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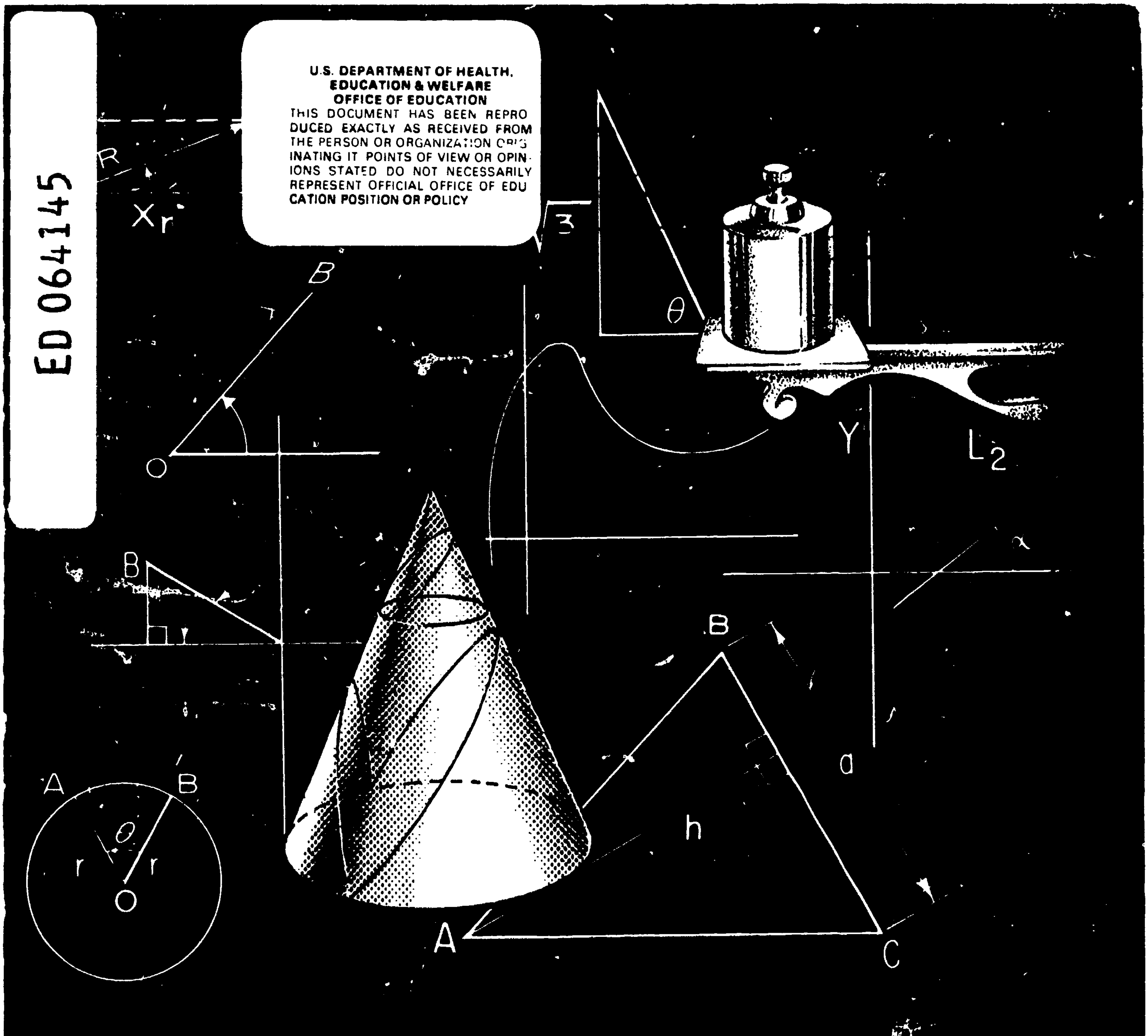
ABSTRACT

The second of three volumes of a mathematics training course for Navy personnel, this document contains material primarily found at the college level. Beginning with logarithms and trigonometry, the text moves into vectors and static equilibrium (physics). Coordinate geometry, conic sections, and the tangents, normals, and slopes of curves follow. Four chapters are devoted to limits, differentiation, and integration; the text concludes with two chapters on combinations, permutations, and probability. Related documents are SE 014 115 and SE 014 117. (JM)



MATHEMATICS, VOL. 2

BUREAU OF NAVAL PERSONNEL

NAVY TRAINING COURSE NAVPERS 10071-B

PREFACE

The purpose of this Navy Training Course is to aid those personnel who need an extension of the knowledge of mathematics gained from Mathematics, Vol. 1, NavPers 10069-C. To serve the wide variety of needs, the text is general in nature and is not directed, therefore, toward any one specific specialty.

The definitions and notations of logarithms followed by computations with logarithms occur early in the text. Trigonometric ratios and analysis and applications along with aids to computations occur next. Vectors and static equilibrium are followed by trigonometric identities and equations.

Straight lines, conic sections, tangents, normals, and slopes precede the introduction to differential and integral calculus.

The introduction to calculus is intended as a survey course prior to a more rigorous study of the subject. The limit concept is followed by a discussion of derivatives and integration. Basic integration formulas follow this discussion.

The last chapter of the course covers combinations and permutations, and gives an introduction to probability.

Numerous examples and practice problems are given throughout the text to aid the understanding of the subject matter.

This training course was prepared by the Navy Training Publications Center, Memphis, Tennessee, for the Bureau of Naval Personnel.

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THE UNITED STATES NAVY

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The United States Navy is responsible for maintaining control of the sea and is a ready force on watch at home and overseas, capable of strong action to preserve the peace or of instant offensive action to win in war.

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Never have our opportunities and our responsibilities been greater.

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REQUIREMENTS *	E1 to E2	E2 to E3	#† E3 to E4	# E4 to E5	† E5 to E6	† E6 to E7	† E7 to E8	† E8 to E9
SERVICE	4 mos. service— or completion of recruit training.	6 mos. as E-2.	6 mos. as E-3.	12 mos. as E-4.	24 mos. as E-5.	36 mos. as E-6. 8 years total enlisted service.	36 mos. as E-7. 8 of 11 years total service must be enlisted.	24 mos. as E-8. 10 of 13 years total service must be enlisted.
SCHOOL	Recruit Training.		Class A for PR3, DT3, PT3, AME 3, HM 3			Class B for AGC MUC, MNC.		
PRACTICAL FACTORS	Locally prepared check-offs.	Records of Practical Factors, NavPers 1414/1, must be completed for E-3 and all PO advancements.						
PERFORMANCE TEST			Specified ratings must complete applicable performance tests before taking examinations.					
ENLISTED PERFORMANCE EVALUATION	As used by CO when approving advancement.		Counts toward performance factor credit in advancement multiple.					
EXAMINATIONS**	Locally prepared tests.	See below.	Navy-wide examinations required for all PO advancements.				Navy-wide, selection board.	
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AUTHORIZATION	Commanding Officer		U.S. Naval Examining Center			Bureau of Naval Personnel		

* All advancements require commanding officer's recommendation.
† 1 year obligated service required for E-5 and E-6; 2 years for E-6, E-7, E-8 and E-9.
\# Military leadership exam required for E-4 and E-5.
** For E-2 to E-3, NAVEXAMCEN exams or locally prepared tests may be used.

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* Recommendation by commanding officer required for all advancements.

† Active duty periods may be substituted for training duty.

CHAPTER 1

LOGARITHMS

The basic definitions and terminology associated with the study of logarithms were discussed in Mathematics, Vol. 1, NavPers 10069-C. Some of these basic topics are reviewed in the following paragraphs, followed by discussion of the use of log tables and natural logarithms.

REVIEW OF DEFINITIONS

The most important definition to remember, concerning logarithms, is that every logarithm is an exponent. For example, since 3^2 is equal to 9, the logarithm of 9 to the base 3 is 2. In order to state a logarithmic relationship, a base must be stated or implied; the various exponents which designate powers of the base are logarithms to that base.

The usual method of expressing the basic definition of logarithms in symbols is as follows:

$$\text{If } b^x = a, \text{ then } x = \log_b a$$

The two forms shown in the foregoing expression are defined as follows:

EXPONENTIAL FORM: $b^x = a$

LOGARITHMIC FORM: $x = \log_b a$

EXAMPLE: Change the expression $2^3 = 8$ to logarithmic form.

SOLUTION: If $b^x = a$, then $\log_b a = x$.

$$\text{Let } b = 2,\ x = 3,\ a = 8$$

Substituting,

$$\log_2 8 = 3$$

EXAMPLE: Change the expression $\log_{10} 100 = 2$ to exponential form.

SOLUTION: If $\log_b a = x$, then $b^x = a$

Substituting,

$$10^2 = 100$$

PRACTICE PROBLEMS:

1. Change $10^3 = 1,000$ to logarithmic form.
2. Change $e^x = N$ to logarithmic form.
3. Change $\log_2 4 = 2$ to exponential form.
4. Change $\log_{10} 3.16 = 1/2$ to exponential form.

ANSWERS:

1. $\log_{10} 1,000 = 3$
2. $\log_e N = x$
3. $2^2 = 4$
4. $10^{1/2} = 3.16$

RULES FOR CALCULATION

Numerical calculation by means of logarithms is performed by using 10 as the base. Therefore, in the discussion which follows, no base designation is used. The expression log A is understood to mean the base 10 logarithm of A.

Two important abilities are necessary for logarithmic calculation, as follows:

1. Recognition of logarithms as exponents.
2. Knowledge of the rules for exponents in multiplication and division of algebraic quantities.

The first of these abilities was discussed in the foregoing section. The second is the subject of the following paragraphs.

Multiplication

Suppose that we wish to multiply A and B, and we know the following:

$$A = 10^m$$
$$B = 10^n$$

The product AB then is

$$AB = 10^m \times 10^n$$
$$= 10^{(m + n)}$$

In logarithmic form,

$$\log A = m$$

$$\log B = n$$

$$\log AB = m + n$$

Assuming that we can find AB if we know its logarithm, the procedure then may be stated as follows:

To multiply two numbers by means of logarithms, add their logarithms and find the number whose logarithm is the sum.

EXAMPLE: Multiply 100 times 1,000 by logarithms.

SOLUTION:

$$100 = 10^2 \therefore \log 100 = 2$$

$$1{,}000 = 10^3 \therefore \log 1{,}000 = 3$$

$$\log (100 \times 1{,}000) = 2 + 3$$

$$= 5$$

$$\therefore 100 \times 1{,}000 = 10^5$$

$$= 100{,}000$$

Division

If A is to be divided by B, and A and B are the same numbers as in the foregoing discussion, then we have

$$\frac{A}{B} = \frac{10^m}{10^n} = 10^{(m - n)}$$

In logarithmic form,

$$\log A = m$$

$$\log B = n$$

$$\log \frac{A}{B} = m - n$$

This may be stated in words as follows:

To divide B into A by logarithms, subtract log B from log A and find the number whose logarithm is the difference.

EXAMPLE: Divide 1,000 by 100 by logarithms.

SOLUTION:

$$\log 1{,}000 = 3$$

$$\log 100 = 2$$

$$\log \frac{1{,}000}{100} = 1$$

$$\frac{1{,}000}{100} = 10^1$$

$$= 10$$

Powers

In order to calculate a power such as A^3 by logarithms, we observe that

$$A^3 = A \times A \times A$$

$$\therefore \log A^3 = \log A + \log A + \log A$$

$$= 3 \log A$$

Stated in words, the power rule is as follows:

To find A^n by logarithms, first express log A^n as n log A. The number whose logarithm is n log A is the desired power, A^n.

EXAMPLE: Find the value of 100^2 by using logarithms.

SOLUTION:

$$\log 100^2 = 2 \log 100$$

$$= 2(2)$$

$$= 4$$

Therefore,

$$100^2 = 10^4$$

$$= 10{,}000$$

Roots

The calculation of roots by logarithms is easily accomplished by first changing the form of the expression to a fractional power. Then the power rule is used.

EXAMPLE: Find $\sqrt{100}$ by logarithms.

SOLUTION:

$$\therefore \sqrt{100} = 100^{1/2}$$

$$\log 100^{1/2} = \frac{1}{2} \log 100$$

$$= \frac{1}{2} (2)$$

$$= 1$$

$$100^{1/2} = 10$$

$$\therefore \sqrt{100} = 10$$

USING LOG TABLES

Tables of logarithms could be constructed using any number as a base. For purposes of calculation, the most logical number for a base is 10, the base of the decimal number system. Logarithms to the base 10 are called COMMON LOGARITHMS.

COMMON LOGARITHMS

Most of the numbers encountered in various calculations are not integral (whole number) powers of 10. For example, the number 316 can be expressed as a power of 10 only if we resort to fractional powers. We find that

$$316 = 10^{5/2}$$

$$= 10^{2\ 1/2} \text{ (approximately)}$$

$$= 10^{2.5}$$

(The exact value of $10^{2.5}$ is very close to 316.23.)

Therefore, in logarithmic form,

$$\log 316 = 2.5 \text{ (approximately)}$$

Every logarithm consists of an integral part and a fractional part. Thus the logarithm of 316 is

$$2.5 = 2 + 0.5$$

The integral part is the CHARACTERISTIC; in this example, the characteristic is 2. The fractional part is the MANTISSA; in this example, the mantissa is 0.5.

Integers

The characteristic for the logarithm of an integer may be determined by inspection. For example, if the integer is between 1 and 10, it is equal to a power of 10 between 0 and 1. This concept is explained fully in Mathematics, Volume 1, NavPers 10069-C.

The numbers in the following list serve to illustrate how the characteristic is determined by the size of the number:

$$\log 3.6 = 0.5563$$

$$\log 36 = 1.5563$$

$$\log 360 = 2.5563$$

$$\log 3{,}600 = 3.5563$$

Since log 1 is 0 and log 10 is 1, we expect the logarithm of 3.6 to be a number between 0 and 1. Therefore, its characteristic is 0. On the other hand, 3,600 is greater than 1,000 and less than 10,000. Therefore its logarithm is between log 1,000 and log 10,000, and its characteristic is 3. In the foregoing tabulation, we find that the complete logarithm of 3,600 is 3.5563.

Scientific notation provides a convenient method for determining the characteristic. For example, 3,600 is written as 3.6×10^3 in scientific notation. Thus we have

$$\log 3{,}600 = \log (3.6 \times 10^3)$$

$$= \log 3.6 + \log 10^3$$

The characteristic of log 3.6 is 0, and the characteristic of log 10^3 is 3. Therefore, the characteristic of log 3,600 is 3, the sum of the characteristics of the two separate logarithms. Any expression written in scientific notation consists of a number between 1 and 10, multiplied by a power of 10. Since the characteristic of a number between 1 and 10 is 0, the power of 10 determines the characteristic of the logarithm.

The exponent that we obtain as the power of 10 in scientific notation is indicated by the number of digits between the actual position of the decimal point in the original number and the standard position of the decimal point. The standard position is immediately after the first nonzero digit in the number. For example, in the number 3,600, the decimal point is understood to be after the second 0 in the original number. This is 3 digits to the right of standard position, so that the exponent of 10 for scientific notation is 3. This exponent is also the characteristic for log 3,600. If the decimal point in the original number had been to the left of standard position, the exponent of 10 (and therefore the characteristic) would have been negative.

Fractions

When the logarithm of a fraction is obtained, a negative characteristic occurs. For example,

$$\log 0.036 = \log (3.6 \times 10^{-2})$$

$$= \log 3.6 + \log 10^{-2}$$

$$= \log 3.6 + (-2)$$

The mantissa for log 3.6 is 0.5563. Therefore,

$$\log 0.036 = 0.5563 - 2$$

Since logarithm tables do not list negative mantissas, we do not subtract 2 from 0.5563 to obtain the final form of log 0.036.

Some tables handle the problem of negative characteristics by placing a negative sign above the characteristic as follows:

$$\log 0.036 = \bar{2}.5563$$

Observe that an entry such as

$$-2.5563$$

would be misleading. It would be interpreted as if the mantissa, as well as the characteristic, were negative.

Perhaps the most universal form for negative characteristics is as follows:

$$\log 0.036 = 8.5563 - 10$$

This form is numerically equal to

$$0.5563 - 2$$

but it has the advantage of presenting the characteristic, as well as the mantissa, as a positive number.

Negative numbers and 0 do not have logarithms. In the case of 0, none is needed for purposes of calculation. However, it is often necessary to multiply and divide negative quantities using logarithms. In order to use logarithms for this purpose, we first determine the sign of the final answer. Then all numbers are treated as positive, and the predetermined sign is affixed.

PRACTICE PROBLEMS:

Determine the characteristic of the logarithm for each of the following numbers:

1. 32
2. 476
3. 0.25
4. 0.0074

ANSWERS:

1. 1
2. 2
3. -1 or 9 - 10
4. -3 or 7 - 10

Mantissa

Tables of logarithms normally contain only mantissas. Since these are understood to be the decimal parts of the logarithms which they represent, the decimal points are often omitted in the printed table. Appendix II of this course is constructed in this way, and the user is expected to supply the characteristic and the decimal point with each mantissa. Table 1-1 is an excerpt from appendix II.

Observe that the table of logarithms has headings consisting of the abbreviation No. (representing Number) and the digits 0 through 9. The first two digits of any number whose logarithm we seek are found in the No. column. The third digit is found as one of the column headings, 0 through 9. The mantissa for the logarithm of any three-digit number is found opposite the first two digits, which are located in the No. column, and below the third digit.

EXAMPLE: Find the mantissa for the logarithm of 124.

SOLUTION:

1. In the No. column, table 1-1, find the number 12.

2. Move to the right, staying in the row of mantissas opposite 12, until you reach the column with the digit 4 as a heading.

3. Read the mantissa, .0934. Notice that we supply the decimal point.

EXAMPLE: Find the complete logarithm of 13.7.

Table 1-1.—Example of common logarithm table.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732

SOLUTION: Always determine the characteristic before entering the table.

1. The characteristic of 13.7 is 1. We may now disregard the decimal point and enter the table with just the digits 137.
2. In the No. column, table 1-1, find the number 13.
3. Move to the right, staying in the 13 row, until you reach the 7 column.
4. The mantissa is .1367.
5. The logarithm of 13.7 is 1.1367.

PRACTICE PROBLEMS:

Find the logarithms of the following numbers:

1. 118
2. 342 (See appendix II.)
3. 14.6
4. 5.48

ANSWERS:

1. 2.0719
2. 2.5340
3. 1.1644
4. 0.7388

INTERPOLATION

Interpolation is the process of calculating the mantissa for the logarithm of a number having one more digit than the table entries. For example, to find the logarithm of 1125 it would be necessary to interpolate in table 1-1.

The logarithm of 1125 is halfway between the logarithms of 1120 and 1130. Therefore, we find the mantissas for the logarithms of these two numbers and then determine the mantissa that is halfway between them. The work is arranged as follows:

		NUMBERS	MANTISSAS	
10	5	1120	.0492	.0039
		1125		
		1130	.0531	

We analyze the foregoing tabulation in terms of the difference between the numbers and the difference between the mantissas. The large bracket on the numbers indicates a difference of 10, and the small bracket shows that our number is $\frac{5}{10}$ of the way between the two numbers in the table. Therefore, the mantissa corresponding to our number should be $\frac{5}{10}$ of the way between the mantissas in the table.

$$\frac{5}{10}(.0039) = .00195$$

$$= .0020 \text{ (to 4 places)}$$

Adding .0020 to .0492, we obtain the mantissa corresponding to 1125; it is .0512. Therefore,

$$\log 1125 = 3.0512$$

EXAMPLE: Find log 25.67.

SOLUTION:

1. The characteristic is 1.

2. The number lies between 2560 and 2570. Therefore the mantissa lies between .4082 and .4099.

3. Tabulate:

$$10\left[\begin{array}{ll} 7\left[\begin{array}{l} 2560 \\ 2567 \end{array}\right. & .4082 \\ \quad 2570 & .4099 \end{array}\right] .0017$$

4. Our number is $\frac{7}{10}$ of the way between 2560 and 2570. Therefore, the mantissa is $\frac{7}{10}$ of the way between .4082 and .4099.

5. $\frac{7}{10}(.0017) = .00119$

$= .0012$ (to 4 places)

6. $.4082 + .0012 = .4094$

$\therefore \log 25.67 = 1.4094$

PRACTICE PROBLEMS:
Find the logarithms of the following numbers:
1. 0.2345
2. 5.432
3. 473.6
4. 9817

ANSWERS:
1. 9.3702 - 10
2. 0.7352
3. 2.6754
4. 3.9919

TRIGONOMETRIC FUNCTIONS

Logarithms of trigonometric functions may be obtained by first looking up the decimal value of the natural function and then finding the logarithm of this decimal number. However, the process is laborious, and it is rendered unnecessary by the existence of tables of logarithms of the trigonometric functions.

Appendix I of this training course is a sample page from a typical table of trigonometric logarithms. Its construction is similar to that of appendix II, except that 10 is added to each mantissa. Thus the entry for log cot 38° is 10.10719, with the quantity "-10" understood. The complete logarithm is then

$$\log \cot 38° = 10.10719 - 10$$
$$= 0.10719$$

The addition of 10 to each table entry makes the correct form automatic in those cases where a "9 - 10" type of format is involved. For example, the characteristic for log sin 38° is -1, with the following result:

$$\log \sin 38° = 9.78934 - 10$$

ANTILOGARITHMS

The procedure of finding a number when we know its logarithm is called "finding the antilogarithm." The word "antilogarithm" is abbreviated "antilog," and a symbol sometimes used to indicate the antilog is $\log^{-1}$. The -1 in a symbol of this kind tends to be confusing, since it is not an exponent. It is an indicator which emphasizes the INVERSE relationship between logs and antilogs.

The antilogarithm is easily found when the corresponding mantissa is an exact table entry.

EXAMPLE: Find the antilogarithm of 1.1271.

SOLUTION:

1. Find the mantissa .1271 in appendix II. (The decimal point is understood.)

2. The column in which .1271 is found determines the third digit of the antilog. The third digit is 4.

3. The row in which .1271 is located determines the first two digits of the antilog. In that row, and in the No. column, we find the digits 13. Thus the digits of the antilog are 134.

4. Since the characteristic of the original logarithm is 1, the antilogarithm must be a number between 10 and 100. Thus the decimal point must be placed between the 3 and the 4.

5. We conclude that

$$\text{antilog } 1.1271 = 13.4$$

Interpolation must be used when the mantissa whose antilogarithm we seek is not an exact entry in the table.

EXAMPLE: Find the antilogarithm of 8.5124 - 10.

SOLUTION:

1. Find the mantissas nearest to .5124 in appendix I. These are .5119 and .5132. Since .5124 is between .5119 and .5132, its antilog is between the antilogs corresponding to these two mantissas.

2. Tabulating the results of step 1, and letting N represent the antilog corresponding to .5124, we have

$$\begin{array}{cc} \text{NUMBER} & \text{MANTISSA} \\ \left.\begin{array}{c} .0325 \\ N \\ .0326 \end{array}\right. & \left.\begin{array}{r} \left.\begin{array}{r} .5119 \\ .5124 \end{array}\right] .0005 \\ .5132 \quad \end{array}\right] .0013 \end{array}$$

(The numbers .0325 and .0326 are decimals because of the -2 characteristic.)

3. Our tabulation shows that we use .0005 parts out of .0013 in going from .0325 to N. This is the same as 5 out of 13, so that we use $\frac{5}{13}$ of the total difference in the mantissa. Thus, N should be $\frac{5}{13}$ of the way from .0325 to .0326.

4. Multiplying to find $\frac{5}{13}$ of the difference between the numbers, we have

$$\frac{5}{13} \times .0001 = .384 \times .0001$$

$$= .00004 \text{ (approximately)}$$

5. The difference between .0325 and N is added to .0325 to obtain N.

$$N = .0325 + .00004$$

$$= .03254$$

6. We conclude that

$$\text{antilog } 8.5124 - 10 = .03254$$

PRACTICE PROBLEMS:
Find the antilogarithms of the following logarithms:

1. 2.7030
2. 9.3636 - 10
3. 1.8451
4. 0.3842

ANSWERS:

1. 504.7
2. 0.2310
3. 70.00
4. 2.422

NATURAL LOGARITHMS

Natural logarithms are so named because the number e, the base of the natural logarithm system, is involved in the law of nature governing growth and decay. This law is stated in symbols as follows:

$$A = A_0 e^{rt}$$

In the foregoing equation, A represents the total amount after a period of growth, and A_0 represents the amount at the beginning of the growth period. The letter r represents the continuous rate of growth, and t represents the time period during which growth occurs. The same remarks apply for a period of decay.

By means of higher mathematics, the number e is found to have the value

$$e = 2.71828 \text{ (approximately)}$$

This number is the base of the natural logarithm system.

CHANGING BASES

The relationship between the common logarithm of a number and its natural logarithm is as follows:

$$\ln N = 2.3026 \log N$$

Observe that the special abbreviation, ln N, is used to represent $\log_e N$.

The derivation of the foregoing equation is described in the following paragraphs.

Let $e^x = N$, where N is any number. Taking the natural logarithms of both sides, we have

$$x \ln e = \ln N$$

Since ln e means $\log_e e$, ln e is the same as 1. Therefore,

$$x = \ln N$$

This result is also obtainable from the basic definition of logarithms.

Taking common logarithms on both sides in the expression

$$e^x = N$$

we have the following:

$$\log e^x = \log N$$

$$x \log e = \log N$$

$$x = \frac{\log N}{\log e}$$

Equating the two expressions which we have obtained for x, we have

$$\ln N = \frac{\log N}{\log e}$$

From the table of common logarithms, we find that log 2.71828 is 0.4343. Finally,

$$\ln N = \frac{\log N}{0.4343}$$

Since the reciprocal of 0.4343 is 2.3026,

$$\ln N = 2.3026 \log N$$

PRACTICE PROBLEMS:

Find the natural logarithms of the following numbers:

1. 15
2. 80
3. 29
4. 35

ANSWERS.

1. 2.7080
2. 4.3820
3. 3.3673
4. 3.5553

CHAPTER 2

COMPUTATION WITH LOGARITHMS

The use of log tables and an introduction to natural logarithms were discussed in chapter 1. Included in that chapter were the rules for numerical calculations involving logarithms and a review of the laws of exponents.

Additional mention of the laws of exponents will be given in the following paragraphs followed by discussions of the use of logarithms in numerical computations.

Six rules of exponents are shown in table 2-1 for reference and review.

The purpose of the study of logarithms is to enable us to shorten computations with numbers. In many computations involving multiplication, division, powers, roots, or combinations of these, the solutions may be reached more easily by replacing ordinary arithmetical processes with logarithmical processes.

Appendix II gives the common (base 10) logarithms of numbers to four places. All calculations by means of logarithms in this chapter use 10 as the base. In accordance with the convention established in chapter 1 the expression log A is understood to mean the base 10 logarithm of A.

MULTIPLICATION

The logarithm of the product of two or more members is the sum of the logarithms of the separate numbers.

EXAMPLE: Use logarithms to find the product:

$$386 \times 254$$

SOLUTION:

Logarithmic equation	Exponential equation
log 386 = 2.5866	$386 = 10^{2.5866}$
log 254 = 2.4048	$254 = 10^{2.4048}$

Exponential solution:

$$\begin{aligned} 386 \times 254 &= 10^{2.5866} \times 10^{2.4048} \\ &= 10^{2.5866 + 2.4048} \\ &= 10^{4.9914} \end{aligned}$$

Logarithmic solution:

$$\begin{aligned} \log (386 \times 254) &= \log 386 + \log 254 \\ &= 2.5866 + 2.4048 \\ &= 4.9914 \end{aligned}$$

Antilog 4.9914 = 980 + (interpolation correction)

INTERPOLATION:

NUMBER			MANTISSA		
980	?	1	.9912	.0002	.0005
...			.9914		
981			.9917		

$$\frac{.0002}{.0005} \times 1 = .4$$

Antilog of .9914 = 980 + .4 = 980.4

When combined with the characteristic of 4 to place the decimal

Antilog 4.9914 = 98,040
∴ 386 x 254 = 98,040

The exponential solution shown in the example is not a part of normal calculations involving logarithms. It was shown in this first example problem solely for the purpose of reemphasizing

Table 2-1.—Laws of exponents.

OPERATION	ALGEBRAIC	NUMERIC
MULTIPLICATION	$a^m a^n = a^{m+n}$	$4^2 \cdot 4^3 = 4^{2+3} = 4^5$
DIVISION	$\frac{a^m}{a^n} = a^{m-n}$	$5^7 \div 5^3 = 5^{7-3} = 5^4$
POWER OF A POWER	$(a^m)^n = a^{mn}$	$(12^3)^2 = 12^{3\cdot 2} = 12^6$
POWER OF A PRODUCT	$(ab)^m = a^m b^m$	$(5\cdot 3\cdot 2)^4 = 5^4 \cdot 3^4 \cdot 2^4$
POWER OF A QUOTIENT	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$
TRANSPOSING NEGATIVE EXPONENTS	$\frac{1}{a^{-m}} = a^m$	$\frac{1}{5^{-2}} = 5^2$
	$a^{-n} = \frac{1}{a^n}$	$4^{-3} = \frac{1}{4^3}$

the relationship between exponents and logarithms.

EXAMPLE: Use logarithms to find the product of (126) x (-42).

SOLUTION: Recall from chapter 1 that negative numbers do not have logarithms. In using logarithms to solve problems that involve negative numbers, we first determine the sign of the final answer. After this sign is determined, the indicated operations are performed treating all numbers as positive quantities, and the predetermined sign is affixed to the answer.

In our example, dealing first with only signs, we determine the answer to be negative. That is, (+) x (-) results in (-) answer. At this point the problem can be restated—use logarithms to find the product of

$$-(126 \times 42)$$

$$\begin{aligned} -\log(126\cdot 42) &= -(\log 126 + \log 42) \\ &= -(2.1004 + 1.6232) \\ &= -3.7236 \end{aligned}$$

$$-\text{antilog } 3.7236 = -5,291$$

Therefore,

$$(126) \times (-42) = -5,291$$

EXAMPLE: Use logarithms to find the product of 1.73 x 0.0024 x 0.08.

SOLUTION:

$$\log(1.73 \times 0.0024 \times 0.08)$$

$$= \log 1.73 + \log 0.0024 + \log 0.08$$

$$\begin{aligned} \log 1.73 &= 0.2380 \\ \log 0.0024 &= 7.3802 - 10 \\ \log 0.08 &= \underline{8.9031 - 10} \\ \text{sum} &= 16.5213 - 20 \end{aligned}$$

this can be adjusted to

$$\text{sum} = 6.5213 - 10$$

or

$$\log (1.73 \times 0.0024 \times 0.08) = 6.5213 - 10$$

$$\text{antilog } 6.5213 - 10 = .0003322$$

$$1.73 \times 0.0024 \times 0.08 = .0003322$$

It is in problems such as this example that the relationships of logarithms, exponents, and the scientific notation can be used. Go back to the step where

$$\log (1.73 \cdot 0.0024 \cdot 0.08) = 6.5213 - 10$$

Perform the calculations required to go from this step to the next one in which

$$\text{antilog } 6.5213 - 10 = 0.0003322$$

Proper decimal placement is often more difficult in a problem of this type than in a problem which does not involve a negative characteristic. The difficulty may be reduced if the relationships of logarithms, exponents, and scientific notation are utilized. After the step

$$\log (1.73 \cdot 0.0024 \cdot 0.08) = 6.5213 - 10$$

reference to the tables shows that the antilog of the mantissa is between 332 and 333. Interpolation and rounding off produces the digits 3322 as the antilog of the mantissa.

In chapter 1 it was shown that a characteristic could be determined by expressing a number as a number between 1 and 10, multiplied by a power of 10. An antilog can be determined by the converse of this procedure if the digits which represent the antilog of the mantissa are written as a number between 1 and 10 and multiplied by a power of 10. The particular power of 10 to be used is equal to the characteristic of the logarithm in question.

Apply this to the example problem where the digits are 3322 and the logarithm is 6.5213 -10. The power of 10 is -4 since the characteristic is 6 - 10 or -4. Expressing the digits as a number between 1 and 10 multiplied by this power of 10 yields 3.322×10^{-4}, and the antilog of 6.5213 -10 equals 3.322×10^{-4}. Inspection of the example problem indicates the antilog is 0.0003322 and since

$$0.0003322 = 3.322 \times 10^{-4}$$

the results of both methods are the same.

PRACTICE PROBLEMS:

Use logarithms to find the product of the following:

1. $53 \times 76 \times 0.021 \times 153$
2. $1.02 \times 10^{9} \times 4.76 \times 10^{-3}$
3. $0.00432 \times 0.00106 \times 15$
4. $0.102 \times 103.5 \times 76.2$

ANSWERS:

1. 12,942
2. 4,855,555
3. 6.87×10^{-5}
4. 804.4

DIVISION

The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor. As with multiplication, this rule is simply an application of the law of exponents. For example,

$$10^5 \div 10^3 = 10^{5-3} = 10^2$$

EXAMPLE: Find the quotient of $\frac{37.4}{1.7}$ by use of logarithms.

SOLUTION:

$$\log (37.4 \div 1.7) = \log 37.4 - \log 1.7$$

$$\log 37.4 = 1.5729$$

$$- \log 1.7 = \underline{0.2304}$$

$$1.3425$$

$$\log (37.4 \div 1.7) = 1.3425$$

$$\text{antilog } 1.3425 = 22$$

$$\frac{37.4}{1.7} = 22$$

EXAMPLE: Find the quotient of $\frac{16.3}{0.008}$

SOLUTION:

$$\log (16.3 \div 0.008) = \log 16.3 - \log 0.008$$

$$\log 16.3 = 1.2122$$

$$\log 0.008 = 7.9031 - 10$$

In order to prevent the complication of subtracting the characteristic, 7, from the smaller characteristic, 1, we add 10 to and subtract 10 from, the logarithm of the dividend. Note that this does not change the value of the logarithm. Thus,

$$\begin{aligned} \log 16.3 &= 11.2122 - 10 \\ -\log 0.008 &= \underline{7.9031} - 10 \\ &\quad 3.3091 \end{aligned}$$

$$\text{antilog } 3.3091 = 2037$$

$$\frac{16.3}{0.008} = 2037$$

PRACTICE PROBLEMS: Use logarithms to solve the following problems:

1. $635.6 \div 25.4$
2. $0.26 \div 0.061$
3. $0.126 \div 0.00542$
4. $874 \div 26.3$

ANSWERS:

1. 25.03
2. 4.263
3. 23.25
4. 33.23

COLOGARITHMS

Dividing one number by another may be accomplished logarithmically by addition rather than subtraction, if the cologarithm is employed. The cologarithm of a number is the logarithm of the reciprocal of the number. The reason that the cologarithm may be added is explained in the following example:

Evaluate $\frac{15}{12}$

$$\log \frac{15}{12} = \log 15 \left(\frac{1}{12}\right)$$

$$= \log 15 + \log \left(\frac{1}{12}\right)$$

Thus,

$$\log \frac{15}{12} = \log 15 + \text{colog } 12$$

The cologarithm may be easily derived by subtracting the logarithm of the number from the logarithm of 1. Thus,

$$\text{colog } 12 = \log \left(\frac{1}{12}\right)$$

$$= \log 1 - \log 12$$

but $\log 1 = 0$

Therefore,

$$\text{colog } 12 = 0 - \log 12$$

$$= 0 - 1.0792$$

Writing 0 as 10 - 10, we have

$$\text{colog } 12 = (10.0000 - 10) - \log 12$$

$$\text{colog } 12 = (10.0000 - 10) - (1.0792)$$

$$= 8.9208 - 10$$

Since writing the colgarithm is almost as simple as writing the logarithm, it is sometimes advantageous to use cologarithms in complicated problems and thus make the problem one of addition.

Returning to the original problem

$$\log \frac{15}{12} = \log 15 + \text{colog } 12$$

$$\log \frac{15}{12} = \begin{array}{r} 1.1761 \\ \underline{8.9208 - 10} \\ 10.0969 - 10 \end{array}$$

$$\log \frac{15}{12} = 0.0969$$

$$\text{antilog } 0.0969 = 1.25$$

and $1.25 = \frac{15}{12}$

EXAMPLE: Evaluate the following by use of logarithms and colgarithms.

$$\frac{343.8}{592 \times 0.76}$$

SOLUTION:

$$\log \frac{343.8}{592 \times 0.76}$$

$$= \log 343.8 - \log 592 - \log 0.76$$

or

$$= \log 343.8 + \text{colog } 592 + \text{colog } 0.76$$

$$\log 343.8 = 2.5363$$

$$\text{colog } 592 = 7.227 - 10$$

$$\text{colog } 0.76 = \underline{0.1192}$$

$$9.8832 - 10$$

antilog $9.8832 - 10 = 0.7642$.

$$\therefore \frac{343.8}{592 \times 0.76} = 0.7642$$

It should be understood that this problem could be solved without using cologarithms by finding the logarithms of the two numbers in the denominator, adding them and then subtracting their sum from the logarithm of the number in the numerator. Since both methods are equally accurate, the selection of either method becomes one of convenience.

PRACTICE PROBLEMS:

Evaluate the following numbers by use of logarithms and cologarithms:

1. $\frac{210 \times 4.1}{33 \times 0.8754 \times 1.7}$

2. $\frac{14 \times 0.27 \times 36.16}{11 \times 8 \times 17 \times 6.76}$

ANSWERS:

1. 17.54
2. 0.0135

RAISING TO A POWER

To find the log of the power of a number, multiply the logarithm of the number by the exponent of the power. This rule is based on the law of exponents for finding the power of a power. For instance,

$$(4^3)^2 = 4^3 \times 4^3 = 4^{3+3} = 4^6$$

or

$$(4^3)^2 = 4^{3\times 2} = 4^6$$

Also

$$(102.8)^2 = (10^{2.01205})^2 = 10^{4.02410}$$

EXAMPLE: Find the value of $(18.53)^5$.

SOLUTION:

$$\log (18.53)^5 = 5 \log 18.53$$

$$5 \log 18.53 = 1.2679 \times 5$$

$$= 6.3395$$

$$\text{antilog } 6.3395 = 2,185,263$$

$$(18.53)^5 = 2,185,263$$

TAKING A ROOT

To find the log of the root of a number, divide the logarithm of the number by the index of the root.

We recall the law of exponents for taking a root. For instance,

$$\sqrt[3]{8^2} = (8^2)^{1/3} = 8^{2/3}$$

Also,

$$\sqrt[5]{103900} = \sqrt[5]{10^{5.0126}}$$

$$= (10^{5.0126})^{1/5}$$

$$= 10^{1.0025}$$

EXAMPLE: Find the value of $\sqrt[5]{327.6}$

SOLUTION:

$$\log \sqrt[5]{327.6} = \frac{1}{5} \log 327.6$$

$$\log 327.6 = 2.5153$$

$$\frac{1}{5} \log 327.6 = 2.5153 \div 5$$

$$= 0.5031$$

$$\text{antilog } 0.5031 = 3.185$$

$$\sqrt[5]{327.6} = 3.185$$

ROOTS OF QUANTITIES WITH NEGATIVE CHARACTERISTICS

When a logarithm with a negative characteristic is to be divided, it is advisable to add and subtract a number that will, after dividing, leave a minus 10 at the right. This is done to keep the logarithm in standard form. For example, if the problem were $\sqrt[5]{0.0018}$, we would have

$$\log \sqrt[5]{0.0018} = \frac{1}{5} \log 0.0018$$

$$= \frac{1}{5} (7.2553 - 10)$$

Here, to keep a minus 10 in the final logarithm, we must add and subtract 40 before dividing. Thus,

$$\log \sqrt[5]{0.0018} = \frac{1}{5} (47.2553 - 50)$$

$$\log \sqrt[5]{0.0018} = \frac{47.2553}{5} - \frac{50}{5}$$

$$\log \sqrt[5]{0.0018} = 9.4511 - 10$$

PRACTICE PROBLEMS: Evaluate the following by the use of logarithms.

1. $(3.276)^3$
2. $(0.00468)^2$
3. $\sqrt[6]{0.00867}$
4. $\sqrt[5]{237.7}$

ANSWERS:

1. 35.15
2. 0.0000219
3. 0.4532
4. 2.987

ALGEBRAIC OPERATIONS

This chapter has demonstrated the use of logarithms in numerical calculations. Practical applications in many fields involve calculations including algebraic expressions in which logarithms are useful. In these problems both the laws of algebra and the laws of exponents or logarithms hold true and the use of logarithms in algebraic operations is valid. For example:

$$\log [(x + 2)(x + 5)] = \log (x + 2) + \log (x + 5)$$

EXAMPLE: Simplify the following.

$$\frac{x^2 - 5x - 6}{x + 1}$$

SOLUTION: (Using only laws of logarithms.)

$$\log \frac{x^2 - 5x - 6}{x + 1} = \log (x^2 - 5x - 6) - \log (x + 1)$$

SOLUTION: (Using logarithms and laws of algebra.)

$$\log \frac{x^2 - 5x - 6}{x + 1} = \log \frac{(x - 6)(x + 1)}{(x + 1)}$$

$$= \log \frac{(x - 6)\cancel{(x + 1)}}{\cancel{(x + 1)}}$$

$$= \log (x - 6)$$

EXAMPLE: Solve for x.

$$23^x = 250$$

SOLUTION: Take the logarithm of both sides of the equation.

$$\log 23^x = \log 250$$

Then,

$$x \log 23 = \log 250$$

$$x = \frac{\log 250}{\log 23}$$

$$= \frac{2.3979}{1.3617}$$

$$= 1.761$$

In complicated problems it may not be possible to solve for the unknown as directly as we did in the example. In that case we can continue to use our knowledge of logarithms. Return to the step where $x = \frac{2.3979}{1.3617}$. Again take the logarithm of both sides of the equation.

$$\log x = \log \frac{2.3979}{1.3617}$$

$$= \log 2.3979 - \log 1.3617$$

$$= 0.3798 - 0.1341$$

$$= 0.2457$$

Take the antilog of both sides of the equation.

$$x = \text{antilog } 0.2457$$

$$x = 1.761$$

EXAMPLE: Solve for x.

$$x^{\frac{3}{2}} = 729$$

$$\log x^{\frac{3}{2}} = \log 729$$

$$\frac{3}{2} \log x = \log 729$$

$$3 \log x = 2 \log 729$$

$$\log x = \frac{2 \log 729}{3}$$

$$= \frac{2(2.8627)}{3}$$

$$= \frac{5.7254}{3}$$

$$\log x = 1.9084$$

$$x = \text{antilog } 1.9084$$

$$x = 81$$

PRACTICE PROBLEMS: Use logarithms to solve for x in the following problems.

1. $17^x = 31$ 2. $x^{\frac{8}{3}} = 6.3496$

ANSWERS:

1. 0.000121 2. 2.0

APPLICATIONS

The use of logarithms can simplify the solution of many problems encountered in mathematics, science, and engineering. By application of the operations described in this chapter, many complicated equations can be reduced to addition and subtraction problems.

EXAMPLE: Find the volume of a circular cone having a height of 3.71 units and a base radius of 2.71 units.

SOLUTION: The formula for volume of a circular cone is $v = \frac{\pi r 2h}{3}$; where v = volume, r = radius, and h = height.

Take the logarithm of both sides of the equation as the first step in the solution.

$$\log v = \log \left(\frac{\pi r^2 h}{3}\right)$$

$$= \log \pi + \log r^2 + \log h - \log 3$$

$$= \log \pi + 2 \log r + \log h - \log 3$$

$$= 0.4972 + 2(0.4330) + 0.5694 - 0.4771$$

$$= 1.4555$$

$$v = \text{antilog } 1.4555$$

$$v = 28.5$$

For simplicity, physical units of measurement were not included in the example. When problems involving physical units are solved, it is often simpler to solve the problem separately for the units and attach the proper units to the answer. For example, if the base and height in the example problem were given in inches, solution for the proper units in the answers could be as follows:

$$v = \frac{\pi r^2 h}{3}$$

Since pi and 3 are not involved with physical units they are ignored in this solution and the problem is expressed as

v = inches squared x inches

= inches cubed

Therefore, in the solution, v will be expressed in cubic inches.

Many electronic problems can be simplified by using logarithms, and other electronic problems include common logarithms in the basic formulas. An example of a formula that includes a logarithmic expression is the formula for finding gain in decibels where

$$\text{decibels} = 10 \log \frac{P1}{P2}$$

Engineering and electronic problems frequently deal with numbers in the millions and decimal fractions in the millionths. These values are easily expressed as exponentials to the base ten and common logarithms are then a natural and convenient means of simplifying these problems.

EXAMPLE: Find the numerical value of X_C in a circuit where f = 22,000,000 cycles per second, C = .0000000015 farads, and $X_C = \frac{1}{2\pi fC}$.

SOLUTION: The formula is $X_C = \frac{1}{2\pi fC}$

Taking logs on both sides,

$$\log X_C = \log 1 + \text{colog } (2\pi fC)$$

$$= 0 + \text{colog } 6.28 + \text{colog } (2.2 \times 10^7)\ (+ \text{ colog } 1.5 \times 10^{-9})$$

$$= (9.2020 - 10) + (2.6576 - 10) + (8.8239)$$

$$= 20.6835 - 20 = 0.6835$$

$$X_C = \text{antilog } 0.6835$$

$$X_C = 4.825$$

PRACTICE PROBLEMS: Use logarithms to solve for the numerical value of the unknown in the following problems.

1. Find the volume (v) of a sphere having a radius of 7.59. The formula for the volume of a sphere is

$$v = \frac{4\pi r^3}{3}$$

2. Find the value of I in the formula

$$P = I^2 R$$

when P = 217 and R = 550,000.

ANSWERS:

1. 1830 2. 0.0198

CHAPTER 3

TRIGONOMETRIC MEASUREMENTS

This is the first of several chapters in this course which deals with the subject of trigonometry. Reference to the table of contents shows that chapters 4, 5, and 8 also deal directly with triangles and trigonometry. Additionally, chapters 6 and 7 deal with vectors and their application to statics. The study of vectors is so closely related to trigonometry that it is normally included in a trigonometry course, and in this course it is included in the same area.

Mathematics, Vol. 1, NavPers 10069-C, introduces numerical trigonometry and some applications in problem solving. However, trigonometry is not restricted to solving problems involving triangles; it also forms a foundation for some advanced mathematical concepts and subject areas. Trigonometry is both algebraic and geometric in nature, and in this course both of these qualities will be utilized.

MEASURING ANGLES

In Mathematics, Vol. 1, it was pointed out that angles are formed when two straight lines intersect. Before proceeding with measurement of angles, an extension of the concept of angles is required. In this course, an angle is considered to be generated when a line having a set direction is rotated about a point. Figure 3-1 depicts the generation of an angle.

Lay out the line AO, as shown in figure 3-1, as a reference line having a set direction. Use one end of the line as a pivot point and rotate the line from its initial position OA to another position OB, as in opening a door. As the line turns on a pivot point, it is generating the angle AOB. Some of the terminology used in this and subsequent chapters is given in the following:

1. Radius vector—the line which is rotated to generate the angle.
2. Initial position—the original position of the radius vector; corresponds to line OA in figure 3-1. (Also called the initial side of the angle.)
3. Terminal side—the final position of the radius vector; corresponds to line OB in figure 3-1.
4. Positive angle—an angle generated by rotating the radius vector counterclockwise from the initial position.
5. Negative angle—an angle generated by rotating the radius vector clockwise from the initial position.

The convention of identifying angles by use of Greek letters is followed in this text. When only one angle is involved it will be called theta (θ). Other Greek letters will be used when more than one angle is involved. The additional symbols normally used will be phi (ϕ), alpha (α), and beta (β).

One unit of angular measure familiar to most people is the revolution. However, this unit is too large for many uses, and three other units are discussed in following paragraphs.

DEGREE SYSTEM

This is the most common system of angular measurement. In this system a complete revolution is divided into 360 equal parts called degrees (360°). For accurate work each degree is divided into 60' (minutes), and each minute into 60" (seconds). In many cases degree measurements are expressed in degrees and tenths of a degree.

For convenience in working with angles, the 360° is divided into four parts of 90° each, similar to the rectangular coordinate system. The 90° sectors (called quadrants) are numbered according to the convention shown in figure 3-2.

When the radius vector (the line generating the angle) has traveled less than 90° from its starting point in a counterclockwise

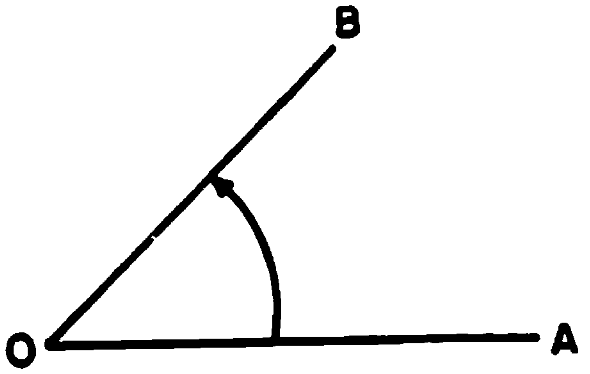


Figure 3-1.—Generation of an angle.

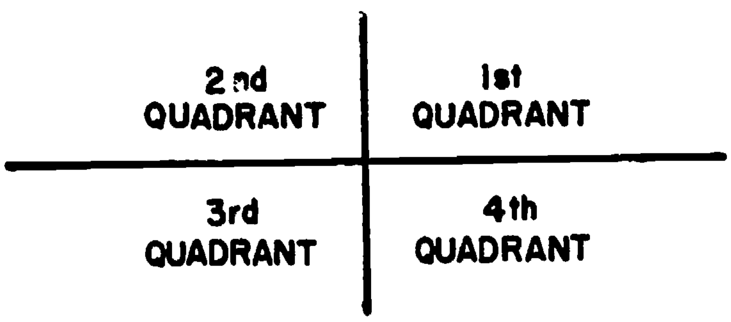


Figure 3-2.—Quadrant positions.

direction (or, as we conventionally call it, in a positive direction), the angle is in the first quadrant. When the radius vector lies between 90° and 180°, the angle is in the second quadrant. Angle between 180° and 270° are said to lie in the third quadrant, while angles greater than 270° and less than 360° are in the fourth quadrant.

When the line generating the angle passes through more than 360°, the quadrant in which the angle lies is found by subtracting from the angle the largest multiple of 360 that the angle contains, and determining the quadrant in which the remainder falls. The original angle lies in the same quadrant in which the remainder angle falls.

EXAMPLE: In which quadrant is the angle 850°?

SOLUTION: The largest multiple of 360° contained in 850° is 720°. Then 850° - 720° = 130°. Since 130° is in quadrant 2, 850° also lies in the second quadrant. This relationship is shown in figure 3-3.

RADIANS

There is another and even more fundamental method of measuring angles. The unit for this type of measurement is the radian. It has certain advantages over the degree method, for it relates the length of arc generated to the size of the angle. Radian measurement also greatly simplifies work with trigonometric functions in calculus. Assume that an angle is generated, as shown in figure 3-4. If we impose the condition that the length of the arc (s) described by the extremity of the line segment generating the angle must equal the length of the line (r), then we would describe an angle exactly one radian in size; that is, for 1 radian, s = r.

Recall from plane geometry that the circumference of a circle is related to the radius by the formula

$$C = 2\pi r$$

This says that the length of the circumference is 2π times the length of the radius. From the relationship of arc length, radius, and radians in the preceding paragraph, this can be extended to say that a circle contains 2π radians.

Since the arc length of the circumference is 2π radians and the circumference encompasses 360° of rotation, it follows that

$$2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = 180^\circ$$

By dividing both sides of this equation by π we find that

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57.2959^\circ$$

In this course we shall use the following conversion factors:

1. 1 radian = 57°17'45" ≐ 57.3° (approximately)
2. 1° = 0.017453 radians (approximately)

If absolute accuracy is desired in a conversion, use the following:

1. To convert radians to degrees multiply by $\frac{180^\circ}{\pi}$.
2. To convert degrees to radians multiply by $\frac{\pi}{180^\circ}$.

It is customary to indicate degrees by the symbol (°) and to indicate radians as a pure number with no name or symbol attached. For example, sin 3 should be understood to represent sine of 3 radians, whereas the sine of 3 degrees would be written sin 3°.

Certain angles occur so frequently in trigonometric problems that it is worthwhile to

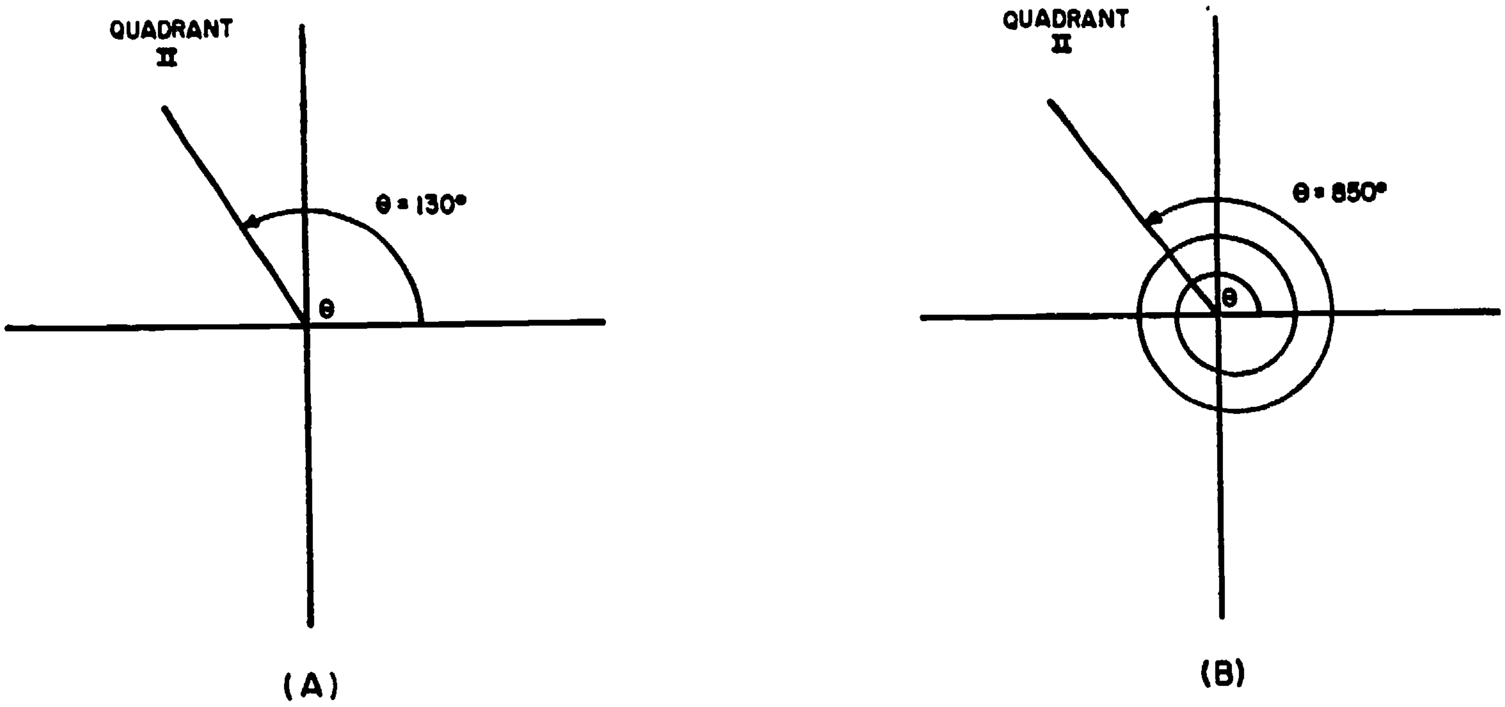


Figure 3-3.—Angle generation. (A) 130°; (B) 850°.

learn the degree and radian equivalences. These are shown in the following list:

Radians	Degrees
$\pi/6$	30
$\pi/4$	45
$\pi/3$	60
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360

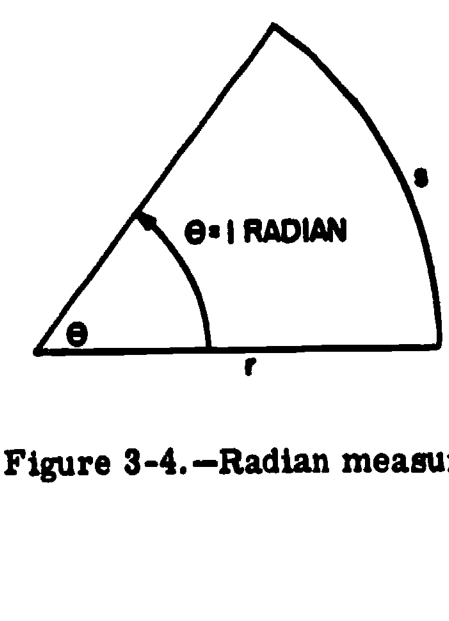


Figure 3-4.—Radian measure.

MILS

The mil is a unit of angular measurement which is not widely used but has some military applications in ranging and sighting. A mil is defined as 1/6400 of the circumference of a circle. This is equavalent to 3'22.5'' or 0.00098 radians.

The importance of the mil in practical approximation is due to the fact that it is approximately 1/1000 of a radian. A circular arc whose length is 1/1000 of the radius will subtend an angle of 1 mil. For very small angles the arc (a) and the chord (c) are nearly equal as shown in figure 3-5.

Since the arc and chord are very nearly equal (ratio of chord to arc nearly 1) for very small angles, we shall consider them as equal for our purposes and develop a method of approximating range (R) to a target of known size (c). Recall from the previous section that a radian is the ratio of arc length to length of radius. In figure 3-5 (with c essentially equal to a), we will consider c as arc length and r as length of radius. Then the size of angle m in radians is

$$m = \frac{c}{r}$$

Then the length (r) is expressed as

$$r = \frac{c}{m}$$

Figure 3-5.—Relationship of arc, chord, and radius.

If we express the angle m in mils (1/1000 of a radian), the formula for range is

$$r = \frac{\frac{c}{m}}{1000}$$

$$r = \frac{1000c}{m}$$

If the range is known and it is desirable to find the length (c) of a target, this formula can be transposed to

$$c = \frac{rm}{1000}$$

These formulas yield good approximations for angles up to several hundred mils and make rapid estimates of range to an object of known size possible.

EXAMPLE: A building known to be 80 feet long, perpendicular to the line of sight, subtends an arc of 100 mils. What is the approximate range to this building?

SOLUTION:

$$r = \frac{1000 \times 80 \text{ ft}}{100}$$

$$r = 800 \text{ ft}$$

PRACTICE PROBLEMS:

Determine the quadrant in which each of the following angles lies.

1. 260°
2. 290°
3. 800°
4. 1,930°

Express the following angles in degrees; the angles are expressed in radians.

5. 20π
6. $\frac{5\pi}{6}$
7. A tower 500 feet away subtends a vertical angle of 250 mils. What is the height of the tower?

ANSWERS:

1. 3rd
2. 4th
3. 1st
4. 2nd
5. 3,600°
6. 150°
7. 125 ft

MEASURES WITH RADIANS

The radian measure of an angle was introduced in previous paragraphs. It was pointed out that radian measure of angles was of particular use when working with trigonometric functions in calculus.

Because of the relationship of the radian to arc length, it has some special applications in measurements of angular velocity and sector area. It was pointed out earlier that in the angle shown in figure 3-4, θ = 1 radian when s = r. This relationship can be generalized by the formula

$$s = r\theta$$

when θ is expressed in radians. This relationship is convenient for solving many types of problems. A few sample problems are included in this section.

EXAMPLE: The human eye cannot clearly distinguish objects if they subtend less than 0.0002 radians at the eye. What is the maximum distance at which a submarine periscope 6 inches in diameter can be picked out from its surroundings? (Of course, this problem disregards the wake made by the periscope).

SOLUTION:
From this information we know that $\theta = 0.0002$ radian and s = 6 in. = 0.5 ft
Substituting in

$$s = r\theta$$

gives

$$0.5 = 0.0002r$$

or

$$r = \frac{5000}{2} = 2{,}500 \text{ feet}$$

ANGULAR VELOCITY

Another type of problem which radian measurement simplifies is that which connects the rotating motion of the wheels of a vehicle to its forward motion. Here we will not be dealing with angles alone but also with angular velocity. Let us analyze this type of motion.

Let us consider the circle at the left in figure 3-6 to indicate the original position of a wheel. As the wheel turns it rolls so that the center moves along the line CC' where C' is the center of the wheel at its final position. The contact point at the bottom of the wheel moves an equal distance PP'; but as the wheel turns through angle θ, the arc s is made to coincide with line PP', so that

$$s = PP'$$

or the length of arc is equal to the forward distance the wheel travels. But since

$$s = r\theta$$

the forward distance d that the wheel travels is

$$d = r\theta$$

Dividing both sides of the equation (2) by t, we have

$$\frac{d}{t} = r\frac{\theta}{t}$$

The forward velocity v of the vehicle is equal to $\frac{d}{t}$, and the angular velocity ω is equal to the angle divided by the time required to describe the angle. Thus,

$$v = r\omega$$

if ω is measured in radians per unit time.

EXAMPLE: Determine the distance a truck will travel in 1 minute if the wheels are 3 feet in diameter and are turning at the rate of 5 revolutions per second.

SOLUTION: In this problem, first convert angular velocity from revolutions per second to radians per second by multiplying the

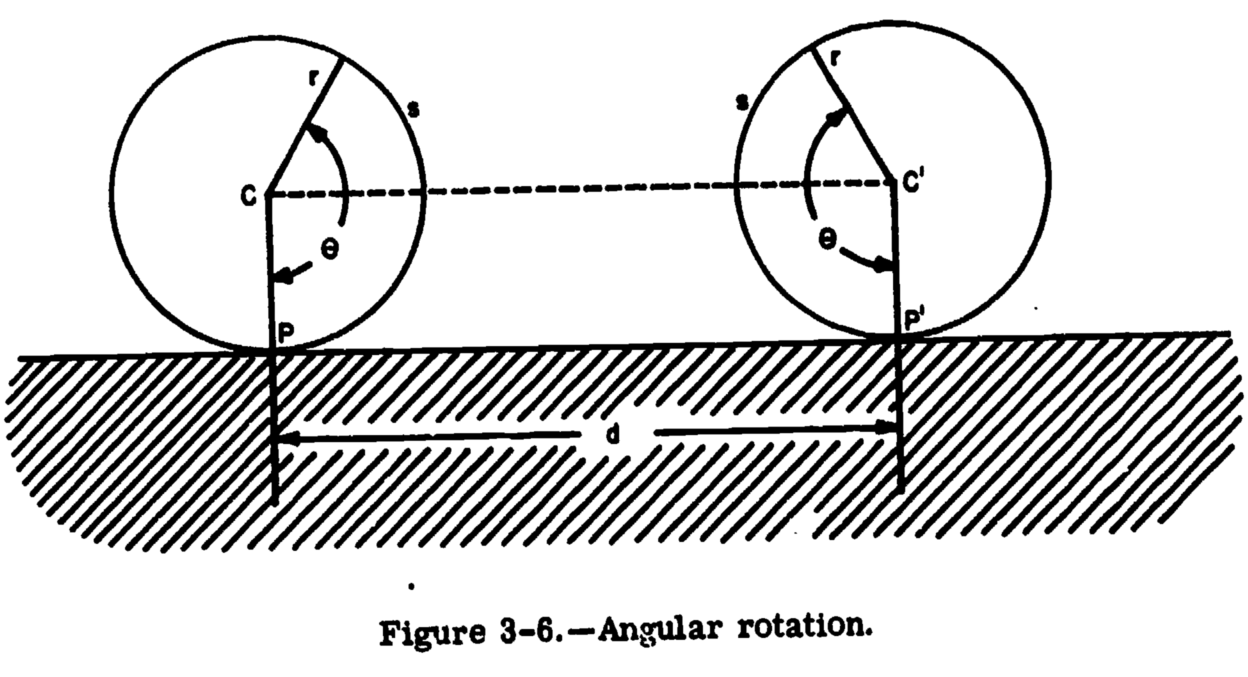


Figure 3-6.—Angular rotation.

revolutions by 2π. (There are 2π radians in one revolution (360°).)

$$2\pi = 6.2832$$

$$5 \text{ rev} \times 6.2832 = 31.416 \text{ rad/sec}$$

Then, radians per second are converted to radians per minute as follows:

$$\text{rad/min} = \text{rad/sec} \times 60$$

$$\text{rad/min} = 31.416 \times 60$$

$$\text{rad/min} = 1{,}884.96$$

Thus, after 1 minute $\theta = 1{,}884.96$ radians and

$$d = r\theta$$

$$d = \frac{3}{2} \text{ ft} \times 1{,}884.96$$

$$d = 2{,}827.44 \text{ ft}$$

EXAMPLE: A car is traveling 40 miles per hour. If the wheel radius is 16 inches, what is the angular velocity of the wheels (a) in radians per minute and (b) in revolutions per minute?

SOLUTION:

(a) First, convert miles per hour to feet per minute.

$$\text{ft/min} = \frac{\text{mph} \times 5280}{60}$$

Thus,

$$40 \text{ mph} = \frac{40}{60} \times 5{,}280 \text{ ft/min}$$

$$40 \text{ mph} = 3{,}520 \text{ ft/min}$$

Then change the radius to feet and

$$r = 1\frac{1}{3} \text{ ft}$$

$$r = \frac{4}{3} \text{ ft}$$

Then,

$$v = r\omega$$

$$\omega = \frac{v}{r}$$

$$\omega = \frac{3{,}520 \text{ ft/min}}{\frac{4}{3} \text{ ft}}$$

$$\omega = \frac{10560}{4}$$

$$\omega = 2{,}640 \text{ radians per minute}$$

(b) $$\text{rpm} = \frac{\text{radians per minute}}{2\pi}$$

$$\omega = \frac{2640 \text{ rad/min}}{2\pi}$$

$$\omega = 420.1 \text{ rpm}$$

AREA OF A SECTOR

From plane geometry we find that the area of the sector of a circle is proportional to the angle enclosed in the sector.

Consider sector AOB of the circle shown in figure 3-7. If θ is increased to a full 360° (or 2π radians), it encompasses the entire circle and the area (A) of the sector equals the area of the circle which is given by the formula

$$A = \pi r^2$$

Multiplying both sides of the equation by θ gives

$$A\theta = \theta\pi r^2$$

We are dealing with a complete circle where $\theta = 2\pi$. Thus, we can substitute 2π for θ in the left member, obtaining

$$2\pi A = \theta\pi r^2$$

Simplifying this by dividing both sides by 2π gives

$$A = \frac{\theta r^2}{2} = \frac{1}{2} r^2\theta$$

Therefore, the area of a sector of a circle can be found by the formula

$$A = \frac{1}{2} r^2\theta$$

with θ expressed in radians.

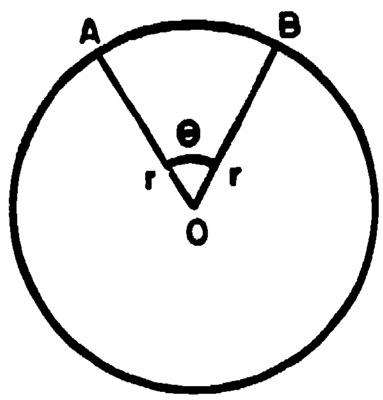


Figure 3-7.—Sector of a circle.

If the radius and arc length are known, the area can be found directly by the formula

$$A = \frac{1}{2} rs$$

This formula is found by the following process:

$$A = \frac{1}{2} r^2\theta$$

$$A = \frac{1}{2} rr\theta$$

Since $s = r\theta$, we substitute in the formula above and have

$$A = \frac{1}{2} rs$$

Of course, if r and s are given, the area could be found by computing θ and using the formula for area in terms of r and θ.

EXAMPLE: Find the area of a sector of a circle with radius of 6 inches having a central angle of 60°.

SOLUTION: First convert 60° to radians:

$$\text{radians} = \text{degress} \times \frac{\pi}{180°}$$

$$\theta = 60° \times \frac{\pi}{180°}$$

$$\theta = \frac{\pi}{3}$$

$$A = \frac{1}{2} r^2 \theta$$

$$A = \frac{1}{2} \times (6 \text{ inches})^2 \times \frac{\pi}{3}$$

$$A = \frac{36\pi \text{ inches}^2}{6}$$

$$A = 6\pi \text{ sq in.}$$

This answer can be converted by multiplying 6 by 3.14159. However, in trigonometry and higher mathematics the conversion is, in many cases, not carried out. In many instances it is less cumbersome to carry π through computations rather than convert it to a decimal number.

PRACTICE PROBLEMS:

1. How far does a car travel in 1 minute if the radius of the wheels is 18 inches and the angular velocity of the wheels is 1,000 radians per minute?
2. A car travels 2,000 feet in 1 minute. The radius of the wheels is 18 inches. What is the angular velocity of the wheels in radians per minute?
3. What is the diameter of a circle if a sector of this circle has an arc length of 9 inches and an area of 18 square inches?

ANSWERS:

1. 1,500 feet per minute
2. 1,333 radians per minute
3. 8 inches

PROPERTIES OF TRIANGLES

Mathematics Vol. 1 contains information on the trigonometric ratios and other properties of triangles. This section reviews the trigonometric functions for acute angles and restates some of the properties of triangles for review and reference.

PYTHAGOREAN THEOREM

This theorem states that in any right triangle, the sum of the squares of the sides adjacent to the right angle is equal to the square of the hypotenuse (side opposite the right angle). In the triangle shown in figure 3-8 this relationship is expressed as

$$x^2 + y^2 = r^2$$

This relationship is useful in solving many problems and in developing other trigonometric concepts. The following are examples of the application of this relationship as applied to the triangles in figures 3-9 and 3-10.

EXAMPLE: **The legs of a right triangle, shown in figure 3-9, which are adjacent to the right angle are 3 and 4 inches in length. How long is the hypotenuse?**

SOLUTION:

$$r^2 = x^2 + y^2$$

$$r^2 = 4^2 \text{ (inches}^2) + 3^2 \text{ (inches}^2)$$

$$r^2 = 16 \text{ (inches}^2) + 9 \text{ (inches}^2)$$

$$r^2 = 25 \text{ (inches}^2)$$

$$r = \sqrt{25 \text{ (inches}^2)}$$

$$r = 5 \text{ inches}$$

This, as pointed out in Mathematics, Vol. 1, is the 3-4-5 triangle, whose side relationships hold also for multiples of 3, 4, and 5.

EXAMPLE: Figure 3-10 shows a plot of ground in the shape of a right triangle. The longest side of the plot is 40 feet. One of the other sides is 10 feet long. How long is the remaining side (x)?

SOLUTION:

$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

$$x^2 = 40^2 \text{ (ft}^2) - 10^2 \text{ (ft}^2)$$

$$x^2 = 1,600 \text{ (ft}^2) - 100 \text{ (ft}^2)$$

$$x^2 = 1,500 \text{ (ft}^2)$$

$$x = \sqrt{1,500 \text{ (ft}^2)}$$

$$x = 38.7 \text{ ft}$$

SIMILAR TRIANGLES

Another relationship of triangles that is useful in trigonometry concerns similar triangles. Whenever the angles of one triangle are equal to the corresponding angles in another triangle, the two triangles are said to be similar.

For example, triangle (A) in figure 3-11 is similar to triangle (B). Since the two triangles

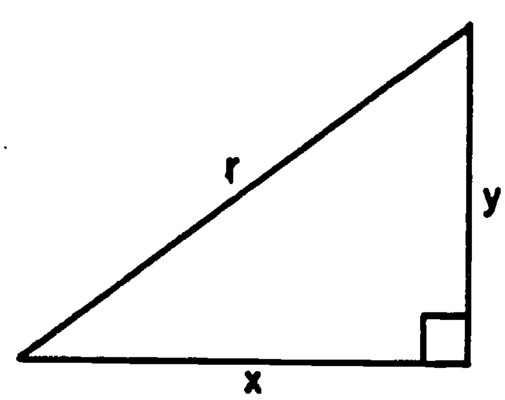


Figure 3-8.—Pythagorean relationship.

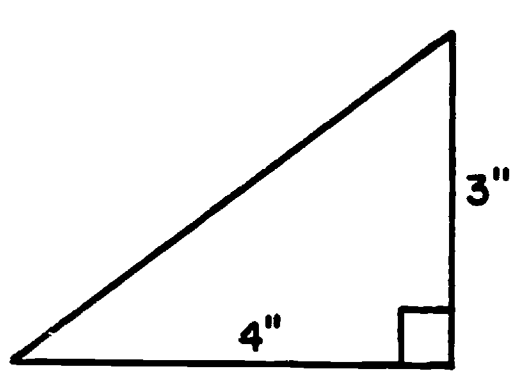


Figure 3-9.—Right triangle with hypotenuse unknown.

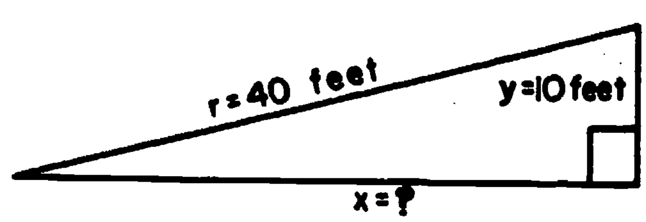


Figure 3-10.—Right triangle with one side unknown.

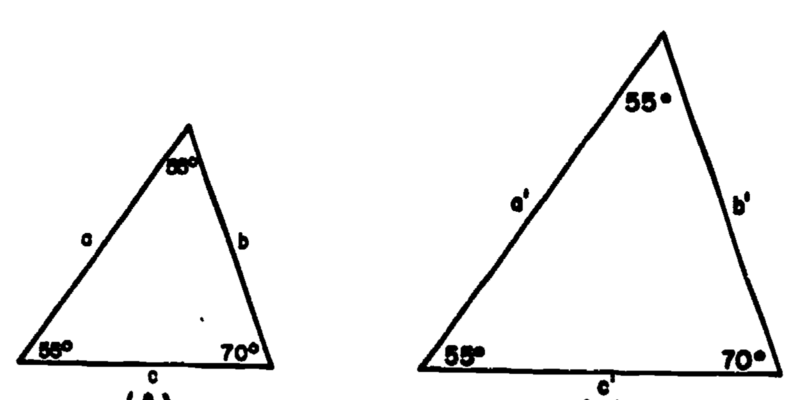


Figure 3-11.—Similar triangles.

are similar by definition, the following proportion involving the lengths of the sides is true:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

This relationship can be used to find the length of unknown sides in similar triangles. The following is an example of this method using the triangles shown in figure 3-12.

EXAMPLE: Triangles (A) and (B) in figure 3-12 are similar with lengths as shown. Find the length of side b' and side c'.

SOLUTION:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

$$\frac{10}{7} = \frac{11.18}{b'} = \frac{5}{c'}$$

Solve the first two of the equal ratios for b'.

$$\frac{10}{7} = \frac{11.18}{b'}$$

$$b' = \frac{11.18 \times 7}{10}$$

$$b' = \frac{78.26}{10}$$

$$b' = 7.826$$

Solve for side c' using the first and third of the equal ratos.

$$\frac{10}{7} = \frac{5}{c'}$$

$$c' = \frac{5 \times 7}{10}$$

$$c' = 3.5$$

NOTE: Side c' could have been determined by using the second and third of the equal ratios, and the reader should verify this. The selection of the first and third ratios in the example was only because it was obvious that the numerical calculations would be simpler without decimals.

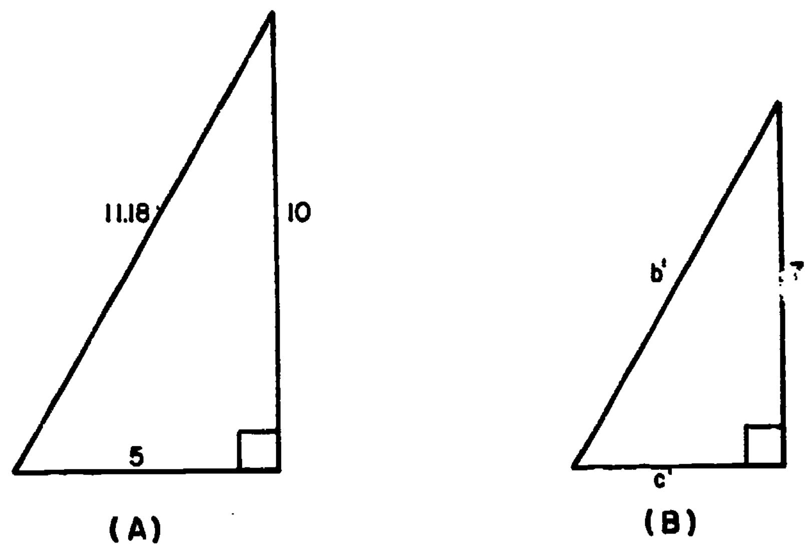


Figure 3-12.—Similar triangles, solution example.

Similar Right Triangles

Recall from plane goemetry that the sum of the interior angles of any triangle is equal to 180°. Using this fact it follows that two triangles are similar if two angles of one are equal to two angles of the other. The remaining angle in any triangle must be equal to 180° minus the sum of the other two angles. Keep these principles in mind during the following discussion of similar right triangles.

Two RIGHT TRIANGLES are similar if an acute angle of one triangle is equal to an acute angle of the other triangle. If these angles are equal, the triangles are similar because the right angles in the two triangles are also equal to each other. Whenever one acute angle of a right triangle is given, the other acute angle may easily be found. Assuming that the acute angle given is 25 degrees, the other acute angle will be

$$180° - 90° - 25° = 65°$$

But 65° and 25° are complements of each other. Thus, if one acute angle of a right triangle is equal to θ, the other acute angle will be $(90° - \theta)$.

Many of the practical uses of trigonometry are based on the fact that two right triangles are similar if one acute angle of one triangle is known to be equal to one angle of the other triangle.

Thus, in figure 3-13, we have two similar right triangles, so we may write

$$\frac{x}{x'} = \frac{y}{y'}$$

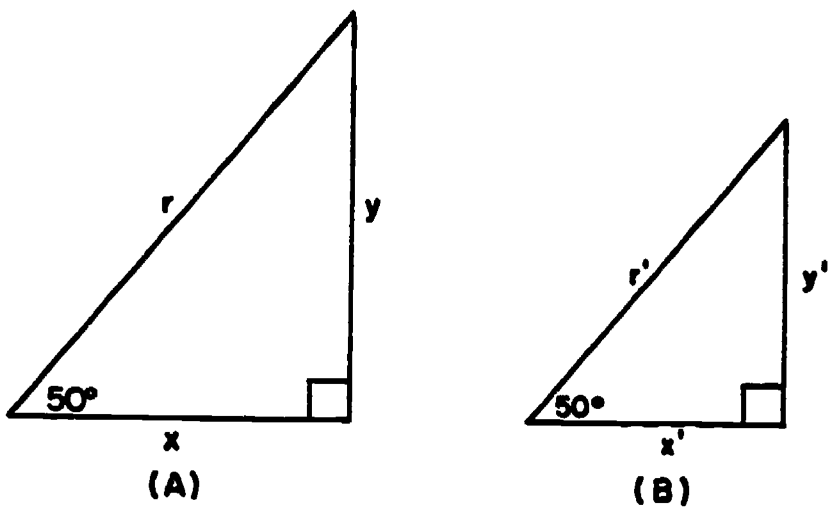


Figure 3-13.—Similar right triangles.

or interchanging the mean terms in the proportion, we have

$$\frac{x}{y} = \frac{x'}{y'}$$

and in like manner

$$\frac{y}{r} = \frac{y'}{r'} \text{ and } \frac{x}{r} = \frac{x'}{r'}$$

This is one of the main principles of numerical trigonometry.

PRACTICE PROBLEMS: Refer to figure 3-14 in solving the following problems.

1. Use the Pythagorean theorem to calculate the missing sides in triangles (A) and (B).
2. Find sides a and b of triangle (D), assuming that triangles (C) and (D) are similar triangles.

ANSWERS:

1. (a) $\sqrt{5}$

   (b) $2\sqrt{2}$

2. (a) 3 3/4

   (b) 1 1/2

TRIGONOMETRIC FUNCTIONS AND TABLES

The properties of triangles given in the previous section provide a means for solving many practical problems. Certain practical problems, however, require knowledge of right triangle relationships other than the Pythagorean theorem or the relationships of similar triangles before solutions can be found. Examples of two of the problems which require this additional knowledge are as follows:

1. Find the value of the missing sides and angles in a right triangle when the value of one side and one acute angle are given.
2. Find the value of the missing side and the value of the angles in a right triangle when two sides are known.

The additional relationships, between the sides and ~~angles of~~ a right triangle, are called trigonometric functions or trigonometric ratios. These ratios were introduced in Mathematics, Vol. 1, and are reviewed in the following paragraphs. The basic foundations of trigonometry rest upon these functions.

RATIOS FOR ACUTE ANGLES

There are six ratios of the sides of a right triangle; these ratios form the six trigonometric functions. We use the acute angle θ in figure 3-15 and define the trigonometric ratios.

In figure 3-15 the three sides x, y, and r are used two at a time to form six ratios. These ratios and the trigonometric function name associated with each are as follows:

1. $\frac{y}{r}$ is the sine of θ, written sin θ
2. $\frac{x}{r}$ is the cosine of θ, written cos θ
3. $\frac{y}{x}$ is the tangent of θ, written tan θ
4. $\frac{r}{y}$ is the cosecant of θ, written csc θ
5. $\frac{r}{x}$ is the secant of θ, written sec θ
6. $\frac{x}{y}$ is the cotangent of θ, written cot θ

The functions of a right triangle in any position are made easier to remember by the convention of naming the sides, as in figure 3-16, and defining the functions by means of these names.

In any right triangle the side y is the side opposite the angle whose function we are seeking. The side x is always adjacent to the angle. Thus, we can summarize the information in this section

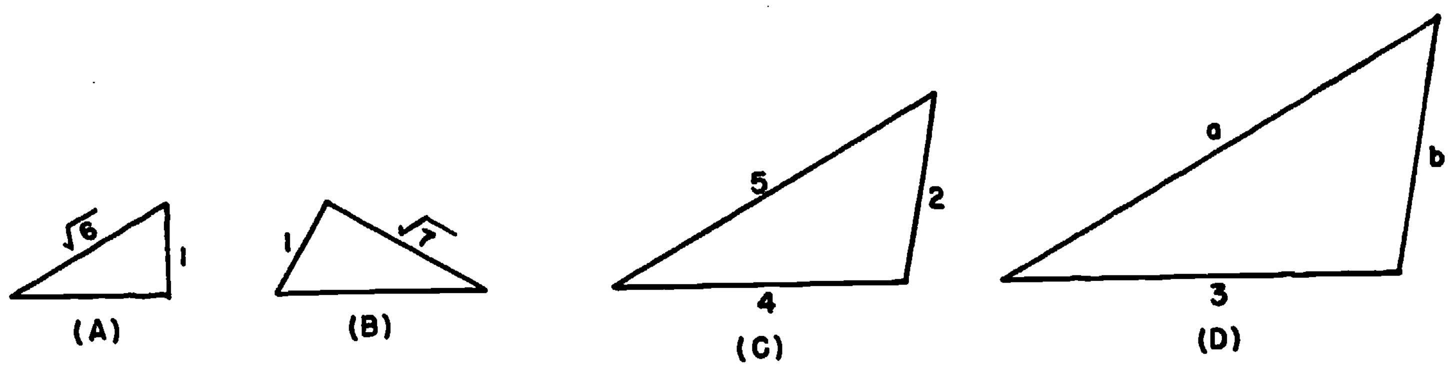


Figure 3-14.—Triangles for practice problems.

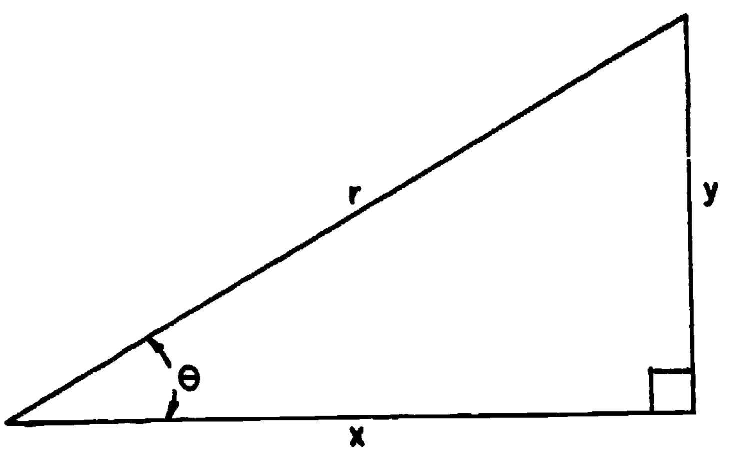


Figure 3-15.—Right triangle for determining ratios.

in the following manner by reference to figure 3-16:

$$\sin\ \theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\ \theta = \frac{x}{r} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\ \theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\csc\ \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\sec\ \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\cot\ \theta = \frac{x}{y} = \frac{\text{adjacent side}}{\text{opposite side}}$$

When the lengths of the sides of a right triangle are known, as in figure 3-17 (A), the six trigonometric ratios can be computed directly.

The values of all the trigonometric functions of the angle θ in the triangle in figure 3-17 (A) are as follows:

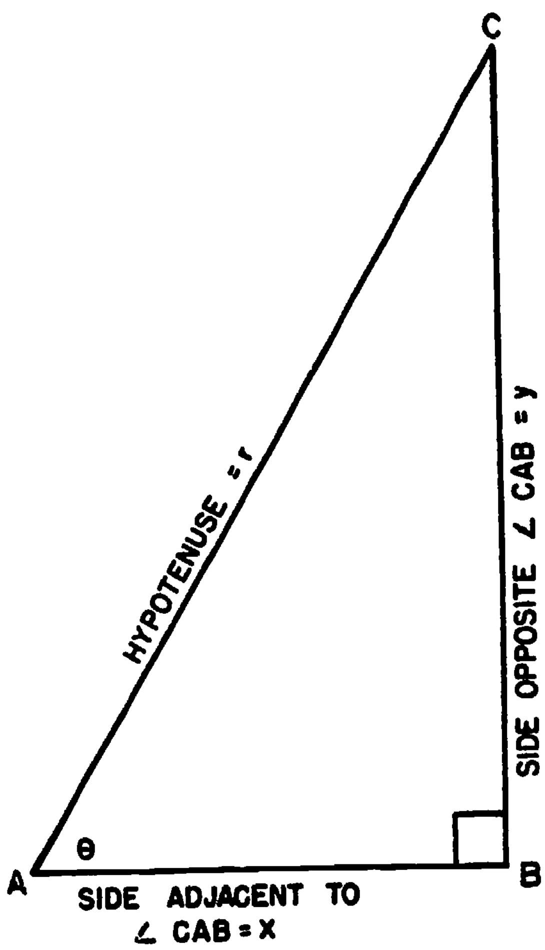


Figure 3-16.—Names of sides of a right triangle.

$$\sin\ \theta = \frac{y}{r} = \frac{3}{5} = 0.600$$

$$\cos\ \theta = \frac{x}{r} = \frac{4}{5} = 0.800$$

$$\tan\ \theta = \frac{y}{x} = \frac{3}{4} = 0.750$$

$$\sec\ \theta = \frac{r}{x} = \frac{5}{4} = 1.250$$

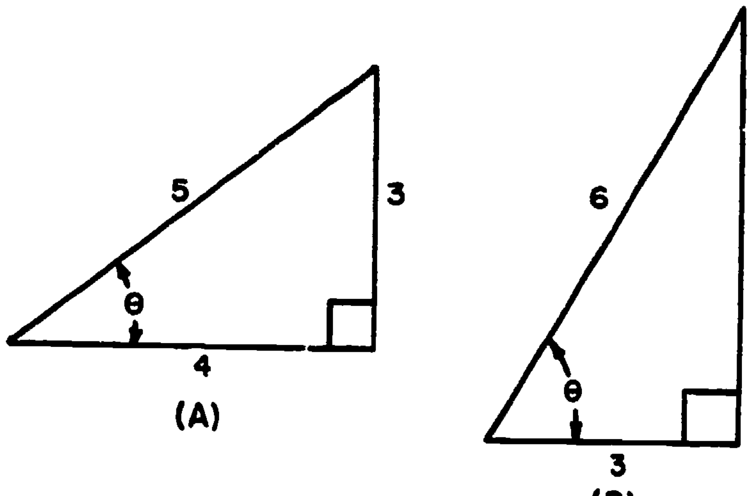


Figure 3-17.—Practice triangles.

$$\csc \theta = \frac{r}{y} = \frac{5}{3} = 1.667$$

$$\cot \theta = \frac{x}{y} = \frac{4}{3} = 1.333$$

EXAMPLE: Give the values of all the trigonometric functions of the angle θ in the triangle of figure 3-17 (B).

SOLUTION: Here only two sides are given. To find the third side, use the Pythagorean theorem.

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y^2 = 6^2 - 3^2 = 36 - 9 = 27$$

$$y = \sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3} = 5.196$$

Now, using these values of x, y, and r

$$\sin \theta = \frac{y}{r} = \frac{5.196}{6} = 0.8660$$

$$\cos \theta = \frac{x}{r} = \frac{3}{6} = 0.500$$

$$\tan \theta = \frac{y}{x} = \frac{5.196}{3} = 1.732$$

$$\csc \theta = \frac{r}{y} = \frac{6}{5.196} = 1.1547$$

$$\sec \theta = \frac{r}{x} = \frac{6}{3} = 2.000$$

$$\cot \theta = \frac{x}{y} = \frac{3}{5.196} = 0.5773$$

TABLES

Trigonometric tables are lists of the numerical values of the ratios of sides of right triangles. If we desire to know the ratio of two sides of a right triangle containing a known acute angle θ, we look for the angle θ in a table and thus find the desired ratio. The table in appendix III provides the sine and cosine of angles from 0° to 90°. Appendix IV gives natural tangents and cotangents of angles from 0° to 90°.

The tables in the appendixes give the trigonometric functions in degrees and minutes. The format used in these tables is fairly standard for trigonometric tables. Refer to appendix III and the first page of values. To find the sine or cosine of angles less than 5°, enter the table at the appropriate degree value listed at the top of the table. The minute column which is used with these values is the first column at the left of the table and is read from top to bottom.

Values of the functions between 85° and 90° are also listed on this page. To determine the sine or cosine of an angle in this range, enter the table at the degree value at the bottom of the page. The minute values which correspond to these angles are found in the last column of the table, and are read from bottom to top. The column headings (sin and cos) are seen to change from the top to the bottom of the column. The correct name of the function is the one which appears at the degree value in use. The tables of the tangent and cotangent ratios in appendix IV are laid out in the same format as the sine and cosine tables.

Most tables list the sine, cosine, tangent, and cotangent of angles from 0° to 90°. Very few give the secant and cosecant since these are seldom used. When needed, they may be found from the values of the sine and cosine. The reciprocal of the sine gives the value of the cosecant.

$$\csc \theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin \theta}$$

The reciprocal of the cosine gives the value of the secant.

$$\sec \theta = \frac{r}{x} = \frac{1}{\frac{x}{r}} = \frac{1}{\cos \theta}$$

The other functions, the tangent and cotangent, can also be expressed in terms of sine and cosine as follows:

$$\tan\,\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\,\theta}{\cos\,\theta}$$

$$\cot\,\theta = \frac{x}{y} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{\cos\,\theta}{\sin\,\theta}$$

In addition, the cotangent can be determined directly as the reciprocal of the tangent.

$$\cot\,\theta = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan\,\theta}$$

These relationships are the fundamental trigonometric identities and will be used extensively in solving the more complex identities which are the topic of chapter 8.

PRACTICAL USE OF THE TRIGONOMETRIC RATIOS

Proper use of the trigonometric ratios and the other principles of triangles furnish powerful tools for use in problem solving. The knowledge of when (and how) to use which of the ratios is an important part of the problem. This knowledge comes with experience and practice; however, each of the functions is more applicable to certain type problems than to others. This section points out some of the situations in which a specific function is used most advantageously.

Tangent of an Angle

The ratio $\frac{y}{x}$ of $\frac{\text{opposite side}}{\text{adjacent side}}$ can be used whenever one of these sides and an acute angle are given and it is desired to find the other side.

EXAMPLE: Find the length of side y in figure 3-18 (A).

SOLUTION: From the figure

$$\tan 35^\circ = \frac{y}{x} = \frac{y}{20}$$

From the tables

$$\tan 35^\circ = 0.70021$$

Here there are two expressions for tan 35°, so that

$$\frac{y}{20} = 0.70021$$

$$y = 14.0042$$

Sine of an Angle

To find y or r, when either of these and an acute angle are given, use the sine to find the unknown.

EXAMPLE: Fine the value of r in figure 3-18 (B).

SOLUTION: From the figure

$$\sin 65^\circ = \frac{5}{r}$$

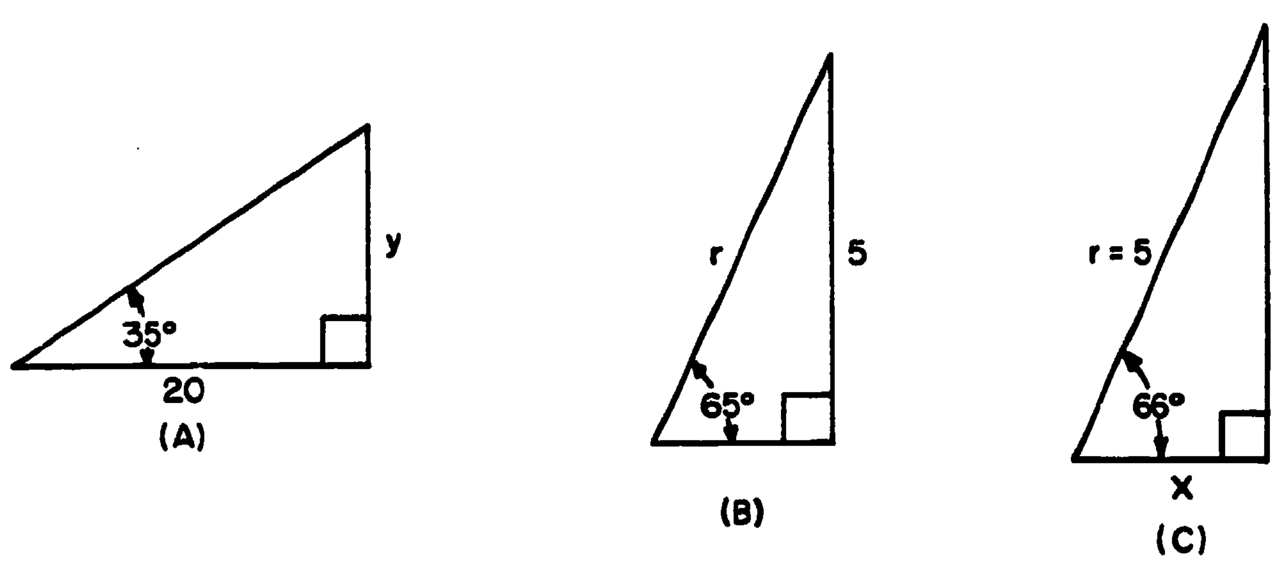


Figure 3-18.—Practical use of ratios.

