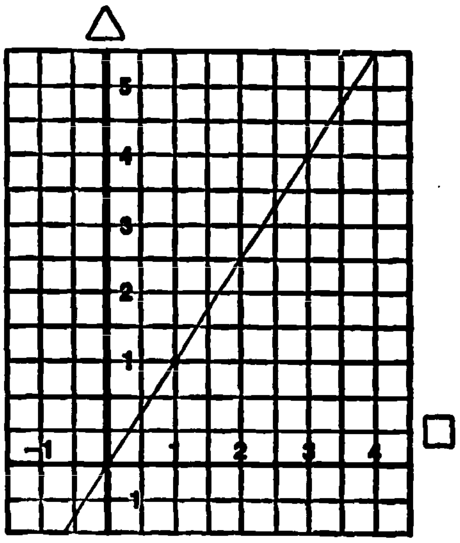
2. Answers vary.

3. $\frac{3}{7} > \frac{2}{5}$, since $\frac{6}{14} > \frac{6}{15}$ $\left(\text{also, } \frac{15}{35} > \frac{14}{35}\right)$.

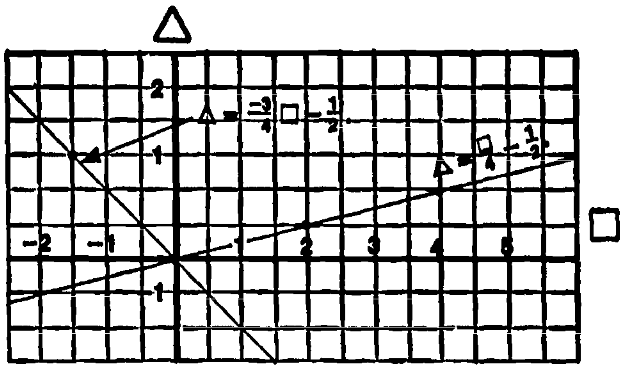
4. $\frac{9}{10} > \frac{6}{7} > \frac{4}{5}$.

Exercise Set 5, p. 329

1. Some pairs are $(0, -\frac{1}{2})$, $(1,1)$, $(3,4)$, and $(4, 5\frac{1}{2})$. The graph is shown below. The slope is $\frac{3}{2}$.

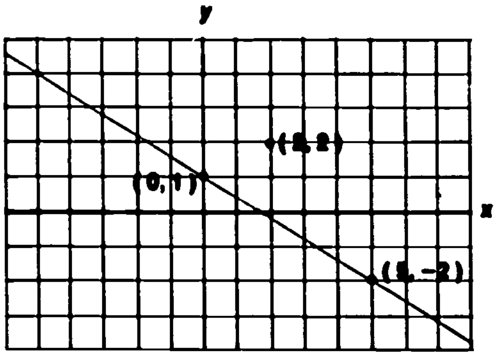


2.



Exercise Set 6, pp. 330–31

1. Some number pairs that satisfy the open sentence are (0,1) and (5,−2), shown on the graph below.



2. The number pair (2,2), for example, added to the graph above, does not satisfy $y = \frac{-3x}{5} + 1$. It does satisfy $y > \frac{-3x}{5} + 1$.

3. Answers vary.

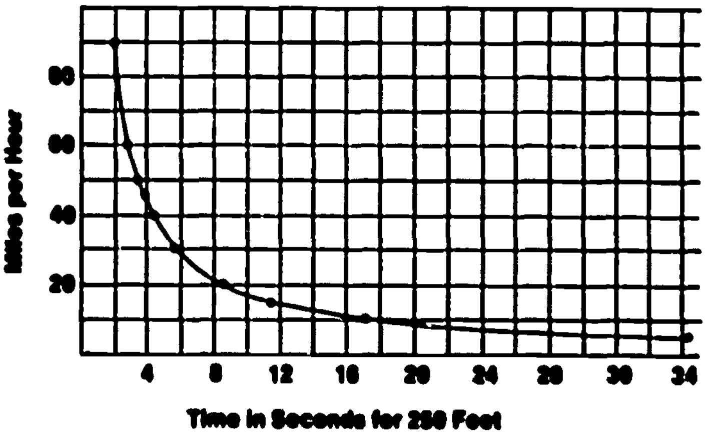
4. Points below and to the left satisfy $y < \frac{-3x}{5} + 1$.

Exercise Set 7, pp. 336–37

1.

Miles per Hour	*Feet per Second*	*Time in Seconds to Travel 250 Feet*
90	132	1.9
60	88	2.8
50	73.3	3.4
45	66	3.8
40	58.7	4.3
30	44	5.7
20	29.3	8.5
15	22	11.4
10	14.7	17.0
5	7.3	34.1

2. The graph for converting time to travel 250 feet to speed in miles per hour is shown below.



The time would be more than 40 seconds if the rate is less than 2 miles per hour. If the time is 15 seconds, the speed is 11 miles per hour.

GLOSSARY

The particular words and symbols in this glossary have been selected to help clear up any possible misunderstanding. Usually a description rather than a precise definition is given. Moreover, not all the meanings are given but only those needed for the text materials. Examples are provided to clarify meanings still further.

Abacus. An ancient device (still used today) for computing. A common type consists of a frame with parallel rods. The rods are usually matched with the ones place, the tens place, the hundreds place, and so on. Movable counters along the rods record numbers and are used to carry out computation.

Absolute value. The absolute value of any number r is the greater of the numbers r and ^-r. It is symbolized by "$|r|$". Thus,

$|r|$ is the greater of r and ^-r.

If $r = 0$, $|r| = |0| = 0$.

$|r|$ is the distance of the point for 0 from the point for r.