From the tables

$$\sin 65^\circ = 0.90631$$

Thus,

$$\frac{5}{r} = 0.90631$$

$$r = 5.517$$

Cosine of an Angle

To find x or r, when either of these and an acute angle are given, use the cosine to find the desired part of the triangle.

EXAMPLE: Find the value of x in the triangle of figure 3-18 (C).

SOLUTION:

$$\frac{x}{5} = \cos 66^\circ$$

$$x = 5 \cos 66^\circ$$

$$= 5(0.40674)$$

$$= 2.0335$$

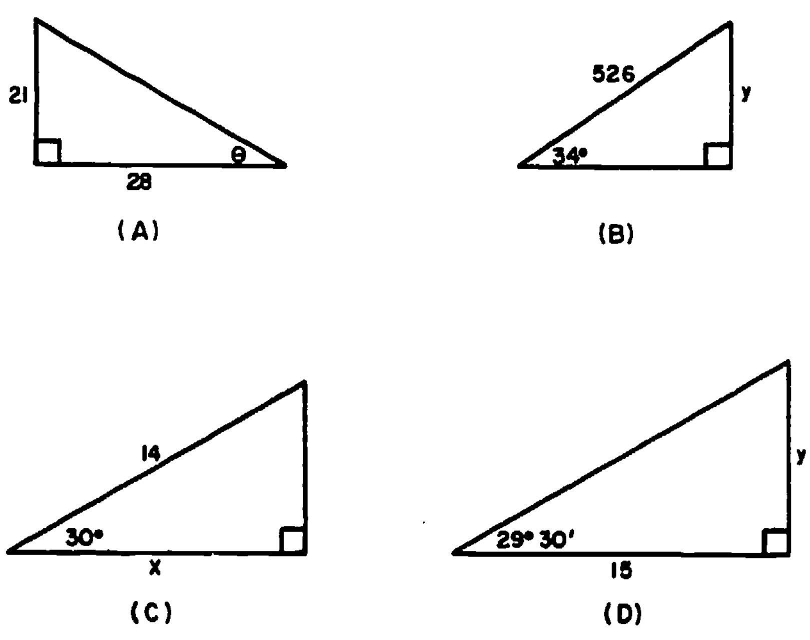


Figure 3-19.—Triangles for practice problems.

PRACTICE PROBLEMS: Refer to figure 3-19 in working problems 1 through 4.

1. a. Find the values of the sine, cosine, and tangent of θ in triangle (A).

b. What is the value of θ to the nearest degree?

2. Using the sine function, find the value of y in triangle (B).

3. Using the cosine, find the value of x in triangle (C).

4. Using the tangent, find the value of y in triangle (D).

5. A navigator on a ship notes that two points on either side of a strait are 5 miles apart and subtend an angle of 40° as shown in figure 3-20. How far from the strait is the ship if it is equidistant from both points? NOTE: The 40° angle can be divided into two 20° angles, and this will form two right triangles.

ANSWERS:

1. a. $\sin \theta = 3/5 = 0.60000$
   $\cos \theta = 4/5 = 0.80000$
   $\tan \theta = 3/4 = 0.75000$
   b. $\theta = 36^\circ$
2. 294
3. 12.12
4. 8.4866
5. 6.87 miles

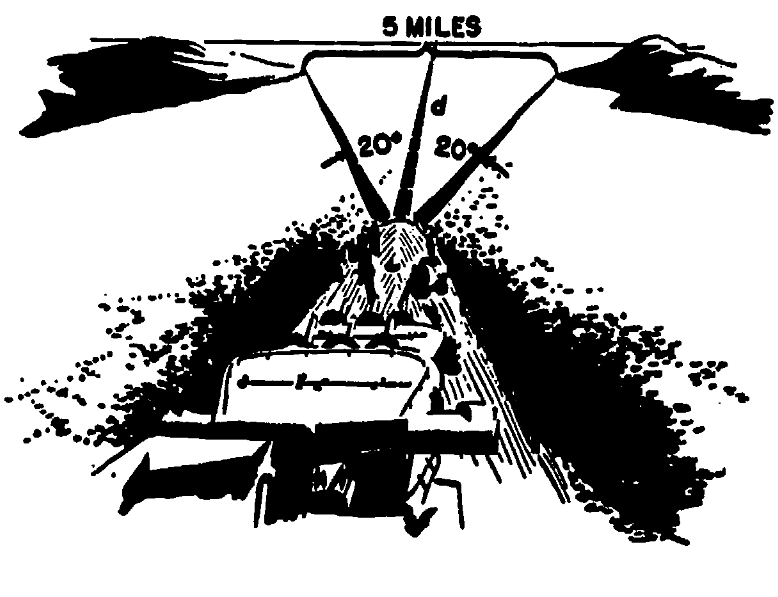


Figure 3-20.—Ship approaching strait.

CHAPTER 4

TRIGNOMETRIC ANALYSIS

This chapter is a continuation of the broad topic of trigonometry introduced in chapter 3. The subject is expanded in this chapter to allow analysis of angles greater than 90°. The chapter is intended as a foundation for analysis of the generalized angle, an angle of any number of degrees. Additionally, the chapter introduces the concept of both positive and negative angles.

RECTANGULAR COORDINATES

The rectangular or Cartesian coordinate system introduced in Mathematics, Vol. 1, NavPers 10069-C, is used here. In Vol. 1 the coordinate system was used in solving equations; in this chapter it is used for analyzing the generalized angle. The following is a brief review of important facts about the coordinate system:

1. The vertical axis (Y axis in fig. 4-1) is considered positive above the origin and negative below the origin.

2. The horizontal axis (X axis) is positive to the right of the origin and negative to the left of the origin.

3. A point anywhere in the plane may be located by two numbers, one showing the distance of the point from the Y axis, and the other showing the distance of the point from the X axis. These points are called coordinates.

4. In notation used to locate points, it is conventional to place the coordinates in parentheses and separate them with a comma. The X coordinate is always written first. Thus, point P in figure 4-1 would have the notation P(4,-5). The general form of this notation is P(x,y).

5. The X coordinate is positive in the first and fourth quadrants, negative in the second and third. The Y coordinate is positive in the first and second quadrants, negative in the third and fourth. The signs of the coordinates are shown in parentheses in figure 4-1. The algebraic signs of the coordinates of a point are used in this chapter for determining the algebraic signs of trigonometric functions.

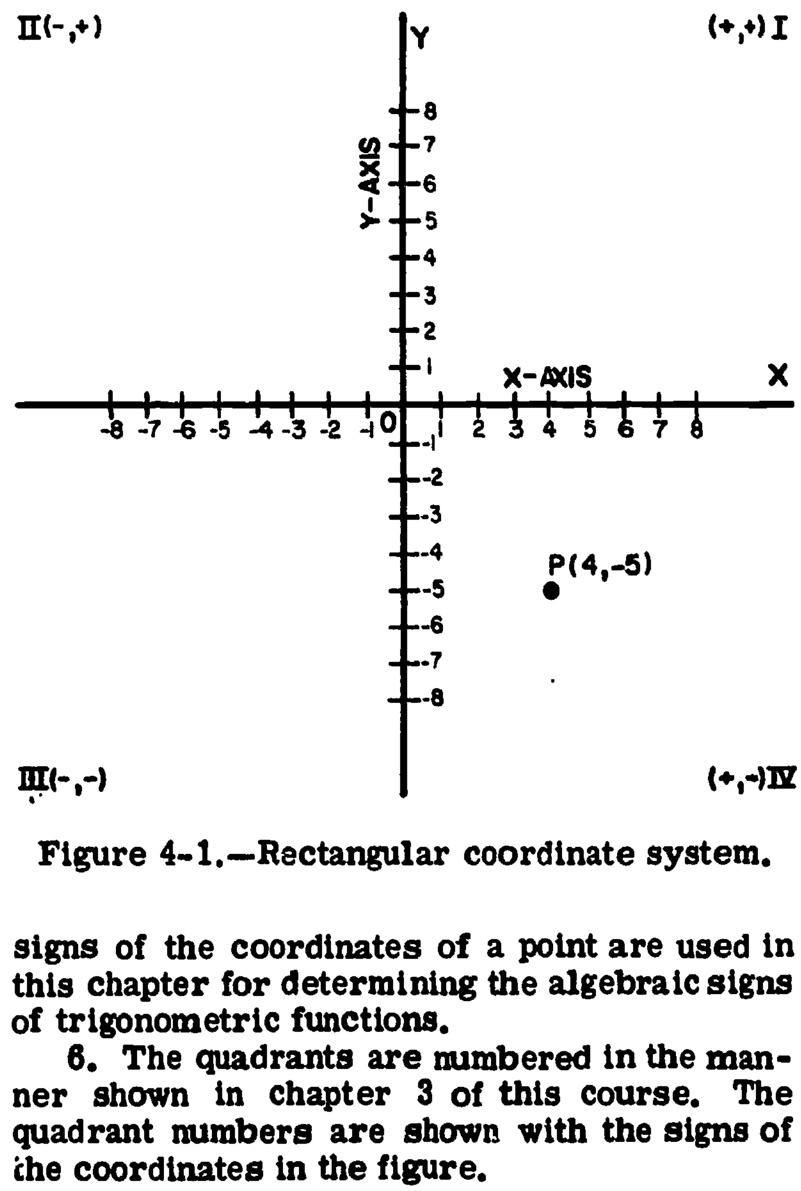


Figure 4-1.—Rectangular coordinate system.

6. The quadrants are numbered in the manner shown in chapter 3 of this course. The quadrant numbers are shown with the signs of the coordinates in the figure.

ANGLES IN STANDARD POSITION

An angle is said to be in standard position when certain conditions are met. To construct an angle in standard position, first lay out a rectangular coordinate system. The angle (θ) is then drawn with the vertex at the origin of the coordinate system, and the original side lying along the positive X axis as shown in figure 4-2. In standard position, the terminal side

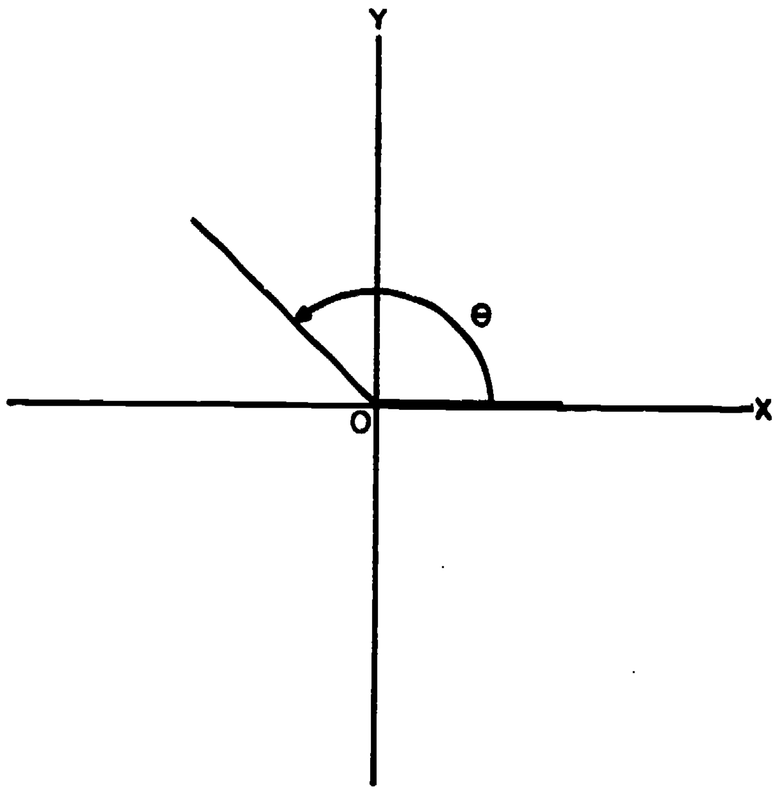


Figure 4-2.—Standard position.

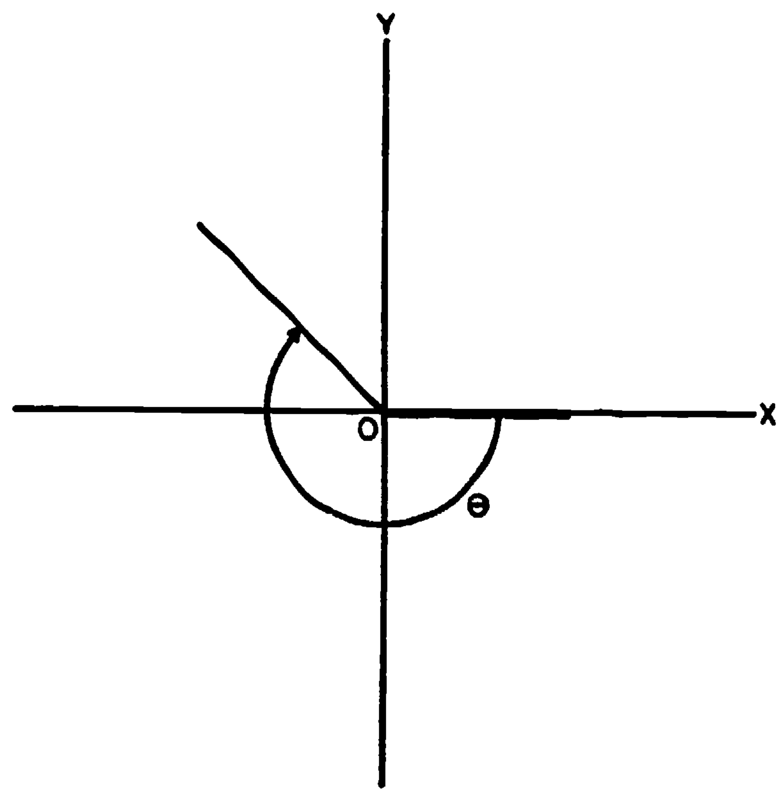


Figure 4-3.—Negative angle in quadrant II.

(radius vector) of the angle may lie in any of the quadrants, or on one of the axes which separate the quadrants. When the terminal side falls on an axis a special case exists, in which the angles are called quadrantal angles and are discussed separately later in this chapter.

The quadrant in which an angle lies is determined by the terminal side. When an angle is placed in standard position the angle is said to lie in the quadrant which contains the terminal side. For example, the negative angle θ, shown in standard position in figure 4-3, is said to lie in the second quadrant.

Coterminal Angles

When two or more angles in standard position have their terminal sides located at the same position they are said to be coterminal. If θ is any general angle then θ plus or minus an integral multiple of 360° yields a coterminal angle. For example, the angles θ, ϕ, and α in figure 4-4 are said to be coterminal angles. If θ is 45° then

$$\phi = \theta - 360^\circ$$

$$\phi = 45 - 360^\circ$$

$$\phi = -315^\circ$$

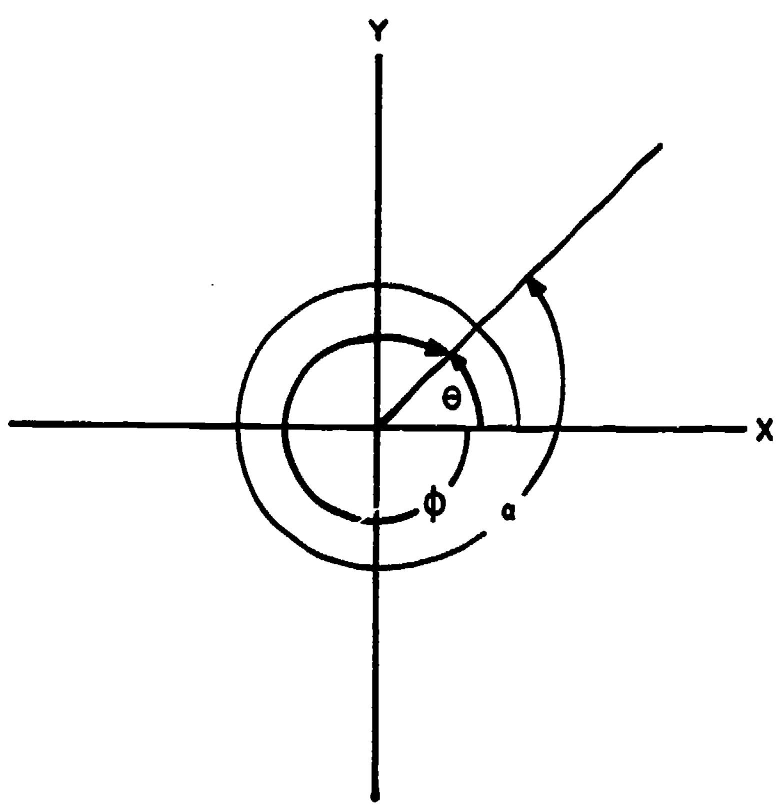


Figure 4-4.—Coterminal angles.

and,

$$\alpha = \theta + 360^\circ$$

$$\alpha = 45^\circ + 360^\circ$$

$$\alpha = 405^\circ$$

The relationship of coterminal angles can be stated in a general form. For any general angle θ, measured in degrees, the angles ϕ coterminal with θ can be found by the following

$$\phi = \theta + n(360^\circ)$$

where n is any integer, positive, negative or zero; that is,

$$n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The principle of coterminal angles is used in developing other trigonometric relationships and in other phases of trigonometric analysis. An expansion of this principle states that the trigonometric functions (ratios) of coterminal angles have the same value. This fact is the basis for part of a discussion in a later section of this chapter.

PRACTICE PROBLEMS: Which of the following sets of angles are coterminal?

1. 60°, -300°, 420°.
2. 735°, -345°, -705°.
3. 45°, -45°, 345°.
4. 0°, 360°, 180°.

ANSWERS:

1. Coterminal
2. Coterminal
3. Not coterminal
4. Not coterminal

Definition Of Functions

The trigonometric functions (ratios) are defined in chapter 3 of this course in two forms as follows:

1. By means of the sides labeled x, y, and r.
2. By means of the names of the sides: opposite side, adjacent side, and hypotenuse.

In this chapter a third form is introduced, using the nomenclature of the coordinate system. The three systems are not different systems, they merely define the same functions by using different terminology. Defining the same ratios in three different sets of terms no doubt has the appearance of complicating a relatively simple operation. However, as progress is made through this course and in advance mathematics courses, particular problems or situations arise where it is natural to think of the ratios in the specific terms used in one of the definitions. It is the intention of this course to introduce the three terminology groups and show that the three definitions are synonymous and interchangeable.

To arrive at the definitions, construct an angle in standard position in respect to a coordinate system as shown in figure 4-5 (A). Choose a point P with coordinates (x,y) as a point on the radius vector. The distance OP is denoted by the positive number r (radius vector).

NOTE: In this chapter the conventional designation of the radius vector as always positive is followed.

The trigonometric functions of the general angle θ in figure 4-5 (A) are defined as follows:

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{radius vector}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{radius vector}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

The remaining functions are reciprocals of the ones given and can be written in this terminology by inverting the given functions.

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{radius vector}}{\text{ordinate}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{radius vector}}{\text{abscissa}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{abscissa}}{\text{ordinate}}$$

The values of the functions are dependent on the angle θ alone and are not dependent upon the selection of a particular point P. If a different point is chosen, the length of r, as well as the values of the X and Y coordinates, will change proportionally and the ratio will be unchanged.

Comparing (A) and (B) in figure 4-5 it is seen that the X and Y values of the point P in (A) correspond to the lengths of sides x and y in

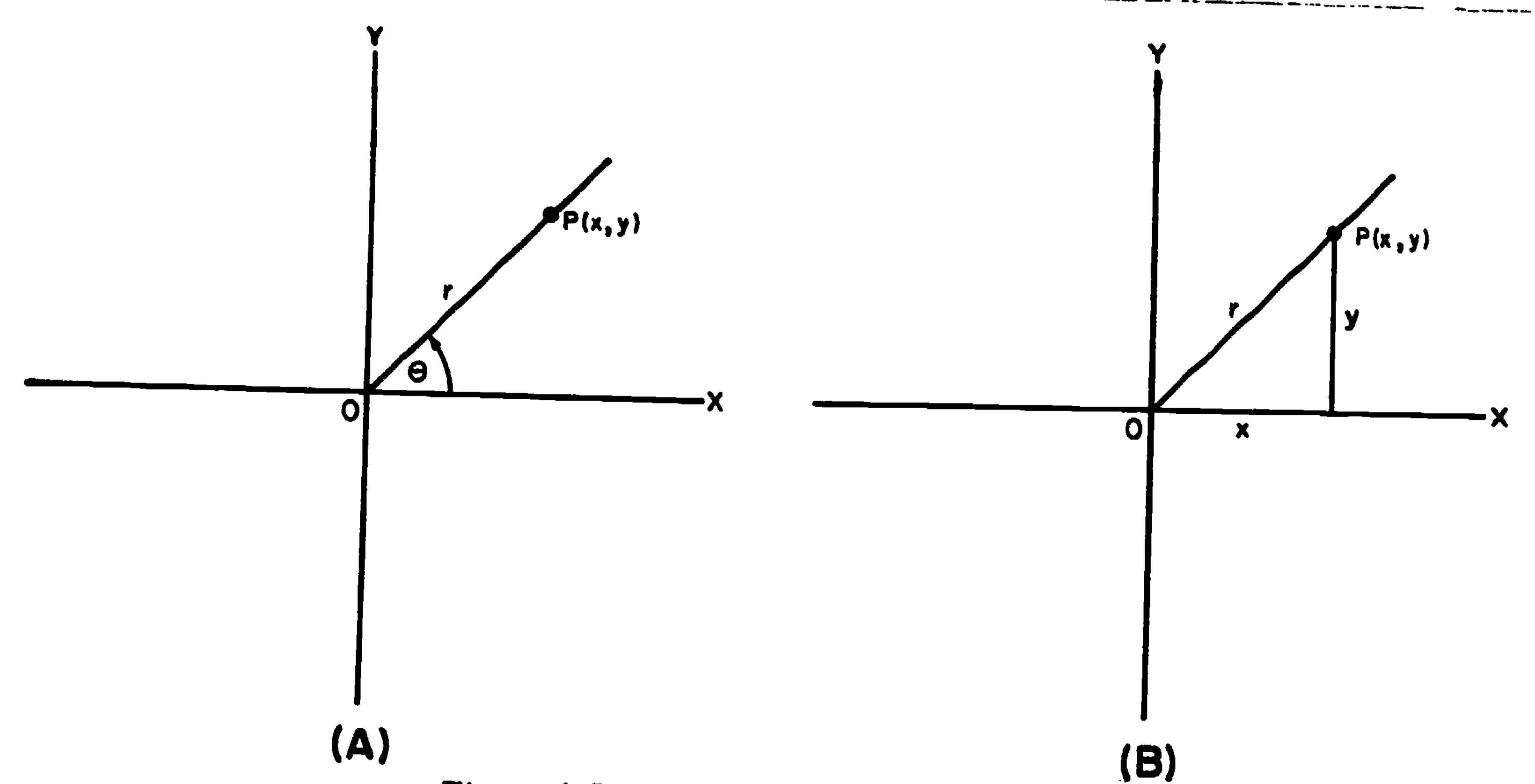


Figure 4-5.—Functions of general angles.

(B). Therefore, the ratios defined here are the same ratios defined in chapter 3 of this course; the only change is in terminology. The procedure given here allows for finding the functions of an angle when only a point on the radius vector is given.

EXAMPLE: Find the sine and cosine of the angle shown in figure 4-5 (A) when the point P (x,y) has the value P(3,4).

SOLUTION: To determine the sine and cosine it is necessary to find the value of r. Since the values of the X and Y coordinates correspond to the lengths of the sides x and y in figure 4-5 (B), we can determine the length of r by use of the Pythagorean theorem or by recalling from Mathematics, Vol. 1 the 3-4-5 triangle. In either case, the length of r is 5 units. Then,

$$\sin \theta = \frac{\text{ordinate}}{\text{radius vector}}$$

$$\sin \theta = \frac{4}{5}$$

$$\cos \theta = \frac{\text{abscissa}}{\text{radius vector}}$$

$$\cos \theta = \frac{3}{5}$$

NOTE: In the remainder of this chapter all angles are understood to be in standard position, unless specifically stated to the contrary.

PRACTICE PROBLEMS: Without using tables, find the sine, cosine, and tangent of the angles whose radius vectors pass through the points given below.

1. P(5,12)
2. P(1,1)
3. P(1, $\sqrt{3}$)
4. P(3,2)

NOTE: In problems such as these it is often helpful to construct a graphic illustration of the problem similar to figure 4-5.

ANSWERS:

1. $\sin = \frac{12}{13}$

   $\cos = \frac{5}{13}$

   $\tan = \frac{12}{5}$

2. $\sin = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos = \frac{\sqrt{2}}{2}$

$\tan = 1$

3. $\sin = \frac{\sqrt{3}}{2}$

$\cos = \frac{1}{2}$

$\tan = \sqrt{3}$

4. $\sin = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

$\cos = \frac{3\sqrt{13}}{13}$

$\tan = \frac{2}{3}$

QUADRANT SYSTEM

The quadrants formed in the rectangular coordinate system are used to determine the algebraic signs of the trigonometric functions. The quadrants in figure 4-6 show the algebraic signs of the sine, cosine, and tangent in the various quadrants.

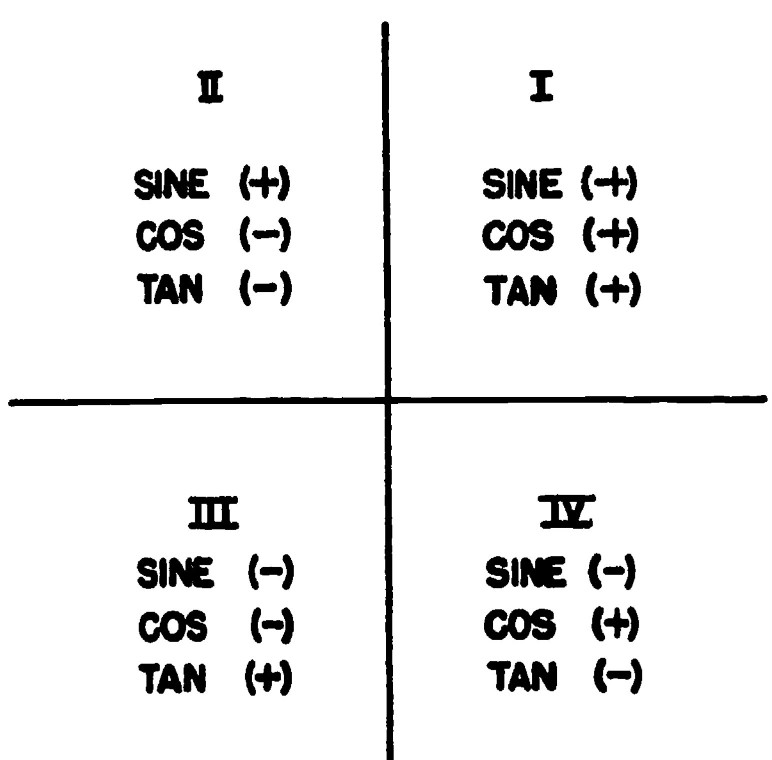


Figure 4-6.—Signs of functions.

A ready recall of the algebraic signs of the functions in particular quadrants will be of assistance in many problems. It is not mandatory, however, that the information in figure 4-6 be memorized. The information will be learned in some cases by extensive usage and, in addition, the following section on reference triangles supplies a means for rapidly determining the algebraic signs of all of the functions for any angle in standard position.

The algebraic signs of the remaining functions, while not shown on the figure, can be determined from the signs of the given functions. In all quadrants the cosecant has the same sign as the sine, the secant has the same sign as the cosine, and the cotangent has the same sign as the tangent.

The last group of practice problems involved angles in the first quadrant only, where all of the functions are positive. When the signs that the functions take on in each quadrant are known, more complicated problems of this type can be solved.

EXAMPLE: Find all of the trigonometric functions of θ if $\tan \theta = \frac{5}{12}$, $\sin \theta < 0$, and $r = 13$.

SOLUTION:

Reference to figure 4-6 shows that an angle with a positive tangent and a negative sine can occur only in the third quadrant.

Then, since

$$\tan \theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}$$

and

$$\tan \theta = \frac{5}{12}$$

it is possible to determine a point P(x,y) which lies on the radius vector of this angle. The value of the Y coordinate is the value of the ordinate and the value of the X coordinate is the value of the abscissa. Thus, it appears that the point is P(12,5). However, in the Cartesian coordinate system, both of the coordinates are negative in the third quadrant so the point is P(-12,-5). This does not conflict with the value of the tangent that was given, however, since

$$\frac{-5}{-12} = \frac{5}{12}$$

and the tangent is positive in a quadrant where both coordinates are negative.

Figure 4-7 shows the angle θ constructed using the following information which was derived from an analysis of the previous paragraph. The point

$$P(x,y) = P(-12,-5)$$

lies on a radius vector in the third quadrant with $r = 13$.

Now the functions can be read from the figure.

$$\sin\theta = \frac{\text{ordinate}}{\text{radius vector}} = \frac{-5}{13} = -\frac{5}{13}$$

$$\cos\theta = \frac{\text{abscissa}}{\text{radius vector}} = \frac{-12}{13} = -\frac{12}{13}$$

$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{-5}{-12} = \frac{5}{12}$$

$$\cot\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{-12}{-5} = \frac{12}{5}$$

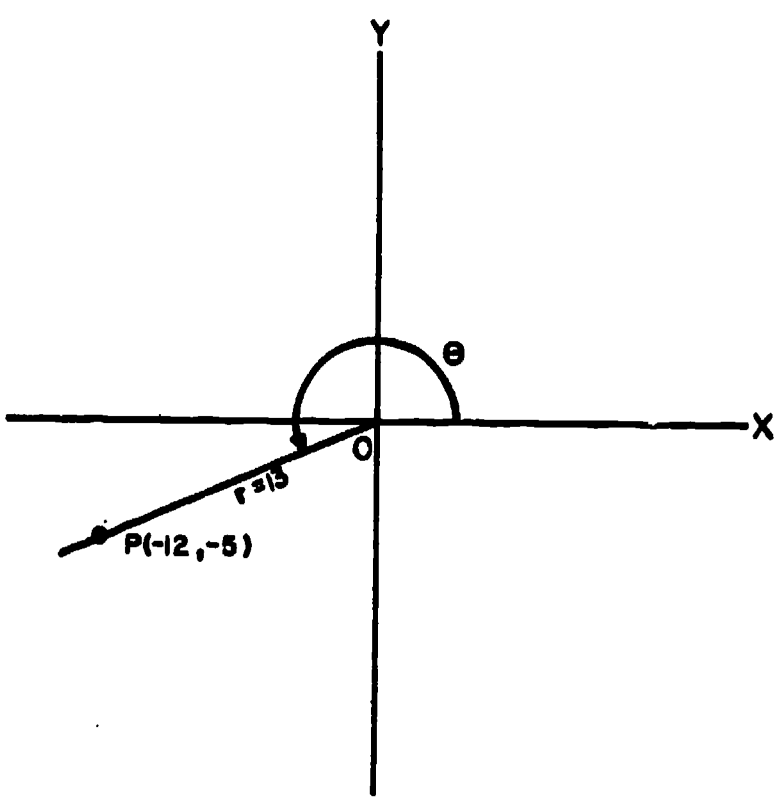


Figure 4-7.—Finding the trigonometric functions for a third-quadrant angle.

$$\sec\theta = \frac{\text{radius vector}}{\text{abscissa}} = \frac{13}{-12} = -\frac{13}{12}$$

$$\csc\theta = \frac{\text{radius vector}}{\text{ordinate}} = \frac{13}{-5} = -\frac{13}{5}$$

Reference to figure 4-6 shows that the functions have the correct algebraic signs for the third quadrant. The solution also meets the specifications of the problem for $\tan\theta = \frac{5}{12}$ and and sine $\theta < 0$ as the sine of θ is $\frac{-5}{13}$.

PRACTICE PROBLEMS: Without using tables, find the sine, cosine, and tangent of θ under the following conditions:

1. $\tan\theta = \frac{3}{4}$, $r = 5$, and θ is not in the first quadrant.
2. $\tan\theta = \frac{-21}{20}$, $r = 29$, and $\cos\theta > 0$.

ANSWERS:

1. $\sin\theta = \frac{-3}{5}$

   $\cos\theta = \frac{-4}{5}$

   $\tan\theta = \frac{3}{4}$

2. $\sin\theta = \frac{-21}{29}$

   $\cos\theta = \frac{20}{29}$

   $\tan\theta = \frac{-21}{20}$

Reference Angle

The reference angle for any angle ϕ in standard position is the smallest positive angle between the radius vector of ϕ and the X axis. In general the reference angle for ϕ is $n\pi \pm \phi$, where n is an integer. Expressed in another form $\phi' = n(180°) \pm \phi$, where ϕ' is the reference angle for ϕ and again n is some integer.

In trigonometric analysis the reference angle is used to form a reference triangle. This triangle is used to find the functions of an angle when less information is given than was available in the problems of the previous section.

For a geometrical explanation of the reference triangle construct an angle in standard position, such as the angle θ in figure 4-8. If $P(x,y)$ is any point on the radius vector a perpendicular from P to the point A on the X axis forms a right triangle with sides OA, OP, and AP. Then the X coordinate equals $\pm$ the distance OA, the Y coordinate equals $\pm$ the distance AP, and r equals the distance OP. The signs of x and y depend on the quadrant in which the radius vector falls. The distances OA and AP are not only the X and Y coordinates, they are also the lengths of the sides of the reference triangle AOP.

In the previous section it was possible to find the functions if the tangent, value of r, and some means of determining the quadrant were known. Using the reference triangle, the range of solvable problems of this type is extended.

EXAMPLE: Find the sin, cos, and tan of θ when $\csc \theta = -\frac{17}{15}$, $\cos \theta < 0$.

SOLUTION: The cosecant is negative where the sine is negative; i.e., quadrants 3 and 4. The cosecant and cosine are both negative only in quadrant 3. Construct an angle in standard position in the third quadrant as in figure 4-9. Drop a perpendicular to the negative X axis and label the angle formed by the X axis and the radius vector θ'.

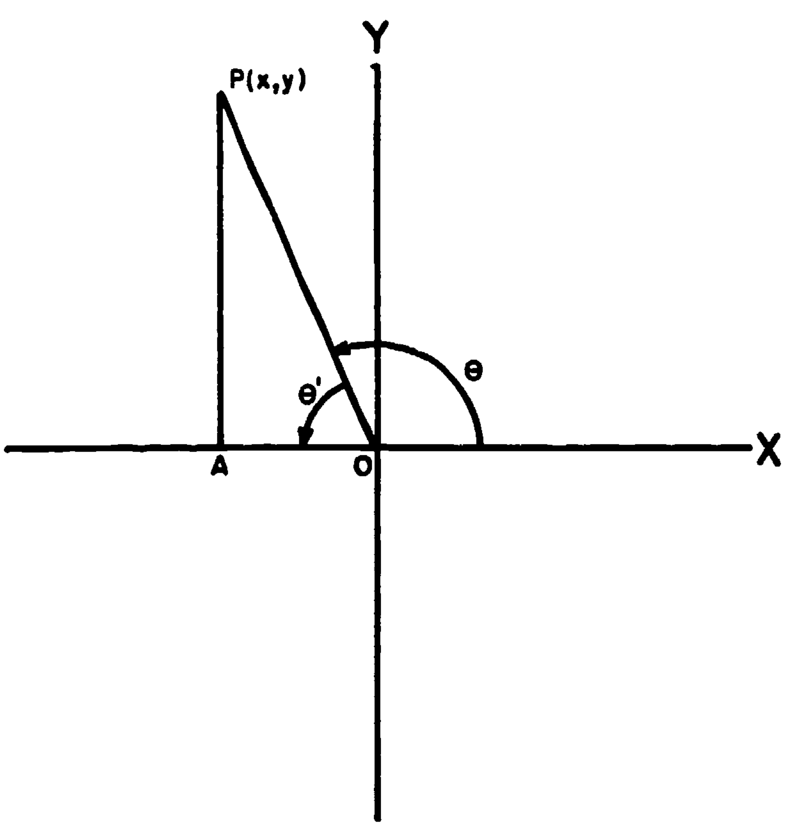


Figure 4-8.—Reference triangle.

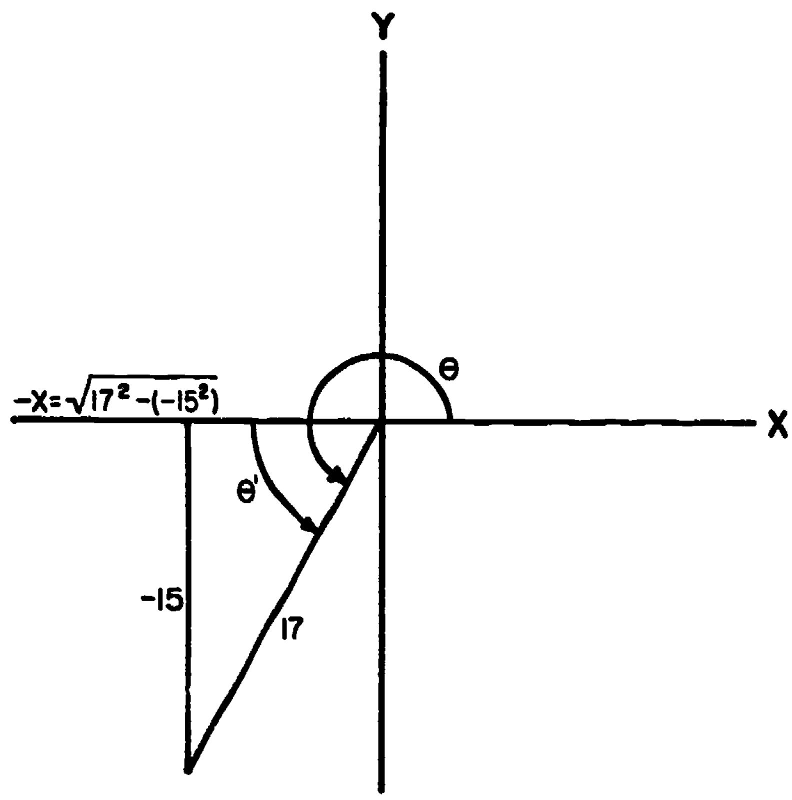


Figure 4-9.—Reference triangle in quadrant 3.

It can be seen in figure 4-9 that angles θ and θ' have the same terminal side. Therefore, the functions of θ and θ' are identical, and $\csc \theta' = \frac{17}{-15}$. Since $\csc \theta' = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$, the hypotenuse is labeled 17 and the opposite side is labeled -15.

NOTE: The fraction $-\frac{17}{15}$ indicates that either the numerator or denominator is negative, but not both. In this case, we know that the denominator is the negative member from the following:

1. By convention, r is always positive.
2. The Y coordinate is negative in quadrant 3.

From the Pythagorean theorem the value of x is found to be 8 and the negative x value is -8. From this information ($y = -15$, $x = -8$, $r = 17$) the functions can be written as follows:

$$\sin \theta = \sin \theta' = \frac{-15}{17}$$

$$\cos \theta = \cos \theta' = \frac{8}{17}$$

$$\tan \theta = \tan \theta' = \frac{-15}{-8} = \frac{15}{8}$$

EXAMPLE: Find the value of the six trigonometric functions of θ if $\sec \theta' = -\frac{25}{24}$ and $0 < \theta < \pi$.

SOLUTION: The secant is negative in quadrants 2 and 3; since θ must lie between 0 and π radians, or 0° and 180°, the angle must lie in quadrant 2. Construct and label the reference triangle as shown in figure 4-10 (A). Use the Pythagorean theorem to determine the y value,

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{25^2 - (-24)^2}$$

$$y = \sqrt{625 - 576}$$

$$y = \sqrt{49}$$

$$y = 7$$

With the reference triangle labeled as in figure 4-10 (B), the functions can be read directly from the figure.

$$\sin \theta = \frac{7}{25}$$

$$\cos \theta = -\frac{24}{25}$$

$$\tan \theta = -\frac{7}{24}$$

$$\cot \theta = -\frac{24}{7}$$

$$\sec \theta = -\frac{25}{24}$$

$$\csc \theta = \frac{25}{7}$$

EXAMPLE: Find the sine, cosine, and tangent of θ when $\cot \theta = 1$.

SOLUTION: This example requires analysis of two points that were not encountered in previous examples. First, with only the cotangent given, how is it possible to determine which of two quadrants (where the cotangent is positive) will contain the angle? The cotangent is positive in quadrants one and three; from the given

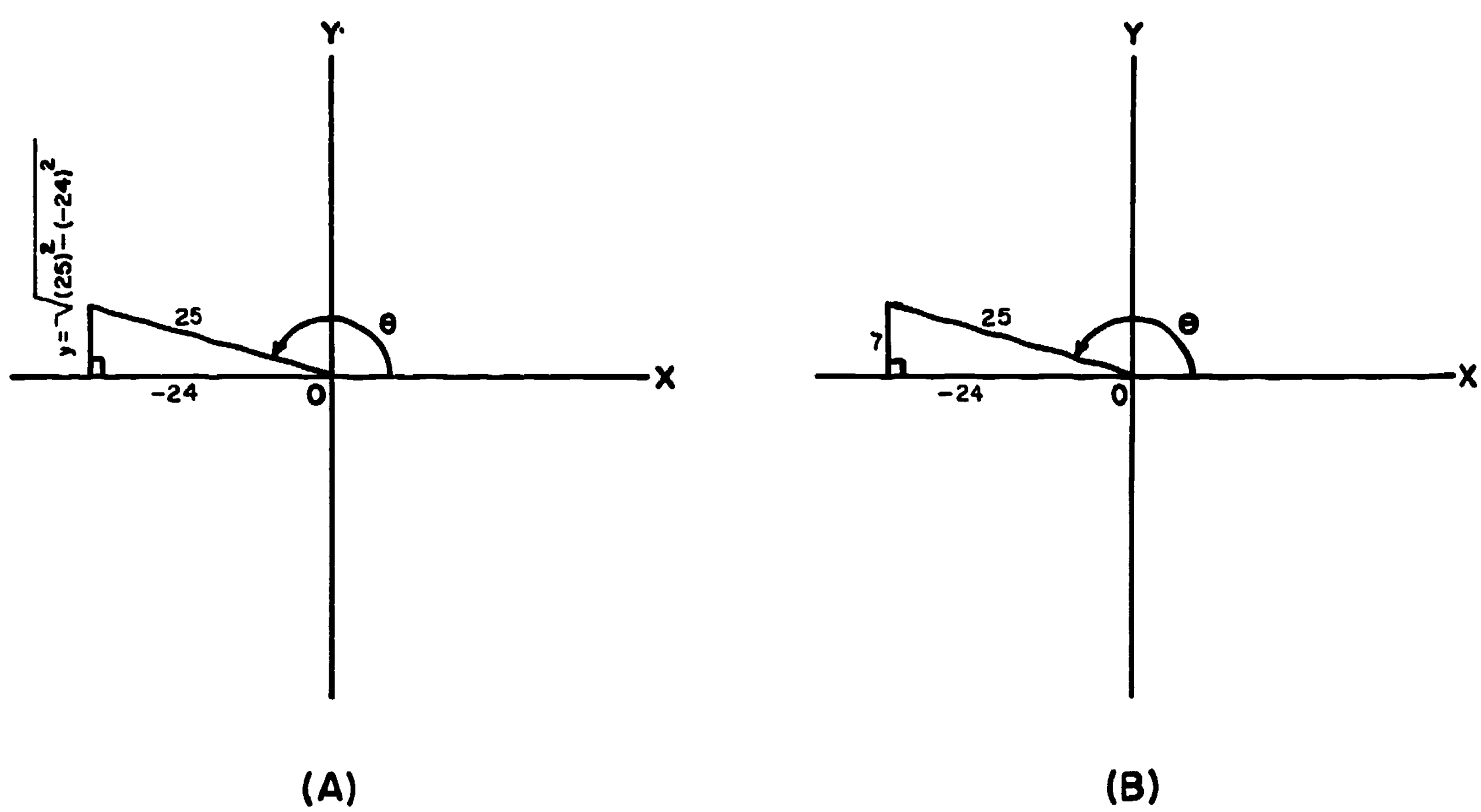


Figure 4-10.—Reference triangles in quadrant 2.

information it is not possible to determine in which quadrant the angle falls. In this example we are dealing with two angles. The relationships of the functions of these two angles should be observed when they are determined.

The second new point to consider is the value of the function. There are an infinite number of side lengths that give a value of 1 for the cotangent. Any right triangle which has equal adjacent and opposite sides will result in this situation. This does not really present a problem, however, because the angle with a cotangent of 1 has the same value whether the sides are 1 and 1, 14 and 14, or any other equal side lengths. For simplicity, we consider the function to be the ratio of $\frac{1}{1}$ in this problem and construct reference triangles as in figure 4-11.

From the figure the functions of θ_1, and θ_2 can be written directly.

$$\sin \theta_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos \theta_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

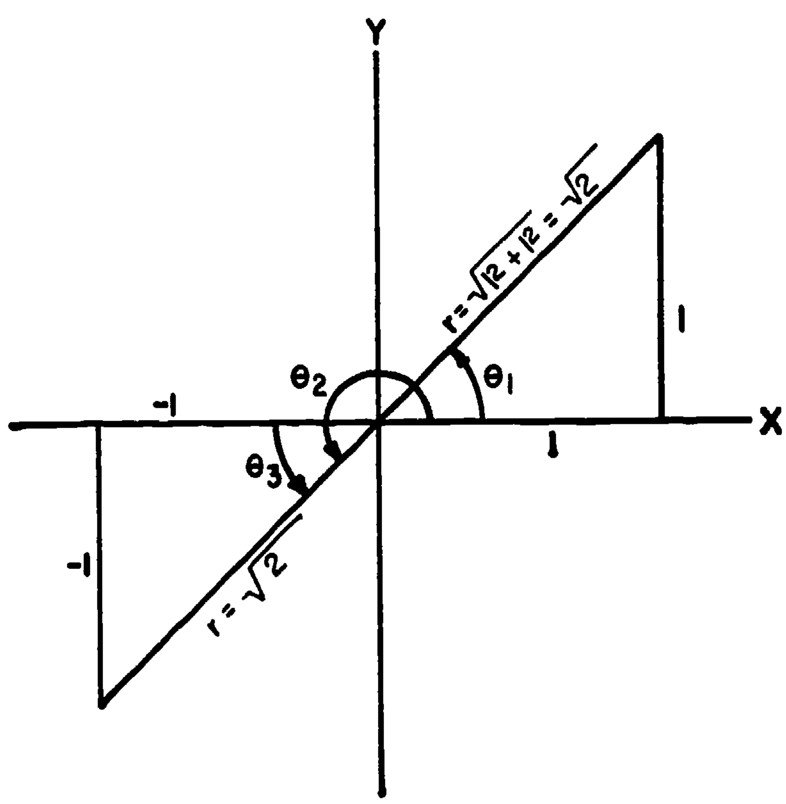


Figure 4-11.—45° reference angles.

$$\tan \theta_1 = \frac{1}{1} = 1$$

and

$$\sin \theta_2 = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\cos \theta_2 = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\tan \theta_2 = \frac{-1}{-1} = 1$$

It is shown here that the functions for these angles are identical, or differ only by algebraic sign. This is as expected, for the angle θ_1 and the reference angle θ_3 are equal. The solution to this problem would normally consider the general angle θ and be given in the following form:

$$\sin \theta = \pm \frac{\sqrt{2}}{2}$$

$$\cos \theta = \pm \frac{\sqrt{2}}{2}$$

$$\tan \theta = 1$$

Use of Tables

The tables of trigonometric functions normally contain only values for the functions of angles between 0° and 90°. Use of reference angles provide a means of using the tables to find the values of the functions for angles greater than 90°. The principle involved here is the following: Consider θ as any angle and θ' as its reference angle. Any trigonometric function of θ is equal to the same function of θ', with the sign of the quadrant in which θ terminates attached.

This is shown geometrically in the following manner. Construct the angle θ and the reference triangle with inclosed angle θ' as shown in figure 4-12. The triangle AOB is the reference triangle and it can be seen that the point B is also a point on the radius vector of θ. We derive the functions of the two angles to show that they are equal except perhaps for the algebraic sign.

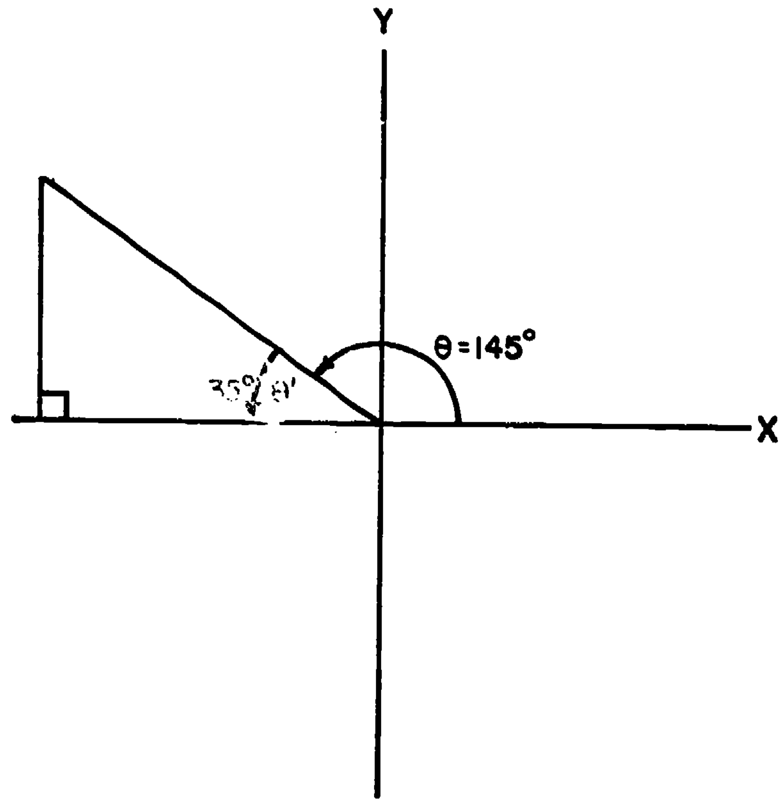


Figure 4-12.–Angles θ and θ'.

First, consider the triangle AOB. The lengths of the sides of this triangle are OA and AB, and the length of the hypotenuse is OB. If we disregard any signs and treat the angle θ' as any positive acute angle the sine function is determined as follows.

$$\sin \theta' = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{OB}$$

Considering angle θ, we determine the coordinates of point B to be x = OA and y = OB (if we disregard signs). Considering the signs, it should be clear that the coordinates of B are given by x = ±OA and y = ±OB, where the signs are determined by the quadrant in which θ terminates. This allows the sine of θ, for any size of θ, to be determined as

$$\sin \theta = \frac{y}{r} = \frac{\pm AB}{OB} = \pm \frac{AB}{OB}$$

By definition, the radius vector OB is always positive.

Hence, we have determined the sine of θ to be $\pm \frac{OA}{OB}$ and the sine of θ' to be $\frac{AB}{OB}$. These functions values differ only by algebraic sign as stated earlier.

EXAMPLE: Find the six trigonometric functions of 145°.

SOLUTION: The angle 145° in standard position is shown in figure 4-13. The reference angle is the smallest positive angle between the terminal side (or ray) and the X axis or, in this case, an angle of 35°. Since 145° lies in the second quadrant the sine and cosecant are positive and all other functions are negative.

Referring to the tables in appendixes III and IV, and utilizing the principle explained in this chapter we have

$$\sin 145° = \sin 35° = 0.57358$$
$$\cos 145° = -\cos 35° = -0.81915$$
$$\tan 145° = -\tan 35° = -0.70021$$
$$\cot 145° = -\cot 35° = -1.42815$$

Recall that earlier in this chapter it was pointed out that certain functions were reciprocals of other functions. Using this we can determine

$$\csc 145° = \csc 35° = \frac{1}{\sin 35°} = \frac{1}{0.57358} = 1.7434$$

$$\sec 145° = -\sec 35° = \frac{1}{-\cos 35°} = \frac{1}{-0.81915} =$$
$$-1.2208$$

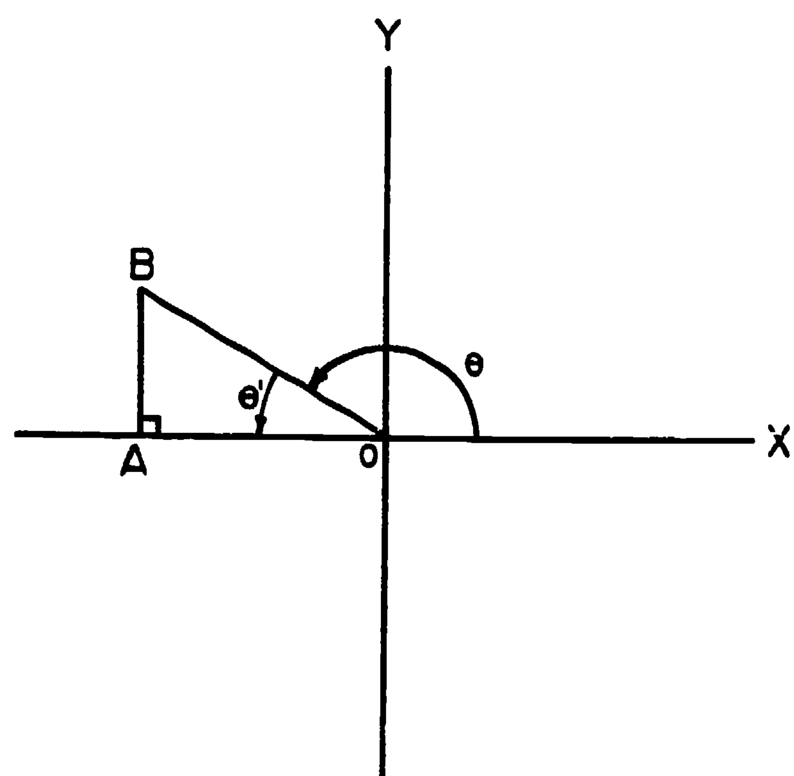


Figure 4-13.—145° angle and reference angle.

EXAMPLE: Find the sine and cosine of an angle of 690°.

SOLUTION: An angle of 690° is coterminal with an angle of 330°

$$690° - 360° = 330°$$

and both have a reference angle of 30° as shown in figure 4-14.

The angle terminates in the third quadrant so the sine is negative and the cosine positive and

$$\sin 690° = -\sin 30° = -0.5000$$

$$\cos 690° = \cos 30° = 0.86603$$

PRACTICE PROBLEMS: Without using tables, find the six trigonometric functions of θ under the conditions given in problems 1, 2, and 3.

1. $\cos\theta = -\frac{3}{5}$, θ not in quadrant 2.

2. $\sin\theta = -\frac{5}{13}$

3. $\tan\theta = \frac{8}{15}$, $\pi < \theta < 2\pi$

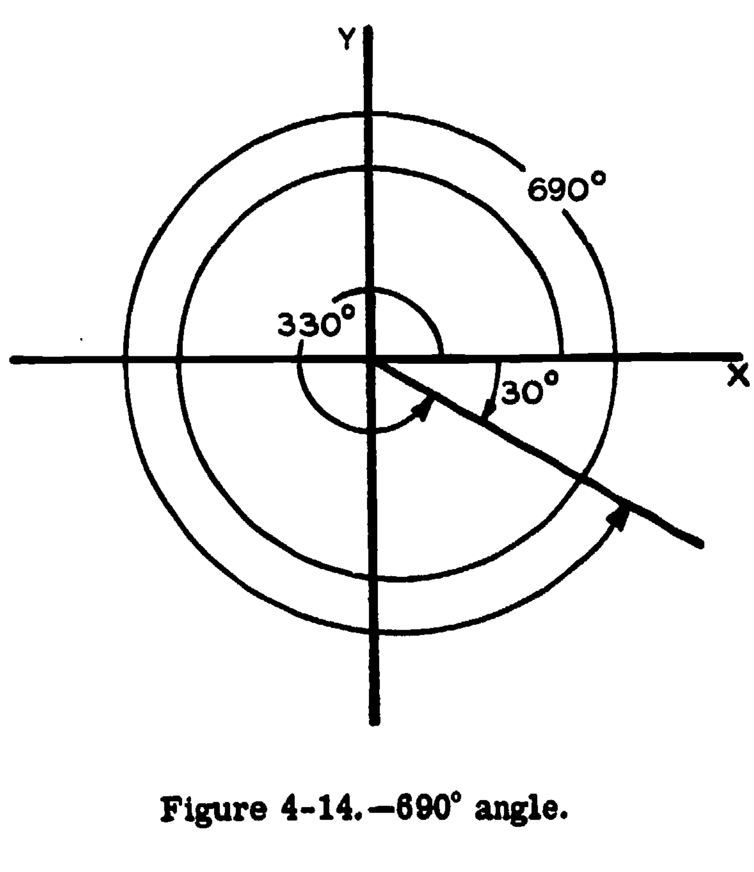


Figure 4-14.—690° angle.

4. In which quadrant must θ fall when:

   a. $\sin\theta > 0$, $\cos\theta < 0$

   b. $\cos\theta < 0$, $\tan\theta < 0$

   c. $\sec\theta > 0$, $\csc\theta < 0$

5. Use tables to find the sine, cosine, and tangent of 281°.

ANSWERS:

1. $\sin\theta = -\frac{4}{5}$

   $\cos\theta = -\frac{3}{5}$

   $\tan\theta = \frac{3}{4}$

   $\cot\theta = \frac{4}{3}$

   $\sec\theta = -\frac{5}{3}$

   $\csc\theta = -\frac{5}{4}$

2. $\sin\theta = -\frac{5}{13}$

   $\cos\theta = \pm\frac{12}{13}$

   $\tan\theta = \pm\frac{5}{12}$

   $\cot\theta = \pm\frac{12}{5}$

   $\sec\theta = \pm\frac{13}{12}$

   $\csc\theta = -\frac{13}{5}$

3. $\sin\theta = -\frac{8}{17}$

   $\cos\theta = -\frac{15}{17}$

$\tan \theta = \frac{8}{15}$

$\cot \theta = \frac{15}{8}$

$\sec \theta = -\frac{17}{15}$

$\csc \theta = -\frac{17}{8}$

4. a. 2
   b. 2
   c. 4

5. $\sin 281° = -0.98163$
   $\cos 281° = 0.19081$
   $\tan 281° = 5.14455$

SPECIAL ANGLES

There are two groups of angles considered in this section. The first group considered contains angles which occur so frequently in problems that their functions are normally considered separately.

The second group considered contains those angles whose radius vectors fall on one of the coordinate axes. These angles cannot be considered as falling in one of the quadrants, and the group is treated as a special case.

FREQUENTLY USED ANGLES

As stated previously, the approximate values of the trigonometric functions for any angle can be read directly from tables or can be determined from the tables by the use of principles stated in this course. However, there are certain frequently used simple angles for which the exact function values are often used because these exact values can easily be determined geometrically. In the following paragraphs the geometrical determination of these functions is shown.

30° - 60° Angles

To determine the functions of these angles geometrically, first construct an equilateral triangle with the side lengths of 2 units. The functions to be determined are not dependent on the lengths of these sides being two units; this size was selected for convenience. The ratios for given values of angles will be constant for all side lengths. Triangle OYA in figure 4-15 is a triangle constructed as described above.

If a perpendicular is dropped from angle Y to the base at point X, two right triangles are formed. Consider the right triangle YOX, formed by the perpendicular, which also bisects angle Y forming a 30° angle. This triangle contains a 60° and a 30° angle. It is seen in figure 4-15 that the side adjacent to the 60° angle is one-half the length of the hypotenuse, or in this case 1. Using the Pythagorean theorem with right triangle YOX, the length of the side opposite the 60° angle (the length of the perpendicular) is found to be $\sqrt{3}$.

Figure 4-16(A) shows the right triangle YOX of figure 4-15 transferred to a rectangular coordinate system, with the 60° angle positioned at the origin of the coordinate system.

The triangle now is set as a reference triangle, and inspection of figure 4-16 shows that the adjacent side is one-half the length of the hypotenuse. This relationship holds for any 60° triangle, regardless of the lengths of the sides. From the figure it is seen that the cosine of 60° is $\frac{1}{2}$. The remaining functions can be read from the figure.

Consider now the right triangle XYA of figure 4-15. If this triangle is placed on a coordinate system with the 30° angle positioned at the origin, the situation is as shown in figure 4-16 (B). In this figure it is seen that the side opposite

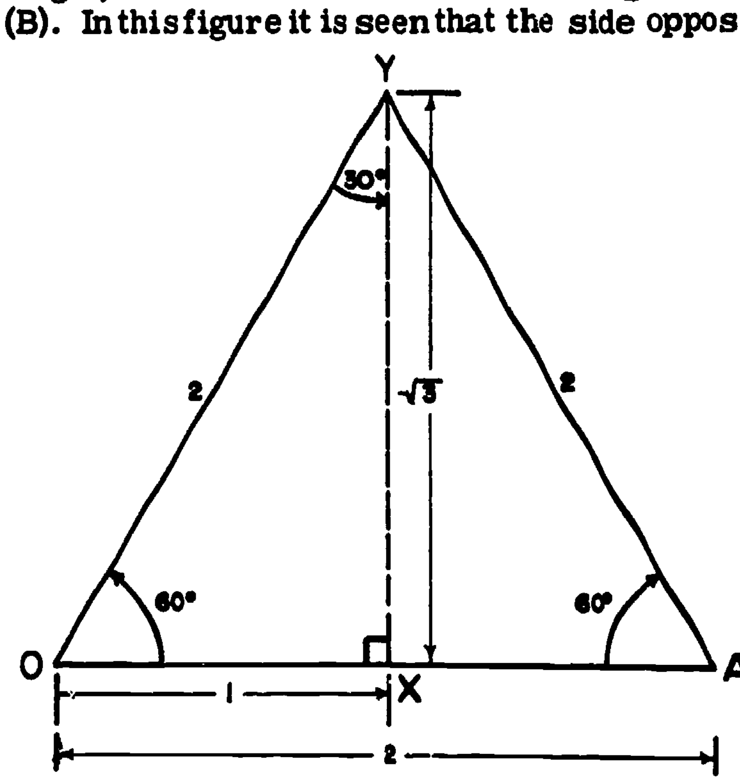
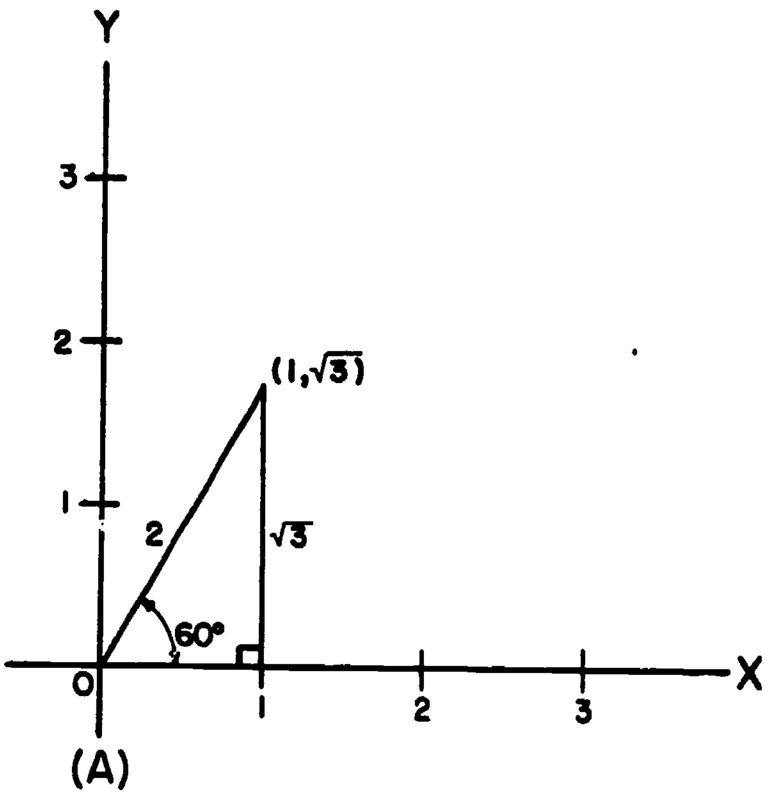


Figure 4-15.—Equilateral triangle.

the 30° angle is one-half the length of the hypotenuse. Therefore, the sine of 30° is $\frac{1}{2}$.



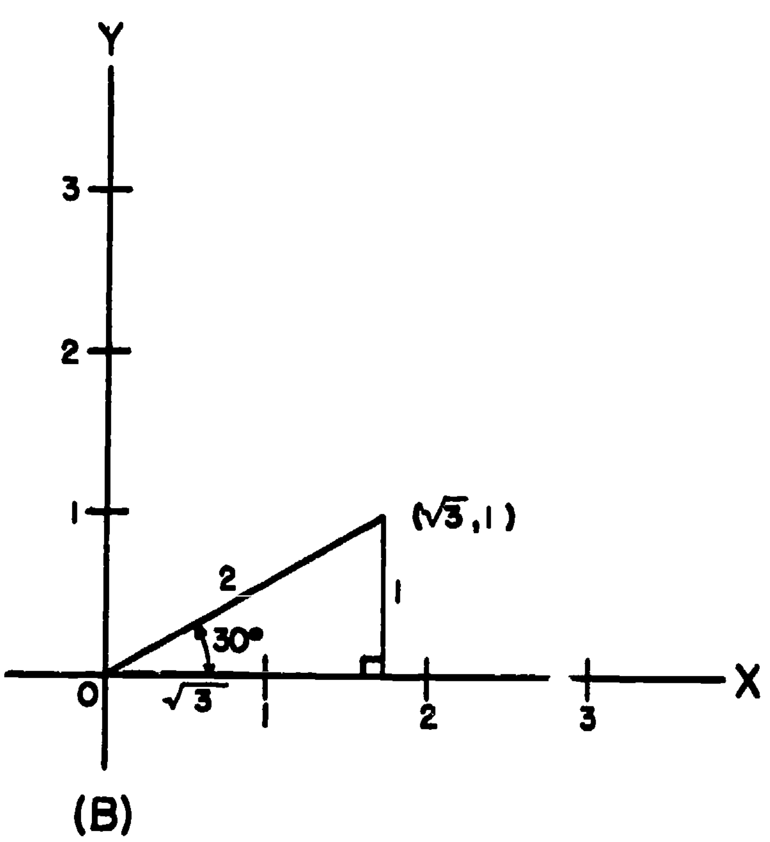


Figure 4-16.—Triangle on coordinate system.

In dealing with 30°-60°-90° triangles it is not necessary to memorize the functions of the 30° and 60° angles. These angles will occur in pairs in right triangles and if one remembers that the side opposite the 30° angle is one-half the length of the hypotenuse it is a simple matter to construct a reference triangle with side lengths of 1, 2, and $\sqrt{3}$ to derive the functions.

45° Angles

For a geometrical derivation of the functions of 45° angles construct a square, one unit length on a side, as shown in figure 4-17. If a diagonal is drawn from point 0 to point Y, two right triangles are formed. Inspection of the figure shows that triangle XOY contains two 45° angles with side lengths of 1 unit. These conditions are also met in the reference triangle in figure 4-18. Here it is seen that the two legs of a 45° triangle must be equal. This relationship is true of all 45° triangles and is not altered by the lengths of the legs.

Reference to figure 4-18 allows the functions for 45° to be written directly. These functions, like those for 30° and 60°, need not be memorized. It is only necessary to remember that the legs of a 45° triangle are equal; from this, it is a simple matter to construct a reference triangle with sides of unit length and all of the functions can be read from the reference figure.

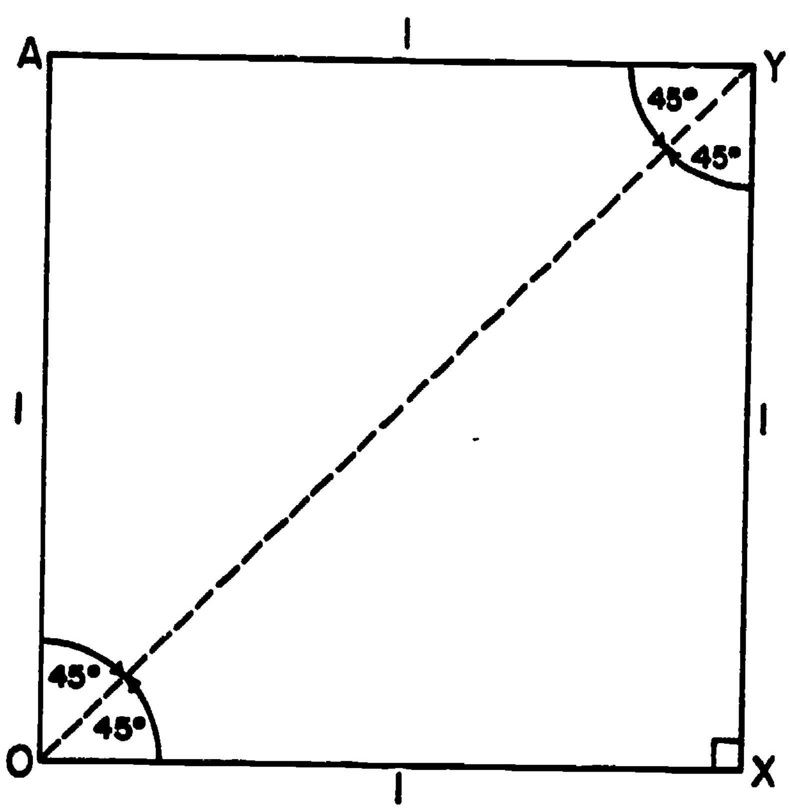


Figure 4-17.—Determining 45° angles.

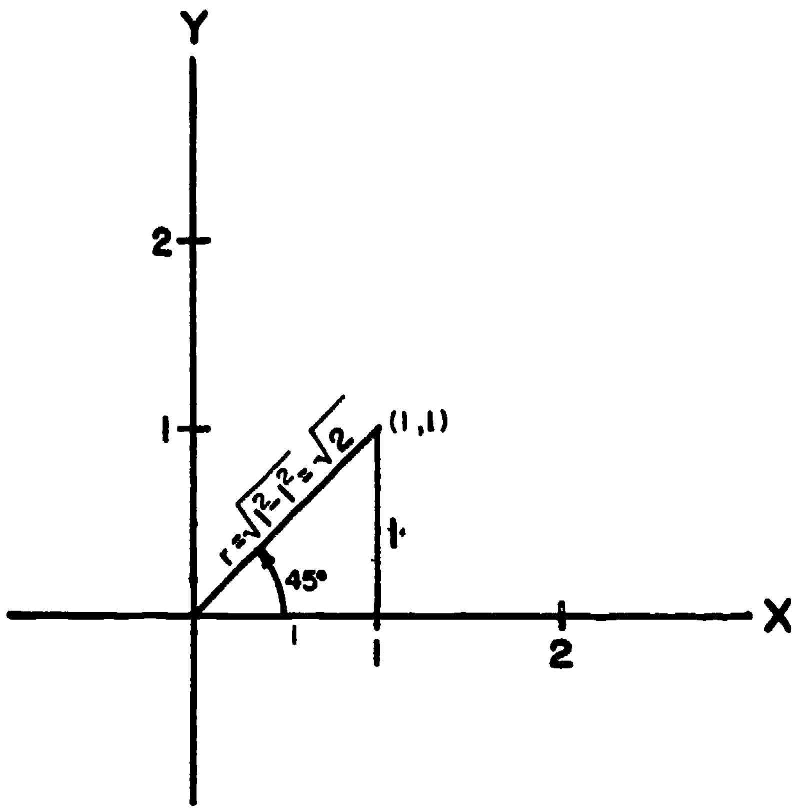


Figure 4-18.—45° reference triangle.

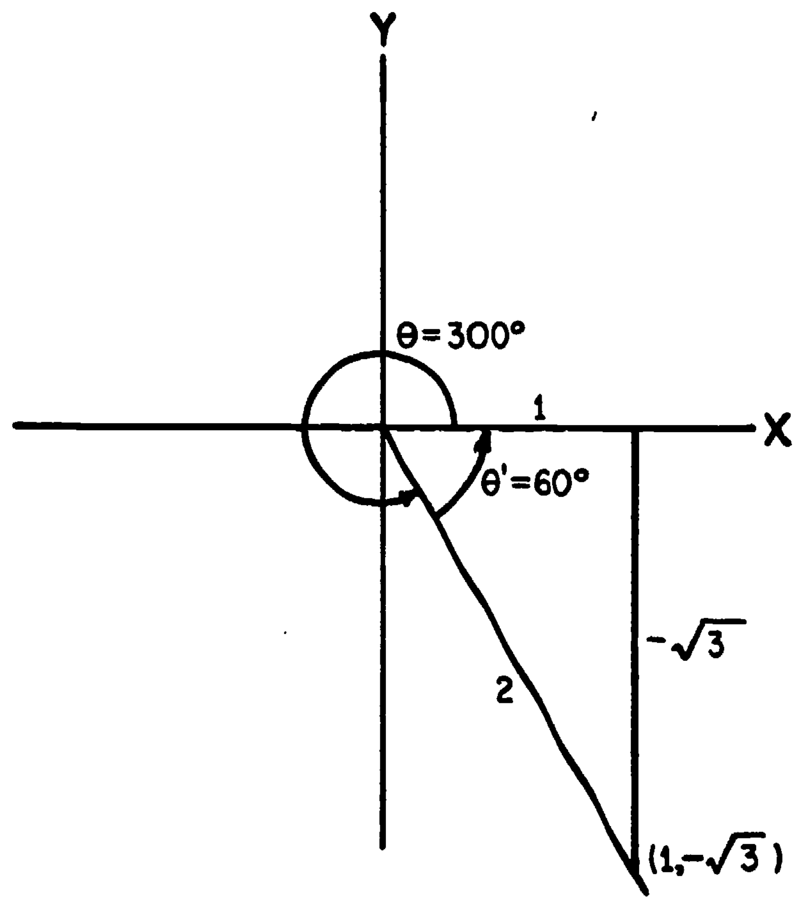


Figure 4-19.—300° angle in standard position.

The trigonometric functions for 30°, 60°, and 45° are summarized in table 4-1 for ready reference. The function values shown are also applicable for any angle which has one of these for a reference angle, upon proper consideration of the appropriate algebraic signs of x and y in the various quadrants.

EXAMPLE: Find the six trigonometric functions of 300°.

SOLUTION: With the angle drawn in standard position as shown in figure 4-19, choose the point $(1, -\sqrt{3})$ on the radius vector with $r = 2$. The functions of 300° are then

$$\sin 300° = \frac{y}{r} = -\frac{\sqrt{3}}{2}$$

$$\cos 300° = \frac{y}{r} = \frac{1}{2}$$

$$\tan 300° = \frac{y}{x} = -\sqrt{3}$$

$$\cot 300° = \frac{x}{y} = -\frac{\sqrt{3}}{3}$$

Table 4-1.—Trigonometric functions of special angles.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$

$$\sec 300^\circ = \frac{r}{x} = \quad 2$$

$$\csc 300^\circ = \frac{r}{y} = -\frac{2\sqrt{3}}{3}$$

QUADRANTAL ANGLES

In previous sections of this chapter those angles which are exact multiples of 90° have been ignored, in order to simplify the explanations. The functions of angles which are exact multiples of 90° (0°, 90°, 180°, 270°, -90°, etc.) are considered a special case and the angles are called quadrantal angles. By definition, an angle which has its radius vector falling along one of the coordinate axes, when the angle is in standard position, is a quadrantal angle.

The trigonometric functions of the quadrantal angles are defined in the same manner as for other angles, except for the restriction that a function is undefined when the denominator of the ratio is zero.

To derive the functions of quadrantal angles we choose points on the terminal sides where r = 1, as shown in figure 4-20. In each angle in this figure it is seen that r = 1. Then either x or y is zero and the other one is ±1.

Consider the case where $\theta = 90^\circ$ in figure 4-20 (A). To derive the functions use the point P (0, 1) and r = 1.

$$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0$$

$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} \text{ is undefined}$$

$$\cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

$$\sec 90^\circ = \frac{r}{x} = \frac{1}{0} \text{ is undefined}$$

$$\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1$$

The functions for the other quadrantal angles can be determined from the other parts of figure 4-20. The functions are summarized in table 4-2 for ready reference. The values of the functions of the special angles, quadrantal and 30°-60°-45°, are used frequently and for that reason are important. However, additional importance is attached to the quadrantal angles for they serve as key values in the graphs of trigonometric functions.

PRACTICE PROBLEMS: Without using tables, determine the trigonometric functions of θ in problems 1 through 4.

1. $\theta = 210^\circ$

2. $\theta = 360^\circ$

3. $\theta = 585^\circ$

4. $\theta = -180^\circ$

5. Without reference to tables or drawings determine the value of θ described in the following, where x and y are points on the radius vector r.

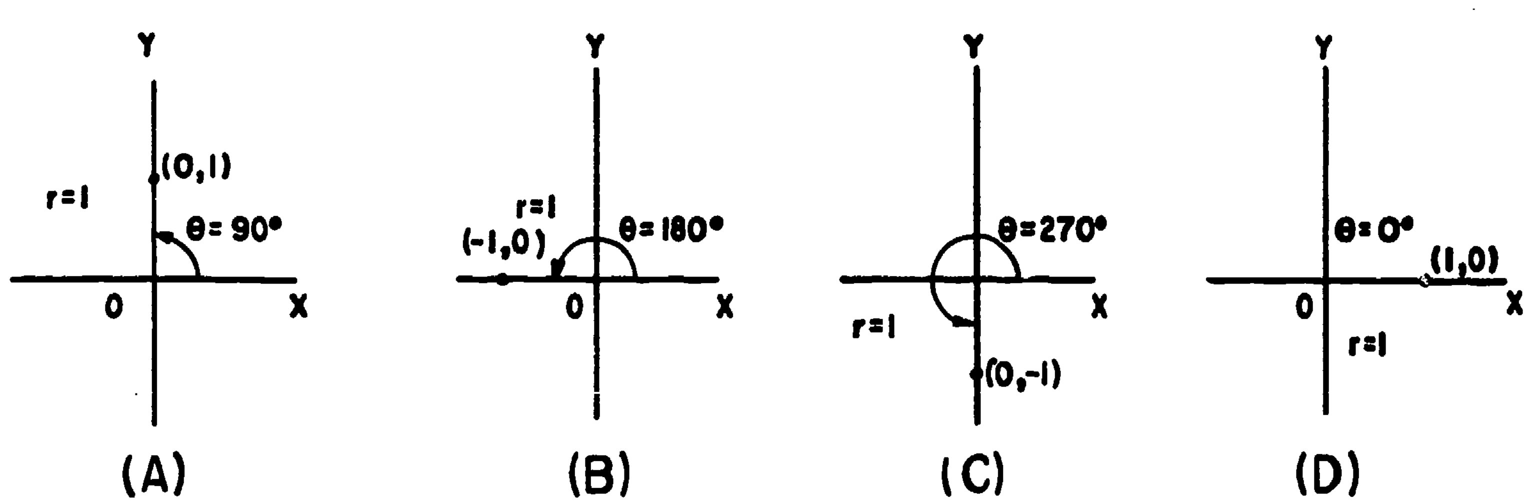


Figure 4-20.—Functions of quadrantal angles.

Table 4-2.—Functions of quadrantal angles.

θ Deg.	θ Rad.	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	-1	0	undefined	-1	undefined
270°	$\frac{3\pi}{2}$	-1	0	undefined	0	undefined	-1

a. $x = y = 3$

b. $x = \frac{r}{2}$

c. $y = -r, x = 0$

Without using tables, determine the numerical value of the functions of indicated angles and verify the statements in problems 6 through 8.

6. $\sin^2 150° + \cos^2 150° = 1$

7. $\sin 340° = 2 \sin 170° \cos 170°$

8. $\cos 270° = \cos^2 135° - \sin^2 135°$

NOTE: Terms such as $\sin^2 \theta$ are used to indicate $(\sin \theta)^2$.

1. $\sin 210° = -\frac{1}{2}$

   $\cos 210° = -\frac{\sqrt{3}}{2}$

   $\tan 210° = \frac{\sqrt{3}}{3}$

   $\cot 210° = \sqrt{3}$

   $\sec 210° = -\frac{2}{\sqrt{3}}$

   $\csc 210° = -2$

2. $\sin 360° = 0$

   $\cos 360° = 1$

   $\tan 360° = 0$

   $\cot 360°$ = undefined

   $\sec 360° = 1$

   $\csc 360°$ = undefined

3. $\sin 585° = -\frac{\sqrt{2}}{2}$

   $\cos 585° = -\frac{\sqrt{2}}{2}$

   $\tan 585° = 1$

   $\cot 585° = 1$

   $\sec 585° = -\sqrt{2}$

   $\csc 585° = -\sqrt{2}$

4. $\sin (-180°) = 0$

   $\cos (-180°) = -1$

   $\tan (-180°) = 0$

   $\cot (-180°)$ is undefined

   $\sec (-180°) = -1$

   $\csc (-180°)$ is undefined

5. a. $\theta = 45°$

   b. $\theta = 60°$

   c. $\theta = 270°$

FUNCTIONS OF ANGLES GREATER THAN 90°

In previous sections of this chapter a means of dealing with angles greater than 90°, by use of reference triangles, was developed. When using reference triangles, it was found that any function of an angle is, except possibly for the algebraic sign, numerically equal to the same function of the reference angle.

In addition to the reference triangle, there are formulas for determining the signs of functions of any angle. These are normally called reduction formulas. This section shows the geometrical development of some of the most commonly used reduction formulas. In general, reduction formulas provide a means of reducing the functions of any angle to an equivalent expression for the function in terms of a positive acute angle. In the discussion in the following paragraphs, this acute angle is designated θ. The reduction formulas can be used in the solution of some trigonometric identities and in other applications which require analysis of trigonometric functions.

SINE OF AN ANGLE

The numerical value of the sine of an angle is equal to the projection of a unit radius on the Y axis. Figure 4-21 shows a circle with a unit radius. According to the definition of the sine of an angle

$$\sin \theta = \frac{y}{r}$$

But r in figure 4-21 is equal to 1 so that

$$\sin \theta = y$$

Note that y is equal to y', which is the projection of the unit radius on the Y axis. Therefore,

$$\sin \theta = y'$$

does not contradict the definition of the sine of an angle less than 90°.

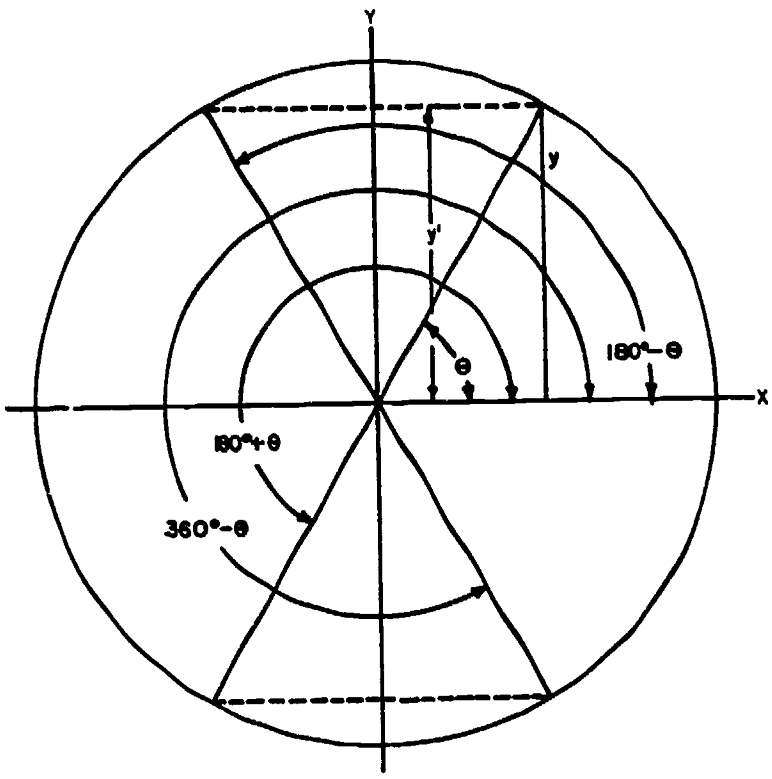


Figure 4-21.—Circle diagram, sine of an angle.

Now, consider an angle in the second quadrant. In figure 4-21 we have constructed the angle (180° - θ). The sine of this angle is evidently the same as the sine of the angle in the first quadrant or

$$\sin (180° - \theta) = \sin \theta$$

In the third quadrant the projection of unit radius on the Y axis becomes negative. The sine of the angle (180° + θ) is the same in magnitude but of opposite sign to that of an angle in the first quadrant, so that

$$\sin (180° + \theta) = - \sin \theta$$

These formulas give the value of the sine of any angle between 90° and 270° in terms of the sine of an angle less than 90°. The sine of an angle in the fourth quadrant is found from the formula

$$\sin (360° - \theta) = \sin (-\theta) = - \sin \theta$$

COSINE OF AN ANGLE

The numerical value of the cosine of an angle is equal to the projection of a unit radius on the X axis.

Using the same method of analysis that was used for the sine, we develop the formulas which are used to evaluate the cosine of any angle.

In figure 4-22, notice that the projections of the unit radius on the X axis, or in other words, the cosine of the angles θ and $180° - \theta$ are the same length but of different sign.

$$\cos (180° - \theta) = -\cos \theta$$

Also, the cosine of an angle in the third quadrant is found from the relation

$$\cos (180° + \theta) = -\cos \theta$$

And lastly, the cosine of an angle in the fourth quadrant is found from the relation

$$\cos (360° - \theta) = \cos (-\theta) = \cos \theta$$

TANGENT OF AN ANGLE

The value of the tangent of an angle is equal to the length of that part of the tangent to the unit circle at 0° between y = 0 and the intersection of the continuation of the unit radius with the tangent line.

Let us clarify this definition by studying figure 4-23. The line MN is the tangent to the unit circle at 0°. The continuation of the unit radius CD cuts the tangent at M.

According to this definition, MA is the tangent of the angle θ. This new definition does not contradict the previous rule where we found tan θ from triangle CMA as follows:

$$\tan \theta = \frac{MA}{CA}$$

Therefore, because CA is equal to 1

$$\tan \theta = MA$$

For angles in the second quadrant we write

$$\tan (180° - \theta) = -\tan \theta$$

For angles in the third quadrant we have

$$\tan (180° + \theta) = \tan \theta$$

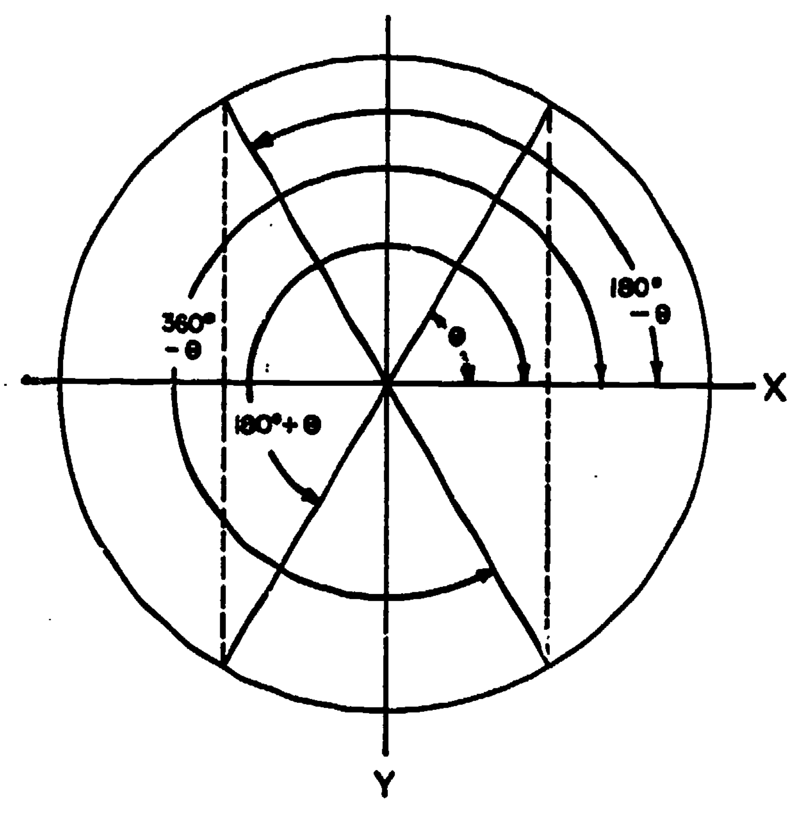


Figure 4-22.—Circle diagram, cosine of an angle.

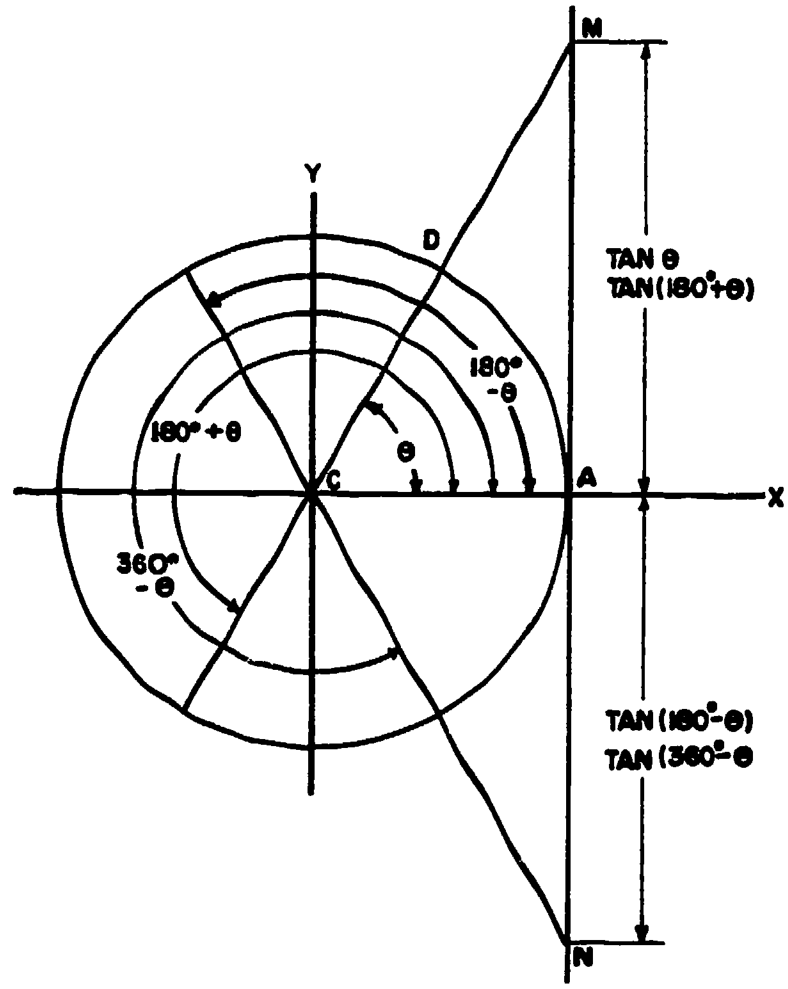


Figure 4-23.—Circle diagram, tangent of an angle.

For angles in the fourth quadrant the relation is

$$\tan (360° - \theta) = \tan (-\theta) = -\tan \theta$$

PRACTICE PROBLEMS: Express each of the following as trigonometric functions of θ.

1. $\sin (180° - \theta)$
2. $\cos (360° - \theta)$
3. $\cos (720° - \theta)$
4. $\tan (180° + \theta)$
5. $\tan (-\theta)$

ANSWERS:

1. $\sin \theta$
2. $\cos \theta$
3. $\cos \theta$
4. $\tan \theta$
5. $-\tan \theta$

This section has developed reduction formulas for the sine, cosine, and tangent of angles. The reduction formulas apply as well to the other three functions. The formulas developed in this section and the corresponding formulas for the remaining functions are summarized in the following paragraphs.

Functions of $180° - \theta$:

$$\sin (180° - \theta) = \sin \theta$$

$$\cos (180° - \theta) = -\cos \theta$$

$$\tan (180° - \theta) = -\tan \theta$$

$$\cot (180° - \theta) = -\cot \theta$$

$$\sec (180° - \theta) = -\sec \theta$$

$$\csc (180° - \theta) = \csc \theta$$

Functions of $180° + \theta$:

$$\sin (180° + \theta) = -\sin \theta$$

$$\cos (180° + \theta) = -\cos \theta$$

$$\tan (180° + \theta) = \tan \theta$$

$$\cot (180° + \theta) = \cot \theta$$

$$\sec (180° + \theta) = -\sec \theta$$

$$\csc (180° + \theta) = -\csc \theta$$

Functions of $360° - \theta$:

$$\sin (360° - \theta) = -\sin \theta$$

$$\cos (360° - \theta) = \cos \theta$$

$$\tan (360° - \theta) = -\tan \theta$$

$$\cot (360° - \theta) = -\cot \theta$$

$$\sec (360° - \theta) = \sec \theta$$

$$\csc (360° - \theta) = -\csc \theta$$

Functions of $-\theta$:

$$\sin (-\theta) = -\sin \theta$$

$$\cos (-\theta) = \cos \theta$$

$$\tan (-\theta) = -\tan \theta$$

$$\cot (-\theta) = -\cot \theta$$

$$\sec (-\theta) = \sec \theta$$

$$\csc (-\theta) = -\csc \theta$$

EXAMPLE: Use reduction formulas and tables to find the sine of 220°.

SOLUTION: 220° is in the third quadrant and can be considered a function of $180° + \theta$.

$$\sin (180° + \theta) = -\sin \theta$$

$$\sin 220° = \sin (180° + 40°)$$

$$\sin 220° = -\sin 40°$$

$$\sin 220° = -0.64279$$

EXAMPLE: Find the tangent of -350°.

SOLUTION:

First,

$$\tan (-\theta) = -\tan \theta$$

$$\tan -350° = -\tan 350°$$

then,

$$\tan (360° - \theta) = -\tan \theta$$

$$\tan 350° = \tan (360° - 350°)$$

$$\tan 350° = -\tan 10°$$

combining the two,

$$\tan -350° = -\tan 350°$$

$$\tan 350° = -\tan 10°$$

therefore,

$$\tan -350° = -\tan 350° = -(-\tan 10°)$$

$$\tan -350° = \tan 10°$$

$$\tan -350° = 0.17633$$

COMPOSITE CIRCLE DIAGRAM

All the trigonometric functions can be shown as lengths of lines in a circle diagram as illustrated in figure 4-24. The circle has a unit radius. A number of things may be learned from this diagram. For example, the three sides of the right triangle OAB are sin θ, cos θ, and 1, so that from the Pythagorean theorem

$$\sin^2\theta + \cos^2\theta = 1$$

In the same way, in triangle ORM

$$1 + \tan^2\theta = \sec^2\theta$$

and in triangle ONP

$$1 + \cot^2\theta = \csc^2\theta.$$

These three equations are true for any angle.

These equations are trigonometric identities and will be used in chapter 8 of this course.

PRACTICE PROBLEMS: Use reduction formulas and tables to find the values of the sine, cosine, and tangent of θ in the following problems.

1. $\theta = 137°$
2. $\theta = 214°$

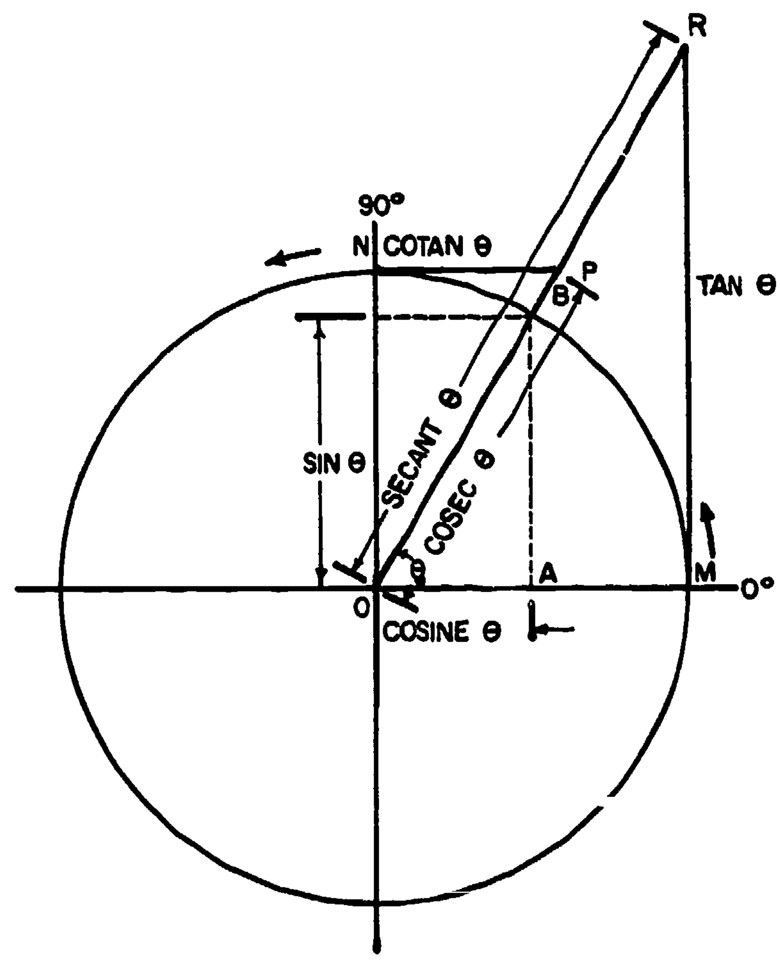


Figure 4-24.—Composite circle diagram.

3. $\theta = 325°$
4. $\theta = -290°$

ANSWERS:

1. $\sin 137° = \sin 43° = 0.68200$

   $\cos 137° = -\cos 43° = -0.73135$

   $\tan 137° = -\tan 43° = -0.93252$

2. $\sin 214° = -\sin 34° = -0.55919$

   $\cos 214° = -\cos 34° = -0.82904$

   $\tan 214° = \tan 34° = 0.67451$

3. $\sin 325° = -\sin 35° = -0.57358$

   $\cos 325° = \cos 35° = 0.81915$

   $\tan 325° = -\tan 35° = -0.70021$

4. $\sin -290° = -\sin 290° = \sin 70° = 0.93969$

$\cos -290° = \cos 290° = \cos 70° = 0.34202$

$\tan -290° = -\tan 290° = \tan 70° = 2.74748$

PERIODICITY OF FUNCTIONS

The trigonometric functions exhibit a property which is also possessed by some other mathematical functions. These functions exhibit a regular repetition of the values which each function has in a certain range.

The importance of the functions which are periodic is that once the values are known for one period of the variable, the values of the functions are then known for all values the variable takes on in its range.

GRAPH OF THE SINE

Figure 4-25 shows the graph of the sine function. The angle is plotted on the horizontal axis, increasing to the right, and the corresponding value of the sine function is plotted on the vertical axis. Two complete revolutions are plotted. It can be seen on the graph that the value of the sine varies between +1 and -1 and never goes beyond these limits as the angle varies. The graph also shows that the sine increases from 0 at 0° to a maximum positive at 90° ($\frac{\pi}{2}$) and then decreases back to 0 at 180° (π). The value of the sine continues decreasing to a maximum negative at 270° ($\frac{3\pi}{2}$) and then increases to a value of 0 at 360°(2π). If the second revolution is analyzed it is found to repeat the variations of the first revolution for the corresponding points. Therefore, the period of the sine function is 360° or 2π radians.

GRAPH OF THE COSINE

The cosine also has a period of 2π, as seen in figure 4-26. The range of values which the cosine can take on also lies between +1 and -1. However, as seen on the graph, the cosine varies from a value of 1 at 0° to a value of 0 at 90° and continues decreasing to reach a maximum negative value at 180°(π). The cosine increases from the 180° point and reaches a maximum positive value (equal to 1) at the 360° (2π) point.

GRAPH OF THE TANGENT

The graph of the tangent, figure 4-27, shows a special kind of discontinuity called an infinite

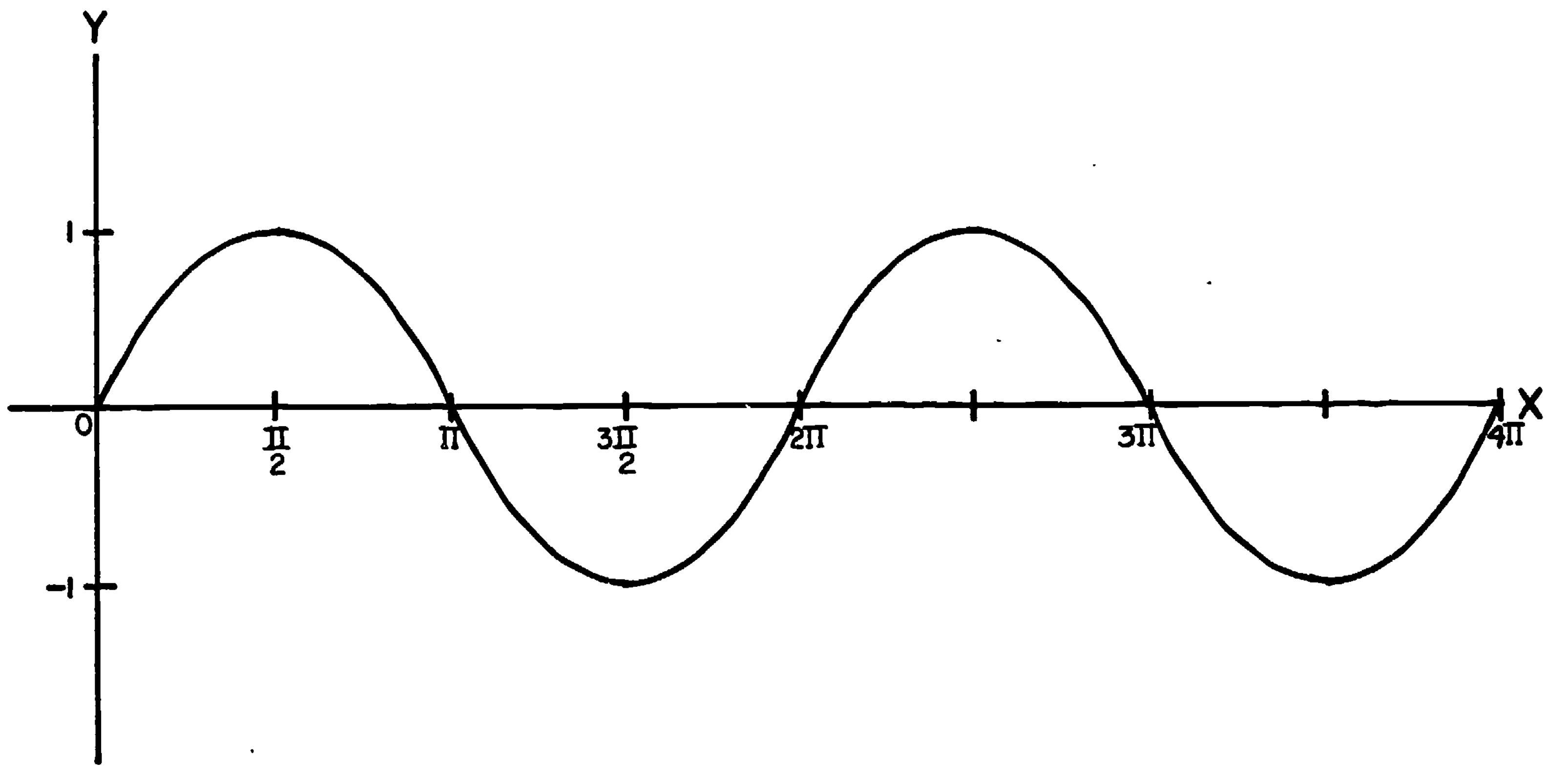


Figure 4-25.—Graph of the sine.

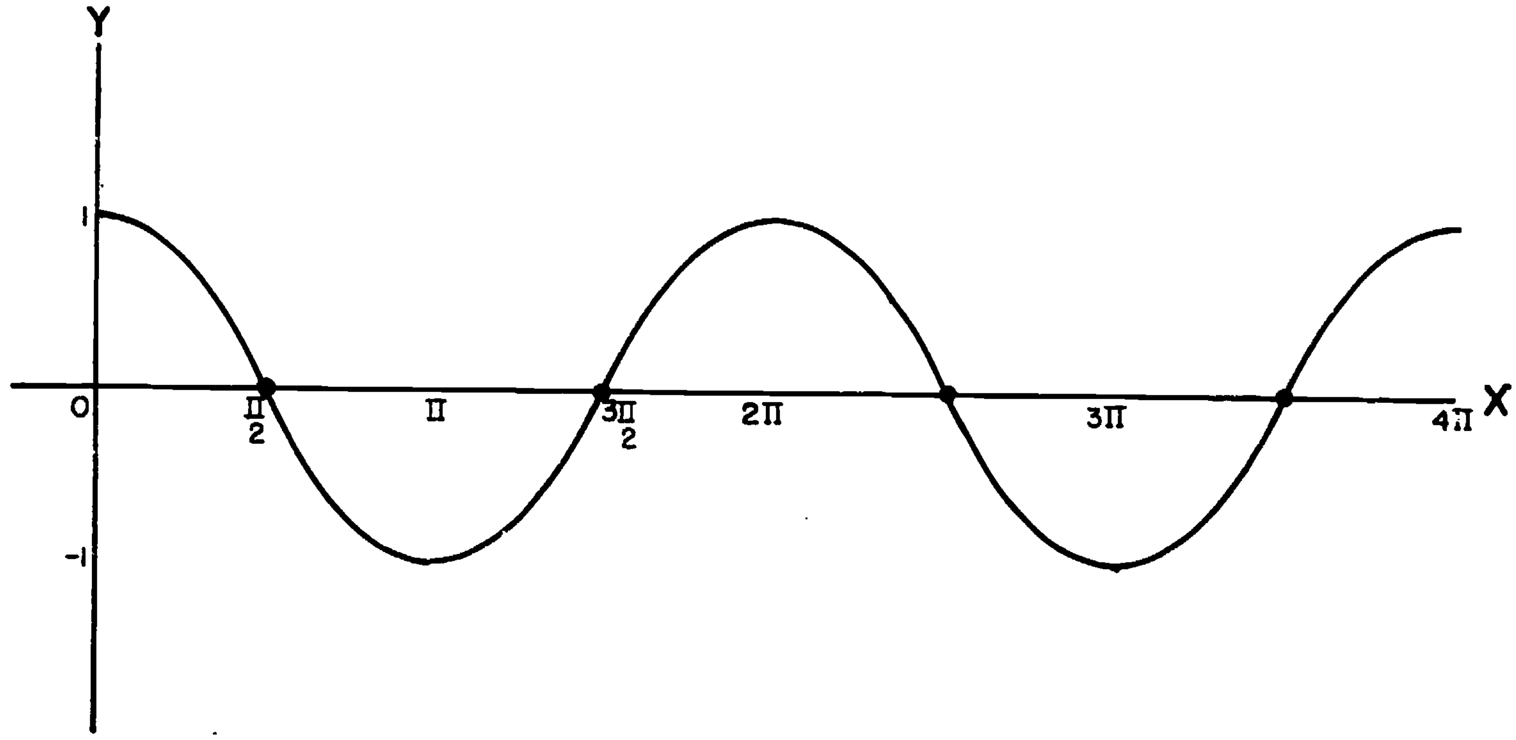


Figure 4-26.—Graph of the cosine.

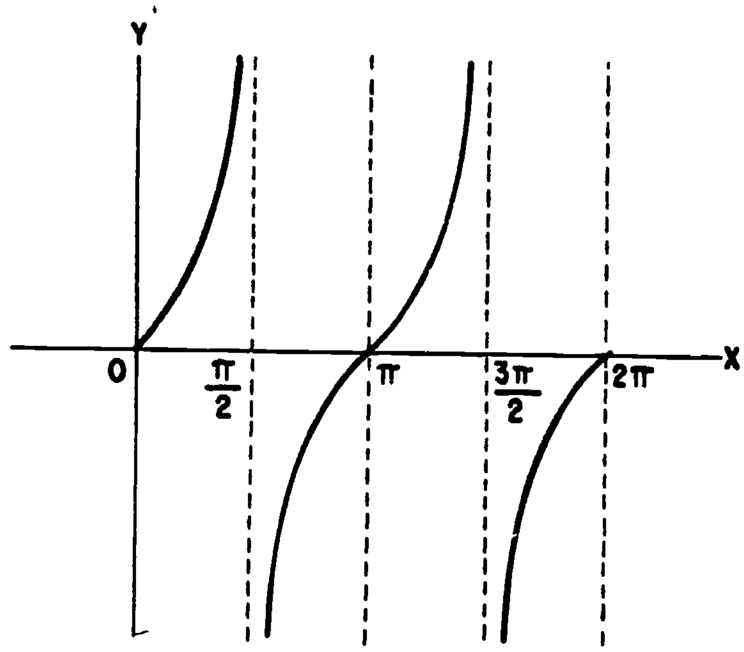


Figure 4-27.—Graph of the tangent.

discontinuity. When the angle is slightly less than 90°, ($\frac{\pi}{2}$), the value of the tangent will be very large and positive. When the angle is exactly $\frac{\pi}{2}$, the tangent curve and the Y axis are parallel. Thus, the tangent of $\frac{\pi}{2}$ is plus or minus infinity. When the angle becomes slightly greater than $\frac{\pi}{2}$, the tangent assumes very large negative values. Thus, the tangent goes from large positive values through infinity to large negative values at $\frac{\pi}{2}$. The same occurs at $\frac{3\pi}{2}$. This is called an infinite discontinuity. The period for this function is 180° (π radians).

PRACTICE PROBLEMS: In the problems listed below use the graphs of figures 4-25, 4-26, and 4-27 to answer the questions; then use the tables in appendixes III and IV to verify the answers.

For what values of θ

1. is cos θ increasing if $0 \leq \theta \leq \pi$?
2. do sin θ and cos θ decrease together if $0 \leq \theta \leq 2\pi$?
3. do cos θ and tan θ increase together if $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$?
4. do sin θ, cos θ, and tan θ increase together if $0 \leq \theta \leq 2\pi$?

ANSWERS:

1. None
2. $\frac{\pi}{2} \leq \theta \leq \pi$
3. $\pi \leq \theta \leq \frac{3\pi}{2}$
4. $\frac{3\pi}{2} \leq \theta \leq 2\pi$

COFUNCTIONS AND COMPLEMENTARY ANGLES

Inspect the two triangles in figure 4-28. Triangle (B) is exactly the same as triangle (A) except that it has been turned on the side.

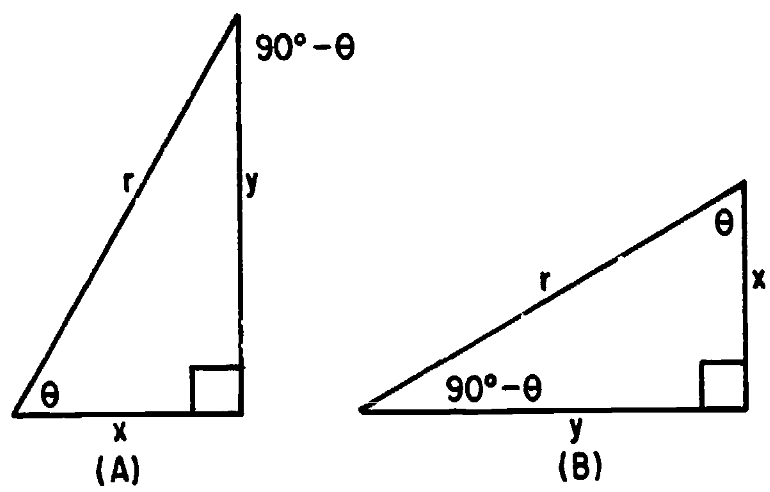


Figure 4-28.—Complementary angles.

In triangle (A)	In triangle (B)
$\sin \theta = \frac{y}{r}$	$\cos (90^\circ - \theta) = \frac{y}{r}$
$\cos \theta = \frac{x}{r}$	$\sin (90^\circ - \theta) = \frac{x}{r}$
$\tan \theta = \frac{y}{x}$	$\cot (90^\circ - \theta) = \frac{y}{x}$
$\sec \theta = \frac{r}{x}$	$\csc (90^\circ - \theta) = \frac{r}{x}$
$\csc \theta = \frac{r}{y}$	$\sec (90^\circ - \theta) = \frac{r}{y}$
$\cot \theta = \frac{x}{y}$	$\tan (90^\circ - \theta) = \frac{x}{y}$

The ratio $\frac{y}{r}$ in triangle (A) is equal to ratio $\frac{y}{r}$ in triangle (B); therefore

$$\sin \theta = \cos (90^\circ - \theta)$$

In the same way,

$$\cos \theta = \sin (90^\circ - \theta)$$

$$\tan \theta = \cot (90^\circ - \theta)$$

$$\sec \theta = \csc (90^\circ - \theta)$$

$$\csc \theta = \sec (90^\circ - \theta)$$

$$\cot \theta = \tan (90^\circ - \theta)$$

The trigonometric function of an angle is equal to the cofunction of its complement. The six trigonometric functions consist of three pairs of cofunctions. The functions are arranged in pairs so that the name of one can be obtained from the other by adding or deleting the prefix "co." For example: sine, cosine; tangent, cotangent; secant, cosecant.

The cofunction principle accounts for the format of tables of trigonometric functions similar to the one in appendix III. Refer to the page of this table that contains the function of 21°. Enter the table at the top of the sin 21° column and go down the column to 21°30' as determined by the minute column on the left. The value found for sin 21°30' (or 21.5°) is 0.36650. Looking now to the bottom of the page and reading up the minute column on the extreme right, find the cosine of 68°30' (or 68.5°). It is seen that cos 68°30' is also 0.36650. Since

$$21^\circ 30' + 68^\circ 30' = 90^\circ$$

21°30' and 68°30' are complementary angles and the cofunctions are numerically equal. If the relationship of complementary angles and cofunctions did not hold true, a table of values of functions from 0° to 90° would require 90 columns each for the sine and cosine instead of the 45 columns that are required.

PRACTICE PROBLEMS: Express the following as a function of the complementary angle.

1. sin 27°
2. tan 38°17'
3. csc 41°
4. cos 16°30'22"

Express the following as an acute angle less than 45°.

5. sec 79°37'16"
6. cos 56°
7. sin 438°
8. tan 48°

ANSWERS:

1. cos 63°
2. cot 51°43'

3. sec 49°
4. sin 73°29'38"
5. csc 10°22'44"
6. sin 34°
7. cos 12°
8. cot 42°

CHAPTER 5

OBLIQUE TRIANGLES

The two previous chapters considered right angles and angles which could be calculated by using triangles which included right angles. This chapter considers oblique triangles which are, by definition, triangles containing no right angles.

In chapter 19 of Mathematics, Vol. 1, NavPers 10069-C, a method for solving problems involving oblique triangles was introduced. This method employed the procedure of dividing the original triangle into two or more right triangles, and using the right triangles to solve the problem involved. It was also pointed out at that time that there were direct methods of dealing with oblique triangles.

This chapter develops two methods of dealing directly with oblique triangles. These two methods, or laws, are developed in the first section of the chapter as aids to calculations. The chapter also contains example and practice problems for solving oblique triangles considered in four standard cases. In this chapter "solving a triangle" is defined as finding the three sides a, b, and c and the three angles A, B, and C of an oblique triangle, when some of these six parts are given.

Also included in this chapter are some problems using logarithms in solving oblique triangles (where another law is introduced) and problems concerning the area of a triangle which combine the area formula of plane geometry with the laws developed in this chapter.

AIDS TO CALCULATION

The aids to calculations, or aids in solving oblique triangles, developed in this chapter are two theorems, known as the law of sines and the law of cosines. This section is concerned with the development and proof of these laws; subsequent sections will be concerned with using them in calculations.

LAW OF SINES

The law of sines states that the lengths of the sides of any triangle are proportional to the sines of their opposite angles. If a triangle is constructed and labeled as shown in figure 5-1 the law of sines can be written

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

PROOF: For the proof of the law of sines we redraw the triangle of figure 5-1 and drop a perpendicular from A to the opposite side as shown in figure 5-2 (A).

Reference to the triangle shown in figure 5-2 (A) shows that the perpendicular from A to the opposite side has divided the triangle into two right triangles and the trigonometric functions previously developed are used here. Considering these two right triangles we obtain

$$\sin B = \frac{h}{c} \text{ or } h = c \sin B$$

and,

$$\sin C = \frac{h}{b} \text{ or } h = b \sin C$$

Here we have two expressions for h which are equal to each other, so

$$c \sin B = b \sin C$$

or in another form

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

In figure 5-2 (B) the triangle is redrawn with a perpendicular from C to an extension of the

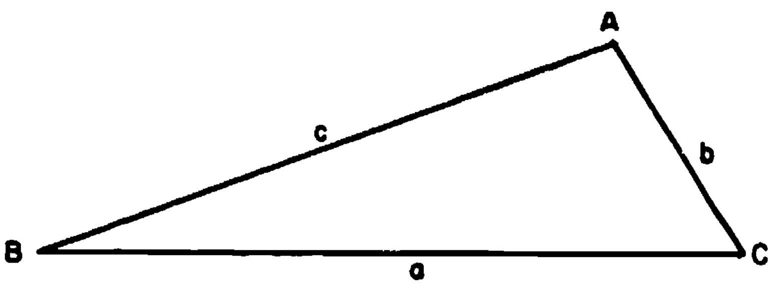


Figure 5-1.—Triangle ABC.

opposite side. Considering the right triangle BCD thus formed, it is seen that

$$\sin B = \frac{h'}{a} \text{ or } h' = a \sin B$$

and in triangle ACD

$$\sin (180^\circ - A) = \frac{h'}{b} \text{ or } h' = b \sin (180^\circ - A)$$

From chapter 4 of this training course, recall that

$$\sin (180^\circ - \theta) = \sin \theta$$

so

$$\sin (180^\circ - A) = \sin A$$

then

$$\sin A = \frac{h'}{b} \text{ or } h' = b \sin A$$

Now equate

$$h' = a \sin B$$

and

$$h' = b \sin A$$

to form

$$a \sin B = b \sin A$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin P}$$

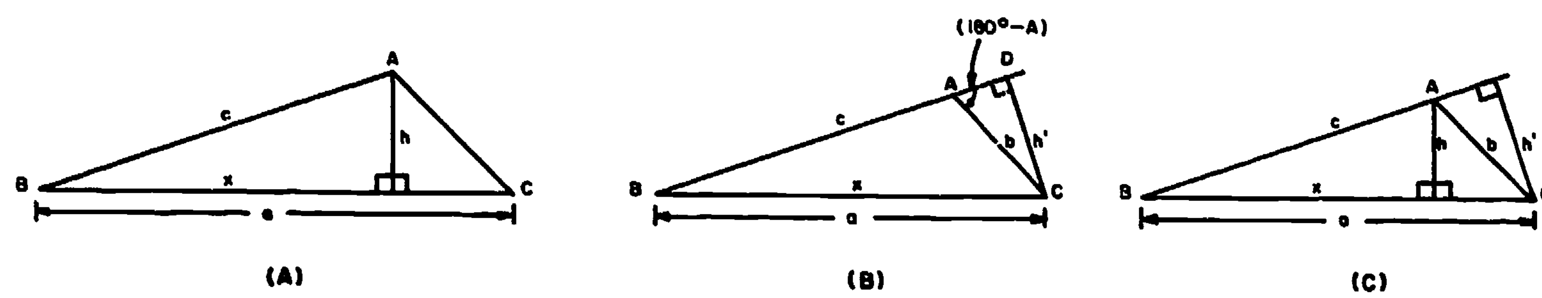


Figure 5-2.—Proving the law of sines.

Two separate triangles were used in proving the law of sines simply for clarity of explanation. If the two triangles are combined in one figure, as shown in figure 5-2 (C), it is seen that the two laws could be derived from this one illustration. Here it is obvious that the angle B which appears in both of the ratio pairs is the same angle; thus the ratios

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

and

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

can be combined to form the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

as previously stated.

LAW OF COSINES

The second of the laws to be developed in this section is the law of cosines which states: In any triangle the square of one side is equal to the sum of the squares of the other two sides minus twice the product of these two sides multiplied by the cosine of the angle between them. For the triangle in figure 5-3 (A), the law of cosines can be stated as

$$a^2 = b^2 + c^2 - 2bc \cos A$$

PROOF: Consider the triangle in figure 5-3 (B) with a perpendicular dropped from B to side b to form two right triangles. To prove

$$a^2 = b^2 + c^2 - 2bc \cos A$$

consider first the triangle ABD and note that

$$\cos A = \frac{x}{c} \qquad (1)$$

or

$$x = c \cos A \qquad (2)$$

and also

$$h^2 = c^2 - x^2 \qquad (3)$$

Substituting in (3) the value of x given in (2) results in

$$h^2 = c^2 - c^2 \cos^2 A \qquad (4)$$

In triangle BDC

$$h^2 = a^2 - (b - x)^2 \qquad (5)$$

which expands to

$$h^2 = a^2 - b^2 + 2bx - x^2 \qquad (6)$$

Substituting again for x the value given in (2) results in

$$h^2 = a^2 - b^2 + 2bc \cos A - c^2 \cos^2 A \qquad (7)$$

Equating the two values of h^2 in (4) and (7) gives

$$c^2 - c^2 \cos^2 A = a^2 - b^2 + 2bc \cos A - c \cos^2 A \qquad (8)$$

Canceling like terms and rearranging,

$$c^2 - \cancel{c^2 A} = a^2 - b^2 + 2bc \cos A - \cancel{c^2 A}$$

$$-a^2 = -b^2 - c^2 + 2bc \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad (9)$$

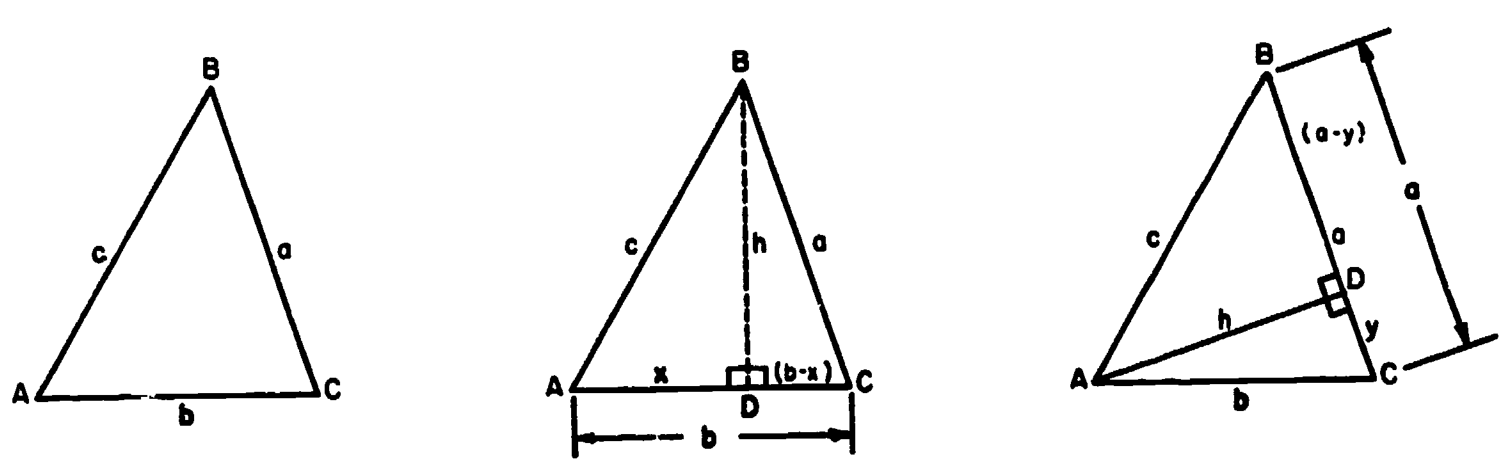
The proof is thus complete.

To prove another form of the law of cosines

$$c^2 = a^2 + b^2 - 2ac \cos C$$

refer to figure 5-3 (C) and note that in the right triangle CAD

$$\cos C = \frac{y}{b}$$



(A) (B) (C)

Figure 5-3.—Proving law of cosines.

or

$$y = b \cos C$$

and also

$$h^2 = b^2 - y^2$$

Substituting for the value of y gives

$$h^2 = b^2 - b^2 \cos^2 C$$

In triangle ABD

$$h^2 = c^2 - (a - y)^2$$

which expands to

$$h^2 = c^2 - a^2 + 2ay - y^2$$

With additional substitution

$$h^2 = c^2 - a^2 + 2ab \cos C - b^2 \cos^2 C$$

Equating the two h^2 values which are representative of the two right triangles results in

$$b^2 - b^2 \cos^2 C$$

$$= c^2 - a^2 + 2ab \cos C - b^2 \cos^2 C$$

Canceling and rearranging yields

$$c^2 = a^2 + b^2 - 2ab \cos C$$

completing the proof.

The same procedures can be applied to prove the remaining form of the law of cosines. In summary, the three forms of the law of cosines are

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

The law of sines and the law of cosines are used mainly to solve oblique triangles, as will be shown in the following sections. In addition, these laws also hold true for right triangles. The trigonometric functions or other methods previously noted are normally more effective in dealing with right triangles; however, application of these laws can be used in an analysis of some trigonometric principles and identities.

EXAMPLE: Show that

$$c^2 = a^2 + b^2 - 2ab \cos C$$

holds true in the right triangle shown in figure 5-4.

SOLUTION: In the figure it is shown that C = 90°. Recall from the graph of the cosine in chapter 4 of this course (or from appendix III) that cos 90° = 0. Therefore, the formula

$$c^2 = a^2 + b^2 - 2ab \cos C$$

can be reduced to

$$c^2 = a^2 + b^2 - 2ab \cos 90°$$

$$c^2 = a^2 + b^2 - (2ab)\ (0)$$

$$c^2 = a^2 + b^2$$

Reference to the figure shows that c is the hypotenuse of a right triangle and from the Pythagorean theorem

$$c^2 = a^2 + b^2$$

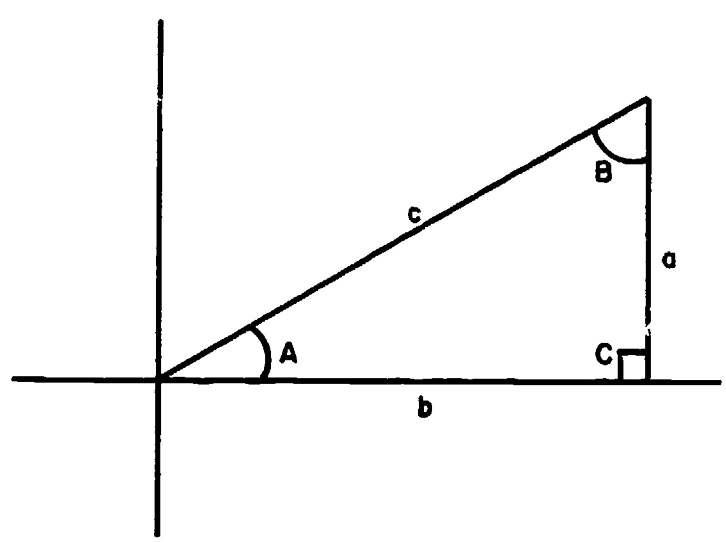


Figure 5-4.—Example problem, law of cosines.

Thus, the law of cosines would provide a proper solution to the right triangle.

In this problem the law of cosines reduces to the Pythagorean theorem. However, in working with oblique triangles remember that while the law of cosines applies to all triangles, the Pythagorean theorem can only be used when dealing with right triangles.

PRACTICE PROBLEMS:

1. Refer to figure 5-5 and prove that $b^2 = a^2 + c^2 - 2ac \cos B$.

2. Assume that the triangle in figure 5-5 is such that a=b=c=2. Transpose the formula in problem 1 and solve for cos B, then refer to the table of functions in the appendix and verify that B = 60°.

FOUR STANDARD CASES

It was stated previously that the solution of a triangle consists of finding the six parts (sides a, b, and c; and the angles A, B, and C) when some of these values are known. If three of these parts are known, at least one of which is the length of a side, the remaining parts can normally be calculated by one of the methods discussed in the following paragraphs. For convenience, the methods for solving oblique triangles are developed by considering the triangles in four categories as follows:

1. Two of the angles and one of the sides are known.
2. The three sides are known.
3. Two of the sides and the angle between them are known.

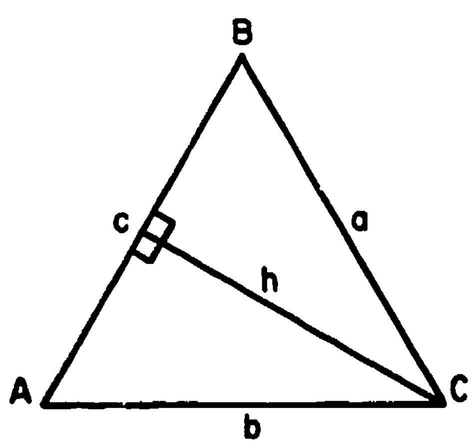


Figure 5-5.—Practice problem, law of cosines.

4. Two of the sides and an angle that is not between them are known.

The last situation described in the preceding list is known as the AMBIGUOUS CASE for, under certain conditions, two triangles which are not congruent can contain the same three known parts.

Recall from plane geometry that two triangles are congruent (having the same shape and size) if one of the following conditions is met:

1. Three sides of one triangle are equal to the corresponding sides of a second triangle.

2. Two sides and the included angle of one triangle are equal to the corresponding parts of a second triangle.

3. Two angles and a side of one triangle are equal to the corresponding parts of a second triangle.

It is seen here that the ambiguous case is the only one that does not parallel a plane geometry theorem for congruent triangles. The first and fourth (ambiguous) cases (or categories) of triangles will employ the law of sines in the solutions and cases 2 and 3 will be solved by using the law of cosines.

TWO ANGLES AND ONE SIDE

When two angles and a side are known, the remaining angle can be determined so easily that this case could be assumed to be one in which one side and all angles are known. (The third angle is equal to the difference between 180° and the sum of the known angles.) The law of sines is then used twice to find the length of the remaining sides. To find either of the unknown sides, select the ratio pair which includes the ratio involving the unknown side and the one which considers the known side.

EXAMPLE: Using the law of sines, find the length of the lettered sides in the triangle in figure 5-6 (A).

SOLUTION: From the law of sines

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{5}{\sin 97.5^\circ} = \frac{b}{\sin 30^\circ}$$

Since

$$\sin \theta = \sin (180^\circ - \theta),$$

$$\sin 97.5^\circ = \sin 82.5^\circ = 0.99144$$

Also

$$\sin 30° = 0.5000$$

so,

$$\frac{5}{0.99144} = \frac{b}{0.5000} \text{ or } \frac{2.50000}{0.99144} = b$$

or

$$b = 2.5216$$

Angle A is equal to

$$180° - B - C = 52.5°$$

Again from the law of sines,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{5}{\sin 97.5°} = \frac{a}{\sin 52.5°}$$

$$\sin 52.5° = 0.79335$$

$$\frac{5}{0.99144} = \frac{a}{0.79335}$$

$$a = \frac{5(0.79335)}{0.99144}$$

$$a = 4.001$$

EXAMPLE: Figure 5-6 (B) shows a flagpole standing vertically on a hill which is inclined 15 degrees with the horizontal. A man climbing the hill notes that at one point his line of sight to the top of the pole makes an angle of 40° with the horizontal. At another point, 200 feet further up the hill, this angle has increased to 55°. How high is the flagpole? Solve using only the law of sines.

SOLUTION: First, define all the angles in the triangles OAB and OBD. In triangle OAB

$$\angle BAO = 40° - 15° = 25°$$

$$\angle OBA = 180° - (55° - 15°)$$

$$= 180° - 40° = 140°$$

$$\angle AOB = 180° - 140° - 25° = 15°$$

In triangle OBD

$$\angle DBO = 55° - 15° = 40°$$

$$\angle BDO = 90° + 15° = 105°$$

$$\angle BOD = 90° - 55° = 35°$$

These two triangles have OB as a common side. We can use the law of sines to find BO in triangle OAB and then apply the law again in triangle OBD to find the length of side OD which is the height of the flagpole. Thus,

$$\frac{AB}{\sin AOB} = \frac{OB}{\sin BAO}$$

$$\frac{200}{\sin 15°} = \frac{OB}{\sin 25°}$$

$$OB = \frac{200 \sin 25°}{\sin 15°} = 326.57 \text{ ft}$$

And in triangle OBD

$$\frac{OB}{\sin BDO} = \frac{OD}{\sin DBO}$$

$$\frac{326.57}{\sin 105°} = \frac{OD}{\sin 40°}$$

$$OD = \frac{326.57 \sin 40°}{\sin 105°}$$

$$\sin \theta = \sin (180° - \theta)$$

$$\sin 105° = \sin 75°$$

$$OD = \frac{326.57 \sin 40°}{\sin 75°}$$

$$OD = 217.3 \text{ ft}$$

PRACTICE PROBLEMS: Refer to figure 5-7 in solving the following problems where the figures (A), (B), and (C) are to be used respectively with problems 1, 2, and 3. Use the law of sines in solving these problems.

1. Find a and b using the values given in the table of functions of special angles (chapter 4 of

this course). Leave answers in radical form where applicable.

2. Find sides d and f to two decimal places.
3. Find the length of a to two decimal places.

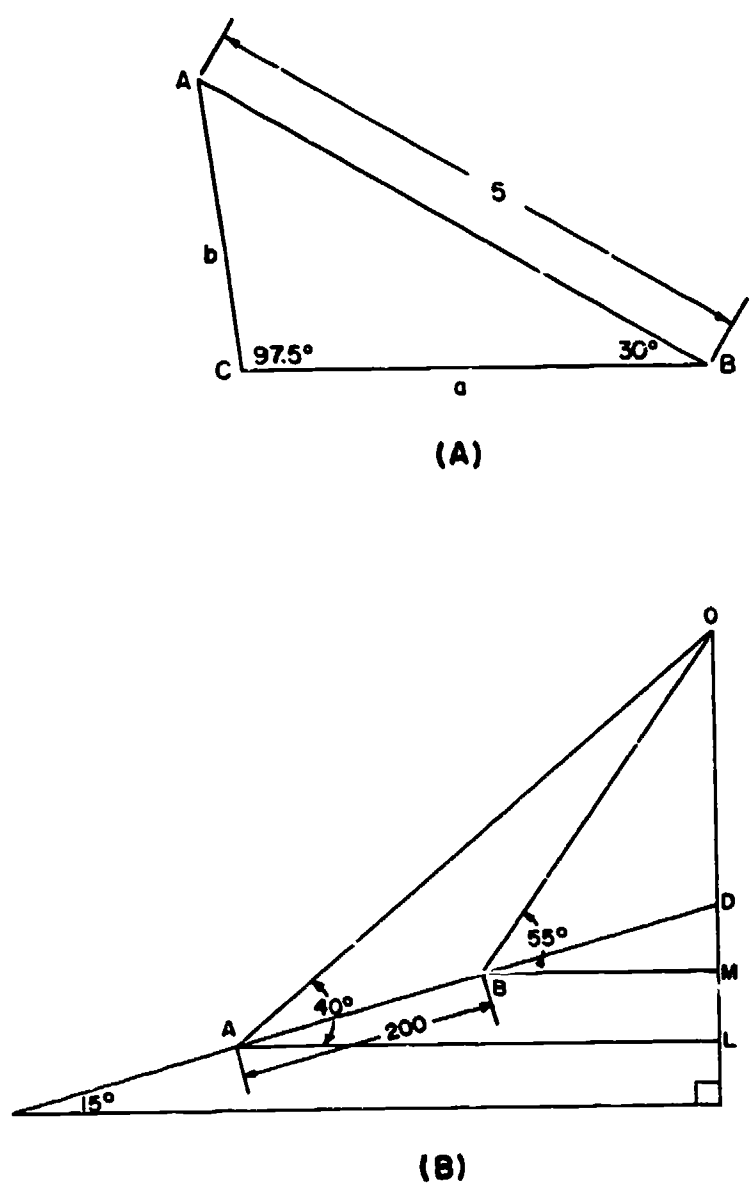


Figure 5-6.—Case 1, example problems.

ANSWERS:

1. $a = 3$
   $b = 3\sqrt{3}$
2. $d = 6.07$
   $f = 3.96$
3. 8.39

THREE SIDES

When the three sides of a triangle are given, the triangle can be solved by three successive applications of the law of cosines. Each application yields the value of one angle. The order of determining the angles is not important; any of the three angles may be determined first. A particular angle is found by using the form of the law of cosines in which the cosine of the angle in questions appears. When the three angles have been found, the solution is checked by verifying that $A + B + C \approx 180°$.

EXAMPLE: Solve the triangle ABC, given $a = 7$, $b = 13$, and $c = 14$. Determine the size of the angles to the nearest degree.

SOLUTION: To simplify the procedure solve the law of cosines algebraically for cos A.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2 bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

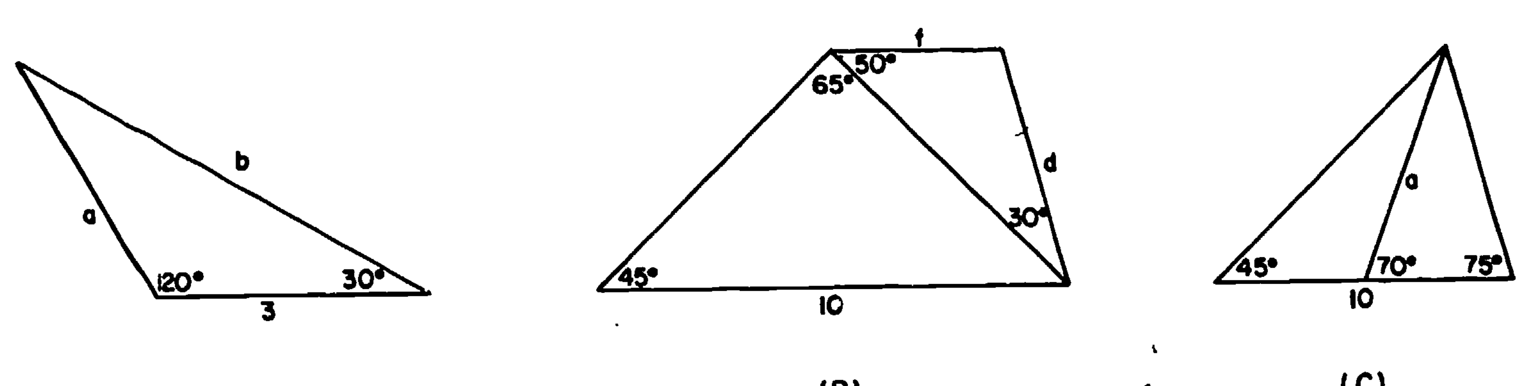


Figure 5-7.—Case 1 practice problems.

The remaining forms of the law can be solved in the same manner and the results are

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Now in the given problem

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{13^2 + 14^2 - 7^2}{2 \times 13 \times 14}$$

$$\cos A = \frac{169 + 196 - 49}{364}$$

$$\cos A = \frac{316}{364}$$

$$\cos A = 0.86813$$

$$A = 30^\circ$$

Then

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{49 + 196 - 169}{2 \times 7 \times 14}$$

$$\cos B = \frac{76}{196}$$

$$\cos B = 0.38776$$

$$B = 67^\circ$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{49 + 169 - 196}{182}$$

$$\cos C = \frac{22}{182}$$

$$\cos C = 0.12088$$

$$C = 83^\circ$$

Checking the calculations gives

$$A + B + C = 30^\circ + 67^\circ + 83^\circ = 180^\circ$$

It may appear that the best method to use in solving a triangle when three sides are given would be to calculate two angles and find the third angle by subtracting the sum of the two from 180°. While this method shortens the computation, it also destroys the check on the calculations, and is not recommended.

EXAMPLE: Solve the triangle ABC when $a = 8$, $b = 13$, and $c = 17$. Express the angles to the nearest degree.

SOLUTION:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{169 + 289 - 64}{442}$$

$$\cos A = \frac{394}{442}$$

$$\cos A = 0.89140$$

$$A = 27^\circ$$

Then

$$\cos B = \frac{64 + 289 - 169}{272}$$

$$\cos B = 0.67647$$

$$B = 47^\circ$$

and finally

$$\cos C = \frac{64 + 169 - 289}{208}$$

$$\cos C = \frac{-56}{208}$$

$$\cos C = -0.26927$$

$$\cos C = -\cos 74°$$

$$-\cos C = \cos 74°$$

reference to reduction formulas gives

$$-\cos C = \cos (180° - C)$$

and

$$C = 106°$$

then, checking,

$$A + B + C = 27° + 47° + 106° = 180°$$

PRACTICE PROBLEMS: The side lengths of triangle ABC are given in the following problems. Use the law of cosines to determine the sizes of the angles to the nearest degree.

1. a = 3, b = 4, c = 5
2. a = 2, b = 3, c = 4
3. a = 7, b = 14, c = 11

ANSWERS:

1. A = 37°
   B = 53°
   C = 90°
2. A = 29°
   B = 47°
   C = 104°
3. A = 29°
   B = 100°
   C = 51°

TWO SIDES AND THE INCLUDED ANGLE

Where two sides and the angle between them are given, the triangle is solved most easily by repeated use of the law of cosines. Using the given parts, solve first for the unknown side. Then, with three sides known, solve for the remaining angles in the same manner as in case two.

EXAMPLE: Using the law of cosines, solve the triangle ABC shown in figure 5-8, angle accuracy to the nearest degree.

SOLUTION: First find the unknown side.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 7^2 + 5^2 - 2(5)(7) \cos 19°$$

$$a^2 = 49 + 25 - 70 \cos 19°$$

$$a^2 = 74 - 70(0.94552)$$

$$a^2 = 74 - 66.1864$$

$$a^2 = 7.8136$$

$$a = \sqrt{7.8136}$$

$$a = 2.795$$

To compute the angles, round the values given above to

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{7.8 + 25 - 49}{2 \times 2.8 \times 5}$$

$$\cos B = -\frac{16.2}{28}$$

$$\cos B = -0.57857$$

$$B = 125°$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{7.8 + 49 - 25}{2 \times 2.8 \times 7} = \frac{31.8}{39.2}$$

$$\cos C = 0.81633$$

$$C = 35°$$

then, checking,

$$A + B + C = 19° + 125° + 35° = 179°$$

$$A + B + C \approx 180°$$

This is acceptable with the accuracy required here.

EXAMPLE: Two ships leave port at the same time; one (ship A) sailed on a course of 050° at a speed of 10 knots, the second (ship B) sails on a course of 110° at 12 knots. How far apart are the two ships at the end of 3 hours?

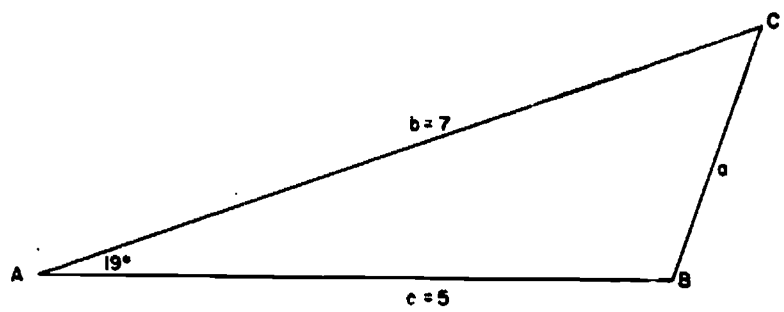


Figure 5-8.—Case 3, example problem.

SOLUTION: A good rule in any problem of this type is to first draw a picture to show the problem. In figure 5-9 a coordinate system oriented to compass headings is constructed, and the given information is plotted. From the figure it is seen that the desired answer is the distance AB opposite the angle labeled C, where C = 110° - 50° = 60°.

Using the law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 36^2 + 30^2 - 2(30 \times 36) \cos 60°$$

$$c^2 = 1296 + 900 - \cancel{2}\,(1080) \times \frac{1}{\cancel{2}}$$

$$c^2 = 2196 - 180 = 1116$$

$$c = \sqrt{1116}$$

$$c = 33.5 \text{ approximately}$$

The ships are approximately 33.5 miles part at the end of 3 hours.

PRACTICE PROBLEMS: Use the law of cosines to solve the triangles described below. Express angles to nearest degree and sides to two decimal places.

1. a = 10, b = 7, C = 25°
2. b = 11, c = 17, A = 20°
3. a = 12, c = 26, B = 140°

ANSWERS:

1. c = 4.69
   A = 116°
   B = 39°
2. a = 7.65
   B = 29°
   C = 131°

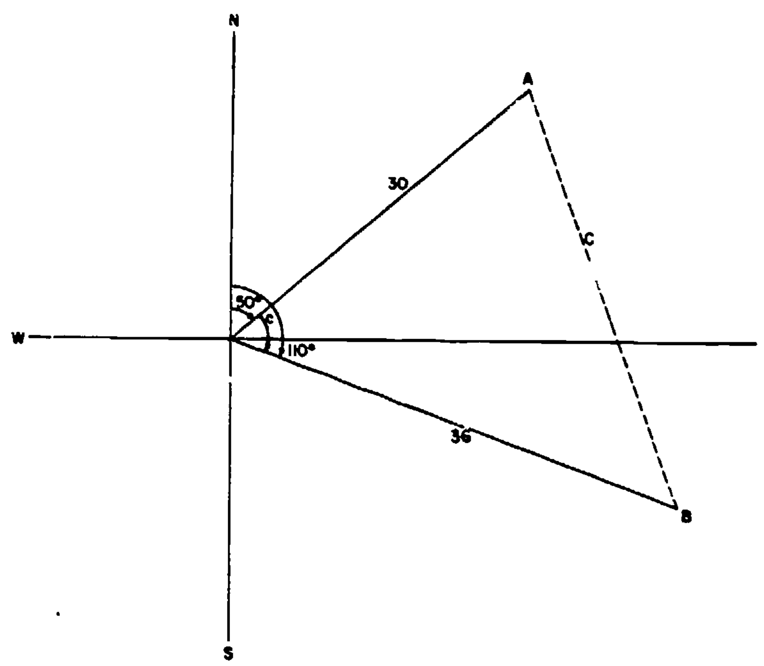


Figure 5-9.—Plot of ship's courses.

3. b = 36
   A = 12°
   C = 28°

TWO SIDES AND AN OPPOSITE ANGLE

When two sides and a nonincluded angle are given, the triangle falls in the ambiguous category and one of the following cases will exist:

1. There is no solution.
2. There are two solutions.
3. There is one solution.

The category is called ambiguous for the given parts cannot always establish the shape and size of one triangle. There may be two triangles which are not congruent, but still contain the given parts. The ambiguity of this category can be seen if we assume that three parts (B, b, and c) are given, and we attempt to construct the triangle from this data. We consider the possibilities as follows:

1. If angle B is obtuse as in figure 5-10 (A), the side b must be larger than side c for a triangle to exist. In this case, b > c, there is only one triangle which exists and only one solution.

2. If B is a right angle, as in figure 5-10 (B), side b must be larger than side c for a triangle to exist and there is only one triangle and one solution.

3. Figure 5-11 shows the situations which can exist if B is an acute angle. In (A) a figure is constructed with $A < 90^\circ$ and $b < c$; that is, a line drawn from vertex A is too short to reach the line BC. In this case, no triangle exists and there is no solution.

4. In figure 5-11 (B), the line from A (side b) is exactly the distance from A to the line BC. In this situation, only one triangle exists and it is a right triangle with one solution.

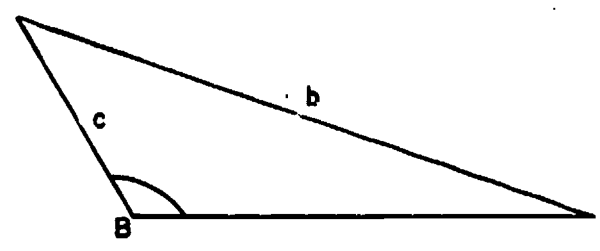
5. Figure 5-11 (C) shows a triangle where the line from A touches the line BC in two places. In this case the sides b and b' are longer than the side b in figure 5-11 (B), but still shorter than line c. In this category there are two triangles, BAC' and BAC, which contain the given parts. With the law of sines two solutions can be found for this possibility.

6. The last possibility considered is one shown in figure 5-11 (D). In this figure the line b is longer than c and again would touch the line BC in two places to form two triangles. However, only the triangle ABC is considered since the triangle ACC' does not include angle B as an interior angle. This situation is considered to have only one solution. If $b = c$, then c and c' coincide and there is only one triangle.

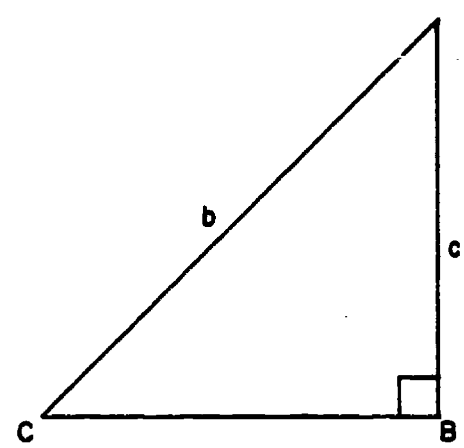
Certain relationships of angle, side, and function values can be found to determine in advance which of the possibilities previously listed exists for a given triangle. However, this knowledge is not required before the solution is attempted. It can be determined in the process of attempting a solution, as will be shown in the example problems, or a drawing can be made from the data given.

The triangles presented in this section have had value sizes only in relation to each other or in relation to 90°. Figure 5-12 points out the ambiguity which can exist when a triangle is described by giving two sides and an angle opposite one of these sides. This figure shows two triangles constructed with given data of $A = 30^\circ$, $a = 4$, and $c = 6$. Solution of these two triangles will show that B, C, and b are not the same for the two, and this is a case where two solutions arise.

A good approach to solving triangles when two sides and an opposite angle (ambiguous category) are given is to first use the law of sines to find the unknown angle opposite a given side. Then the third angle can easily be determined and the law of sines can be used again, to compute the unknown side. In the following examples and practice problems desired



(A)



(B)

Figure 5-10.—Ambiguous case, $B \geq 90^\circ$.

accuracy is in degrees to the nearest minute and sides to two decimal places.

EXAMPLE: Solve the triangle (or triangles) ABC when $B = 45^\circ$, $b = 3$, $c = 7$.

SOLUTION: First use the law of sines to find angle C

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\sin C = \frac{c \sin B}{b}$$

$$\sin C = \frac{7 \sin 45^\circ}{3}$$

$$\sin C = \frac{7 \times 0.70711}{3}$$

$$\sin C = \frac{4.94977}{3}$$

$$\sin C = 1.64992$$

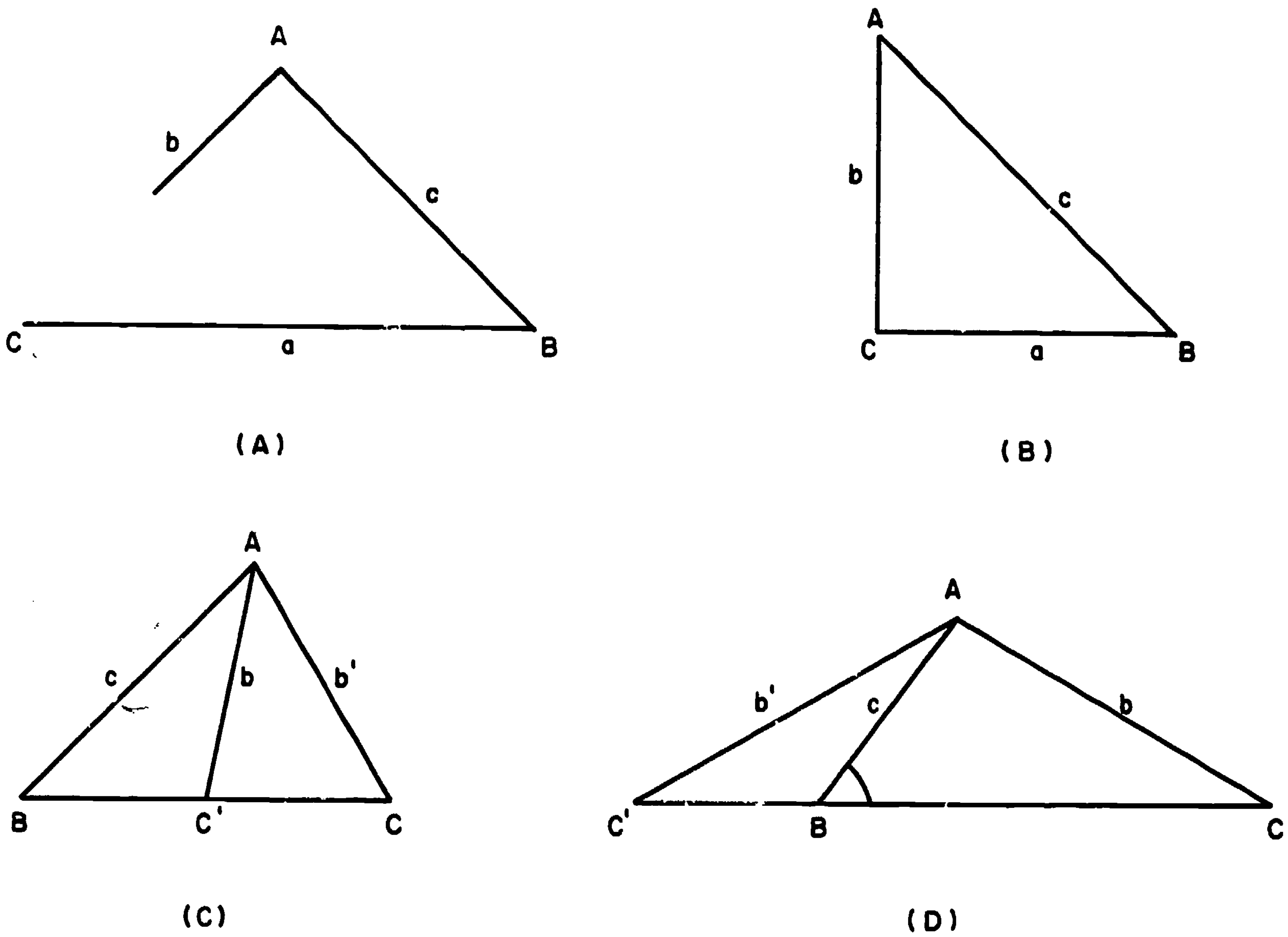


Figure 5-11.—Ambiguous case, B < 90°

The calculations show that

$$\sin C > 1$$

However, reference to tables or a graph of the sine function shows that the sine is never greater than 1, so this is the case where no triangle or solution exists.

EXAMPLE: Solve the triangle (or triangles) ABC when A = 22°, a = 5.4, c = 14.

SOLUTION: Apply the law of sines to determine angle C.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\sin C = \frac{c \sin A}{a}$$

$$\sin C = \frac{14 \sin 22°}{5.4}$$

$$\sin C = \frac{14 \times 0.37461}{5.4}$$

$$\sin C = 0.97121$$

$$C = 76° \ 13'$$

Since the side opposite the known angle is smaller than the other given side, there are two angles to consider. Since

$$\sin (180° - C) = \sin C$$

the other angle (C') is 103° 47'.

Continue the solution considering two triangles, ABC and A'B'C'. In ABC;

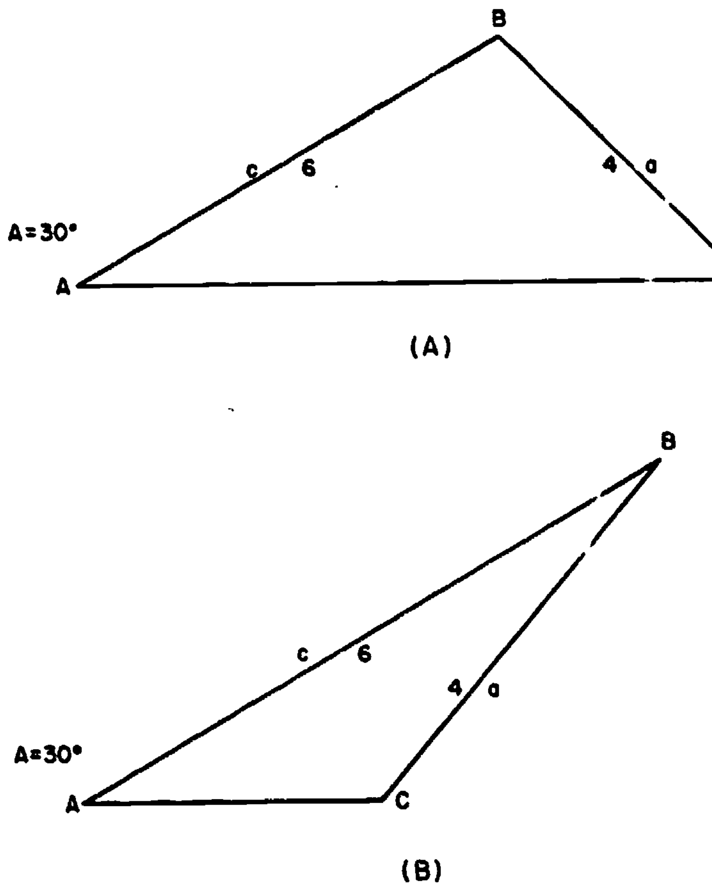


Figure 5-12.—Two different triangles derived from identical given data.

$A = 22°$, $a = 5.4$, $c = 14$, and $C = 76° 13'$

In A'B'C'; $A' = A$, $a' = a$, $c' = c$, and

$C' = 103° 47'$.

Solving ABC first, angle B is found by

$$B = 180° - (A + C)$$

$$B = 180° - (22° + 76° 13')$$

$$B = 180° - 98° 13'$$

$$B = 81° 47'$$

Then by the law of sines

$$b = \frac{a \sin B}{\sin A}$$

$$b = \frac{5.4 \times 0.98973}{0.37461}$$

$$b = 14.27$$

This completes the solution of triangle ABC.

In the second triangle, angle B' is found first by using values previously calculated

$$B' = 180° - (A' + C')$$

$$B' = 180° - (22° + 103° 47')$$

$$B' = 180° - 125° 47'$$

$$B' = 54° 13'$$

Then from the law of sines

$$b' = \frac{a' \sin B'}{\sin A'}$$

$$b' = \frac{5.4 \times \sin 54° 13'}{\sin 22°}$$

$$b' = \frac{5.4 \times 0.81123}{0.37461}$$

$$b' = 11.70$$

This completes the solution for both possible triangles. Figure 5-13 shows a scale drawing of the triangle of this example. The angle A was constructed with one side of length 14 units terminating at B, and the other line to form an angle of 22°. A compass was set to 5.4 units and an arc was struck using B as the center. As shown in the figure, the arc intersected the line from A at two points. Therefore, there are two triangles which satisfy the data.

EXAMPLE: Solve the triangle (or triangles) when $A = 35°$, $a = 10$, $b = 8$.

SOLUTION: From the law of sines

$$\sin B = \frac{b \sin A}{a}$$

$$\sin B = \frac{8 \sin 35°}{10}$$

$$\sin B = 0.45886$$

$$B = 27° 19'$$

Then

$$C = 180° - (A + B)$$

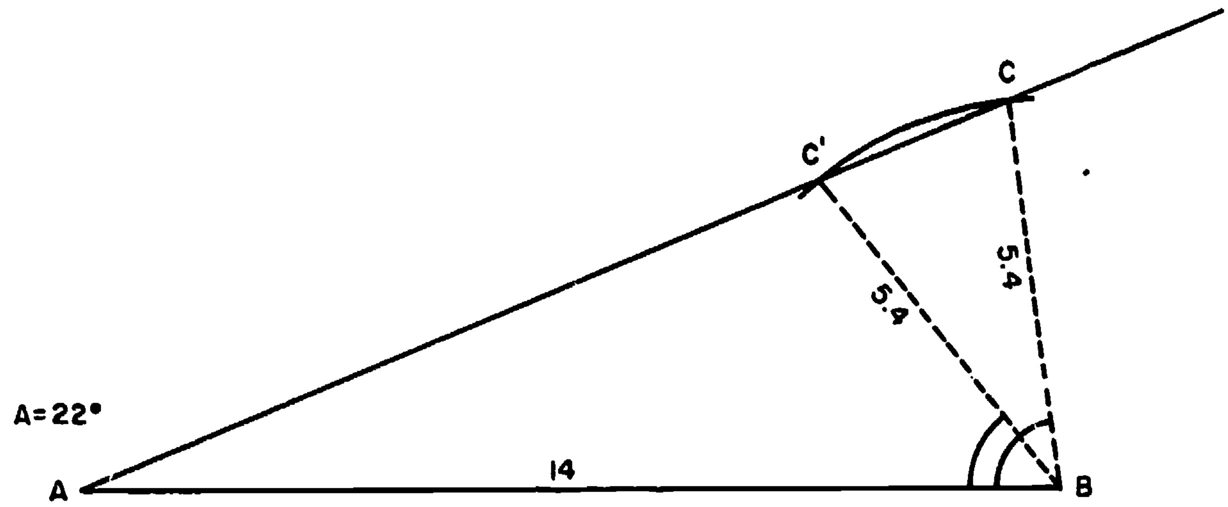


Figure 5-13.—Scale drawing of example triangles.

$$C = 180° - (35° + 27° 19')$$

$$C = 117° 41'$$

Apply the law of sines again to find c

$$c = \frac{a \sin C}{\sin A}$$

$$c = \frac{10 \sin 117° 41'}{\sin 35°}$$

$$c = \frac{10 \times 0.88553}{0.57358}$$

$$c = 15.44$$

This solves one triangle and since the side opposite the given angle is larger than the other given side, there should be only one solution. However, if this point is overlooked and the solution is continued in an attempt to find a second solution as in the previous example

$$\sin C' = \sin (180° - C)$$

$$\sin C' = \sin (180° - 117° 41')$$

$$\sin C' = \sin 62° 19'$$

$$C' = 62° 19'$$

Now if this angle is contained in a second triangle described by the given data, it is known that

$$A + B + C' = 180°$$

but

$$35° + 27° 19' + 62° 19' = 124° 38'$$

so

$$A + B + C' \neq 180°$$

and ABC' is not a triangle described by the given data.

PRACTICE PROBLEMS: In the following problems use the given data to solve the triangle or triangles involved.

1. $C = 100°$, $c = 46$, $b = 30$
2. $A = 40°$, $a = 25$, $b = 30$
3. $B = 42°$, $b = 2$, $c = 4$
4. $B = 30°$, $b = 10$, $a = 10$

ANSWERS:

1. $B = 39° 58'$, $A = 40° 02'$, $a = 30.04$
2. $B = 50° 29'$, $C = 89° 31'$, $c = 38.89$ and $B = 129° 31'$, $C = 10° 29'$, $c = 7.08$
3. No solution
4. $A = 30°$, $C = 120°$, $c = 17.32$

LOGARITHMIC SOLUTIONS

Computations involving triangle solutions are often concerned with the multiplication or division of trigonometric functions, which contain values given in four or five decimal places, and other values which may also contain numerous digits. There are many opportunities for arithmetic errors in these computations and, in many cases, the errors are the result of the multiplication and division by large decimals. In logarithmic solution the computations are reduced to addition problems, and the number of errors is frequently reduced.

By the combined use of tables of trigonometric functions and tables of logarithms one could solve the triangles by first finding the value for the function and converting this to a logarithm or by converting the logarithm of a function value to a decimal and then converting this to an angle. However, the logarithm equivalents of the principal natural functions have long since been worked out, and are available in tables, so that the necessary multiplication or division may be performed by the use of logarithms.

A table of "common logarithms of trigonometric functions" usually lists the logs for the sine, cosine, tangent, and cotangent of angles from 0° through 180°. Appendix I shows a sample page from such a table; reference to the appendix shows that both the characteristic and the mantissa are listed. In addition, for each value listed, a characteristic of -10 at the end of the log is understood.

Take the log listed for the sine of 38° 00' 00", for example. This is listed as 9.78934. What this actually means is 9.78934-10, which in turn means that the log of this function is actually -1+.78934. On the other hand, the log listed for the tangent of 51° 10' 00" is 10.09422. What this means is 10.09422-10; in other words, the log of this function is 0.09422. The logs are printed in this manner simply to avoid the necessity for printing minus characteristics. Note that, even when a characteristic is minus, the mantissa is considered as plus.

A complete table of the logarithms of trigonometric functions is not included in this course. For purposes of the examples and practice problems in this course, a short table of values for the sine, cosine, and tangent is given in table 5-1. The values in this table are given for each 5° from 0° to 90°. The complete logarithm (both characteristic and mantissa) is given in this table. The problems in this section will be worked with an accuracy of side lengths to three digits and angles to the nearest 5°.

The solution of oblique triangles was considered in four cases. In logarithmic solutions the solutions are also considered in four cases. In cases 1 and 4 the law of sines was used for solutions. Since the law of sines fits well with logarithmic solutions, cases 1 and 4 also use the law of sines for logarithmic solutions. Cases 2 and 3 were solved using the law of cosines; however, the law of cosines involves addition and subtraction and does not lend itself to logarithmic solutions. For cases 2 and 3, we will use methods other than the law of cosines for logarithmic solutions.

CASES 1 AND 4

As previously stated, the solution of these two cases by the law of sines adapts readily to logarithmic solution since the law of sines involves multiplication and division. Example solutions of the cases are given in the following paragraphs.

EXAMPLES: Use logarithms to solve the triangle ABC when A = 110°, B = 25°, and c = 125.

SOLUTION: Find the unknown angle as the first step

$$C = 180° - A - B$$

$$C = 180° - 110° - 25°$$

$$C = 45°$$

Using the law of sines with ratios involving a and c find the value of a as follows:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c}{\sin C} \times \sin A$$

Taking the logarithm of both sides of the equation gives

$$\log a = (\log c - \log \sin C) + \log \sin A$$

The logarithms of trigonometric functions given in table 5-1 include only angles from 0° to 90° so the equation above becomes

$$\log a = (\log c - \log \sin C) + \log \sin (180° - A)$$

$$\log a = (\log 125 - \log \sin 45°) + \log \sin 70°$$

Refer to table 5-1 and appendix II and convert the values to logarithms. One method of simplifying the computation is to convert each logarithm to one with an end characteristic of -10 and use the following procedure

$$\begin{aligned} \log 125 &= 12.0969 - 10 \\ \log \sin 45° &= \underline{9.8495 - 10} \quad \text{subtract} \end{aligned}$$

Table 5-1.—Logarithms of trigonometric functions.

Degrees	Log sin	Log cos	Log tan
0	——	.0000	——
5	8.9403 - 10	9.9983 - 10	8.9420 - 10
10	9.2397 - 10	9.9934 - 10	9.2463 - 10
15	9.4130 - 10	9.9849 - 10	9.4281 - 10
20	9.5341 - 10	9.9730 - 10	9.5611 - 10
25	9.6260 - 10	9.9573 - 10	9.6687 - 10
30	9.6990 - 10	9.9375 - 10	9.7614 - 10
35	9.7586 - 10	9.9134 - 10	9.8452 - 10
40	9.8081 - 10	9.8843 - 10	9.9238 - 10
45	9.8495 - 10	9.8495 - 10	0.0000
50	9.8843 - 10	9.8081 - 10	0.0762
55	9.9134 - 10	9.7586 - 10	0.1548
60	9.9375 - 10	9.6990 - 10	0.2386
65	9.9573 - 10	9.6260 - 10	0.3313
70	9.9730 - 10	9.5341 - 10	0.4389
75	9.9849 - 10	9.4130 - 10	0.5720
80	9.9934 - 10	9.2397 - 10	0.7537
85	9.9983 - 10	8.9403 - 10	1.0580
90	.0000	——	——

$$\log (c/\sin C) = 2.2474$$

$$\log (125/\sin 45°) = 12.2474 - 10 \quad \text{add}$$

$$\log \sin 70° = \underline{9.9730 - 10}$$

$$\log a = 22.2204 - 20$$

$$\log a = 2.2204$$

Taking the antilog,

$$a = 166$$

To complete the solution, find side b using the same procedure

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$b = \frac{a}{\sin A} \times \sin B$$

$$\log b = (\log a - \log \sin A) + \log \sin B$$

$$\log b = (\log 166 - \log \sin 70°) + \log \sin 25°$$

log 166	= 12.2204 - 10		
log sin 70°	= 9.9730 - 10	subtract	
log (a/sin A) =	2.2474		

log (a/sin A)	= 12.2474 - 10	
log sin 25°	= 9.6260 - 10	add
log b	= 21.8734 - 20	

$$\log b = 1.8734$$

$$b = 74.7$$

This completes the logarithmic solution for a triangle in case 1.

EXAMPLE: Solve the triangle (or triangles) when $A = 40°$, $a = 3$, $b = 4$.

This is an ambiguous case and with the side opposite the given angle smaller than the other given side there are two possibilities: either there are two solutions or side a is too short to reach the baseline and there are no solutions. Recall from earlier examples that there is no solution when the sine of the angle opposite the second given side (angle B in this case) is greater than one. In logarithmic solutions the condition for no solution is when the log sin of the angle is greater than zero; this corresponds to a sine greater than one. Reference to table 5-1 shows that all of the values listed for log sin are less than zero (negative characteristic).

SOLUTION: Solve first for the unknown angle opposite a given side using the law of sines

$$\sin B = \frac{b \sin A}{a}$$

this in logarithmic form becomes

$$\log \sin B = \log b + \log \sin A - \log a$$

Evaluation of the logarithms gives

log b	= 10.6021 - 10	
log sin A	= 9.8081 - 10	add
log (b x sin A)	= 20.4102 - 20	

log (b x sin A)	= 20.4102 - 20	
log a	= 10.4771 - 10	subtract
log sin B	= 9.9331 - 10	

We note that log sin B is less than 0, so there are two angles to consider, say B and B', where $\sin(180° - B) = \sin B'$.

Now

$$\log \sin B = 9.9331 - 10$$

$$B = 60°$$

$$B' = 120°$$

Then the corresponding angles, C and C', are

$$C = 180° - (A + B)$$

$$C = 180° - (40° + 60°)$$

$$C = 80°$$

and

$$C' = 180° - (A + B')$$

$$C' = 180° - (40° + 120°)$$

$$C' = 20°$$

To complete the solution find sides c and c'. Consider first side c and use the following form of the law of sines

$$c = \frac{a \sin C}{\sin A}$$

or, in logarithmic form,

$$\log c = \log a + \log \sin C - \log \sin A$$

Evaluate the logarithms

$$\begin{array}{lrl} \log a & = & 10.4771 - 10 \\ \log \sin C & = & \underline{9.9934 - 10} \quad \text{add} \\ \log (a \times \sin C) & = & 20.4705 - 20 \\ \log \sin A & = & \underline{9.8081 - 10} \quad \text{subtract} \\ \log c & = & 10.6624 - 10 \end{array}$$

Taking the antilog,

$$c = 4.6$$

Finally, to find c' use

$$c' = \frac{a \sin C'}{\sin A}$$

$$\log c' = \log a + \log \sin C' - \log \sin A$$

$$\begin{array}{lrl} \log a & = & 10.4771 - 10 \\ \log \sin C' & = & \underline{9.5341 - 10} \quad \text{add} \\ \log (a \times \sin C') & = & 20.0112 - 20 \\ \log (a \times \sin C') & = & \underline{10.0112 - 10} \quad \text{subtract} \\ \log \sin A & = & 9.8081 - 10 \\ \log c' & = & 0.2031 \end{array}$$

Therefore,

$$c' = 1.6$$

PRACTICE PROBLEMS: Use logarithms to solve the triangle (or triangles) described by the following data.

1. $A = 70°$, $B = 100°$, $c = 50$
2. $A = 60°$, $a = 11$, $b = 18$
3. $A = 40°$, $a = 25$, $b = 30$

ANSWERS:

1. $C = 10°$, $a = 271$, $b = 284$
2. No solution
3. $B = 50°$, $C = 90°$, $c = 39.8$
4. $B' = 130°$, $C' = 10°$, $c = 6.75$

LAW OF TANGENTS

The law of cosines does not lend itself to logarithmetic solutions. The two cases in which we used the law of cosines are solved by logarithms using two different methods. The first method to solve triangles in case 3, where two sides and the included angle are given.

The law of tangents is expressed in words as follows:

In any triangle the difference between two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

For any pair of sides—such as side a and side b—the law may be expressed as follows:

$$\frac{a - b}{a + b} = \frac{\tan 1/2\,(A - B)}{\tan 1/2\,(A + B)}$$

The law may be expressed in a form that includes other combinations of sides and angles by systematically changing the letters in the formula.

In solving case 3 by the law of tangents, select the formula which includes the given sides, say a and b; then angle C is also given. The sum of the unknown angles, A + B, is found as

$$A + B = 180° - C$$

and the law of tangents is used to find A - B. After the sum and difference of A and B are determined, the angles themselves can be found. With the angles known, the law of sines is used to find the unknown side.

EXAMPLE: Solve the triangle ABC when $A = 25°$, $b = 10$, $c = 7$.

SOLUTION: With b and c given and $b > c$ use the law of tangents in the form

$$\frac{b - c}{b + c} = \frac{\tan 1/2\,(B - C)}{\tan 1/2\,(B + C)}$$

First, determine the sum of the unknown angles

$$B + C = 180° - A$$

$$B + C = 180° - 25°$$

$$B + C = 155°$$

Then $$1/2(B + C) = 1/2(155°)$$

$$1/2(B + C) = 77.5°$$

and

$$b - c = 10 - 7 = 3$$

$$b + c = 10 + 7 = 17$$

At this point, the only unknown in the formula for the law of tangents is tan 1/2(B - C). The next step is to transpose the formula and solve for this unknown.

$$\frac{b - c}{b + c} = \frac{\tan 1/2(B - C)}{\tan 1/2(B + C)}$$

$$(b - c)(\tan 1/2(B + C)) = (\tan 1/2(B - C))(b + c)$$

$$\frac{(b - c)(\tan 1/2(B + C))}{b + c} = \tan 1/2(B - C)$$

$$\tan 1/2(B - C) = \frac{3 \tan 77.5^\circ}{17}$$

Rounding the angle to 80° for use with the given table, the following logarithmic equation can be written:

$$\log \tan 1/2(B - C) = \log 3 + \log \tan 80^\circ - \log 17$$

then

$$\begin{aligned} \log 3 &= 10.4771 - 10 \\ \log \tan 80^\circ &= \underline{10.7537 - 10} \quad (+) \\ \log (3 \times \tan 80^\circ) &= 21.2308 - 20 \\ \log 17 &= \underline{11.2304 - 10} \quad (-) \\ \log \tan 1/2(B - C) &= 10.0004 - 10 \\ \log \tan 1/2(B - C) &= 0.0004 \\ 1/2(B - C) &= 45^\circ \\ B - C &= 90^\circ \end{aligned}$$

There are now two equations for B and C, (B + C) and (B - C); these are solved simultaneously to find B and C.

First the two are added to find B

$$\begin{aligned} B + C &= 155^\circ \\ \underline{B - C} &= \underline{90^\circ} \\ 2B &= 240^\circ \\ B &= 120^\circ \end{aligned}$$

Next, subtract the two to find C

$$\begin{aligned} B + C &= 155^\circ \\ \underline{-B + C} &= \underline{-90^\circ} \\ 2C &= 65^\circ \\ C &= 32.5^\circ \end{aligned}$$

To fit the given table round C to 35°, then

$$A + B + C = 25^\circ + 120^\circ + 35^\circ = 180^\circ$$

In the final part of the solution, use the law of sines to find side a.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{b \sin A}{\sin B}$$

$$\log a = \log b + \log \sin A - \log \sin B$$

$$\log a = \log 10 + \log \sin 25^\circ - \log \sin 120^\circ$$

$$\begin{aligned} \log 10 &= 11.0000 - 10 \\ \log \sin 25^\circ &= \underline{9.6260 - 10} \quad (+) \\ \log (b \sin A) &= 20.6260 - 20 \\ \log \sin B &= \underline{9.9375 - 10} \quad (-) \\ \log a &= 10.6885 - 10 \\ \log a &= 0.6885 \\ a &= 4.88 \end{aligned}$$

There are six forms of the law of tangents, two of which have been shown. The remaining four are formed, as previously stated, by replacing the corresponding letters in the formula.

PRACTICE PROBLEMS: Use logarithms and the law of tangents to solve the triangles described by the given data.

1. $a = 4$, $b = 3$, $C = 60°$
2. $a = 0.0316$, $b = 0.0132$, $C = 50°$

ANSWERS:

1. $A = 70°$, $B = 50°$, $c = 3.69$
2. $A = 105°$, $B = 25°$, $c = 0.0239$

CASE 2

The logarithmic solution of oblique triangles when three sides are given involves formulas derived from the law of cosines. These formulas are called half-angle formulas and, in these solutions, are expressed in terms of the semiperimeter of the triangle and the radius of a circle inscribed in the triangle.

The half-angle formulas expressed in s (semiperimeter) and r (radius of inscribed circles) are not presented in this course. The logarithmic solutions of oblique triangles in this course are limited to cases 1, 3, and 4.

AREA FORMULA

In this section two formulas for finding the area of oblique triangles are given. These formulas are used to find the area of triangles in cases 1 and 3 from the given parts.

Recall from plane geometry that the area of a triangle is found by the formula

$$A = \frac{1}{2}bh$$

where b is any side of the triangle and h is the altitude drawn to that side. To avoid confusion between A for area and A as an angle in the triangle the word "area" will be used in this chapter. Then the formula is stated as

$$\text{area} = \frac{1}{2}\,bh$$

Reference to figure 5-3 (B) shows that the length of the altitude h can be found by

$$h = c \sin A$$

Substituting this value of h in the area formula results in

$$\text{area} = \frac{1}{2}\,bc \sin A$$

$$\text{area} = \frac{bc \sin A}{2}$$

This area formula is stated in words as follows: the area of a triangle is equal to one-half the product of two sides and the sine of the angle between them. This formula is used for solutions of triangles when two sides and the included angle are given.

A second formula for area can be derived from the law of sines and the previous formula. From the law of sines

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{c \sin B}{\sin C}$$

Substituting this value of b in the area formula

$$\text{area} = \frac{bc \sin A}{2}$$

results in

$$\text{area} = \left(\frac{c \sin B}{\sin C}\right)\left(\frac{c \sin A}{2}\right)$$

$$\text{area} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

This formula can be used to solve triangles when two angles and one side are given since, when two angles are given, the third angle can be found directly. It can be seen that the area formulas can be easily adapted to logarithmic solutions, as well as to normal solutions.

PRACTICE PROBLEMS: Derive the area formulas most applicable when the following are given.

1. a, b, C
2. a, c, B
3. A, C, b
4. B, C, a

ANSWERS:

1. $\text{area} = \frac{ab \sin C}{2}$

2. $\text{area} = \dfrac{ac \sin B}{2}$

3. $\text{area} = \dfrac{b^2 \sin A \sin C}{2 \sin B}$

4. $\text{area} = \dfrac{a^2 \sin B \sin C}{2 \sin A}$

EXAMPLE: Find the area of triangle ABC when $A = 40°$, $b = 13$, and $c = 9$.

SOLUTION: Use the first area formula given in this section:

$$\text{area} = \frac{bc \sin A}{2}$$

$$\text{area} = \frac{13 \times 9 \times \sin 40°}{2}$$

$$\text{area} = \frac{13 \times 9 \times 0.64279}{2}$$

$$\text{area} = 37.6 \text{ (square units)}$$

This formula (as well as the other area formula) adapts easily to logarithmic solutions. In a logarithmic solution the formula

$$\text{area} = \frac{bc \sin A}{2}$$

is written

$$\log \text{area} = \log b + \log c + \log \sin A - \log 2$$

EXAMPLE: Find the area of triangle ABC when $A = 25°$, $B = 105°$, $c = 12$.

SOLUTION: First determine angle C.

$$C = 180° - (A + B)$$

$$C = 180° - 130°$$

$$C = 50°$$

then apply the area formula

$$\text{area} = \frac{C^2 \sin A \sin B}{2 \sin C}$$

$$\text{area} = \frac{(12)^2 \sin 25° \sin 105°}{2 \sin 50°} \qquad (1)$$

$$\text{area} = \frac{144 \times 0.42262 \times 0.96593}{2 \times 0.76604}$$

With the specific values involved in this problem, a logarithmic solution should simplify the arithmetical process. To write the logarithmic equation go back to equation (1). Recall that the logarithm of 12^2 is 2 log 12 and that $\sin (180° - 105°) = \sin 75°$ and write the equation as

$$\log \text{area} = 2 \log 12 + \log \sin 25° + \log \sin 75° - (\log 2 + \log 50°)$$

$$\begin{aligned} 2 \log 12 &= 2.1584 \\ \log \sin 25° &= 9.6260 - 10 \\ \log \sin 75° &= \underline{9.9849 - 10} \quad \text{add} \\ \log (c^2 \sin A \sin B) &= 21.7693 - 20 \\ \log (c^2 \sin A \sin B) &= 11.7693 - 10 \end{aligned}$$

and

$$\begin{aligned} \log 2 &= 0.3010 \\ \log \sin 50° &= \underline{9.8843 - 10} \quad \text{add} \\ \log (2 \sin C) &= 10.1853 - 10 \end{aligned}$$

subtracting these sums

$$\begin{aligned} \log (c^2 \sin A \sin B) &= 11.7693 - 10 \\ \log (2 \sin C) &= \underline{10.1853 - 10} \quad \text{subtract} \\ \log \text{area} &= 1.5840 \\ \text{area} &= 38.37 \end{aligned}$$

PRACTICE PROBLEMS: Find the area of the triangles described by the given data.

1. Find the area of the triangle given in the second example problem ($A = 25°$, $B = 105°$, $c = 12$), without using logarithms.

2. A = 25°, B = 45°, c = 24

3. A = 42°, b = 4.4, c = 3

4. A = 120°, b = 8, c = 12

ANSWERS
1. 38.369
2. 91.6
3. 4.4
4. 41.5

CHAPTER 6

VECTORS

Vector quantities, which we will now concern ourselves with, are different from scalar quantities. Navigation involves the use of vector quantities, surveyors use vectors in their work, as do structural engineers, and electrical and electronic technicians. Many of the applications of electricity and electronics involve the use of vector quantities.

DEFINITIONS AND TERMS

In this section we will make a distinction between a scalar quantity and a vector. We will also define the coordinate systems used in working with vectors and show some of the symbols used.

SCALARS

Heretofore, we have been concerned with scalar quantities, which are measurements or quantities having only magnitude, in the appropriate units. Examples of scalar quantities are: 10 pounds, 4 miles, 17 feet, and 28.2 pounds per square inch.

VECTORS

A vector, in contrast to a scalar, has direction as well as magnitude. Examples of vector quantities are: 6 miles due north, 9 blocks toward the west, and 250 knots at 30°. Notice that the vector quantities have both a magnitude and a direction.

SYMBOLS

The letters A, B, C, and D have been previously used to represent scalar quantities in algebra. In vector algebra, a notation is used to denote scalar symbols in relation to vector symbols. A dash over the letters, for example $\bar{A}$ and $\bar{B}$, denotes vectors.

A vector can conveniently be represented by a straight line. The length of this straight line represents the magnitude, and its position in space represents the direction of the vector quantity. In figure 6-1 the vectors $\bar{A}$, $\bar{B}$, and $\bar{C}$ are equal because they have the same magnitude and direction and vector $\bar{D}$ is not equal to either vector $\bar{A}$, $\bar{B}$, or $\bar{C}$ because although it has the same magnitude it does not have the same direction.

In navigational problems, a coordinate system is used in which the compass points serve as indicators of direction, and magnitude is given by lengths of the lines. For example, using the origin of the coordinate system as the point of departure from the harbor, figure 6-2 represents two ships heading out to sea. Vector $\bar{A}$ represents a ship bearing 45° from due north at a speed of 20 knots, and vector $\bar{B}$ represents a ship bearing 60° from due north at a speed of 25 knots. Notice that directions in this coordinate system are measured clockwise from due north.

When we use the trigonometric system in designating angles or giving a direction to a magnitude we use the Cartesian coordinate system which includes the abscissa (measurement on the X axis) and the ordinate (measurement on the Y axis). Directions in this coordinate system are measured counterclockwise from the X axis. For example, using the origin of the coordinate system as the point of departure from the harbor, figure 6-3 represents a ship heading out to sea bearing 30° at a speed of 25 knots. This is represented by vector $\bar{B}$ and is the same representation as vector $\bar{B}$ in figure 6-2 but is shown on a different coordinate system.

Angle measurements will be referenced from the vertical in the compass coordinate system and will be referenced from the horizontal in the Cartesian coordinate system.

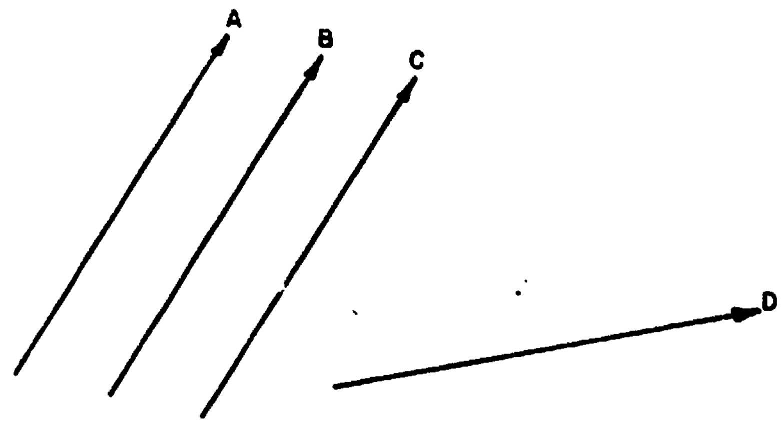


Figure 6-1.—Comparison of vectors.

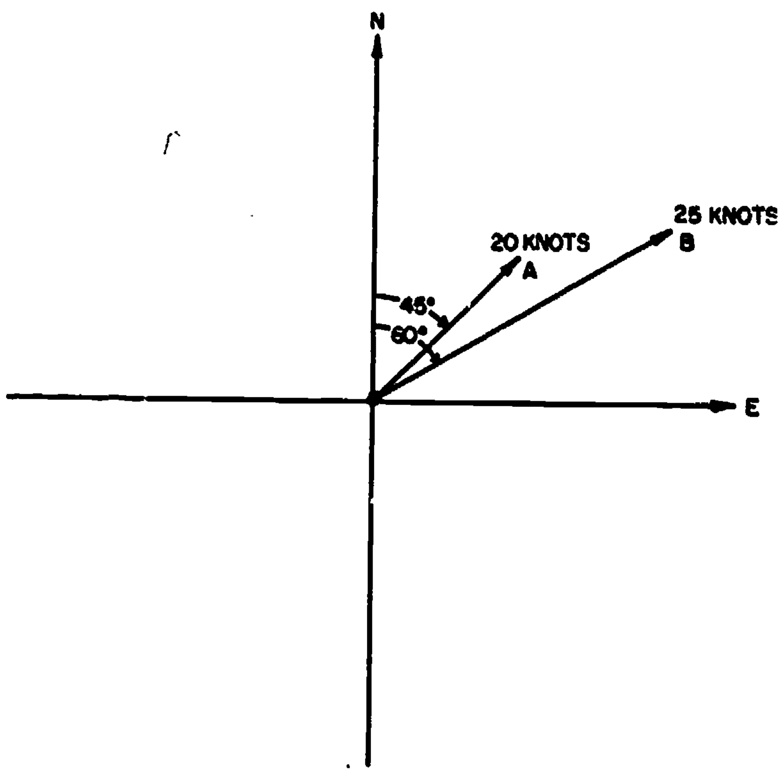


Figure 6-2.—Compass coordinate system.

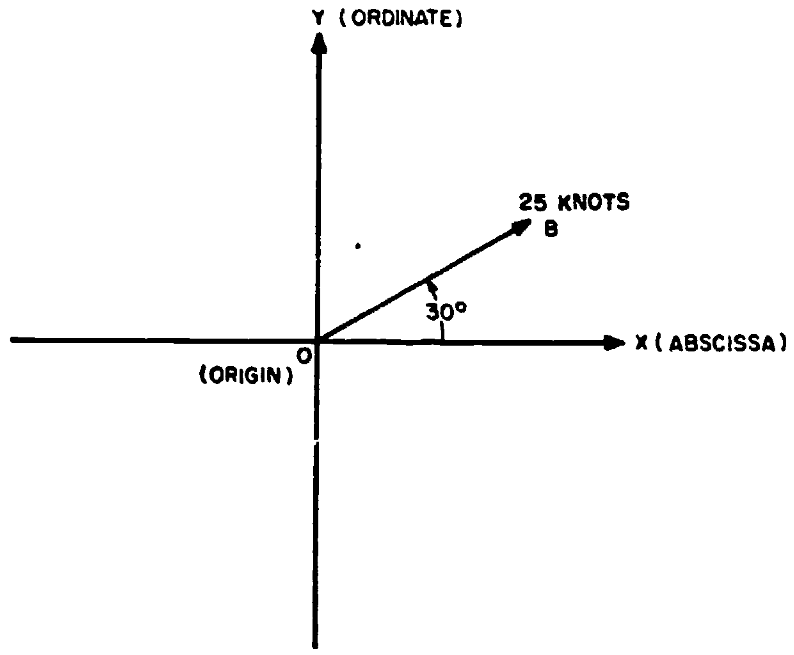


Figure 6-3.—Cartesian coordinate system.

COMBINING VECTORS

If a vector $\overline{A}$ represents the displacement of a particle or the force acting on the particle, it is convenient to let $-\overline{A}$ represent the displacement of a particle in the opposite direction or to represent a force in the direction opposite to $\overline{A}$. Thus, vectors $\overline{A}$ and $-\overline{A}$ are equal in magnitude but are opposite in direction.

ADDITION

The resultant of two vectors acting in the same direction or acting in opposite directions is the algebraic sum of their magnitudes. An example of this is walking due east four steps and then walking due west one step. The resultant is three steps due east. If one travels from his home to his place of employment he may have several choices for his route. We will assume two of these routes as indicated in figure 6-4. He may move east to point A then north to point B or he may move north to point C then east to point B. In either case he arrives at his place of employment. If we assume his travel in both directions as forces acting on him, we can call the direct distance from home to place of employment, in figure 6-4, the RESULTANT of these two forces and refer to it as vector $\overline{R}$. Notice that either path taken results in vector $\overline{R}$.