Addend. One of the numbers added to determine a sum. When a pair of numbers is associated with a sum under addition, each number of the pair is called an addend of the sum. In the sentence $6 + 7 = 13$, the numbers 6 and 7 are addends. In more general terms, $a + b$ is the sum of its addends a and b. In the sentence $6 + \square = 13$, one of the addends is "missing." See **Missing addend.**

Addition. With every pair of numbers a and b, addition associates the sum $a + b$. For example, with the pair 13 and 6, addition associates $13 + 6$, or 19. The sum $a + b$ may be determined in the following way:

If A and B are disjoint sets such that $n(A) = a$ *and* $n(B) = b$, *then*

$$a + b = n(A \cup B).$$

Addition property of zero. For every rational number b, $b + 0 = b$ and $0 + b = b$. Informally stated, the sum of every rational number and zero is the given rational number. See **Identity element for addition.**

Additive inverse. If $a + b = 0$, a is the additive inverse of b and b is the additive inverse of a. Example: $^{-}3$ is the additive inverse of 3, and 3 is the additive inverse of $^{-}3$.

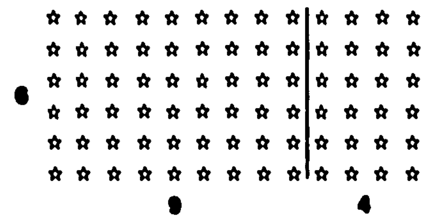
Additive property of numeration systems. Each symbol in a numeral stands for a number. The sum of these numbers is the value of the numeral. In "XXIII", the individual symbols stand for 10, 10, 1, 1, and 1. Because of the additive property, the number represented by "XXIII" is 10 + 10 + 1 + 1 + 1, or 23.

Algorithm (algorism). A systematic, step-by-step procedure for reaching some goal. An algorithm for a subtraction computation is a systematic procedure for obtaining a standard name for a difference. See **Compute.**

One of the possible algorithms for computing 624 − 397 yields the following steps:

$$\text{(a)}\quad \begin{array}{r} 6\;2^{1}4 \\ -3\;9_{1}7 \\ \hline 7 \end{array} \qquad \text{(b)}\quad \begin{array}{r} 6^{1}2^{1}4 \\ -3_{1}9_{1}7 \\ \hline 2\;7 \end{array} \qquad \text{(c)}\quad \begin{array}{r} 6^{1}2^{1}4 \\ -3_{1}9_{1}7 \\ \hline 2\;2\;7 \end{array}$$

Array, rectangular. A rectangular arrangement of objects in rows and columns. The array below, viewed both as a whole and as split into two parts, illustrates the distributive property.



The large array has 6 rows and 13 columns. We say that it is a 6-by-13 or 6 × 13 array. The 6 × 13 array is shown partitioned into two arrays, a 6 × 9 array and a 6 × 4 array, showing that $6 \times (9 + 4) = (6 \times 9) + (6 \times 4)$.

Associative property of addition. (Also called the **grouping property of addition.**) Whenever a, b, and c are rational numbers, $a + (b + c) = (a + b) + c$. That is, when numbers are added, the grouping of the numbers does not affect the sum. An instance of this property is the fact that $6 + (9 + 4) = (6 + 9) + 4$. Subtraction, on the other hand, is not associative. A single exception, although there are many, suffices to show this:

$$(8 - 5) - 2 \neq 8 - (5 - 2).$$

Associative property of multiplication. (Also called the **grouping property of multiplication.**) Whenever a, b, and c are rational numbers, $a \times (b \times c) = (a \times b) \times c$. That is, when numbers are multiplied, the grouping of the factors does not affect the product. An instance of this property is the fact that $3 \times (7 \times 5) = (3 \times 7) \times 5$. Division, on the other hand, is not associative. A single exception, although there are many, suffices to show this:

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2).$$

Base. A number used as a repeated factor. In the expression $10^3 = 10 \times 10 \times 10$, for example, the base is shown to be 10. We refer to "3" as the exponent. In the expression "4^3", 4 is the base. In general, "b" is the base for "b^n". See **Exponent** and **Factor.**

The symbol "10" is a name for the number ten in our Hindu-Arabic decimal numeration system, but it is not a name for the number ten in systems with other bases.

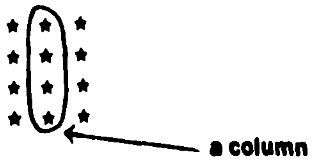
Base-sixty system. A system of writing numerals designed to represent ones, sixties, sixty sixties, and so on.

Base-ten system. A system of writing numerals based upon ones, tens, ten tens, and so on. The Egyptian system of numeration is a base-ten system, as is the Hindu-Arabic system.

Cancellation property of addition. If $r + s = t + s$, then $r = t$. Example: If $n + 3 = 7 + 3$, then $n = 7$.

Cancellation property of multiplication (restricted). In general terms, if $rs = ts$ and $s \neq 0$, then $r = t$.

Column. A vertical line of objects in an array. The array below has three columns.



Commutative property of addition. Also called the **order property of addition.** Whenever a and b are rational numbers, $a + b = b + a$. That is, when two numbers are added, the order in which the numbers are added or the order of the addends does not affect the sum. An instance of this property is the fact that $9 + 4 = 4 + 9$.

Commutative property of multiplication. Also called the **order property of multiplication.** Whenever a and b are rational numbers, $a \times b = b \times a$. That is, when two numbers are multiplied, the order in which they are multiplied or the order of the factors does not affect the product. An instance of this property is the fact that $6 \times 14 = 14 \times 6$.

Composite. An integer n is composite if it has an integer divisor other than 1, -1, n, and $-n$. Examples: 4, -6, 33. The divisors of -6 are 1, -1, 2, -2, 3, -3, 6, -6,

Computation. A process for finding the standard numeral for a sum, a product, etc.; a process for finding a standard name.

Compute. To find a standard numeral for a sum, a product, etc. To compute the sum of 34 and 8 means to find the standard numeral for $34 + 8$, namely "42". To find a standard name.