We may now state that $\overline{OA} \rightarrow\!\!\!+ \overline{AB} = \overline{OC} \rightarrow\!\!\!+ \overline{CB} = \overline{R}$. The symbol ($\rightarrow\!\!\!+$) is used to indicate that vector $\overline{OA}$ is added vectorially to vector $\overline{AB}$. From this it is apparent that vectors may be added in either order with the same results.

If several vectors $\overline{A}$, $\overline{B}$, and $\overline{R}$ are to be resolved into components, $\overline{X}_r$, $\overline{X}_a$, $\overline{X}_b$, and $\overline{Y}_r$, $\overline{Y}_a$, $\overline{Y}_b$ are used to denote these components, as figure 6-5 portrays.

In the addition of vectors, the initial point of vector $\overline{B}$ must be placed directly on the

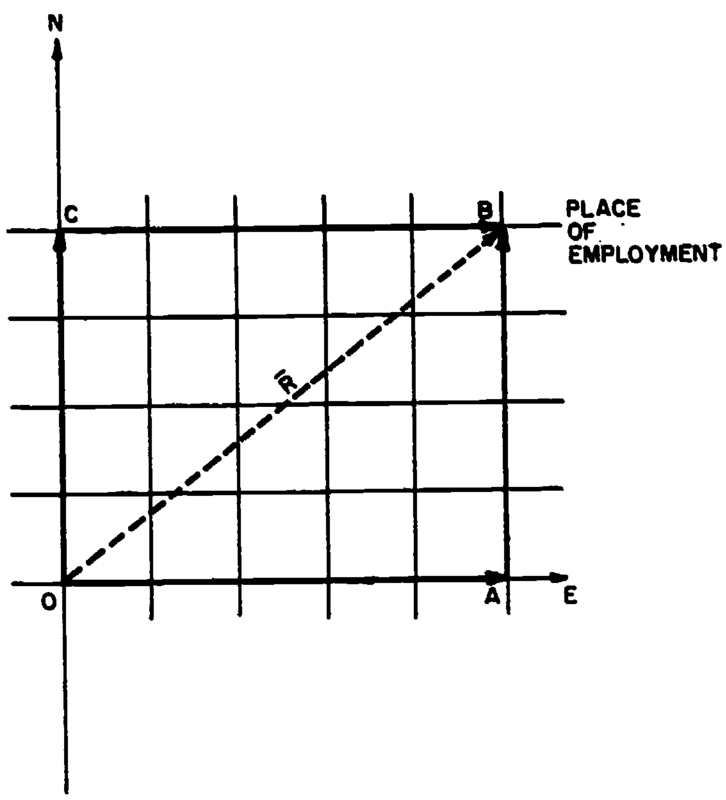


Figure 6-4.—Right angle vectors.

terminal point of vector $\overline{A}$, and so on for any number of vectors. Then vector $\overline{R}$, which joins the initial point of vector $\overline{A}$ with the terminal point of the last vector $\overline{N}$, is the result of adding vectors $\overline{A}$ and $\overline{B}$ and $\overline{C}$ through $\overline{N}$ vectorially. Hence $\overline{A} +\!\!\!\!\rightarrow \overline{B} +\!\!\!\!\rightarrow \overline{C} +\!\!\!\!\rightarrow \overline{D} +\!\!\!\!\rightarrow \ldots +\!\!\!\!\rightarrow \overline{N} = \overline{R}$. Here it may be shown that the commutative and associative principles apply, which means that it makes no difference which vector is used first and which order is followed when adding vectors. (The commutative and associative laws may be reviewed in Mathematics, Vol. 1, NavPers 10069-C.)

SUBTRACTION

Subtracting a vector is defined as adding a negative vector:

$$\overline{A} - \overline{B} = \overline{A} +\!\!\!\!\rightarrow (-\overline{B})$$

It follows that if

$$\overline{A} +\!\!\!\!\rightarrow \overline{B} = 0$$

Then

$$\overline{A} = -\overline{B}$$

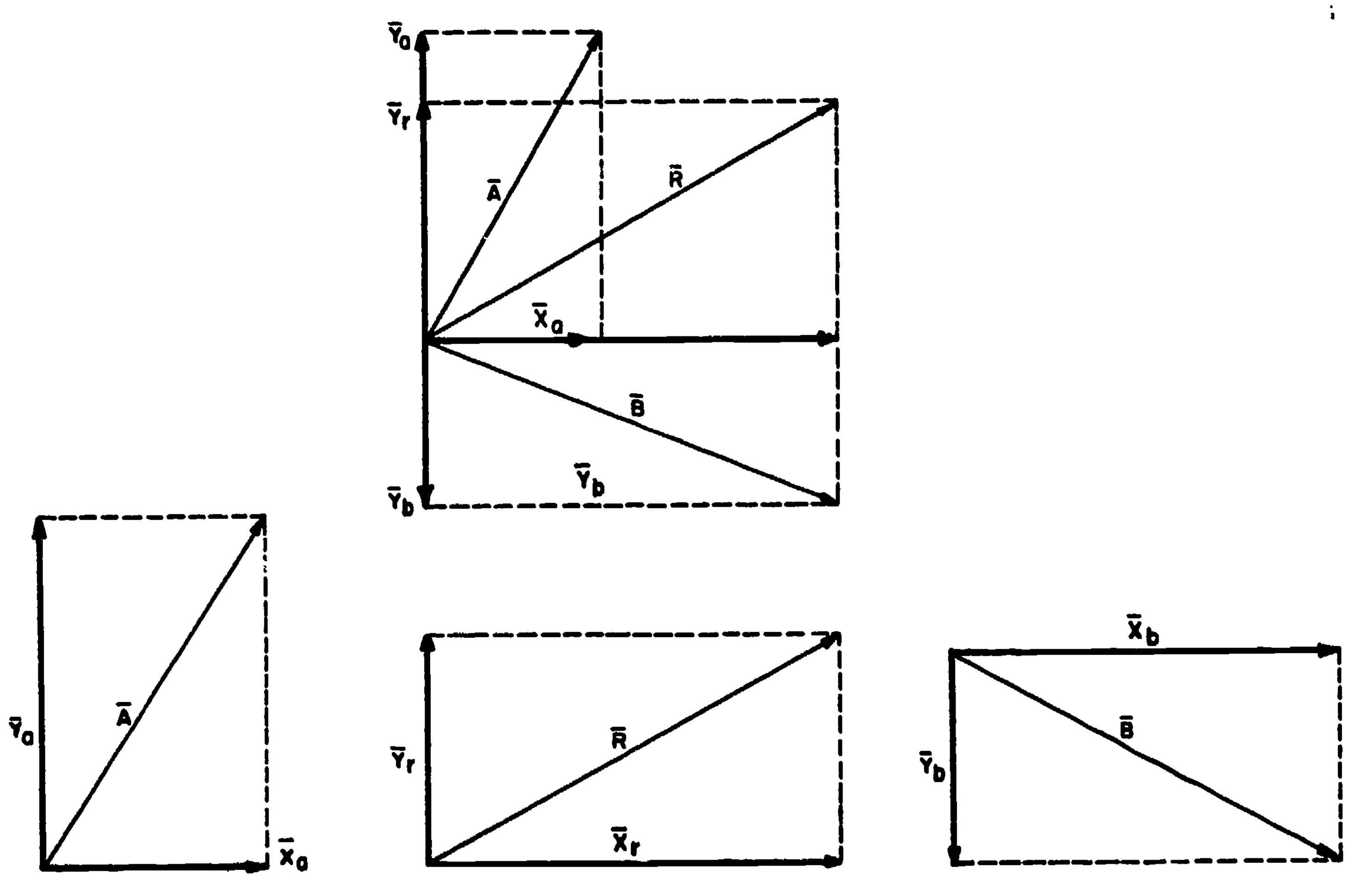


Figure 6-5.—Components of vectors.

A careful study of figure 6-4 will show that if

$$\overline{OA} \rightarrow\!\!\!\!\!+\; \overline{AB} = \overline{R}$$

Then

$$\overline{R} \rightarrow\!\!\!\!\!+\; (-\overline{AB}) = \overline{R} - \overline{AB} = \overline{OA}$$

VECTOR SOLUTIONS

In the previous sections we agreed that:

$$\overline{A} \rightarrow\!\!\!\!\!+\; \overline{B} \rightarrow\!\!\!\!\!+\; \overline{C} \rightarrow\!\!\!\!\!+\; \overline{D} \rightarrow\!\!\!\!\!+\; \ldots \rightarrow\!\!\!\!\!+\; \overline{N} = \overline{R}$$

Approaching this relation from a graphical standpoint, one can understand exactly what this means.

GRAPHICAL

As an example of the graphical method, six vectors may be used to represent the path taken by a man looking for a lost golf ball. He stands at position P_o, hits the ball and does not notice where the ball went. Vectors $\overline{A}$ through $\overline{F}$ in figure 6-6 represent the path he takes in an attempt to find the ball. P_f is the position where the ball is found and we will call it the termination point. Figure 6-7 shows another of the many different arrangements of the six vectors. The dotted lines from P_o to P_f indicate the resultant vector, and this resultant has the same magnitude and direction regardless of the arrangement of the vectors we use. This method of graphically solving vector problems is called the polygon method, and is used in civil engineering problems involving structures such as bridges; it is also used in logic problems of everyday living.

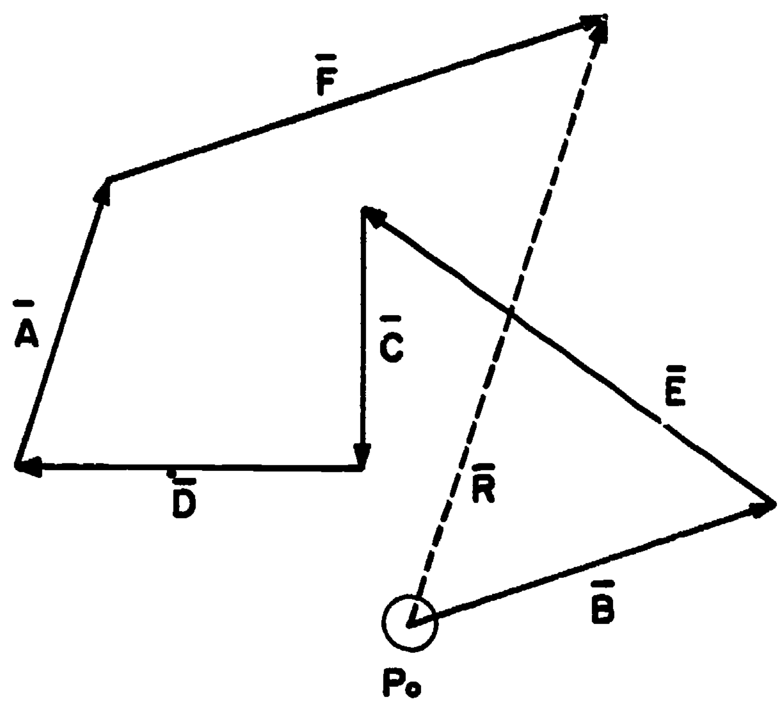


Figure 6-7.—Polygon example number 2.

If two vectors are to be resolved into a single resultant, this may be done graphically by the parallelogram method. Given any two vectors $\overline{A}$ and $\overline{B}$ lying in a plane (fig. 6-8) form a parallelogram by projecting $\overline{B}$ onto $\overline{A}$, initial point to terminal point, and $\overline{A}$ onto $\overline{B}$, initial point to terminal point, thus forming a parallelogram which has as a diagonal the resultant vector $\overline{R}$.

This process can be reversed in order to find the components of a vector as shown in figure 6-9. Vector $\overline{R}$ is given and the problem is to find the rectangular components of this

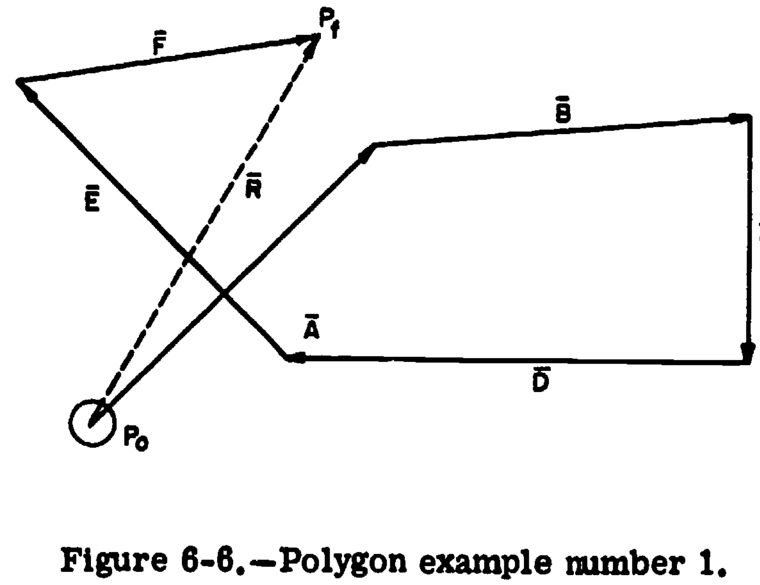


Figure 6-6.—Polygon example number 1.

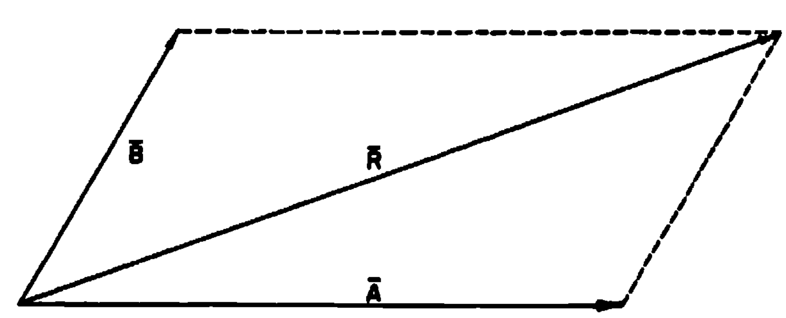


Figure 6-8.—Parallelogram example 1.

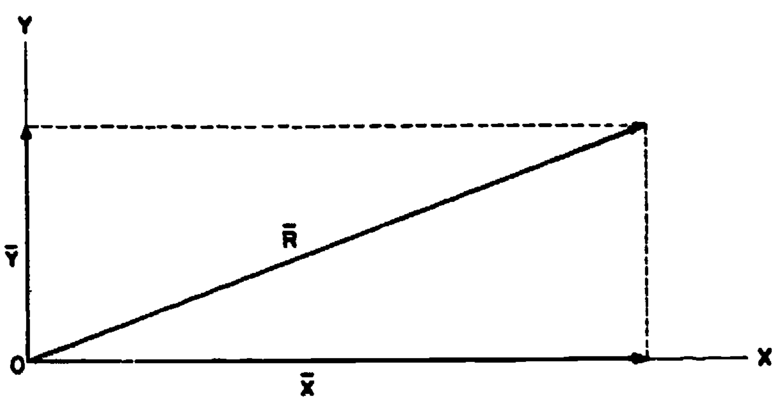


Figure 6-9.—Parallelogram example 2.

vector. In this case the parallelogram is a rectangle and the projections of $\bar{R}$ on the X axis and the Y axis show the components. Generally, the graphical method of resolving vectors will be used to check the validity of an analytical method of solution.

ANALYTICAL

The trigonometric functions are used to solve vector problems analytically.

EXAMPLE: Find the resultant of two vectors at right angles to each other. Vector $\bar{A}$ represents 90 pounds of force and vector $\bar{B}$ represents 60 pounds of force.

SOLUTION: Vector $\bar{A}$ is directed vertically and $\bar{B}$ lies on the reference line, as shown in figure 6-10. In this case the angle θ is unknown and the resultant is required. The Pythagorean theorem is sufficient to solve for the magnitude of the resultant. This is the case only if we establish a right triangle from our vector and its components.

Since the magnitude of $\bar{R}$ is a scalar quantity we will designate it by r. As you recall from Mathematics, Vol. 1, NavPers 10069-C, the Pythagorean theorem of right triangles states: $x^2 + y^2 = r^2$. We apply this to our figure and find that:

$$r^2 = (90)^2 + (60)^2$$

$$r = \sqrt{(90)^2 + (60)^2}$$

$$= \sqrt{11,700}$$

$$= 108.2 \text{ pounds}$$

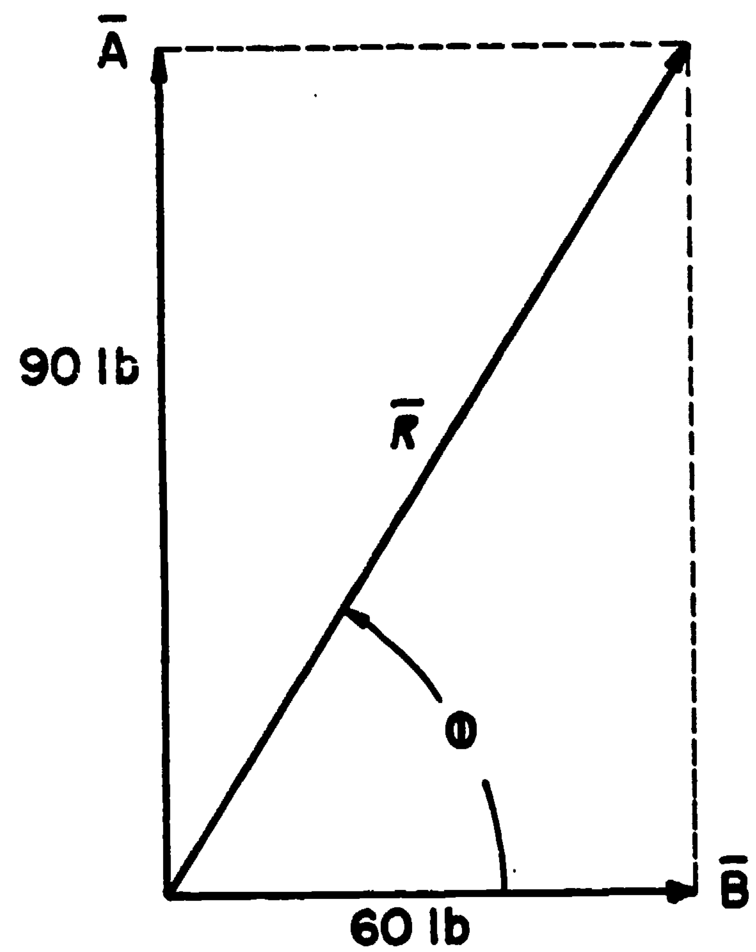


Figure 6-10.—Vector sum.

If we desire the angle θ of the resultant vector $\bar{R}$ we may use the trigonometric function for the tangent of an angle; that is,

$$\tan \theta = \frac{y}{x} = \frac{90}{60} = 1.50000$$

Then,

$$\theta = 56° \ 19'$$

EXAMPLE: Resolve the vector $\bar{R}$ into its components. In figure 6-11, $\bar{R}$ represents 50 mph at 30°. We are to find the $\bar{X}_r$ and $\bar{Y}_r$ components.

SOLUTION: Recalling that the trigonometric functions for sin θ and cos θ are:

$$\sin \theta = \frac{y}{r}$$

and

$$\cos \theta = \frac{x}{r}$$

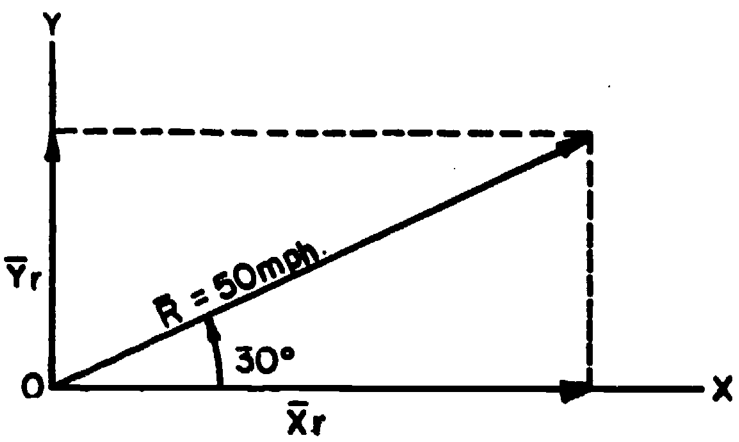


Figure 6-11.—Vector resolution.

we use these and find that

$$\sin 30° = \frac{y}{50}$$

then

$$50 \sin 30° = y$$
$$y = 25 \text{ mph}$$

and

$$\cos 30° = \frac{x}{50}$$

then

$$50 \cos 30° = x$$
$$x = 43.3 \text{ mph}$$

Vector components acting in the same direction or in opposite directions may be added or subtracted algebraically. Vector components in the form $\overline{R} = \overline{X}_r + \overline{Y}_r$ fulfill this statement.

EXAMPLE: Add the following vectors given by their rectangular components.

SOLUTION:

If

$$\overline{A} = \overline{5} + \overline{2}$$

$$\overline{B} = \overline{6} - \overline{1}$$

Then $$\overline{A} + \overline{B} = \overline{11} + \overline{1}$$

Observe that the first component of each pair is the X component, and the second is the Y component. If vector $\overline{A}$ is to be added to $\overline{B}$, the X component of the resultant is the sum of the X components of $\overline{A}$ and $\overline{B}$. The same reasoning applies to the Y components. Note especially that an X component is never added to a Y component or vice versa.

PRACTICE PROBLEMS: Add the following rectangular form vectors.

1. $\overline{15} - \overline{5}$ and $\overline{3} + \overline{2}$
2. $\overline{3.96} + \overline{2.87}$ and $\overline{1.21} + \overline{3.11}$
3. $\overline{9.3} + \overline{4.8}$ and $\overline{0.2} - \overline{3.1}$
4. $\overline{182} + \overline{312}$ and $\overline{76} - \overline{81}$

ANSWERS:

1. $\overline{18} - \overline{3}$
2. $\overline{5.17} + \overline{5.98}$
3. $\overline{9.5} + \overline{1.7}$
4. $\overline{258} + \overline{231}$

EXAMPLE: Subtract the following vectors given by their rectangular components.

If

$$\overline{A} = \overline{4.2} + \overline{3.1}$$

$$\overline{B} = \overline{8.1} + \overline{6.2}$$

Then subtract $\overline{A}$ from $\overline{B}$.

Thus

$$\overline{B} = \overline{8.1} + \overline{6.2}$$

$$\underline{\overline{A} = \overline{4.2} + \overline{3.1}}$$

$$\overline{B} - \overline{A} = \overline{3.9} + \overline{3.1}$$

PRACTICE PROBLEMS: Subtract the following rectangular form vectors.

1. Subtract $\overline{4.2} + \overline{3.1}$ from $\overline{8.1} + \overline{6.2}$
2. Subtract $\overline{57} + \overline{28}$ from $\overline{103} - \overline{35}$
3. Subtract $\overline{32.3} - \overline{8.3}$ from $\overline{15.3} + \overline{10.2}$
4. Subtract $\overline{-6.2} + \overline{2.9}$ from $\overline{-3.1} - \overline{2.6}$

ANSWERS:

1. $\overline{3.9} + \overline{3.1}$
2. $\overline{46} - \overline{63}$
3. $\overline{-17} + \overline{18.5}$
4. $\overline{3.1} - \overline{5.5}$

The notation up to this point has involved the regular rectangular coordinates. The form $\overline{R} = \overline{X}_r + \overline{Y}_r$ implies that a number of horizontal units and a number of vertical units combine to determine the end point of a vector. A second method commonly used describes a vector in terms of polar coordinates.

POLAR COORDINATES

If the length of the vector is known, all that is required to locate the vector is the angle through which it has been rotated. Measured from the reference line, the notation used is

$$\overline{R} = r \angle \theta$$

where r is the magnitude and θ defines the direction. Thus, r is a scalar quantity. For instance, a vector 10 units long at 30° would be written $10\angle 30°$.

If $\overline{X}_r$ and $\overline{Y}_r$ are known, the scalar quantity r can be found by using the Pythagorean theorem:

$$r = \sqrt{(x_r)^2 + (y_r)^2}$$

The angle can then be found by using the tangent, thus:

$$\tan \theta = \frac{y_r}{x_r}$$

Now we have a method whereby we can change from the rectangular form to the polar coordinate form when working with vectors.

EXAMPLE: Change the vector $\overline{3} + \overline{4}$ into polar form. (NOTE: x_r is always placed first.)

SOLUTION:

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\tan \theta = \frac{4}{3} = 1.33333$$

$$\theta = 53° \; 8'$$

If

$$\overline{R} = \overline{3} + \overline{4}$$

Then

$$\overline{R} = 5\angle 53° \; 8'$$

PRACTICE PROBLEMS: Change the rectangular form into polar form.

1. $\overline{1} + \overline{2}$
2. $\overline{8} + \overline{6}$
3. $\overline{\sqrt{3}} + \overline{1}$
4. $\overline{18.3} + \overline{2.8}$

ANSWERS:

1. $2.24\angle 63° \; 26'$
2. $10\angle 36° \; 52'$
3. $2\angle 30°$
4. $18.5\angle 8° \; 42'$

This method may be reversed and it is possible to change a vector from polar to rectangular coordinates.

EXAMPLE: Change the vector $30\angle 65°$ into rectangular form.

SOLUTION:

If

$$\overline{R} = 30\angle 65°$$

$$\overline{Y}_r = r \sin \theta$$

Then

$$\overline{Y}_r = 30 \sin 65°$$

$$= 27.18930$$

And

$$\overline{X}_r = 30 \cos 65°$$

$$\overline{X}_r = r \cos \theta$$

$$= 12.67860$$

Thus

$$\overline{R} = \overline{12.68} + \overline{27.19}$$

PRACTICE PROBLEMS: Change the polar form to rectangular form.

1. $5\angle 25°$
2. $83\angle 72°$
3. $20\angle 63°$
4. $8.2\angle 31° \; 23'$

ANSWERS:

1. $\overline{4.53} + \overline{2.11}$
2. $\overline{25.6} + \overline{78.9}$
3. $\overline{9.08} + \overline{17.82}$
4. $\overline{7.00} + \overline{4.27}$

If we are to combine two vectors, proceed as follows:

EXAMPLE: Find the resultant of two vectors $\overline{A}$ and $\overline{B}$ if $\overline{A} = 12\angle 102°$ and $\overline{B} = 5\angle 12°$ in figure 6-12.

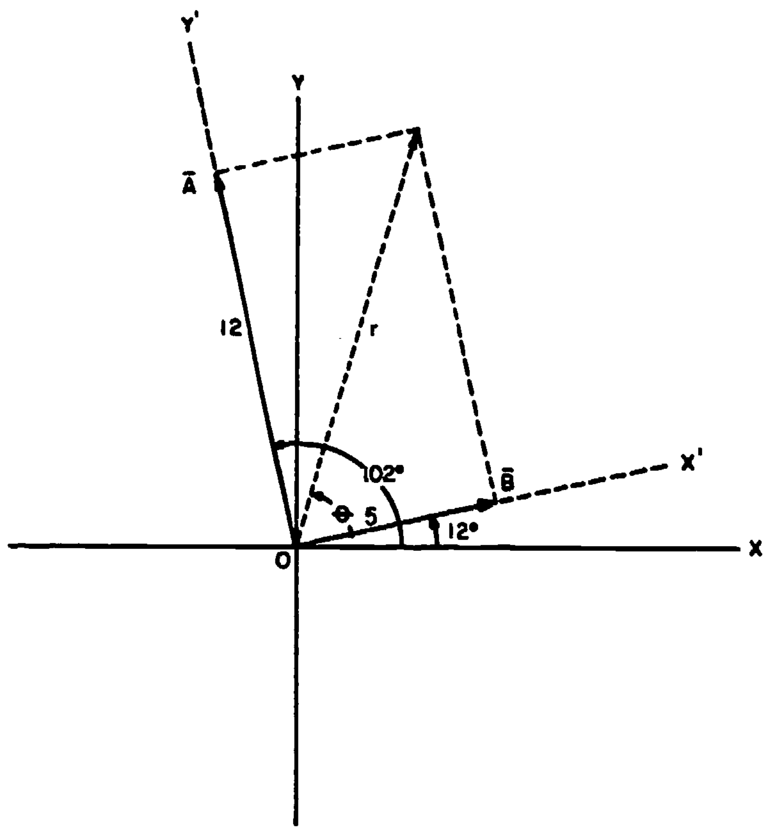


Figure 6-12.—A new reference.

SOLUTION: In this problem we do not have either vector to coincide with the X axis or the Y axis. We may choose a new frame of reference, that is, X' and Y' to determine the magnitude of the resultant. Because $\bar{A}$ differs in direction by 90° from $\bar{B}$, we may still use the properties of right triangles.

Therefore $r^2 = (12)^2 + (5)^2$

And $r = \sqrt{(12)^2 + (5)^2}$

$= \sqrt{169}$

$= 13$

We now have the magnitude of our resultant and need only to find its direction. As we are concerned with only two vectors we may approach this problem in either of two ways. We may find our direction from our new reference then add the angle our new reference makes with the standard X axis and Y axis reference. In the new frame of reference we find the resultant to be:

$$\tan \theta = \frac{y'_r}{x'_r} = \frac{12}{5} = 2.40000$$

Therefore

$$\theta = 67° \ 23'$$

Now, the direction of the resultant is 67° 23' from the X' axis but the X' axis is 12° from the X axis so the resultant is 67° 23' + 12° or 79° 23' from the X axis.

Another approach to this problem is by resolving each of our vectors into their X axis and Y axis components and then adding these components algebraically. We have $5\underline{/12°}$ which resolves into the following (fig. 6-13):

$$\sin \theta = \frac{y}{r}$$

$$= \frac{y}{5}$$

Therefore

$$\bar{y} = 5 \sin 12°$$

$$= 5(0.20791)$$

$$= 1.03955$$

And

$$\cos \theta = \frac{x}{r}$$

$$= \frac{x}{5}$$

Therefore

$$\bar{x} = 5 \cos 12°$$

$$= 5(0.97815)$$

$$= 4.89075$$

We have now determined the X axis and Y axis components of one of the vectors. We will proceed to find the components of the

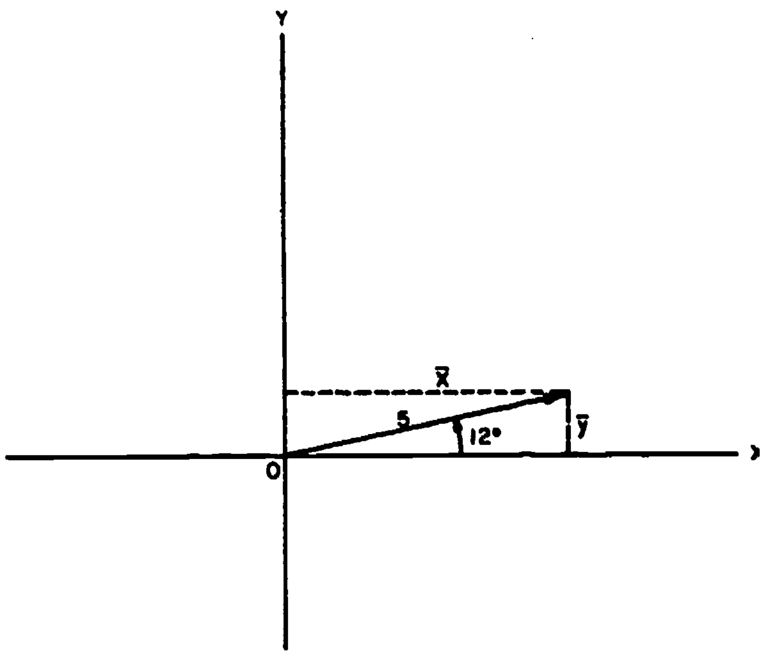


Figure 6-13.—Components for $5\underline{/12°}$.

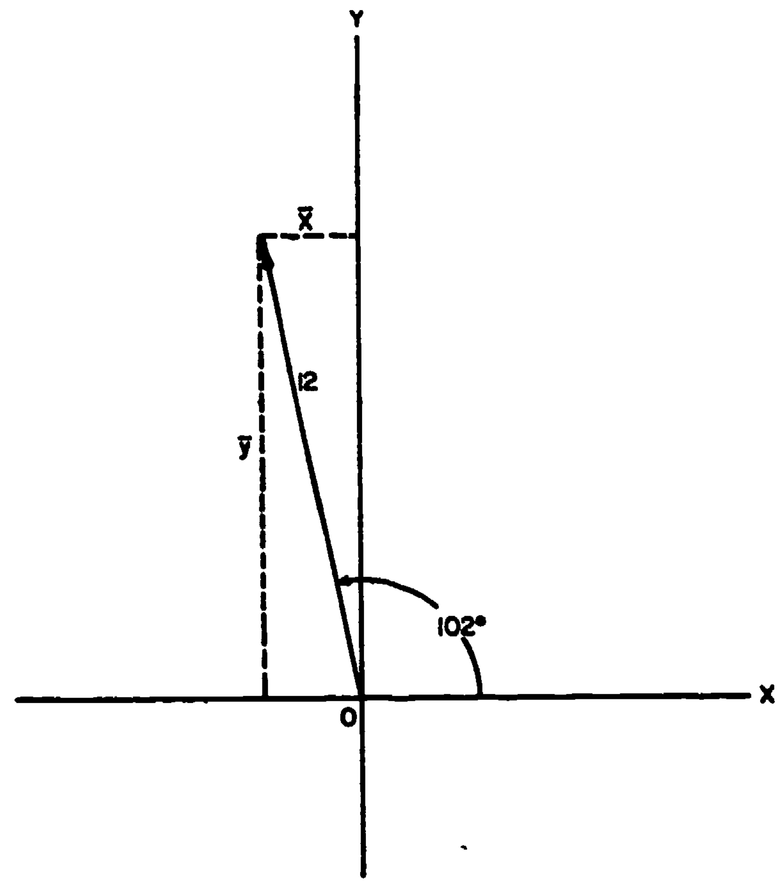


Figure 6-14.—Components for $12\underline{/102°}$.

other vector, $12\underline{/102°}$, as shown in figure 6-14. The Y axis component is as follows:

If

$$\sin\ \theta = \frac{y}{r}$$

$$\sin\ \theta = \frac{y}{12}$$

Then

$$\overline{Y} = 12\ \sin\ 102°$$

And, since

$$\sin\ (180° - \theta) = \sin\ \theta$$

Then

$$\sin\ 102° = \sin\ (180° - 78°)$$

$$= \sin\ 78°$$

Therefore

$$\overline{Y} = 12\ \sin\ 78°$$

$$= 12(0.97815)$$

$$= 11.73780$$

We now find the X axis components as follows:

If

$$\cos\ \theta = \frac{x}{r}$$

$$\cos\ \theta = \frac{x}{12}$$

Then

$$\overline{X} = 12\ \cos\ 102°$$

And, since

$$\cos\ (180° - \theta) = -\cos\ \theta$$

Then

$$\cos\ 102° = \cos\ (180° - 78°)$$

$$= -\cos\ 78°$$

Therefore

$$\overline{X} = 12(-0.20791)$$

$$= -2.49492$$

We must now add the X axis components and the Y axis components of the two vectors algebraically as follows:

$$\overline{Y} = 1.03955 + 11.73780$$

$$= 12.77735$$

And

$$\overline{X} = 4.89075 + (-2.49492)$$

$$= 2.39583$$

We now have $\overline{x}_r$ and $\overline{y}_r$ in rectangular form and may use the Pythagorean theorem to find the resultant in scalar measurement as shown in figure 6-15. This is as follows:

If

$$r^2 = (2.39583)^2 + (12.77735)^2$$

Then

$$r = \sqrt{5.74030 + 163.25173}$$

$$= \sqrt{168.99203}$$

$$= 13$$

This is in agreement with the result found by using the method of finding the scalar resultant of two vectors.

We must now find the direction of $\overline{R}$, as follows:

Since

$$\tan \theta = \frac{y}{x}$$

$$= \frac{12.77738}{2.39583}$$

$$= 5.33480$$

Therefore

$$\theta = 79° \ 23'$$

This direction agrees with the direction found when we used the first method of finding the direction of the resultant of two vectors.

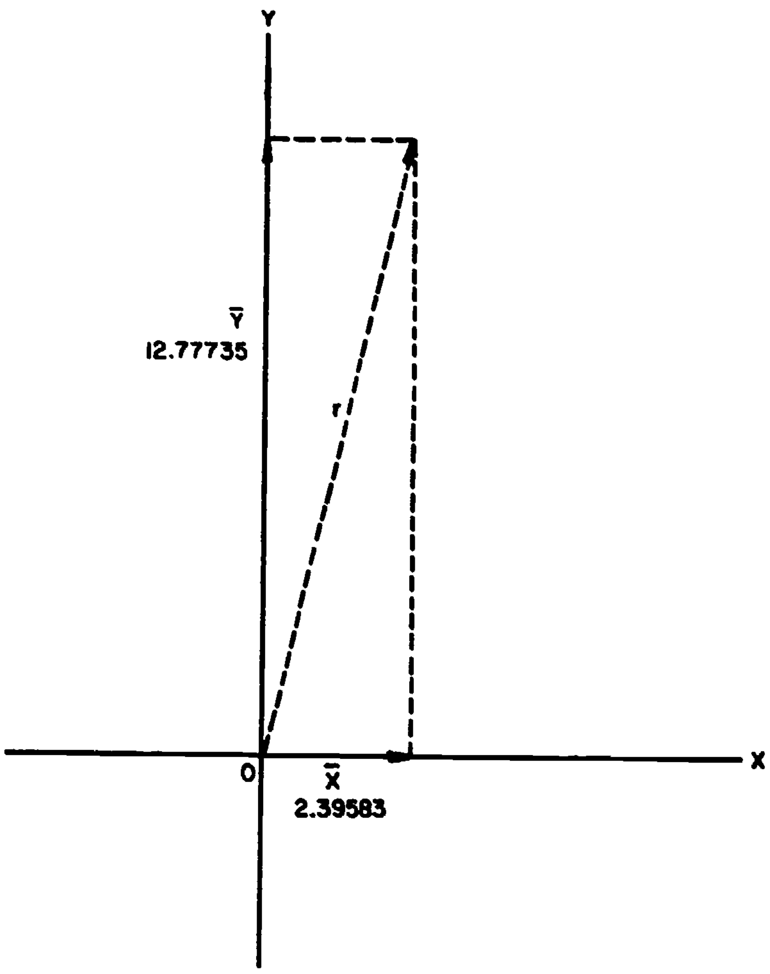


Figure 6-15.—Resultant of $\overline{X}$ and $\overline{Y}$.

PRACTICE PROBLEMS: Find the resultant of two vectors at right angles to each other.

1. $\overline{A} = 5\underline{/0°}$
   $\overline{B} = 10\underline{/90°}$
2. $\overline{A} = 7.5\underline{/90°}$
   $\overline{B} = 6.3\underline{/180°}$
3. $\overline{A} = 131\underline{/185°}$
   $\overline{B} = 60\underline{/275°}$
4. $\overline{A} = 65\underline{/45°}$
   $\overline{B} = 120\underline{/135°}$

ANSWERS:

1. $11.18\underline{/63° \ 26'}$
2. $9.8\underline{/130° \ 2'}$
3. $144.1\underline{/209° \ 36'}$
4. $136.5\underline{/106° \ 36'}$

Let us examine a problem of adding several vectors. We will use the method last described. The method may be used to find the addition of any number of vectors. We will consider a problem of the addition of several vectors, as follows:

EXAMPLE: Find the resultant of the vectors in figure 6-16, analytically.

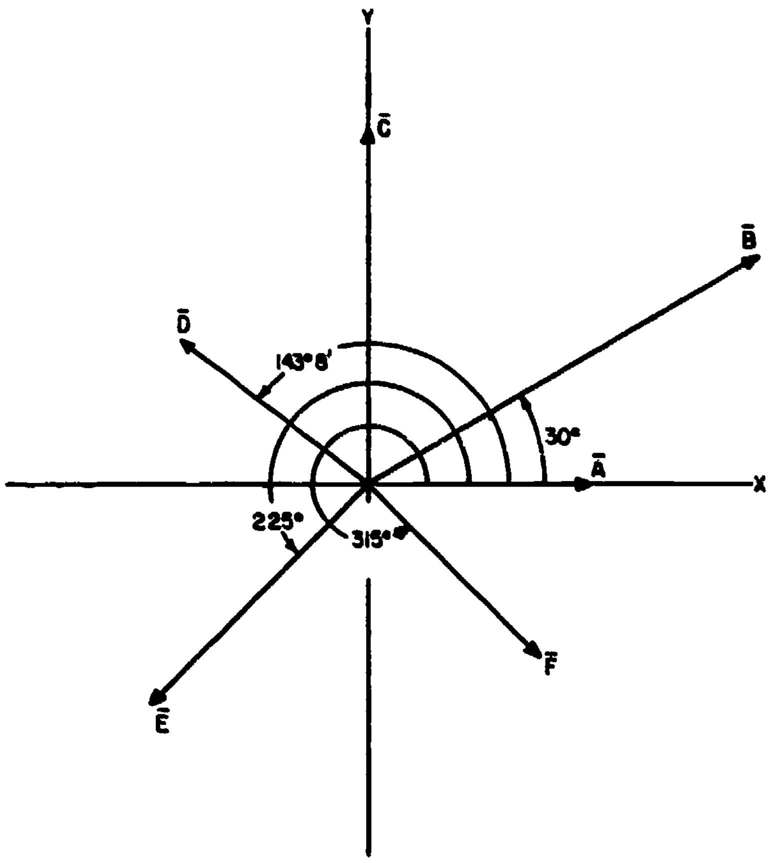


Figure 6-16.—Resultant of several vectors.

SOLUTION: The vectors are given as follows:

$\bar{A}$ is $50\underline{/0^\circ}$

$\bar{B}$ is $100\underline{/30^\circ}$

$\bar{C}$ is $75\underline{/90^\circ}$

$\bar{D}$ is $50\underline{/143^\circ\ 8'}$

$\bar{E}$ is $70.7\underline{/225^\circ}$

$\bar{F}$ is $55\underline{/315^\circ}$

We will use the method of resolving each vector into its component X axis and Y axis coordinates. We set up the coordinate system and place each vector so that it radiates from the origin. Then we find

$$\begin{aligned}\bar{x}_a &= 50 \cos 0^\circ \\ &= 50(1) \\ &= 50\end{aligned}$$

$$\begin{aligned}\bar{y}_a &= 50 \sin 0^\circ \\ &= 50(0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\bar{x}_b &= 100 \cos 30^\circ \\ &= 100(0.86603) \\ &= 86.6\end{aligned}$$

$$\begin{aligned}\bar{y}_b &= 100 \sin 30^\circ \\ &= 100(0.50000) \\ &= 50\end{aligned}$$

$$\begin{aligned}\bar{x}_c &= 75 \cos 90^\circ \\ &= 75(0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\bar{y}_c &= 75 \sin 90^\circ \\ &= 75(1) \\ &= 75\end{aligned}$$

$$\begin{aligned}\bar{x}_d &= 50 \cos 143^\circ\ 8' \\ &= 50(-\cos 36^\circ\ 52') \\ &= 50(-0.80003) \\ &= -\ 40.002\end{aligned}$$

$$\begin{aligned}\bar{y}_d &= 50 \sin 143^\circ\ 8' \\ &= 50(\sin 36^\circ\ 52') \\ &= 50(0.59995) \\ &= 30\end{aligned}$$

$$\begin{aligned}\bar{x}_e &= 70.7 \cos 225^\circ \\ &= 70.7\ (-\cos 45^\circ) \\ &= 70.7\ (-0.70711) \\ &= -\ 50\end{aligned}$$

$$\begin{aligned}\bar{y}_e &= 70.7 \sin 225^\circ \\ &= 70.7(-\sin 45^\circ) \\ &= 70.7(-0.77011) \\ &= -\ 50\end{aligned}$$

$$\begin{aligned}\bar{x}_f &= 55 \cos 315^\circ \\ &= 55\ (\cos 45^\circ) \\ &= 55\ (0.77011) \\ &= 38.9\end{aligned}$$

$$\begin{aligned}\bar{y}_f &= 55 \sin 315^\circ \\ &= 55(-\sin 45^\circ) \\ &= 55(-0.77011) \\ &= -\ 38.9\end{aligned}$$

We now collect the X axis components and the Y axis components, as follows:

Vector	$\bar{X}$	$\bar{Y}$
$\bar{A}$	50	0
$\bar{B}$	86.6	50
$\bar{C}$	0	75
$\bar{D}$	- 40	30
$\bar{E}$	- 50	- 50
$\bar{F}$	38.9	- 38.9

Adding the X axis components and the Y axis components, we find the magnitudes of $\bar{X}$ and $\bar{Y}$ as follows:

$$\bar{x}_r = 85.5$$

$$\bar{y}_r = 66.1$$

The magnitude of the resultant $\bar{R}$ is

$$r = \sqrt{(85.5)^2 + (66.1)^2}$$

$$= \sqrt{11680.67}$$

$$= 108$$

The direction is given by using the tangent function, as follows:

$$\tan \theta = \frac{66.1}{85.5}$$

$$= 0.77309$$

Therefore

$$\theta = 37° \ 42'$$

MULTIPLICATION

Before we discuss the mechanics of multiplication and division of vectors, in polar form, we will multiply and divide vectors in rectangular form. This will serve as an intuitive explanation of why the mechanics of polar form multiplication and division may be used.

As explained in Mathematics, Vol. 1, NavPers 10069-C, we may express the following rectangular form vectors as complex numbers, as follows:

If

$$\bar{R}_1 = \bar{3} + \bar{4}$$

$$\bar{R}_2 = \bar{8} + \bar{5}$$

Then

$$\bar{R}_1 = 3 + 4i$$

$$\bar{R}_2 = 8 + 5i$$

And

$$\begin{array}{rl} (\bar{R}_1)(\bar{R}_2) = & 3 + 4i \\ & \underline{8 + 5i} \\ & 24 + 32i \\ & \underline{\quad + 15i + 20i^2} \\ & 24 + 47i + 20i^2 \\ = & 24 + 47i + 20(-1) \\ = & 4 + 47i \end{array}$$

Thus

$$(\bar{R}_1)(\bar{R}_2) = \bar{4} + \overline{47}$$

We now find, as shown previously, the polar form of $\bar{4} + \overline{47}$ which is as follows:

$$r = \sqrt{(4)^2 + (47)^2}$$

$$= \sqrt{2225}$$

$$= 47.2$$

And

$$\tan \theta = \frac{47}{4} = 11.75000$$

$$\theta = 85° \ 8'$$

Then

$$(\bar{R}_1)(\bar{R}_2) = 47.2 \underline{/85° \ 8'}$$

In multiplying vectors $\bar{R}_1$ and $\bar{R}_2$, in polar form, we first change to polar form as follows:

If

$$\bar{R}_1 = \bar{3} + \bar{4}$$

$$\bar{R}_2 = \bar{8} + \bar{5}$$

Then

$$\bar{R}_1 = 5\underline{/53^\circ\ 8'}$$

$$\bar{R}_2 = 9.43\underline{/32^\circ}$$

The multiplication of $\bar{R}_1$ and $\bar{R}_2$ results in a product which we will label $\bar{R}_{1,2}$. The following rule will be used:

To multiply two vectors find the product of the scalar quantities and the sum of the angles through which they have been rotated.

In our example

$$\bar{R}_1 = 5\underline{/53^\circ\ 8'}$$

and

$$\bar{R}_2 = 9.43\underline{/32^\circ}$$

The product of the scalar quantities is

$$(5)(9.43) = 47.2$$

and the sum of the angles is

$$(53^\circ \underline{/8'}) + (32^\circ) = 85^\circ\ 8'$$

We now have the product of $\bar{R}_1$ and $\bar{R}_2$ which is $\bar{R}_{1,2}$ and is equal to $47.2\underline{/85^\circ\ 8'}$. This result is the same as the result of multiplying the vectors in rectangular form and we intuitively understand why the mechanics of polar multiplication may be used.

DIVISION

We will now divide vector $\bar{R}_1$ by $\bar{R}_2$ in rectangular form as follows:

If

$$\bar{R}_1 = \overline{40} + \overline{30}$$

$$\bar{R}_2 = \overline{8} + \overline{5}$$

Then

$$\bar{R}_1 = 40 + 30i$$

$$\bar{R}_2 = 8 + 5i$$

Thus

$$\left(\frac{40 + 30i}{8 + 5i}\right)\left(\frac{8 - 5i}{8 - 5i}\right)$$

$$= \frac{320 + 40i - 150i^2}{64 - 25i^2}$$

$$= \frac{470 + 40i}{89}$$

$$= \frac{470}{89} + \frac{40}{89}i$$

$$= 5.28 + 0.449i$$

$$= \overline{5.28} + \overline{0.449}$$

And

$$r = \sqrt{(5.28)^2 + (0.440)^2}$$

$$= \sqrt{27.10}$$

$$= 5.3$$

If

$$\tan\theta = \frac{0.449}{5.28}$$

$$= 0.08504$$

Then

$$\theta = 4^\circ\ 52'$$

In dividing vector $\bar{R}_1$ by $\bar{R}_2$, in polar form, we first change to polar form as follows:

If

$$\bar{R}_1 = \overline{40} + \overline{30}$$

$$\bar{R}_2 = \overline{8} + \overline{5}$$

Then

$$\bar{R}_1 = 50\underline{/36^\circ\ 52'}$$

$$\bar{R}_2 = 9.43\underline{/32^\circ}$$

In division of vectors in polar form we will use the following rule:

To divide two vectors, in polar form, find the quotient of their scalar quantities and the difference between the angles through which they have been rotated.

Thus

$$\frac{\overline{R}_1}{\overline{R}_2} = \frac{50\underline{/36^\circ\ 52'}}{9.43\underline{/32^\circ}}$$

$$= 5.3\underline{/4^\circ\ 52'}$$

This result is the same as the result obtained by dividing vector $\overline{R}_1$ by $\overline{R}_2$, in rectangular form, and we intuitively understand why the mechanics of polar division may be used.

PRACTICE PROBLEMS: Multiply the following vectors:

1. $(5\underline{/10^\circ})\ (10\underline{/5^\circ})$
2. $(8.3\underline{/6^\circ})\ (1.1\underline{/73^\circ})$
3. $(6.2\underline{/52^\circ})\ (8\underline{/200^\circ})$
4. $(100\underline{/45^\circ})\ (30\underline{/20^\circ})$

ANSWERS:

1. $50\underline{/15^\circ}$
2. $9.13\underline{/79^\circ}$
3. $49.6\underline{/252^\circ}$
4. $3000\underline{/65^\circ}$

PRACTICE PROBLEMS: Perform the indicated division:

1. $\dfrac{64\underline{/24^\circ}}{8\underline{/24^\circ}}$
2. $\dfrac{300\underline{/24^\circ}}{20\underline{/8^\circ}}$
3. $\dfrac{620\underline{/154^\circ}}{5\underline{/142^\circ}}$
4. $\dfrac{64\underline{/18^\circ}}{16\underline{/27^\circ}}$

ANSWERS:

1. $8\underline{/0^\circ}$
2. $15\underline{/16^\circ}$
3. $124\underline{/12^\circ}$
4. $4\underline{/-9^\circ}$

It follows that a vector can be raised to any integral or fractional power. To square a vector, square the scalar quantity and multiply the angle by 2.

EXAMPLE: Square the vector $8\underline{/32^\circ}$

SOLUTION: $(8\underline{/32^\circ})^2$

$$= (8)^2\underline{/32^\circ\ (2)}$$

$$= 64\underline{/64^\circ}$$

To cube a vector, cube the scalar quantity and multiply the angle by 3.

EXAMPLE: Cube the vector $3\underline{/4^\circ}$

SOLUTION: $(3\underline{/4^\circ})^3$

$$= (3)^3\underline{/4^\circ\ (3)}$$

$$= 27\underline{/12^\circ}$$

To find the square root of a vector, extract the square root of the scalar quantity and divide the angle by 2.

EXAMPLE: Find the square root of the vector $16\underline{/70^\circ}$.

SOLUTION: $\sqrt{16\underline{/70^\circ}}$ or $(16\underline{/70^\circ})^{1/2}$

$$= \sqrt{16}\underline{/70^\circ\ \div\ 2}$$

$$= 4\underline{/35^\circ}$$

To find the cube root of a vector, extract the cube root of the scalar quantity and divide the angle by 3.

EXAMPLE: Find the cube root of the vector $27\underline{/33^\circ}$.

SOLUTION: $\sqrt[3]{27\underline{/33^\circ}}$ or $(27\underline{/33^\circ})^{1/3}$

$$= \sqrt[3]{27}\underline{/33^\circ\ \div\ 3}$$

$$= 3\underline{/11^\circ}$$

PRACTICE PROBLEMS: Perform the indicated operations:

1. $(10\underline{/20°})^2$
2. $(4\underline{/10°})^3$
3. $(64\underline{/90°})^{1/2}$
4. $(64\underline{/90°})^{1/3}$

ANSWERS:

1. $100\underline{/40°}$
2. $64\underline{/30°}$
3. $8\underline{/45°}$
4. $4\underline{/30°}$

CHAPTER 7

STATIC EQUILIBRIUM

Statics is a branch of physics that deals with bodies at rest. In this chapter we will make use of the previous investigation of vectors to establish the mathematical basis necessary for an understanding of static equilibrium. Since forces acting upon bodies have magnitude and direction, they may be represented by vectors.

DEFINITIONS AND TERMS

The following paragraphs include definitions and terms which will be used in this chapter. The definitions used will clarify the meanings of the discussions on static equilibrium.

EQUILIBRIUM

If a body undergoes no change in its motion, it is said to be in a state of equilibrium. We will discuss a body at rest as indicated by the term static equilibrium. Balanced forces may act upon a body in static equilibrium, but no motion, neither translatory nor rotary, will occur. In order for bodies to be in static equilibrium, two conditions are required. These two conditions are (1) the body must not have translatory motion, and (2) the body must not have rotary motion.

TRANSLATION

Translation, as defined, is motion independent of rotation. Attention is called to the fact that translation involves magnitude and direction of motion and hence can be described by vectors.

ROTATION

Rotation, as defined, is the turning motion of a body, such as a wheel turning. Rotation is independent of translation.

TRANSLATIONAL EQUILIBRIUM

If a body at rest is acted upon by an unbalanced force, it will be set into motion. This motion is called translation. All particles of the moving body will have at any instant the same velocity and direction of motion. Two forces acting in the same line upon a body must be equal in magnitude, but opposite in direction, if the body is to remain in equilibrium. We will consider our translations to be confined to the XY plane.

FIRST CONDITION

The first condition of equilibrium may be stated as follows: For a particle to be in equilibrium the sum of the vectors (forces) acting in any direction upon that particle must equal zero.

In figure 7-1, an iron block is resting on a table. The weight of the block is directed downward; thus the table must exert a force equal and opposite to the weight of this iron block. The block has weight, W, and the table exerts a push, P, upward against the block. Since the bodies are in equilibrium, there can be no unbalanced force. Thus,

$$W = P \text{ and } \overline{W} = \overline{P}$$

The weight of the block can be represented by a vector because we know the force and direction exerted by the weight. The magnitude and direction of the push (P) by the table is also known, and it can be represented by a vector. The vectors $\overline{W}$ and $\overline{P}$ are shown in figure 7-1. The weight is in equilibrium because the sum of the vectors acting upon it is equal to zero. We call these two forces parallel concurrent forces. We will also call $\overline{P}$ the equilibrant of $\overline{W}$. It is relatively easy to find the equilibrant of two or more vectors which are acting upon the same point. We first find the resultant of the vectors. The equilibrant of the resultant will have the same magnitude but will be opposite in direction. In figure 7-2 the resultant of $\overline{A}$ and $\overline{B}$ is $\overline{R}$. The magnitude of $\overline{R}$ is 18 and the direction is 56°18'. The magnitude of $\overline{C}$, the equilibrant of $\overline{R}$, is 18 and the direction is 236°18'. The sum of $\overline{R}$ and $\overline{C}$ is zero; therefore, a point 0 is in equilibrium.

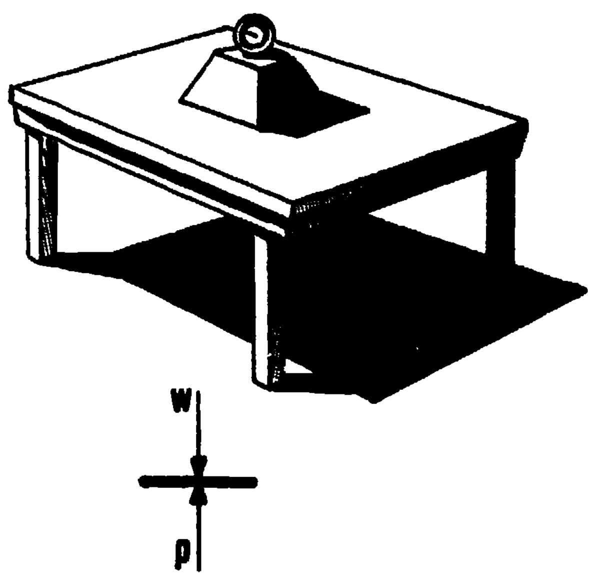


Figure 7-1.—Table and weight.

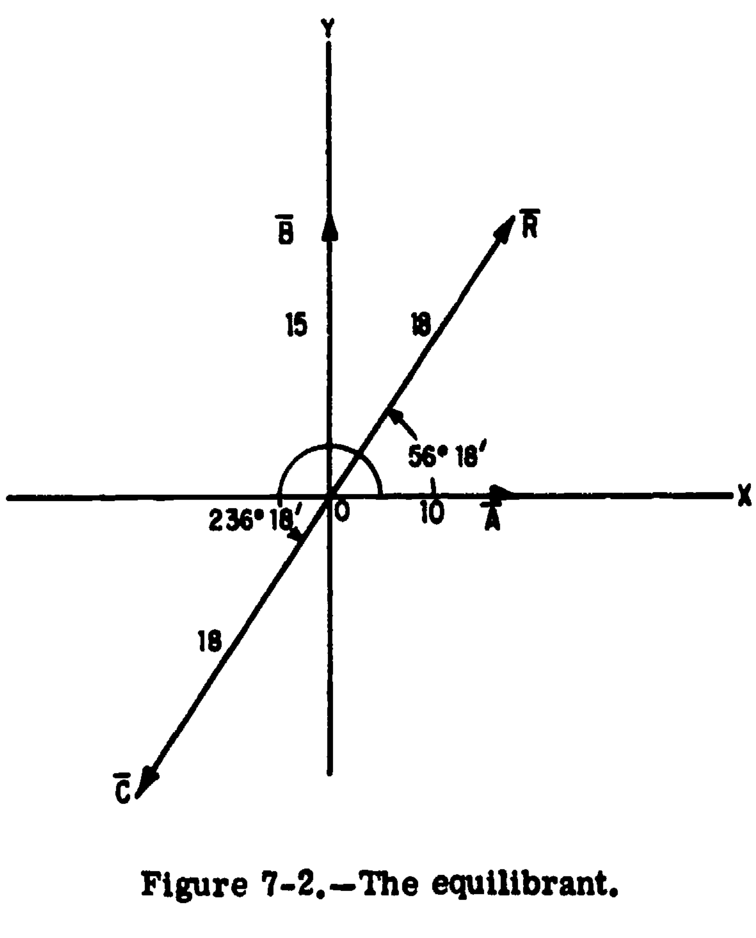


Figure 7-2.—The equilibrant.

In figure 7-2, the vectors $\overline{A}$ and $\overline{B}$ are called nonparallel concurrent vectors. All vectors can be resolved into horizontal and vertical components. Since the sum of all forces acting on a particle must be equal to zero, to satisfy the condition of equilibrium, we can say that the sum of all vertical components must equal zero, and the sum of all the horizontal components must equal zero. The symbols for this condition of equilibrium are:

$$\Sigma\overline{X} = 0$$

and

$$\Sigma\overline{Y} = 0$$

The symbol Σ is the Greek letter, sigma, and means "the sum of." Thus, $\Sigma\overline{X}$ equals 0 means that the sum of the vectors along the X axis equals zero.

We may show graphically that a particle P is in equilibrium while being acted upon by $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$, and $\overline{E}$, as in figure 7-3, by drawing a polygon of forces. If the polygon of forces is closed, there is no resultant force acting upon particle $\overline{P}$, and that particle is in equilibrium.

We will now examine the condition of equilibrium of a point which is acted upon by nonparallel concurrent forces.

EXAMPLE: We are to find the force of $\overline{A}$, in figure 7-4, in order that point 0 will remain in equilibrium while being acted upon by $\overline{B}$ and $\overline{C}$.

SOLUTION: We are looking for the equilibrant of the resultant of $\overline{B}$ and $\overline{C}$. The resultant of $\overline{B}$ and $\overline{C}$, called $\overline{R}$, is found as follows:

$$\overline{y}_b = 5 \sin 30°$$
$$= 5(0.50000)$$
$$= 2.5$$
$$\overline{x}_b = 5 \cos 30°$$
$$= 5(0.86603)$$
$$= 4.3$$
$$\overline{y}_c = 8 \sin 270°$$
$$= 8(-1.00000)$$
$$= -8$$
$$\overline{x}_c = 8 \cos 270°$$
$$= 8(0.00000)$$
$$= 0$$

S8

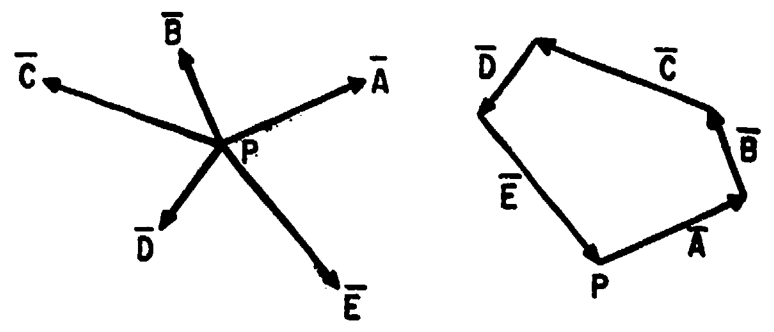


Figure 7-3.—Polygon of forces.

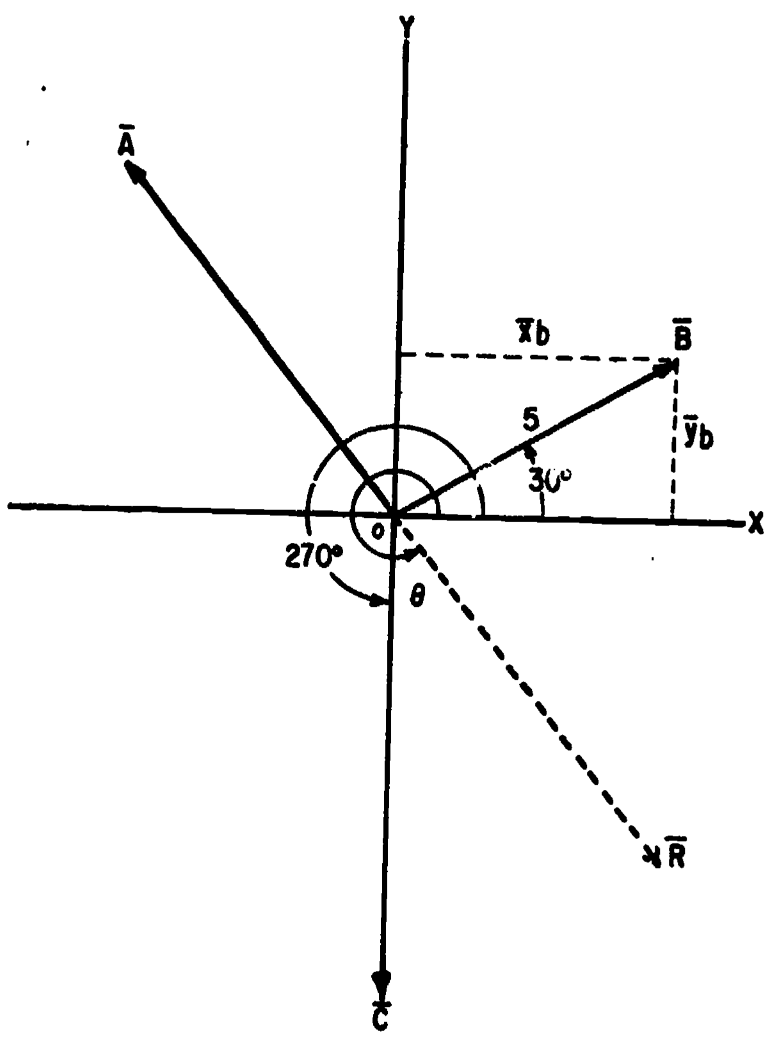


Figure 7-4.—Resultant and equilibrant.

We now add the X axis components and the Y axis components and find that

$$\bar{Y} = 2.5 + (-8)$$

$$= -5.5$$

$$\bar{X} = 4.3 + 0$$

$$= 4.3$$

and

$$\tan\theta = \frac{-5.5}{4.3}$$

$$= -1.27907$$

thus

$$\theta = 308°\,1'$$

and

$$r = \sqrt{(4.3)^2 + (-5.5)^2}$$

$$= \sqrt{48.74}$$

$$= 6.9$$

therefore our resultant is

$$6.9\underline{/308°\,1'}$$

and the equilibrant is

$$6.9\underline{/128°\,1'}$$

In some cases we are given the vectors by the problem and can easily find our solution.

EXAMPLE: A boy in a swing, as shown in figure 7-5, weighs 70 pounds and is pulled backward with a force of 30 pounds. Find the force the ropes exert on the swing; also find the angle the ropes make with the horizontal axis.

SOLUTION: We must find the resultant of the two vectors given and then find the equilibrant of the resultant. This is done as follows:

$$\bar{x}_a = -30$$

$$\bar{y}_a = 0$$

$$\bar{x}_b = 0$$

$$\bar{y}_b = -70$$

and

$$\tan\theta = \frac{-70}{-30} = 2.33333$$

thus

$$\theta = 246°\,48'$$

and

$$r = \sqrt{(-30)^2 + (-70)^2}$$

$$= \sqrt{5800}$$

$$= 76.2$$

therefore the resultant is

$$76.2\underline{/246^\circ\ 48'}$$

and the equilibrant is

$$76.2\underline{/66^\circ\ 48'}$$

Thus the ropes exert a 76.2-pound force at 66°48' on the swing.

PRACTICE PROBLEMS: Find the equilibrant of the following vectors.

1. $35\underline{/0^\circ}$ and $60\underline{/90^\circ}$
2. $7\underline{/35^\circ}$ and $9\underline{/125^\circ}$
3. $12\underline{/15^\circ}$ and $7\underline{/25^\circ}$
4. $9\underline{/55^\circ}$ and $10\underline{/100^\circ}$

ANSWERS:

1. $69.4\underline{/239^\circ\ 45'}$
2. $11.4\underline{/267^\circ\ 7'}$
3. $18.7\underline{/198^\circ\ 43'}$
4. $17.5\underline{/257^\circ\ 16'}$

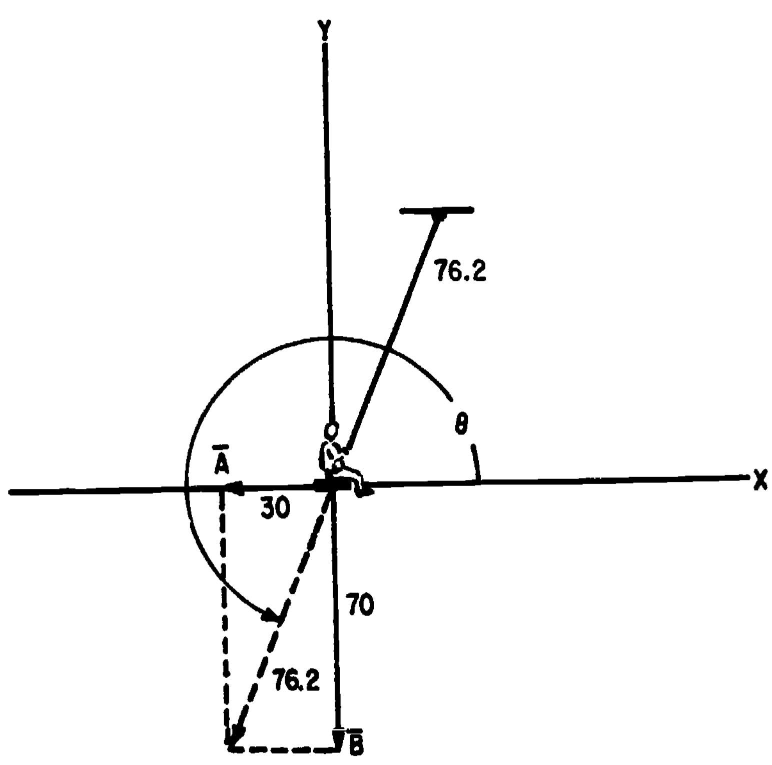


Figure 7-5.—Boy in a swing.

FREE BODY DIAGRAMS

One of the distinct advantages of vectors is that a vector may be substituted for the cable or member of a mechanism it is going to represent. As seen in figure 7-6 vectors may be substituted for the cables holding the weight W. Starting at point 0, a vector representing the tension in cable MO can be drawn, and vectors may also be drawn for the tension in cables NO and OW. This will give us the forces acting on particle O. Figure 7-6 (B) is called a free body diagram, and vector $\overline{A}$ represents the tension in cable MO. Vectors $\overline{B}$ and $\overline{W}$ represent the tensions in NO and WO, respectively.

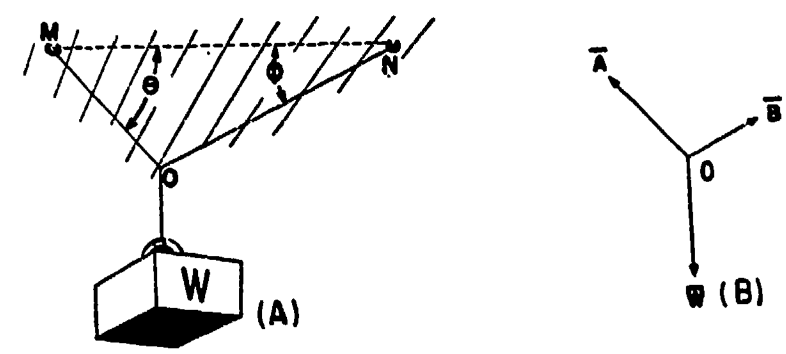


Figure 7-6.—Single weight.

Free body diagrams are very important in mechanics and the student should learn to draw these diagrams with ease. In a free body diagram, a member of a mechanism is replaced by a vector representing the force in that member and acting in the same direction as the member. The student should pay particular attention to the magnitude of the vector which represents a member of the mechanism. In figure 7-6, notice that the vector representation for MO is longer and the vector representation for NO is shorter in the free body diagram (B) than they appear in the pictorial view (A).

We may use the free body diagram to graphically verify our mathematical solution to a problem. (Refer again to fig. 7-5.) We find the boy in the swing to be in equilibrium and we will use a free body diagram to verify this. We draw our diagram as shown in figure 7-7 (A) where vector $\overline{C}$ is the equilibrant of the resultant. We draw the vectors $\overline{A}$, $\overline{B}$, and $\overline{C}$, initial point to terminal point, as shown in figure 7-7 (B). If the vectors form a closed loop, we have the sum of the vectors equal to zero and have present a state of equilibrium.

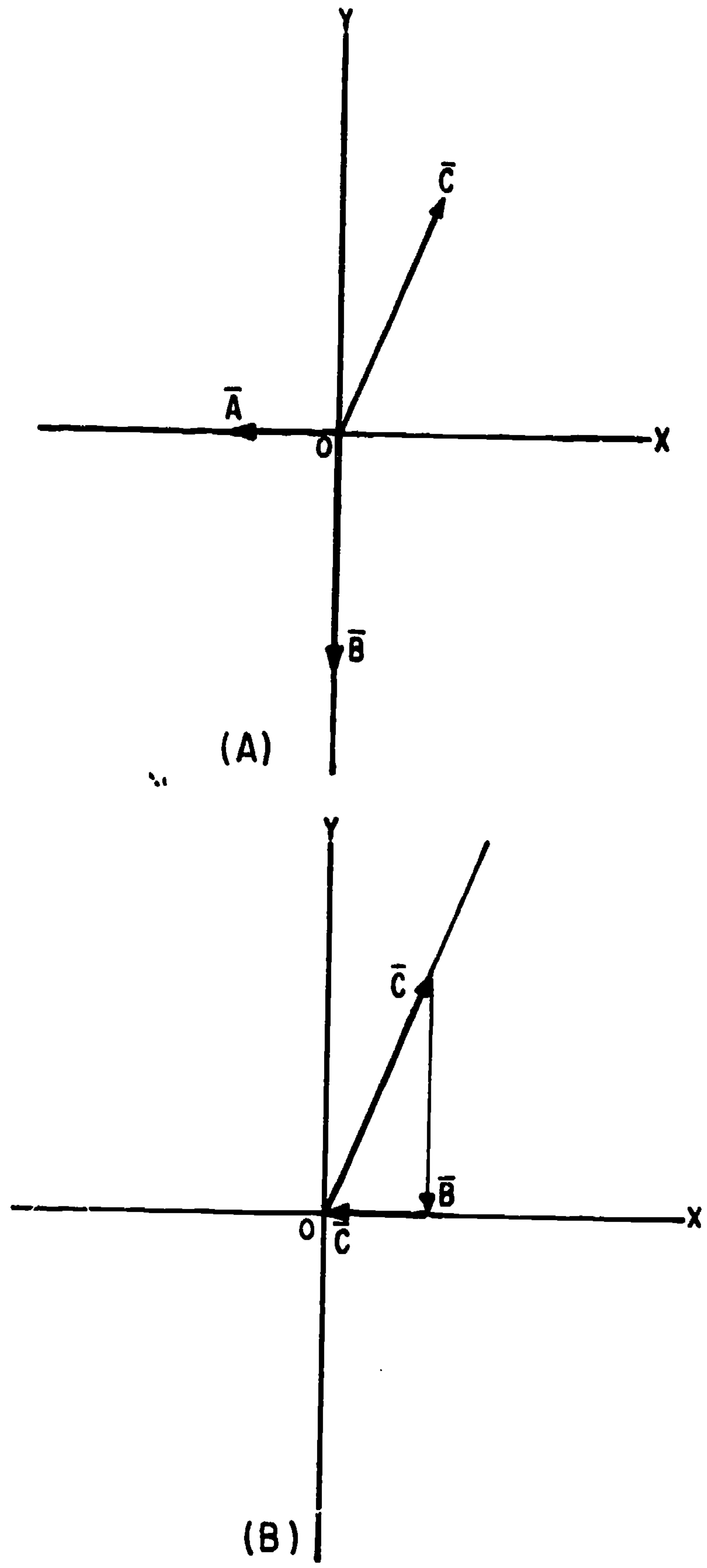


Figure 7-7.—Closed loop.

We have discussed parallel concurrent forces and nonparallel concurrent forces. In the following paragraphs we will discuss noncurrent parallel forces, remembering that we are still under the requirements for the first condition of equilibrium.

In figure 7-8 (A), we find a board balanced on and supported by a fulcrum. Draw the free body diagram as shown in figure 7-8 (B). Assume the board weighs so little that it is insignificant. Consider forces in a downward direction to be negative (-) and those upward to be positive (+). For equlibrium, we must have ΣX equal to 0 and ΣY equal to 0. We have no X axis components, therefore ΣX equals 0. The ΣY equals 0 because we have a state of equilibrium. Therefore

$$-\bar{A} - \bar{B} + \bar{C} = 0$$

and

$$\bar{A} = 42 \text{ lbs}$$

therefore

$$\bar{B} = 18 \text{ lbs}$$

$$\bar{C} = 60 \text{ lbs}$$

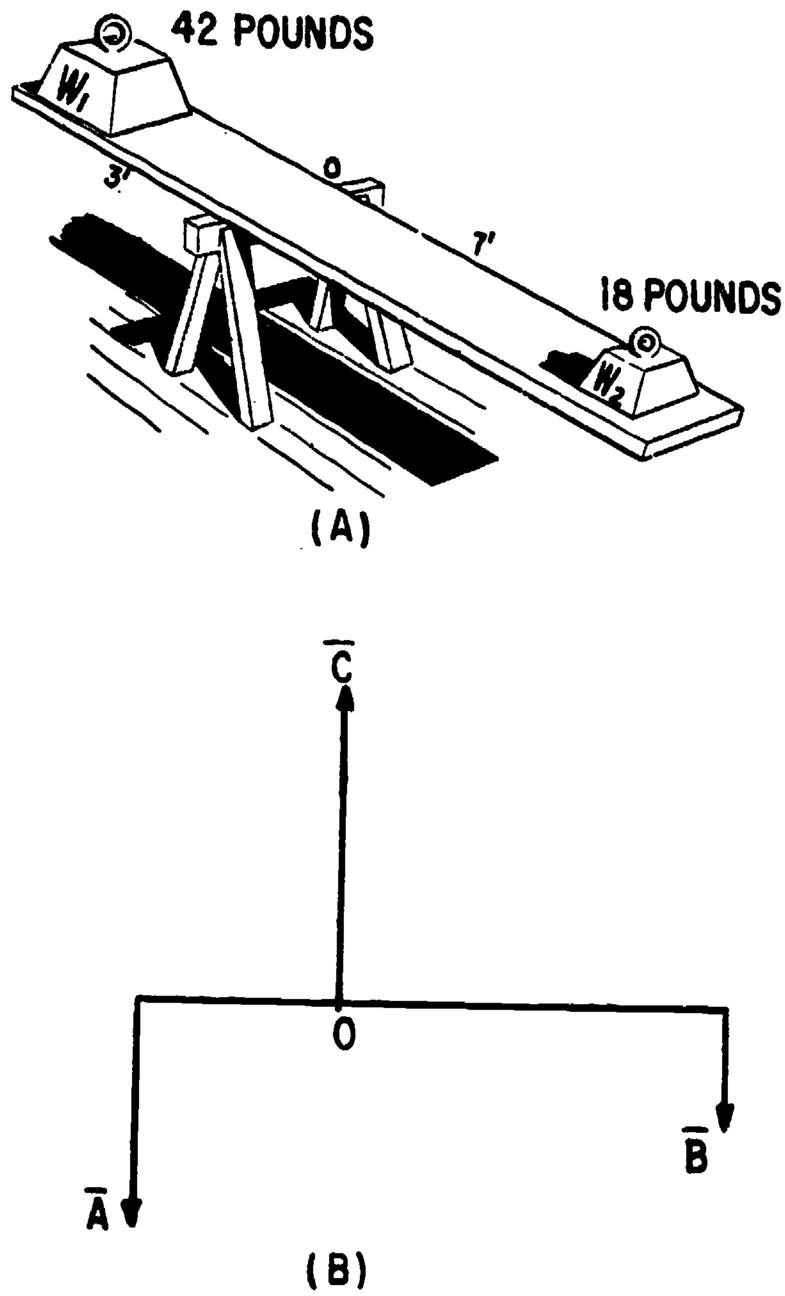


Figure 7-8.—Parallel nonconcurrent forces.

PRACTICE PROBLEMS: Draw the free body diagrams for the member indicated in the following figures (emphasize direction and not magnitude):

1. Figure 7-9 (A) (Bar)
2. Figure 7-9 (B) (Boom)
3. Figure 7-9 (C) (Bar)
4. Figure 7-9 (D) (Point 0)

ANSWERS:

1. Figure 7-10 (A)
2. Figure 7-10 (B)
3. Figure 7-10 (C)
4. Figure 7-10 (D)

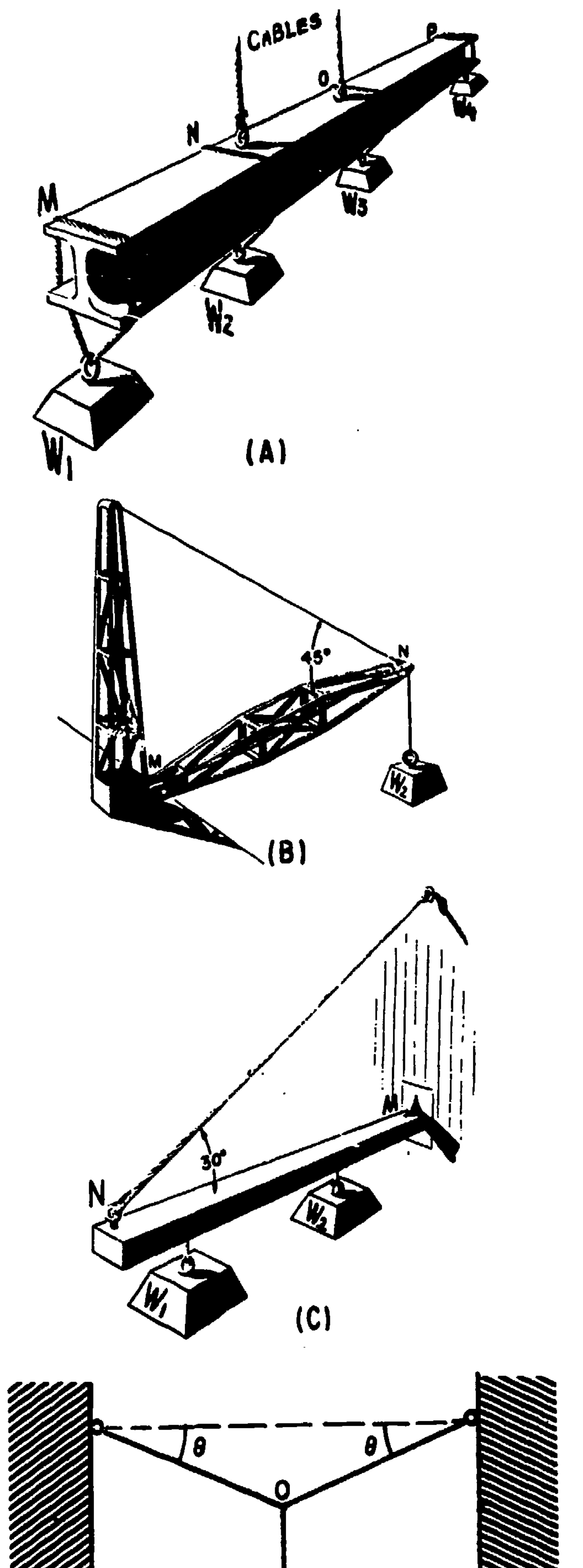


Figure 7-9.—Free body practice problems.

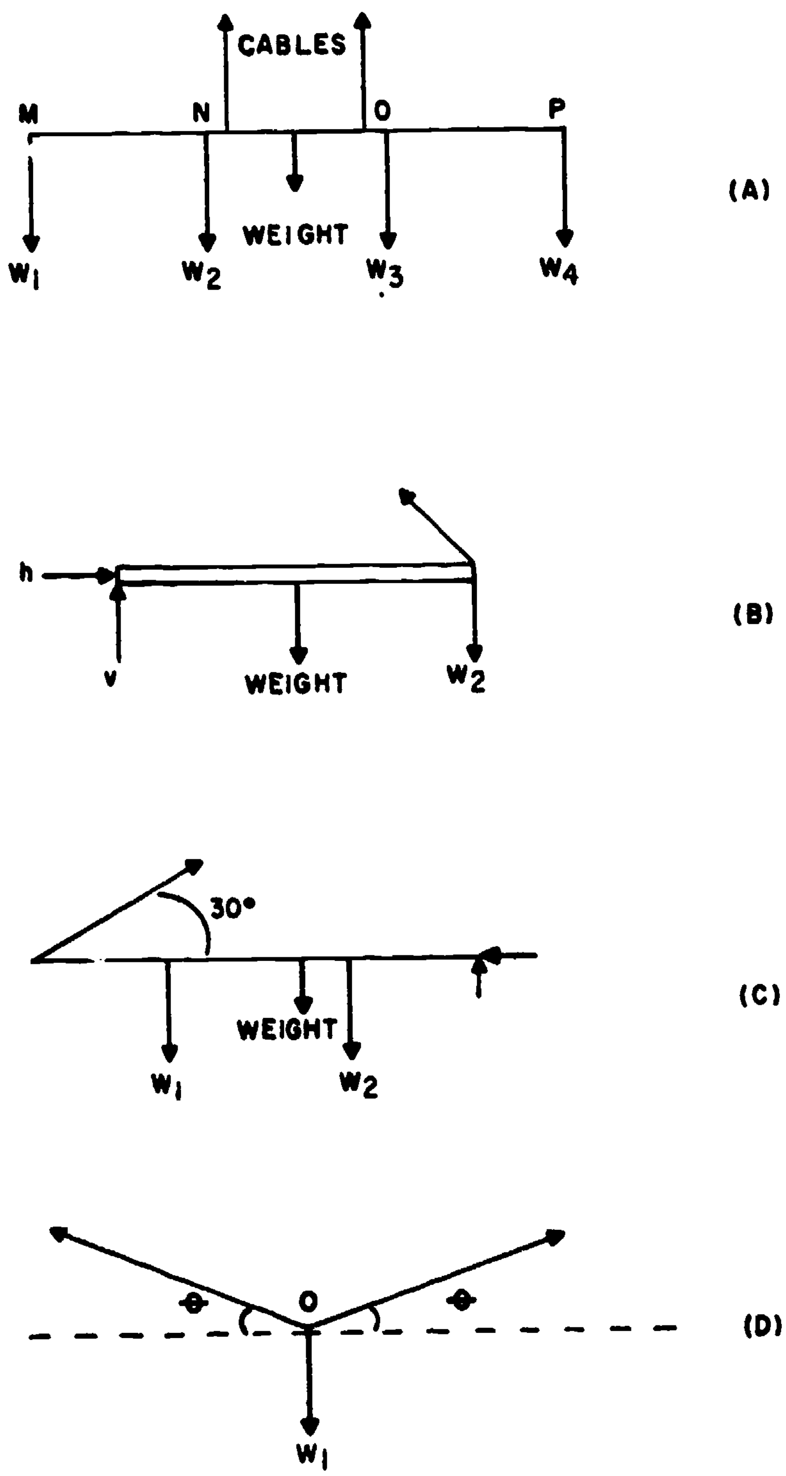


Figure 7-10.—Free body answers.

ROTATIONAL EQUILIBRIUM

The first condition of equilibrium guaranteed that there would be only translatory motion. It was stated that there was a distinction between the motion of translation and rotation. The second condition of equilibrium concerns the forces tending to rotate a body.

SECOND CONDITION

Figure 7-11 shows a body acted upon by two equal and opposite forces, F_1 and F_2. The sum of the forces in the horizontal direction equals zero, and there is no translatory motion. It is clear that there will be rotation of the body. These two equal and opposite forces not acting along the same line constitute a couple and cause a moment to be produced. The term couple is defined as two equal forces acting on a body but in opposite directions and not along the same line. For a body acted upon by a couple to remain in equilibrium, it must be acted upon by another couple equal in magnitude but opposite in direction.

The magnitude of a couple is the perpendicular distance between the forces multiplied by one of the forces. This product is called the moment of the couple. We will use M to indicate a moment and we can say, to fulfill the conditions of equilibrium, that ΣM equals 0. That is, the sum of all the moments acting upon a body must equal zero to maintain equilibrium. Clockwise moments, such as in figure 7-11, are positive and counterclockwise moments are negative. Our statement that the sum of all the moments acting upon a body must be zero, that is, ΣM equals 0, is called the second condition for equilibrium.

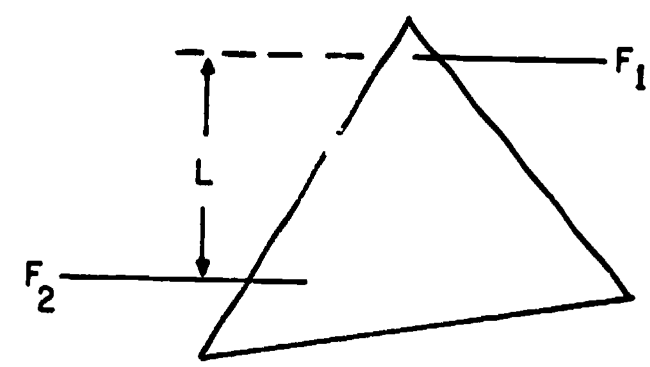
Assume that the body shown in figure 7-11 will rotate about a point halfway between the two forces. A moment, defined as a force acting on a lever arm L, is present for both of the forces. The moment acting on the body in figure 7-11 will be

$$F_1\left(\frac{L}{2}\right) + F_2\left(\frac{L}{2}\right)$$

Since $F_1 = F_2$

then

$$M = F_1\left(\frac{L}{2}\right) + F_1\left(\frac{L}{2}\right)$$
$$= F_1 L$$



If it were assumed that the body were to rotate about the point upon which F_2 acted, then the lever arm would be L for F_1 and zero for F_2. And again M equals F_1L. Hence, the moment of a couple is one of the forces multiplied by the distance between them. The definition for moment of a couple holds. The dimensions of a moment will include a distance as well as a force. The effect of a force upon the rotation is the perpendicular distance from the rotation point to the line of action of the force. In the English system the most used term is foot-pounds.

EXAMPLE: Calculate the moment of a couple consisting of two forces, F_1 (equal to 20 pounds) and F_2 (equal to 20 pounds), acting directly opposite to each other at a distance of 3 feet. The moment of this couple is M equals FL or

$$M = (20)(3) = 60 \text{ ft-lb}$$

Notice that in this example there is no balance of moments; that is, ΣM does not equal 0, and the conditions for equilibrium are not met.

We now put to use the first and second conditions for equilibrium. That is,

$$\Sigma Y = 0$$
$$\Sigma X = 0$$
$$\Sigma M = 0$$

In figure 7-12 (A) we have a board balanced on a fulcrum and we are to find the weight W_2 and the force F on the fulcrum if W_1 equals 12 pounds and the distances are as shown. There are no horizontal forces; therefore ΣX equals 0. We draw the free body diagram as shown in figure 7-12 (B) and find the solution as follows: If

$$W_1 = 12 \text{ lb}$$

$$L_1 = 2 \text{ ft}$$

$$W_2 = \text{unknown}$$

$$L_2 = 8 \text{ ft}$$

$$F = \text{unknown}$$

then

$$L_1W_1 = L_2W_2$$

$$= (12 \text{ lb})\ (2 \text{ ft})$$

$$= 24 \text{ ft-lb}$$

Therefore

$$L_2W_2 = 24 \text{ ft-lb}$$

$$(8 \text{ ft})\ (W_2) = 24 \text{ ft-lb}$$

$$W_2 = 3 \text{ lb}$$

Thus

$$\Sigma M = 0$$

and

$$W_1 + W_2 - F = 0$$

$$F = 12 \text{ lb} + 3 \text{ lb}$$

$$= 15 \text{ lb}$$

Thus

$$\Sigma Y = 0$$

A very useful theorem that originates from the second condition of equilibrium states that: If three nonparallel forces acting upon a body produce equilibrium, their lines of action must pass through a common point. In other words, the three conditions ΣX equals 0, ΣY equals 0, and ΣM equals 0 must be satisfied for equilibrium; and, in order for three nonparallel forces to produce zero moment, the lines of action of the forces must pass through a common point, thus having zero lever arm.

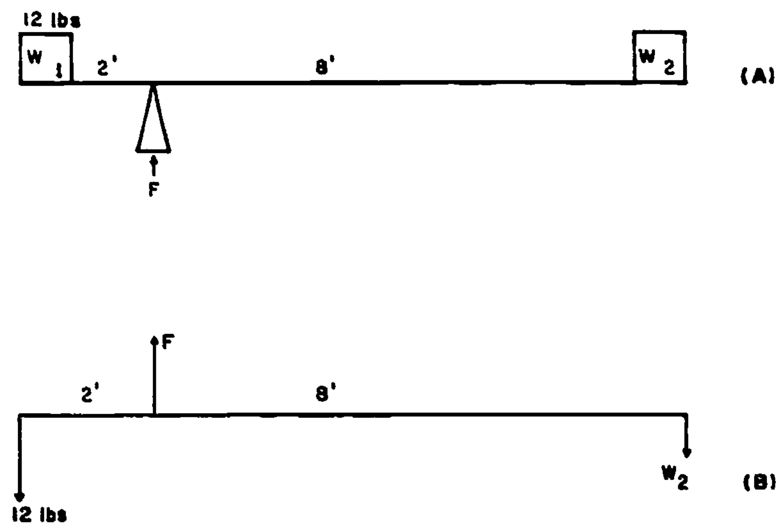


Figure 7-12.—Forces and moments.

CENTER OF GRAVITY

The earth's gravitational field attracts each particle in a body and the weight of that body is regarded as a system of parallel forces acting upon each particle of the body. All of these parallel forces can be replaced by a single force equal to their sum. The point of application of this single force is called the center of gravity (or C. G.) of the body. For bodies of simple shape and uniform density, the C. G. is at the geometric center and can be found by inspection.

The C. G. of an irregularly shaped body can be found by suspending the body from three different points on the body. In each case the body will come to rest (equilibrium) with its C. G. directly beneath the point of suspension. The intersection of any two of these lines will determine the C. G. The third line should also pass through this intersection and thus may be used to check the result. Figure 7-13 (A) shows this simple system for finding the C. G. Figure 7-13 (B) shows that in some cases the C. G. may fall outside of the body.

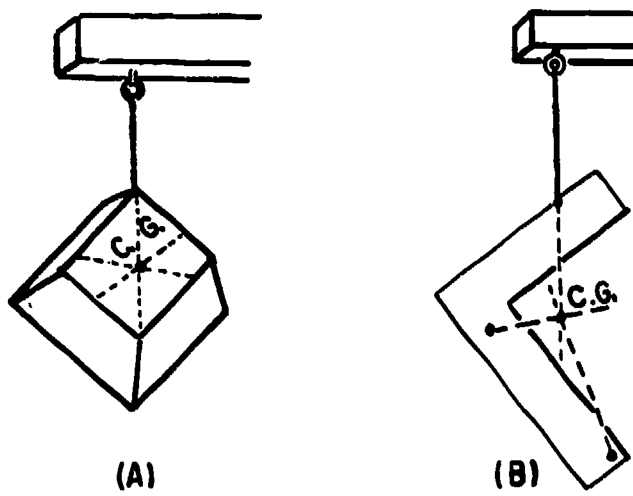


Figure 7-13.—Center of gravity.

APPLICATIONS

The greatest difficulty encountered in solving problems dealing with equilibrium is finding all of the forces acting upon a body. The use of a free body diagram will aid in eliminating this difficulty.

The procedure recommended for solving static equilibrium problems is as follows:

1. Sketch the system, taking into account all known facts, and assign symbols to all of the knowns and unknowns.
2. Select a member that involves one or more of the unknowns and construct a free body diagram.
3. Write the equations obtained from ΣX equals 0, ΣY equals 0, and ΣM equals 0.
4. Solve these equations for the unknowns.
5. Continue the process from one side of a structure to the other side.