Correspondence. A pairing of the members of two sets whereby each member of the first set is paired with a member of the second set and never with more than one member of the second set. See also **One-to-one correspondence** and **Operation.**

Counting. The process of pairing the elements of a set with the counting numbers taken one after another in order of "size" and starting with 1. If this process stops, the last counting number used is the number of elements in the set being counted. When this happens, the set is said to be a *finite set*, and the number associated with the set a *finite number*. Every whole number is a finite number.

Counting number. Any whole number other than 0. (Some authors include 0 among the counting numbers.)

Cross product. The cross product of a pair of sets is the set of all ordered pairs whose first element is from the first set and whose second element is from the second set. The cross product of $\{a, b\}$ and $\{x, y, z\}$ is $\{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$. See **Symbol,** "$\times$".

Decimal. Pertaining to ten (from the Latin word *decima*, meaning "tithe" or "a tenth part"). Also, a numeral in base-ten numeration consisting of a string of digits with a decimal point. (Also called a decimal numeral.) Examples: 1.7, 2.35, 2.03005010, .46666 . . . .

Decimal numeration system. A system for naming numbers based on tens. See **Hindu-Arabic system of numeration.**

Denominator. The number named by the numeral written below the fraction bar of a fraction is the fraction's denominator. For example, the denominator for each of the following fractions is 7:

$$\frac{2}{7}, \quad \frac{-3}{7}, \quad \frac{9}{7}.$$

Difference. A number assigned to pairs of rational numbers by subtraction. $18 - 11$, or 7, is the difference of 18 and 11. See **Missing addend.** $a - b$ is the difference of a and b. If a set A has a elements and one of its subsets, B, has b elements, then the number of elements in A but not in B is $a - b$. Alternately, the difference $a - b$ is the missing addend in $\square + b = a$.

Digits. The basic symbols in a numeration system. In the Hindu-Arabic system the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Direct measurement. A process for finding a number associated with a property of an object by seeing how many times a specified unit can be used to obtain this property of the object. A measurement in which the actual unit is used to obtain the measure.

Disjoint sets. Two sets are disjoint if they have no elements in common. If $a, b, c, d,$ and e are distinct objects, then the sets $\{a, b\}$ and $\{c, d, e\}$, are disjoint; but the sets $\{a, b\}$ and $\{b, c\}$ are not disjoint.

Distributive property, or **distributive property of multiplication over addition.** Whenever a, b, and c are rational numbers,

$$a \times (b + c) = (a \times b) + (a \times c).$$

An instance of this property is the fact that

$$13 \times (8 + 7) = (13 \times 8) + (13 \times 7).$$

Because multiplication is commutative, the distributive property may also be written in the form $(b + c) \times a = (b \times a) + (c \times a)$.

Distributive principle for division. Division distributes over addition from the right. In general terms,

$$(r + s) \div t = (r \div t) + (s \div t).$$

Dividend. In the sentence $a \div b = q$, the number a is called the dividend. The number a is also called the dividend in the sentence $a = (q \times b) + r$, with $r < b$. In the sentence $15 \div 3 = 5$, the number 15 is the dividend. In the sentence $15 = (2 \times 7) + 1$, the number 15 is again the dividend.

Division. With pairs of rational numbers a and b, $b \neq 0$, division associates the quotient, $a \div b$. Division assigns to pairs of rational numbers a and b a unique rational number $a \div b$. Such a unique rational number $a \div b$ exists provided $b \neq 0$ and there is a number c such that $c \times b = a$. For example, $51 \div 3 = 17$ because $17 \times 3 = 51$. (Of course $3 \neq 0$.) Division assigns $72 \div 9$, or 8, to the pair 72 and 9. The standard name for $a \div b$ can be obtained in three ways:

1. If a set of a elements can be partitioned into disjoint subsets of b elements each, then the number of subsets thus formed is $a \div b$.
2. If an array has a elements and b rows, then the number of columns of the array is $a \div b$. If an array has a elements and b columns, then the number of rows is $a \div b$.
3. If a and b are whole numbers, the rational number that correctly completes the sentence $b \times \square = a$, or $\square \times b = a$, is $a \div b$, provided there is exactly one such rational number.

Division by zero. Division by zero has no meaning. The expressions $5 \div 0$, $18 \div 0$, $0 \div 0$, $1 \div 0$, etc., do not name numbers. Division by 0 is meaningless because there are no rational numbers that fit sentences like

$$0 \times \square = 5, \quad \square \times 0 = 18, \quad 0 \times \square = 1, \quad \text{etc.,}$$

and because every rational number fits the sentence $0 \times \square = 0$.

Division with a remainder assigns a quotient and a remainder to a pair of whole numbers. If a and b are whole numbers ($b \neq 0$), then there are whole numbers, a quotient q and a remainder r (with $r < b$), which satisfy the equation

$$a = (b \times q) + r.$$

If $a = 23$, $b = 4$, division with a remainder determines $q = 5$ and $r = 3$.

$$23 = (4 \times 5) + 3.$$

Divisor. In the sentence $a \div b = q$, the number b is called the divisor. The number b is also called the divisor in the sentence $a = (q \times b) + r$. For example, in the sentence $15 \div 3 = 5$, the number 3 is the divisor; in the sentence $15 = (3 \times 4) + 3$, or $15 = (\square \times 4) + \triangle$, the number 4 is the divisor.

Element of a set. Each object in any nonempty set of objects is an element of the given set. For example, the set {New York, California, Michigan} has three elements: New York, California, and Michigan.

Empty set. The set that has no elements, the null set. Often designated by either the symbol "{ }" or "∅". Examples: the set of people 30 feet tall; the set of female presidents of the United States.