EXAMPLE: Consider the ladder standing against a building in figure 7-14 (A) and making an angle of 60° with the ground. The ladder is 16 feet long and weighs 50 pounds.

SOLUTION: We sketch the free body diagram as shown in figure 7-14 (B) and assign symbols to the known and unknown forces. The arrows indicate the directions of the forces and h and v represent horizontal and vertical components of a force. The frictional force f holds the ladder from slipping, h is the horizontal force of the wall pushing against the ladder, and v is the vertical force which the ground exerts on the ladder. We assume all the weight of the ladder to be located at the center of gravity and assign the letter W to indicate this weight. The ladder is in a state of equilibrium and we have the following:

$$\Sigma X = 0$$

$$\Sigma Y = 0$$

$$\Sigma M = 0$$

therefore

$$f - h = 0$$

$$W - v = 0$$

We use trigonometry to find $\overline{AB}$ and $\overline{BC}$, as follows:

$$\begin{aligned} \overline{AB} &= r \cos \phi \\ &= 16 \cos 60^\circ \\ &= 16\ (0.50000) \\ &= 8 \end{aligned}$$

and

$$\begin{aligned} \overline{BC} &= r \sin \phi \\ &= 16 \sin 60^\circ \\ &= 16\ (0.86603) \\ &= 13.85 \end{aligned}$$

Using similar triangles we find that W is located directly above the midpoint of $\overline{AB}$.

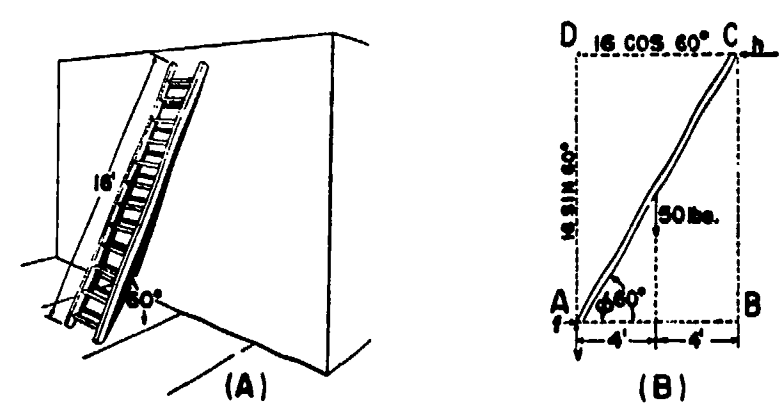


Figure 7-14.—Ladder problem.

Next, take the moments about the bottom of the ladder and in that way the two forces (f and v) have zero lever arm and are eliminated. The moments (clockwise) are as follows:

$$\Sigma M = 0$$

$$= 4W - h(16 \sin 60°)$$

Notice that we used the perpendicular distances from point A to where the forces were applied.

We now have equations as follows:

$$f - h = 0$$

$$W - v = 0$$

$$4W - h\,(16 \sin 60°) = 0$$

and substituting known values, then solving, we find

$$W - v = 0$$

$$50 \text{ lb} - v = 0$$

$$v = 50 \text{ lb}$$

and

$$4W - h\,(16 \sin 60°) = 0$$

$$4\,(50 \text{ lb}) - h\,(13.85) = 0$$

$$h = 14.4 \text{ lb}$$

and

$$f - h = 0$$

$$f - 14.4 = 0$$

$$f = 14.4 \text{ lb}$$

EXAMPLE: Determine the forces acting upon the members of the A-type frame as shown in figure 7-15 (A). The horizontal surface is considered smooth and no horizontal force can be exerted on the legs of the frame. A weight of 1,000 pounds hangs from the crossbar. The frame is considered as having no weight.

SOLUTION: Draw the free body diagrams and assign symbols as shown in figures 7-15(B) and (C). Since the system is symmetrical, the reactions at A and C are equal, and each is equal to 500 pounds (each carries half the load). Thus, A_v equals 500 pounds and C_v equals 500 pounds. The forces D_v and E_v can be found from the diagram of the crossbar in figure 7-15 (B) by taking ΣM about D.

Thus

$$\Sigma M = 3W - E_v(6) = 0$$

$$= 3(1000) - E_v(6)$$

$$E_v = 500 \text{ lb}$$

and from symmetry

$$D_v = 500 \text{ lb}$$

These forces E_v and D_v are upward to oppose the weight on the bar; thus this member must exert the same forces downward on the inclined member.

It is apparent that BC pushes upward against AB. This force is unknown, but it does have a vertical and horizontal component.

Using the two conditions of equilibrium we find the following:

$$\Sigma \overline{X} = D_h - B_h = 0$$

$$\Sigma \overline{Y} = A_v + B_v - D_v = 0$$

$$\Sigma M = 5A_v - 3D_v - (6 \sin 60°)D_h = 0$$

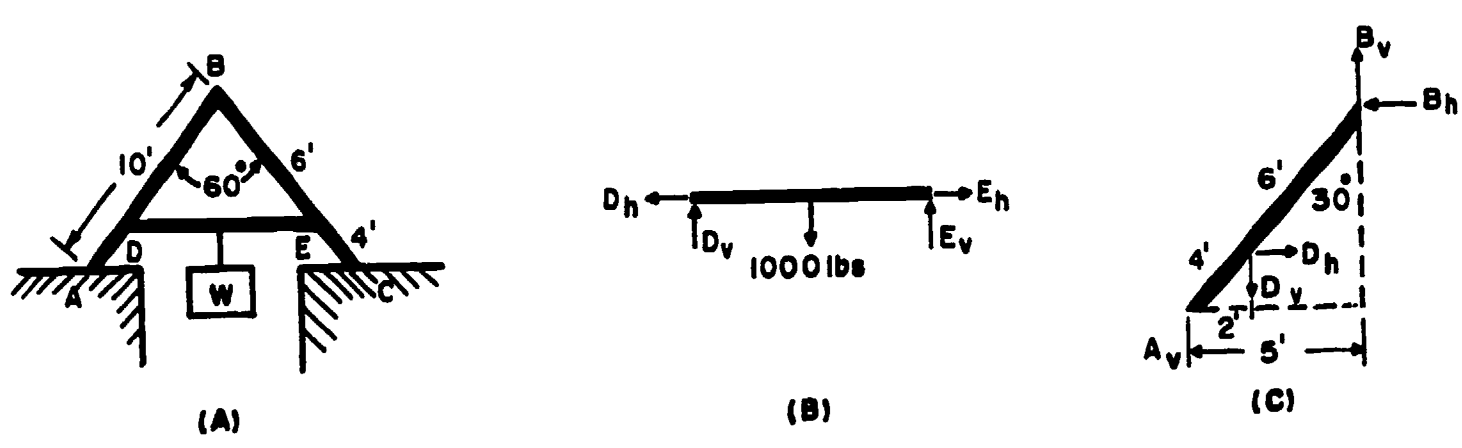


Figure 7-15.—A-type frame.

Thus

$$\Sigma\overline{Y} = 500 + B_v - 500 = 0$$

$$B_v = 0$$

$$\Sigma M_b = 5(500\text{ lb}) - 3(500\text{ lb}) - 6(0.86603) = 0$$

$$D_h = 192\text{ pounds}$$

$$\Sigma\overline{X} = 192\text{ lb} - B_h = 0$$

$$B_h = 192\text{ lb}$$

From symmetry we find the following:

$$E_h = D_h = 192\text{ pounds}$$

$$E_v = D_v = 500\text{ pounds}$$

The magnitude and direction of the X axis and Y axis forces may be used to find the forces and their directions.

One important thing to remember when taking ΣM equal to 0 is to take the moments about some point that will eliminate one or more of the unknowns. In the last example, the moment equation was taken about point B and eliminated the forces B_h and B_v. The moment equations can be taken about any point and more than one moment equation can be taken, if necessary.

PRACTICE PROBLEMS: Find the required information in the following:

1. Two boys pull a wagon, each with a force of 35 pounds. The angle between the ropes on which the boys are pulling is 30°. What is the resultant pull on the wagon?
2. A large portrait weighs 100 pounds, and is supported by a wire 10 feet long which is hooked to the picture at two points 5 feet apart. Find the tension in the wire.
3. A 180-pound man is standing half-way up a 20-foot, 20-pound ladder. The bottom of the ladder is 4 feet from the base of the vertical wall it is leaning against. Find the forces exerted on the ladder. (Use same symbols as shown in fig. 7-14 (B).)
4. A bar of uniform weight, 12 feet long and weighing 7 pounds, is supported by a fulcrum which is 4 feet from the left end. If a 10-pound weight is hung from the left end, find the weight needed at the right end to hold the bar in equilibrium and find the force with which the fulcrum pushes against the bar.

ANSWERS:

1. 67.6 pounds
2. 57.8 pounds
3. f = 20.4 pounds, h = 20.4 pounds, v = 200 pounds
4. 3.25 pounds and 20.25 pounds

CHAPTER 8

TRIGONOMETRIC IDENTITIES AND EQUATIONS

This is the final chapter in the section dealing directly with trigonometry and trigonometric relationships. This chapter includes the basic identities, formulas for additional identities involving multiples of an angle, and formulas for identities which involve more than one angle. Methods and examples of the use of the identities in simplifying expressions are given, and practice problems in simplification are included.

Also included in the chapter are methods for solving equations involving trigonometric functions. In many cases, the verification or simplification of an identity is an integral part of the solution of an equation. An additional topic considered in the chapter is the inverse trigonometric functions. Examples and problems involving equations and the inverse functions are also given.

FUNDAMENTAL IDENTITIES

In earlier chapters it was shown that all of the trigonometric functions of an angle could be determined if one function or certain related information was given. This seems to indicate that there are certain special relationships among the functions. These relationships are called identities and are independent of any particular angle. Many of the identities which will be considered in this section were established in earlier chapters and will be used here to change the form of an expression. In many problems, especially in calculus and other branches of mathematics, one particular method of expressing a function is more useful than any of the others. In these instances, the identities are used to put the expression in the desired form.

An equality which is true for all values of an unknown is called an identity. Identities are familiar in algebra, although they are not always specifically identified as such. A factoring process such as

$$(x^2 - 1) = (x - 1)(x + 1)$$

involves expression of an identity since it is true for all values of the variable. In trigonometric identities, the same situation must hold; that is, the equality must be true for all values of the variable.

Problems in identities are often given as equalities, and the identity is established by changing either one or both sides of the equality until both sides are the same. The fundamental rule in proving identities is as follows: NEVER WORK ACROSS THE EQUALITY SIGN. The algebraic rules for cross multiplication are never used. In this course all problems will be solved by working on only one side of the equality; that is, one side of the equality will be reduced or expanded until it is identical to the other side.

There are no hard and fast steps or methods to use in solving identities. However, there are some basic procedures or hints which will normally prove helpful in verification of the identities. Some of these hints are as follows:

1. Reduce the complex expressions to simple expressions, rather than building up from a simpler to a more complex one.
2. When possible, change the expression to one containing only sines and cosines.
3. If the expression contains fractions, it may help to change the form of the fractions.
4. Factoring an expression may suggest a subsequent step.
5. Keep the other member of the equality in mind. Since we are striving to change one expression to another, the form of the desired expression may suggest the steps to be taken.

RECIPROCALS AND RATIOS

One group of identities was presented in part in chapter 4 of this course. Recall that

$$\csc \theta = \frac{1}{\sin \theta}$$

Since

$$\sin \theta = \frac{y}{r}$$

and

$$\csc \theta = \frac{r}{y}$$

the cosecant function can be written as

$$\csc \theta = \frac{1}{\frac{y}{r}}$$

and simplified to

$$\csc \theta = \frac{1}{\sin \theta}$$

There are six reciprocal identities, one for each function. The numbers assigned these and subsequent identities in this chapter do not constitute any rules as to order or precedence. They are used only to simplify the explanation of steps in the example problems. The reciprocal formulas follow directly from the definitions of the trigonometric functions

$$\sin \theta = \frac{1}{\csc \theta} \qquad (1)$$

$$\cos \theta = \frac{1}{\sec \theta} \qquad (2)$$

$$\tan \theta = \frac{1}{\cot \theta} \qquad (3)$$

$$\csc \theta = \frac{1}{\sin \theta} \qquad (4)$$

$$\sec \theta = \frac{1}{\cos \theta} \qquad (5)$$

$$\cot \theta = \frac{1}{\tan \theta} \qquad (6)$$

There are also identities involving the sine, cosine, tangent, and cotangent of an angle which are sometimes called ratio identities. These identities also result directly from the functions, and one of these expresses the tangent in terms of the sine and cosine, as follows:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad (7)$$

Since

$$\sin \theta = \frac{y}{r}$$

and

$$\cos \theta = \frac{x}{r}$$

substituting these values in (7) gives

$$\tan \theta = \frac{\frac{y}{r}}{\frac{x}{r}}$$

which can be simplified to

$$\tan \theta = \frac{y}{x}$$

This is the definition of the tangent function given in an earlier chapter. The following identity for the cotangent,

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \qquad (8)$$

can be shown to reduce identically to the definition of the cotangent function in terms of x, y, and r. The reciprocal and ratio identities are used to simplify trigonometric expressions as shown in the following example problems.

EXAMPLE: Simplify the expression

$$\sin \theta \cos \theta \tan \theta$$

to an expression containing only the sine function.

SOLUTION: One method of accomplishing this is to apply identity (7) to the given expression; then it becomes

$$\sin \theta \cos \theta \left(\frac{\sin \theta}{\cos \theta}\right)$$

or

$$\frac{\sin \theta \cos \theta \sin \theta}{\cos \theta}$$

Simplifying the $\cos \theta$ terms in both the numerator and denominator of the fraction results in

$$\frac{\sin \theta \sin \theta}{1}$$

or

$$\sin^2 \theta$$

This is the desired form and is identical, for all values of θ, to the original expression.

EXAMPLE: Use fundamental identities to verify the identity

$$\csc \theta + \cot \theta = \frac{1 + \cos \theta}{\sin \theta}$$

SOLUTION: Since the right-hand member of the identity contains sines and cosines, use (4) and (8) to change the left member to sines and cosines.

Then

$$\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta}$$

Change the sum of fractions in the left member to a single fraction as in the following

$$\frac{1 + \cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta}$$

and the identity is verified

Observe that the right-hand member of the identity was not altered throughout the entire process. This is in accordance with our stated intention of working on just one side of the equality sign.

If we desire to verify this identity by retaining the left member and operating on the right member, the following steps may be used.

$$\csc \theta + \cot \theta = \frac{1 + \cos \theta}{\sin \theta}$$

Change the fraction in the right member to the sum of two fractions

$$\csc \theta + \cot \theta = \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}$$

Next, apply (4) and (8) to the right member

$$\csc \theta + \cot \theta = \csc \theta + \cot \theta$$

and the identity is verified.

SQUARED RELATIONSHIPS

Another group of fundamental identities involves the squares of the functions. These, in some texts, are called Pythagorean identities since the Pythagorean theorem is used in their development. Consider the Pythagorean theorem

$$x^2 + y^2 = r^2$$

and divide both sides by r^2,

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

Write this in the form

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

and consider that

$$\cos \theta = \frac{x}{r}$$

and

$$\sin \theta = \frac{y}{r}$$

If $\cos \theta$ and $\sin \theta$ are substituted for $\frac{x}{r}$ and $\frac{y}{r}$ then

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

This is rewritten as

$$\cos^2 \theta + \sin^2 \theta = 1$$

which is a fundamental squared or Pythagorean identity.

NOTE: The practice of writing an expression such as $(\sin \theta)^2$ in the form $\sin^2 \theta$ is common, and is the preferred method.

In the same manner, dividing both sides of the equation

$$x^2 + y^2 = r^2$$

by x^2 (where x is not equal to 0) gives

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

or

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

Then, since

$$\tan\theta = \frac{y}{x}$$

and

$$\sec\theta = \frac{r}{x}$$

substitution gives

$$1 + (\tan\theta)^2 = (\sec\theta)^2$$

or

$$1 + \tan^2\theta = \sec^2\theta \qquad (10)$$

which is another fundamental identity.

The identity

$$1 + \cot^2\theta = \csc^2\theta \qquad (11)$$

is derived in a similar manner.

The three squared identities can be transposed algebraically to other forms with the following results:

$$\cos^2\theta = 1 - \sin^2\theta \qquad (12)$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad (13)$$

$$\tan^2\theta = \sec^2\theta - 1 \qquad (14)$$

$$\sec^2\theta - \tan^2\theta = 1 \qquad (15)$$

$$\cot^2\theta = \csc^2\theta - 1 \qquad (16)$$

$$\csc^2\theta - \cot^2\theta = 1 \qquad (17)$$

In addition to the fundamental identities, there are many complicated identities involving the trigonometric functions. In the majority of cases, these identities can be proved by use of the laws of algebra and the fundamental identities.

EXAMPLE: Verify the identify

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$$

SOLUTION: Reduce the left member to equal the right member. First, change each function to sines or cosines as follows:
Apply (4) to the denominator of the first fraction to obtain

$$\frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\sec\theta} = 1$$

Simplification of the first fraction gives

$$\frac{\sin^2\theta}{1} + \frac{\cos\theta}{\sec\theta} = 1$$

Applying (5) to the remaining fraction gives

$$\sin^2\theta + \frac{\cos\theta}{\frac{1}{\cos\theta}} = 1$$

Simplification gives

$$\sin^2\theta + \cos^2\theta = 1$$

Then applying (9) to the left member results in

$$1 = 1$$

and the identity is verified.

EXAMPLE: Verify the following identity:

$$1 + \cot^2 2x = \frac{1}{\sin^2 2x}$$

SOLUTION: As a first step, apply (8) to the term $\cot^2 2x$.

Then

$$1 + \frac{\cos^2 2x}{\sin^2 2x} = \frac{1}{\sin^2 2x}$$

Combine the left term into a single fraction with a denominator of $\sin^2 2x$

$$\frac{\sin^2 2x + \cos^2 2x}{\sin^2 2x} = \frac{1}{\sin^2 2x}$$

Applying (9) to the numerator of the left member gives

$$\frac{1}{\sin^2 2x} = \frac{1}{\sin^2 2x}$$

and the identity is verified.

PRACTICE PROBLEMS: Verify the following identities:

1. $\dfrac{1}{\tan^2 x + 1} = \cos^2 x$
2. $\csc x - \sin x = \cos x \cot x$
3. $\dfrac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos \theta$
4. $\cos \theta (\sec \theta - \cos \theta) = \sin^2 \theta$
5. $\tan^2 x (1 - \sin^2 x) = 1 - \cos^2 x$
6. $\sin^3 x = \dfrac{1 - \cos^2 x}{\csc x}$
7. $\dfrac{1}{2 + \cot^2 x} = \dfrac{1}{2\csc^2 x - \cot^2 x}$

REDUCTION FORMULAS

In chapter 4 of this course, reduction formulas were developed for dealing with angles greater than 90°. These reduction formulas can be combined into a general category of identities which also includes the formulas developed in chapter 4 for dealing with cofunctions and complementary angles. The formulas, of the type

$$\sin(90° - \theta) = \cos \theta$$

or

$$\sec(180° + \theta) = -\sec \theta$$

are listed in chapter 4 and the listings will not be repeated in this chapter.

The formulas from chapter 4 will be used to simplify expressions, in the same manner as the other identities, in the following examples.

EXAMPLE: Simplify the expression

$$\sin(180° - \theta)\tan(90° - \theta)\cot(180° - \theta)$$

into an expression containing functions of θ alone.

SOLUTION: From chapter 4, the following formulas are chosen

$$\sin(180° - \theta) = \sin \theta$$

$$\tan(90° - \theta) = \cot \theta$$

$$\cot(180° - \theta) = -\cot \theta$$

Substitution of these values in the expression

$$\sin(180° - \theta)\tan(90° - \theta)\cot(180° - \theta)$$

results in the expression

$$\sin \theta \cot \theta (-\cot \theta)$$

Rewrite this in the form

$$-\sin \theta \cot \theta \cot \theta$$

and apply identity (8) to one of the cot θ factors. Then,

$$-\sin \theta \left(\frac{\cos \theta}{\sin \theta}\right) \cot \theta$$

results and this can be simplified to

$$-\cos \theta \cot \theta$$

to complete the problem.

EXAMPLE: Express the following as an expression containing the least possible number of functions of θ.

$$\sin(360° - \theta)\tan(90° - \theta)\csc \theta$$

SOLUTION: The following formulas are given in chapter 4:

$$\sin(360° - \theta) = \sin(-\theta) = -\sin \theta$$

and

$$\tan(90° - \theta) = \cot \theta$$

Substitution of these values in the original expression results in

$$-\sin \theta \cot \theta \csc \theta$$

Rewrite this as

$$-\cot \theta \sin \theta \csc \theta$$

and apply identity (1) to the factor sin θ to arrive at

$$-\cot\theta \csc\theta \frac{1}{\csc\theta}$$

or

$$-\cot\theta$$

which is an expression in terms of one function of θ.

PRACTICE PROBLEMS: Express the following as expressions containing the least number of functions of θ.

1. $\sin(180^\circ - \theta)\sin\theta$
2. $\sin(180^\circ + \theta)\sec(180^\circ - \theta)$
3. $\cos(360^\circ - \theta)\cot(90^\circ - \theta)\csc(90^\circ - \theta)$

ANSWERS:

1. $\sin^2\theta$
2. $\tan\theta$
3. $\tan\theta$

INVERSE TRIGONOMETRIC FUNCTIONS

In this section we will discuss the definitions which apply to the inverse trigonometric functions along with the principal values of these functions. Relations among these functions will be examined by the use of examples and practice problems.

DEFINITIONS

It is often convenient and useful to turn a trigonometric function around so that instead of writing

$$\tan\theta = A$$

we write

$$\theta = \text{the angle whose tangent is A}$$

Rather than write out the last statement, mathematicians use either of the following notations:

$$\theta = \arctan A$$

or

$$\theta = \tan^{-1} A$$

In the last notation we do not mean $^{-1}$ to represent an algebraic exponent and $\tan^{-1} A$ does denote $\frac{1}{\tan A}$. If we meant $\tan^{-1}$ equals $\frac{1}{\tan A}$, we would have written $(\tan A)^{-1}$ equals $\frac{1}{\tan A}$.

In this course, the preferred notation is arctan A.

PRINCIPAL VALUES

For any angle there is one and only one function which corresponds to it; but to any value of a trigonometric function, there are numerous angles which will satisfy the value. For instance,

$$\theta = \arctan 1$$

can be written

$$\tan\theta = 1$$

but 1 is the tangent of many angles such as 45°, 225°, 405°, 585°, and others. Any angle θ which satisfies $(45^\circ + n \cdot 180^\circ)$, where n is an integer, satisfies the expression

$$\tan\theta = 1$$

For any inverse trigonometric function there are two angles less than 360° which satisfy it. Thus,

θ = arccos(0.500) refers to 60° and 300°

θ = arccos(-0.500) refers to 120° and 240°

θ = arcsin(0.707) refers to 45° and 135°

θ = arcsec(2.000) refers to 60° and 300°

Since a given inverse trigonometric function has many values, one of these values is selected as its principal value.

To denote principal values, we will capitalize the first letter in the name and we will use the ranges for principal values as follows:

$$-90^\circ \le \text{Arcsin } x \le 90^\circ$$
$$0^\circ \le \text{Arccos } x \le 180^\circ$$
$$-90^\circ < \text{Arctan } x < 90^\circ$$
$$-90^\circ \le \text{Arccsc } x \le 90^\circ$$
$$0^\circ \le \text{Arcsec } x \le 180^\circ$$
$$0^\circ < \text{Arccot } x < 180^\circ$$

All principal values lie between -90° and 180° proceeding counterclockwise form -90°. The principal values for positive numbers are between 0° and 90°. For negative numbers, principal values of the inverse sine, tangent, and cosecant lie between -90° and 0°, while principal values of the inverse cosine, cotangent, and secant lie between 90° and 180°. We will use, for understanding, the examples which follow.

EXAMPLE: Find the principal value of the angle in the function

$$\theta = \text{Arccos } (0.4472)$$

SOLUTION: Using the trigonometric tables, we find the angle whose cosine is 0.4472 is 63° 26' or 296° 34'. We choose 63° 26' as this is the first quadrant angle and is the principal value. We reject 296° 34' because it is a fourth quadrant angle and the principal values for the Arccos function are limited to the first and second quadrants.

EXAMPLE: Find the principal value of the angle in the function

$$\theta = \text{Arccos } (-0.5000)$$

SOLUTION: Using the trigonometric tables, we find the angle whose cosine is (-0.5000) is 120° or 240°. We choose 120° because 240° is in the third quadrant and does not satisfy the value we agreed on as the principal value for the cosine function.

EXAMPLE: Find the principal value of the angle in the function

$$\theta = \text{Arctan } 1$$

SOLUTION: The angle whose tangent is 1 is 45° or 225°. We reject 225°, a third quadrant angle, and select 45° because it is a first quadrant angle.

EXAMPLE: Find the principal value of the angle in the function

$$\theta = \text{Arcsec } 2.236$$

SOLUTION: If the trigonometric tables do not list secant values, the function may be changed by the following:
Since

$$\sec \theta = \frac{1}{\cos \theta}$$

then

$$\text{Arcsec } (2.236) = \text{Arccos } \frac{1}{2.236}$$

$$= \text{Arccos } (0.4472)$$

and we find that the angle whose cosine is 0.4472 is 63° 26' and the principal value of the angle whose secant is 2.236 is 63° 26'.

PROBLEMS: Find the principal values of the angles in the following functions:

1. $\theta = \text{Arccos } (0.9135)$
2. $\theta = \text{Arcsin } (0.8829)$
3. $\theta = \text{Arctan } (11.430)$
4. $\theta = \text{Arccot } (-0.1169)$
5. $\theta = \text{Arcsec } (1.0075)$
6. $\theta = \text{Arctan } (-0.1228)$

ANSWERS:

1. 24°
2. 62°
3. 85°
4. 96° 40'
5. 7°
6. -7°

In dealing with the inverse trigonometric functions, we may be presented the problem of finding the principal value of the function $\theta = \text{Arcsin } \left(\frac{\sqrt{2}}{2}\right)$. In this case we could solve it as follows:

Draw a right triangle as shown in figure 8-1. This expression $\frac{\sqrt{2}}{2}$ can be rewritten as $\frac{1}{\sqrt{2}}$ by the following steps:

$$\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Recall that $\sin \theta$ equals $\frac{y}{r}$. The triangle is a 45°-90° triangle as shown in Mathematics, Vol. 1, NavPers 10069-C. Now, the function

$$\theta = \text{Arcsin } \frac{\sqrt{2}}{2}$$

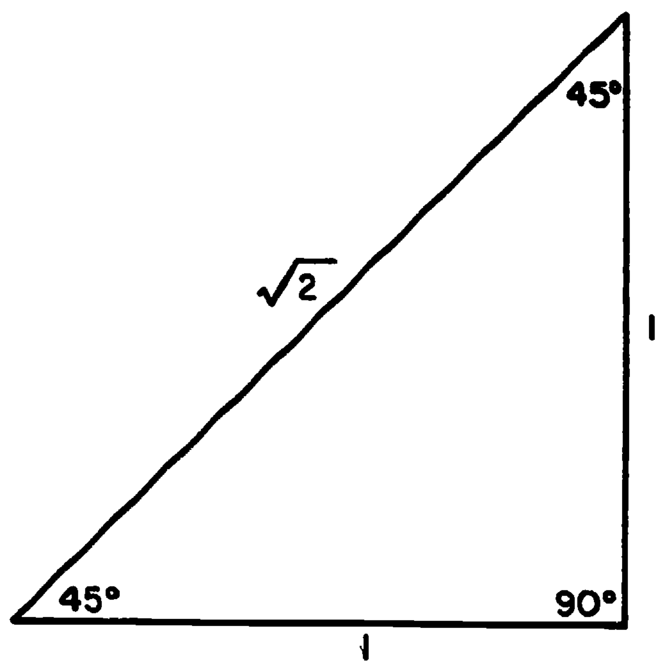


Figure 8-1.—45° - 90° triangle.

becomes

$$\theta = \text{Arcsin}\ \frac{1}{\sqrt{2}}$$

and we find that

$$\theta = 45°$$

Use this same approach to answer the following questions.

PROBLEMS: Find the principal values of the angles in the following functions. (Hint; draw the two special triangles as shown in Mathematics, Vol. 1, NavPers 10069-C):

1. $\theta = \text{Arccos}\left(\frac{\sqrt{2}}{2}\right)$
2. $\theta = \text{Arctan}\ (\sqrt{3})$
3. $\theta = \text{Arccot}\ (-\sqrt{3})$
4. $\theta = \text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$
5. $\theta = \text{Arcsin}\left(\frac{\sqrt{3}}{2}\right)$
6. $\theta = \text{Arccot}\left(-\frac{1}{\sqrt{3}}\right)$.

ANSWERS:

1. 45°
2. 60°
3. 150°
4. 30°
5. 60°
6. 120°

If we are to find, using the principal values, the value of the expression Arctan $\sqrt{3}$ - Arctan $\frac{1}{\sqrt{3}}$ in radians, we proceed as follows:

$$\text{Arctan}\ \sqrt{3} = 60°$$

and

$$\text{Arctan}\ \frac{1}{\sqrt{3}} = 30°$$

thus

$$\text{Arctan}\ \sqrt{3} - \text{Arctan}\ \frac{1}{\sqrt{3}} = 60° - 30°$$

and

$$1° = \frac{\pi}{180}\ \text{radians}$$

then

$$30° = \frac{\pi}{180}\left(\frac{30}{1}\right)\text{radians}$$

$$= \frac{\pi}{6}\ \text{radians}$$

PROBLEMS: Using the principal values, give the values of the following expressions in radians:

1. Arcsin $\frac{1}{2}$ - Arccos $\frac{1}{2}$
2. Arccos $\frac{\sqrt{3}}{2}$ - Arcsin $\frac{\sqrt{3}}{2}$
3. Arctan 1 - Arctan $\frac{1}{\sqrt{3}}$
4. Arctan $\sqrt{3}$ - Arcsin $\frac{1}{2}$

ANSWERS:

1. $-\frac{\pi}{6}$
2. $-\frac{\pi}{6}$
3. $\frac{\pi}{12}$
4. $\frac{\pi}{6}$

RELATIONS AMONG INVERSE FUNCTIONS

In order to understand the relations among the inverse functions, we will start by drawing a triangle. If θ equals Arcsin x, we can write sin θ equals x. We now draw a triangle which contains the angle whose sine is x and assume the hypotenuse equal to one. The remaining side of the triangle will be, from the Pythagorean theorem, $\sqrt{1 - x^2}$. This is shown in figure 8-2.

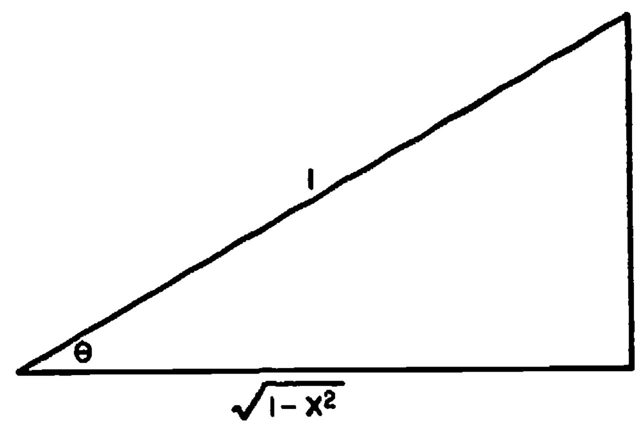


Figure 8-2.—Triangle containing Arcsin x.

Now, we can write all of the functions and inverse functions of the angle θ in terms of the sides of the triangle as follows:

$$\sin \theta = x, \qquad \text{or } \theta = \text{Arcsin } x$$

$$\cos \theta = \sqrt{1 - x^2}, \text{ or } \theta = \text{Arccos } \sqrt{1 - x^2}$$

$$\tan \theta = \frac{x}{\sqrt{1 - x^2}}, \text{ or } \theta = \text{Arctan } \frac{x}{\sqrt{1 - x^2}}$$

$$\csc \theta = \frac{1}{x}, \qquad \text{or } \theta = \text{Arccsc } \frac{1}{x}$$

$$\sec \theta = \frac{1}{\sqrt{1 - x^2}}, \text{ or } \theta = \text{Arcsec } \frac{1}{\sqrt{1 - x^2}}$$

$$\cot \theta = \frac{\sqrt{1 - x^2}}{x}, \text{ or } \theta = \text{Arccot } \frac{\sqrt{1 - x^2}}{x}$$

All of the inverse functions are equal to θ; therefore, they are equal to each other. We will use this type of analysis to solve a few problems.

EXAMPLE: Using principal values, find the tangent of the angle whose sine is $\frac{\sqrt{3}}{2}$; that is,

$$\tan \text{Arcsin } \frac{\sqrt{3}}{2} = ?$$

SOLUTION: Draw the triangle containing the Arcsin $\frac{\sqrt{3}}{2}$ as shown in figure 8-3. The remaining side will be given by

$$x = \sqrt{2^2 - (\sqrt{3})^2} = \sqrt{1} = 1$$

Using the tangent ratio we have

$$\tan \theta = \sqrt{3}$$

and

$$\theta = 60°$$

EXAMPLE: Using principal values find

$$\sin\left(\text{Arccos } \frac{3}{5} - \text{Arcsin } \frac{4}{5}\right)$$

SOLUTION: Draw the triangle as shown in figure 8-4. The missing side of the triangle containing Arccos $\frac{3}{5}$ is given by

$$y = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$$

Notice that this triangle also contains Arcsin $\frac{4}{5}$, so that

$$\text{Arccos } \frac{3}{5} = \text{Arcsin } \frac{4}{5}$$

and the original expression becomes

$$\sin 0°$$

which is zero.

PRACTICE PROBLEMS: Evaluate the following expressions:

1. $\sin\left(\text{Arccos } \frac{3}{5}\right)$
2. $\tan\left(\text{Arcsin } \frac{1}{10}\right)$
3. $\cot\left(\text{Arcsin } \frac{x}{1 + x}\right)$
4. $\sec\left(\text{Arctan } \frac{x^2}{x^2 - 1}\right)$
5. $\cos\left(\text{Arcsin } \frac{1}{x^2 - 1}\right)$

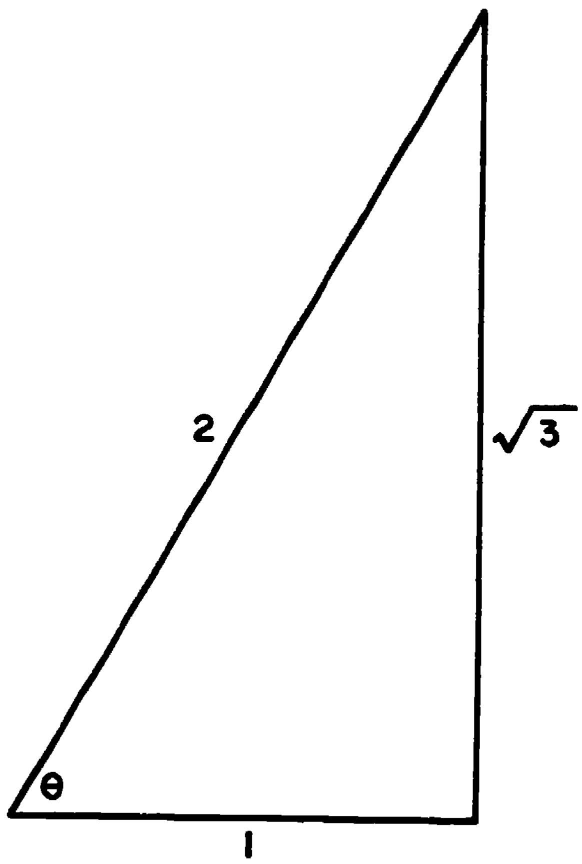


Figure 8-3.—Triangle containing Arcsin $\frac{\sqrt{3}}{2}$.

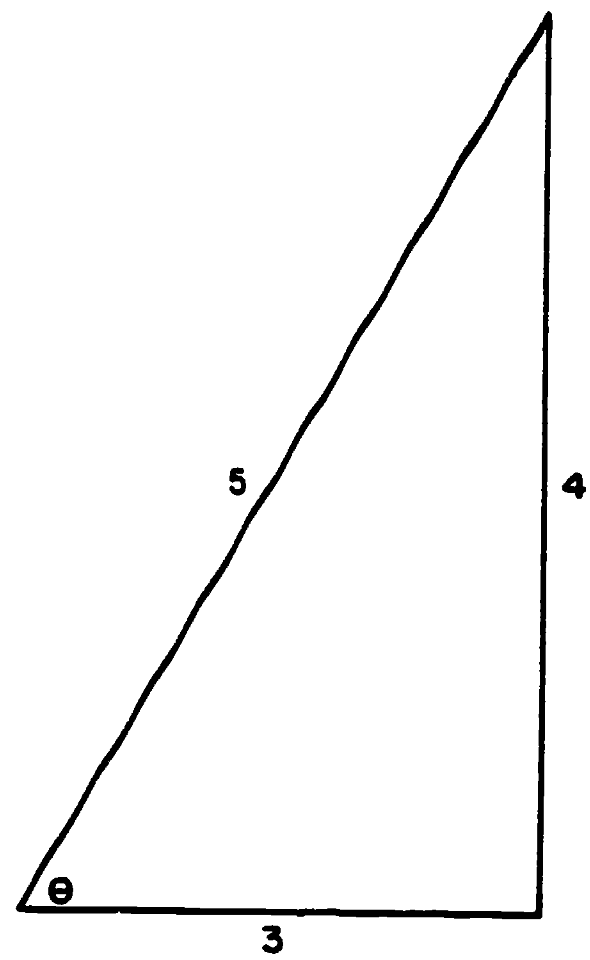


Figure 8-4.—Triangle containing Arccos $\frac{3}{5}$.

ANSWERS:

1. $\frac{4}{5}$
2. $\frac{1}{9.95}$
3. $\frac{\sqrt{2x + 1}}{x}$
4. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2 - 11}$
5. $\frac{x\sqrt{x^2 - 2}}{x^2 - 1}$

FORMULAS

In this section we will discuss the trigonometric formulas for addition and subtraction of angles, half angles, double angles, and transcendental functions. We will use examples for better understanding and in some instances we will derive formulas.

ADDITION AND SUBTRACTION FORMULAS

Four formulas express the sine or cosine of the sum and difference of two angles as a function of the sines and cosines of the single angles. They are very important because they are the basis for much of trigonometric analysis. From the following four formulas we may derive all of the formulas in the following sections:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \quad (1)$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \quad (2)$$

$$\sin (A - B) = \sin A \cos B - \cos A \sin B \quad (3)$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B \quad (4)$$

We will prove these four formulas for angles whose sum is less than 90°. They are actually true for all angles.

In figure 8-5 we have indicated the sum of two angles, A and B. The hypotenuse of the triangle containing angle B has been set equal to 1 so that the legs of this triangle have the values sin B and cos B.

Thus,

$$\frac{PL}{1} = \sin B$$

and

$$\frac{OL}{1} = \cos B$$

In the triangle containing angle A, we can see that

$$\cos A = \frac{ON}{OL} = \frac{ON}{\cos B}$$

Therefore,

$$ON = \cos A \cos B$$

and

$$\sin A = \frac{NL}{OL} = \frac{NL}{\cos B}$$

Therefore

$$NL = \sin A \cos B$$

Now, let us add a construction to the figure, as in figure 8-6, and calculate more lines in

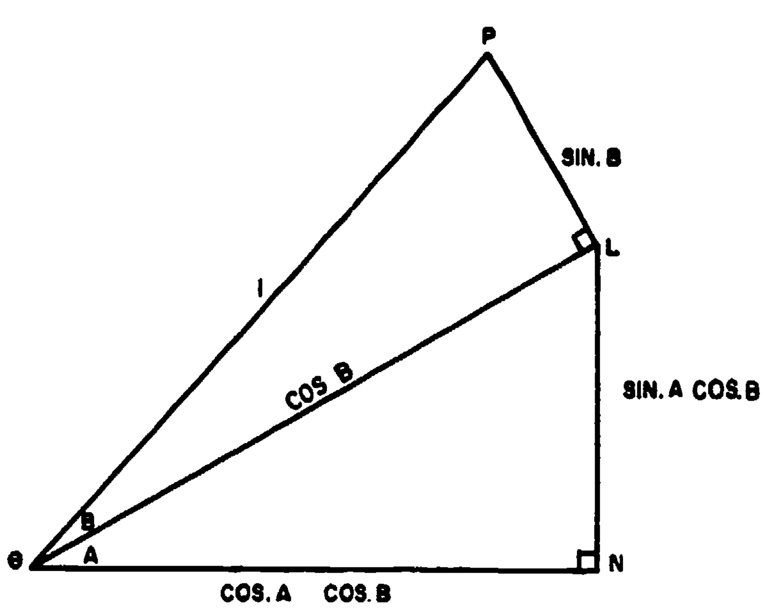


Figure 8-5.—Sum of two angles, part one.

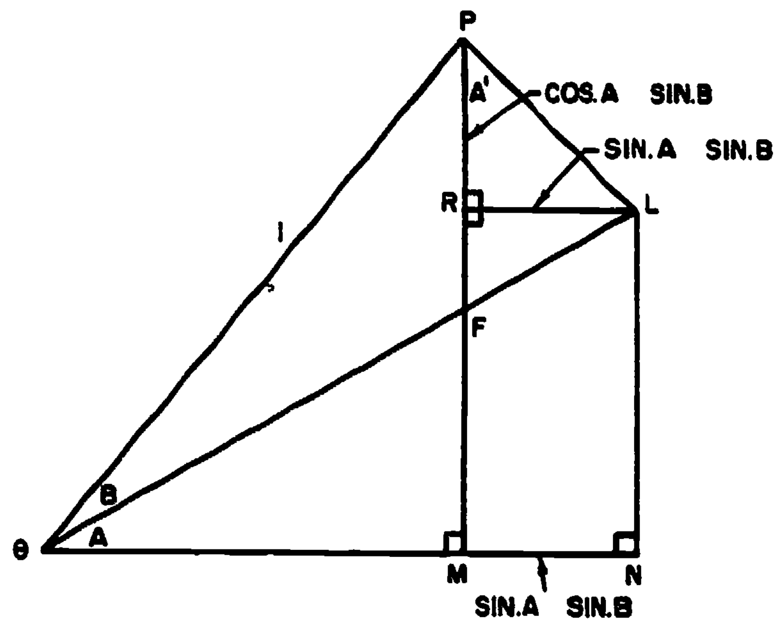


Figure 8-6.—Sum of two angles, part two.

terms of the sine and cosine of angles A and B. First, let us prove that angle A is equal to angle A'.

Triangle OMF is similar to triangle PFL, and angles A and A' are corresponding angles. Therefore, angle A equals angle A'. In triangle PRL

$$\cos A' = \frac{PR}{\sin B}$$

and

$$\begin{aligned} PR &= \cos A' \sin B \\ &= \cos A \sin B \qquad (7) \end{aligned}$$

Also,

$$\sin A' = \frac{RL}{\sin B}$$

and

$$\begin{aligned} RL &= \sin A' \sin B \\ &= \sin A \sin B \qquad (8) \end{aligned}$$

Furthermore,

$$RL = MN = \sin A \sin B \qquad (9)$$

Now, in triangle OMP we can write

$$\sin (A + B) = \frac{PM}{1} = PR + RM \qquad (10)$$

but

$$RM = LN \tag{11}$$

so

$$\sin(A+B) = PR + LN = LN + PR \tag{12}$$

Also

$$\cos(A+B) = \frac{OM}{1} = ON - MN \tag{13}$$

Substituting from equations (6) and (7) into equation (12)

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \tag{14}$$

Substituting from equations (5) and (9) into equation (13)

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{15}$$

EXAMPLE: Use the addition formulas to find cos 75°.

SOLUTION: Use cos(A + B) equal to cos A cos B - sin A sin B. From this we write cos 75° equals cos (45° + 30°) and substitute as follows:

$$\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

The subtraction formulas can be derived from the addition formulas by substituting (-B) for (+B). Now, we have

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B) \tag{16}$$

and

$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B) \tag{17}$$

But, the cosine of a negative angle is equal to the cosine of the angle. The sine of a negative angle, however, is equal to minus the sine of the angle; that is,

$$\cos(-B) = \cos B \tag{18}$$

$$\sin(-B) = -\sin B \tag{19}$$

Substitution of these values in equations (16) and (17) gives

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{20}$$

and

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \tag{21}$$

EXAMPLE: Show that

$$\sin(45° - \theta) = \frac{\cos\theta - \sin\theta}{\sqrt{2}}$$

SOLUTION: Applying equation (20)

$$\sin(45° - \theta) = \sin 45° \cos\theta - \cos 45° \sin\theta$$

but

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$$

Substituting these values we have

$$\sin(45° - \theta) = \frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{2}}$$

$$= \frac{\cos\theta - \sin\theta}{\sqrt{2}}$$

We can use these four formulas, (14), (15), (20), and (21), to drive a number of other important ones. One of these is the tangent addition formula.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{22}$$

In order to prove this formula, proceed as follows:

Taking the ratio of equalities (14) and (15), we have

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \tag{23}$$

Dividing both numerator and denominator of the right side of this equation by cos A cos B we have

$$\tan(A + B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}} \tag{24}$$

which reduces to

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{25}$$

Another important formula is the subtraction formula for the tangent. To find tan(A - B) replace B by (-B). Therefore,

$$\tan(A - B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \qquad (26)$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad (27)$$

EXAMPLE: Use subtraction formulas to find tan 15°

SOLUTION: Use the special triangles as previously discussed.

$$\tan 15^\circ = \tan(45^\circ - 30^\circ)$$

and

$$\tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + (1)\frac{\sqrt{3}}{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}}\left(\frac{3 - \sqrt{3}}{3 - \sqrt{3}}\right)$$

$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= 2 - \sqrt{3}$$

EXAMPLE: Given tan 45° equals 1 and tan 60° equals $\sqrt{3}$. Find the tangent of 105°

SOLUTION: Applying this knowledge to equation (22), we have

$$\tan(45^\circ + 60^\circ) = \tan 105^\circ = \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

It is easy to evaluate a fraction in this form by rationalizing the denominator.

Multiply numerator and denominator by the same numbers as are in the denominator but connected by the opposite sign.

$$\frac{(1 + \sqrt{3})^2}{(1 - \sqrt{3})(1 + \sqrt{3})} = \frac{1 + 2\sqrt{3} + 3}{1 - 3}$$

$$= \frac{4 + 2\sqrt{3}}{-2}$$

$$= -(2 + \sqrt{3})$$

$$= -3.732$$

Therefore,

$$\tan 105^\circ = -3.732$$

PROBLEMS: Use the addition and subtraction formulas to find the values of the following without tables:

1. sin 75°
2. cos 15°
3. tan 75°
4. cot 165° Hint: recall $\cot(180 - \theta) = -\cot\theta$ and $\cot\theta = \frac{1}{\tan\theta}$

ANSWERS:

1. $\frac{\sqrt{6} + \sqrt{2}}{4}$
2. $\frac{\sqrt{6} + \sqrt{2}}{4}$
3. $2 + \sqrt{3}$
4. $\frac{1}{\sqrt{3} - 2}$ or $-\sqrt{3} - 2$

DOUBLE ANGLE FORMULAS

The addition formulas may be used to derive the double angle formulas.

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad (28)$$

In equations (14) and (15), if B equals A, we can write

$$\sin 2A = \sin A \cos A + \cos A \sin A \quad (29)$$

$$\cos 2A = \cos A \cos A - \sin A \sin A$$

from which we obtain

$$\sin 2A = 2 \sin A \cos A \qquad (30)$$

and, using

$$\sin^2 A + \cos^2 A = 1$$

then

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \qquad (31) \\ &= 1 - 2 \sin^2 A \end{aligned}$$

Substituting A for B in equation (25), we obtain

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad (32)$$

EXAMPLE: Evaluate sin 15° cos 15°.

SOLUTION: Since

$$2 \sin A \cos A = \sin 2A$$

and

$$\sin A \cos A = \frac{1}{2} \sin 2A$$

then

$$\begin{aligned} \sin 15^\circ \cos 15^\circ &= \frac{1}{2} \sin 2(15^\circ) \\ &= \frac{1}{2} \sin 30^\circ \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

EXAMPLE: Find the three first quadrant angles which satisfy the trigonometric equation

$$\sin 4x = \cos 2x$$

SOLUTION: From the double angle formulas, we can write

$$2 \sin 2x \cos 2x = \cos 2x$$

or

$$\cos 2x(2 \sin 2x - 1) = 0$$

The solutions of this equation may be obtained by setting the factors equal to zero and making use of inverse trigonometric functions. We may write

$$\begin{aligned} \cos 2x &= 0 \\ 2x &= \text{Arccos } 0 \\ &= 90^\circ \\ x &= 45^\circ \end{aligned}$$

and

$$\begin{aligned} 2 \sin 2x - 1 &= 0 \\ \sin 2x &= \frac{1}{2} \\ 2x &= \text{Arcsin } \frac{1}{2} \\ 2x &= 30^\circ,\ 150^\circ \\ x &= 15^\circ,\ 75^\circ \end{aligned}$$

The equation has three solutions, x equals 15°, 45°, and 75°. Notice that in writing down the values of the inverse functions, it was necessary to include 150° since, when divided by 2, this gives an angle in the first quadrant.

HALF ANGLE FORMULAS

Dividing all the angles in equation (31) by 2 we obtain

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \qquad (33)$$

Using equation (33), we can derive two useful formulas. Adding and subtracting $\sin^2 \frac{A}{2}$ on the right side of equation (33) we have

$$\cos A = (\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}) - 2 \sin^2 \frac{A}{2} \qquad (34)$$

Observe that the methods necessary for simplifying trigonometric identities and equations often include operations which may at first appear to be pointless. In the preceding sentence we referred to "adding and subtracting $\sin^2 \frac{A}{2}$ on the right side of equation (33)." The advantage of adding $\sin^2 \frac{A}{2}$ becomes obvious when we group it with $\cos^2 \frac{A}{2}$. The expression $(-\sin^2 \frac{A}{2})$ is added to the right-hand side along with $(\sin^2 \frac{A}{2})$ in order to avoid changing the overall value.

The quantity in the parentheses in equation (34) is equal to 1, so

$$\cos A = 1 - 2\sin^2\frac{A}{2} \qquad (35)$$

Rearranging equation (35) and taking the square root of both sides, we have

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \qquad (36)$$

Adding and subtracting $\cos^2\frac{A}{2}$ on the right side of equation (33), we have

$$\cos A = 2\cos^2\frac{A}{2} - \left(\cos^2\frac{A}{2} + \sin^2\frac{A}{2}\right) \qquad (37)$$

but, the quantity within the parenthesis is equal to 1 so that

$$\cos A = 2\cos^2\frac{A}{2} - 1 \qquad (38)$$

Rearranging equation (38) and taking the square root of both sides gives us

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad (39)$$

Taking the ratio of equations (36) and (39), we have

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \qquad (40)$$

Notice that the use of (+) or (-) is dependent upon the quadrant of angle termination.

EXAMPLE: Find cos 15°, if cos 30° equals 0.866 or $\frac{\sqrt{3}}{2}$.

SOLUTION: From equation (39) we have

$$\cos\frac{30^\circ}{2} = \cos 15^\circ = \sqrt{\frac{1+0.866}{2}} = \sqrt{0.933}$$

Thus

$$\cos 15^\circ = 0.9659$$

The solution using $\frac{\sqrt{3}}{2}$ follows:

$$\cos\frac{30^\circ}{2} = \cos 15^\circ = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

PRACTICE PROBLEMS: Use the half angle formulas to find the exact value of the following:

1. sin 15°
2. cos 135°
3. tan 22.5°
4. tan 195°

ANSWERS:

1. $\pm\frac{\sqrt{2-\sqrt{3}}}{2}$
2. $\pm\sqrt{\frac{1}{2}}$
3. $\pm\sqrt{3-2\sqrt{2}}$
4. $\pm\sqrt{7-4\sqrt{3}}$

TRANSCENDENTAL FUNCTIONS

To define transcendental functions we state that any functions other than algebraic functions will be classified, for the purposes of this course, as transcendental functions. This group of transcendental functions includes such functions as trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

Later in calculus we will prove that the sine and cosine of an angle can be calculated from the following series, if the angle is expressed in radian measure:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

NOTE: 3! is read 3 factorial and is equal to 1 x 2 x 3 or 6. 5! is read 5 factorial and is equal to 1 x 2 x 3 x 4 x 5 or 120.

At the same time the expansion of the function e^x where e is the number 2.71828. . . , the base of the system of natural logarithms, is as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

Moreover, by using the notation $i = \sqrt{-1}$ two similar expressions are obtained:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

and

$$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \cdots$$

Adding e^{ix} to e^{-ix} term by term and dividing by two we obtain an expression equal to cos x.

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x$$

Subtracting e^{-ix} from e^{ix} term by term and dividing by 2i we have an expression equal to sin x.

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

In calculus, trigonometric and logarithmic functions are grouped together and called transcendental functions, partly because they cannot be expressed by a simple algebraic formula, and partly because they both are related to the number e. This relation becomes important in derivations in advanced and applied calculus.

EQUATIONS

A trigonometric equation is an equality which is true for some values but may not be true for all values of the variable. The principles and processes used to solve algebraic equations may be used to solve trigonometric equations. The identities and reduction formulas previously studied may also be used in solving trigonometric equations. There are so many different approaches to solving these equations that we will use several examples for better understanding.

MULTIPLE SOLUTIONS

We will use the following examples and practice problems to show the multiple solutions of a trigonometric equation.

EXAMPLE: Find the value of θ if

$$\tan \theta = 1$$

SOLUTION: We must find the angle or angles having a tangent equal to 1. Using the inverse trigonometric functions, we may write

$$\tan \theta = 1$$

$$\theta = \arctan 1$$

$$\theta = 45°$$

and recalling that

$$\tan(180° + \theta) = \tan \theta$$

then

$$\tan(180° + 45°) = \tan 45°$$

$$\tan 225° = \tan 45°$$

Thus,

$$\arctan 1 = 45°$$

and

$$\arctan 1 = 225°$$

Therefore, the solutions for the equation are 45° and 225°. The 45° angle is a first quadrant angle and the 225° angle is a third quadrant angle and both have a positive sign and the same trigonometric value; thus we have a multiple solution.

Notice that the two solutions of the equation differ from the investigation of the inverse trigonometric functions in which we were required to find the PRINCIPAL value. The PRINCIPAL value solution was found to be in a particular quadrant. In multiple solutions we will use the term PRIMARY to indicate that we are searching for solutions which are restricted to the range

$$0° \leq \theta < 360°$$

Notice also that if we remove the restriction of the term PRIMARY we may write

$$\tan \theta = \tan(\theta + n \cdot 360°)$$

and we find, if n is an integer, that there are many solutions to the equation.

EXAMPLE: Find the primary solutions to the equation

$$\cos \theta = \frac{\sqrt{3}}{2}$$

SOLUTION: We first use the inverse trigonometric function to write

$$\cos\theta = \frac{\sqrt{3}}{2}$$

then

$$\theta = \arccos\frac{\sqrt{3}}{2}$$
$$= 30°$$

but

$$\cos\theta = \cos(360° - \theta)$$
$$\cos 30° = \cos 330°$$

Therefore, the solutions are 30° and 330°.

EXAMPLE: Find the primary solutions to the equation

$$\csc\theta = 2$$

SOLUTION: As in the previous example, we write

$$\csc\theta = 2$$

and

$$\csc\theta = \frac{1}{\sin\theta}$$
$$\frac{1}{\sin\theta} = 2$$
$$\sin\theta = \frac{1}{2}$$

then

$$\theta = \arcsin\frac{1}{2}$$
$$= 30°$$

but

$$\sin\theta = \sin(180° - \theta)$$
$$\sin 30° = \sin 150°$$

Therefore, the solutions are 30° and 150°

PROBLEMS: Find the primary values of θ in the following equations:

1. $\tan\theta = -1$
2. $\sec\theta = 2$
3. $\sin\theta = -\frac{\sqrt{3}}{2}$
4. $\cos\theta = \frac{1}{2}$

ANSWERS:

1. 135° and 315°
2. 60° and 300°
3. 240° and 300°
4. 60° and 300°

The following examples show how to find the solution of more difficult equations.

EXAMPLE: Find the primary values of the equation

$$\sin\theta + \sin\theta\cot\theta = 0$$

SOLUTION: We factor the equation and find that

$$\sin\theta + \sin\theta\cot\theta = 0$$

and

$$\sin\theta\,(1 + \cot\theta) = 0$$

Setting each factor equal to zero, we have the two equations

$$\sin\theta = 0$$

and

$$1 + \cot\theta = 0$$

Solve each as follows:

$$\sin\theta = 0$$

then

$$\theta = \arcsin 0$$
$$= 0° \text{ and } 180°$$

and

$$1 + \cot\theta = 0$$
$$\cot\theta = -1$$

Therefore,

$$\theta = \text{arccot } -1$$

$$= 135° \text{ and } 315°$$

Therefore, the solutions to the equation are 0°, 135°, 180°, and 315°.

EXAMPLE: Find the primary values of the equation

$$2 \sin^2\theta + \sin\theta - 3 = 0$$

SOLUTION: Factor the equation and then set each factor equal to zero, as follows:

$$2 \sin^2\theta + \sin\theta - 3 = 0$$

and

$$(2 \sin\theta - 1)(\sin\theta - 1) = 0$$

then

$$2 \sin\theta - 1 = 0$$

and

$$\sin\theta - 1 = 0$$

Solve each as follows:

$$2 \sin\theta - 1 = 0$$

$$2 \sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

then

$$\theta = \arcsin\frac{1}{2}$$

and

$$\theta = 30° \text{ and } 150°$$

Also,

$$\sin\theta - 1 = 0$$

$$\sin\theta = 1$$

then

$$\theta = \arcsin 1$$

$$= 90°$$

Therefore, the solutions are 30°, 90°, and 150°.

PRACTICE PROBLEMS: Find the primary value of the following equations:

1. $2 \cos^2\theta = 3 \cos\theta - 1$
2. $\tan^2\theta = 3$
3. $2 \sin^2\theta - 3 \sin\theta = -1$

ANSWERS:

1. 0°, 60°, 300°
2. 60°, 120°, 240°, 300°
3. 30°, 90°, 150°

LIMITED SOLUTIONS

We will consider two types of possible solutions to fall within the category of limited solutions. The following examples show both of these types.

The first type of limited solution occurs when an equation is solved and one of the solutions is not true upon inspection.

EXAMPLE: Find the primary values of θ in the equation

$$\sin^2\theta \sec\theta = \sec\theta$$

SOLUTION: We first rearrange, and then factor the equation as follows:

$$\sin^2\theta \sec\theta = \sec\theta$$

$$\sec\theta - \sin^2\theta \sec = 0$$

$$\sec\theta (1 - \sin^2\theta) = 0$$

Set each factor equal to zero

$$1 - \sin^2\theta = 0$$

and

$$\sec\theta = 0$$

Solving the first equation

$$1 - \sin^2\theta = 0$$

$$\sin^2\theta = 1$$

$$\sin\theta = \pm 1$$

then

$$\theta = \arcsin \pm 1$$

$$= 90° \text{ and } 180°$$

Solving the second equation

$$\sec \theta = 0$$

then

$$\theta = \text{arcsec } 0$$

However, there is no angle for which sec θ equals zero and we reject this false solution. Therefore, the primary solutions for the original equation are 90° and 180°.

The second type of limited solutions occur when introducing a radical into an equation by substitution or by squaring both members of an equation in solving the equation. These solutions are called extraneous roots. Solutions of this type must be substituted into the original equation for verification.

EXAMPLE: Find the primary values of θ for the equation

$$\tan \theta - \sec \theta + 1 = 0$$

SOLUTION: We first rearrange the equation to read

$$\tan \theta + 1 = \sec \theta$$

Square both sides:

$$\tan^2\theta + 2 \tan \theta + 1 = \sec^2\theta$$

Rearrange again:

$$2 \tan \theta = \sec^2\theta - (\tan^2\theta + 1)$$

This gives

$$2 \tan \theta = 0$$

and

$$\tan \theta = 0$$

Therefore,

$$\theta = \arctan 0$$

$$= 0° \text{ and } 180°$$

These seem to be the values of the equation, but we squared both sides of the equation and we must now substitute these values into the original equation to verify the values. Upon substituting we find

$$\tan \theta - \sec \theta + 1 = 0$$

This implies that

$$\tan 0° - \sec 0° + 1 = 0$$

and since

$$0 - 1 + 1 = 0$$

the value 0° holds true.

For 180° we find

$$\tan \theta - \sec \theta + 1 = 0$$

$$\tan 180° - \sec 180° + 1 = 0$$

but

$$1 -(-1) + 1 \neq 0$$

and we say 180° is an extraneous root.

PRACTICE PROBLEMS: Find the primary values for θ in the following equations:

1. $\tan \theta \cos^2\theta = \sin^2\theta$
2. $\cos^2\theta \sin \theta = \sin \theta + 1$
3. $2 \sec \theta + 1 - \cos \theta = 0$
4. $\cot \theta - \csc \theta - \sqrt{3} = 0$

ANSWERS:

1. 0°, 45°, 180°, 225°
2. 270°
3. 180°
4. 240°

CHAPTER 9

STRAIGHT LINES

The study of straight lines provides an excellent introduction to analytic geometry. As its name implies, this branch of mathematics is concerned with geometrical relationships. However, in contrast to plane and solid geometry, the study of these relationships in analytic geometry is accomplished by algebraic analysis.

The invention of the rectangular coordinate system made algebraic analysis of geometrical relationships possible. Rene Descartes, a French mathematician, is credited with this invention, and the coordinate system is often designated as the Cartesian coordinate system in his honor.

Recalling our study of the rectangular coordinate system in Mathematics, Vol. 1, NavPers 10069-C, we review the following definitions and terms:

1. Distances measured along, or parallel to, the X axis are ABSCISSAS. They are positive if measured to the right of the origin; they are negative if measured to the left of the origin. (See fig. 9-1.)

2. Distances measured along, or parallel to, the Y axis are ORDINATES. They are positive if measured above the origin; they are negative if measured below the origin.

3. Any point on the coordinate system is designated by naming its abscissa and ordinate. For example, the abscissa of point P (fig. 9-1) is 3 and the ordinate is -2. Therefore, the symbolic notation for P is

$$P(3, -2)$$

In using this symbol to designate a point, the abscissa is always written first, followed by a comma. The ordinate is written last. Thus the general form of the symbol is

$$P(x, y)$$

4. The abscissa and ordinate of a point are its COORDINATES.

DISTANCE BETWEEN TWO POINTS

The distance between two points, P_1 and P_2, can be expressed in terms of their coordinates by using the Pythagorean Theorem. From our study of Mathematics, Vol. 1, NavPers 10069-C, we recall that this theorem is stated as follows:

In a right triangle, the square of the length of the hypotenuse (longest side) is equal to the sum of the squares of the lengths of the two shorter sides.

Let the coordinates of P_1 be (x_1,y_1) and let those of P_2 be (x_2,y_2), as in figure 9-2. By the Pythagorean Theorem,

$$d = \sqrt{(P_1N)^2 + (P_2N)^2}$$

where d represents the distance from P_1 to P_2. We can express the length of P_1N in terms of x_1 and x_2 as follows:

$$P_1N = x_2 - x_1$$

Likewise,

$$P_2N = y_2 - y_1$$

By substitution in the formula developed previously for d, we reach the following conclusion:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Although we have demonstrated the formula for the first quadrant only, it can be proved

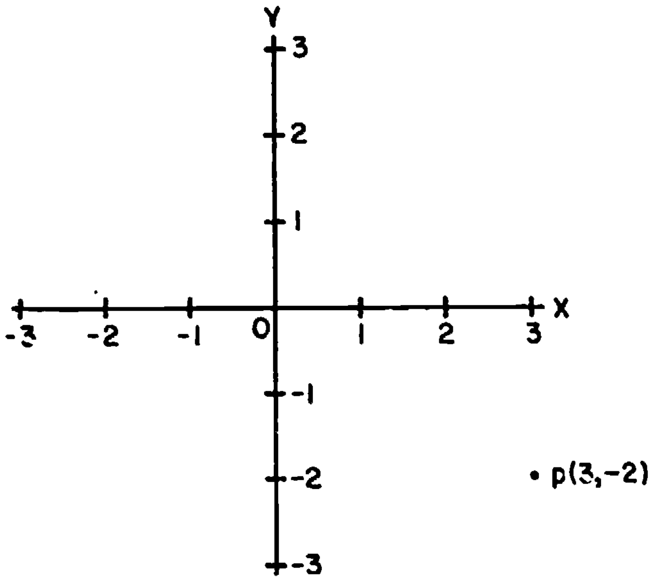


Figure 9-1. Rectangular coordinate system.

for all quadrants and all pairs of points.

EXAMPLE: In figure 9-2, $x_1 = 2$, $x_2 = 6$, $y_1 = 2$, and $y_2 = 5$. Find the length of d.

SOLUTION: $d = \sqrt{(6-2)^2 + (5-2)^2}$

$= \sqrt{4^2 + 3^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

This result could have been foreseen by observing that triangle P_1NP_2 is a 3-4-5 triangle.

EXAMPLE: Find the distance between P_1 (4, 6) and P_2(10, 4).

SOLUTION: $d = \sqrt{(10-4)^2 + (4-6)^2}$

$d = \sqrt{36 + 4}$

$d = 2\sqrt{10}$

DIVISION OF A LINE SEGMENT

Many times it becomes necessary to find the coordinates of a point which is some known fraction of the distance between P_1 and P_2.

In figure 9-3, P is a point lying on the line joining P_1 and P_2 so that

$$\frac{P_1P}{P_1P_2} = k$$

If P should lie one-quarter of the way between P_1 and P_2, then k would equal 1/4.

Triangles P_1MP and P_1NP_2 are similar. Therefore,

$$\frac{P_1M}{P_1N} = \frac{P_1P}{P_1P_2}$$

Since $\frac{P_1P}{P_1P_2}$ is the ratio that defines k,

$$\frac{P_1M}{P_1N} = k$$

Therefore,

$$P_1M = k\,(P_1N)$$

Referring again to figure 9-3, observe that P_1N is equal to $x_2 - x_1$. Likewise, P_1M is equal to $x - x_1$. Therefore, replacing P_1M and P_1N with their equivalents in terms of x, the foregoing equation becomes

$$x - x_1 = k(x_2 - x_1)$$

$$x = x_1 + k(x_2 - x_1)$$

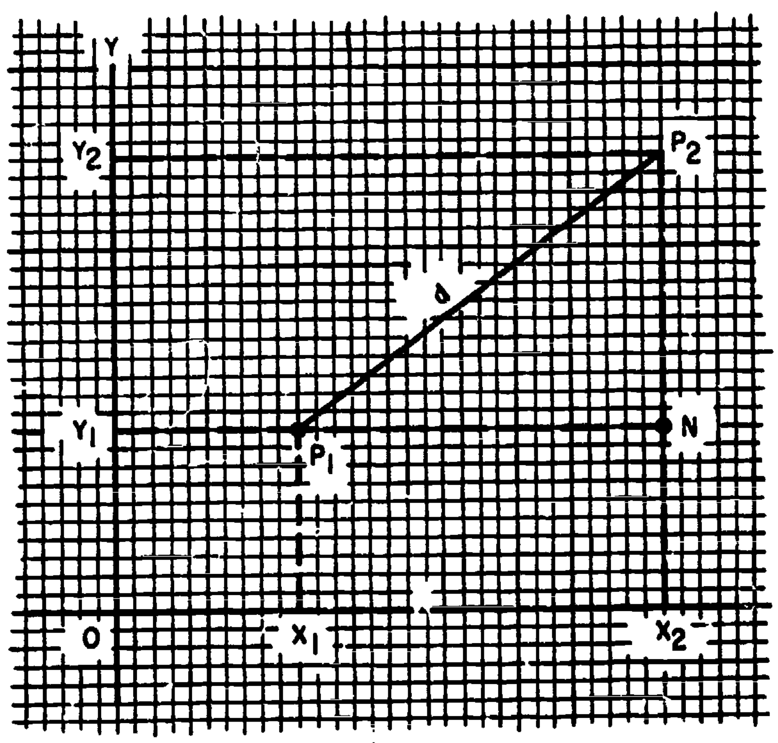


Figure 9-2.—Distance between two points.

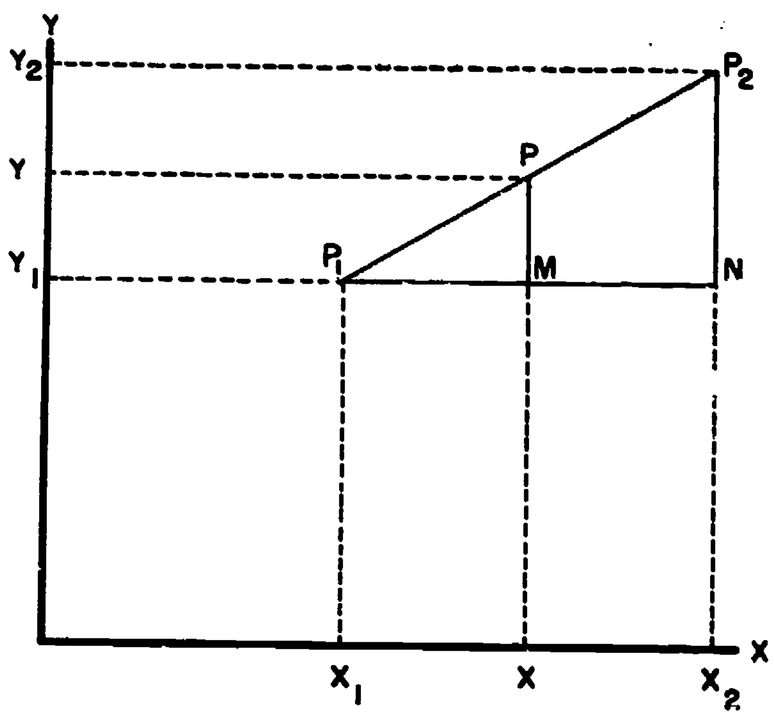


Figure 9-3.—Division of a line segment.

By similar reasoning,

$$y = y_1 + k(y_2 - y_1)$$

The x and y found as a result of the foregoing discussion are the coordinates of the desired point, whose distances from P_1 and from P_2 are determined by the value of k.

EXAMPLE: Find the coordinates of a point 1/4 of the way from P_1 (2,3) to P_2 (4,1).

SOLUTION:

$$k = \frac{1}{4}, \; x_2 - x_1 = 2, \; y_2 - y_1 = -2$$

$$x = 2 + \frac{1}{4}(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$y = 3 + \frac{1}{4}(-2) = 3 - \frac{1}{2} = \frac{5}{2}$$

The point P is $(\frac{5}{2}, \frac{5}{2})$.

FINDING THE MIDPOINT

When the midpoint of a line segment is to be found, the value of k is 1/2. Therefore,

$$x = x_1 + \frac{1}{2}(x_2 - x_1)$$

$$= x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1$$

$$= \frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$= \frac{1}{2}(x_1 + x_2)$$

By similar reasoning,

$$y = \frac{1}{2}(y_1 + y_2)$$

EXAMPLE: Find the midpoint of the line between P_1 (2,4) and P_2 (4,6).

SOLUTION: $k = \frac{1}{2}$

$$x = \frac{1}{2}(2 + 4)$$

$$= 3$$

$$y = \frac{1}{2}(4 + 6)$$

$$= 5$$

The midpoint is (3,5).

INCLINATION AND SLOPE

A line drawn on the rectangular coordinate system and crossing the X axis forms a positive acute angle with the X axis. This angle, shown in figure 9-4 as angle α, is called the angle of inclination of the line.

The slope of any line, such as AB in figure 9-4, is equal to the tangent of its angle of inclination. Slope is denoted by the letter m. Therefore, for line AB,

$$m = \tan \alpha$$

If the axes are in their conventional positions, a line sloping upward to the right has a positive slope. A line sloping downward to the right has a negative slope.

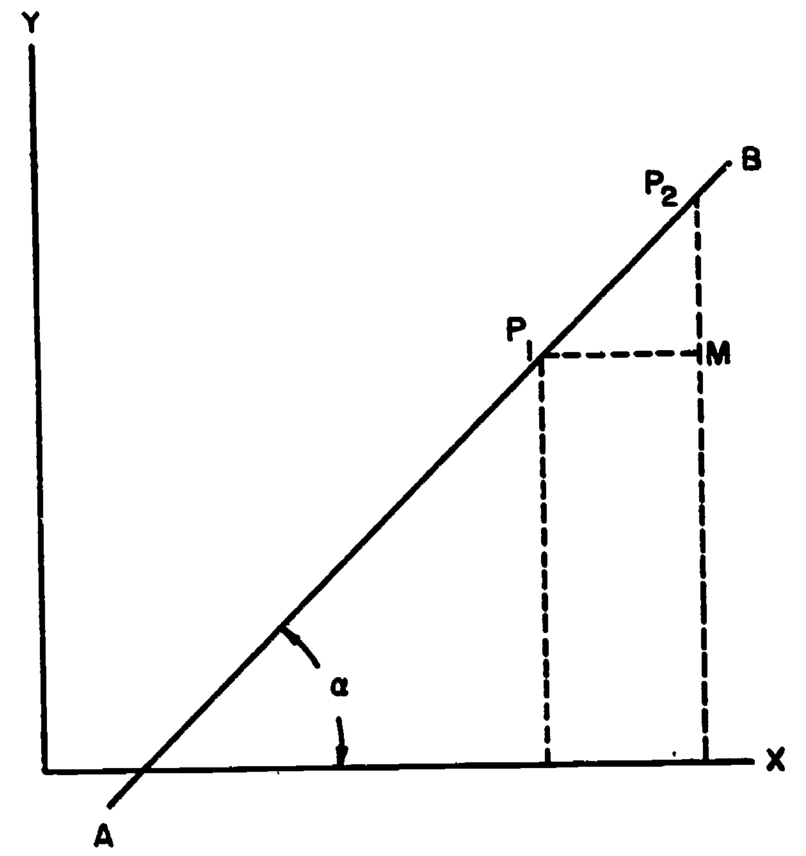


Figure 9-4.—Angle of inclination.

Since the tangent of α is the ratio of P_2M to P_1M, we can relate the slope of line AB to the points P_1 and P_2 as follows:

$$m = \tan \alpha = \frac{P_2M}{P_1M}$$

Designating the coordinates of P_1 as (x_1,y_1), and those of P_2 as (x_2,y_2), we recall that

$$P_2M = y_2 - y_1$$

$$P_1M = x_2 - x_1$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The quantities $(x_2 - x_1)$ and $(y_2 - y_1)$ represent changes that occur in the values of the x and y coordinates as a result of changing from P_2 to P_1 on line AB. The symbol used by mathematicians to represent an increment of change is the Greek letter delta (Δ). Therefore, Δx means "the change in x" and Δy means "the change in y." The amount of change in the x coordinate, as we change from P_2 to P_1, is $x_2 - x_1$. Therefore,

$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

We use this notation to express the slope of line AB, as follows:

$$m = \frac{\Delta y}{\Delta x}$$

EXAMPLE: Find the slope of the line connecting P_2 (7,6) and P_1 (-1,-4).

SOLUTION:

$$m = \frac{\Delta y}{\Delta x}$$

$$\Delta y = y_2 - y_1 = 6 - (-4) = 10$$

$$\Delta x = x_2 - x_1 = 7 - (-1) = 8$$

$$m = \frac{10}{8} = \frac{5}{4}$$

It is important to realize that the choice of labels for P_1 and P_2 is strictly arbitrary. If we had chosen the point (7,6) to be P_1 in the foregoing example, and the point (-1,-4) to be P_2, the following calculation would have resulted:

$$m = \frac{\Delta y}{\Delta x}$$

$$\Delta y = y_2 - y_1 = -4 - 6 = -10$$

$$\Delta x = x_2 - x_1 = -1 - 7 = -8$$

$$m = \frac{-10}{-8} = \frac{5}{4}$$

This is the same result as in the foregoing example.

The slope of 5/4 means that a point moving along this line would move vertically +5 units for every horizontal movement of +4 units. This result is consistent with the previously stated meaning of positive slope; i.e., sloping upward to the right.

If line AB in figure 9-4 were parallel to the X axis, y_1 and y_2 would be equal and the difference $(y_2 - y_1)$ would be 0. Therefore,

$$m = \frac{0}{x_2 - x_1} = 0$$

Thus we conclude that the slope of a horizontal line is 0. This conclusion can also be reached by noting that angle α (fig. 9-4) is 0 when the line is parallel to the X axis. Since the tangent of 0° is 0,

$$m = \tan\ \alpha = 0$$

The slope of a line that is parallel to the Y axis becomes meaningless. The tangent of the angle α increases indefinitely as α approaches 90°. It is sometimes said that $m \rightarrow \infty$ (m approaches infinity) when α approaches 90°.

PARALLEL AND PERPENDICULAR LINES

If we are given two lines that are parallel, their slopes must be equal. Each line will cut the X axis at the same angle α, so that

$$m_1 = \tan\ \alpha,\ m_2 = \tan\ \alpha$$

Therefore, $m_1 = m_2$

We conclude that two lines which are parallel have the same slope.

Suppose that two lines are perpendicular to each other, as lines L_1 and L_2 in figure 9-5. The slope and inclination of L_1 are m_1 and α_1, respectively. The slope and inclination of L_2 are m_2 and α_2, respectively. Then the following is true:

$$m_1 = \tan\ \alpha_1$$

$$m_2 = \tan\ \alpha_2$$

It can be shown geometrically that α_2 (fig. 9-5) is equal to α_1, plus 90°. Therefore,

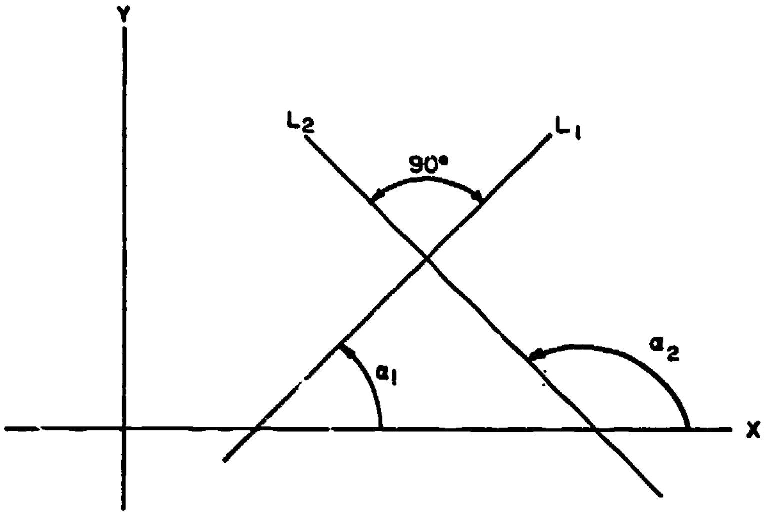


Figure 9-5.—Slopes of perpendicular lines.

$$\tan\ \alpha_2 = \tan\ (\alpha_1 + 90°)$$

$$= -\cot\ \alpha_1$$

$$= -\frac{1}{\tan\ \alpha_1}$$

Replacing $\tan\ \alpha_1$ and $\tan\ \alpha_2$ by their equivalents in terms of slope, we have

$$m_2 = -\frac{1}{m_1}$$

We conclude that, if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

Conversely, if the slopes of two lines are negative reciprocals of each other, the lines are perpendicular.

EXAMPLE: In figure 9-6, show that line L_1 is perpendicular to line L_2. Line L_1 passes through points P_1 (0,5) and P_2 (-1,3). Line L_2 passes through points P_2 (-1,3) and P_3 (3,1).

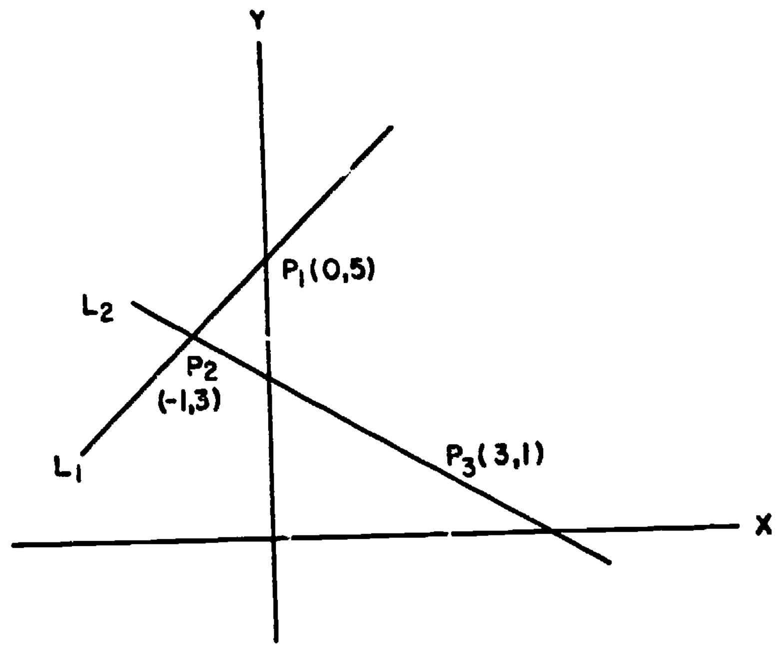


Figure 9-6.—Proving lines perpendicular.

SOLUTION: Let m_1 and m_2 represent the slopes of lines L_1 and L_2, respectively. Then we have

$$m_1 = \frac{5 - 3}{0 - (-1)} = 2$$

$$m_2 = \frac{1 - 3}{3 - (-1)} = \frac{-2}{4} = -\frac{1}{2}$$

Since their slopes are negative reciprocals of each other, the lines are perpendicular.

PRACTICE PROBLEMS:

1. Find the distance between P_1 (5,3) and P_2 (6,7).
2. Find the distance between P_1 (1/2,1) and P_2 (3/2,5/3).
3. Find the midpoint of the line connecting P_1 (5,2) and P_2 (-1,-3).
4. Find the slope of the line joining P_1 (-2,-5) and P_2 (2,5).

ANSWERS:

1. $\sqrt{17}$
2. $\frac{\sqrt{13}}{3}$
3. $(2, -\frac{1}{2})$
4. $\frac{5}{2}$

ANGLE BETWEEN TWO LINES

When two lines intersect, the angle between them is defined as the smallest angle through which one of the lines must be rotated to make it coincide with the other line. For example, the angle ϕ in figure 9-7 is the angle between lines L_1 and L_2.

Referring to figure 9-7,

$$\alpha_2 = \alpha_1 + \phi$$

$$\therefore \phi = \alpha_2 - \alpha_1$$

It is possible to determine the value of ϕ directly from the slopes of lines L_1 and L_2, as follows:

$$\tan \phi = \tan (\alpha_2 - \alpha_1)$$

$$= \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}$$

This result is obtained by use of the trigonometric identity for the tangent of the difference between two angles. Trigonometric

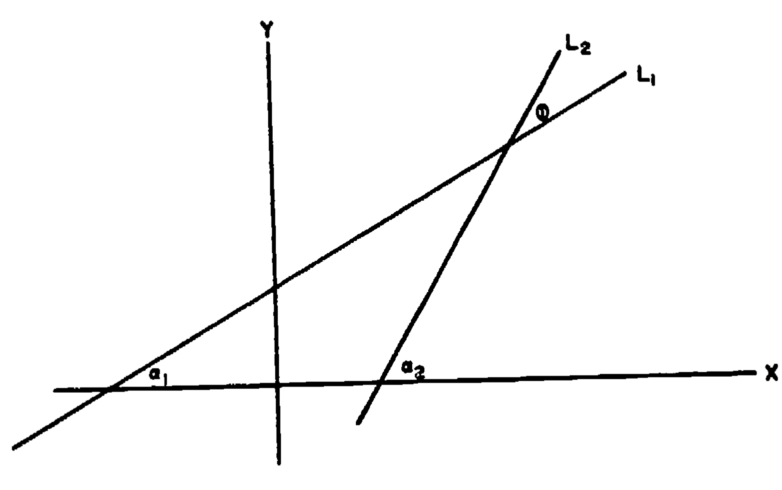


Figure 9-7.—Angle between two lines.

identities are discussed in chapter 8 of this training course.

Recalling that the tangent of the angle of inclination is the slope of the line, we have

$$\tan \alpha_1 = m_1 \text{ (the slope of } L_1)$$

$$\tan \alpha_2 = m_2 \text{ (the slope of } L_2)$$

Substituting these expressions in the tangent formula derived in the foregoing discussion, we have

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If one of the lines were parallel to the Y axis, its slope would be infinite. This would render the slope formula for tan ϕ useless, because an infinite value in both the numerator and denominator of the fraction $\frac{m_2 - m_1}{1 + m_1 m_2}$ produces an indeterminate form. However, if one of the lines is known to be parallel to the Y axis the tangent of ϕ may be expressed by another method.

Suppose that L_2 (fig. 9-7) were parallel to the Y axis. Then we would have

$$\alpha_2 = 90°$$

$$\phi = 90° - \alpha_1$$

$$\tan \phi = \cot \alpha_1$$

$$= \frac{1}{m_1}$$

PRACTICE PROBLEMS:

1. Find the angle between the two lines which have $m_1 = 3$ and $m_2 = 7$ for slopes.
2. Find the angle between two lines whose slopes are $m_1 = 0$, $m_2 = 1$. ($m_1 = 0$ signifies that line L_1 is horizontal and the formula still holds).
3. Find the angle between the Y axis and a line with a slope of $m = -8$.
4. Find the obtuse angle between the X axis and line with a slope of $m = -8$.

ANSWERS:
1. 10°18'
2. 45°
3. 7°7'
4. 97°7'

EQUATION OF A STRAIGHT LINE

In Mathematics, Volume 1, NavPers 10069-C, equations such as

$$2x + y = 6$$

are designated as linear equations, and their graphs are shown to be straight lines. The purpose of the present discussion is to study the relationship of slope to the equation of a straight line.

POINT-SLOPE FORM

Suppose that we desire to find the equation of a straight line which passes through a known point and has a known slope. Let (x,y) represent the coordinates of any point on the line, and let (x_1,y_1) represent the coordinates of the known point. The slope is represented by m.

Recalling the formula defining slope in terms of the coordinates of two points, we have

$$m = \frac{y - y_1}{x - x_1}$$

$$\therefore y - y_1 = m(x - x_1)$$

EXAMPLE: Find the equation of a line passing through the point (2,3) and having a slope of 3.

SOLUTION:

$$x_1 = 2 \text{ and } y_1 = 3$$

$$y - y_1 = m(x - x_1)$$

$$\therefore y - 3 = 3(x - 2)$$

$$y - 3 = 3x - 6$$

$$y - 3x = -3$$

The point-slope form may be used to find the equation of a line through two known points. The values of x_1, x_2, y_1, and y_2 are first used to find the slope of the line, and then either known point is used with the slope in the point-slope form.

EXAMPLE: Find the equation of the line through the points (-3,4) and (4,-2).
SOLUTION:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - 4}{4 + 3} = \frac{-6}{7}$$

Letting (x,y) represent any point on the line, and using (-3,4) as a known point, we have

$$y - 4 = \frac{-6}{7}\,[x - (-3)]$$

$$7(y - 4) = -6(x + 3)$$

$$7y - 28 = -6x - 18$$

$$7y + 6x = 10$$

SLOPE-INTERCEPT FORM

Any line which is not parallel to the Y axis intersects the Y axis in some point. The x coordinate of the point of intersection is 0, because the Y axis is vertical and passes through the origin. Let the y coordinate of the point of intersection be represented by b. Then the point of intersection is (0,b), as shown in figure 9-8. The y coordinate, b, is called the y intercept.

The slope of the line in figure 9-8 is $\frac{\Delta y}{\Delta x}$. The value of Δy in this expression is $y - b$, where y represents the y coordinate of any point on the line. The value of Δx is equal to the x coordinate of P(x,y), so that

$$m = \frac{\Delta y}{\Delta x} = \frac{y - b}{x}$$

$$mx = y - b$$

$$y = mx + b$$

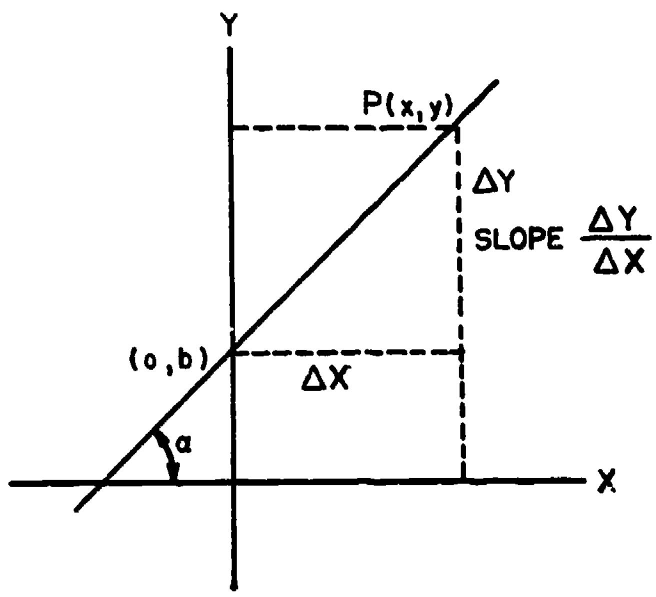


Figure 9-8.—Slope-intercept form.

This is the standard slope-intercept form of a straight line.

EXAMPLE: Find the equation of a line that intersects the Y axis at the point (0,3) and has a slope of 5/3.
SOLUTION:

$$v = mx + b$$

$$y = \frac{5}{3}x + 3$$

$$3y = 5x + 9$$

PRACTICE PROBLEMS:

Write equations for lines having points and slopes as follows:

1. P(3,5), $m = -2$
2. P(-2,-1), $m = \frac{1}{3}$
3. $P_1(2,2)$ and $P_2(-4,-1)$
4. Y intercept = 2, m = 3

ANSWERS:

1. $y = -2x + 11$
2. $3y = x - 1$

3. $2y = x + 2$

4. $y = 3x + 2$

NORMAL FORM

Methods for determining the equation of a line usually depend upon some knowledge of a point or points on the line. We now consider a method which does not require advance knowledge concerning any of the line's points. All that is known about the line is its perpendicular distance from the origin and the angle between the perpendicular and the x axis.

In figure 9-9, line AB is a distance p away from the origin, and line OM forms an angle θ with the X axis. We select any point P(x,y) on line AB and develop the equation of line AB in terms of the x and y of P. Since P represents ANY point on the line, the x and y of the equation will represent EVERY point on the line and therefore will represent the line itself.

PR is constructed perpendicular to OB at point R. NR is drawn parallel to AB, and PN is parallel to OB. PS is perpendicular to NR and to AB. Since right triangles OMB and RSP have their sides mutually perpendicular, they are similar; therefore, angle PRS is equal to θ. Finally, the x distance of point P is equal to OR, and the y distance of P is equal to PR.

In order to relate the distance p to x and y, we reason as follows:

$$ON = (OR)(\cos \theta)$$

$$= x \cos \theta$$

$$PS = (PR)(\sin \theta)$$

$$= y \sin \theta$$

$$OM = ON + PS$$

$$p = ON + PS$$

$$p = x \cos \theta + y \sin \theta$$

This final equation is the NORMAL FORM. The word "normal" in this usage refers to the perpendicular relationship between OM and AB. "Normal" frequently means "perpendicular" in mathematical and scientific usage.

The distance p is considered to be always positive, and θ is any angle between 0° and 360°.

EXAMPLE: Find the equation of a line that is 5 units away from the origin, if the perpendicular from the line to the origin forms an angle of 30° with the positive side of the X axis.

SOLUTION:

$$p = 5; \theta = 30°$$

$$p = x \cos \theta + y \sin \theta$$

$$5 = x \cos 30° + y \sin 30°$$

$$5 = x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right)$$

$$10 = x\sqrt{3} + y$$

PARALLEL AND PERPENDICULAR LINES

The general equation of a straight line is often written with capital letters for coefficients, as follows:

$$Ax + By + C = 0$$

These literal coefficients, as they are called, represent the numerical coefficients encountered in a typical linear equation.

Suppose that we are given two equations which are duplicates except for the constant term, as follows:

$$Ax + By + C = 0$$

$$Ax + By + D = 0$$

By placing these two equations in slope-intercept form, we can show that their slopes are equal, as follows:

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{D}{B}\right)$$

Thus the slope of each line is - A/B.

Since lines having equal slopes are parallel, we reach the following conclusion: In any two

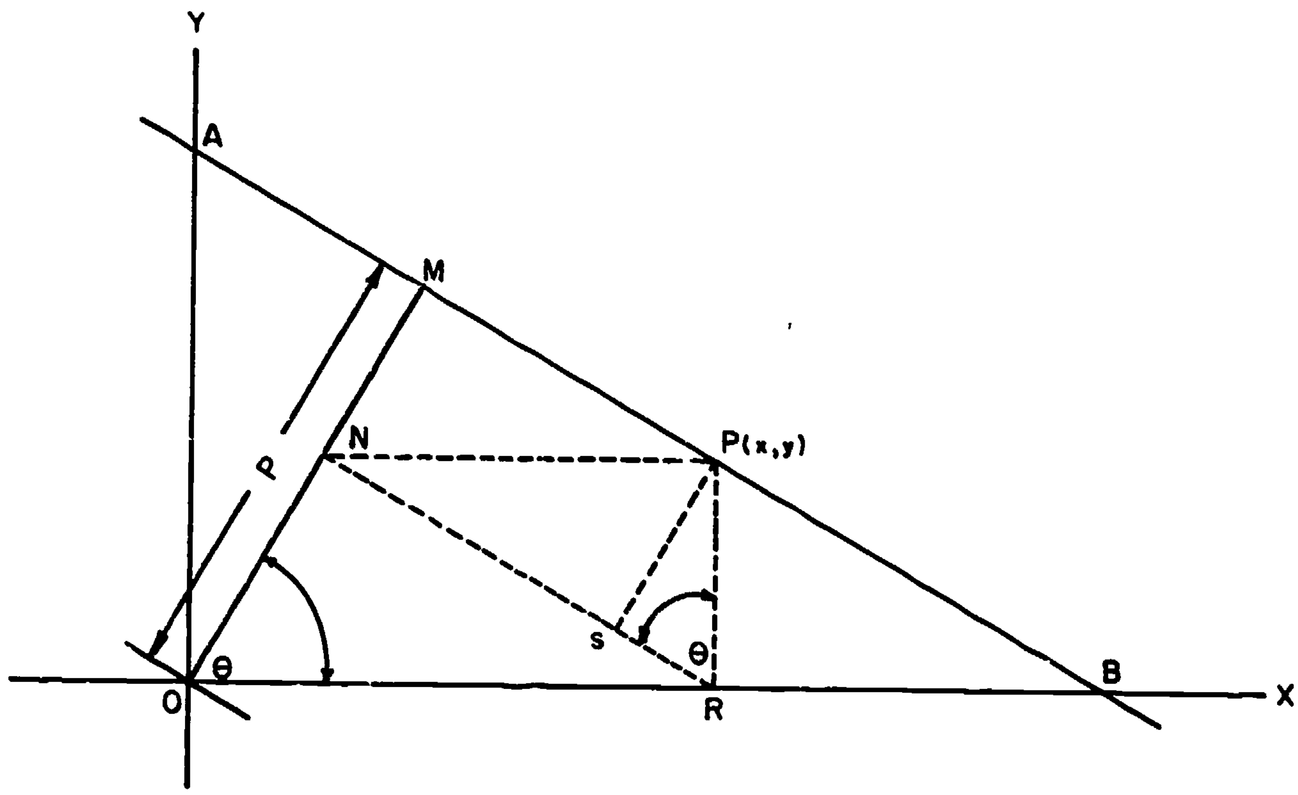


Figure 9-9.—Normal form.

linear equations, if the coefficients of the x and y terms are identical in value and sign, then the lines represented by these equations are parallel.

EXAMPLE: Write the equation of a line parallel to $3x - y - 2 = 0$ and passing through the point (5,2).

SOLUTION: The coefficients of x and y in the desired equation are the same as those in the given equation. Therefore, the equation is

$$3x - y + D = 0$$

Since the line passes through (5,2), the values $x = 5$ and $y = 2$ must satisfy the equation. Substituting these, we have

$$3(5) - (2) + D = 0$$

$$D = -13$$

Thus the required equation is

$$3x - y - 13 = 0$$

A situation similar to that prevailing with parallel lines involves perpendicular lines. For example, consider the equations

$$Ax + By + C = 0$$

$$Bx - Ay + D = 0$$

Transposing into the slope-intercept form, we have

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

$$y = \left(\frac{B}{A}\right)x + \left(\frac{D}{A}\right)$$

Since the slopes of these two lines are negative reciprocals, the lines are perpendicular.

The conclusion derived from the foregoing discussion is as follows: If a line is to be perpendicular to a given line, the coefficients of x and y in the required equation are found by interchanging the coefficients of x and y in the given equation and changing the sign of one of them.

EXAMPLE: Write the equation of a line perpendicular to the line $x + 3y + 3 = 0$ and having a y intercept of 5.

SOLUTION: The required equation is

$$3x - y + D = 0$$

Notice the interchange of coefficients and the change of sign. At the point where the line crosses the Y axis, the value of x is 0 and the value of y is 5. Therefore, the equation is

$$3(0) - (5) + D = 0$$

$$D = 5$$

The required equation is

$$3x - y + 5 = 0$$

PRACTICE PROBLEMS:

Find the equations of the following lines:

1. Through (1,1) and parallel to $5x - 3y = 9$.
2. Through (-3,2) and perpendicular to $x + y = 5$.
3. Through (2,3) and perpendicular to $3x - 2y = 7$.
4. Through (2,3) and parallel to $3x - 2y = 7$.

ANSWERS:

1. $5x - 3y = 2$
2. $x - y = -5$
3. $2x + 3y = 13$
4. $3x - 2y = 0$

DISTANCE OF A POINT FROM A LINE

It is frequently necessary to express the distance of a point from a line in terms of the coefficients in the equation of the line. In order to do this, we compare the two forms of the equation of a straight line, as follows:

General equation: $Ax + By + C = 0$
Normal form: $x \cos \theta + y \sin \theta - p = 0$

The general equation and the normal form represent the same straight line. Therefore, A (the coefficient of x in general form) is proportional to $\cos \theta$ (the coefficient of x in the normal form). By similar reasoning, B is proportional to $\sin \theta$, and C is proportional to -p. Recalling that quantities proportional to each other form ratios involving a constant of proportionality, let k be this constant. Thus we have

$$\frac{\cos \theta}{A} = k$$

$$\frac{\sin \theta}{B} = k$$

$$\cos \theta = kA$$

$$\sin \theta = kB$$

Squaring both sides of these two expressions and then adding, we have

$$\cos^2 \theta + \sin^2 \theta = k^2 (A^2 + B^2)$$

$$1 = k^2 (A^2 + B^2)$$

$$k^2 = \frac{1}{A^2 + B^2}$$

$$k = \frac{1}{\pm \sqrt{A^2 + B^2}}$$

The coefficients in the normal form, expressed in terms of A, B, and C, are as follows:

$$\cos\theta = \frac{A}{\pm\sqrt{A^2 + B^2}}$$

$$\sin\theta = \frac{B}{\pm\sqrt{A^2 + B^2}}$$

$$-p = \frac{C}{\pm\sqrt{A^2 + B^2}}$$

The sign of $\sqrt{A^2 + B^2}$ is chosen so as to make p (a distance) always positive.

The conversion formulas developed in the foregoing discussion are used in finding the distance from a point to a line. Let p represent the distance of line L from the origin. (See fig. 9-10.) In order to find d, the distance of point P_1 from line L, we construct a line through P_1 and parallel to L. The distance of this line from the origin is OS, and the difference between OS and p is d.

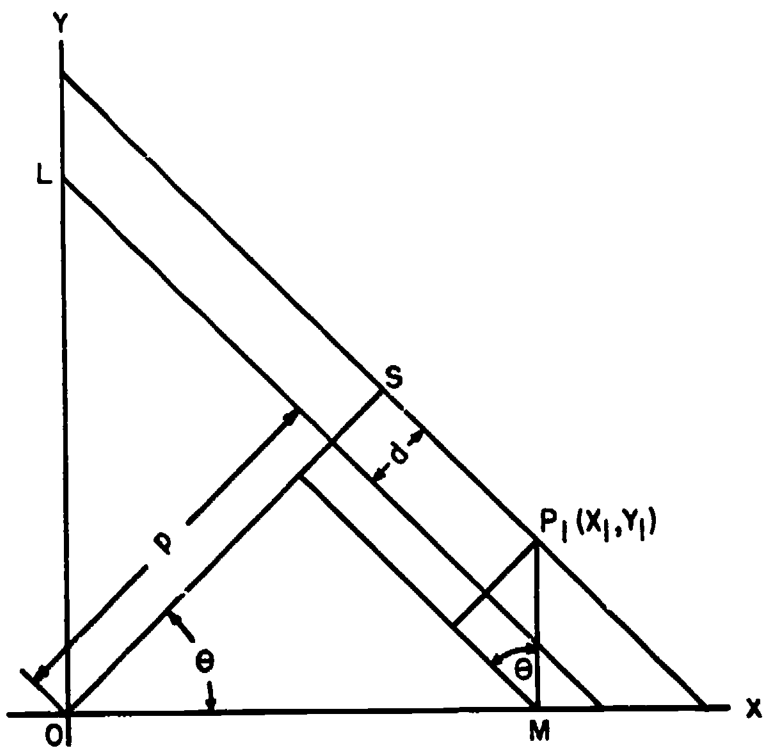


Figure 9-10.—Distance from a point to a line.

We obtain an expression for d, based on the coordinates of P_1, as follows:

$$OS = x_1\cos\theta + y_1\sin\theta$$

$$d = OS - p$$

$$= x_1\cos\theta + y_1\sin\theta - p$$

Returning to the expressions for $\sin\theta$, $\cos\theta$, and $-p$ in terms of A, B, and C (the coefficients in the general equation), we have

$$d = x_1\left(\frac{A}{\pm\sqrt{A^2 + B^2}}\right) + y_1\left(\frac{B}{\pm\sqrt{A^2 + B^2}}\right) + \frac{C}{\pm\sqrt{A^2 + B^2}}$$

The denominator in each of the expressions comprising the formula for d is the same. Therefore we may combine as follows:

$$d = \left|\frac{x_1A + y_1B + C}{\sqrt{A^2 + B^2}}\right|$$

We use the absolute value, since d is a distance, and thus avoid any confusion arising from the ± radical.

EXAMPLE: Find the distance from the point (2,1) to the line $4x + 2y + 7 = 0$.

SOLUTION:

$$d = \left|\frac{(4)(2) + (2)(1) + 7}{\sqrt{4^2 + 2^2}}\right|$$

$$= \frac{8 + 2 + 7}{\sqrt{20}}$$

$$= \frac{17}{2\sqrt{5}}$$

$$= \frac{17\sqrt{5}}{10}$$

PRACTICE PROBLEMS:

In each of the following problems, find the distance from the point to the line:

1. $(5,2)$, $3x - y + 6 = 0$
2. $(3,-5)$, $2x + y + 4 = 0$
3. $(3,-4)$, $4x + 3y = 10$
4. $(-2,5)$, $3x + 4y - 9 = 0$

ANSWERS:

1. $\frac{19\sqrt{10}}{10}$
2. $\sqrt{5}$
3. 2
4. 1

CHAPTER 10

CONIC SECTIONS

This chapter is a continuation of the study of analytic geometry. The figures presented in this chapter are plane figures which are included in the general class of conic sections or simply "conics."

Conic sections are so named because they are all plane sections of a right circular cone. A circle can be formed by cutting a cone perpendicular to its axis. An ellipse is produced when the cone is cut obliquely to the axis and the surface. A hyperbola results when the cone is intersected by a plane parallel to the axis, and a parabola is the result when the intersecting plane is parallel to an element of the surface. These are illustrated in figure 10-1.

When the curve produced by cutting the cone is placed on a coordinate system it may be defined as follows:

A conic section is the locus of a point that moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line. The fixed point is the focus, and the fixed line is the directrix.

The ratio referred to in the definition is called the eccentricity. If the eccentricity (e) is less than one, the curve is an ellipse. If e is greater than one, the curve is a hyperbola. If e is equal to 1, the curve is a parabola. A circle is a special case having an eccentricity equal to zero, and may be defined by the distance from a point. It is actually a limiting case of an ellipse in which the eccentricity approaches zero. Thus, if

$e = 0$, it is a circle

$e < 1$, it is an ellipse

$e = 1$, it is a parabola

$e > 1$, it is a hyperbola

The eccentricity, focus, and directrix are used in the algebraic analysis of conic sections and the corresponding equations. The concept of the locus of an equation also enters into analytic geometry; this concept is discussed before the individual conic sections are studied.

THE LOCUS OF AN EQUATION

In chapter 9 of this course, methods for analysis of linear equations are presented. If a group of x and y values (or ordered pairs, P (x,y)) which satisfy a given linear equation are plotted on a coordinate system, the resulting graph is a straight line.

When higher ordered equations such as

$$x^2 + y^2 = 1 \text{ or } y = \sqrt{2x + 3}$$

are encountered, the resulting graph is not a straight line. However, the points whose coordinates satisfy most of the equations in x and y are normally not scattered in a random field. If the values are plotted they will seem to follow a line or curve (or a combination of lines and curves). In many texts the plot of an equation is called a curve, even when it is a straight line. This curve is called the locus of the equation. The locus of an equation is a curve containing those points, and only those points, whose coordinates satisfy the equation.

At times the curve may be defined by a set of conditions rather than by an equation, though an equation may be derived from the given conditions. Then the curve in question would be the locus of all points which fit the conditions. For instance a circle may be said to be the locus of all points in a plane which lie a fixed distance from a fixed point. A straight line may be defined as the locus of a point that moves in a plane so that it is at all times equidistant from two fixed points. The method of expressing a set

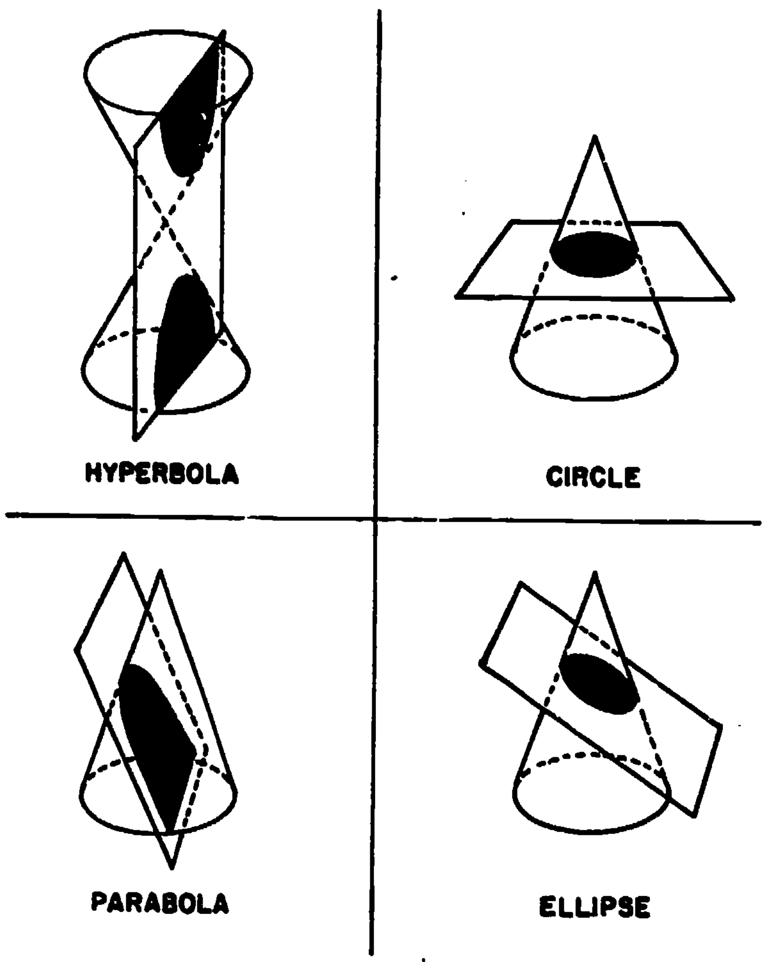


Figure 10-1.—Conic sections.

of conditions in analytical form gives an equation. Let us draw up a set of conditions and translate them into an equation.

EXAMPLE: What is the equation of the curve which is the locus of all points which are equidistant from the two points (5, 3) and (2,1)?

SOLUTION: First, as in figure 10-2, choose some point having coordinates (x, y). Recall from chapter 9 of this course that the distance between this point and (2, 1) is given by:

$$\sqrt{(y - 1)^2 + (x - 2)^2}$$

The distance between point (x, y) and (5, 3) will be given by

$$\sqrt{(y - 3)^2 + (x - 5)^2}$$

Equating these distances, since the point is to be equidistant from the two given points, we have

$$\sqrt{(y - 1)^2 + (x - 2)^2} = \sqrt{(y - 3)^2 + (x - 5)^2}$$

Squaring both sides

$$(y - 1)^2 + (x - 2)^2 = (y - 3)^2 + (x - 5)^2$$

Expanding

$$y^2 - 2y + 1 + x^2 - 4x + 4$$

$$= y^2 - 6y + 9 + x^2 - 10x + 25$$

Canceling and collecting terms:

$$4y + 5 = -6x + 34$$

$$4y = -6x + 29$$

$$y = -\frac{3}{2}x + 7.25$$

This is the equation of a straight line with a slope of minus 3/2, and a Y intercept of +7.25.

EXAMPLE: Find the equation of the curve which is the locus of all points which are equidistant from the line x = -3 and the point (3, 0).

SOLUTION: The distance from the point (x, y) on the curve to the line will be (x - (-3)) or

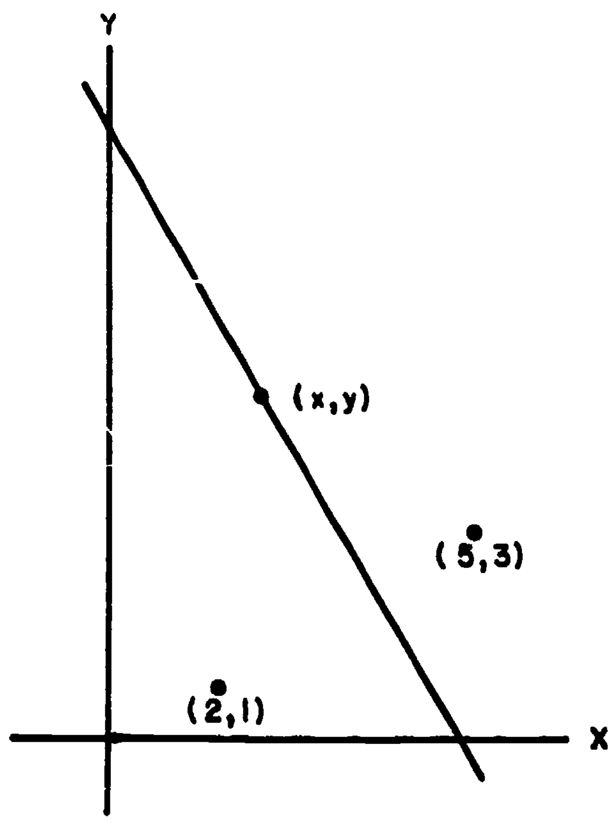


Figure 10-2.—Locus of points equidistant from two given points.

(x + 3). Refer to figure 10-3. The distance from the point (x, y) to the point (3, 0) is

$$\sqrt{(y - 0)^2 + (x - 3)^2}$$

Equating the two distances,

$$x + 3 = \sqrt{y^2 + (x - 3)^2}$$

Squaring both sides,

$$x^2 + 6x + 9 = y^2 + x^2 - 6x + 9$$

Canceling and collecting terms,

$$y^2 = 12x$$

which is the equation of a parabola.

EXAMPLE: What is the equation of the curve the locus of which is a point which moves so that at all times the ratio of its distance from the point (3, 0) to its distance from the line x = 25/3 is equal to 3/5? Refer to figure 10-4.

SOLUTION: The distance from a point (x, y) to the point (3, 0) is given by

$$d_1 = \sqrt{(x - 3)^2 + (y - 0)^2}$$

The distance from the same point (x, y) to the line is

$$d_2 = \frac{25}{3} - x$$

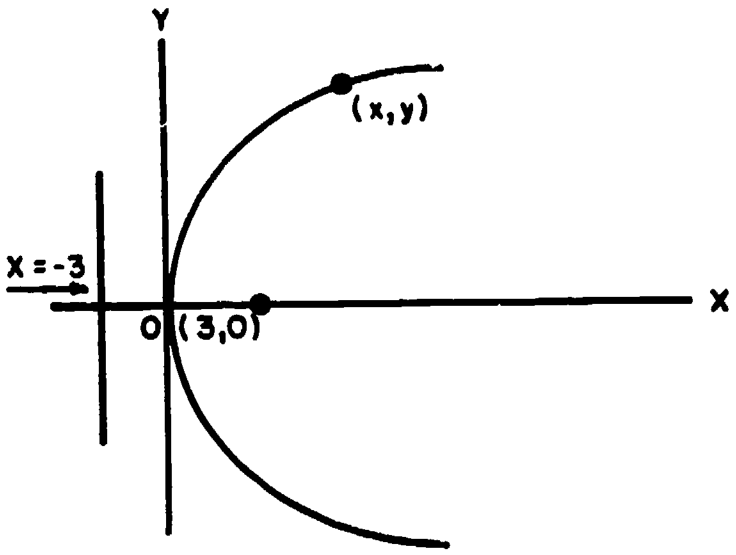


Figure 10-3.—Parabola.

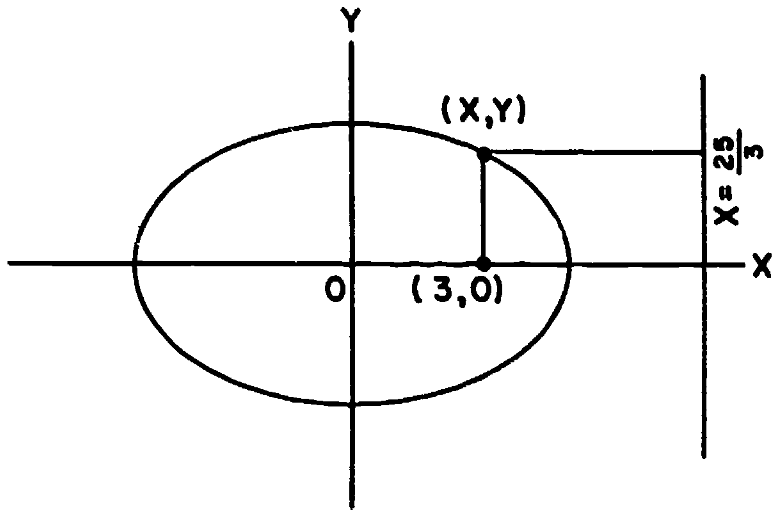


Figure 10-4.—Ellipse

Since

$$\frac{d_1}{d_2} = \frac{3}{5} \text{ or } d_1 = \frac{3}{5} d_2$$

then

$$\sqrt{(x - 3)^2 + y^2} = \frac{3}{5}\left(\frac{25}{3} - x\right)$$

Squaring both sides and expanding,

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}\left(x^2 - \frac{50}{3} x + \frac{625}{9}\right)$$

$$x^2 - 6x + 9 + y^2 = \frac{9}{25} x^2 - 6x + 25$$

Collecting terms and transposing

$$\frac{16}{25} x^2 + y^2 = 16$$

Dividing through by 16

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

This is the equation of an ellipse.

PRACTICE PROBLEMS: Find the equation which is the locus of the point which moves so that it is at all times:

1. Equidistant from the points (0, 0) and (5, 4).

2. Equidistant from the points (3, -2) and (-3, 2).

3. Equidistant from the line $x = -4$ and the point (3, 4).

4. Equidistant from the point (4, 5) and the line $y = 5x - 4$. HINT: Use the standard distance formula to find the distance from the point P (x, y) and the point P (4, 5). Then use the formula for finding distance from a point to a line, given in chapter 9 of this course, to find the distance from P (x, y) to the given line. Put the equation of the line in the form $Ax + By + C = 0$.

ANSWERS:

1. $y = -1.25x + \frac{41}{8}$

2. $2y = 3x$

3. $y^2 - 8y = 14x - 9$

4. $x^2 + 10xy + 25y^2 - 168x - 268y + 1050 = 0$

THE CIRCLE

A circle is the locus of a point which is always a fixed distance from a fixed point called the center.

The fixed distance spoken of here is the radius of the circle.

The equation of a circle with its center at the origin (figure. 10-5) is, from the definition:

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

where (x, y) is a point on the circle and r is the radius and replaces d in the standard distance formula. Then

$$\sqrt{x^2 + y^2} = r$$

or

$$x^2 + y^2 = r^2 \tag{1}$$

If the center of a circle, figure 10-6, is at some point $x = h$, $y = k$, the distance of the moving point from the center will be constant and equal to

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or

$$(x - h)^2 + (y - k)^2 = r^2 \tag{2}$$

Equations (1) and (2) are the standard forms for the equation of a circle. Equation (1) is merely a special case of equation (2) in which h and k are equal to zero.

The equation of a circle may also be expressed in the form:

$$x^2 + y^2 + Bx + Cy + D = 0 \tag{3}$$

where B, C, and D are constants.

THEOREM: An equation of the second degree in which the coefficients of the x^2 and y^2 terms are equal, and there is no (xy) term, represents a circle.

Whenever we find an equation in the form of equation (3), it is best to convert it to the form of equation (2), so that we have the coordinates of the center of the circle and the radius as part of the equation. This may be done as shown in the following example problems.

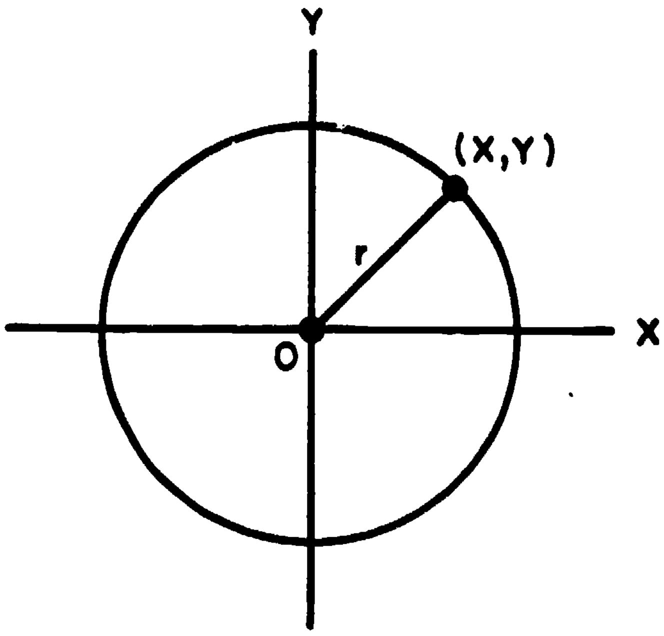


Figure 10-5.—Circle with center at the origin.

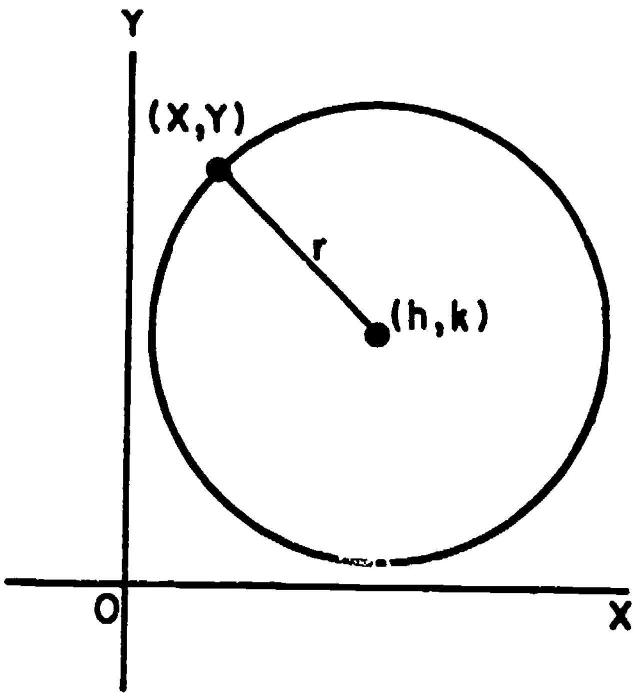


Figure 10-6.—Circle with center at (h, k).

EXAMPLE: Find the coordinates of the center and the radius of the circle which is described by the following equation:

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

SOLUTION: First rearrange the terms

$$x^2 - 4x + y^2 - 6y + 9 = 0$$

and complete the square in both x and y. Completing the square is discussed in the chapter on quadratic solutions in Mathematics, Vol. 1, NavPers 10069-C. The procedure consists basically of adding certain quantities to both sides of a second degree equation to form a perfect square trinomial. When both the first and second degree members are known, the square of one-half the coefficient of the first degree term is added to both sides of the equation. This will allow the quadratic equation to be factored into a perfect square trinomial. To complete the square in x in the given equation

$$x^2 - 4x + y^2 - 6y + 9 = 0$$

add the square of one-half the coefficient of x to both sides of the equation

$$x^2 - 4x + (2)^2 + y^2 - 6y + 9 = 0 + (2)^2$$

then

$$(x^2 - 4x + 4) + y^2 - 6y + 9 = 4$$

$$(x-2)^2 + y^2 - 6y + 9 = 4$$

completes the square in x.
For y then

$$(x - 2)^2 + y^2 - 6y + (3)^2 + 9 = 4 + (3)^2$$

$$(x - 2)^2 + (y^2 - 6y + 9) + 9 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 + 9 = 4 + 9$$

completes the square in y.

Transpose all constant terms to the right-hand side and simplify

$$(x - 2)^2 + (y - 3)^2 = 4 + 9 - 9$$

$$(x - 2)^2 + (y - 3)^2 = 4$$

and the equation is in the standard form of equation (2). This represents a circle with the center at (2, 3) and with a radius equal to $\sqrt{4}$ or 2.

EXAMPLE: Find the coordinates of the center and the radius of the circle given by the equation

$$x^2 + y^2 + \frac{1}{2}x - 3y - \frac{27}{16} = 0$$

SOLUTION: Rearrange and complete the squares in x and y

$$x^2 + \frac{1}{2}x + y^2 - 3y - \frac{27}{16} = 0$$

$$(x^2 + \frac{1}{2}x + \frac{1}{16}) + (y^2 - 3y + \frac{9}{4}) - \frac{27}{16} = \frac{1}{16} + \frac{9}{4}$$

Transposing all constant terms to the right-hand side and adding,

$$(x^2 + \frac{1}{2}x + \frac{1}{16}) + (y^2 - 3y + \frac{9}{4}) = 4$$

Reducing to standard form

$$(x + \frac{1}{4})^2 + (y - \frac{3}{2})^2 = (2)^2$$

Thus, the equation represents a circle with its center at (-1/4, 3/2) and a radius equal to 2.

PRACTICE PROBLEMS: Find the coordinates of the center and the radius for the circles described by the following equations.

1. $x^2 - \frac{4}{5}x + y^2 - 4y + \frac{29}{25} = 0$

2. $x^2 + 6x + y^2 - 14y = 23$

3. $x^2 - 14x + y^2 + 22y = -26$

4. $x^2 + y^2 + \frac{2}{5}x + \frac{2}{3}y = \frac{2}{25}$

5. $x^2 + y^2 - 1 = 0$

ANSWERS:

1. Center $(\frac{2}{5}, 2)$, radius $\sqrt{3}$

2. Center (-3, 7), radius 9

3. Center (7, -11), radius 12

4. Center $(-\frac{1}{5}, -\frac{1}{3})$, radius $\frac{2\sqrt{13}}{15}$

5. Center (0, 0), radius 1

THE CIRCLE DEFINED BY THREE POINTS

In certain situations it is convenient to consider the following standard form of a circle

$$x^2 + y^2 + Bx + Cy + D = 0$$

as the equation of a circle in which the specific values of the constants B, C, and D are to be determined. In this problem the unknowns to be found are not x and y, but the values of the constants B, C, and D. The conditions which define the circle are used to form algebraic relationships between these constants. For example, if one of the conditions imposed on the circle is that it pass through the point (3, 4) then the standard form is written with x and y replaced by 3 and 4 respectively; thus

$$x^2 + y^2 + Bx + Cy + D = 0$$

is rewritten as

$$(3)^2 + (4)^2 + B(3) + C(4) + D = 0$$

$$3B + 4C + D = -25$$

There are three independent constants in the equation of a circle; therefore, there must be three conditions given to define a circle. Each of these conditions will yield an equation with B, C, and D as the unknowns. These three equations are then solved simultaneously to determine the values of the constants which satisfy all of the equations. In an analysis, the number of independent constants in the general equation of a curve indicate how many conditions must be set before a curve can be completely defined. Also, the number of unknowns in an equation indicates the number of equations which must be solved simultaneously to find the values of the unknowns. For example, if B, C, and D are unknowns in an equation, three separate equations involving these variables are required for a solution.

A circle may be defined by three noncollinear points, that is, by three points which do not lie on a straight line. There is only one possible circle through any three noncollinear points. To find the equation of the circle determined by the three points substitute the x, y values of each of the given points into a general equation to form three equations with B, C, and D as the unknowns. These equations are then solved simultaneously to find the values of B, C, and D in the equation which satisfies the three given conditions.

The solution of simultaneous equations in two variables is discussed in Mathematics Vol. 1. Systems involving three variables use an extension of the same principles, but with three equations instead of two. Step-by-step explanations of the solution will be given in the example problems.

EXAMPLE: Write the equation of the circle which passes through the points (2, 8), (5, 7), and (6, 6).

SOLUTION: The method used in this solution corresponds to the addition-subtraction method used for solution of equations in two variables. However, the method or combination of methods used will depend on a particular problem. No one method is best suited to all problems.

First, write a general equation of the form

$$x^2 + y^2 + Bx + Cy + D = 0$$

for each of the given points, substituting the given values for x and y and rearranging

For (2, 8) $4 + 64 + 2B + 8C + D = 0$

$$2B + 8C + D = -68$$

For (5, 7) $25 + 49 + 5B + 7C + D = 0$

$$5B + 7C + D = -74$$

For (6, 6) $36 + 36 + 6B + 6C + D = 0$

$$6B + 6C + D = -72$$

To aid in the explanation we number the three resulting equations

$$2B + 8C + D = -68 \qquad (1)$$

$$5B + 7C + D = -74 \qquad (2)$$

$$6B + 6C + D = -72 \qquad (3)$$

The first step is to eliminate one of the unknowns and have two equations and two unknowns remain. The coefficient of D is the same in all three equations and is the one most easily eliminated by addition and subtraction. This is done in the following manner. Subtract (2) from (1)

$$2B + 8C + D = -68 \qquad (1)$$

$$\underline{5B + 7C + D = -74} \qquad (-)\ (2)$$

$$-3B + C = 6 \qquad (4)$$

Subtract (3) from (2)

$$5B + 7C + D = -74 \qquad (2)$$

$$\underline{6B + 6C + D = -72} \qquad (3)$$

$$-B + C = -2 \qquad (5)$$

This gives two equations, (4) and (5), in two unknowns which can be solved simultaneously. Since the coefficient of C is the same in both equations it is the most easily eliminated variable.

To eliminate C, subtract (4) from (5)

$$-B + C = -2 \qquad (5)$$

$$\underline{-3B + C = 6} \qquad (4)$$

$$2B = -8$$

$$B = -4 \qquad (6)$$

To find the value of C substitute the value found for B in (6) in equation (5)

$$-B + C = -2 \qquad (5)$$

$$-(-4) + C = -2$$

$$C = -6 \qquad (7)$$

Now the values of B and C can be substituted in any one of the original equations to determine the value of D.

If the values are substituted in (1)

$$2B + 8C + D = -68 \qquad (1)$$

$$2(-4) + 8(-6) + D = -68$$

$$-8 - 48 + D = -68$$

$$D = -68 + 56$$

$$D = -12 \qquad (8)$$

The solution of the system of equations gave values for three independent constants in the general equation

$$x^2 + y^2 + Bx + Cy + D = 0$$

When the constant values are substituted the equation takes the form of

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

Rearranging and completing the square in x and y,

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) - 12 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 = 25$$

which corresponds to a circle with the center at (2, 3) with a radius of 5. This is the circle described by the three given conditions and is shown in figure 10-7 (A).

The previous example problem showed one method for determining the equation of a circle when three points are given. The next example shows another method for solving the same problem. One of the most important things to keep in mind when studying analytic geometry is that many problems may be solved by more than one method. Each problem should be analyzed carefully to determine what relationships exist between the given data and the desired results of the problem. Relationships such as distance from one point to another, distance from a point to a line, slope of a line, the Pythagorean theorem, etc., will be used to solve various problems.

EXAMPLE: Find the equation of the circle described by the three points (2, 8), (5, 7), and (6, 6). Use a method other than that used in the previous example problem.

SOLUTION: A different method of solving this problem results from the reasoning in the following paragraphs.

The center of the desired circle will be the intersection of the perpendicular bisectors of the chords connecting points (2, 8) with (5, 7) and (5, 7) with (6, 6), as shown in figure 10-7(B).

The perpendicular bisector of the line connecting two points is the locus of a point which moves so that it is always equidistant from the two points. Using this analysis we can get the equations of the perpendicular bisectors of the two lines.

Equating the distance formulas which describe the distances from a point (x,y), which is equidistant from the points (2,8) and (5,7), gives

$$\sqrt{(x-2)^2 + (y-8)^2} = \sqrt{(x-5)^2 + (y-7)^2}$$

Squaring both sides gives

$$(x-2)^2 + (y-8)^2 = (x-5)^2 + (y-7)^2$$

or

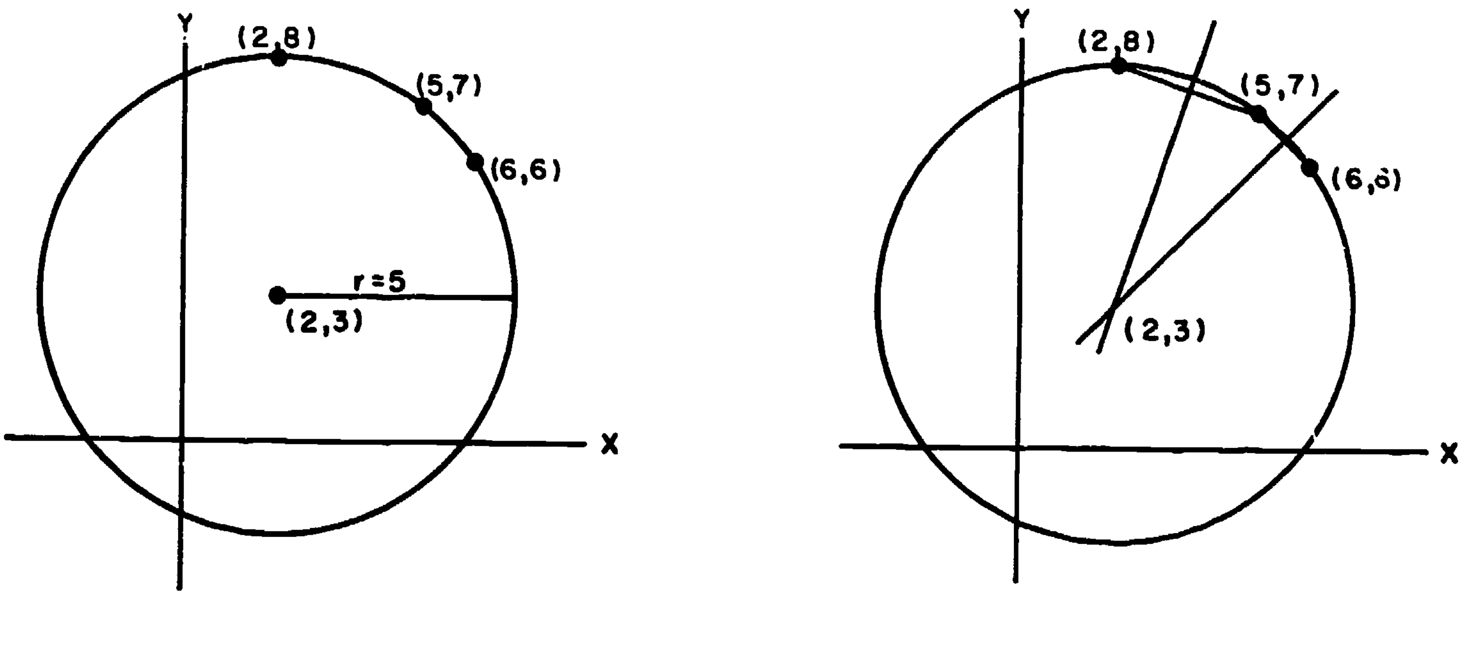
$$x^2 - 4x + 4 + y^2 - 16y + 64 =$$
$$x^2 - 10x + 25 + y^2 - 14y^2 + 49$$

Canceling and combining terms results in

$$6x - 2y = 6$$

or

$$3x - y = 3$$



(A) (B)

Figure 10-7.—Circle described by three points.

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Follow the same procedure for the points (5, 7) and (6, 6).

$$\sqrt{(x - 5)^2 + (y - 7)^2} = \sqrt{(x - 6)^2 + (y - 6)^2}$$

Squaring each side gives

$$(x - 5)^2 + (y - 7)^2 = (x - 6)^2 + (y - 6)^2$$
$$x^2 - 10x + 25 + y^2 + 14y + 49 =$$
$$x^2 - 12x + 36 + y^2 - 12y + 36$$

Canceling and combining terms gives a second equation in x and y.

$$2x - 2y = -2$$

or

$$x - y = -1$$

Solving the equations simultaneously will give the coordinates of the intersection of the two perpendicular bisectors; this is the center of the circle.

$$\begin{array}{rl} 3x - y = 3 & \\ \underline{x - y = -1} & \text{(Subtract)} \\ 2x \quad = 4 & \\ x = 2 & \end{array}$$

Substitute the value $x = 2$ in one of the equations to find the value of y.

$$x - y = -1$$
$$2 - y = -1$$
$$-y = -3$$
$$y = 3$$

Thus, the center of the circle is the point (2,3).

The radius will be the distance between the center (2,3) and one of the three given points. Using point (2, 8) we obtain.

$$r = \sqrt{(2 - 2)^2 + (8 - 3)^2} = \sqrt{25} = 5$$

The equation of this circle is

$$(x - 2)^2 + (y - 3)^2 = 25$$

as was found in the previous example.

If a circle is to be defined by three points the points must be noncollinear. In some cases it is obvious that the three points are noncollinear. Such is the case with points (1,1), (-2, 2), and (-1, -1), since the points are in quadrants 1, 2, and 3 respectively and cannot be connected by a straight line. However, there are many cases in which it is difficult to determine by inspection whether or not the points are collinear, and a method for determining this analytically is needed. In the followng example an attempt is made to find the circle described by three points, when the three points are collinear

EXAMPLE: Find the equation of the circle which passes through the points (1,1), (2, 2), (3,3).

SOLUTION: Substitute the given values of x and y in the standard form of the equation of a circle to get three equations in three unknowns.

$$x^2 + y^2 + Bx + Cy + D = 0$$

For (1,1) $\quad 1 + 1 + B + C + D = 0$

$$B + C + D = -2 \qquad (9)$$

For (2,2) $\quad 4 + 4 + 2B + 2C + D = 0$

$$2B + 2C + D = -8 \qquad (10)$$

For (3,3) $\quad 9 + 9 + 3B + 3C + D = 0$

$$3B + 3C + D = -18 \qquad (11)$$

To eliminate D, first subtract (9) from (10).

$$\begin{array}{rl} 2B + 2C + D = -8 & \\ \underline{B + C + D = -2} & \text{(Subtract)} \\ B + C = -6 & \qquad (12) \end{array}$$

Next subtract (10) from (11).

$$\begin{array}{rl} 3B + 3C + D = -18 & \\ \underline{2B + 2C + D = -8} & \\ B + C = -10 & \qquad (13) \end{array}$$

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Then subtract (13) from (12) to eliminate one of the unknowns.

$$B + C = -6$$
$$\underline{B + C = -10}$$
$$0 + 0 = 4$$
$$0 = 4$$

This solution is not valid and there is no circle through the three given points. The reader should attempt to solve (12) and (13) by the substitution method. When the three given points are collinear an inconsistent solution of some type will result.

If we attempt to solve the problem by eliminating both B and C at the same time (to find D) another type of inconsistent solution results. With the given coefficients it is not difficult to eliminate both A and B at the same time. First, multiply (10) by 3 and (11) by -2 and add the resultant equations.

$$6B + 6C + 3D = -24$$
$$\underline{-6B - 6C - 2D = 36} \quad (+)$$
$$D = 12$$

Then multiply (9) by -2 and add the resultant to (10)

$$-2B - 2C - 2D = 4$$
$$\underline{2B + 2C + D = -8}$$
$$-D = -4$$
$$D = 4$$

This gives two values for D and is inconsistent since each of the constants must have a unique value consistent with the given conditions. The three points are on the straight line $y = x$.

PRACTICE PROBLEMS: In each of the problems below find the equation of the circle which passes through the three given points.

1. (14,0), (12,4), and (3,7)
2. (10,3), (11,8), and (7,14)
3. (1,1), (0,0), and (-1,-1)
4. (12,-5), (-9,-12), and (-4,3)

ANSWERS:

1. $x^2 + y^2 - 10x + 4y = 56$
2. $x^2 + y^2 - 6x - 14y = 7$
3. No solution; the given points describe the straight line $y = x$.
4. $x^2 + y^2 - 2x + 14y = 75$

THE PARABOLA

The parabola is the locus of all points which are equidistant from a fixed point, called the focus, and a fixed line called the directrix. In the parabola shown in figure 10-8, the point V, which lies halfway between the focus of the directrix is called the VERTEX of the parabola. In this figure and in many of the parabolas discussed in the first portion of this section, the vertex of the parabola will fall at the origin; however, the vertex of the parabola, like the center of the circle, can fall at any point in the plane.

In figure 10-8, the distance from the point (x, y) on the curve to the focus (a, 0) is

$$\sqrt{(x - a)^2 + y^2}$$

The distance from the point (x, y) to the directrix is

$$x + a$$

Since by definition these two distances are equal we may set them equal

$$\sqrt{(x - a)^2 + y^2} = x + a$$

Squaring both sides

$$(x - a)^2 + y^2 = (x + a)^2$$

Expanding

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

Canceling and combining terms we have an equation for the parabola

$$y^2 = 4ax$$

For every positive value of x in the equation of the parabola there are two values of y. But when x becomes negative the values of y are imaginary. Thus, the curve must be entirely to the right of the Y axis when the equation is in this form and a is positive. If the equation is

$$y^2 = -4ax$$

(a negative) the curve lies entirely to the left of the Y axis.

If the form of the equation is

$$x^2 = 4ay$$

the curve will open upward and the focus will be a point on the Y axis. For every positive value of y there will be two values of x and the curve will be entirely above the X axis. When the equation is in the form

$$x^2 = -4ay$$

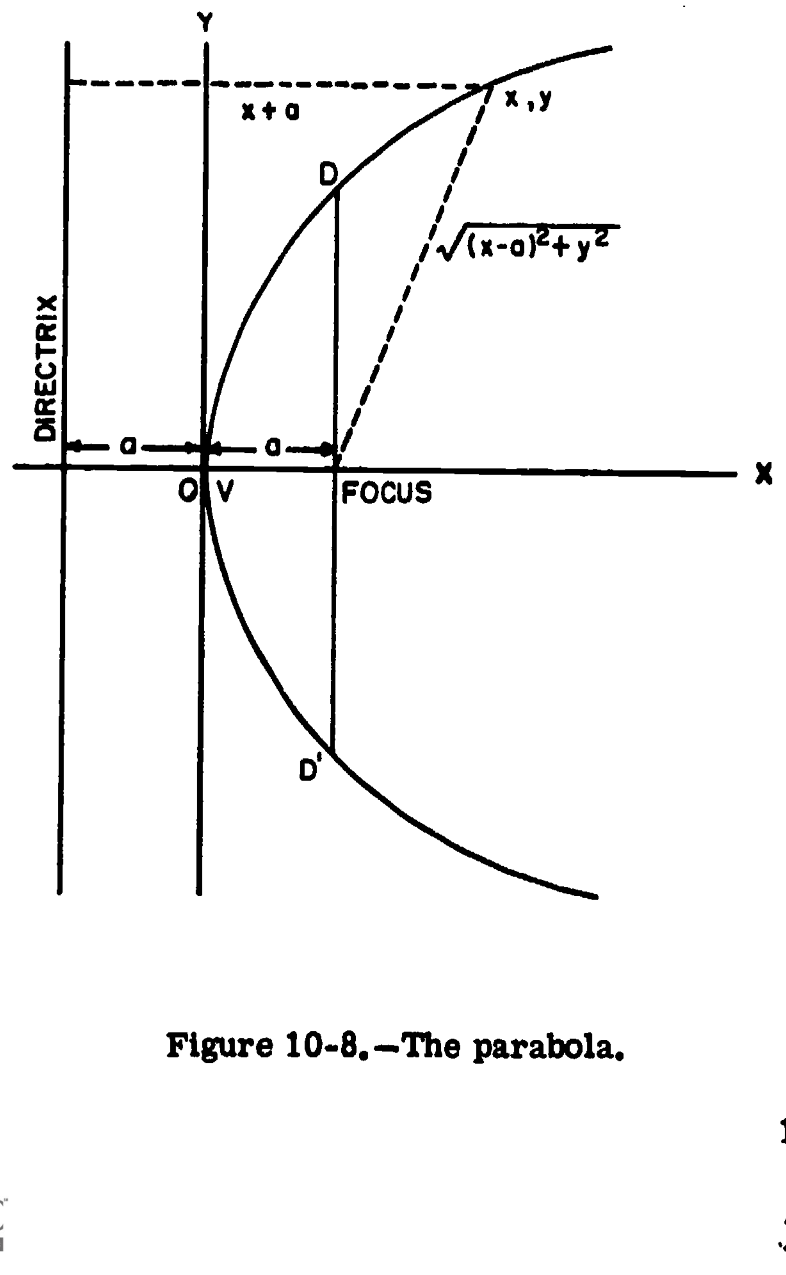


Figure 10-8.—The parabola.

the curve will open downward, be entirely below the X axis, and have as its focus a point on the negative Y axis. Parabolas which are representative of the four cases given here are shown in figure 10-9.

When x is equal to a in the equation

$$y^2 = 4ax$$

it follows that

$$y^2 = 4a^2$$

and

$$y = \pm 2a$$

This value of y is the height of the curve at the focus or the distance from the focus to point D in figure 10-8. The width of the curve at the focus is the distance from point D to point D' in the figure and is equal to 4a. This width is called the LATUS RECTUM in many texts; however, a more descriptive term is FOCAL CHORD and both terms will be used in this course. The latus rectum is one of the properties of a parabola which is used in the analysis of a parabola or in the sketching of a parabola.

EXAMPLE: Give the value of a, the length of the focal chord, and the equation of the parabola which is the locus of all points equidistant from the point (3, 0) and the line x = -3.

SOLUTION: First plot the given information on a coordinate system as shown in figure 10-10 (A). Reference to figure 10-8 shows that the point (3, 0) corresponds to the position of the focus and that the line x = -3 is the directrix of the parabola. Figure 10-8 also shows that the value of a is equal to one half of the distance from the focus to the directrix or, in this problem, one half the distance from x = -3 to x = 3. Thus, the value of a is 3.

The second value required by the problem is the length of the focal chord. As stated previously, the focal chord length is equal to 4a. The value of a was found to be 3 so the length of the focal chord is 12. Reference to figure 10-8 shows that one extremity of the focal chord will be a point on the curve which is 2a or 6 units above the focus, and the other extremity is a second point 2a or 6 units below the focus. Using this information and recalling that the vertex is one-half the distance

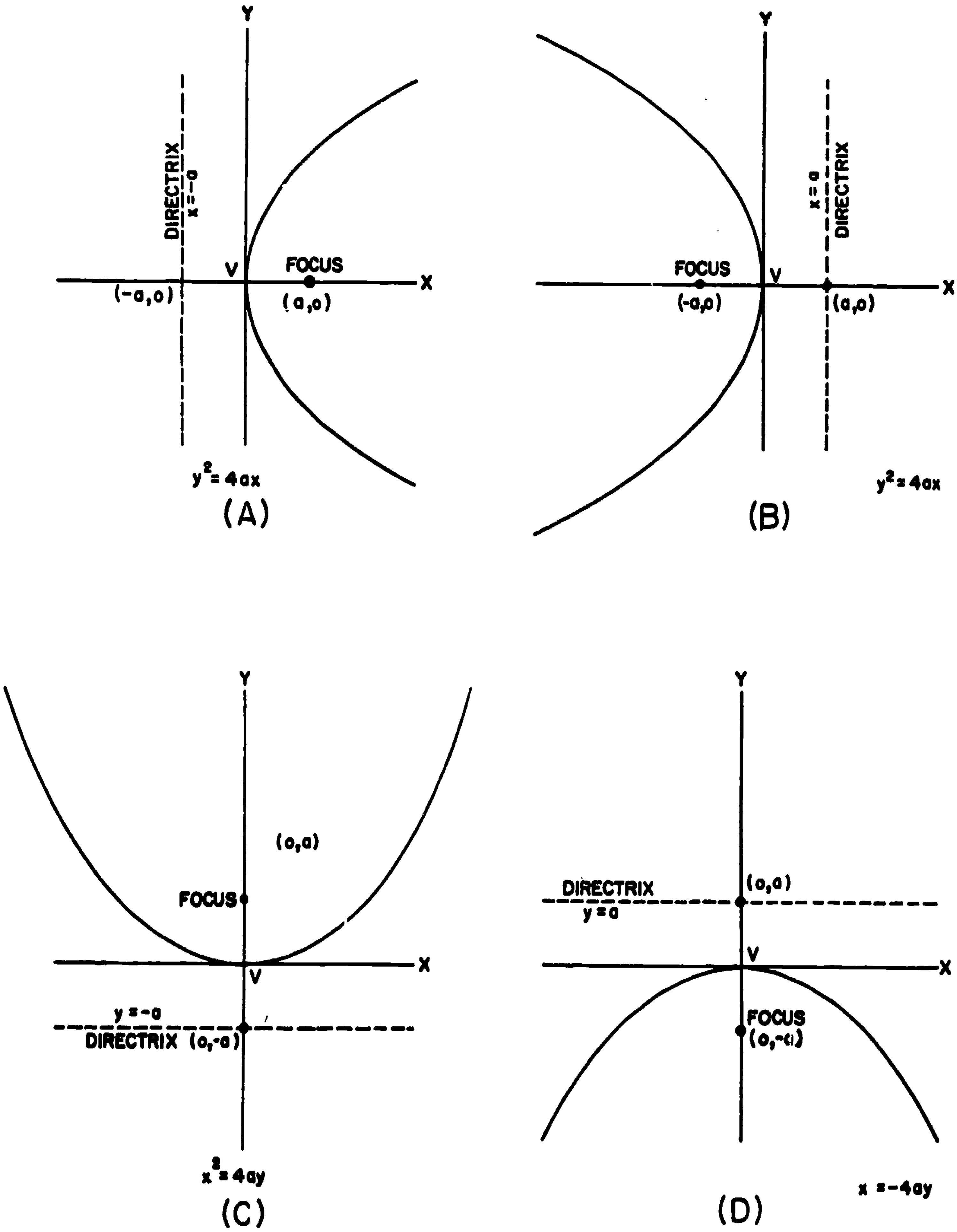


Figure 10-9.—Parabolas corresponding to four forms of the equation.

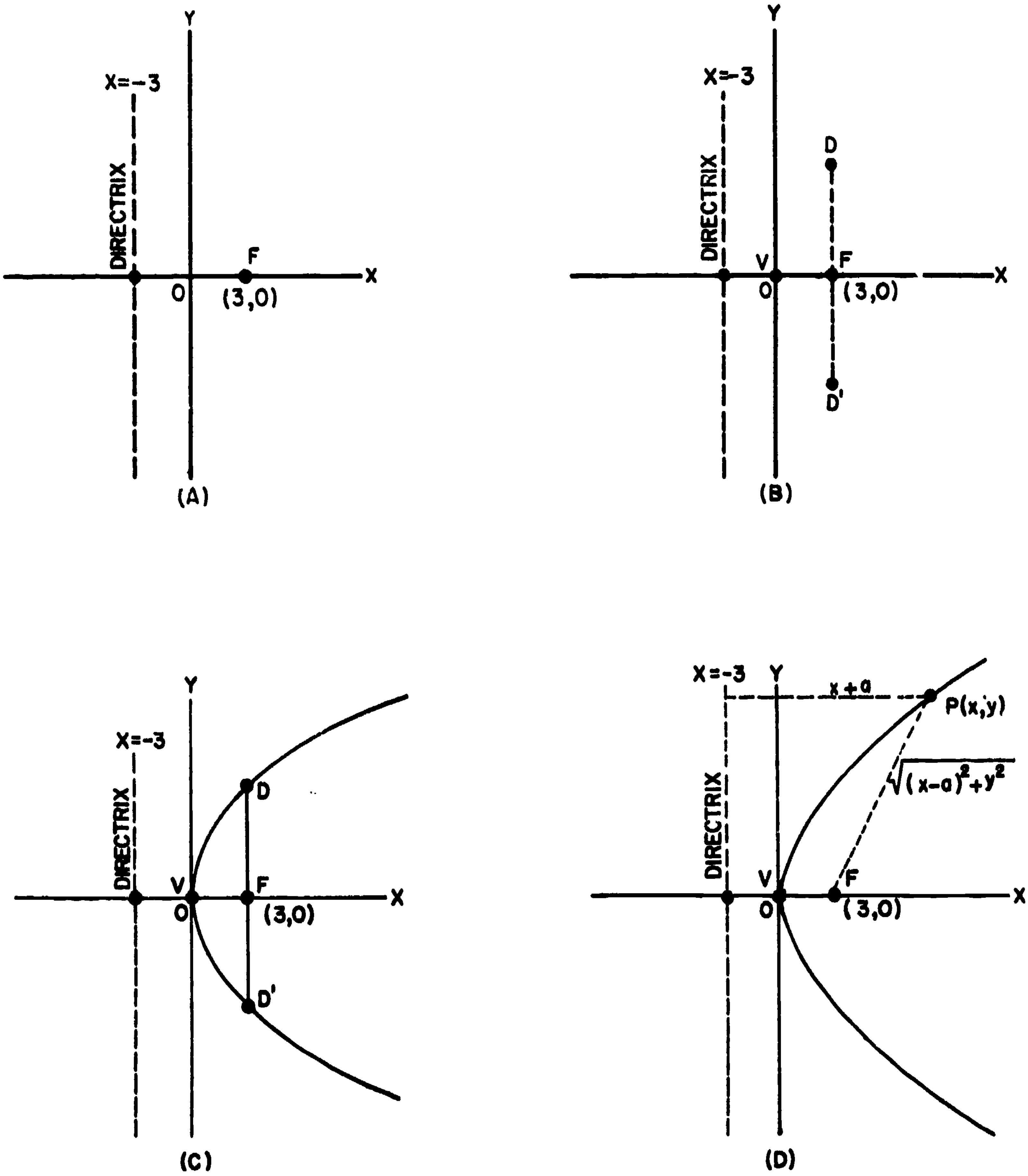


Figure 10-10.—Sketch of a parabola.

from the focus to the directrix, plot three more points as shown in figure 10-10 (B).

Now a smooth curve through the vertex and the two points that are the extremities of the focal chord is a sketch of the parabola in this problem. (See fig. 10-10 (C).)

To find the equation of the hyperbola refer to figure 10-10 (D) and use the procedure used earlier. We know by definition that any point P(x, y) on the parabola is equidistant from the focus and directrix.

Thus we equate these two distances and

$$\sqrt{(x-a)^2+y^2} = x + a$$

However, we have found the distance a to be equal to 3 so we substitute and

$$(x-3)^2 + y^2 = x + 3$$

Square both sides

$$(x-3)^2 + y^2 = (x+3)^2$$

Expand

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$

Cancel and combine terms to obtain the equation of the parabola

$$y^2 = 12x$$

If we check the consistency of our findings, we see that the form of the equation and the sketch agree with figure 10-9 (A). Also, the 12 in the right side of the equation corresponds to the 4a in the general form and is correct since we determined that the value of a was 3.

NOTE: When the focus of a parabola lies on the Y axis, the equated distance equation is

$$\sqrt{(y-a)^2+x^2} = y + a$$

PRACTICE PROBLEMS: Give the equation, the value of a, and the length of the focal chord for the parabola which is the locus of all points equidistant from the point and line given in the following problems.

1. The point (-2,0) and the line $x = 2$
2. The point (0,4) and the line $y = -4$
3. The point (0,-1) and the line $y = 1$
4. The point (1,0) and the line $x = -1$

ANSWERS:

1. $y^2 = -8x$, $a = -2$, f.c. = 8
2. $x^2 = 16y$, $a = 4$, f.c. = 16
3. $x^2 = -4y$, $a = -1$, f.c. = 4
4. $y^2 = 4x$, $a = 1$, f.c. = 4

FORMULA GENERALIZATION

All of the parabolas in the preceding section had the vertex at the origin and the corresponding equations were in one of four forms as follows:

1. $y^2 = 4ax$
2. $y^2 = -4ax$
3. $x^2 = 4ay$
4. $x^2 = -4ay$

In this section we will present four more forms of the equation of a parabola, generalized to consider a parabola with a vertex at point V (h,k). When the vertex is moved from the origin to a point V(h,k) the x and y terms of the equation are replaced by (x - h) and (y - k). Then the general equation for the parabola which opens to the right (fig. 10-9 (A)) is

$$(y-k)^2 = 4a(x-h)$$

The four general forms of the equations for parabolas with vertex at the point V(h,k) are as follows:

1. $(y-k)^2 = 4a(x-h)$, corresponding to $y^2 = 4ax$, parabola opens to the right

2. $(y-k)^2 = -4a(x-h)$, corresponding to $y^2 = -4ax$, parabola opens to the left

3. $(x - h)^2 = 4a(y - k)$, corresponding to $x^2 = 4ay$, parabola opens upward

4. $(x - h)^2 = -4a(y - k)$, corresponding to $x^2 = -4ay$, parabola opens downward.

The method for reducing an equation to one of these standard forms is similar to the methods used for reducing the equation of a circle.

EXAMPLE: Reduce the equation

$$y^2 - 6y - 8x + 1 = 0$$

to standard form.

SOLUTION: Rearrange the equation so that the second degree term and any first degree terms of the same unknown are on the left side. Then group the unknown term which appears only in the first degree and all constants on the right.

$$y^2 - 6y = 8x - 1$$

Then complete the square in y

$$y^2 - 6y + 9 = 8x - 1 + 9$$

$$(y - 3)^2 = 8x + 8$$

To get the equation in the form

$$(y - k)^2 = 4a(x - h)$$

factor an 8 out of the right side. Thus

$$(y - 3)^2 = 8(x + 1)$$

is the equation of the parabola.

PRACTICE PROBLEMS: Reduce the equations given in the following problems to standard form.

1. $x^2 + 4 = 4y$
2. $y^2 - 4x = 6y + 9$
3. $4x + 8y + y^2 + 20 = 0$
4. $4x - 12y + 40 + x^2 = 0$

ANSWERS:

1. $x^2 = 4(y - 1)$
2. $(y - 3)^2 = 4x$
3. $(y + 4)^2 = -4(x + 1)$
4. $(x + 2)^2 = 12(y - 3)$

THE ELLIPSE

An ellipse is a conic section with an eccentricity less than one.

Referring to figure 10-11, let

$$PO = a$$

$$FO = c$$

$$OM = d$$

where F is the focus, 0 is the center, and P and P' are points on the ellipse. Then from the definition of eccentricity,

$$\frac{a - c}{d - a} = e \qquad a - c = ed - ea$$

$$\frac{a + c}{d + a} = e \qquad a + c = ed + ea$$

Addition and subtraction of the two equations give:

$$2c = 2ae \text{ or } c = ae$$

$$2a = 2de \text{ or } d = \frac{a}{e} \tag{14}$$

Place the center of the ellipse at the origin so that one focus lies at (-ae,0) and one directrix is the line $x = -a/e$.

Referring to figure 10-12, there will be a point on the Y axis which will satisfy the conditions for an ellipse. Let

$$P''O = b$$

$$FO = c$$

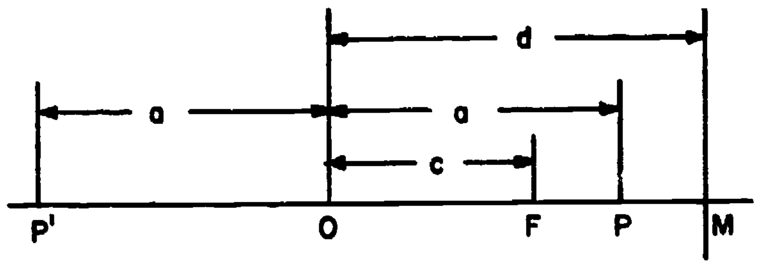


Figure 10-11.—Development of focus and directrix.

Then

$$P''F = \sqrt{b^2 + c^2}$$

and the ratio of the distance of P'' from the focus and the directrix is e so that

$$\frac{\sqrt{b^2 + c^2}}{\frac{a}{e}} = \epsilon$$

Multiplying both sides by a/e gives

$$\sqrt{b^2 + c^2} = a$$

or

$$b^2 + c^2 = a^2$$

so that

$$b = \pm\sqrt{a^2 - c^2} \qquad (15)$$

Now combining equations (14) and (15) gives

$$b = \pm\sqrt{a^2 - a^2e^2}$$

or

$$b = \pm a\sqrt{1 - e^2} \qquad (16)$$

Refer to figure 10-13. If the point (x, y) is on the ellipse, the ratio of its distance from F to its distance from the directrix will be e: The distance from (x, y) to the focus (-ae,0) will be

$$\sqrt{(x + ae)^2 + y^2}$$

and the distance from (x, y) to the directrix $x = -\frac{a}{e}$ is

$$x + \frac{a}{e}$$

The ratio of these two distances is equal to e so that

$$\frac{\sqrt{(x + ae)^2 + y^2}}{x + \frac{a}{e}} = e$$

or

$$\sqrt{(x + ae)^2 + y^2} = e\left(x + \frac{a}{e}\right)$$

$$= ex + a$$

Squaring both sides gives

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

Canceling like terms and transposing terms in x to the left-hand side of the equation gives

$$x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$

Removing a common factor,

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2) \qquad (17)$$

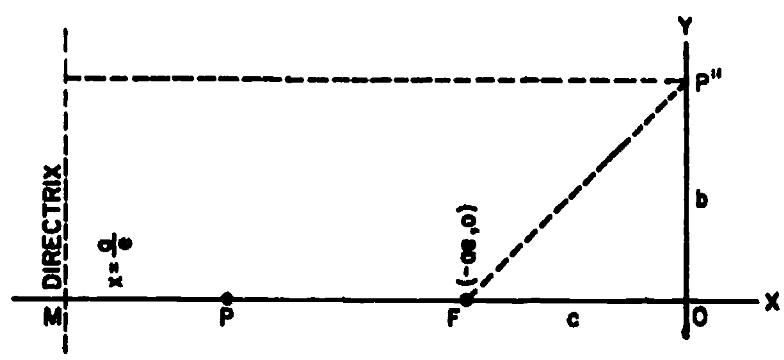


Figure 10-12.—Focus directrix, and point P''.

Dividing equation (17) through by the right-hand member,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

From equation (16) we obtain

$$b = \pm a\sqrt{1 - e^2}$$

$$a^2(1 - e^2) = b^2$$

so that the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (18)$$

This is the equation of an ellipse in standard form. In figure 10-14, a is the length of the semimajor axis and b is the length of the semiminor axis.

The curve is symmetrical with respect to the x and y axes, so that it is easily seen that it has another focus at (ae,0) and a corresponding directrix x = a/e.

The distance from the center through the focus to the curve is always designated a and is called the semimajor axis. This axis may be in either the x or y direction. When it is in the y direction, the directrix is a line with the equation

$$y = k$$

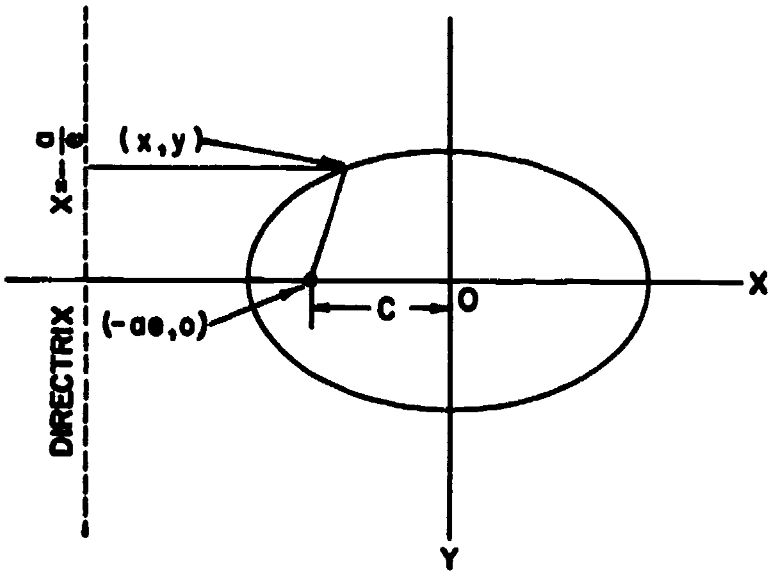


Figure 10-13.—The ellipse.

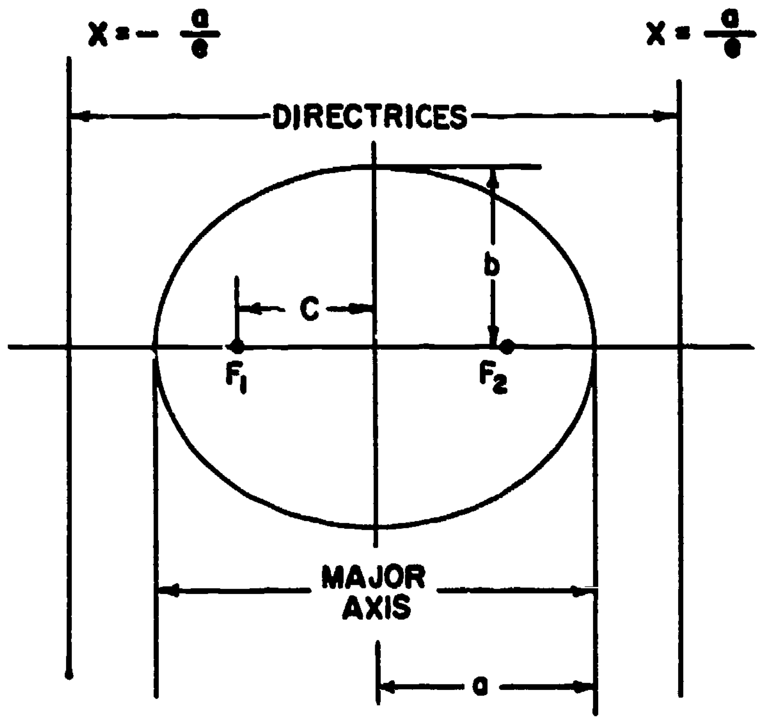


Figure 10-14.—Ellipse showing axes.

In the case we have studied, the directrix was denoted by the formula

$$x = k$$

where k is a constant equal to - a/e.

The perpendicular distance from the midpoint of the major axis to the curve is called the semiminor axis and is always signified by b.

The distance from the center of the ellipse to the focus is called c and in any ellipse the following relations hold for a, b, and c

$$c = \sqrt{a^2 - b^2}$$

$$b = \sqrt{a^2 - c^2}$$

$$a = \sqrt{b^2 + c^2}$$

Whenever the directrix is a line with the equation y = k the major axis will be in the y direction and the equation of the ellipse will be as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (19)$$

Otherwise everything remains as before and the equation is given by (18).

In an ellipse the position of the a^2 and b^2 terms indicate the orientation of the ellipse axis. As shown in figure 10-14 the value is the semimajor or longer axis.

In the previous paragraphs formulas were given which related a, b, and c and, in the first portion of this discussion, a formula relating a, c, and the eccentricity was given. These relationships will be used to find the equation of an ellipse in the following example.

EXAMPLE: Find the equation of the ellipse with center at the origin and having foci at $(\pm 2\sqrt{6}, 0)$ and an eccentricity equal to $\frac{2\sqrt{6}}{7}$.

SOLUTION: With the focal points on the x axis the ellipse is oriented as in figure 10-14 and the standard form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With the center at origin the numerators of the fractions on the left are x^2 and y^2 so the problem is to find the values of a and b.

The distance from the center to either of the foci is the value c (fig. 10-14) so in this problem

$$c = \pm 2\sqrt{6}$$

from the given coordinates of the foci.

The values of a, c, and e (eccentricity) are related by

$$c = ae$$

or

$$a = \frac{c}{e}$$

From the known information, substitute the values of c and e

$$a = \frac{\pm 2\sqrt{6}}{\frac{2\sqrt{6}}{7}}$$

$$a = \pm 2\sqrt{6} \times \frac{7}{2\sqrt{6}}$$

and

$$a = \pm 7$$

$$a^2 = 49$$

Then, using the formula

$$b = \sqrt{a^2 - c^2}$$

or

$$b^2 = a^2 - c^2$$

and substituting for a^2 and c^2

$$b^2 = 49 - (\pm 2\sqrt{6})2$$

$$b^2 = 49 - (4 \times 6)$$

$$b^2 = 49 - 24$$

gives the final required value of

$$b^2 = 25$$

Then, the equation of the ellipse is

$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

PRACTICE PROBLEMS: Find the equation of the ellipse with center at the origin and for which the following properties are given.

1. Foci at $(\pm\sqrt{7}, 0)$ and an eccentricity of $\frac{\sqrt{7}}{4}$
2. $b = 5$, $e = \frac{\sqrt{11}}{6}$
3. $a = 7$, $e = \frac{3\sqrt{5}}{7}$

ANSWERS:

1. $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$
2. $\frac{x^2}{36} + \frac{y^2}{25} = 1$
3. $\frac{x^2}{7^2} + \frac{y^2}{2^2} = 1$

ELLIPSE AS A LOCUS OF POINTS

An ellipse may be defined as the locus of a point which moves so that the sum of its distances from two fixed points is a constant equal to 2a.

Let the foci be F_1 and F_2 at ($\pm$ae,0), as shown in figure 10-15, and let the directrices be

$$x = \frac{\pm a}{e}$$

Then

$$F_1P = e\left(\frac{a}{e} - x\right) = a - ex$$

$$F_2P = e\left(\frac{a}{e} + x\right) = a + ex$$

so that

$$F_1P + F_2P = a - ex + a + ex$$

$$F_1P + F_2P = 2a$$

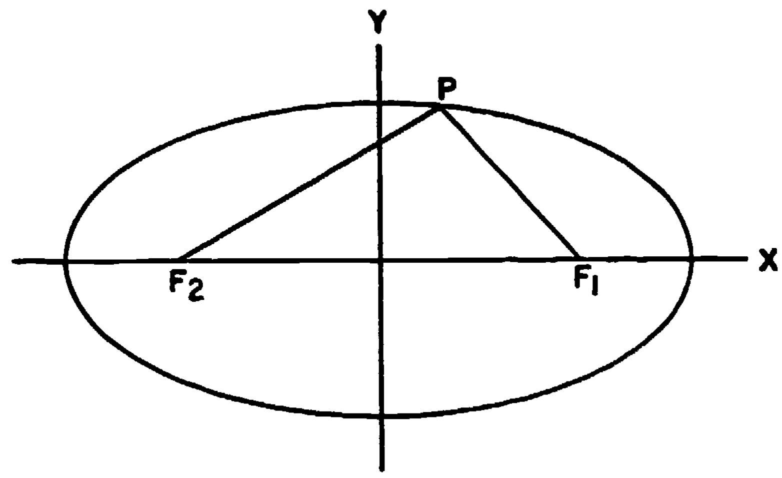


Figure 10-15.—Ellipse, center at origin.

Whenever the center of the ellipse is at some point other than (0,0), say at the point (h,k), figure 10-16, the equation of the ellipse must be modified to the following form

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad (20)$$

Subtracting h from the value of x reduces the value of the term (x - h) to the value which x would have if the center were at the origin. The term (y - k) is identical in value to the value of y if the center were at the origin.

REDUCTION TO STANDARD FORM

Whenever we have an equation in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad (21)$$

where the capital letters refer to independent constants and A and C have the same sign, we can reduce the equation to the standard form for an ellipse. Completing the squares in x and y and performing a few simple algebraic transformations will change the form to that of equation (20).

THEOREM: An equation of the second degree, in which the xy term is missing and the coefficients of x^2 and y^2 are different but have the same sign, represents an ellipse with axes parallel to the coordinate axes.

EXAMPLE: Reduce the equation

$$4x^2 + 9y^2 - 40x - 54y + 145 = 0$$

to the standard form of an ellipse.

SOLUTION: Collect terms in x and y and remove the common factors of these terms.

$$4x^2 - 40x + 9y^2 - 54y + 145 = 0$$

$$4(x^2 - 10x) + 9(y^2 - 6y) + 145 = 0$$

Transpose the constant terms and complete the squares in x and y. When there are factored terms involved in completing the square, as in this example, an error is frequently made. The

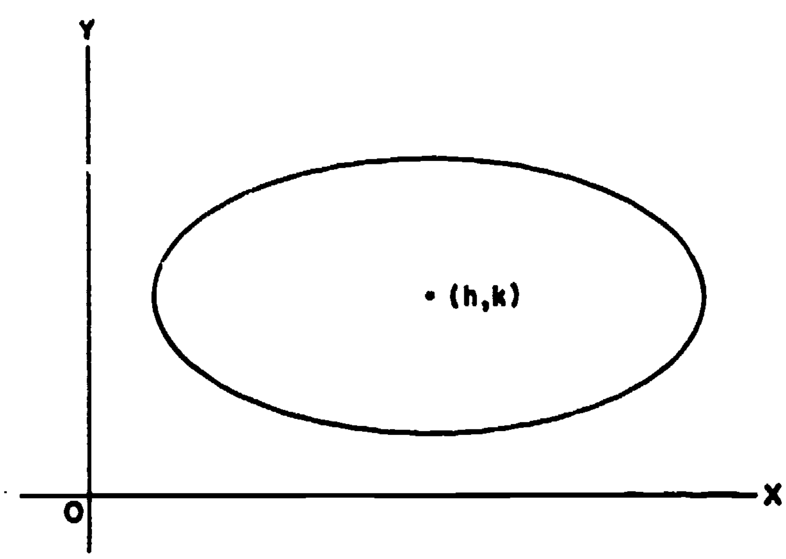


Figure 10-16.—Ellipse, center at (h,k).

factored value operates on the term added inside the parentheses as well as the original terms. Therefore, the values added to the right side of the equation will be the product of the factored value and the term added to complete the square.

$$4(x^2 - 10x + 25) + 9(y^2 - 6y + 9)$$

$$= -145 + 4(25) + 9(9)$$

$$= -145 + 100 + 81$$

$$= 36$$

$$4(x - 5)^2 + 9(y - 3)^2 = 36$$

Divide through by the right-hand (constant) term. This reduces the right member to 1 as required by the standard form.

$$\frac{4(x - 5)^2}{36} + \frac{9(y - 3)^2}{36} = 1$$

$$\frac{(x - 5)^2}{9} + \frac{(y - 3)^2}{4} = 1$$

This reduces to the standard form

$$\frac{(x - 5)^2}{(3)^2} + \frac{(y - 3)^2}{(2)^2} = 1$$

Corresponding to equation (20) and represents an ellipse with the center at (5,3), its semimajor axis (a) equal to 3, and its semiminor axis (b) equal to 2.

EXAMPLE: Reduce the equation

$$3x^2 + y^2 + 20x + 32 = 0$$

to the standard form of an ellipse.

SOLUTION: First, collect terms in x and y. As in the previous example, the coefficients of x^2 and y^2 must be reduced to 1 in order to facilitate completing the square. Thus the coefficient of the x^2 term is divided out of the two terms containing x, as follows:

$$3x^2 + 20x + y^2 + 32 = 0$$

$$3\left(x^2 + \frac{20x}{3}\right) + y^2 = -32$$

Complete the square in x noting that there will be a product added to the right side

$$3\left(x^2 + \frac{20x}{3} + \frac{100}{9}\right) + y^2 = -32 + 3\left(\frac{100}{9}\right)$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = -32 + \frac{300}{9}$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{-288 + 300}{9}$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{12}{9}$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{4}{3}$$

Divide through by the right-hand term.

$$\frac{3\left(x + \frac{10}{3}\right)^2}{\frac{4}{3}} + \frac{y^2}{\frac{4}{3}} = 1$$

$$\frac{9\left(x + \frac{10}{3}\right)^2}{4} + \frac{y^2}{\frac{4}{3}} = 1$$

$$\frac{\left(x + \frac{10}{3}\right)^2}{\left(\frac{4}{9}\right)} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1$$

This reduces to the standard form

$$\frac{\left(x + \frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2} + \frac{y^2}{\left(\frac{2}{\sqrt{3}}\right)^2} = 1$$

$$\frac{\left(x + \frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2} = 1$$

PRACTICE PROBLEMS: Express the following equations in the standard form for an ellipse.

1. $5x^2 - 110x + 4y^2 + 425 = 0$
2. $x^2 - 14x + 36y^2 - 216y + 337 = 0$
3. $9x^2 - 54x + 4y^2 + 16y + 61 = 0$
4. $3x^2 - 14x + 4y^2 + 11 = 0$

ANSWERS:

1. $\frac{(x - 11)^2}{(6)^2} + \frac{y^2}{(3\sqrt{5})^2} = 1$
2. $\frac{(x - 7)^2}{(6)^2} + \frac{(y - 3)^2}{1} = 1$
3. $\frac{(x - 3)^2}{(2)^2} + \frac{(y + 2)^2}{(3)^2} = 1$
4. $\frac{\left(x + \frac{7}{3}\right)^2}{\left(\frac{4}{3}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2} = 1$

THE HYPERBOLA

A hyperbola is a conic section with an eccentricity greater than one.

The formulas

$$c = ae$$

and

$$d = \frac{a}{e}$$

developed in the section concerning the ellipse were derived so that they hold true for any value of eccentricity. Thus, they hold true for the hyperbola as well as for an ellipse. Since e is greater than one for a hyperbola, then

$$c = ae \text{ and } c > a$$

$$d = \frac{a}{e} \text{ and } d < a$$

Therefore $c > a > d$.

According to this analysis, if the center of symmetry of a hyperbola is the origin, the foci will lie farther from the origin than the directrices. An inspection of figure 10-17 shows that the curve will never cross the Y axis. Thus, the solution for the value of b, the semiminor axis of the ellipse will yield no real value for b. In other words, b will be an imaginery number. This can easily be seen from the equation

$$b = \sqrt{a^2 - c^2}$$

since $c > a$ for a hyperbola.

However, we can square both sides of the above equation, and since the square of an imaginary number is a negative real number we write

$$-b^2 = a^2 - c^2$$

or

$$b^2 = c^2 - a^2$$

and, since $c = ae$,

$$b^2 = a^2e^2 - a^2 = a^2(e^2 - 1)$$

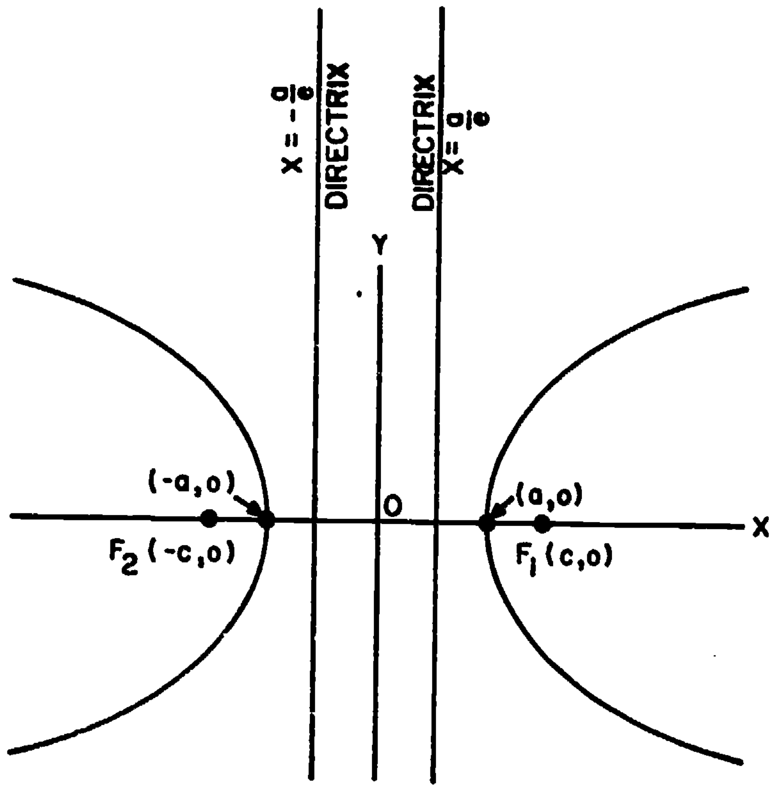


Figure 10-17.—The hyperbola.

Now we can use this equation to obtain the equation of a hyperbola from the following equation which was developed in the section on the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

and since

$$a^2(1 - e^2) = -a^2(e^2 - 1) = -b^2$$

we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is a standard form for the equation of a hyperbola. The solution of this equation for y gives

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

which shows that y is imaginary only when $x^2 < a^2$. The curve, therefore, lies entirely beyond the two lines $x = \pm a$ and crosses the x axis at $x = \pm a$.

The two straight lines

$$bx + ay = 0 \text{ and } bx - ay = 0 \qquad (22)$$

can be used to illustrate an interesting property of a hyperbola. The distance from the line $bx - ay = 0$ to a point (x_1, y_1) on the curve is given by

$$d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}} \qquad (23)$$

Since (x_1, y_1) is on the curve, its coordinates satisfy the equation

$$b^2x_1^{\ 2} - a^2y_1^{\ 2} = a^2b^2$$

which may be written

$$(bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2$$

or

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}$$

Now substituting this value into equation (23), gives us

$$d = \frac{a^2b^2}{\sqrt{a^2 + b^2}}\left(\frac{1}{bx_1 + ay_1}\right)$$

As the point (x_1, y_1) is chosen farther and farther from the center of the hyperbola, the absolute values for x_1 and y_1 will increase and the distance d will approach zero. A similar result can easily be derived for the line $bx + ay = 0$.

The lines of equation (22) which are usually written

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

are called the asymptotes of the hyperbola. They are very important in tracing a curve and studying its properties. The asymptotes of a hyperbola, figure 10-18, are the diagonals of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve. The latus rectum of a hyperbola is equal to $\frac{2b^2}{|a|}$

Another definition of a hyperbola is the locus of a point that moves so that the difference of its distances from two fixed points is constant. The fixed points are the foci and the constant difference is 2a.

The nomenclature of the hyperbola is slightly different from that of an ellipse. The transverse axis is 2a or the distance between the intersections of the hyperbola with its focal axis. The conjugate axis is 2b and is perpendicular to the transverse axis.

Whenever the foci are on the Y axis and the directrices are lines of the form $y = \pm k$, where k is a constant, the equation of the hyperbola will read

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

This equation represents a hyperbola with its transverse axis on the Y axis. Its asymptotes are the lines $by - ax = 0$ and $by + ax = 0$.

ANALYSIS OF THE EQUATION

The properties of the hyperbola most often used in analysis of the curve are the foci, directrices, length of the latus rectum, and the equations of the asymptotes.

Reference to figure 10-17 shows that the foci are given by the points F_1 (c,0) and F_2 (-c,0) when the equation of the hyperbola is in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If the equation were

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

the foci would be the points (0,c) and (0,-c). The value of c is either determined from the formula

$$c^2 = a^2 + b^2$$

or the formula

$$c = ae$$

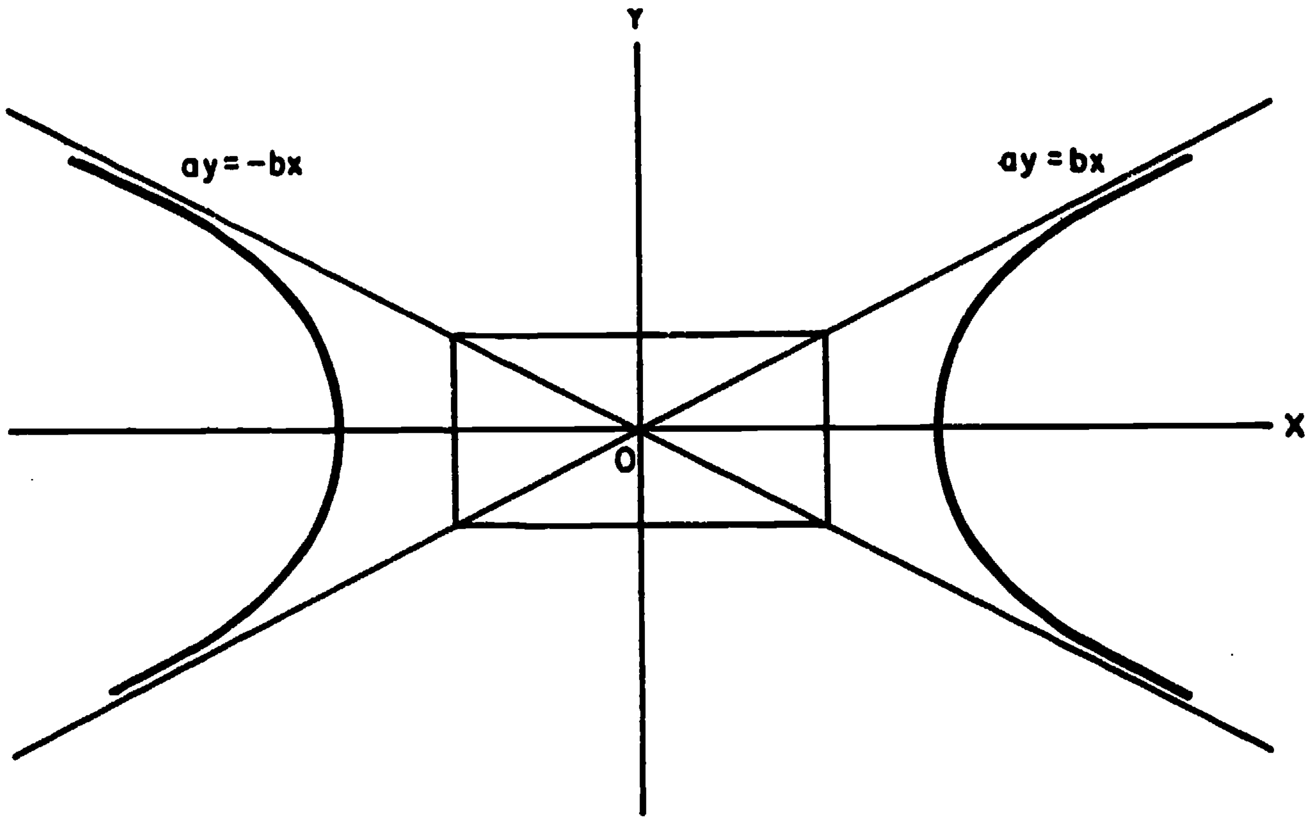


Figure 10-18.—Using asymptotes to sketch a hyperbola.

Figure 10-17 also shows that the directrices are the lines $x = \pm\frac{a}{e}$ or, in the case where the hyperbolas open upward and downward, $y = \pm\frac{a}{e}$. This is also given earlier in this discussion as $d = \frac{a}{e}$.

The equations of the asymptotes are given earlier as

$$bx + ay = 0 \text{ and } bx - ay = 0$$

or

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

It was also pointed out that the length of the latus rectum is equal to $\frac{2b^2}{|a|}$

The properties of a hyperbola can be determined from the equation of a hyperbola or the equation can be written given certain properties, as shown in the following examples. In these examples and in the practice problems immediately following, all of the hyperbolas considered have their centers at the origin.

EXAMPLE: Find the equation of the hyperbola with an eccentricity of 3/2, directrices $x = \pm 4/3$, and foci at $(\pm 3,0)$.

SOLUTION: The foci lie on the X axis at the points (3,0) and (-3,0) so the equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This fact is also shown by the equation of the directrices.

Before proceeding with the problem one point should be emphasized: in the basic formula for the hyperbola the a^2 term will always be the denominator for the x^2 term and the b^2 term the denominator for the y^2 term. The orientation of the axis of symmetry is not dependent on the size of a^2 and b^2 as in the ellipse; it lies along or parallel to the axis of the positive x^2 or y^2 term. Since we have determined the form of the equation and since the center of the curve in this section is restricted to the origin the problem is reduced to finding the values of a^2 and b^2

First, the foci are given as $(\pm 3,0)$ and since the foci are also the points $(\pm c,0)$ it follows that

$$c = \pm 3$$

The eccentricity is given and the value of a^2 can be determined from the formula

$$c = ae$$

$$a = \frac{c}{e}$$

$$a = \frac{\pm 3}{\frac{3}{2}}$$

$$a = \frac{\pm 6}{3}$$

$$a = \pm 2$$

$$a^2 = 4$$

The relationship of a, b, and c for the hyperbola is

$$b^2 = c^2 - a^2$$

and

$$b^2 = (\pm 3)^2 - (\pm 2)^2$$

$$b^2 = 9 - 4$$

$$b^2 = 5$$

When these values are substituted in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

results and is the equation of the hyperbola.

The equation could also be found by the use of other relationships which utilize the given information.

The directrices are given as

$$x = \pm\frac{4}{3}$$

and, since

$$d = \frac{a}{e}$$

or

$$a = de$$

Substituting the values given for d and e results in

$$a = \pm\frac{4}{-3}\left(\frac{3}{2}\right)$$

When $d > 0$

$$a = \frac{4}{3}\left(\frac{3}{2}\right)$$

$$a = 2$$

When $d < 0$

$$a = -\frac{4}{3}\left(\frac{3}{2}\right)$$

$$a = -2$$

therefore

$$a = \pm 2$$

and

$$a^2 = 4$$

While the value of c can be determined by the given information in this problem, it could also be computed since

$$c = ae$$

and a has been found to equal ± 2 and e is given as $\frac{3}{2}$, then

$$c = \pm 2\left(\frac{3}{2}\right)$$

and, when $a > 0$

$$c = 2\left(\frac{3}{2}\right)$$

$$c = 3$$

For $a < 0$

$$c = -2\left(\frac{3}{2}\right)$$

$$c = -3$$

Then

$$c = \pm 3$$

With values for a and c computed, the value of b is found as before and the equation can be written.

EXAMPLE: Find the foci, directrices, eccentricity, length of the latus rectum, and equations of the asymptotes of the hyperbola described by the equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

SOLUTION: This equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the values for a and b are determined by inspection to be

$$a^2 = 9$$

$$a = \pm 3$$

and

$$b^2 = 16$$

$$b = \pm 4$$

With a and b known, find c by using the formula

$$b^2 = c^2 - a^2$$

$$c^2 = a^2 + b^2$$

$$c = \pm \sqrt{a^2 + b^2}$$

$$c = \pm \sqrt{9 + 16}$$

$$c = \pm \sqrt{25}$$

$$c = \pm 5$$

From the form of the equation we know that the foci are at the points

$$F_1(c, 0)$$

and

$$F_2(-c, 0)$$

so the foci = $(\pm 5, 0)$.

The eccentricity is found by the formula

$$e = \frac{c}{a}$$

$$e = \frac{\pm 5}{\pm 3}$$

$$e = \frac{5}{3}$$

Reference to figure 10-17 shows that with the center at the origin, c and a will have the same sign.

The directrix is found by the formula

$$d = \frac{a}{e}$$

or, since this equation will have directrices parallel to the Y axis, use the formula

$$x = \frac{a}{e}$$

Then

$$x = \frac{\pm 3}{\frac{5}{3}}$$

$$x = \pm 3 \left(\frac{3}{5}\right)$$

When $a > 0$

$$x = 3 \left(\frac{3}{5}\right)$$

$$x = \frac{9}{5}$$

and when $a < 0$

$$x = -3 \left(\frac{3}{5}\right)$$

$$x = -\frac{9}{5}$$

so the directrices are the lines

$$x = \frac{\pm 9}{5}$$

The latus rectum (l. r.) is found by

$$\text{l. r.} = \frac{2b^2}{|a|}$$

$$\text{l. r.} = \frac{2(16)}{3}$$

$$\text{l. r.} = \frac{32}{3}$$

Finally, the equations of the asymptotes are the equation of the two straight lines

$$bx + ay = 0$$

and

$$bx - ay = 0$$

In this problem, substituting the values of a and d in the equation gives

$$4x + 3y = 0$$

and

$$4x - 3y = 0$$

or

$$4x \pm 3y = 0$$

The equations of the lines asymptotic to the curve can also be written in the form

$$y = \frac{b}{a}x$$

and

$$y = -\frac{b}{a}x$$

In this form the lines are

$$y = \frac{4}{3}x$$

and

$$y = -\frac{4}{3}x$$

or

$$y = \pm\frac{4}{3}x$$

If we think of this equation as a form of the slope intercept formula

$$y = mx + b$$

from chapter 9, the lines would have slopes of $\pm\frac{b}{a}$ and each would have its Y intercept at the origin as shown in figure 10-18.

PRACTICE PROBLEMS:

1. Find the equation of the hyperbola with an eccentricity of $\sqrt{2}$, directrices $x = \pm\frac{\sqrt{2}}{2}$, and foci at $(\pm\sqrt{2}, 0)$.