Equal addition principle. Adding the same number to both the sum and the given addend does not change the missing addend. In general terms,

$$a - b = (a + c) - (b + c).$$

Equal multiplication principle. Multiplying a product and a given factor by the same number does not change the missing factor. In general terms,

$$a \div b = (ac) \div (bc) \quad \text{for} \quad c \neq 0.$$

Equal sign. See **Symbol.**

Equivalence class. The equivalence class for a rational number r is the set of all fractions that name r where each fraction indicates a quotient of integers with each integer expressed by a standard numeral. Example: $\frac{1}{2}$ has the equivalence class

$$\left\{\frac{1}{2}, \frac{-1}{-2}, \frac{2}{4}, \frac{-2}{-4}, \ldots\right\}.$$

Equivalent. If there is a one-to-one correspondence between two sets, then the sets are said to be equivalent. Sets that are equivalent are assigned the same number. Sets that are not equivalent are not assigned the same number. Examples of equivalent sets are $\{a, b\}$ and {blue, green}.

Equivalent numeral. If two numerals name the same number, they are called equivalent numerals. Examples:

$$\frac{3}{2} \quad \text{and} \quad 1.5$$

$\frac{2}{3}$ and $\frac{4}{6}$ (which are also equivalent fractions)

$.\overline{9}$ and 1

$.24\overline{9}$ and $.25$ (which are also equivalent decimals)

Expanded form. An expanded form of a decimal numeral is a numeral that shows explicitly the place value of the digits in a decimal numeral. Expanded forms of the numeral 456 include:

$$400 + 50 + 6$$
$$(4 \times 100) + (5 \times 10) + (6 \times 1)$$
$$(4 \times 10^2) + (5 \times 10) + (6 \times 1)$$
$$(4 \times 10^2) + (5 \times 10^1) + (6 \times 10^0)$$

Exponent. A number used to indicate a repeated factor. The repeated factor is called the base. In 10^2, "2" is the exponent and "10" is the base. 10^2 means 10 × 10. In 10^3, "3" is the exponent and "10" is the base. 10^3 means 10 × 10 × 10. See **Base.**

Factor. One of the numbers multiplied to determine a product. When a pair of numbers is associated with a product under multiplication, each number of the pair is called a factor of the product. In the sentence $3 \times 4 = 12$, 3 and 4 are factors of 12. In general terms, if $a \times b = c$, a and b are factors of c. In the sentence $3 \times \square = 12$, one of the factors is "missing." See **Missing factor.**

Family. A collection of sets every two of which are equivalent.

Finite decimal. See **Terminating decimal.**

Four-in-a-Row. A game in which two teams play, taking turns to mark points in a plane having integer number pairs. The team first to obtain four consecutive marked points vertically, horizontally, or diagonally wins.

Fraction. A symbol consisting of a numeral written over a bar (usually horizontal), which is over another numeral. Examples:

$$\frac{-4}{3}, \quad \frac{\sqrt{2}}{5}, \quad \frac{2 \longleftarrow \text{numerator}}{3 \longleftarrow \text{denominator}} \text{— fraction bar}$$

Fraction in lowest terms. If both the numerator and the denominator of a fraction are integers and have for their greatest common divisor 1 or −1, then the fraction is in lowest terms. Also called a reduced fraction. Examples:

$\frac{4}{9}$ is a fraction in lowest terms.

$\frac{4}{6}$ is not in lowest terms.

Frame. A shape in which a symbol is to be written. Some of the frames most frequently used are shown in the following sentences:

$\square + 3 = 5.$ $\qquad$ $4 + 7 \bigcirc 22.$

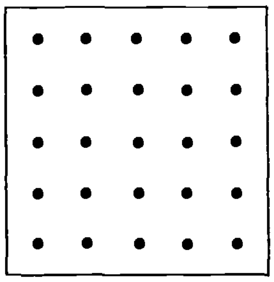
⌓ − ⌂ > 6. $\qquad$ △ − ▽ < 10.

4 ▽ 5 > 7.

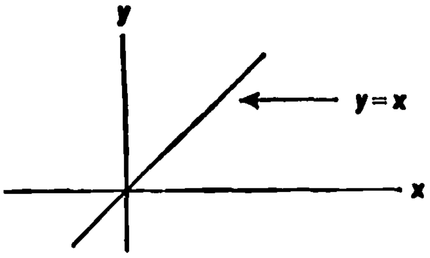
Function. Any nonempty set of ordered pairs no two of which have the same first number.

Fundamental theorem of arithmetic. Every positive composite integer can be expressed as a product of positive prime numbers in exactly one way except that the order of the factors may vary.

Geobaord. A visual aid consisting of a board containing n^2 nails or pegs placed at corners of squares. An example is shown at the right.



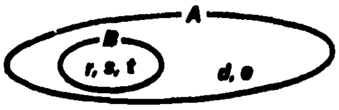
Graph. A graph pictures numerical quantities. The graph for $y = x$ is a straignt line, as shown here.



To graph is to picture with a graph.

Greater than.

1. Whole number a is greater than whole number b if there is a whole number c, other than 0, such that $a = b + c$. For example, 5 is greater than 3 because there is the whole number 2 such that $5 = 3 + 2$. Also holds for rational numbers in general.
2. Whole number a is greater than whole number b if there are two sets A and B such that set A contains all the elements of set B, A has at least one element not in B, a is the number of elements in set A, and b is the number of elements in set B. For example, 5 is greater than 3 because (see diagram) $n(A) = 5$, $n(B) = 3$, and set A has elements d and e which are not in set B.



3. Let whole number a be the number of elements in set A. Let whole number b be the number of elements in set B. We say that a is "greater than" b if and only if set B can be matched with a proper subset of set A. (In this case, we say that set A has "more" elements than set B.)

4. Integer a is greater than integer b, symbolized by "$a > b$", if and only if the point for a on the number line is to the right of the point for b.
5. If $c > 0$, then $\frac{a}{c} > \frac{b}{c}$ if and only if $a > b$.
6. Rational number r is greater than rational number q if and only if the point for r on the number line is to the right of the point for q.

Greatest common divisor (GCD). The greatest common divisor of a set of two or more integers is the greatest integer that divides each integer in the set. Example: The GCD of {12, 18, 60} is 6.

Grouping property of addition. See **Associative property of addition.**
Grouping property of multiplication. See **Associative property of multiplication.**

Guess My Rule. A game played by a leader and the class. A student supplies a number, which is entered in a table. The leader enters his reply number using his rule. The class tries to guess his rule. The student who guesses his rule becomes the leader, and a new game begins. The rule for the table shown could be $\triangle \times 3 = \square$.

Student's number △	Leader's reply number □
1	3
4	12
6	18

Hindu-Arabic system of numeration. Our decimal system for naming numbers. All whole numbers can be expressed using ten digits and the idea of place value.

Hypsometer. An instrument for obtaining an indirect measurement for a length, based on the use of similar triangles.

Identity element for addition. The number 0 is the identity element for addition of rational numbers because whenever b is a rational number, $b + 0 = b$ and $0 + b = b$. That is, when 0 is an addend, the sum is the same number as the other addend. The number 0 is sometimes called the *neutral* element for addition. Examples: $4 + 0 = 4$, $0 + 56 = 56$, etc.

Identity element for multiplication. The number 1 is the identity element for multiplication of rational numbers because whenever b is a rational number, $b \times 1 = b$ and $1 \times b = b$. That is, when 1 is a factor, the product is the same as the other factor. The number 1 is sometimes called the *neutral* element for multiplication. Examples: $4 \times 1 = 4$, $1 \times 56 = 56$, etc.

Indirect measurement. Any measurement in which the unit itself is not used to obtain the measure.

Integer. Any member of the set of integers

$$\{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

Any number that can be obtained by adding or substracting ones.

Irrational number. Any real number that is not a rational number. It is assumed that for each point on a number line there is a corresponding number. It can be shown that some points do not correspond to rational numbers, that is, the rational numbers do not "use up" all the points on a line. Any point that does not correspond to a rational number can be thought of as corresponding to an irrational number. The decimal representation of an irrational number must be nonterminating and nonrepeating. Examples:

$$\sqrt{2}, \quad \sqrt{3}, \quad \pi, \quad .101100111000111110000\ldots$$

Known addend. In a sentence such as $\square + 8 = 14$, 8 is the known addend, or given addend. See **Missing addend.**

Known factor. In a sentence such as $\square \times 3 = 12$, 3 is the known factor, or given factor. See **Missing factor.**

Least common multiple (LCM). The least common multiple of a set of two or more integers is the smallest positive integer that is a multiple of each integer in the set. Example: The LCM of $\{4, 6, 9\}$ is 36.

Less than. a is less than b means b is greater than a. See **Greater than.**

Match. See **One-to-one correspondence.**

Measure. The number obtained from a measurement.

Measurement. The process of associating a number with a particular property of an object.

Member of a set. In any nonempty set of objects each object is a member of the set. Synonymous with **element of a set.** For example, the set consisting of the elements a, b, c has for its members a, b, c. It follows that a is a member of this set, b is a member of this set, and c is a member of this set. The members of the set $\{(3,7), 9\}$

are (3,7) and 9. The members of a set may be ordered pairs as well as numbers.

Minus. The name for the subtraction symbol. "−". See under **Symbol.**

Missing addend. In a sentence such as $8 + \square = 12$, one of the addends is not given, or is "missing." The $\square$, called a frame, provides a place in which to name the missing addend. Determination of the missing addend in $8 + \square = 12$ corresponds to subtracting 8 from 12. That is, since $8 + 4 = 12$, $4 = 12 - 8$.

Missing factor. In a sentence such as $\square \times 8 = 40$, one of the factors is not given, or is "missing." The $\square$, called a frame, provides a place in which to name the missing factor. Determination of the missing factor in $\square \times 8 = 40$ corresponds to dividing 40 by 8. That is, since $5 \times 8 = 40$, $5 = 40 \div 8$.

Mixed numeral. Composed of a fraction and a standard numeral for an integer. The fraction appears immediately to the right of the integer numeral. The value of a mixed numeral is the sum of the values of its integer numeral and its fraction. The fraction part must express a quotient of a nonnegative integer and a positive integer. Examples:

$$2\frac{3}{4} \text{ has the value } 2 + \frac{3}{4}.$$

$$-2\frac{3}{4} \text{ has the value } -\left(2 + \frac{3}{4}\right) \quad \text{or} \quad -2 + \left(-\frac{3}{4}\right).$$

Multiple. A number which is the product of a given integer and another integer. A whole number a is a multiple of a whole number b if there is a whole number c such that $a = b \times c$. For example, 30 is a multiple of 10 because $30 = 10 \times 3$. 28 is a multiple of 7 because $28 = 7 \times 4$. The multiples of 10 are 0, 10, 20, 30, 40, ... The set of multiples of a nonzero number is an infinite set.

Multiplication. With every pair of rational numbers a and b multiplication associates the product $a \times b$. For example, with the pair 7 and 9 multiplication assigns the product 7×9, or 63. The product $a \times b$ can be computed in the following ways:

1. If set A contains a elements and set B contains b elements, then $a \times b = n(A \times B)$, the number of elements in the cross-product set, $A \times B$.
2. Choose a sets, disjoint from each other, with b elements in each of the a sets. Then $a \times b$ is the number of elements in the union of these sets.

3. The number of elements in an a-by-b array is $a \times b$.
4. On a number line, the product $a \times b$ is the number of units in a single "jump" that covers the same distance as a jumps with b units in each jump.
5. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Multiplication property of one. For any rational number a, $a \times 1 = a$ and $1 \times a = a$. Informally stated, the product of any rational number and 1 is that rational number. See **One in division.**

Multiplication property of zero. For any rational number a, $a \times 0 = 0$ and $0 \times a = 0$. Informally stated, the product of any rational number and 0 is 0. See **Division by zero.**

$n(A)$ is the number of elements in set A. See "$n(A)$". **Symbol.**

Natural number. Each of the numbers 1, 2, 3, 4, 5, . . . ; any whole number except 0. (Some authors include 0 as a natural number, but we do not.)