2. Find the equation of the hyperbola with an eccentricity of 5/3, foci at $(\pm5,0)$, and directrices $x = \pm 9/5$.

Find the foci, directrices, eccentricity, equations of the asymptotes, and length of the latus rectum of the hyperbolas given in problems 3 and 4.

3. $\frac{x^2}{9} - \frac{y^2}{9} = 1$

4. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

ANSWERS:

1. $x^2 - y^2 = 1$

2. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

3. foci $= (\pm3\sqrt{2}, 0)$; directrices, $x = \frac{\pm3}{\sqrt{2}}$, eccentricity $= \sqrt{2}$; l.r. = 6; asymptotes $x \pm y = 0$.

4. foci $= (\pm\sqrt{13}, 0)$; directrices $x = \frac{\pm9}{\sqrt{13}}$; eccentricity $= \frac{\sqrt{13}}{3}$; l. r. $= \frac{8}{3}$; asymptotes $2x \pm 3y = 0$.

The hyperbola can be represented by an equation in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have different signs. These equations can be reduced to standard form in the same manner in which similar equations for the ellipse were reduced to standard form. The general forms of these standard equations are given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

and

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

POLAR COORDINATES

So far we have located a point in a plane by giving the distances of the point from two perpendicular lines. The location of a point can be defined equally well by noting its distance and bearing. This method is commonly used aboard ship to show the position of another ship or target. Thus, 3 miles at 35° locates the position of a ship relative to the course of the ship making the reading. We can use this method to develop curves and bring out their properties. Assume a fixed direction OX and a fixed point 0 on the line in figure 10-19. The position of any point P is fully determined, if we know the directed distance from 0 to P and the angle that the line 0P makes with reference line 0X. The line 0P is called the radius vector and the angle

POX is the polar angle. The radius vector is designed ρ while θ is the angle designation.

Point 0 is the pole or origin. As in conventional trigonometry, the polar angle is positive when measured counterclockwise and negative when measured clockwise. However, unlike the convention established in trigonometry, the radius vector for polar coordinates is positive only when it is laid off on the terminal side of the angle. When the radius vector is laid off on the terminal side of the ray produced beyond the pole (the given angle plus 180°) a negative value is assigned the radius vector. For this reason, there may be more than one equation in polar coordinates to describe a given locus. The concept of a negative radius vector is utilized in some advanced mathematics. For purposes of this course the concept is not explained or used. It is sufficient that the reader remember that the convention of an always positive radius vector is not followed in some branches of mathematics.

TRANSFORMATION FROM CARTESIAN TO POLAR COORDINATES

At times it will be simpler to work with the equation of a curve in polar coordinates than in cartesian coordinates. Therefore, it is important to know how to change from one system to the other. Sometimes both forms are useful, for some properties of the curve may be more apparent from one form of the equation and other properties more evident from the other.

Transformations are made by applying the following equations which can be derived from figure 10-20.

$$x = \rho \cos \theta \qquad (24)$$

$$y = \rho \sin \theta \qquad (25)$$

$$\rho^2 = x^2 + y^2 \qquad (26)$$

$$\tan \theta = \frac{y}{x} \qquad (27)$$

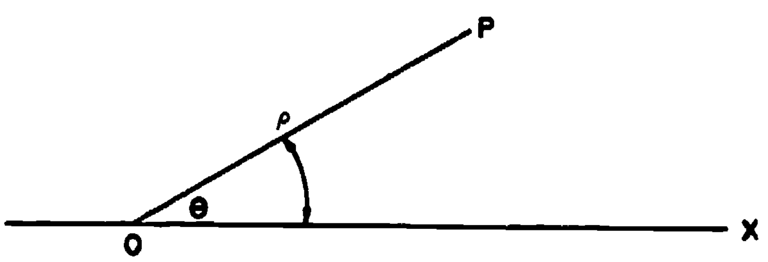


Figure 10-19.—Defining the polar coordinates.

EXAMPLE: Change the equation

$$y = x^2$$

from rectangular to polar coordinates.

SOLUTION: Substitute $\rho \cos\theta$ for x and $\rho \sin \theta$ for y so that we have

$$\rho \sin\theta = \rho^2 \cos^2 \theta$$

$$\sin \theta = \rho \cos^2 \theta$$

or

$$\rho = \frac{\sin\theta}{\cos^2\theta}$$

$$\rho = \tan\theta \sec\theta$$

EXAMPLE: Express the equation of the circle with its center at (a,0) and with a radius a, as shown in figure 10-21.

$$(x - a)^2 + y^2 = a^2$$

in polar coordinates.

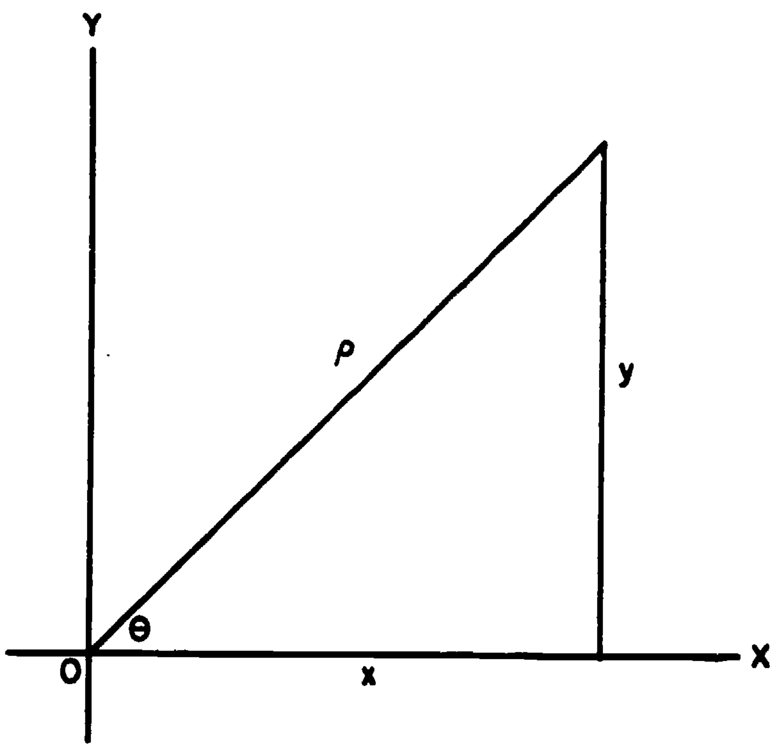


Figure 10-20.—Cartesian and polar relationship.

SOLUTION: First, expanding this equation gives us

$$x^2 - 2ax + a^2 + y^2 = a^2$$

Rearranging terms we have

$$x^2 + y^2 = 2ax$$

The use of equation (26) gives us

$$\rho^2 = 2ax$$

and applying the value of x given by equation (24), results in

$$\rho^2 = 2a\rho \cos\theta$$

Dividing through by ρ we have the equation of a circle with its center at (a,0) and radius a in polar coordinates

$$\rho = 2a \cos\theta$$

TRANSFORMATION FROM POLAR TO CARTESIAN COORDINATES

In order to transform to an equation in cartesian or rectangular coordinates from an equation in polar coordinates use the following equations which can be derived from figure 10-22.

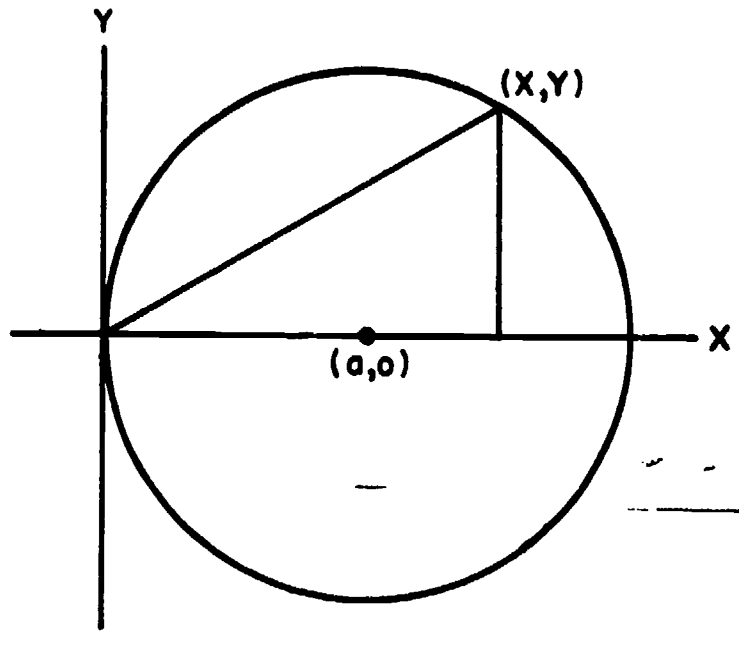


Figure 10-21.—Circle with center (a,0).

$$\rho = \sqrt{x^2 + y^2} \qquad (28)$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad (29)$$

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad (30)$$

$$\tan\theta = \frac{y}{x} \qquad (31)$$

$$\sec\theta = \frac{\sqrt{x^2 + y^2}}{x} \qquad (32)$$

$$\csc\theta = \frac{\sqrt{x^2 + y^2}}{y} \qquad (33)$$

$$\cot\theta = \frac{x}{y} \qquad (34)$$

EXAMPLE: Change the equation

$$\rho = \sec\theta \tan\theta$$

to an equation in rectangular coordinates

SOLUTION: Applying relations (28), (31), and (32) to the above equation gives

$$\sqrt{x^2 + y^2} = \frac{\sqrt{x^2 + y^2}}{x} \left(\frac{y}{x}\right)$$

Dividing both sides by $\sqrt{x^2 + y^2}$, we obtain

$$1 = \left(\frac{y}{x^2}\right)$$

or

$$y = x^2$$

which is the equation we set out to find.

EXAMPLE: Change the following equation to an equation in rectangular coordinates.

$$\rho = \frac{3}{\sin\theta - 3\cos\theta}$$

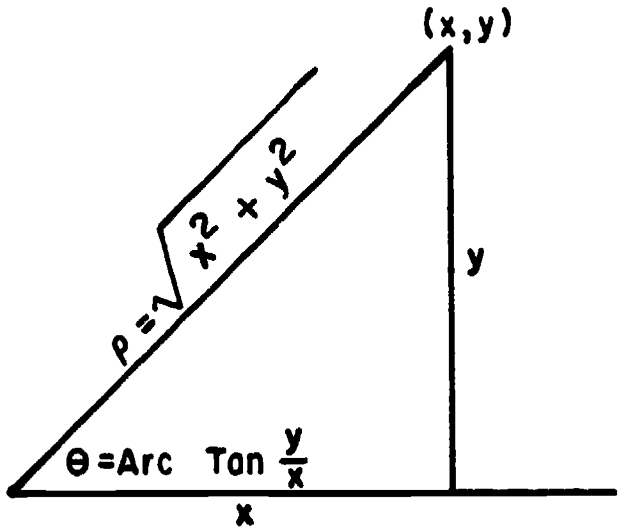


Figure 10-22.—Polar to cartesian relationship.

SOLUTION: Written without a denominator the polar equation is

$$\rho \sin\theta - 3\rho \cos\theta = 3$$

Using the transformations

$$\rho \sin\theta = y$$

$$\rho \cos\theta = x$$

we have

$$y - 3x = 3$$

as the equation in rectangular coordinates.

PRACTICE PROBLEMS: Change the equation in problems 1 through 4 to equations having polar coordinates.

1. $x^2 + y^2 = 4$
2. $\left(x^2 + y^2\right) = a^3 x^2$
3. $3y - 7x = 10$
4. $y = 2x - 3$

Change the equations in the following problems to equations having Cartesian coordinates.

5. $\rho = 4 \sin\theta$
6. $\rho = \sin\theta + \cos\theta$
7. $\rho = a^2$

ANSWERS:

1. $\rho = \pm 2$
2. $\rho = a\sqrt{a \cos\theta}$
3. $\rho = \dfrac{10}{3 \sin\theta - 7 \cos\theta}$
4. $\rho = \dfrac{-3}{\sin\theta - 2 \cos\theta}$
5. $x^2 + y^2 - 4y = 0$
6. $x^2 + y^2 = y + x$
7. $x^2 + y^2 = a^4$

CHAPTER 11

TANGENTS, NORMALS, AND SLOPES OF CURVES

In chapter 9, the notation $\frac{\Delta y}{\Delta x}$ was introduced to represent the slope of a line. The straight line discussed has a constant slope and the symbol Δy was defined as $(y_2 - y_1)$ and Δx was defined as $(x_2 - x_1)$. In this chapter we will discuss the slope of curves at specific points on the curves. We will do this with as little calculus as possible but our discussion will be directed toward the study of calculus.

SLOPE OF A CURVE AT A POINT

In figure 11-1, the slope of the curve is represented at two different places by $\frac{\Delta y}{\Delta x}$. The value of $\frac{\Delta y}{\Delta x}$ taken on the lower part of the curve will be extremely close to the actual slope at P_1 because P_1 lies on a nearly straight portion of the curve. The value of the slope at P_2 will be less accurate than the slope near P_1 because P_2 lies on a portion of the curve which has more curvature than the portion of the curve near P_1. In order to obtain an accurate measure of the slope of the curve at each point, as small a portion of the curve as possible should be used. When the curve is nearly a straight line a very small error will occur when finding the slope regardless of the value of the increments Δy and Δx. If the curvature is great and large increments are used when finding the slope of a curve, the error will become very large.

Thus, it follows that the error can be reduced to an infinitesimal if the increments are chosen infinitely small. Whenever the slope of a curve at a given point is desired, the increments Δy and Δx should be extremely small. Consequently, the arc of the curve can be replaced by a straight line, which determines the slope of the tangent at that point.

It must be understood that when we speak of the tangent to a curve at a specific point we are really considering the secant line, which cuts a curve in at least two points. This secant line is to be decreased in length, keeping the end points on the curve, to such a small value that it may be considered to be a point. This point is then extended to form the tangent to the curve at that specific point. Figure 11-2 shows this concept.

DIRECTION OF A CURVE

If we allow

$$y = f(x)$$

to represent the equation of a curve, then $\frac{\Delta y}{\Delta x}$ is the slope of the line tangent to the curve at P (x, y).

The direction of a curve is defined as the direction of the tangent line at any point on the curve. Let θ equal the inclination of the tangent line; then the slope equals $\tan \theta$ and

$$\frac{\Delta y}{\Delta x} = \tan \theta$$

is the slope of the curve at any point P (x, y). The angle θ_1 is the inclination of the tangent to the curve at P_1 in figure 11-3. This angle is acute and the value of $\tan \theta_1$ is positive. Hence the slope is positive at point P_1. The angle θ_2 is an obtuse angle and $\tan \theta_2$ is negative and the slope at point P2 is negative. All lines which lean to the right have positive slopes and all lines which lean to the left have negative slopes. At point P_3 the tangent to the curve is horizontal and θ equals 0. This means that

$$\frac{\Delta y}{\Delta x} = \tan 0° = 0$$

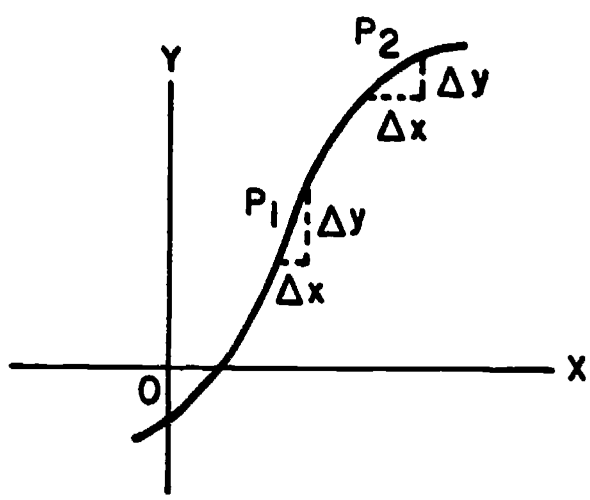


Figure 11-1.—Curve with increments Δy and Δx.

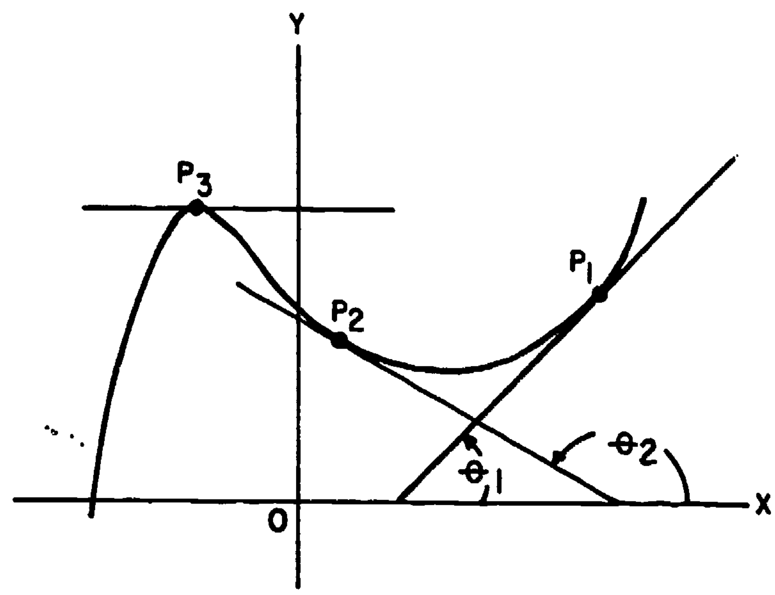


Figure 11-3.—Curve with tangent lines.

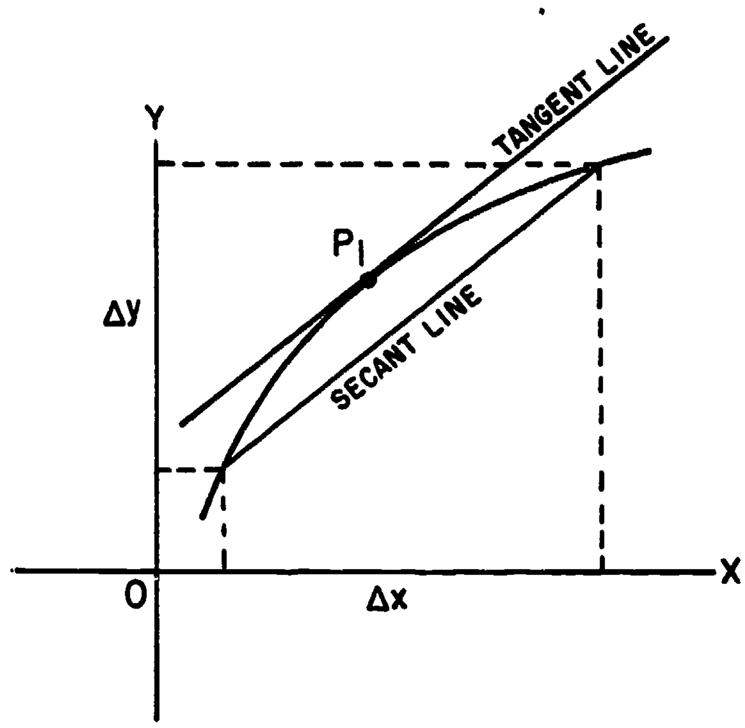


Figure 11-2.—Curve with secant line and tangent line.

The fact that the slope of a curve is zero when the tangent to the curve at that point is horizontal is of great importance in calculus when determining the maximum or minimum points of a curve. Whenever the slope of a curve is zero, the curve may be at either a maximum or a minimum.

Whenever the inclination of the tangent to a curve at a point is 90°, the tangent line is vertical and parallel to the Y axis. This results in an infinitely large slope

$$\frac{\Delta y}{\Delta x} = \tan 90^\circ = \infty$$

TANGENT AT A GIVEN POINT ON THE STANDARD PARABOLA

The standard parabola is represented by the equation

$$y^2 = 4ax$$

Let P_1 with coordinates (x_1, y_1) be a point on the curve. Choose P' on the curve, figure 11-4, near the given point so that the coordinates of P' are

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Since P' is a point on the curve

$$y^2 = 4ax$$

the values of its coordinates may be substituted for x and y. This gives

$$(y_1 + \Delta y)^2 = 4a(x_1 + \Delta x)$$

or

$$y_1^2 + 2y_1 \Delta y + (\Delta y)^2 = 4ax_1 + 4a \Delta x \qquad (1)$$

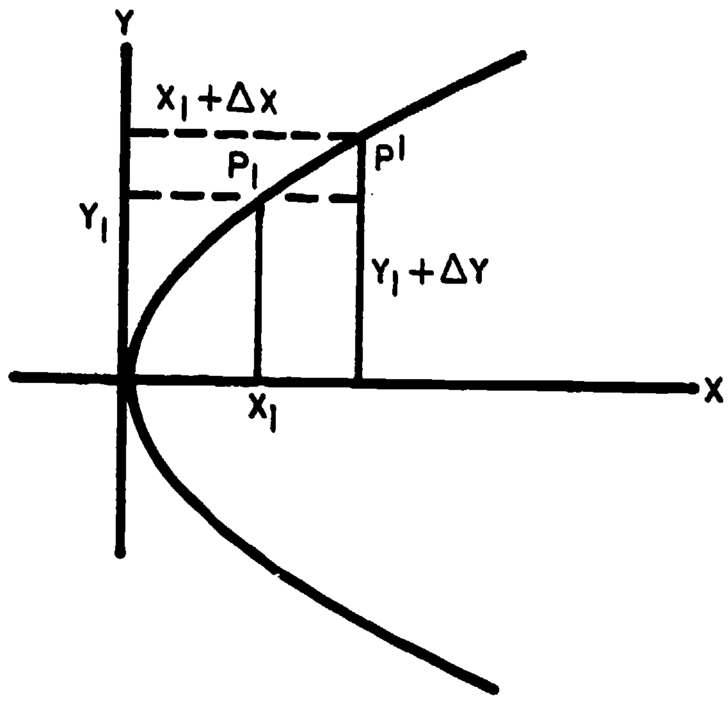


Figure 11-4.--Parabola.

The point $P_1(x_1, y_1)$ also lies on the curve and we have

$$y_1^2 = 4ax_1$$

Substituting this value for y_1^2 into equation (1) transforms it into

$$4ax_1 + 2y_1 \Delta y + (\Delta y)^2 = 4ax_1 + 4a\,\Delta x$$

Simplifying we obtain

$$2y_1\Delta y + (\Delta y)^2 = 4a\Delta x \qquad (2)$$

Divide through by Δx, obtaining

$$\frac{2y_1\Delta y}{\Delta x} + \frac{(\Delta y)^2}{\Delta x} = \frac{4a\Delta x}{\Delta x}$$

which gives

$$2y_1 \frac{\Delta y}{\Delta x} = 4a - \frac{(\Delta y)^2}{\Delta x}$$

Solving for $\frac{\Delta y}{\Delta x}$ we find

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{4a}{2y_1} - \frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1} \\ &= \frac{2a}{y_1} - \frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}\end{aligned} \qquad (3)$$

Before proceeding, a discussion of the term

$$\frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}$$

in equation (3) is in order. If we solve equation (2) for Δy we find

$$2y_1\,\Delta y + (\Delta y)^2 = 4a\Delta x$$

then

$$\Delta y(2y_1 + \Delta y) = 4a\Delta x$$

and

$$\Delta y = \frac{4a\Delta x}{2y_1 + \Delta y}$$

Since the denominator contains a term not dependent upon Δy or Δx, as we let Δx approach zero Δy will also approach zero.

NOTE: We may find a value for Δx that will make Δy less than 1 and then when Δy is squared it will approach zero at least as rapidly as Δx does.

We now refer to equation (3) again and make the statement that we may disregard $\frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}$ since it approaches zero when Δx approaches zero.

Then

$$\frac{\Delta y}{\Delta x} = \frac{2a}{y_1} \qquad (4)$$

The quantity $\frac{\Delta y}{\Delta x}$ is the slope of the line connecting P_1 and P'. From figure 11-4, it is obvious that the slope of the curve at P_1 is different from the slope of the line connecting P_1 and P'.

As Δx and Δy approach zero, the ratio $\frac{\Delta y}{\Delta x}$ will approach more and more closely the true slope of the curve at P_1. We designate the slope by (m). Thus, as Δx approaches zero, equation (4) becomes

$$m = \frac{2a}{y_1}$$

The equation for a straight line in the point slope form is

$$y - y_1 = m(x - x_1)$$

Substituting $\frac{2a}{y_1}$ for m gives

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

Clearing fractions we have

$$yy_1 - y_1^2 = 2ax - 2ax_1 \qquad (5)$$

but

$$y_1^2 = 4ax_1 \qquad (6)$$

Adding equations (5) and (6) yields

$$yy_1 = 2ax + 2ax_1$$

Dividing by y_1 gives

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

which is an equation of a straight line in the slope intercept form. This is the equation of the tangent line to the parabola

$$y^2 = 4ax$$

at the point (x_1, y_1).

EXAMPLE: Given the equation

$$y^2 = 8x$$

find the slope of the curve and the equation of the tangent line at the point (2, 4).

SOLUTION: Put the equation in standard form as follows: Solve for (a) by letting

$$y^2 = 8x$$

have the form

$$y^2 = 4ax$$

Then

$$4a = 8$$

$$a = 2$$

and

$$2a = 4$$

The slope m at point (2, 4) becomes

$$m = \frac{2a}{y_1}$$

$$= \frac{2(2)}{4} = 1$$

The slope of the line is 1 and the equation of the tangent to the curve at the point (2, 4) is

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

$$= \frac{(2)\ (2)\ (x)}{4} + \frac{(2)\ (2)\ (2)}{4}$$

$$= x + 2$$

This method, used to find the slope and equation of the tangent for a standard parabola, can be used to find the slope and equation of the tangent to a curve at any point regardless of the type of curve. The method can be used to find these relationships for circles, hyperbolas, ellipses, and general algebraic curves.

This general method is outlined as follows: To find the slope (m) of a given curve at the point $P_1(x_1, y_1)$ choose a second point P' on the curve so that it has coordinates $(x_1 + \Delta x, y_1 + \Delta y)$ and substitute the coordinates of P' in the equation of the curve and simplify. Divide through by Δx and eliminate terms which contain powers of Δy higher than the first power, as previously discussed. Let Δx approach zero and $\frac{\Delta y}{\Delta x}$ will approach the absolute value for the slope (m) at point P_1. Finally solve for (m).

When the slope and coordinates of a point on the curve are known, the equation of the tangent line can be found by using the point slope method.

EXAMPLE: Using the method outlined, find the slope and equation of the tangent line to the curve

$$x^2 + y^2 = r^2 \text{ at } (x_1, y_1) \tag{7}$$

SOLUTION: Choose a second point P_1 such that it has coordinates

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Substitute into equation (7) and

$$(x_1 + \Delta x)^2 + (y_1 + \Delta y)^2 = r^2$$

thus

$$x_1^2 + 2x_1\Delta x + (\Delta x)^2 + y_1^2 + 2y_1\Delta y + (\Delta y)^2 = r^2$$

then

$$2x_1\Delta x + (\Delta x)^2 + 2y_1\Delta y + (\Delta y)^2 = r^2 - x_1^2 - y_1^2$$

$$= r^2 - (x_1^2 + y_1^2)$$

$$= 0$$

Divide through by Δx

$$\frac{2x_1\Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} + \frac{2y_1\Delta y}{\Delta x} + \frac{(\Delta y)^2}{\Delta x} = 0$$

and eliminating $(\Delta y)^2$ results in

$$2x_1 + \Delta x + 2y_1 \frac{\Delta y}{\Delta x} = 0$$

Solve for $\frac{\Delta y}{\Delta x}$ as follows:

$$\frac{\Delta y}{\Delta x} = \frac{-2x_1 - \Delta x}{2y_1}$$

but

$$\frac{\Delta y}{\Delta x} = m$$

and

$$m = \frac{-2x_1 - \Delta x}{2y_1}$$

Let Δx approach zero and

$$m = \frac{-x_1}{y_1}$$

Now use the point slope form of a straight line with the slope equal to $\frac{-x_1}{y_1}$ and find at point (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$= \frac{-x_1}{y_1}(x - x_1)$$

Rearranging

$$yy_1 - y_1^2 = -x_1x + x_1^2$$

and

$$yy_1 = -x_1x + x_1^2 + y_1^2$$

but

$$x_1^2 + y_1^2 = r^2$$

Then, by substitution

$$yy_1 = -x_1x + r^2$$

and

$$y = \frac{-x_1x}{y_1} + \frac{r^2}{y_1}$$

which is the general equation of the tangent line to the curve

$$x^2 + y^2 = r^2 \text{ at } (x_1, y_1)$$

EXAMPLE: Using the given method, with minor changes, find the slope and equation of the tangent line to the curve

$$x^2 - y^2 = k^2 \text{ at } (x_1, y_1) \qquad (8)$$

SOLUTION: Choose a second point P_1 such that it has coordinates

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Substitute into equation (8) and

$$(x_1 + \Delta x)^2 - (y_1 + \Delta y)^2 = k^2$$

and

$$x_1^2 + 2x_1\Delta x + (\Delta x)^2 - y_1^2 - 2y_1\Delta y - (\Delta y)^2 = k^2 \quad (9)$$

then subtract equation (8) from (9) and obtain

$$2x_1\Delta x + (\Delta x)^2 - 2y_1\Delta y - (\Delta y)^2 = 0$$

then divide by Δx and we have

$$2x_1 + \Delta x - 2y_1 \frac{\Delta y}{\Delta x} - \frac{(\Delta y)^2}{\Delta x} = 0$$

Let Δx approach zero and

$$2x_1 - 2y_1 \frac{\Delta y}{\Delta x} = 0$$

Solving for $\frac{\Delta y}{\Delta x}$ results in

$$\frac{\Delta y}{\Delta x} = m = \frac{x_1}{y_1}$$

Use the point slope form of a straight line to find the equation of the tangent line at point (x_1, y_1) as shown in the following:

$$y - y_1 = m(x - x_1)$$

Substitute $\frac{x_1}{y_1}$ for m

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

Multiply through by y_1 and

$$yy_1 - y_1^2 = x_1x - x_1^2$$

Rearrange to obtain

$$yy_1 = x_1x - x_1^2 + y_1^2$$
$$= x_1x - \left(x_1^2 - y_1^2\right)$$

Substitute $x_1^2 - y_1^2$ for k^2 and

$$yy_1 = x_1x - k^2$$

Divide through by y_1 to obtain

$$y = \frac{x_1x}{y_1} - \frac{k^2}{y_1}$$

which is the equation and slope desired.

PROBLEMS: Find the slope (m) and equation of the tangent line, in problems 1 through 6, at the given points.

1. $y^2 = \frac{4}{3}x$ at (3, 2)
2. $y^2 = 12x$ at (3, 6)
3. $x^2 + y^2 = 25$ at (-3, 4)
4. $x^2 + y^2 = 100$ at (6, 8)
5. $x^2 - y^2 = 9$ at (5, 4)
6. $x^2 - y^2 = 3$ at (2, 1)
7. Find the slope of $y = x^2$ at (2, 4)
8. Find the slope of $y = 2x^2 - 3x + 2$ at (2, 4)

ANSWERS:

1. $y = \frac{x}{3} + 1$, $m = \frac{1}{3}$
2. $y = x + 3$, $m = 1$
3. $y = \frac{3x}{4} + \frac{25}{4}$, $m = \frac{3}{4}$
4. $y = \frac{-3x}{4} + \frac{25}{2}$, $m = \frac{-3}{4}$
5. $y = \frac{5x}{4} - \frac{9}{4}$, $m = \frac{5}{4}$
6. [illegible] $- 3$, $m = 2$
7. $m = 4$
8. $m = 5$

EQUATIONS OF TANGENTS AND NORMALS

In figure 11-5, the coordinates of point P_1 on the curve are (x_1, y_1). Let the slope of the tangent to the curve at point P_1 be denoted by m_1. Knowing the slope and a point through which the tangent line passes, the equation of that tangent line can be determined by using the point slope form.

Thus, the equation of the tangent line (MP_1) is

$$y - y_1 = m(x - x_1)$$

The normal to a curve at a point (x_1, y_1) is the line which is perpendicular to the tangent line at that point. The slope of the normal line is then $-\frac{1}{m_1}$ where, as before, the slope of the tangent line is m_1. This is shown in the following:

If

$$m_1 = \tan \theta$$

then

$$m_2 = \tan (\theta + 90^\circ)$$

$$= -\tan [180^\circ - (\theta + 90^\circ)]$$

$$= -\tan (90^\circ - \theta)$$

$$= -\cot \theta$$

$$= -\frac{1}{\tan \theta}$$

$$= -\frac{1}{m_1}$$

therefore

$$m_2 = -\frac{1}{m_1}$$

The equation of the normal through P_1 is

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$

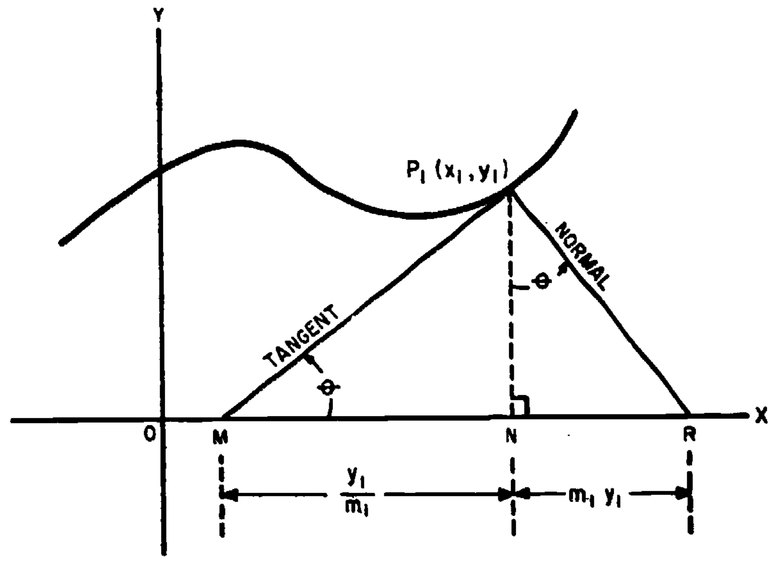


Figure 11-5.—Curve with tangent and normal lines.

Notice that if the slope of the tangent is m_1 and the slope of the normal to the tangent is m_2 and

$$m_2 = -\frac{1}{m_1}$$

then the product of the slopes of the tangent and normal equals -1. The relationship between the slopes of the tangent and normal stated more formally is: The slope of the normal is the negative reciprocal of the slope of the tangent.

Another approach to show the relationship between the slopes of the tangent and normal follows: The inclination of one line must be 90° greater than the other. Then

$$\theta_2 = \theta_1 + 90^\circ$$

If

$$\tan \theta_2 = m_2$$

and

$$\tan \theta_2 = \tan (\theta_1 + 90^\circ)$$

then

$$\tan(\theta_1 + 90°) = \frac{\sin(\theta_1 + 90°)}{\cos(\theta_1 + 90°)}$$

$$= \frac{\sin\theta_1 \cos 90° + \cos\theta_1 \sin 90°}{\cos\theta_1 \cos 90° - \sin\theta_1 \sin 90°}$$

$$= -\frac{\cos\theta_1}{\sin\theta_1}$$

$$= -\cot\theta_1$$

$$= -\frac{1}{\tan\theta_1}$$

therefore

$$\tan\theta_2 = -\frac{1}{\tan\theta_1}$$

LENGTHS OF SUBTANGENT AND SUBNORMAL

The length of the tangent is defined as that portion of the tangent line between the point $P_1(x_1, y_1)$ and the point where the tangent line crosses the X axis. In figure 11-5, the length of the tangent is (MP_1).

The length of the normal is defined as that portion of the normal line between the point P_1 and the X axis. That is (P_1R) which is perpendicular to the tangent.

The projections of these lines on the X axis are known as the length of the subtangent (MN) and the length of the subnormal (NR).

The relationships between the slope of the tangent and the lengths of the subtangent and subnormal follows:

From the triangle MP_1N, in figure 11-5,

$$\tan\theta = m_1 = \frac{P_1N}{MN}$$

and

$$MN = \frac{P_1N}{m_1}$$

The line segment (MN) is the length of the subtangent and (P_1N) is equal to the vertical coordinate y_1. Therefore, the length of the subtangent is $\frac{y_1}{m_1}$.

In the triangle NP_1R,

$$\tan\theta = \frac{NR}{NP_1}$$

but

$$\tan\theta = m_1$$

and

$$NR = m_1 NP_1$$

$$= m_1 y_1$$

Therefore, the length of the subnormal is m_1y_1.

From this, as shown in figure 11-5, the length of the tangent and normal may be found by using the Pythagorean theorem.

NOTE: If the subtangent lies to the right of point M, it is considered positive, if to the left it is negative. Likewise, if the subnormal extends to the right of N it is positive, to the left it is negative.

EXAMPLE: Find the equation of the tangent, the equation of the normal, the length of the subtangent, the length of the subnormal, and the lengths of the tangent and normal of

$$y^2 = \frac{4}{3}x, \text{ at } (3, 2)$$

SOLUTION: Find the value of 2a from

$$y^2 = 4ax$$

Since

$$y^2 = \frac{4}{3}x$$

then

$$4a = \frac{4}{3}$$

$$a = \frac{1}{3}$$

$$2a = \frac{2}{3}$$

The slope is

$$m = \frac{2a}{y_1} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

Using the point slope form

$$y - y_1 = m(x - x_1)$$

then, at point (3, 2)

$$y - 2 = \frac{1}{3}(x - 3)$$

$$= \frac{x}{3} - 1$$

and

$$y = \frac{x}{3} + 1$$

which is the equation of the tangent line.

Use the negative reciprocal of the slope to find the equation of the normal as follows:

$$y - 2 = -3(x - 3)$$

$$= -3x + 9$$

then

$$y = -3x + 11$$

The length of the subtangent is

$$\frac{y_1}{m_1} = \frac{2}{\frac{1}{3}} = 6$$

and the length of the subnormal is

$$y_1 m_1 = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

To find the length of the tangent we use the Pythagorean theorem. Thus, the length of the tangent is

$$\sqrt{\left(\frac{y_1}{m_1}\right)^2 + (y_1)^2}$$

$$= \sqrt{(6)^2 + (2)^2}$$

$$= \sqrt{40}$$

$$= 6.32$$

The length of the normal is equal to

$$\sqrt{(y_1 m_1)^2 + (y_1)^2}$$

$$= \sqrt{\left(\frac{2}{3}\right)^2 + (2)^2}$$

$$= \sqrt{\frac{40}{9}}$$

$$= \frac{6.32}{3}$$

PROBLEMS: Find the equation of the tangent and normal, and the lengths of the subtangent and subnormal in the following:

1. $y^2 = 12x$, at (3, 6)
2. $x^2 + y^2 = 25$, at (-3, 4)
3. $x^2 - y^2 = 9$, at (5, 4)
4. $y = 2x^2 - 3x + 2$, at (1, 1)

ANSWERS:

1.	Equation of tangent	$y = x + 3$
	Equation of normal	$y = -x + 9$
	Length of subtangent	6
	Length of subnormal	6
2.	Equation of tangent	$y = \frac{3x}{4} + \frac{25}{4}$
	Equation of normal	$y = \frac{-4x}{3}$
	Length of subtangent	$\frac{16}{3}$
	Length of subnormal	5
3.	Equation of tangent	$y = \frac{5x}{4} - \frac{9}{4}$
	Equation of normal	$y = \frac{-4x}{5} + 8$
	Length of subtangent	$\frac{5}{16}$
	Length of subnormal	5

4. Equation of tangent $y = x$
   Equation of normal $y = -x + 2$
   Length of subtangent 1
   Length of subnormal 1

PARAMETRIC EQUATIONS

Equations used previously have been functions involving two unknowns such as x and y. The functions have been in either Cartesian or polar coordinates and have been defined by one equation. If a third variable is introduced it is called a parameter. When two equations are used, each containing the parameter, the equations are called parametric equations.

MOTION IN A STRAIGHT LINE

To illustrate the application of a parameter we will assume that an aircraft takes off from a field which we will call the origin. Figure 11-6 shows the diagram we will use. The aircraft is flying on a compass heading of due north. There is a wind blowing from the west at 20 miles per hour and the airspeed of the aircraft is 400 miles per hour. Let the direction of the positive Y axis be due north and the positive X axis be due east as shown in figure 11-6. Use the scales as shown.

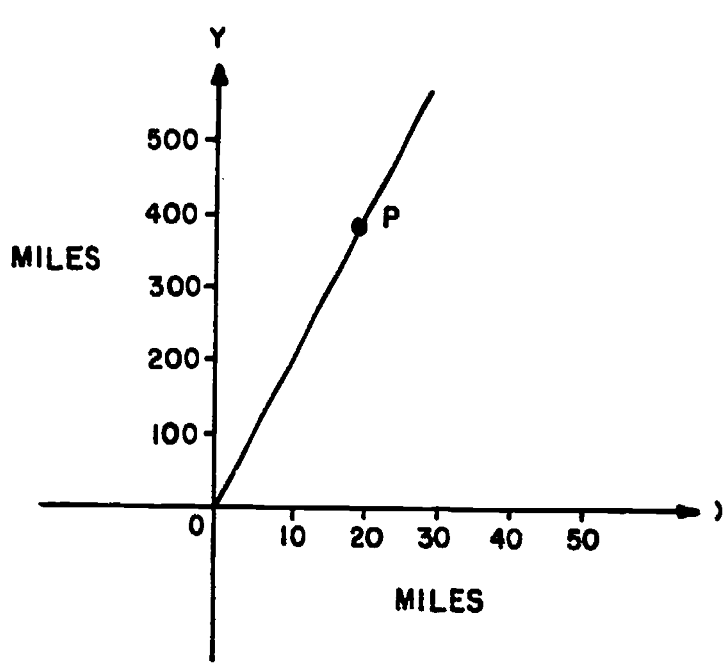


Figure 11-6.—Aircraft position.

One hour after takeoff the position of the aircraft, represented by point P, is 400 miles north and 20 miles east of the origin. If we use t as the parameter, then at any time t the aircrafts position (x, y) will be given by x equals 20t and y equals 400t.
The equations are

$$x = 20t$$

and

$$y = 400\ t$$

and are called parametric equations. Notice that time is not plotted on the graph of figure 11-6. The parameter t is used only to plot the position (x, y) of the aircraft.

We may eliminate the parameter t to obtain a direct relationship between x and y as follows:

If

$$t = \frac{x}{20}$$

then

$$y = 400\left(\frac{x}{20}\right)$$

$$y = 20x$$

and we find the graph to be a straight line. When we eliminated the parameter the result was the rectangular coordinate equation of the line.

MOTION IN A CIRCLE

Consider the parametric equations

$$x = r\ \cos\ t$$

and

$$y = r\ \sin\ t$$

These equations describe the position of a point (x, y) at any time t. They can be transposed into a single equation by squaring both sides of each equation to obtain

$$x^2 = r^2\ \cos^2 t$$

$$y^2 = r^2\ \sin^2 t$$

and adding

$$x^2 + y^2 = r^2 \cos^2 t + r^2 \sin^2 t$$

Rearranging we have

$$x^2 + y^2 = r^2 (\cos^2 t + \sin^2 t)$$

but

$$\cos^2 t + \sin^2 t = 1$$

then

$$x^2 + y^2 = r^2$$

which is the equation of a circle.

This means that if various values were assigned to t and the corresponding values of x and y were calculated and plotted, the result would be a circle. In other words, the point (x, y) moves in a circular path.

Using this example again, that is

$$x = r \cos t$$

and

$$y = r \sin t$$

and given that

$$m_2 = \frac{\Delta x}{\Delta t} = -r \sin t$$

and

$$m_1 = \frac{\Delta y}{\Delta t} = r \cos t$$

we are able to express the slope at any point on the circle in terms of t.

NOTE: These expressions for $\frac{\Delta x}{\Delta t}$ and $\frac{\Delta y}{\Delta t}$ may be found by using calculus, but we will accept them for the present without proof.

If we know $\frac{\Delta y}{\Delta t}$ and $\frac{\Delta x}{\Delta t}$, we may find $\frac{\Delta y}{\Delta x}$ which is the slope of a curve at any point.

That is,

$$m = \frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}$$

By substituting we find

$$\frac{\Delta y}{\Delta x} = \frac{r \cos t}{-r \sin t} = -\cot t$$

Comparing this result with equation (7) of the previous section, we find that in rectangular coordinates the slope is given as

$$m = -\frac{x_1}{y_1}$$

while in terms of a parameter it is

$$m = -\cot t$$

OTHER PARAMETRIC EQUATIONS

EXAMPLE: Find the equation of the tangent and the normal and the length of the subtangent and the subnormal for the curve represented by

$$x = t^2$$

and

$$y = 2t + 1$$

at

$$t = 1$$

given that

$$\frac{\Delta x}{\Delta t} = 2t$$

and

$$\frac{\Delta y}{\Delta t} = 2$$

SOLUTION: Since t equals 1 we write

$$x = 1$$

and

$$y = 3$$

and

$$\frac{\Delta y}{\Delta x} = \frac{2}{2t} = \frac{1}{t}$$

then

$$m = \frac{1}{t}$$

The equation of the tangent line when t is equal to 1 is

$$y - 3 = 1(x - 1)$$

$$y = x + 2$$

The equation of the normal line is

$$y - 3 = -1(x - 1)$$

$$y = -x + 4$$

The length of the subtangent is

$$\frac{y_1}{m} = \frac{3}{1} = 3$$

The length of the subnormal

$$y_1 m = (1)(3) = 3$$

EXAMPLE: Find the equation of the tangent and normal and the lengths of the subtangent and subnormal to the curve represented by the parametric equations

$$x = 2 \cos \theta$$

and

$$y = 2 \sin \theta$$

at the point where

$$\theta = 45^\circ$$

given that

$$\frac{\Delta x}{\Delta \theta} = -2 \sin \theta$$

and

$$\frac{\Delta y}{\Delta \theta} = 2 \cos \theta$$

SOLUTION: We know that

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta \theta}}{\frac{\Delta x}{\Delta \theta}} = \frac{2 \cos \theta}{-2 \sin \theta} = - \cot \theta$$

Then at the point where

$$\theta = 45^\circ$$

we have

$$m = \frac{\Delta y}{\Delta x} = - \cot 45^\circ = -1$$

In order to find (x_1, y_1) we substitute

$$\theta = 45^\circ$$

in the parametric equations and

$$x_1 = 2 \cos 45^\circ = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

$$y_1 = 2 \sin 45^\circ = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

The equation of the tangent is

$$y - y_1 = m(x - x_1)$$

Substituting we have

$$y - \sqrt{2} = -1(x - \sqrt{2})$$

or

$$x + y = 2\sqrt{2}$$

The equation of the normal is

$$y - \sqrt{2} = 1(x - \sqrt{2})$$

or

$$x - y = 0$$

The length of the subtangent is

$$\frac{y_1}{m} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

The length of the subnormal is

$$y_1 m = (\sqrt{2})(-1) = -\sqrt{2}$$

The horizontal and vertical tangents of a curve can be found very easily when the curve is represented by parametric equations. The slope of a curve at any point equals zero when the tangent is parallel to the x axis. In parametric equations the horizontal and vertical tangents can be found easily by setting

$$\frac{\Delta y}{\Delta t} = 0$$

and

$$\frac{\Delta x}{\Delta t} = 0$$

For the horizontal tangent solve $\frac{\Delta y}{\Delta t}$ equals zero for t and for the vertical tangent solve $\frac{\Delta x}{\Delta t}$ equals zero for t.

EXAMPLE: Find the points of contact of the horizontal and vertical tangents to the curve represented by the parametric equations

$$x = 3 - 4 \sin \theta$$

and

$$y = 4 + 3 \cos \theta$$

Plot the graph of the function by taking θ from 0° to 360° in increments of 30°.
Given that

$$\frac{\Delta x}{\Delta \theta} = -4 \cos \theta$$

and

$$\frac{\Delta y}{\Delta \theta} = -3 \sin \theta$$

SOLUTION: The graph of the function shows that the figure is an ellipse, figure 11-7, and consequently there will be two horizontal and two vertical tangents. The coordinates of the horizontal tangent points are found by first setting

$$\frac{\Delta y}{\Delta \theta} = 0$$

This gives

$$-3 \sin \theta = 0$$

Then

$$\sin \theta = 0$$

and

$$\theta = 0° \text{ or } 180°$$

Substituting 0° we have

$$\begin{aligned} x &= 3 - 4 \sin 0° \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} y &= 4 + 3 \cos 0° \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Substituting 180° we obtain

$$\begin{aligned} x &= 3 - 4 \sin 180° \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} y &= 4 + 3 \cos 180° \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

The coordinates of the points of contact of the horizontal tangents to the ellipse are (3, 1) and (3, 7).

The coordinates of the vertical tangent points of contact are found by setting

$$\frac{\Delta y}{\Delta \theta} = 0$$

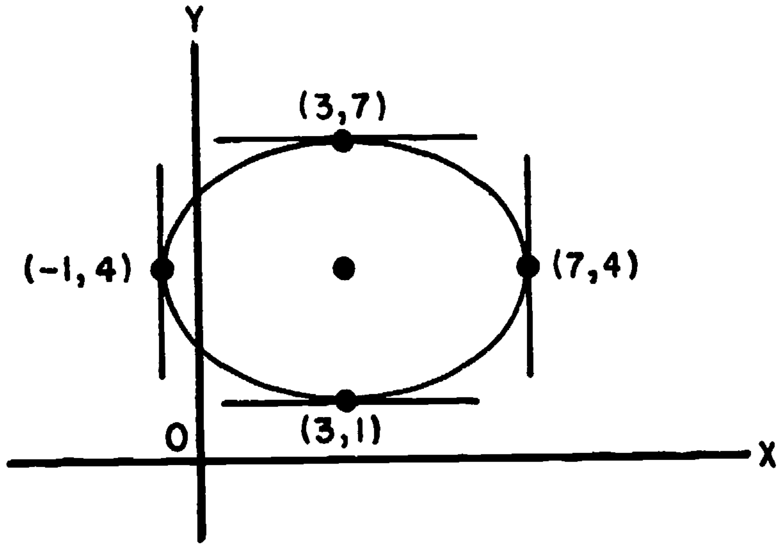


Figure 11-7.—Ellipse.

We find

$$-4\cos\theta = 0$$

from which

$$\theta = 90^\circ \text{ or } 270^\circ$$

Substituting 90° we obtain

$$x = 3 - 4\sin 90^\circ$$
$$= 3 - 4$$
$$= -1$$

and

$$y = 4 + 3\cos 90^\circ$$
$$= 4 + 0$$
$$= 4$$

Substituting 270° gives

$$x = 3 - 4\sin 270^\circ$$
$$= 3 + 4$$
$$= 7$$

and

$$y = 4 + 3\cos 270^\circ$$
$$= 4 + 0$$
$$= 4$$

The coordinates of the points of contact of the vertical tangents to the ellipse are (-1, 4) and (7, 4).

PROBLEMS: Find the equations of the tangent and the normal and the lengths of the subtangents and the subnormal for each of the following curves at the point indicated.

1. $x = t^3$

   $y = 3t$

   at $t = -1$

   given $\frac{\Delta x}{\Delta t} = 3t^2$ and $\frac{\Delta y}{\Delta t} = 3$

2. $x = t^2 + 8$

   $y = t^2 + 1$

   at $t = 2$

   given $\frac{\Delta x}{\Delta t} = 2t$ and $\frac{\Delta y}{\Delta t} = 2t$

3. $x = t$

   $y = t^2$

   at $t = 1$

   given $\frac{\Delta x}{\Delta t} = 1$ and $\frac{\Delta y}{\Delta t} = 2t$

4. Find the points of contact of the horizontal and vertical tangents to the curve

$$x = 2\cos\theta$$
$$y = 3\sin\theta$$

given

$$\frac{\Delta x}{\Delta\theta} = -2\sin\theta$$
$$\frac{\Delta y}{\Delta\theta} = 3\cos\theta$$

ANSWERS:

1. Equation of tangent $y = x - 2$
   Equation of normal $y = -x - 4$
   Length of subtangent 3
   Length of subnormal 3

2. Equation of tangent $y = x - 7$
   Equation of normal $y = -x + 17$
   Length of subtangent 5
   Length of subnormal 5

3. Equation of tangent $y = 2x - 1$
   Equation of normal $y = \frac{-x}{2} + \frac{3}{2}$
   Length of subtangent $\frac{1}{2}$
   Length of subnormal 2

4. Coordinates of the points of contact of the horizontal tangent to the ellipse are (0, 3) and (0, -3) and the vertical tangent to the ellipse are (2, 0) and (-2, 0).

CHAPTER 12

LIMITS AND DIFFERENTIATION

Limits and differentiation are the beginning of the study of calculus, which is an important and powerful method of computation.

LIMIT CONCEPT

The study of the limit concept is very important as it is the very heart of the theory and operation of calculus. We will include in this section the definition of limit, some of the indeterminate forms of limits, and some limit formulas, along with example problems.

DEFINITION OF LIMIT

Before we start differentiation there are certain concepts which we must understand. One of these concepts deals with the limit of a function. Many times it will be necessary to find the value of the limit of a function.

The discussion of limits will begin with an intuitive point of view.

We will work with the equation

$$y = f(x) = x^2$$

which is shown in figure 12-1. The point P represents the point corresponding to

$$y = 16$$

and

$$x = 4$$

The behavior of y for given values of x near the point

$$x = 4$$

is the center of the discussion. For the present we will exclude the point P which is encircled on the graph.

We will start with values lying between

$$x = 2$$

and

$$x = 6$$

indicated by line A in figure 12-1. This interval may be written as

$$0 < |x - 4| < 2$$

The corresponding interval for y is between

$$y = 4$$

and

$$y = 36$$

We now take a smaller interval about

$$x = 4 \text{ (line B)}$$

by using values of

$$x = 3$$

and

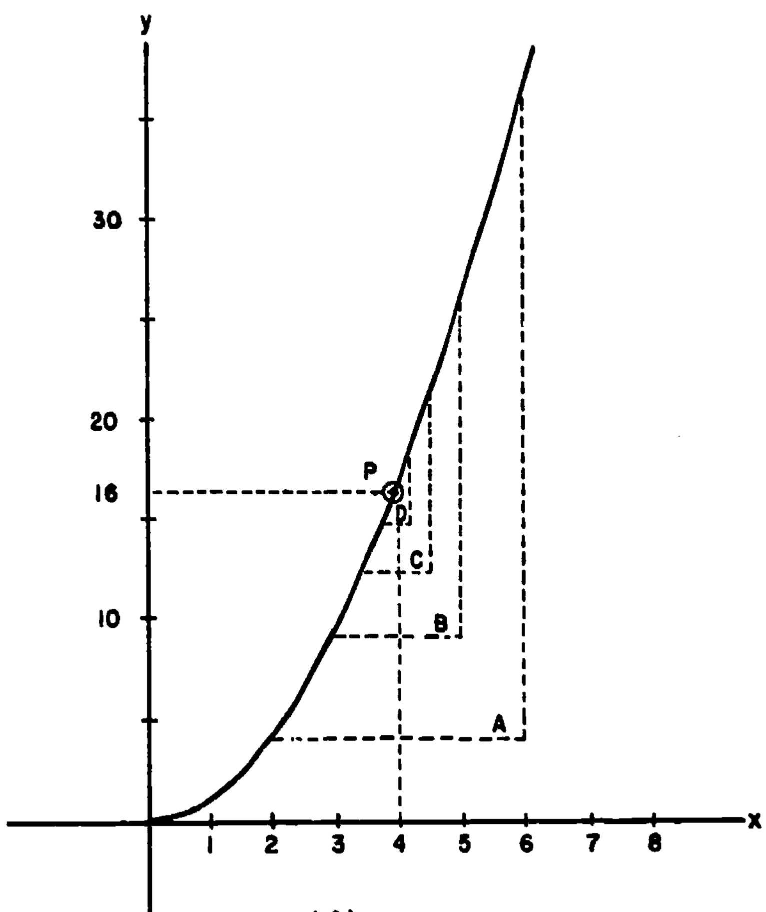
$$x = 5$$

and find the corresponding interval for y to be between

$$y = 9$$

and

$$y = 25$$



(A)

Interval of x		Interval of f(x)
2 - 6	(A)	4 - 36
3 - 5	(B)	9 - 25
3.5 - 4.5	(C)	12.25 - 20.25
3.9 - 4.1	(D)	15.21 - 16.81

(B)

Figure 12-1.—(A) Graph of $y = x^2$; (B) value chart.

These intervals for x and y are written as

$$0 < |x - 4| < 1$$

and

$$9 < y < 25$$

As we diminish the interval of x around

$$x = 4 \text{ (line C and line D)}$$

we find the values of

$$y = x^2$$

to be grouped more and more closely around

$$y = 16$$

This is shown by the chart in figure 12-1.

Although we have used only a few intervals of x in the discussion, it should be apparent that we can make the values about y group as closely as we desire by merely limiting the values assigned to x about

$$x = 4$$

Because the foregoing is true, we may now say that the limit of x^2, as x approaches 4, results in the value 16 for y and we write

$$\lim_{x \to 4} x^2 = 16$$

In the general form we may write

$$\lim_{x \to a} f(x) = L$$

and we mean that as x approaches a, the limit of f(x) will approach L and L is called the limit of f(x) as x approaches a. No statement is made about f(a) for it may or may not exist although the limit of f(x), as x approaches a, is defined.

If f(x) is defined at

$$x = a$$

and for all values of x near a, and if the function is continuous, then

$$\lim_{x \to a} f(x) = f(a)$$

We are now ready to define a limit.

Let f(x) be defined for all x in the interval near

$$x = a$$

but not necessarily at

$$x = a$$

Then there exists a number L such that for every positive number ϵ

$$|f(x) - L| < \epsilon$$

provided that we may find the number δ such that

$$0 < |x - a| < \delta$$

Then we say L is the limit of f(x) as x approaches a and we write

$$\lim_{x \to a} f(x) = L$$

This means that for every challenge number ϵ we must find a number δ in the interval

$$0 < |x - a| < \delta$$

such that the difference between f(x) and L is smaller than the number ϵ.

EXAMPLE: Suppose we are given

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3$$

and the challenge number ϵ is 0.1.

SOLUTION: We must find a number δ such that in the neighborhood of

$$x = 1$$

for all points except

$$x = 1$$

we have the difference between f(x) and 3 smaller than 0.1.

We write

$$\left|\frac{x^2 + x - 2}{x - 1} - 3\right| < 0.1$$

and

$$\frac{x^2 + x - 2}{x - 1} - 3$$

$$= \frac{(x + 2)(x - 1)}{x - 1} - 3$$

and we consider only values where

$$x \neq 1$$

Simplifying the first term, we have

$$\frac{(x + 2)(x - 1)}{x - 1} = x + 2$$

Finally, combine terms as follows:

$$x + 2 - 3 = x - 1$$

and

$$|x - 1| < 0.1$$

or

$$-0.1 < x - 1 < 0.1$$

then

$$0.9 < x < 1.1$$

and we have fulfilled the definition of the limit.

If the limit of a function exists and the function is continuous then

$$\lim_{x \to a} f(x) = f(a)$$

For instance, in order to find the limit of the function $x^2 - 3x + 2$ as x approaches 3, we substitute 3 for x in the function. Then

$$f(3) = 3^2 - 3(3) + 2$$

$$= 9 - 9 + 2$$

$$= 2$$

Since x is a variable it may assume a value as close to 3 as we wish, and the closer we choose the value of x to 3, the closer f(x) will approach the value 2. Therefore, 2 is called the limit of f(x) as x approaches 3 and we write

$$\lim_{x \to 3} (x^2 - 3x + 2) = 2$$

PROBLEMS: Find the limit of each of the following functions.

1. $\lim_{x \to 1} \dfrac{2x^2 - 1}{2x - 1}$
2. $\lim_{x \to 2} (x^2 - 2x + 3)$

3. $\lim_{x \to a} \frac{x^2 - a}{a}$

4. $\lim_{t \to 0} (5t^2 - 3t + 2)$

5. $\lim_{E \to 6} \frac{E^3 - E}{E - 1}$

6. $\lim_{Z \to 0} \frac{Z^2 - 3Z + 2}{Z - 4}$

ANSWERS:

1. 1
2. 3
3. a - 1
4. 2
5. 42
6. $-\frac{1}{2}$

INDETERMINATE FORMS

Whenever the answer obtained by substitution, in searching for the value of a limit, assumes any of the following forms, another method for finding the correct limit must be used.

$$\frac{0}{0}, \frac{\infty}{\infty}, (\infty)0, 0^\circ, \infty^\circ, 1^\infty$$

These are called indeterminate forms.

The proper method for evaluating the limit depends on the problem and sometimes calls for a high degree of ingenuity. We will restrict the methods of solution of indeterminate forms to factoring and division of the numerator and denominator by powers of the variable. Later in the study of limits, L'Hospital's rule will be used as a method of solving indeterminate forms.

Sometimes factoring will resolve an indeterminate form.

EXAMPLE: Find the limit of

$$\frac{x^2 - 9}{x - 3} \text{ as x approaches 3}$$

SOLUTION: By substitution we find

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0}$$

which is an indeterminate form and is therefore excluded as a possible limit. We must now search for a method to find the limit. Factoring is attempted and results in

$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3}$$
$$= x + 3$$

then

$$\lim_{x \to 3} (x + 3) = 6$$

and we have a determinate limit of 6.

Another indeterminate form is often met when we try to find the limit of a function as the independent variable becomes infinite.

EXAMPLE: Find the limit of

$$\frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1}$$

as x becomes infinite.

SOLUTION: If we let x become infinite in the original expression the result will be

$$\lim_{x \to \infty} \frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1} = \frac{\infty}{\infty}$$

which must be excluded as an indeterminate form. However, if we divide both numerator and denominator by x^4, we obtain

$$\lim_{x \to \infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2} + \frac{2}{x^3}}{3 - \frac{2}{x^2} + \frac{1}{x^4}}$$
$$= \frac{1 + 0 - 0 + 0}{3 - 0 + 0}$$
$$= \frac{1}{3}$$

and we have a determinate limit of $\frac{1}{3}$.

PROBLEMS: Find the limit of the following:

1. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

2. $\lim_{x \to \infty} \frac{2x + 3}{7x - 6}$

3. $\lim\limits_{a \to 0} \dfrac{2a^2b - 3ab^2 + 2ab}{5ab - a^3b^2}$

4. $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

5 $\lim\limits_{x \to a} \dfrac{x^4 - a^4}{x - a}$

6. $\lim\limits_{a \to 0} \dfrac{(x - a)^2 - x^2}{a}$

ANSWERS:

1. 4
2. $\frac{2}{7}$
3. $\frac{2 - 3b}{5}$
4. 5
5. $4a^3$
6. $-2x$

LIMIT FORMULAS

To obtain results in calculus we will frequently operate with limits. The proofs of theorems shown in this section will not be given as they are quite long and demand considerable discussion.

The theorems will be stated and examples will be given. Assume that we have three simple functions of x.

$$f(x) = u$$
$$g(x) = v$$
$$h(x) = w$$

Further, let these functions have separate limits such that

$$\lim_{x \to a} u = A$$
$$\lim_{x \to a} v = B$$
$$\lim_{x \to a} w = C$$

Theorem 1. The limit of the sum of two functions is equal to the sum of the limits.

$$\lim_{x \to a} [f(x) + g(x)] = A + B$$
$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

This theorem may be extended to include any number of functions such as

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = A + B + C$$
$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

EXAMPLE: Find the limit of

$$(x - 3)^2 \quad \text{as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} (x - 3)^2 = \lim_{x \to 3} (x^2 - 6x + 9)$$
$$= \lim_{x \to 3} x^2 - \lim_{x \to 3} 6x + \lim_{x \to 3} 9$$
$$= 9 - 18 + 9$$
$$= 0$$

Theorem 2. The limit of a constant c times a function f(x) is equal to the constant c times the limit of the function.

$$\lim_{x \to a} cf(x) = cA = c \lim_{x \to a} f(x)$$

EXAMPLE: Find the limit of

$$2x^2 \quad \text{as } x \to 3$$

SOLUTION: $\lim\limits_{x \to 3} 2x^2 = 2 \lim\limits_{x \to 3} x^2$

$$= (2)(9)$$
$$= 18$$

Theorem 3. The limit of the product of two functions is equal to the product of their limits.

$$\lim_{x \to a} f(x)\, g(x) = AB$$
$$= \left(\lim_{x \to a} f(x)\right) \left(\lim_{x \to a} g(x)\right)$$

EXAMPLE: Find the limit of

$$(x^2 - x)(\sqrt{2x}) \text{ as } x \to 2$$

SOLUTION:

$$\lim_{x\to 2} (x^2 - x)(\sqrt{2x}) = AB$$

$$= \left(\lim_{x\to 2} (x^2 - x)\right) \left(\lim_{x\to 2} \sqrt{2x}\right)$$

$$= (4 - 2)(\sqrt{4})$$

$$= 4$$

Theorem 4. The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}$$

$$= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ if } B \neq 0$$

EXAMPLE: Find the limit of

$$\frac{3x^2 + x - 6}{2x - 5} \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x\to 3} \frac{3x^2 + x - 6}{2x - 5}$$

$$= \frac{\lim_{x\to 3} 3x^2 + x - 6}{\lim_{x\to 3} 2x - 5}$$

$$= 24$$

PROBLEMS: Find the limits of the following, using the theorem indicated.

1. $x^2 + x + 2$ as $x \to 1$ (Theorem 1)
2. $7(x^2 - 13)$ as $x \to 4$ (Theorem 2)
3. $5x^4$ as $x \to 2$ (Theorem 3)
4. $\frac{2x^2 + x - 4}{3x - 7}$ as $x \to 3$ (Theorem 4)

ANSWERS:

1. 4
2. 21
3. 80
4. $\frac{17}{2}$

INFINITESIMALS

In chapter 11, the slope of a curve at a given point was found by taking very small increments of Δy and Δx and the slope was said to be equal to $\frac{\Delta y}{\Delta x}$. This section will be a continuation of this concept.

DEFINITIONS

A variable that approaches 0 as a limit is called an infinitesimal. This may be written as

$$\lim V = 0$$

or

$$V \to 0$$

and means, as recalled from a previous section of this chapter, that the numerical value of V becomes and remains less than any positive challenge number ϵ.

If the

$$\lim V = L$$

then

$$\lim V - L = 0$$

which indicates that the difference between a variable and its limit is an infinitesimal. Conversely, if the difference between a variable and a constant is an infinitesimal, then the variable approaches the constant as a limit.

EXAMPLE: As x becomes increasingly large, is the term $\frac{1}{x^2}$ an infinitesimal?

SOLUTION: By the definition of infinitesimal, $\frac{1}{x^2}$ approaches 0 as x increases in value, $\frac{1}{x^2}$ is an infinitesimal. It does this, and is therefore an infinitesimal.

EXAMPLE: As x approaches 2, is the expression $\frac{x^2 - 4}{x - 2} - 4$ an infinitesimal?

SOLUTION: By the converse of the definition of infinitesimal, if the difference between $\frac{x^2 - 4}{x - 2}$ and 4 approaches 0, as x approaches 2, the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal. By direct substitution we find an indeterminate form; therefore we make use of our knowledge of indeterminates, and write

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)\ (x - 2)}{x - 2} = x + 2$$

and

$$\lim_{x \to 2} (x + 2) = 4$$

The difference between 4 and 4 is 0 and the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal, as x approaches 2.

SUMS

An infinitesimal is a variable that approaches 0 as a limit. We state that ϵ and δ, in figure 12-2, are infinitesimals because they both approach 0 as shown.

Theorem 1. The algebraic sum of any number of infinitesimals is an infinitesimal.

In figure 12-2, as ϵ and δ approach 0, notice that their sum approaches 0 and by definition this sum is an infinitesimal and the truth of theorem 1 has been shown. This approach may be used for the sum of any number of infinitesimals.

PRODUCTS

Theorem 2. The product of any number of infinitesimals is an infinitesimal.

In figure 12-3, the product of two infinitesimals, ϵ and δ, is an infinitesimal as shown. The product of any number of infinitesimals is also an infinitesimal by the same approach as shown for two numbers.

Theorem 3. The product of a constant and an infinitesimal is an infinitesimal.

This may be shown, in figure 12-3, by holding either ϵ or δ constant and noticing their product as the variable approaches 0.

$\delta \backslash \epsilon$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	2	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{65}{64}$	$\frac{257}{256}$	
$\frac{1}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{17}{64}$	$\frac{65}{256}$	
$\frac{1}{16}$	$\frac{17}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{64}$	$\frac{17}{256}$	
$\frac{1}{64}$	$\frac{65}{64}$	$\frac{17}{64}$	$\frac{5}{64}$	$\frac{1}{32}$	$\frac{5}{256}$	
$\frac{1}{256}$	$\frac{257}{256}$	$\frac{65}{256}$	$\frac{17}{256}$	$\frac{5}{256}$	$\frac{1}{128}$	
$\downarrow$ 0						$\searrow$ 0

Figure 12-2.—Sums of infinitesimals.

$\delta \backslash \epsilon$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	
$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	
$\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	$\frac{1}{65536}$	
$\downarrow$ 0						$\searrow$ 0

Figure 12-3.—Products of infinitesimals.

CONCLUSIONS

The term infinitesimal was used to describe the term Δx as it approaches zero. The quantity Δx was called in increment of: where increment was used to imply that we have added a small amount to x. Thus $x + \Delta x$ indicates that we are holding x constant and adding a small but variable amount to which we will call Δx.

A very small increment is sometimes called a differential. A small Δx is indicated by dx. The differential of θ is $d\theta$ and that of y is dy. The limit of Δx as it approaches zero is of course zero, but this does not mean that the ratio of two infinitesimals cannot be a real number or a real function of x. For instance, no matter how small Δx is chosen, the ratio $\frac{dx}{dx}$ will still be equal to 1.

In the section on indeterminate forms, a method for evaluating the form $\frac{0}{0}$ was shown. This form results whenever the limit takes the form of one infinitesimal over another. In every case the limit was a real number.

DISCONTINUITIES

The discussion of discontinuities will be based upon a comparison to continuity which is defined by:

A function f(x) is continuous at

$$x = a$$

if f(x) is defined at

$$x = a$$

and has a limit as $x \to a$, as follows:

$$\lim_{x \to a} f(x) = f(a)$$

Notice that for continuity a function must fulfil the following three conditions:

1. f(x) is defined at $x = a$.
2. The limit of f(x) exists as x approaches a.
3. The value of f(x) at $x = a$ is equal to the limit of f(x) at $x = a$.

If a function f(x) is not continuous at

$$x = a$$

then it is said to be discontinuous at

$$x = a$$

We will use examples to show the above statements.

EXAMPLE: In figure 12-4, is the function

$$f(x) = x^2 + x - 4$$

continuous at f(2)?

SOLUTION:

$$f(2) = 4 + 2 - 4$$
$$= 2$$

and

$$\lim_{x \to 2} x^2 + x - 4 = 2$$

and

$$\lim_{x \to 2} f(x) = f(2)$$

Therefore the curve is continuous at

$$x = 2$$

EXAMPLE: In figure 12-5, is the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

continuous at f(2)?

SOLUTION:

f(2) is undefined at

$$x = 2$$

and the function is therefore discontinuous at

$$x = 2$$

However, by extending the original definition of f(x) to read

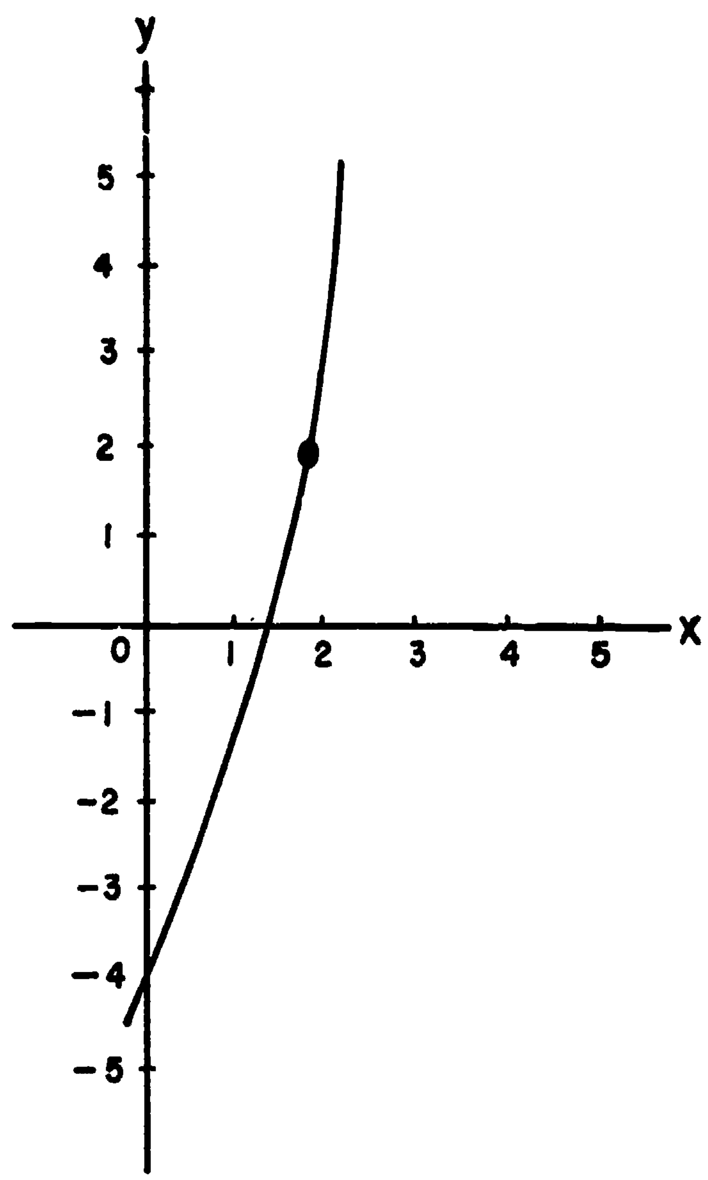


Figure 12-4.—Function $f(x) = x^2 + x - 4$.

$$f_1(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

we will have a continuous function at

$$x = 2$$

NOTE: The value of 4 at x = 2 was found by factoring the numerator of f(x) and then simplifying.

A common kind of discontinuity occurs when dealing with the tangent function of an angle. Figure 12-6 is the graph of the tangent as the angle varies from 0° to 90°; that is, from 0 to $\frac{\pi}{2}$. It should be obvious that the value of the tangent at $\frac{\pi}{2}$ is undefined. Thus the function is said to be discontinuous at $\frac{\pi}{2}$.

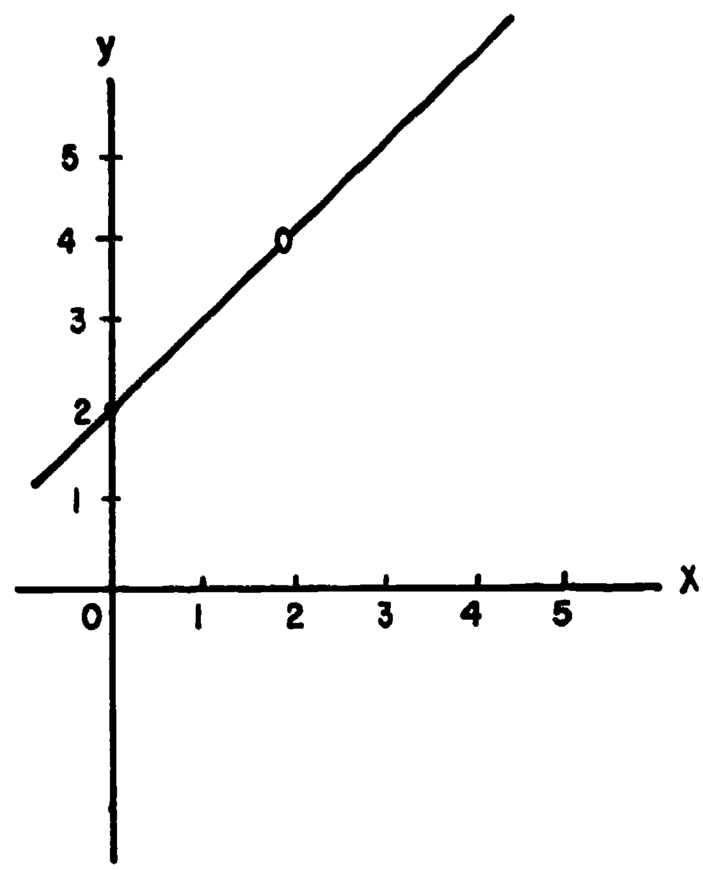


Figure 12-5.—Function $f(x) = \frac{x^2 - 4}{x - 2}$.

PROBLEMS: In the following definitions of the functions find where the functions are discontinuous and then extend the definitions so that the functions are continuous.

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

2. $f(x) = \dfrac{x^2 + 2x - 3}{x + 3}$

3. $f(x) = \dfrac{x^2 + x - 12}{3x - 9}$

ANSWERS:

1. $x = 2$, $f(2) = 3$

2. $x = -3$, $f(-3) = -4$

3. $x = 3$, $f(3) = \frac{7}{3}$

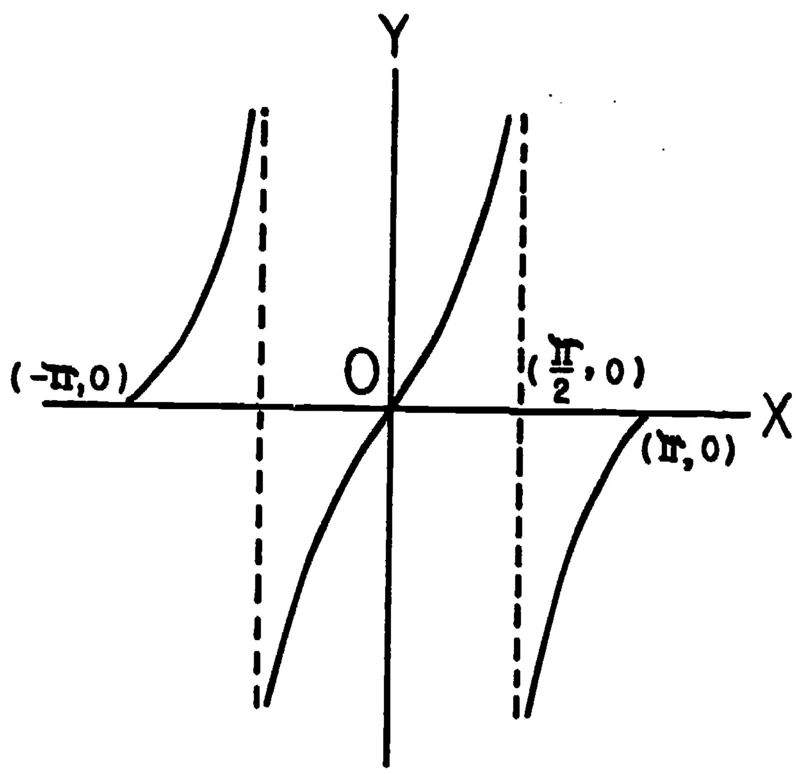


Figure 12-6.—Graph of tangent function.

INCREMENTS AND DIFFERENTIATION

In this section we will extend our discussion of limits and examine the idea of the derivative, the heart of differential calculus.

We will assume we have a particular function of x, such that

$$y = x^2$$

If x is assigned the value 10, the corresponding value of y will be $(10)^2$ or 100. Now, if we increase the value of x by 2, making it 12, we may call this increase of 2 an increment or Δx. This results in an increase in the value of y and we may call this increase an increment or Δy. From this we write

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ &= (10 + 2)^2 \\ &= 144 \end{aligned}$$

As x increases from 10 to 12, y increases from 100 to 144 so that

$$\Delta x = 2$$

$$\Delta y = 44$$

and

$$\frac{\Delta y}{\Delta x} = \frac{44}{2} = 22$$

We are interested in the ratio $\frac{\Delta y}{\Delta x}$ because the limit of this ratio as Δx approaches zero is the derivative of

$$y = f(x)$$

As recalled from the discussion of limits, as Δx is made smaller, Δy gets smaller also, but the radio $\frac{\Delta y}{\Delta x}$ approaches 20. This is shown in table 12-1.

Table 12-1.—Slope values.

Variable	Values of the variable						
Δx	2	1	0.5	0.2	0.1	0.01	0.0001
Δy	44	21	10.25	4.04	2.01	0.2001	0.00200001
$\frac{\Delta y}{\Delta x}$	22	21	20.5	20.2	20.1	20.01	20.0001

There is a much simpler way to find that the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is, in this case, equal to 20. We have two equations

$$y + \Delta y = (x + \Delta x)^2$$

and

$$y = x^2$$

By expanding the first equation so that

$$y + \Delta y = x^2 + 2x\,\Delta x + (\Delta x)^2$$

and subtracting the second from this, we have

$$\Delta y = 2x\,\Delta x + (\Delta x)^2$$

Dividing both sides of the equation by Δx gives

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

Now, taking the limit as Δx approaches zero

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Thus,

$$\frac{dy}{dx} = 2x \qquad (2)$$

NOTE: Equation (2) is one way of expressing the derivative of y with respect to x. Other ways are

$$\frac{dy}{dx} = y' = f'(x) = D(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Equation (2) has the advantage that it is exact and true for all values of x. Thus if

$$x = 10$$

then

$$\frac{dy}{dx} = 2\ (10) = 20$$

and if

$$x = 3$$

then

$$\frac{dy}{dx} = 2(3) = 6$$

This method for obtaining the derivative of y with respect to x is general and may be formulated as follows:

1. Set up the function of x as a function of $(x + \Delta x)$ and expand this function.
2. Subtract the original function of x from the new function $(x + \Delta x)$.
3. Divide both sides of the equation of Δx.
4. Take the limit of all the terms in the equation as Δx approaches zero. The resulting equation is the derivative of f(x) with respect to x.

GENERAL FORMULA

In order to obtain a formula for the derivative of any expression in x, assume the function

$$y = f(x) \qquad (3)$$

so that

$$y + \Delta y = f(x + \Delta x) \qquad (4)$$

Subtracting equation (3) from equation (4) gives

$$\Delta y = f(x + \Delta x) - f(x)$$

and dividing both sides of the equation by Δx we have

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The desired formula is obtained by taking the limit of both sides as Δx approaches zero, so that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

NOTE: The notation $\frac{dy}{dx}$ is not to be considered as a fraction which has dy for the numerator and dx for the denominator. The expression $\frac{\Delta y}{\Delta x}$ is a fraction with Δy as its numerator and Δx as its denominator and $\frac{dy}{dx}$ is a symbol representing the limit approached by $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

EXAMPLES OF DIFFERENTIATION

In this last section of the chapter we will use several examples of differentiation to obtain a firm understanding of the general formula.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ for the function

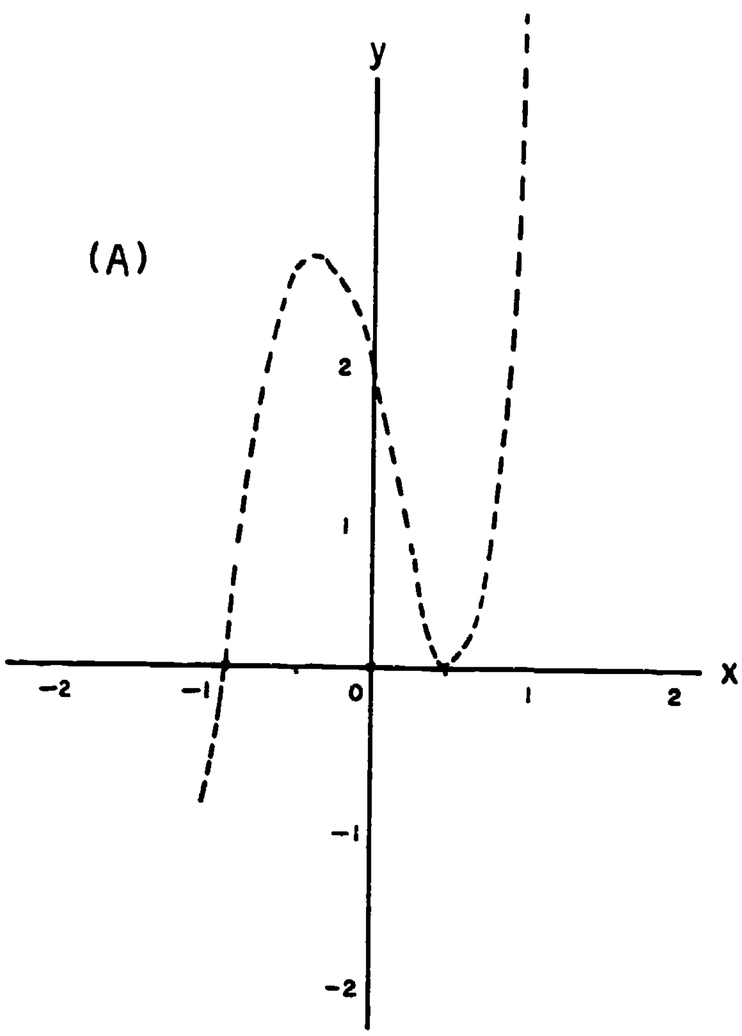
$$y = 5x^3 - 3x + 2$$

and determine the slope of its graph at

$$x = -1,\ -\frac{1}{\sqrt{5}},\ 0,\ \frac{1}{\sqrt{5}},\ 1$$

Draw the graph of the function, as shown in figure 12-7

(A)



(B)

x	-1	$-\frac{1}{\sqrt{5}}$	0	$\frac{1}{\sqrt{5}}$	1
$\frac{dy}{dx}$	12	0	-3	0	12
y	0	2.89	2	1.1	4

Figure 12-7.—(A) Graph of $f(x) = 5x^3 - 3x + 2$; (B) chart of values.

SOLUTION: Finding the derivative by formula, we have

$$f(x + \Delta x) = 5(x + \Delta x)^3 - 3(x + \Delta x) + 2 \qquad (5)$$

and

$$f(x) = 5x^3 - 3x + 2 \qquad (6)$$

Expand equation (5), then subtract equation (6) from equation (5) and simplify to obtain

$$f(x + \Delta x) - f(x)$$

$$= 5\left[3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right] - 3\Delta x$$

Divide through by Δx and we have

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= 5\left[3x^2 + 3x\Delta x + (\Delta x)^2\right] - 3$$

Take the limit of both sides as $\Delta x \to 0$ and

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 15x^2 - 3$$

then

$$\frac{dy}{dx} = 15x^2 - 3$$

Using this derivative let us find the slope of the curve at the points given.

Thus we have a new method of graphing an equation. By substituting different values of x in equation (7) we can find the slope of the curve at the point corresponding to the value of x.

EXAMPLE: Differentiate the function, that is, find $\frac{dy}{dx}$ of

$$y = \frac{1}{x}$$

and then find the slope of the curve at

$$x = 2$$

SOLUTION: Apply the formula for the derivative, and simplify as follows:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \frac{\frac{x - (x + \Delta x)}{x(x + \Delta x)}}{\Delta x}$$

$$= \frac{-1}{x(x + \Delta x)}$$

Now take the limit of both sides as $\Delta x \to 0$ and

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

In order to find the slope of the curve at the point where x has the value 2, substitute 2 for x in the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{1}{2^2}$$

$$= -\frac{1}{4}$$

EXAMPLE: Find the slope of the tangent line on the curve

$$f(x) = x^2 + 4$$

at

$$x = 3$$

SOLUTION: We need to find $\frac{dy}{dx}$ which is the slope of the tangent line at a given point. Apply the formula for the derivative; then,

$$f(x + \Delta x) = (x + \Delta x)^2 + 4 \tag{8}$$

and

$$f(x) = x^2 + 4 \tag{9}$$

Expand equation (8) so that

$$f(x + \Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 + 4$$

then subtract equation (9) from equation (8) and

$$f(x + \Delta x) - f(x) = 2x\Delta x + (\Delta x)^2$$

Now, divide through by Δx and

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x$$

then take the limit of both sides as $\Delta x \to 0$ and

$$\frac{dy}{dx} = 2x$$

Substitute 3 for x in the expression for the derivative to find the slope of the function at

$$x = 3$$

so that

$$\text{slope} = 6$$

In this last example we will set the derivative of the function f(x) equal to zero to find a maximum or minimum point on the curve. By maximum or minimum of a curve we mean the point or points through which the slope of the curve changes from positive to negative or from negative or positive.

NOTE: When the derivative of a function is set equal to zero this does not mean that in all cases we will have found a maximum or minimum point on the curve. A complete discussion of maxima or minima may be found in most calculus texts.

We will require that the following conditions are met:

1. We have a maximum or minimum point.
2. The derivative exists.
3. We are dealing with an interior point on the curve.

When these conditions are met the derivative of the function will be equal to zero.

EXAMPLE: Find the derivative of the function

$$y = 5x^3 - 6x^2 - 3x + 3$$

and set the derivative equal to zero and find the points of maximum and minimum on the curve, then verify this by drawing the graph of the curve.

SOLUTION: Apply the formula for $\frac{dy}{dx}$ as follows:

$$f(x + \Delta x) = \tag{10}$$

$$5(x + \Delta x)^3 - 6(x + \Delta x)^2 - 3(x + \Delta x) + 3$$

and

$$f(x) = 5x^3 - 6x^2 - 3x + 3 \tag{11}$$

Expand equation (10) and subtract equation (11), obtaining

$$f(x + \Delta x) - f(x) = 5(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - 6(2x\Delta x + \Delta x^2) - 3\Delta x$$

Now, divide through by Δx and take the limit as $\Delta x \to 0$, so that

$$\frac{dy}{dx} = 5(3x^2) - 6(2x) - 3$$

$$= 15x^2 - 12x - 3$$

Set $\frac{dy}{dx}$ equal to zero, thus

$$15x^2 - 12x - 3 = 0$$

then

$$3(5x^2 - 4x - 1) = 0$$

and

$$(5x + 1)(x - 1) = 0$$

Set each factor equal to zero and find the points of maximum or minimum are

$$5x = -1$$

$$x = -\frac{1}{5}$$

and

$$x = 1$$

The graph of the function is shown in figure 12-8.

PROBLEMS: Differentiate the functions in problem 1 through 3.

1. $f(x) = x^2 - 3$
2. $f(x) = x^2 - 5x$
3. $f(x) = 3x^2 - 2x + 3$
4. Find the slope of the curve

$$y = x^3 - 3x + 2$$

at the points

$$x = -2, 0, \text{ and } 3$$

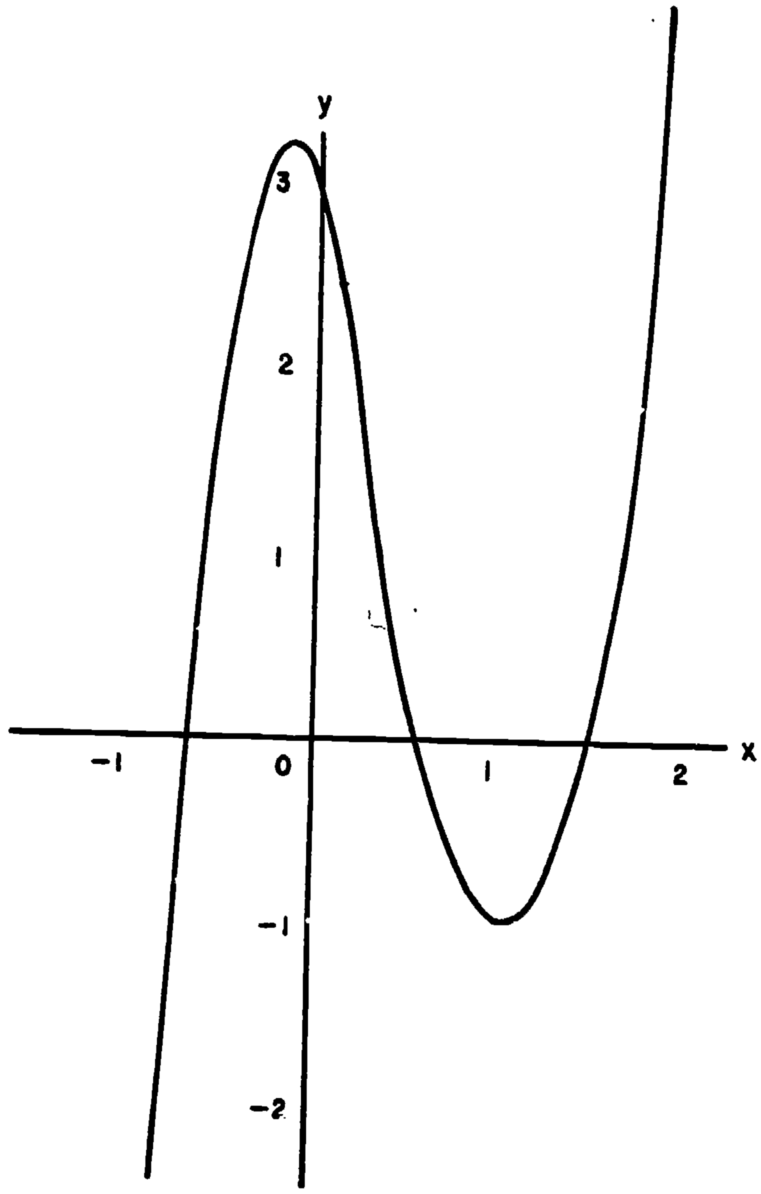


Figure 12-8.—Graph of $5x^3 - 6x^2 - 3x + 3$.

5. Find the values of x where the function

$$f(x) = 2x^3 - 9x^2 - 60x + 12$$

has a maximum or a minimum.

ANSWERS:

1. $2x$
2. $2x - 5$
3. $6x - 2$
4. $m = 9, -3,$ and 24
5. $x = -2, x = 5$

CHAPTER 13

DERIVATIVES

In the previous chapter on limits, we used the delta process to find the limit of a function as Δx approached zero. We called the result of this tedious and in some cases lengthy process the derivative. In this chapter we will examine some rules used to find the derivative of a function without using the delta process.

To find how y changes as x changes, we take the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ and write

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

which is called the derivative of y with respect to x, and we use the symbol $\frac{dy}{dx}$ to indicate the derivative and write

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

In this section we will take up a number of rules which will enable us to easily obtain the derivative of many algebraic functions. In the derivation of these rules, which will be called theorems, we will assume that

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

exists and is finite.

DERIVATIVE OF A CONSTANT

The method used to find the derivative of a constant will be similar to the delta process used in the previous chapter but will include an analytical proof. A diagram is used to give a geometrical meaning of the function.

FORMULA

Theorem 1. The derivative of a constant is zero. Expressed as a formula, this may be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

when y is parallel to the x axis.