Negative number. Any real number whose point on the number line is to the left of the point for zero. Denoted by a minus sign affixed before a numeral for an unsigned or positive number. Zero is neither a negative nor a positive number. Examples:

$$-3, \quad -7.9, \quad -.052, \quad -\frac{22}{7}, \quad -3\frac{1}{7}.$$

Neutral element. Same meaning as identity element. See **Identity element for addition** and **Identity element for multiplication.**

Nonrepeating decimal. See **Nonterminating decimal** and **Irrational number.**

Nonterminating decimal. A decimal numeral that has infinitely many nonzero digits to the right of the decimal point. Examples:

Repeating: $6.8272727\ldots (=6.8\overline{27})$

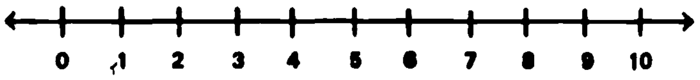
Nonrepeating: $.1011011101111011111\ldots$

Notation, system of. See **Numeration system,**

Null set. See **Empty set.**

Number. See **Counting number, Integer, Natural number, Rational number, Real number,** and **Whole number.**

Number line. A drawing of a line (with arrows to indicate unlimited length) on which a unit length has been selected and marked off consecutively beginning at any fixed place and moving to the right and to the left. The marks are labeled in order "0", "1", "2", "3", "4", "5", and so on to the right; and "−1", "−2", "−3", and so on to the left of "0". The drawing below is an example of a number line for nonnegative numbers.



Numeral. Mark or name for a number; any symbol that names a number. For example, some numerals for the number five are "V," "4 + 1," "five," "5."

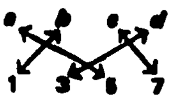
Numeration system. A scheme for naming numbers. Any organized system of using words or marks to denote numbers. Examples: decimal numeration system, Roman numeration system, Egyptian numeration system.

Numerator. The number named by the numeral written above the fraction bar of a fraction. For example, the numerator for each of the following fractions is 7:

$$\frac{7}{2}, \quad \frac{7}{9}, \quad \frac{7}{-2}.$$

One in division. For any rational number a, $a \div 1 = a$ and, if $a \neq 0$, $a \div a = 1$. Informally stated, any rational number divided by 1 is that rational number, and any rational number (except 0) divided by itself is 1.

One-to-one correspondence between two sets. A pairing of the members of the two sets, not necessarily different sets, so that each pair contains exactly one member from each set, and each element of each set is in exactly one pair. For example, one-to-one correspondence between the sets $\{a, b, c, d\}$ and $\{1, 3, 5, 7\}$ is shown by the accompanying diagram. This correspondence can also be shown by listing the pairs: $(a, 5)$, $(b, 1)$, $(c, 7)$, $(d, 3)$.



Operation. A set of associations for elements of two sets, pairing each member of the first set with a member of the second set but never pairing the same member of the first set with more than one member of the second set. In a binary operation the elements of the first set are ordered pairs. According to this definition, addition, subtraction, multiplication, and division are binary operations. An operation is a correspondence.

Mathematicians generally consider a binary operation on, say, set A, to be more restricted: A binary operation on set A is a correspondence between $A \times A$ and A such that *every* member of $A \times A$ has a partner in set A. Under this restricted definition for rational numbers, addition and multiplication are still binary operations while subtraction and division are not.

Opposite of a number. See **Additive inverse.**

Order property of addition. See **Commutative property of addition.**

Order property of multiplication. See **Commutative property of multiplication.**

Ordered pair. Two objects considered together where one of the objects is first in the pair and the other is second in the pair. The ordered pair of numbers (4,7) is different from the ordered pair (7,4). In an ordered pair the first and second elements (also called components) may be the same, as in (7,7).

Ordered set. An example of an ordered set is the set of counting numbers $\{1, 2, 3, 4, 5, \ldots\}$. This particular listing in the braces means that its members are assigned specific positions in the ordering; namely, 1 is the first number, 2 is the second number, 3 is the third, etc. The ordering of the counting numbers used here is according to "size." Each number is 1 less than its successor. This particular ordering is essential for counting. The set of rational numbers is also an ordered set for the relation of "less than." For every two distinct rational numbers r and s, we must have either $r < s$ or $s < r$.

Pair. See **Ordered pair.**

Partial quotient. When a quotient has been computed as a sum, each addend of this sum is called a partial quotient. For instance, in computing 8,972 ÷ 24, we obtain the quotient 300 + 70 + 3. Each of the numbers (300 or 70 or 3) is a partial quotient.

```
24 ) 8,972 |
     7,200 |    300
     ----- |
     1,772 |
     1,680 |     70
     ----- |
        92 |
        72 |      3
        -- |   ----
r =     20 |    373 = q
```

Partition. To partition a set is to split up the set into nonempty disjoint subsets so that every element in the set is in exactly one of the subsets. See **Division** (1).

Place value. The number assigned to each position occupied by a digit in a standard numeral. In the standard numeral "289", the "2" occupies the position to which the value 100 is assigned. We say the "2" is in the hundreds place. The place value of "2" in "289" is 100.

Plus. A name for the symbol "+". See **Symbol, "+"**.

Positional value of digits in a numeral. See **Place value.**

Positive number. Any real number whose point on the number line is to the right of the point for zero. See **Negative number.** It is denoted by a numeral having no sign or the positive sign. Examples:

$$7, \quad +7, \quad 3.4, \quad +3.4, \quad \frac{22}{7}, \quad +\frac{22}{7}, \quad 3\frac{1}{7}, \quad +3\frac{1}{7}.$$

Powers of ten. In this book, *powers of ten* refers to the numbers 1, 10, 100, 1,000, etc. These numbers are also expressed as 10^0, 10^1, 10^2, 10^3, etc.

Prime. An integer n is a prime number if it has exactly 4 divisors: $1, -1, n, -n$. The smallest positive prime number is 2. Its only divisors are $1, -1, 2, -2$. See **Composite.**

Product. With every pair of rational numbers a and b multiplication associates the product "$a \times b$." The product of whole numbers a and b, denoted by $a \times b$, is the number of elements in a cross-product set $A \times B$, where $n(A) = a$ and $n(B) = b$. Alternately, $a \times b$ is the number of elements in the union of a sets, disjoint from each other, with b elements in each. Finally, $a \times b$, the product of a and b, is the number of elements in an array having a rows and b columns. Example:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

See **Multiplication.**

Proper subset. Set A is a proper subset of set B if A is a subset of B while B contains at least one element which is not a member of A. Example: If a, b, c, and y are distinct elements, $\{a, b\}$ is a proper subset of $\{a, b, c, y\}$.

Quotient. A number assigned to pairs of rational numbers by division. In the sentence $a \div b = q$, $b \neq 0$, the number q is called the quotient of a and b. When we try to compute $a \div b$, the unique rational number q for which $a = (q \times b) + r$ with $r < b$ is also called the quotient. Examples: The quotient of 15 and 3 is $15 \div 3$, or 5. In $17 =$

$(2 \times 7) + 3$, the number 2 is the quotient when we regard 7 as the divisor.

Ratio. Numbers a and b are said to be in the ratio of c to d if (for some nonzero number r) $a = rc$ and $b = rd$. Example: 15 and 20 are in the ratio of 3 to 4 because $15 = 5 \times 3$ and $20 = 5 \times 4$.

Rational number. Any number that is expressible as a quotient of integers. An equivalence class of fractions. Its decimal representation must either be terminating or repeating. Every whole number and every integer is a rational number. See **Irrational number.**

Real number. Every point in the number line has a number associated with it. All such numbers are real numbers. Every rational number is a real number. See **Irrational number.**

Remainder. When we try to compute $a \div b$, the unique whole number r less than b for which $a = (q \times b) + r$ is called the remainder. For example, in $15 = (2 \times 7) + 1$, the number 1 is the remainder.

Repeated addition. If m and n are whole numbers,

$$m \times n = \underbrace{n + n + n + \ldots + n + n}_{m \text{ addends}}.$$

Thus, for example,

$$3 \times 4 = 4 + 4 + 4.$$

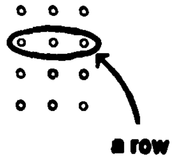
If $m = 1$, the right side is interpreted to mean n. If $m = 0$, the right side is taken to mean 0.

Repeating decimal. A decimal numeral in which a block of one or more digits, at least one of which is not 0, repeats successively infinitely many times to the right. Examples are .333 . . . , which may be written $.\overline{3}$, and 5.27070 . . . , which may be written $5.2\overline{70}$. A repeating decimal is sometimes called a periodic decimal. (Sometimes we regard expressions like $.3\overline{0}$ as being repeating.)

Repetend. The block of repeating digits in a repeating decimal. Examples: The repetend of 5.20707 . . . = $5.2\overline{07}$ is 07. The repetend of 2.49999 . . . = $2.4\overline{9}$ is 9.

Repetitive property. A numeration system has this property provided, when any of its basic symbols is repeated in a numeral, it has the same particular value regardless of its position. In the Egyptian system, each basic symbol in "∩ ∩ ∩" has the same value, 10. In our system, each digit in "333" represents a different number according to its position in the numeral, so our decimal numeration system does not have the repetitive property.

Row. A horizontal line of objects in an array. The array below has four rows.



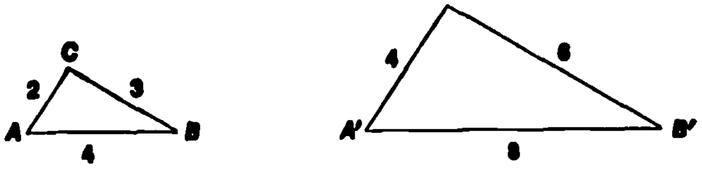
Scale drawing. A scale drawing or a scale model is a representation (or a one-to-one correspondence) having the following property: For every three points A', B', C' of the scale drawing or scale model and the corresponding three points A, B, C of the prototype we have:

1. $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle A'C'B'$, and $\angle BAC = \angle B'A'C'$. That is, corresponding angles have the same measures.
2. $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$. That is, corresponding lengths are in proportion.

Set. Collection or group or aggregate of objects that may be concrete or abstract, similar or dissimilar. One would usually like to be able to decide if any particular object is or is not a member of the set. Mathematicians usually do not define *set*.

Similar figures. Two figures are similar if one is a scale drawing or scale model of the other.

Similar triangles. Two triangles are similar if one triangle is a scale drawing or scale model of the other. Example:



Slope. If (a, b) and (c, d) with $a \neq c$ are number pairs for points on a line l, then the slope of l is the value of the fraction

$$\frac{d - b}{c - a}.$$

If $a = c$, the line is parallel to the y-axis and has no slope. If l has the equation $y = mx + b$, then m is the slope of l.

Standard form of a numeral (Standard numeral, standard name for a number). In the Hindu-Arabic system, a numeral consisting of digits only without any sign of operation. The symbols "0", "1", "2", "3", "4", "5", . . . , "11", "12", "13", ... See **Expanded form.** The standard name for 2 + 3 is "5."

Standard numeral. The following are generally regarded as standard numerals for rational numbers:

1. Any reduced fraction. Examples:

$$\frac{2}{3}, \quad \frac{3}{2}, \quad -\frac{2}{3}, \quad \frac{-2}{3}.$$

2. Any mixed numeral, provided the fraction part is reduced and has a value less than 1.
3. Any decimal, except a repeating decimal with repetend 9.

Subset. Set A is a subset of set B if every member of A is also a member of B. Alternately, set A is a subset of set B if every element not in B is also not in A. As a special case, A may be the entire set B itself. As another special case, A may be the empty set; that is, A may have no elements. Thus, if set A is identical to set B, or if A is the empty set, set A is a subset of set B.

Subtraction. With every pair of rational numbers a and b, subtraction assigns the difference of a and b, denoted by $a - b$. For example, the difference of 8 and 2 is $8 - 2$, or 6.

Successor. If n is a whole number, then $n + 1$ is the successor of n. Examples: The successor of 0 is 1, the successor of 8 is 9, etc.

Sum. With every pair of numbers a and b addition associates the sum $a + b$. The sum of whole numbers a and b, denoted by $a + b$, is the number of elements in the union of sets A and B provided that $n(A) = a$, $n(B) = b$, and sets A and B are disjoint. For example, $4 + 2$, or 6, is the sum of 4 and 2. See **Addition.**

Symbol. A mark, a collection of marks, or an expression that is used to communicate an idea. For example, numerals are symbols for numbers. Some special mathematical symbols follow:

"{ }" Braces. Sometimes called curly brackets. Consist of two symbols used to enclose the names of members of a set or a description of the members of a set, as $\{a, b, c\}$ and {even

	numbers}. If nothing appears between the braces, then the set has no members and is the empty set, designated by { }.
"="	Equal sign. The symbol is used between two expressions to assert that the expressions name the same thing and, in particular, when referring to numbers, name the same number. Examples: $3 + 3 = 4 + 2$, $\{a, b\} = \{b, a\}$.
"$n(A)$"	An abbreviation for any one of the following synonymous expressions: 1. The number of elements in set A 2. The number associated with set A 3. The number property of set A 4. The number of set A 5. The cardinal number of set A
". . ."	Three ellipsis points, or dots, as in 1, 2, 3, 4, 5, . . . , signify that the indicated pattern (in this case, of adding 1) is to continue indefinitely.
"+"	The plus sign, a symbol of addition. The symbol "$a + b$" (read "a plus b") names the sum of numbers a and b. See **Addition.**
"∪"	A symbol for union. $A \cup B$ (read "A union B") names the union of sets A and B. $A \cup B \cup C$ means the union of sets A, B, and C.
">"	means "is greater than." For example, $5 > 3$ means 5 is greater than 3.
"<"	means "is less than." For example, $3 < 5$ means 3 is less than 5.
"−"	The minus sign, the symbol for subtraction. $a - b$ (read "a minus b") names the difference of a and b, that is, the missing addend in the sentence $\square + b = a$. See **Subtraction.**
"$^{-}2$"	Elevated minus sign. Negative 2. Opposite of 2. $^{-}2 = (-1)\cdot 2 = -2.$
"^{-}n"	Elevated minus sign. Opposite of n. $^{-}n = (-1)\cdot n = -n.$ ^{-}n is a positive number if $n < 0$. ^{-}n is a negative number if $n > 0$.
"X"	The symbol for cross product. $A \times B$ (read "A cross B") names the cross product of sets A and B. See **Cross product.**

"×" The times sign, the symbol for multiplication. $a \times b$ (read "a times b") names the product of numbers a and b. See **Multiplication.**

"÷" The symbol for division. $a \div b$ (read "a divided by b") names the result of dividing a by b, that is, the missing factor in $\square \times b = a$ or $b \times \square = a$, that is, the quotient of a and b. See **Division.**

"≠" Means "is not equal to," "does not equal." The symbol is used between two expressions to assert that the expressions do not name the same thing. For example, $5 + 1 \neq 8$ asserts that $5 + 1$ and 8 are different numbers.

"□" A frame for entering a symbol. Examples: Compute the missing number: $\square = 14 + 2$. Determine the missing operation: $2 \square 3 = 6$. When the same frame is repeated in a sentence, the same symbol must be used. If different frames are used, the symbols need not be different.

Terminating decimal. An ordinary decimal expression. This term is used to contrast with a nonterminating or infinite decimal. In a terminating decimal, there are a finite number of nonzero digits to the right of the decimal point.

Union. The union of two sets is the set consisting of all the elements that are in either or both of the two sets. The union of $\{x, y\}$ and $\{y, z, w\}$ is $\{w, x, y, z\}$. See **Symbol,** "∪". If $A = \{x, y\}$, $B = \{y, z, w\}$, then the union of A and B is denoted by $A \cup B$. Thus $A \cup B = \{w, x, y, z\}$. The union of two sets is the set that contains all the elements of each set and no others. In more general terms, the union of any collection of sets is the set consisting of all those elements that are members of at least one of the sets in the given collection.

Uniqueness property of addition. See **Well-defined property of addition.**

Unit. In this book, the word *unit* is used in reference to a representation of a number line. Any length we wish to select is used as a basic length to be marked off consecutively on the illustrated line. For example, we might choose as our unit the segment illustrated by ├─┤ . Then we mark consecutively on the drawing of the number line as many of these lengths as we want.

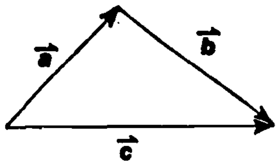
On the above representation of the number line, we have marked off 7 of the selected units.

The word is also used to denote a unit region.

1 unit

Vector. A vector may be represented by a line segment with an arrowhead at one end, an arrow. The length of the vector arrow represents the magnitude of the vector quantity, and the arrow generally indicates its direction. Two vectors for similar vector quantities must add according to the following rule:

$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$. as shown by



Well-defined property of addition. If $r = s$ and $t = u$, then $r + t = s + u$. (Also called the **uniqueness property of addition.**) Example:

$$\frac{1}{2} = .5 \quad \text{and} \quad \frac{3}{4} = .75,$$

so

$$\frac{1}{2} + \frac{3}{4} = .5 + .75.$$

Whole number. One of the numbers 0, 1, 2, 3, ... The set of whole numbers consists of 0 and the counting numbers.