PROOF

In figure 13-1, the graph of

$$y = c \text{ (a constant)}$$

the value of y is the same for all values of x, and any change in x (that is, Δx) does not affect y, then

$$\Delta y = 0$$

and

$$\frac{\Delta y}{\Delta x} = 0$$

and

$$\frac{dy}{dx} = 0$$

Another way of stating this is that when x is equal to x_1 and when x is equal to $x_1 + \Delta x$, y has the same value. Therefore,

$$y = c$$

and

$$y + \Delta y = c$$

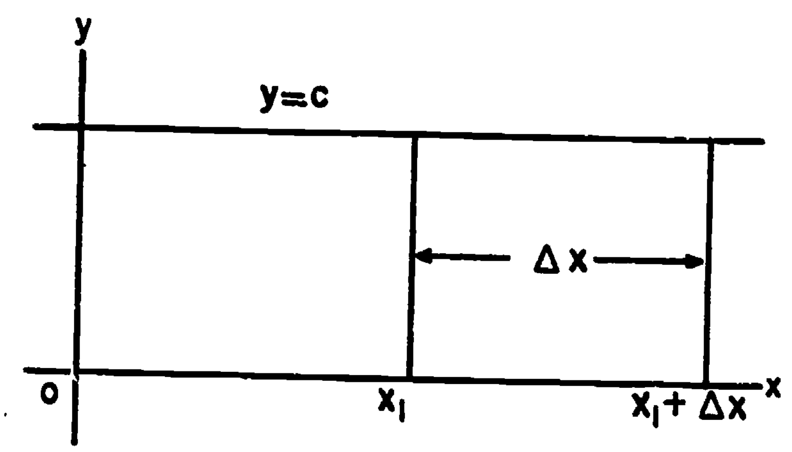


Figure 13-1.—Graph of $y = c$ (a constant).

so that

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{c - c}{\Delta x}$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

The equation

$$y = c$$

represents a straight line parallel to the x axis. The slope of this line will be zero for all values of x. Therefore, the derivative is zero for all values of x.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$y = 6$$

SOLUTION:

$$y = 6$$

and

$$y + \Delta y = 6$$

therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{6 - 6}{\Delta x}$$

$$= 0$$

VARIABLES

In this section on variables, we will extend the theorems of limits covered previously. Recall that a derivative is actually a limit. The proof of the theorems presented here involve the delta process, and only a few of these proofs will be offered.

POWER FORM

Theorem 2. The derivative of the function

$$y = x^n$$

where n is any number is given by

$$\frac{dy}{dx} = nx^{n-1}$$

Proof: By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - (x)^n}{\Delta x}$$

The expression $(x + \Delta x)^n$ may be expanded by the binomial theorem into

$$x^n + nx^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} \Delta x^2 + \ldots + \Delta x^n$$

Substituting in the expression for the derivative, we have

$$\frac{dy}{dx} = \lim_{x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2} x^{n-2}\Delta x^2 + \ldots + \Delta x^n}{\Delta x}$$

Simplifying, this becomes

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} \Delta x + \ldots + \Delta x^{n-1}\right]$$

Letting Δx approach zero, we have

$$\frac{dy}{dx} = nx^{n-1}$$

Thus, the proof is complete.

EXAMPLE: Find the derivative of

$$f(x) = x^5$$

SOLUTION: Apply Theorem 2 and find

$$x^5 = x^n$$

therefore

$$n = 5$$

and

$$n - 1 = 4$$

so that given

$$\frac{dy}{dx} = nx^{n-1}$$

and substituting values for n find that

$$\frac{dy}{dx} = 5x^4$$

EXAMPLE: Find the derivative of

$$f(x) = x$$

SOLUTION: Apply Theorem 2 and find

$$x^n = x$$

therefore

$$n = 1$$

and

$$n - 1 = 0$$

so that

$$\frac{dy}{dx} = x^\circ$$

$$= 1$$

The previous example is a special case of the power form and indicates that the derivative of a function with respect to itself is 1.

EXAMPLE: Find the derivative of

$$f(x) = ax, \ a = \text{constant}$$

SOLUTION:

$$f(x) = ax$$

and

$$f(x + \Delta x) = a(x + \Delta x)$$

$$= ax + a\Delta x$$

so that

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= (ax + a\Delta x) - ax$$

$$= a\Delta x$$

Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x}$$

$$= a$$

The previous example is a continuation of the derivative of a function with respect to itself and indicates that the derivative of a function with respect to itself, times a constant, is that constant.

EXAMPLE: Find the derivative of

$$f(x) = 6x$$

SOLUTION:

$$\frac{dy}{dx} = 6$$

A study of the functions and their derivatives in table 13-1 should further the understanding of this section.

Table 13-1. —Derivatives of functions.

f(x)	3	x	x^2	x^3	x^4	$3x^2$	$9x^3$	x^{-1}	x^{-2}	$3x^{-4}$
$\frac{dy}{dx}$	0	1	2x	$3x^2$	$4x^3$	6x	$27x^2$	$-x^{-2}$	$-2x^{-3}$	$-12x^{-5}$

PROBLEMS: Find the derivatives of the following:

1. $f(x) = 21$
2. $f(x) = x$
3. $f(x) = 21x$
4. $f(x) = 7x^3$
5. $f(x) = 4x^2$
6. $f(x) = 3x^{-2}$

ANSWERS:

1. 0
2. 1
3. 21
4. $21x^2$
5. $8x$
6. $-6x^{-3}$

SUMS

Theorem 3. The derivative of the sum of two or more functions of x is equal to the sum of their derivatives.

Assume two functions of x which we will call u and v, such that

$$u = g(x)$$

and

$$v = h(x)$$

and also

$$y = u + v$$

$$= g(x) + h(x)$$

then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Proof:

$$y = g(x) + h(x) \tag{1}$$

and

$$y + \Delta y = g(x + \Delta x) + h(x + \Delta x) \tag{2}$$

Subtract equation (1) from equation (2) and

$$\Delta y = g(x + \Delta x) + h(x + \Delta x) - g(x) - h(x)$$

Rearrange this equation such that

$$\Delta y = g(x + \Delta x) - g(x) + h(x + \Delta x) - h(x)$$

Divide both sides of the equation by Δx and then take the limit as $\Delta x \to 0$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

but, by definition

$$\lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = \frac{du}{dx}$$

and

$$\lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{dv}{dx}$$

then by substitution

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = x^3 - 8x^2 + 7x - 5$$

SOLUTION: Theorem 3 indicates that we should find the derivative of each term and then show them as a sum; that is, if

$y = x^3, \quad \frac{dy}{dx} = 3x^2$

$y = -8x^2, \quad \frac{dy}{dx} = -16x$

$y = 7x, \quad \frac{dy}{dx} = 7$

$y = -5, \quad \frac{dy}{dx} = 0$

then, if

$$y = x^3 - 8x^2 + 7x - 5$$

then

$$\frac{dy}{dx} = 3x^2 - 16x + 7 + 0$$
$$= 3x^2 - 16x + 7$$

PROBLEMS: Find the derivative of the following:

1. $f(x) = x^2 + x - 1$
2. $f(x) = 2x^4 + 3x + 16$
3. $f(x) = 2x^3 + 3x^2 + x - 3$
4. $f(x) = 3x^3 + 2x^2 - 4x + 2 + 2x^{-1} - 3x^{-3}$

ANSWERS:

1. $2x + 1$
2. $8x^3 + 3$
3. $6x^2 + 6x + 1$
4. $9x^2 + 4x - 4 - 2x^{-2} + 9x^{-4}$

PRODUCTS

Theorem 4. The derivative of the product of two functions of x is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.

If

$$y = uv$$

then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This theorem may be extended to include the product of three functions. The result will be as follows:

If

$$y = uvw$$

then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

EXAMPLE: Find the derivative of

$$f(x) = (x^2 - 2)(x^4 + 5)$$

SOLUTION: The derivative of the first factor is $2x$, and the derivative of the second factor is $4x^3$. Therefore

$$f'(x) = (x^2 - 2)(4x^3) + (x^4 + 5)(2x)$$
$$= 4x^5 - 8x^3 + 2x^5 + 10x$$
$$= 6x^5 - 8x^3 + 10x$$

EXAMPLE: Find the derivative of

$$f(x) = (x^3 - 3)(x^2 + 2)(x^4 - 5)$$

SOLUTION: The derivatives of the three factors, in the order given, are $3x^2$, $2x$, and $3x^2$.

Therefore

$$f'(x) = (x^3 - 3)(x^2 + 2)(4x^3)$$
$$+ (x^2 + 2)(x^4 - 5)(3x^2)$$
$$+ (x^3 - 3)(x^4 - 5)(2x)$$

then

$$f'(x) = 4x^8 + 8x^6 - 12x^5 - 24x^3$$
$$+ 3x^8 + 6x^6 - 15x^4 - 30x^2$$
$$+ 2x^8 - 6x^5 - 10x^4 + 30x$$
$$= 9x^8 + 14x^6 - 18x^5 - 25x^4 - 24x^3 - 30x^2 + 30x$$

PROBLEMS: Find the derivatives of the following:

1. $f(x) = x^3(x^2 - 4)$
2. $f(x) = (x^3 - 3)(x^2 + 2x)$
3. $f(x) = (x^2 - 7x)(x^5 - 4x^2)$
4. $f(x) = (x - 2)(x^2 - 3)(x^3 - 4)$

ANSWERS

1. $5x^4 - 12x^2$
2. $5x^4 + 8x^3 - 6x^{-6}$
3. $7x^6 - 42x^5 - 16x^3 + 84x^2$
4. $6x^5 - 10x^4 - 12x^3 + 6x^2 + 16x + 12$

QUOTIENTS

Theorem 5. The derivative of the quotient of two functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

If

$$y = \frac{u}{v}$$

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{x^2 - 7}{2x + 8}$$

SOLUTION: The derivative of the numerator is 2x, and the derivative of the denominator is 2. Therefore

$$f'(x) = \frac{(2x + 8)(2x) - (x^2 - 7)(2)}{(2x + 8)^2}$$

$$= \frac{4x^2 + 16x - 2x^2 + 14}{(2x + 8)^2}$$

$$= \frac{2x^2 + 16x + 14}{4(x + 4)^2}$$

$$= \frac{x^2 + 8x + 7}{2(x + 4)^2}$$

PROBLEMS: Find the derivatives of the following:

1. $f(x) = \frac{x^4}{x^2 - 2}$
2. $f(x) = \frac{x^2 - 3}{x + 7}$
3. $f(x) = \frac{x^2 + 3x + 5}{x^3 - 4}$

ANSWERS:

1. $\frac{2x^5 - 8x^3}{(x^2 - 2)^2}$
2. $\frac{x^2 + 14x + 3}{(x + 7)^2}$
3. $\frac{-(x^4 + 6x^3 + 15x^2 + 8x + 12)}{(x^3 - 4)^2}$

POWERS OF FUNCTIONS

Theorem 6. The derivative of any function of x raised to the power n, where n is any number, is equal to n times the polynomial function of x to the (n - 1) power times the derivative of the polynomial itself.

If

$$y = u^n$$

where u is any function of x then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = (x^3 - 3x^2 + 2x)^7$$

SOLUTION: Apply Theorem 6 and find

$$\frac{dy}{dx} = 7(x^3 - 3x^2 + 2x)^6 (3x^2 - 6x + 2)$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{(x^2 + 2)^3}{x-1}$$

SOLUTION: This problem involves Theorem 5 and Theorem 6. Theorem 6 is used to find the derivative of the numerator, then Theorem 5 is used to find the derivative of the resulting quotient.

The derivative of the numerator is

$$3(x^2 + 2)^2 (2x)$$

and the derivative of the denominator is 1. Then, by Theorem 5

$$\frac{dy}{dx} = \frac{(x - 1)[3(x^2 + 2)^2 (2x)] - (1)(x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{6x(x^2 + 2)^2 (x - 1) - (x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 [6x(x - 1) - (x^2 + 2)]}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 (5x^2 - 6x - 2)}{(x - 1)^2}$$

PROBLEMS: Find the derivatives of the following:

1. $f(x) = (x^3 + 2x - 6)^2$
2. $f(x) = 5(x^2 + x + 7)^4$
3. $f(x) = \frac{2(x + 3)^3}{3x}$

ANSWERS:

1. $2(x^3 + 2x - 6)(3x^2 + 2)$
2. $20(x^2 + x + 7)^3 (2x + 1)$
3. $\frac{18x(x + 3)^2 - 6(x + 3)^3}{9x^2}$

RADICALS

To differentiate a function containing a radical, replace the radical by a fractional exponent then find the derivative by applying the appropriate theorems.

EXAMPLE: Find the derivative of

$$f(x) = \sqrt{2x^2 - 5}$$

SOLUTION: Replace the radical by the proper fractional exponent, then

$$f(x) = (2x^2 - 5)^{1/2}$$

and by Theorem 6

$$\frac{dy}{dx} = \frac{1}{2} (2x^2 - 5)^{1/2-1} (4x)$$

$$= \frac{1}{2} (2x^2 - 5)^{-1/2} (4x)$$

$$= 2x(2x^2 - 5)^{-1/2}$$

$$= \frac{2x}{\sqrt{2x^2 -5}}$$

$$= \frac{2x \sqrt{2x^2 - 5}}{2x^2 - 5}$$

EXAMPLE: Find the derivative of

$$f(x) = \frac{2x + 1}{\sqrt{3x^2 + 2}}$$

SOLUTION: Replace the radical by the proper fractional exponent, thus

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

At this point a decision is in order. This problem may be solved by either writing

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}} \qquad (3)$$

and applying Theorem 6 in the denominator then applying Theorem 5 for the quotient, or writing

$$f(x) = (2x + 1)(3x^2 + 2)^{-1/2} \qquad (4)$$

and applying Theorem 6 for the second factor then applying Theorem 4 for the product.

The two methods of solution will be completed individually as follows:

Use equation (3)

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

Find the derivative of the denominator

$$\frac{d}{dx}(3x^2 + 2)^{1/2}$$

by applying the power theorem and

$$\frac{d}{dx}(3x^2 + 2)^{1/2} = \frac{1}{2}(3x^2 + 2)^{1/2-1}(6x)$$

$$= 3x(3x^2 + 2)^{-1/2}$$

The derivative of the numerator is

$$\frac{d}{dx}(2x + 1) = 2$$

Now apply Theorem 5 and

$$f'(x) = \frac{(3x^2 + 2)^{1/2}(2) - (2x + 1)[3x(3x^2 + 2)^{-1/2}]}{(3x^2 + 2)}$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{1/2}$$

and simplify, then

$$f'(x) = \frac{2(3x^2 + 2) - 3x(2x + 1)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{6x^2 + 4 - 6x^2 - 3x}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + ?)^{3/2}}$$

To find the same solution, by a different method, use equation (4)

$$f(x) = (2x + 1)(3x^2 + 2)^{-1/2}$$

Find the derivative of each factor

$$\frac{d}{dx}(2x + 1) = 2$$

and

$$\frac{d}{dx}(3x^2 + 2)^{-1/2} = -\frac{1}{2}(3x^2 + 2)^{-1/2-1}(6x)$$

$$= -3x(3x^2 + 2)^{-3/2}$$

Now apply Theorem 4 and

$$f'(x) = (2x+1)\left[-3x(3x^2+2)^{-3/2}\right] + (3x^2+2)^{-1/2}(2)$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{-1/2}$$

and

$$f'(x) = \frac{-3x(2x + 1) + 2(3x^2 + 2)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{-6x^2 - 3x + 6x^2 + 4}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + 2)^{3/2}}$$

which agrees with the solution of the first method used.

PROBLEMS: Find the derivatives of the following:

1. $f(x) = \sqrt{x}$
2. $f(x) = \frac{1}{\sqrt{x}}$
3. $f(x) = \sqrt{3x} - 4$
4. $f(x) = 3\sqrt{4x^2 - 3x + 2}$

ANSWERS:

1. $\frac{1}{2\sqrt{x}}$ or $\frac{\sqrt{x}}{2x}$
2. $-\frac{1}{2}\sqrt{x^3}$ or $-\frac{\sqrt{x^3}}{2x^3}$

3. $\frac{3}{2\sqrt{3x-4}}$ or $\frac{3\sqrt{3x-4}}{2(3x-4)}$

4. $\frac{8x-3}{3\,\sqrt[3]{(4x^2-3x+2)^2}}$ or

$$\frac{(8x-3)\ \sqrt[3]{4x^2-3x+2}}{3(4x^2-3x+2)}$$

CHAIN RULE

A frequently used rule in differential calculus is the chain rule. This rule links together derivatives which have related variables. The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

when the variable y depends on u and u in turn depends on x.

EXAMPLE: Find the derivative of

$$y = (x + x^2)^2$$

SOLUTION: Let

$$u = (x + x^2)$$

and

$$y = u^2$$

Then

$$\frac{dy}{du} = 2u$$

and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\frac{du}{dx} \qquad (5)$$

Now,

$$\frac{du}{dx} = 1 + 2x$$

and substituting into equation (5) gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u(1 + 2x)$$

but

$$u = (x + x^2)$$

therefore,

$$\frac{dy}{dx} = 2(x + x^2)(1 + 2x)$$

EXAMPLE: Find $\frac{dy}{dx}$ where

$$y = 12t^4 + 7t$$

and

$$t = x^2 + 4$$

SOLUTION: By the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx}$$

and

$$\frac{dy}{dt} = 48t^3 + 7$$

and

$$\frac{dt}{dx} = 2x$$

then

$$\frac{dy}{dx} = (48t^3 + 7)(2x)$$

and by substitution

$$\frac{dy}{dx} = [48(x^2 + 4)^3 + 7]\ (2x)$$

PROBLEMS: Find $\frac{dy}{dx}$ in the following:

1. $y = 3t^3 + 8t$ and

   $t = x^3 + 2$

2. $y = 7n^2 + 8n + 3$ and

   $n = 2x^3 + 4x^2 + x$

ANSWERS:

1. $[9(x^3 + 2)^2 + 8]\ (3x^2)$

2. $[14(2x^3 + 4x^2 + x) + 8]\ (6x^2 + 8x + 1)$

INVERSE FUNCTIONS

Theorem 7. The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.

In the equations to this point, x has been the independent variable and y has been the dependent variable. The equations have been in a form such as

$$y = x^2 + 3x + 2$$

Suppose that we have a function like

$$x = \frac{1}{y^2} - \frac{1}{y}$$

and we wish to find the derivative $\frac{dy}{dx}$. Notice that if we solve for y in terms of x, using the quadratic formula, we get the more complicated function

$$y = \frac{-1 \pm \sqrt{1 + 4x}}{2x}$$

If we call this function the direct function, then

$$x = \frac{1}{y^2} - \frac{1}{y}$$

is the inverse function. It is easy to determine $\frac{dy}{dx}$ from the inverse function.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = \frac{1}{y^2} - \frac{1}{y}$$

SOLUTION: Find the derivative $\frac{dx}{dy}$, thus

$$\frac{dx}{dy} = \frac{-2y}{y^4} + \frac{1}{y^2}$$

$$= \frac{-2}{y^3} + \frac{1}{y^2}$$

$$= \frac{-2 + y}{y^3}$$

The reciprocal of $\frac{dx}{dy}$ is the derivative $\frac{dy}{dx}$ of the direct function, and we find

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{y^3}{y - 2}$$

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = y^2$$

SOLUTION: Find $\frac{dx}{dy}$ to be $\frac{d4}{dy} = 2y$

then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y}$$

PROBLEMS: Find the derivative $\frac{dy}{dx}$ of the following functions:

1. $x = 4 - y^2$

2. $x = 9 + y^2$

ANSWERS:

1. $-\frac{1}{2y}$

2. $\frac{1}{2y}$

IMPLICIT FUNCTIONS

In equations containing x and y, it is not always easy to separate the variables. If we do not solve an equation for y, we call y an implicit function of x. In the equation

$$x^2 - 4y = 0$$

y is an implicit function of x, and x is also called an implicit function of y. If we solved this equation for y, that is

$$y = \frac{x^2}{4}$$

then y would be called an explicit function of x. In many cases such a solution would be far too complicated to handle conveniently.

When y is given by an equation such as

$$x^2 + xy^2 = 0$$

y is an implicit function of x.

Whenever we have an equation of this type in which y is a function of x, we can differentiate the function in a straightforward manner. The derivative of each term containing y will be followed by $\frac{dy}{dx}$. Refer to theorem 6.

EXAMPLE: Obtain the derivative $\frac{dy}{dx}$ of the following:

$$x^2 + xy^2 = 0$$

SOLUTION: Find the derivative

$$\frac{d}{dx}(x^2) = 2x$$

and the derivative

$$\frac{d}{dx}(xy^2) = x(2y)\frac{dy}{dx} + (y^2)(1)$$

Therefore,

$$\frac{d}{dx}(x^2 + xy^2) = 2x + 2xy\frac{dy}{dx} + y^2$$

Solving for $\frac{dy}{dx}$ we find that

$$-2xy\frac{dy}{dx} = 2x + y^2$$

and

$$\frac{dy}{dx} = \frac{2x + y^2}{-2xy}$$

$$= -\frac{1}{y} - \frac{y}{2x}$$

Thus, whenever we differentiate an implicit function, the derivative will usually contain terms in both x and y.

PROBLEMS: Find the derivative $\frac{dy}{dx}$ of the following:

1. $x^5 + 4xy^3 - 3y^5 = 2$
2. $x^3y^2 = 3$
3. $x^2y + y^3 = 4$

ANSWERS:

1. $\frac{-5x^4 - 4y^3}{12xy^2 - 15y^4}$
2. $-\frac{3y}{2x}$
3. $\frac{-2xy}{x^2 + 3y^2}$

TRIGONOMETRIC FUNCTIONS

If we are given

$$y = \sin u$$

we may state that, from the general formula,

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\sin(u + \Delta u) - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u \cos\Delta u + \cos u \sin\Delta u - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u(\cos\Delta u - 1)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\sin\Delta u \cos u}{\Delta u} \qquad (6)$$

It can be shown that

$$\lim_{\Delta u \to 0} \frac{\cos\Delta u - 1}{\Delta u} = 0 \qquad (7)$$

and

$$\lim_{\Delta u \to 0} \frac{\sin\Delta u}{\Delta u} = 1 \qquad (8)$$

Therefore, by substituting equations (7) and (8) into equation (6)

$$\frac{dy}{du} = \cos u \qquad (9)$$

Now, we are interested in finding the derivative $\frac{dy}{dx}$ of the function sin u so we apply the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and from the chain rule and equation (9) we find

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In words, this states that to find the derivative of the sine of a function, we use the cosine of the function times the derivative of the function.

By a similar process we find the derivative of the cosine function to be

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

The derivatives of the other trigonometric functions may be found by expressing them in terms of the sine and cosine. That is

$$\frac{d}{dx}(\tan u) = \frac{d}{dx}\left(\frac{\sin u}{\cos u}\right)$$

and by substituting sin u for u, cos u for v, and du for dx in the expression of the quotient theorem

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

we have

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{\sin u}{\cos u}\right)$$

$$= \frac{\cos u \frac{d}{du}(\sin u) - \sin u \frac{d}{du}(\cos u)}{\cos^2 u} \qquad (10)$$

Taking

$$\frac{d}{du}(\sin u) = \cos u$$

and

$$\frac{d}{du}(\cos u) = -\sin u$$

and substituting into equation (10) find that

$$\frac{dy}{du} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u}$$

$$= \sec^2 u \qquad (11)$$

Now, using the chain rule and equation (11) we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \sec^2 u \frac{du}{dx}$$

By stating the other trigonometric functions in terms of the sine and cosine and using similar processes, the following derivatives may be found to be

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = \sin 3x$$

SOLUTION:

$$\frac{dy}{dx} = \cos 3x \frac{d}{dx}(3x)$$

$$= 3 \cos 3x$$

EXAMPLE: Find the derivative of the function

$$y = \tan^2 3x$$

SOLUTION: Use the power theorem and

$$\frac{dy}{dx} = 2 \tan 3x \frac{d}{dx} (\tan 3x)$$

then find

$$\frac{d}{dx} (\tan 3x) = \sec^2 3x \frac{d}{dx} (3x)$$

and

$$\frac{d}{dx} (3x) = 3$$

Combining all of these, we find that

$$\frac{dy}{dx} = (2 \tan 3x)(\sec^2 3x)(3)$$

$$= 6 \tan 3x \sec^2 3x$$

PROBLEMS: Find the derivative of the following:

1. $y = \sin 2x$
2. $y = (\cos x^2)^2$

ANSWERS:

1. $2 \cos 2x$
2. $- 4x \cos x^2 \sin x^2$

CHAPTER 14

INTEGRATION

The two main branches of calculus are differential calculus and integral calculus. Having investigated differential calculus in previous chapters, we now turn our attention to integral calculus. Basically, integration is the inverse of differentiation just as division is the inverse of multiplication, and as subtraction is the inverse of addition.

DEFINITIONS

Integration is defined as the inverse of differentiation. When we were dealing with differentiation, we were given a function F(x) and were required to find the derivative of this function. In integration we will be given the derivative of a function and will be required to find the function. That is, when we are given the function f(x), we will find another function F(x) such that

$$\frac{dF(x)}{dx} = f(x) \qquad (1)$$

In words, when we have the function f(x), we must find the function F(x) whose derivative is the function f(x).

If we change equation (1) to read

$$dF(x) = f(x)\,dx \qquad (2)$$

we have used dx as a differential. An equivalent statement for equation (2) is

$$F(x) = \int f(x)\,dx$$

We call f(x) the integrand, and we say F(x) is equal to the indefinite integral of f(x). The elongated S, that is, $\int$, is the integral sign. This symbol is used because integration may be shown to be the limit of a sum.

INTERPRETATION OF AN INTEGRAL

We will use the area under a curve for the interpretation of an integral. It should be realized, however, that an integral may represent many things, and it may be real or abstract. It may represent plane area, volume, or surface area of some figure.

AREA UNDER A CURVE

In order to find the area under a curve, we must agree on what is desired. In figure 14-1, where f(x) is equal to the constant 4, and the "curve" is the straight line

$$y = 4$$

The area of the rectangle is found by multiplying the height times the width. Thus, the area under the curve is

$$A = 4(b - a)$$

The next problem will be to find a method for determining the area under any curve, provided that the curve is continuous. In figure 14-2, the area under the curve

$$y = f(x)$$

between points x and $x + \Delta x$ is approximately $f(x)\Delta x$. We consider that Δx is small and the area is given to be ΔA. This area under the curve is nearly a rectangle. The area ΔA, under the curve, would differ from the area of the rectangle by the area of the triangle ABC if AC were a straight line.

When Δx becomes smaller and smaller, the area of ABC becomes smaller at a faster rate, and ABC finally becomes indistinguishable from a triangle. The area of this triangle becomes negligible when Δx is sufficiently small.

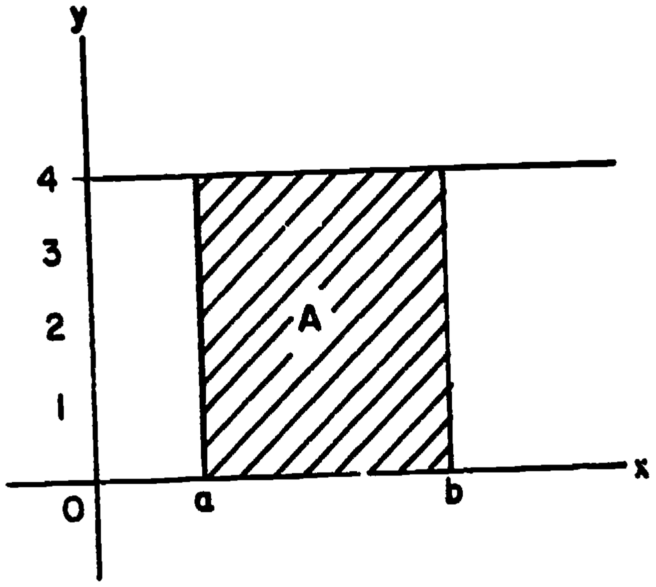


Figure 14-1.—Area of a rectangle.

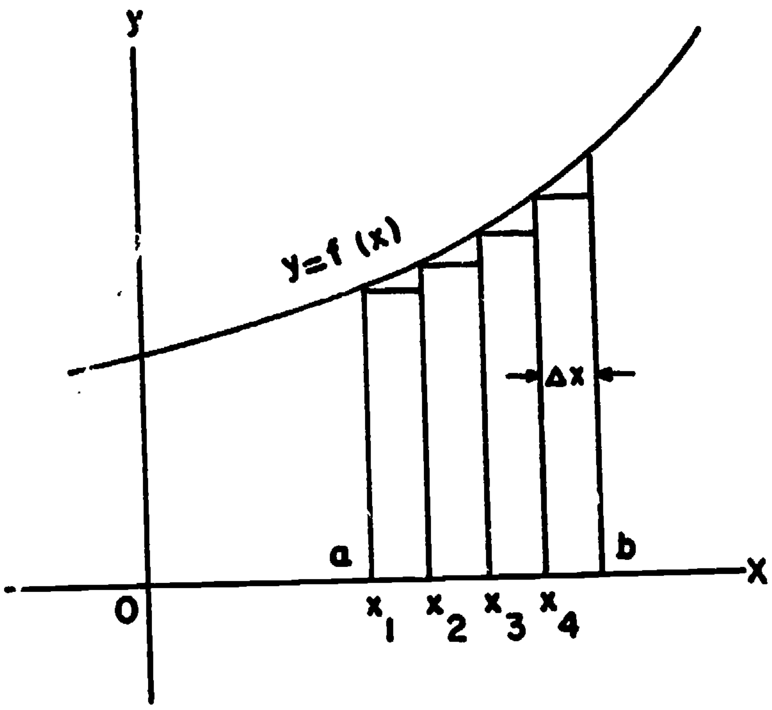


Figure 14-3.—Area of strips.

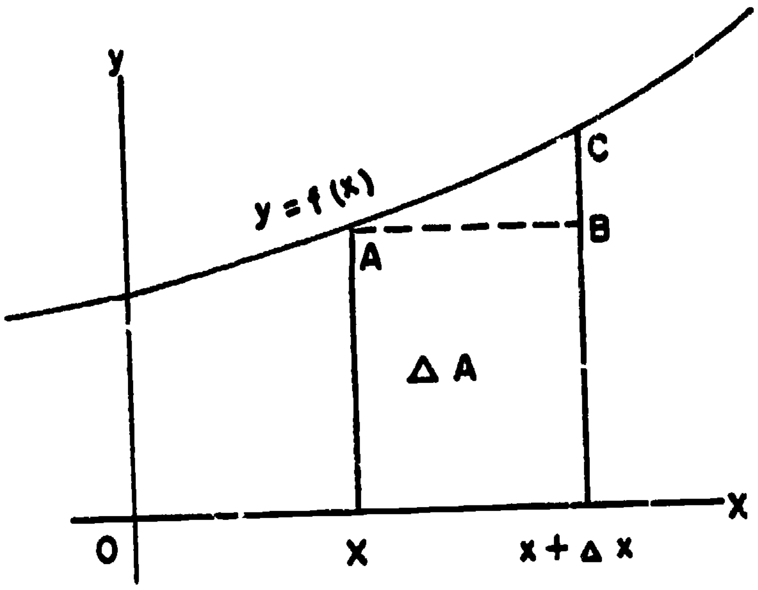


Figure 14-2.—Area ΔA.

Therefore, for sufficiently small values of Δx we can say that

$$\Delta A \approx f(x)\Delta x$$

Now, if we have the curve in figure 14-3, the sum of all the rectangles will be approximately equal to the area under the curve and bounded by the lines at a and b. The difference between the actual area under the curve and the sum of the areas of the rectangles will be the sum of the areas of the triangles above each rectangle.

As Δx is made smaller and smaller, the sum of the rectangular areas will approach the value of the area under the curve. The sum of the areas of the rectangles may be indicated by

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\,\Delta x \qquad (3)$$

where Σ (sigma) is the symbol for sum, n is the number of rectangles, $f(x)\Delta x$ is the area of each rectangle, and k is the designation number of each rectangle. In the particular example just discussed, where we have four rectangles, we would write

$$A = \sum_{k=1}^{4} f(x_k)\,\Delta x$$

and we would have only the sum of four rectangles and not the limiting area under the curve.

When using the limit of a sum, as in equation (3), we are required to use extensive algebraic techniques to find the actual area under the curve.

To this point we have been given a choice of using arithmetic and finding only an approximation of the area under a curve, or we could use extensive algebraic preliminaries and find the actual area.

We will now use calculus to find the area under a curve fairly easily.

In figure 14-4, the area under the curve, from a to b, is shown as the sum of the areas of ${}_aA_c$ and ${}_cA_b$. The notation ${}_aA_c$ means the area under the curve from a to c.

The Intermediate Value Theorem states that

$$ {}_aA_b = f(c)\ (b - a) $$

where f(c) in figure 14-4 is the function at an intermediate point between a and b.

We now modify figure 14-4 as shown in figure 14-5.

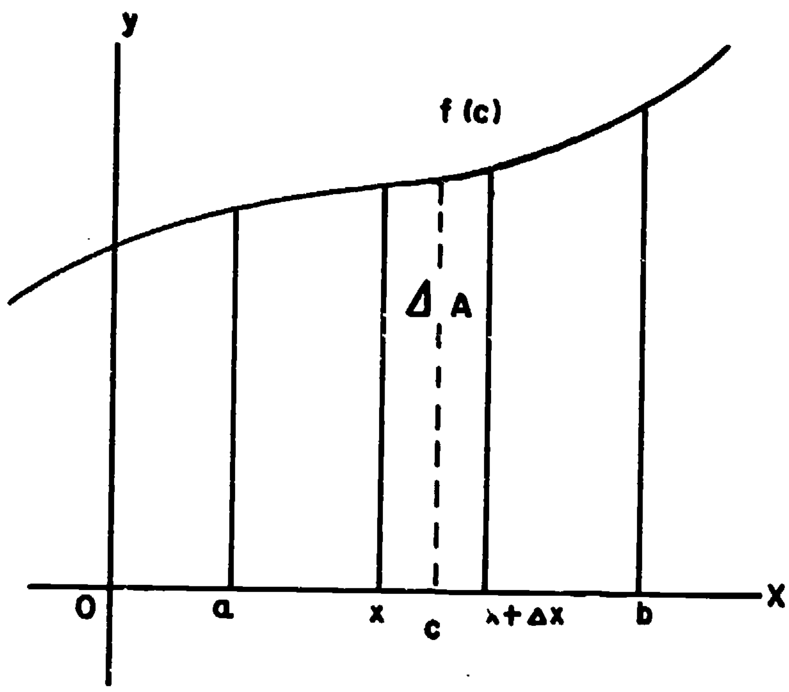


Figure 14-5.—Increments of area at f(c).

When

$$ x = a $$

$$ {}_aA_a = 0 $$

It is seen in figure 14-5 that

$$ {}_aA_x + {}_xA_{(x+\Delta x)} = {}_aA_{(x+\Delta x)} $$

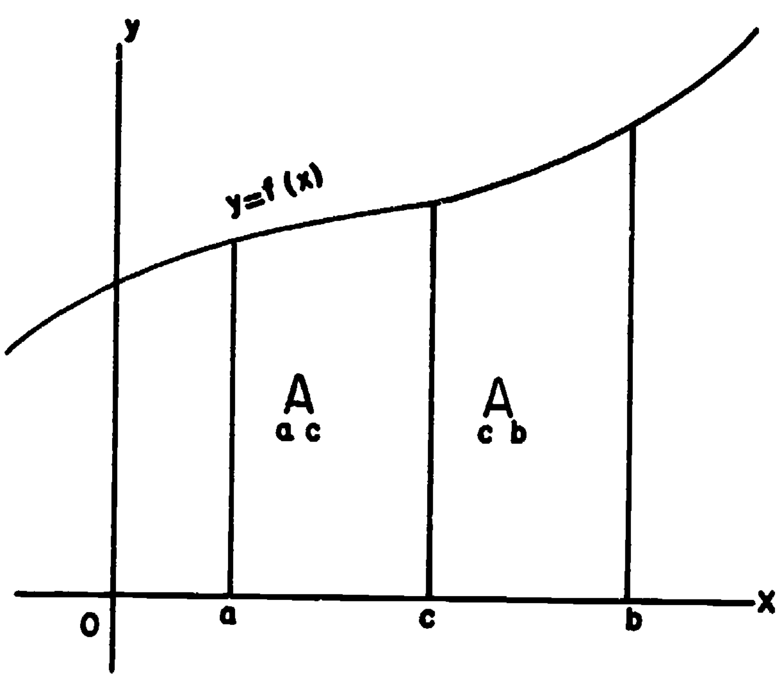


Figure 14-4.—Designation of limits.

therefore, the increase in area, as shown, is

$$ \Delta A = {}_xA_{(x+\Delta x)} $$

but reference to figure 14-5 shows

$$ {}_xA_{(x+\Delta x)} = f(c)\,\Delta x $$

where c is a point between a and b. Then, by substitution

$$ \Delta A = f(c)\,\Delta x $$

or

$$ \frac{\Delta A}{\Delta x} = f(c) $$

and as Δx approaches zero we have

$$ \frac{d(A)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} $$

$$ = \lim_{c \to x} f(c) $$

$$ = f(x) $$

Now, from the definition of integration

$$ {}_aA_x = \int f(x)\,dx \qquad (4)$$
$$ = F(x) + C$$

and

$$ {}_aA_a = F(a) + C$$

but

$$ {}_aA_a = 0$$

therefore

$$F(a) + C = 0$$

and solving for C we have

$$C = -F(a)$$

By substituting -F(a) into equation (4) we find

$$ {}_aA_x = F(x) - F(a)$$

If we let

$$x = b$$

then

$$ {}_aA_b = F(b) - F(a) \qquad (5)$$

where F(b) and F(a) are the integrals of the function of the curve at the values b and a.

The constant of integration C is omitted in equation (5) because when the function of the curve at b and a is integrated C will occur with both F(a) and F(b) and will therefore be subtracted from itself.

EXAMPLE: Find the area under the curve

$$y = 2x - 1$$

in figure 14-6, bounded by the vertical lines at a and b, and the x-axis.

SOLUTION: We know that

$$ {}_aA_b = F(b) - F(a)$$

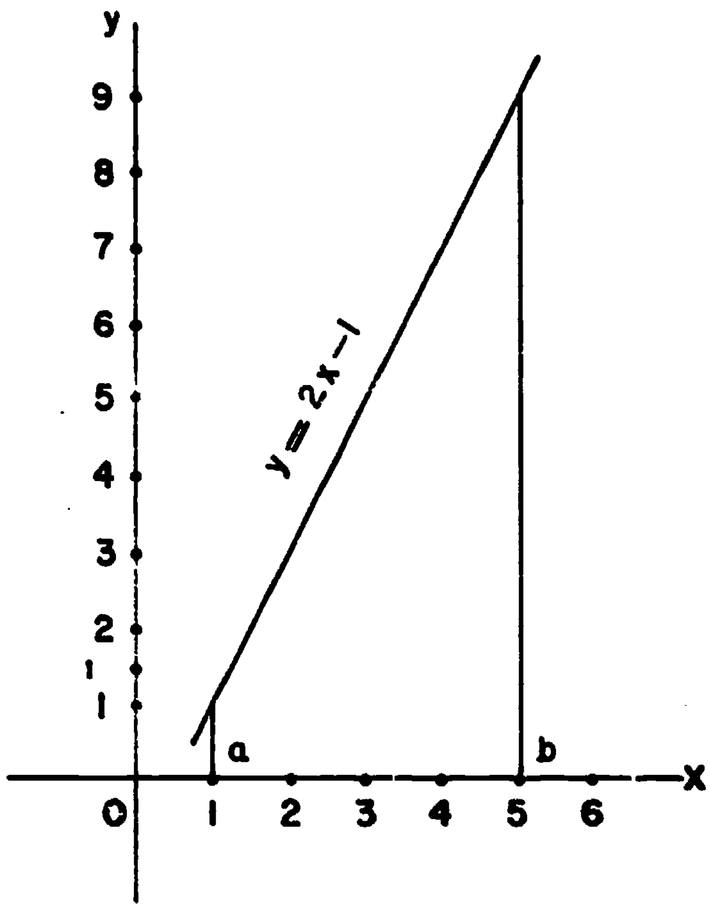


Figure 14-6.—Area of triangle and rectangle.

and find

$$F(x) = \int f(x)dx$$
$$= \int (2x - 1)dx$$
$$= x^2 - x \text{ (this step will be justified later)}$$

Then, substituting the values for a and b into F(x) (that is, $x^2 - x$) find that when

$$x = a$$
$$= 1$$
$$F(a) = 1 - 1$$
$$= 0$$

and when

$$x = b$$
$$= 5$$
$$F(b) = 25 - 5$$
$$= 20$$

Then by substituting these values in

$$ {}_aA_b = F(b) - F(a) $$

find that

$$ {}_aA_b = 20 - 0 $$
$$ = 20 $$

We may verify this by considering figure 14-6 to be a triangle with base 4 and height 8 sitting on a rectangle of height 1 and base 4. By known formulas, we find the area under the curve to be 20.

CONSTANT OF INTEGRATION

A number which is independent of the variable of integration is called a constant of integration. This is to say that two integrals of the same function may differ by the constant of integration.

INTEGRAND

When we are given a differential (or derivative) and we are to find the function whose derivative is the differential we were given, we call the operation integration.

If we have

$$ \frac{dy}{dx} = x^2 $$

and are asked to find the function whose derivative is this value, x^2, we write

$$ dy = x^2 dx $$

then

$$ y = \frac{x^3}{3} + C $$

or we write

$$ y = \int x^2 dx $$
$$ = \frac{x^3}{3} + C $$

The symbol $\int$ is the integral sign, $\int x^2 dx$ is the integral of $x^2 dx$, and x^2 is called the integrand. The C is called the constant of integration.

INDEFINITE INTEGRALS

When we were finding the derivative of a function, we wrote

$$ \frac{dy}{dx} = F(x) $$

or

$$ \frac{dF(x)}{dx} = f(x) $$

where we say the derivative of $F(x)$ is $f(x)$. Our problem is to find $F(x)$ when we are given $f(x)$.

We know that the symbol $\int ...dx$ is the inverse of $\frac{d}{dx}$, or when dealing with differentials, the operator symbols d and $\int$ are the inverse of each other.

That is

$$ F(x) = \int f(x)dx $$

and when the derivative of each side is taken, d annulling $\int$, we have

$$ dF(x) = f(x)dx $$

or where $\int ...dx$ annuls $\frac{d}{dx}$, we have

$$ \frac{dF(x)}{dx} = \frac{d}{dx} \int f(x)dx $$
$$ = f(x) $$

From this, we find that

$$ d(x^3) = 3x^2 dx $$

then

$$ \int 3x^2 dx = x^3 + C $$

Also we find that

$$ d(x^3 + 3) = 3x^2 dx $$

then

$$ \int 3x^2 dx = x^3 + 2 $$

Again, we find that

$$ d(x^3 - 9) = 3x^2 dx $$

then

$$\int 3x^2dx = x^3 - 9$$

This is to say that

$$d(x^3 + C) = 3x^2dx$$

and

$$3x^2dx = x^3 + C$$

where C is any constant of integration. Since C may have infinitely many values, then a differential expression may have infinitely many integrals which differ only by the constant. We assume the differential expression has at least one integral.

Because the integral contains C and C is indefinite, we call

$$F(x) + C$$

an indefinite integral of f(x)dx. In the general form we say

$$\int f(x)dx = F(x) + C$$

With regard to the constant of integration, a theorem and its converse state:

If two functions differ by a constant, they have the same derivative.

If two functions have the same derivative, their difference is a constant.

EVALUATING THE CONSTANT

To evaluate the constant of integration we will use the following approach.

If we are to find the equation of a curve whose first derivative is 2 times the independent variable x, we may write

$$\frac{dy}{dx} = 2x$$

or

$$dy = 2xdx \qquad (6)$$

We may obtain the desired equation for the curve by integrating the expression for dy. That is, integrate both sides of equation (6).

If

$$dy = 2xdx$$

then

$$\int dy = \int 2xdx$$

but

$$\int dy = y$$

and also

$$\int 2xdx = x^2 + C$$

therefore

$$y = x^2 + C$$

We have obtained only a general equation of the curve because a different curve results for each value we assign to C. This is shown in figure 14-7. If we specified that

$$x = 0$$

and

$$y = 6$$

we may obtain a specific value for C and hence a particular curve.

Suppose that

$$y = x^2 + C, \ x = 0, \text{ and } y = 6$$

then

$$6 = 0^2 + C$$

or

$$C = 6$$

By substituting the value 6 into the general equation, the equation for the particular curve is

$$y = x^2 + 6$$

which is curve C of figure 14-7.

The values for x and y will determine the value for C and also determine the particular curve of the family of curves.

In figure 14-7, curve A has a constant equal to -4, curve B has a constant equal to 0, and curve C has a constant equal to 6.

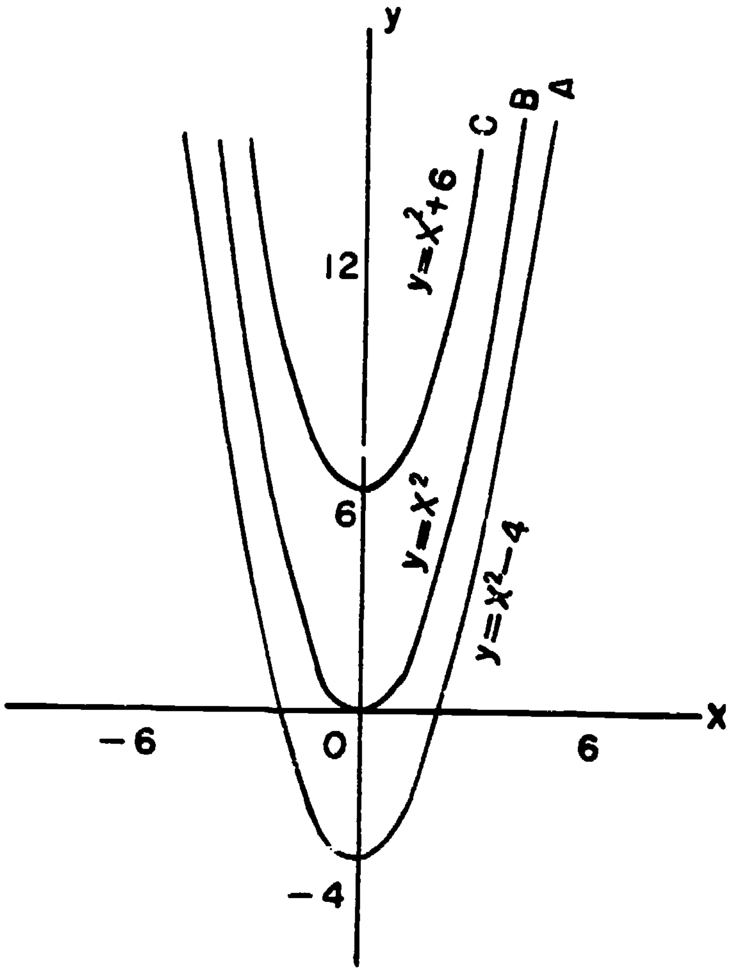


Figure 14-7.—Family of curves.

EXAMPLE: Find the equation of the curve if its first derivative is 6 times the independent variable, y equals 2, and x equals 0.

SOLUTION: We may write

$$\frac{dy}{dx} = 6x$$

and

$$\int dy = \int 6x dx$$

therefore

$$y = 3x^2 + C$$

Solving for C when

$$x = 0$$

and

$$y = 2$$

we have

$$2 = 3(0^2) + C$$

or

$$C = 2$$

and the equation is

$$y = 3x^2 + 2$$

RULES FOR INTEGRATION

Although integration is the inverse of differentiation, and we were given rules for differentiation, we are required to determine the answers in integration by trial and error. There are some formulas which aid us in the determination of the answer.

In this section we will discuss four of the rules and how they are used to integrate standard elementary forms. In the rules we will let u and v denote a differentiable function of a variable such as x. We will let C, n, and a denote constants.

Our proofs will involve remembering that we are searching for a function F(x) whose derivative is f(x)dx.

Rule 1. $\int du = u + C$

The integral of a differential of a function is the function plus a constant.

Proof: If

$$\frac{d(u + C)}{du} = 1$$

then

$$d(u + C) = du$$

and

$$\int du = u + C$$

EXAMPLE: Evaluate the integral

$$\int dx$$

SOLUTION: By Rule 1 we have

$$\int dx = x + C$$

Rule 2. $\int a\,du = a \int du = au + C$

The integral of the product of a constant and a variable is equal to the product of the constant and the integral of the variable. That is, a constant may be moved across the integral sign. NOTE: A variable may NOT be moved across the integral sign.

Proof: If

$$d(au + C) = (a)\,d(u + C)$$

$$= au$$

then

$$\int a\,du = a \int du = au + C$$

EXAMPLE: Evaluate the integral

$$\int 4dx$$

SOLUTION: By Rule 2

$$\int 4dx = 4 \int dx$$

and by Rule 1

$$\int dx = x + C$$

therefore

$$\int 4dx = 4x + C$$

Rule 3. $\int (du + dv + dw) = \int du + \int dv + \int dw$

$$= u + v + w + C$$

The integral of a sum is equal to the sum of the integrals.

Proof: If

$$d(u + v + w + C) = du + dv + dw$$

then

$$\int (du + dv + dw) = (u + C_1) + (v + C_2) + (w + C_3)$$

$$= u + v + w + C$$

where

$$C = C_1 + C_2 + C_3$$

EXAMPLE: Evaluate the integral

$$\int (2x - 5x + 4)dx$$

SOLUTION: We will not combine 2x and -5x. Then, by Rule 3

$$\int (2x - 5x + 4)dx$$

$$= \int 2xdx - \int + 5xdx + \int 4dx$$

$$= 2 \int xdx - 5 \int xdx + 4 \int dx$$

$$= \frac{2x^2}{2} + C_1 - \frac{5x^2}{2} + C_2 + 4x + C_3$$

$$= x^2 - \frac{5}{2}x^2 + 4x + C$$

where C is the sum of C_1, C_2, and C_3. This solution requires knowledge of Rule 4 which follows.

Rule 4. $\int u^n du = \frac{u^{n+1}}{n+1} + C$

The integral of $u^n du$ may be obtained by adding 1 to the exponent, then dividing by this new exponent. NOTE: If n is minus 1, this rule is not valid and another method must be used.

Proof: If

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)u^n}{n+1}\,du$$

$$= u^n du$$

then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

EXAMPLE: Evaluate the integral

$$\int x^3 dx$$

SOLUTION: By Rule 4

$$\int x^3 dx = \frac{x^{3+1}}{3+1} + C$$

$$= \frac{x^4}{4} + C$$

EXAMPLE: Evaluate the integral

$$\int \frac{7}{x^3}\, dx$$

SOLUTION: First write the integral

$$\int \frac{7}{x^3}\, dx$$

as

$$\int 7x^{-3} dx$$

then, by Rule 2 write

$$7 \int x^{-3} dx$$

and by Rule 4

$$7 \int x^{-3} dx$$

$$= 7 \left(\frac{x^{-2}}{-2}\right) + C$$

$$= -\frac{7}{2x^2} + C$$

EXAMPLE: Evaluate the integral

$$\left(\frac{1}{x^2} + \frac{2}{x^3}\right) dx$$

SOLUTION:

$$\int \left(\frac{1}{x^{\frac{1}{2}}} + \frac{2}{x^{\frac{2}{3}}}\right) dx$$

$$= \int x^{\frac{1}{2}} dx + \int x^{\frac{2}{3}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C + \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C$$

$$= \frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{5}{3}}}{5} + C$$

PRACTICE PROBLEMS: Evaluate the following integrals:

1. $\int x^2 dx$
2. $\int 4x dx$
3. $\int (x^3 + x^2 + x) dx$
4. $\int 6 dx$
5. $\int \frac{5}{x^2} dx$

ANSWERS:

1. $\frac{x^3}{3} + C$
2. $2x^2 + C$
3. $\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + C$
4. $6x + C$
5. $-\frac{5}{x} + C$

DEFINITE INTEGRALS

The general form of the indefinite integral is

$$\int f(x)dx = F(x) + C$$

and has two identifying characteristics. First, the constant of integration was required to be added to each integration. Second, the result of integration is a function of a variable and has no definite value, even after the constant of integration is determined, until the variable is assigned a numerical value.

The definite integral eliminates these two characteristics. The form of the definite integral is

$$\int_a^b f(x)dx = F(b) + C - [F(a) + C] \qquad (7)$$
$$= F(b) - F(a)$$

where a and b are given values. Notice that the constant of integration does not appear in the final expression of equation (7). In words, this equation states that the difference of the values of

$$\int_a^b f(x)dx$$

for

$$x = a$$

and

$$x = b$$

gives the area under the curve defined by f(x), the x axis, and the ordinates where

$$x = a$$

and

$$x = b$$

UPPER AND LOWER LIMITS

In figure 14-8, the value of b is the upper limit and the value at a is the lower limit. These upper and lower limits may be any assigned values in the range of the curve. The upper limit is positive with respect to the lower limit in that it is located to the right (positive in our case) of the lower limit.

Equation (7) is the limit of the sum of all the strips between a and b, having areas of $f(x)\Delta x$. That is

$$\lim \sum_{x=a}^{x=b} f(x)\Delta x = \int_a^b f(x)dx$$

To evaluate the definite integral

$$\int_a^b f(x)dx$$

find the function F(x) whose derivative is f(x)dx at the value of b and subtract the function at the value of a. That is

$$\int_a^b f(x)dx = F(x)\Big|_a^b \qquad (8)$$

$$= F(b) - F(a)$$

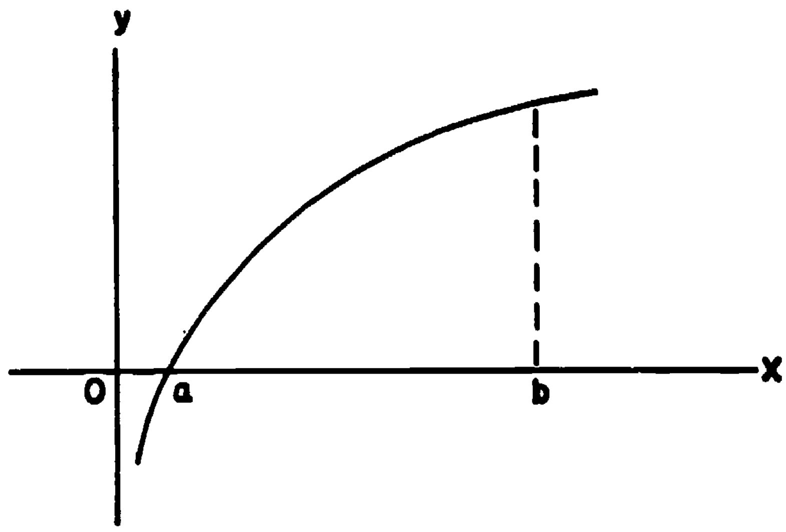


Figure 14-8.—Upper and lower limits.

where b is the upper limit and a is the lower limit.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the x axis, and the ordinates

$$x = 2$$

and

$$x = 3$$

as shown in figure 14-9.

SOLUTION: Substituting into equation (8)

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$= \int_2^3 x^2 dx$$

$$= \frac{x^3}{3}\Big|_2^3$$

$$= \frac{3^2}{3} - \frac{2^3}{3}$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3}$$

$$= 6\frac{1}{3}$$

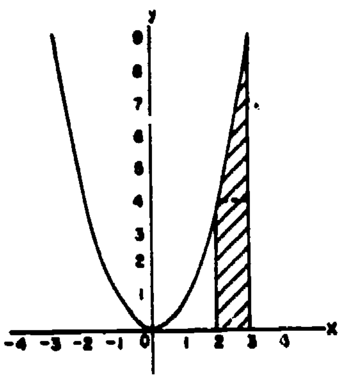


Figure 14-9.—Area from x = 2 to x = 3.

We may make an estimate of this solution by considering the area desired in figure 14-9 as being a right triangle resting on a rectangle. The triangle has an approximate area of

$$A = \frac{1}{2} bh$$

$$= \frac{1}{2} (1)(5)$$

$$= \frac{5}{2}$$

and the area of the rectangle is

$$A = bh$$

$$= (1)(4)$$

$$= 4$$

and

$$4 + \frac{5}{2} = \frac{13}{2} = 6\frac{1}{2}$$

which is a close approximation of the area found by the process of integration.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the x axis, and the ordinates

$$x = -2$$

and

$$x = 2$$

as shown in figure 14-10.

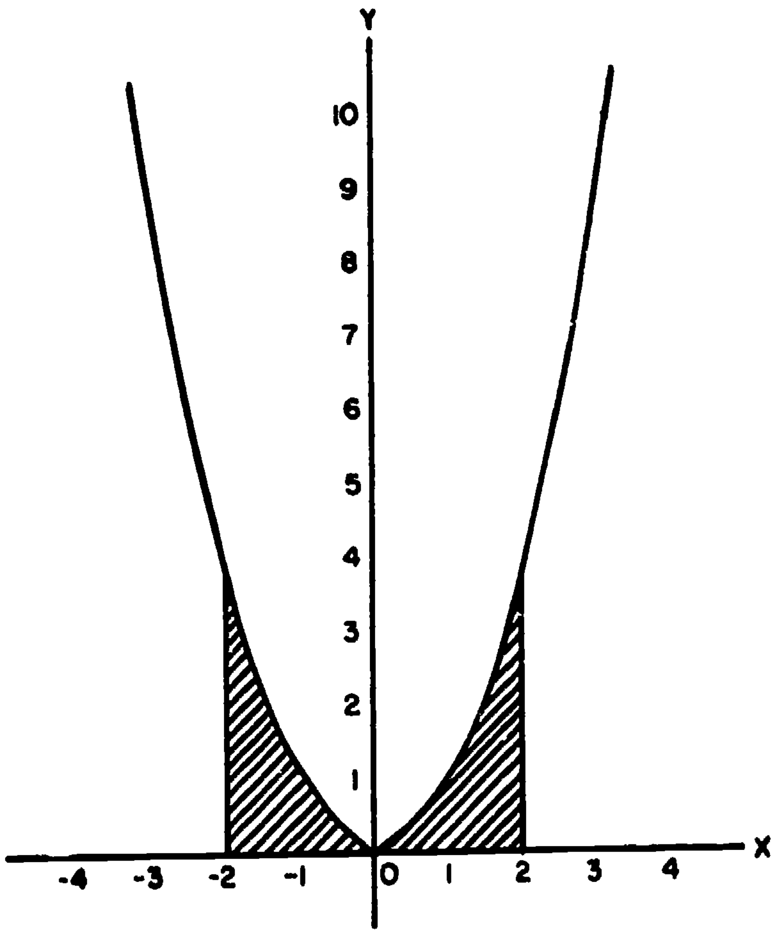


Figure 14-10.—Area under a curve.

SOLUTION: Substituting into equation (8)

$$\int_a^b f(x)dx = F(x) \Big|_{-2}^{2}$$

$$= F(2) - F(-2)$$

$$= \int_{-2}^{2} x^2 dx$$

$$= \frac{x^3}{3} \Big|_{-2}^{2}$$

$$= \frac{8}{3} - \left[-\frac{8}{3}\right]$$

$$= \frac{16}{3}$$

$$= 5\frac{1}{3}$$

The area above a curve and below the x axis, as shown in figure 14-11, will through integration furnish a negative answer.

Then when dealing with area as shown in figure 14-12, each of the areas shown must be found separately. The areas thus found are then added together, with area considered as the absolute value.

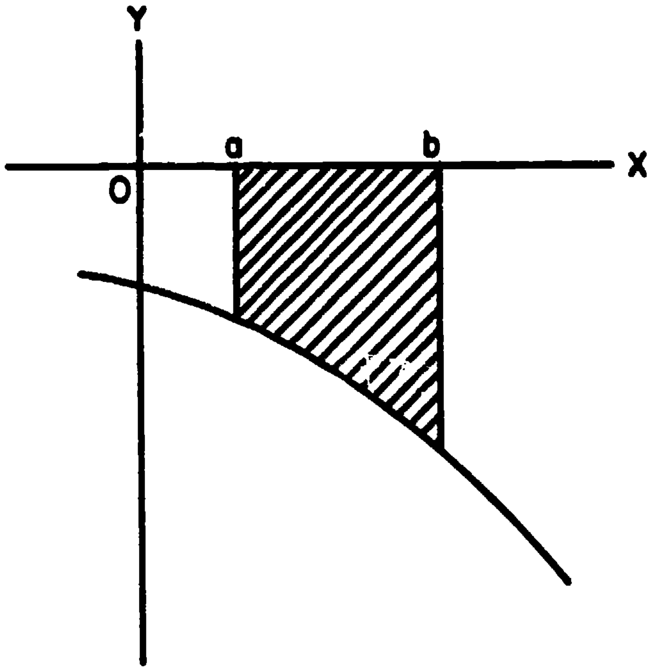


Figure 14-11.—Area above a curve.

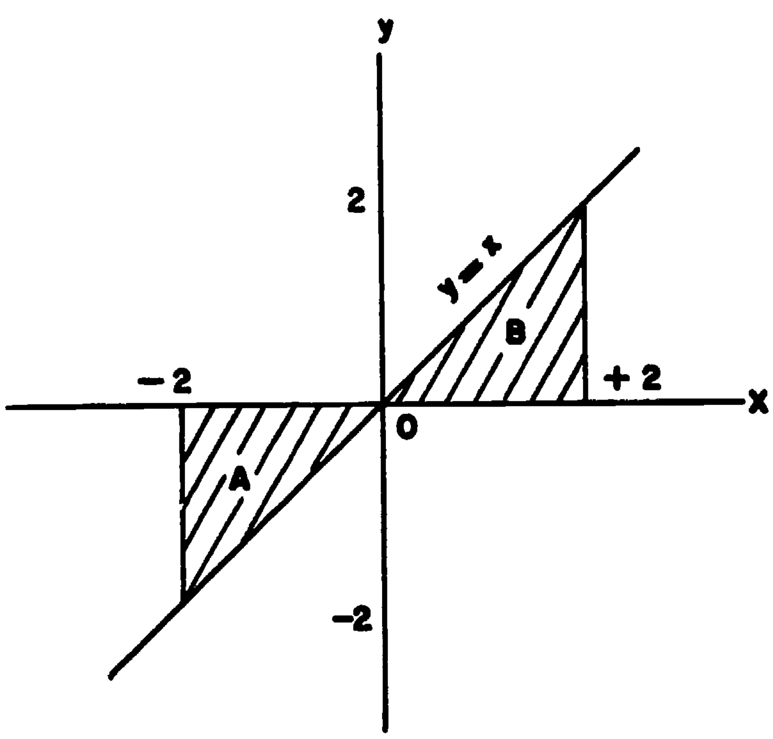


Figure 14-12.—Negative and positive value areas

EXAMPLE: Find the area between the curve

$$y = x$$

and the x axis, bounded by the lines

$$x = -2$$

and

$$x = 2$$

SOLUTION: These areas must be computed separately; therefore we write

$$\text{Area A} = \int_{-2}^{0} f(x)dx$$

$$= \int_{-2}^{0} xdx$$

$$= \frac{x^2}{2}\Big|_{-2}^{0}$$

$$= 0 - \left[\frac{4}{2}\right]$$

$$= -2$$

and the absolute value of -2 is

$$|-2| = 2$$

Then

$$\text{Area B} = \int_{0}^{2} f(x)dx$$

$$= \frac{x^2}{2}\Big|_{0}^{2}$$

$$= \frac{4}{2} - [0]$$

$$= 2$$

and adding the two areas A and B we find

$$A + B = 2 + 2$$

$$= 4$$

NOTE: INCORRECT SOLUTION: If the function is integrated from -2 to 2 the following incorrect result will occur

$$\text{Area} = \int_{-2}^{2} f(x)dx$$

$$= \int_{-2}^{2} xdx$$

$$= \frac{x^2}{2}\Big|_{-2}^{2}$$

$$= \frac{4}{2} - \left[\frac{4}{2}\right]$$

$$= 0 \text{ (INCORRECT SOLUTION)}$$

This is obviously not the area shown in figure 14-12. Such an example emphasizes the value of making a commonsense check on every solution. A sketch of the function will aid this commonsense judgment.

EXAMPLE: Find the total area bounded by the curve

$$y = x^3 - 9x$$

the x axis, and the lines

$$x = -3$$

and

$$x = 3$$

as shown in figure 14-13.

SOLUTION: The area desired is both above and below the x axis; therefore we need to find the areas separately, then add them together using their absolute values.
Therefore

$$A_1 = \int_{-3}^{0} (x^3 - 9x)dx$$

$$= \frac{x^4}{4} - \frac{9}{2}x^2\Big|_{-3}^{0}$$

$$= 0 - \left[\frac{81}{4} - \frac{81}{2}\right]$$

$$= \frac{81}{4}$$

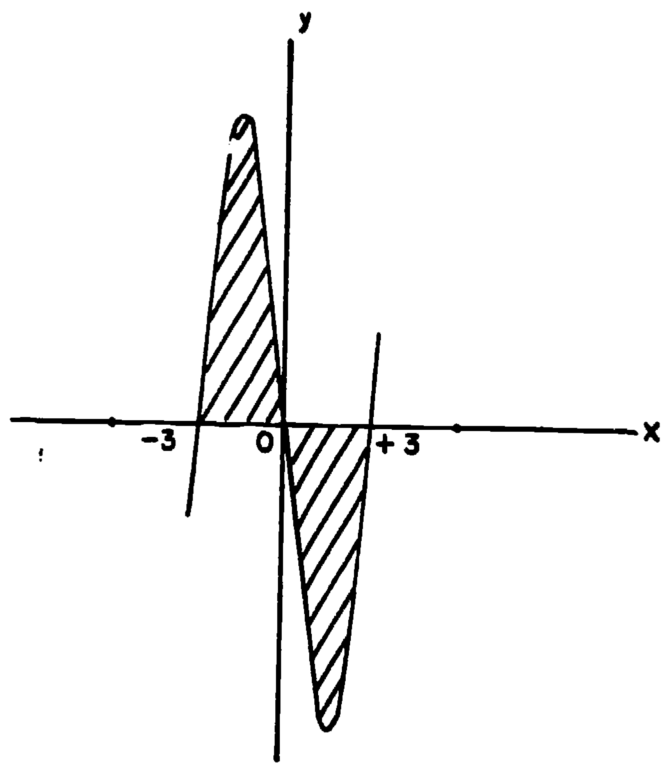


Figure 14-13.—Positive and negative value areas.

The area

$$A_2 = \int_{0}^{3} (x^3 - 9x)dx$$

$$= \frac{x^4}{4} - \frac{9}{2}x^2\Big|_{0}^{3}$$

$$= \frac{81}{4} - \frac{81}{2} - [0]$$

$$= -\frac{81}{4}$$

and

$$\left|-\frac{81}{4}\right| = \frac{81}{4}$$

Then

$$A_1 + A_2 = \frac{81}{4} + \frac{81}{4}$$

$$= \frac{162}{4}$$

$$= 40\frac{1}{2}$$

PROBLEMS:

1. Find, by integration, the area under the curve

$$y = x + 4$$

bounded by the x axis and the lines

$$x = 2$$

and

$$x = 7$$

Verify this by a geometric process.

2. Find the area under the curve

$$y = 3x^2 + 2$$

bounded by the x axis and the lines

$$x = 0$$

and

$$x = 2$$

3. Find the area between the curve

$$y = x^3 - 12x$$

and the x axis, from

$$x = -1$$

to

$$x = 3$$

ANSWERS:

1. $42 \frac{1}{2}$

2. 12

3. 39 1/4

CHAPTER 15

INTEGRATION FORMULAS

In this chapter several of the integration formulas and proofs are discussed and examples are given. Some of the formulas from the previous chapter are repeated because they are considered essential for the understanding of integration. The formulas in this chapter are basic and should not be considered a complete collection of integration formulas. Integration is so complex that tables of integrals have been published for use as reference sources.

In the following formulas and proofs, u, v, and w are considered functions of a single variable.

POWER OF A VARIABLE

The integral of a variable to a power is the variable to a power increased by one and divided by the new power.

Formula:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

Proof:

$$d\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{(n+1)x^{n+1-1}}{(n+1)}\,dx$$

$$= \frac{(n+1)x^n}{(n+1)}\,dx$$

$$= x^n\,dx$$

therefore

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

EXAMPLE: Evaluate

$$\int x^5\,dx$$

SOLUTION:

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C$$

$$= \frac{x^6}{6} + C$$

EXAMPLE: Evaluate

$$\int x^{-5}\,dx$$

SOLUTION:

$$\int x^{-5}\,dx = \frac{x^{-4}}{-4} + C$$

PRODUCT OF CONSTANT AND VARIABLE

When the variable is multiplied by a constant, the constant may be written either before or after the integral sign.

Formula:

$$\int a\,du = a\int du = au + C$$

Proof:

$$d(au + C) = a\,d\left(u + \frac{C}{a}\right)$$

$$= a\,du$$

therefore

$$\int a\,du = a\int du = au + C$$

EXAMPLE: Evaluate

$$\int 17\,dx$$

SOLUTION:

$$\int 17\,dx = 17 \int dx$$
$$= 17x + C$$

EXAMPLE: Evaluate

$$\int 3x^4\,dx$$

SOLUTION:

$$\int 3x^4\,dx = 3 \int x^4\,dx$$
$$= (3)\,\frac{x^5}{5} + C$$
$$= \frac{3x^5}{5} + C$$

SUMS

The integral of an algebraic sum of differentiable functions is the same as the algebraic sum of the integrals of these functions taken separately.

Formula:

$$\int (du + dv + dw) = \int du + \int dv + \int dw$$

Proof:

$$d(u + v + w + C) = du + dv + dw$$

therefore

$$\int du + \int dv + \int dw = u + C_1 + v + C_2 + w + C_3$$

where

$$C_1 + C_2 + C_3 = C$$

Then

$$\int du + \int dv + \int dw = u + v + w + C$$

and

$$\int (du + dv + dw) = \int du + \int dv + \int dw$$
$$= u + v + w + C$$

EXAMPLE: Evaluate

$$\int (3x^2 + x)\,dx$$

SOLUTION:

$$\int (3x^2 + x)\,dx = \int 3x^2\,dx + \int x\,dx$$
$$= x^3 + C_1 + \frac{x^2}{2} + C_2$$
$$= x^3 + \frac{x^2}{2} + C$$

EXAMPLE: Evaluate

$$\int (x^5 + x^{-3})\,dx$$

SOLUTION:

$$\int (x^5 + x^{-3})\,dx = \int x^5\,dx + \int x^{-3}\,dx$$
$$= \frac{x^6}{6} + C_1 + \frac{x^{-2}}{-2} + C_2$$
$$= \frac{x^6}{6} - \frac{x^{-2}}{2} + C$$
$$= \frac{x^6}{6} - \frac{1}{2x^2} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int x^6\,dx$
2. $\int x^{-4}\,dx$
3. $\int 17x^2\,dx$
4. $\int \pi r\,dr$
5. $\int 7x^{1/2}\,dx$
6. $\int (x^7 + x^6 + 3x^3)\,dx$
7. $\int (6 - x^3)\,dx$

ANSWERS:

1. $\frac{x^7}{7} + C$
2. $-\frac{1}{x^3} + C$

3. $\frac{17}{3}x^3 + C$

4. $\frac{\pi r^2}{2} + C$

5. $\frac{14}{3}x^{3/2} + C$

6. $\frac{x^8}{8} + \frac{x^7}{7} + \frac{3}{4}x^4 + C$

7. $6x - \frac{x^4}{4} + C$

POWER OF A FUNCTION OF x

The integral of a function raised to a power and multiplied by the derivative of that function is found by the following steps:

1. Increase the power of the function by 1.
2. Divide the result of step 1 by this increased power.
3. Add the constant of integration.

Formula:

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$$

Proof:

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)u^n}{n+1}\,du$$

$$= u^n\,du$$

therefore

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

NOTE: Recall that

$$\frac{d}{dx}\left[\frac{2}{3}(2x-3)^3\right] = (3)\left(\frac{2}{3}\right)(2x-3)^2$$

$$= 2(2x-3)^2$$

EXAMPLE: Evaluate

$$\int (2x-3)^2\,(2)\,dx$$

SOLUTION: Let

$$u = (2x-3)$$

and

$$du = 2\,dx$$

and

$$n = 2$$

Then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

$$= \frac{u^3}{3} + C$$

and by substitution

$$\int (2x-3)^2\,(2)dx = \frac{(2x-3)^3}{3} + C$$

When using this formula the integral must contain precisely du. If du is not present it must be placed in the integral and then compensation must be made.

EXAMPLE: Evaluate

$$\int (3x+5)^2\,dx$$

SOLUTION: Let

$$u = (3x+5)$$

and

$$du = 3\,dx$$

We find dx in the integral but not 3 dx. A 3 must be included in the integral in order to fulfill the requirements of du.

In words, this means the integral

$$\int (3x+5)^2\,dx$$

needs du in order that the formula may be used. Therefore, we write

$$\frac{3}{3}\int (3x+5)^2\,dx$$

and recalling that a constant may be carried across the integral sign, we write

$$\frac{3}{3}\int (3x+5)^2\,dx = \frac{1}{3}\int (3x+5)^2\,3\,dx$$

Notice that we needed 3 in the integral for du and we included 3 in the integral, then compensated for the 3 by multiplying the integral by 1/3.
Then

$$\frac{1}{3}\int (3x+5)^2\, 3\, dx = \left(\frac{1}{3}\right)\frac{(3x+5)^3}{3} + C$$

$$= \frac{1}{9}(3x+5)^3 + C$$

EXAMPLE: Evaluate

$$\int x(2+x^2)^2\, dx$$

SOLUTION: Let

$$u = (2+x^2)$$

and

$$du = 2x\, dx$$

Then

$$\int x(2+x^2)^2\, dx = \frac{2}{2}\int x(2+x^2)^2\, dx$$

$$= \frac{1}{2}\int 2x(2+x^2)^2\, dx$$

$$= \left(\frac{1}{2}\right)\frac{(2+x^2)^3}{3} + C$$

$$= \frac{(2+x^2)^3}{6} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int (x^2+6)(2x)\, dx$
2. $\int x^2(7+x^3)^2\, dx$
3. $\int (3x^2+2x)^2(6x+2)\, dx$
4. $\int (6x^3+2x)^{1/2}(9x^2+1)\, dx$
5. $\int (x^2+7)^{-2}\, x\, dx$

ANSWERS:

1. $\frac{(x^2+6)^2}{2} + C$
2. $\frac{(7+x^3)^3}{9} + C$
3. $\frac{(3x^2+2x)^3}{3} + C$
4. $\frac{(6x^3+2x)^{3/2}}{3} + C$
5. $\frac{1}{2(x^2+7)} + C$

QUOTIENT

In this section three methods of integrating quotients are discussed but only the second method will be proved.

The first method is to put the quotient into the form of the power of a function. The second method results in operations with logarithms. The third method is a special case in which the quotient must be simplified in order to use the sum rule.

METHOD 1

If we are given the integral

$$\int \frac{2x}{(9-4x^2)^{1/2}}\, dx$$

we observe that this integral may be written as

$$\int 2x(9-4x^2)^{-1/2}\, dx$$

and by letting

$$u = (9-4x^2)$$

and

$$du = -8x\, dx$$

the only requirement for this to fit the form

$$\int u^n\, du$$

is the factor for du of -4. We accomplish this by multiplying 2x dx by -4, giving -8x dx which is du. We then compensate for the factor -4 by multiplying the integral by - 1/4.

Then

$$\int \frac{2x}{(9 - 4x^2)^{1/2}}\, dx = \int 2x(9 - 4x^2)^{-1/2}\, dx$$

$$= -\frac{1}{4} \int (-4)(2x)(9 - 4x^2)^{-1/2}\, dx$$

$$= -\frac{1}{4} \int -8x(9 - 4x^2)^{-1/2}\, dx$$

$$= \left(-\frac{1}{4}\right) \frac{(9 - 4x^2)^{1/2}}{\frac{1}{2}} + C$$

$$= -\frac{(9 - 4x^2)^{1/2}}{2} + C$$

EXAMPLE: Evaluate

$$\int \frac{x}{(3 + x^2)^{1/2}}\, dx$$

SOLUTION: Write

$$\int \frac{x}{(3 + x^2)^{1/2}}\, dx = \int x(3 + x^2)^{-1/2}\, dx$$

Then, let

$$u = (3 + x^2)$$

and

$$du = 2x\, dx$$

The factor 2 is used in the integral to give du and is compensated for by multiplying the integral by 1/2.
Therefore

$$\int x(3 + x^2)^{-1/2}\, dx = \frac{1}{2} \int 2x(3 + x^2)^{-1/2}\, dx$$

$$= \left(\frac{1}{2}\right) \frac{(3 + x^2)^{1/2}}{\frac{1}{2}} + C$$

$$= (3 + x^2)^{1/2} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int \frac{x}{(2 + x^2)^{1/2}}\, dx$

2. $\int \frac{dx}{\sqrt[3]{3x + 1}}$

3. $\int \frac{dx}{(3x + 2)^5}$

ANSWERS:

1. $(2 + x^2)^{1/2} + C$

2. $\frac{(3x + 1)^{2/3}}{2} + C$

3. $\frac{-1}{12(3x + 2)^4} + C$

METHOD 2

In the previous examples, if the exponent of u was -1, that is

$$\int u^n\, du$$

where

$$n = -1$$

we would have applied the following.

Formula:

$$\int \frac{du}{u} = \ln u + C, \quad u > 0$$

Proof:

$$d(\ln u + C) = \frac{1}{u}\, du$$

therefore

$$\int \frac{du}{u} = \ln u + C$$

EXAMPLE: Evaluate the integral

$$\int \frac{1}{x}\, dx$$

SOLUTION: If we write

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

we find we are unable to evaluate

$$\int x^{-1}\,dx$$

by use of the power of a variable rule so we write

$$\int \frac{1}{x}\,dx = \ln x + C$$

because the 1 dx in the numerator is precisely du and we have fulfilled the requirements for

$$\int \frac{du}{u} = \ln u + C$$

EXAMPLE: Evaluate

$$\int \frac{2}{2x + 1}\,dx$$

SOLUTION: Let

$$u = 2x + 1$$

and

$$du = 2\,dx$$

then we have the form

$$\int \frac{du}{u} = \ln u + C$$

therefore

$$\int \frac{2}{2x + 1}\,dx = \ln(2x + 1) + C$$

EXAMPLE: Evaluate

$$\int \frac{2}{3x + 1}\,dx$$

SOLUTION: Let

$$u = 3x + 1$$

and

$$du = 3\,dx$$

We find we need 3 dx but we have 2 dx. Therefore, we need to change 2 dx to 3 dx. We do this by writing

$$\begin{aligned}\int \frac{2}{3x + 1}\,dx &= \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\int \frac{2}{3x + 1}\,dx \\ &= \frac{2}{3}\int \frac{3}{2}\,\frac{2}{3x + 1}\,dx \\ &= \frac{2}{3}\int \frac{3}{3x + 1}\,dx \\ &= \frac{2}{3}\ln(3x + 1) + C\end{aligned}$$

where the 2/3 is used to compensate for the 3/2 used in the integral.

Therefore

$$\int \frac{2}{3x + 1}\,dx = \frac{2}{3}\ln(3x + 1) + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int \frac{dx}{3x + 2}$
2. $\int \frac{dx}{5 - 2x}$
3. $\int \frac{x}{2 - 3x^2}\,dx$
4. $\int \frac{2x^3}{3 + 2x^4}\,dx$

ANSWERS:

1. $\frac{1}{3}\ln(3x + 2) + C$
2. $-\frac{1}{2}\ln(5 - 2x) + C$
3. $-\frac{1}{6}\ln(2 - 3x^2) + C$
4. $\frac{1}{4}\ln(3 + 2x^4) + C$

METHOD 3

In the third method that we will discuss, for solving integrals of quotients, we find that to integrate an algebraic function which has a numerator which is not of lower degree than the denominator we proceed as follows.

Change the integrand into a polynomial plus a fraction by dividing the denominator into the numerator. After this is accomplished, apply the rules available.

EXAMPLE: Evaluate

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx$$

SOLUTION: Divide the denominator into the numerator, then

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx = \int \left(8x - 6 - \frac{2}{2x + 1}\right) dx$$

$$= \int 8x\, dx - \int 6\, dx - \int \frac{2}{2x+1}\, dx$$

and, integrating each separately, we have

$$\int 8x\, dx = 4x^2 + C_1$$

and

$$-\int 6\, dx = -6x + C_2$$

and

$$-\int \frac{2}{2x + 1}\, dx = -\ln(2x + 1) + C_3$$

Then, by substitution, find that

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx = 4x^2 - 6x - \ln(2x + 1) + C$$

where

$$C = C_1 + C_2 + C_3$$

EXAMPLE: Evaluate

$$\int \frac{x}{x + 1}\, dx$$

SOLUTION: The numerator is not of lower degree than the denominator; therefore we divide and find that

$$\int \frac{x}{x + 1}\, dx = \int 1 - \frac{1}{x + 1}\, dx$$

$$= \int dx - \int \frac{1}{x + 1}\, dx$$

Integrating separately,

$$\int dx = x + C_1$$

and

$$-\int \frac{1}{x + 1}\, dx = -\ln(x + 1) + C_2$$

therefore

$$\int \frac{x}{x + 1}\, dx = x - \ln(x + 1) + C$$

where

$$C = C_1 + C_2$$

PROBLEMS: Evaluate the following integrals:

1. $\int \frac{2x^2 + 6x + 5}{x + 1}\, dx$

2. $\int \frac{3x - 8}{x}\, dx$

3. $\int \frac{6x^3 + 13x^2 + 20x + 23}{2x + 3}\, dx$

4. $\int \frac{21x + 16x + 4}{3x + 1}\, dx$

ANSWERS:

1. $x^2 + 4x + \ln(x + 1) + C$

2. $3x - 8\ln(x) + C$

3. $x^3 + x^2 + 7x + \ln(2x + 3) + C$

4. $\frac{7}{2}x^2 + 3x + \frac{1}{3}\ln(3x + 1) + C$

CONSTANT TO A VARIABLE POWER

In this section a discussion of two forms of a constant to a variable power is presented. The two forms are a^u and e^u where u is the variable and a and e are the constants.

Formula:

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

Proof:

$$d(a^u + C_1) = a^u \ln a\, du$$

then

$$\int a^u \ln a\, du = a^u + C_1$$

but ln a is a constant, then

$$\int a^u \ln a\, du = \ln a \int a^u\, du$$

and

$$\ln a \int a^u\, du = a^u + C_1$$

Then, by dividing both sides by ln a, we have

$$\frac{\ln a}{\ln a} \int a^u\, du = \frac{a^u}{\ln a} + \frac{C_1}{\ln a}$$

and letting

$$C = \frac{C_1}{\ln a}$$

we have

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

EXAMPLE: Evaluate

$$\int 3^x\, dx$$

SOLUTION: Let

$$u = x$$

and

$$du = 1\, dx$$

therefore, by knowing that

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

and using substitution, we find that

$$\int 3^x\, dx = \frac{3^x}{\ln 3} + C$$

EXAMPLE: Evaluate

$$\int 3^{2x}\, dx$$

SOLUTION: Let

$$u = 2x$$

and

$$du = 2\, dx$$

The integral should contain a factor of 2 in order that

$$du = 2dx$$

Thus we add a factor of 2 in the integral and compensate by multiplying the integral by 1/2.

Then

$$\int 3^{2x}\, dx = \frac{1}{2} \int (2)\, 3^{2x}\, dx$$

$$= \frac{1}{2} \int 3^{2x}\, 2\, dx$$

therefore

$$\frac{1}{2} \int 3^{2x}\, 2\, dx = \left(\frac{1}{2}\right) \frac{3^{2x}}{\ln 3} + C$$

$$= \frac{3^{2x}}{2 \ln 3} + C$$

EXAMPLE: Evaluate

$$\int 7xb^{x^2}\, dx$$

SOLUTION: Let

$$u = x^2$$

and

$$du = 2x\, dx$$

In order to use

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

the integral must be in the form of

$$\int b^{x^2}\, 2x\, dx$$

but we have

$$\int b^{x^2}\, 7x\, dx$$

therefore we remove the 7 and insert a 2 by writing

$$\int 7xb^{x^2}\,dx = \int \left(\frac{7}{2}\right)\left(\frac{2}{7}\right) 7xb^{x^2}\,dx$$

$$= \frac{7}{2}\int \frac{2}{7}\, 7xb^{x^2}\,dx$$

$$= \frac{7}{2}\int 2xb^{x^2}\,dx$$

$$= \frac{7}{2}\int b^{x^2}\, 2x\,dx$$

$$= \frac{7}{2}\frac{b^{x^2}}{\ln b} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int 10^{2x}\,dx$
2. $\int 7^{3x}\,dx$
3. $\int 9^{x^2}\,x\,dx$
4. $\int 2^{(3x^2+1)}\,x\,dx$

ANSWERS:

1. $\dfrac{10^{2x}}{2 \ln 10} + C$
2. $\dfrac{7^{3x}}{3 \ln 7} + C$
3. $\dfrac{9^{x^2}}{2 \ln 9} + C$
4. $\dfrac{2^{(3x^2+1)}}{6 \ln 2} + C$

We will now discuss the second form of the integral of a constant to a variable power.
Formula:

$$\int e^u\,du = e^u + C$$

Proof:

$$d(e^u + C) = e^u\,du$$

therefore

$$\int e^u\,du = e^u + C$$

EXAMPLE: Evaluate

$$\int e^x\,dx$$

SOLUTION: Let

$$u = x$$

and

$$du = 1\,dx$$

The integral is in the correct form to use;

$$\int e^u\,du = e^u + C$$

therefore, using substitution, we find

$$\int e^x\,dx = e^x + C$$

EXAMPLE: Evaluate

$$\int e^{2x}\,dx$$

SOLUTION: Let

$$u = 2x$$

and

$$du = 2\,dx$$

We need a factor of 2 in the integral and write

$$\int e^{2x}\,dx = \frac{2}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2}\int e^{2x}\,2dx$$

$$= \frac{1}{2}e^{2x} + C$$

EXAMPLE: Evaluate

$$\int xe^{2x^2}\,dx$$

SOLUTION: Let

$$u = 2x^2$$

and

$$du = 4x\,dx$$

Here a factor of 4 is needed in the integral, therefore

$$\int xe^{2x^2}\,dx = \int \frac{4}{4} xe^{2x^2}\,dx$$

$$= \frac{1}{4}\int 4xe^{2x^2}\,dx$$

$$= \frac{1}{4} e^{2x^2} + C$$

EXAMPLE: Evaluate

$$\int \frac{x^2}{e^{x^3}}\,dx$$

SOLUTION: Write the integral

$$\int \frac{x^2}{e^{x^3}}\,dx = \int x^2 e^{-x^3}\,dx$$

and let

$$u = -x^3$$

and

$$du = -3x^2\,dx$$

therefore

$$\int x^2 e^{-x^3}\,dx = -\frac{1}{3}\int -3x^2 e^{-x^3}\,dx$$

$$= -\frac{1}{3} e^{-x^3} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int -2xe^{-x^2}\,dx$
2. $\int e^{4x}\,dx$
3. $\int e^{(2x - 1)}\,dx$
4. $\int \frac{2x}{e^{x^2}}\,dx$

ANSWERS:

1. $e^{-x^2} + C$
2. $\frac{1}{4} e^{4x} + C$
3. $\frac{1}{2} e^{(2x - 1)} + C$
4. $-e^{-x^2} + C$

TRIGONOMETRIC FUNCTIONS

Trigonometric functions, which comprise one group of transcendental functions, may be differentiated and integrated in the same fashion as the other functions. We will limit our proofs to the sine, cosine, and secant functions, but will list several others.

Formula:

$$\int \sin u\,du = -\cos u + C$$

Proof:

$$d(\cos u + C) = -\sin u\,du$$

and

$$d(-\cos u + C) = \sin u\,du$$

therefore

$$\int \sin u\,du = -\cos u + C$$

Formula:

$$\int \cos u\,du = \sin u + C$$

Proof:

$$d(\sin u + C) = \cos u\,du$$

therefore

$$\int \cos u\,du = \sin u + C$$

Formula:

$$\int \sec^2 u\,du = \tan u + C$$

Proof:

$$d(\tan u + C) = d\left(\frac{\sin u}{\cos u} + C\right)$$

and by the quotient rule

$$d\left(\frac{\sin u}{\cos u}\right) + C = \frac{\cos u(\cos u) - \sin u(-\sin u)}{\cos^2 u} \, du$$

$$= \left(\frac{\cos^2 u + \sin^2 u}{\cos^2 u}\right) du$$

$$= \left(\frac{1}{\cos^2 u}\right) du$$

$$= \sec^2 u \, du$$

therefore

$$\int \sec^2 u \, du = \tan u + C$$

To this point we have considered integrals of trigonometric functions which result in functions of the sine, cosine, and tangent. Those integrals which result in functions of the cotangent, secant, and cosecant are included in the following list of elementary integrals.

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

EXAMPLE: Evaluate

$$\int \sin 3x \, dx$$

SOLUTION: We need the integral in the form of

$$\int \sin u \, du = -\cos u + C$$

therefore, we let

$$u = 3x$$

and

$$du = 3 \, dx$$

but we do not have 3 dx. Therefore, we multiply the integral by 3/3 and rearrange as follows:

$$\int \sin 3x \, dx = \frac{3}{3} \int \sin 3x \, dx$$

$$= \frac{1}{3} \int \sin 3x \; 3 \, dx$$

then

$$\frac{1}{3} \int \sin 3x \; 3 \, dx = \frac{1}{3} (-\cos 3x) + C$$

$$= -\frac{1}{3} \cos 3x + C$$

EXAMPLE: Evaluate

$$\int \cos(2x + 4) \, dx$$

SOLUTION: Let

$$u = (2x + 4)$$

and

$$du = 2 \, dx$$

Therefore,

$$\int \cos(2x + 4) \, dx = \frac{2}{2} \int \cos(2x + 4) \, dx$$

$$= \frac{1}{2} \int \cos(2x + 4) \; 2 \, dx$$

$$= \frac{1}{2} \sin(2x + 4) + C$$

EXAMPLE: Evaluate

$$\int (3 \sin 2x + 4 \cos 3x) \, dx$$

SOLUTION: We use the rule for sums and write

$$\int (3 \sin 2x + 4 \cos 3x) \, dx$$
$$= \int 3 \sin 2x \, dx + \int 4 \cos 3x \, dx$$

Then, in the integral

$$\int 3 \sin 2x \, dx$$

let

$$u = 2x$$

and

$$du = 2\ dx$$

but we have

$$3\ dx$$

To change 3 dx to 2 dx we divide by 3 and multiply by 2, with proper compensation, as follows:

$$\int 3 \sin 2x\ dx = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right) \int 3 \sin 2x\ dx$$

$$= \frac{3}{2} \int \frac{2}{3} (3 \sin 2x\ dx)$$

$$= \frac{3}{2} \int 2 \sin 2x\ dx$$

$$= \frac{3}{2} (-\cos 2x) + C_1$$

$$= -\frac{3}{2} \cos 2x + C_1$$

The second integral

$$\int 4 \cos 3x\ dx$$

with

$$u = 3x$$

and

$$du = 3\ dx$$

is evaluated as follows:

$$\int 4 \cos 3x\ dx = \left(\frac{4}{3}\right)\left(\frac{3}{4}\right) \int 4 \cos 3x\ dx$$

$$= \frac{4}{3} \int 3 \cos 3x\ dx$$

$$= \frac{4}{3} (\sin 3x) + C_2$$

Then, by combining the two solutions, we have

$$\int (3 \sin 2x + 4 \cos 3x)\ dx$$

$$= -\frac{3}{2} \cos 2x + C_1 + \frac{4}{3} \sin 3x + C_2$$

$$= -\frac{3}{2} \cos 2x + \frac{4}{3} \sin 3x + C$$

where

$$C_1 + C_2 = C$$

EXAMPLE: Evaluate

$$\int \sec^2 3x\ dx$$

SOLUTION: Let

$$u = 3x$$

and

$$du = 3\ dx$$

We need 3 dx so we write

$$\int \sec^2 3x\ dx = \frac{3}{3} \int \sec^2 3x\ dx$$

$$= \frac{1}{3} \int \sec^2 3x\ 3\ dx$$

$$= \frac{1}{3} (\tan 3x) + C$$

EXAMPLE: Evaluate

$$\int \csc 2x \cot 2x\ dx$$

SOLUTION: Let

$$u = 2x$$

and

$$du = 2\ dx$$

We require du equal to 2 dx so we write

$$\int \csc 2x \cot 2x\ dx = \frac{2}{2} \int \csc 2x \cot 2x\ dx$$

$$= \frac{1}{2} \int 2 \csc 2x \cot 2x\ dx$$

$$= -\frac{1}{2} \csc 2x + C$$

EXAMPLE: Evaluate

$$\int \csc^2 3x\ dx$$

SOLUTION: Let

$$u = 3x$$

and

$$du = 3\ dx$$

then

$$\int \csc^2 3x\ dx = \frac{1}{3}\int 3 \csc^2 3x\ dx$$

$$= -\frac{1}{3}\cot 3x + C$$

EXAMPLE: Evaluate

$$\int \sec \frac{x}{2} \tan \frac{x}{2}\ dx$$

SOLUTION: Let

$$u = \frac{x}{2}$$

and

$$du = \frac{1}{2}\ dx$$

then

$$\int \sec \frac{x}{2} \tan \frac{x}{2}\ dx = 2\int \frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2}\ dx$$

$$= 2 \sec \frac{x}{2} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int \cos 4x\ dx$
2. $\int \sin 5x\ dx$
3. $\int \sec^2 6x\ dx$
4. $\int 3 \cos(6x + 2)\ dx$
5. $\int x \sin (2x^2)\ dx$
6. $\int 2 \csc^2 5x\ dx$
7. $\int 3 \sec \frac{x}{3} \tan \frac{x}{3}\ dx$

ANSWERS:

1. $\frac{1}{4} \sin 4x + C$
2. $-\frac{1}{5} \cos 5x + C$
3. $\frac{1}{6} \tan 6x + C$
4. $\frac{1}{2} \sin(6x + 2) + C$
5. $-\frac{1}{4} \cos(2x^2) + C$
6. $-\frac{2}{5} \cot 5x + C$
7. $9 \sec \frac{x}{3} + C$

TRIGONOMETRIC FUNCTIONS OF THE FORM $\int u^n\ du$

The integrals of powers of trigonometric functions will be limited to those which may, by substitution, be written in the form

$$\int u^n\ du$$

EXAMPLE: Evaluate

$$\int \sin^4 x \cos x\ dx$$

SOLUTION: Let

$$u = \sin x$$

and

$$du = \cos x\ dx$$

By substitution

$$\int \sin^4 x \cos x\ dx = \int u^4\ du$$

$$= \frac{u^5}{5} + C$$

Then, by substitution again, find that

$$\frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C$$

therefore

$$\int \sin^4 x \cos x\ dx = \frac{\sin^5 x}{5} + C$$

EXAMPLE: Evaluate

$$\int \cos^3 x(-\sin x)\ dx$$

SOLUTION: Let

$$u = \cos x$$

and

$$du = -\sin x \, dx$$

Write

$$\int u^3 \, du = \frac{u^4}{4} + C$$

and by substitution

$$\frac{u^4}{4} + C = \frac{\cos^4 x}{4} + C$$

PROBLEMS: Evaluate the following integrals:

1. $\int \sin^2 x \cos x \, dx$
2. $\int \sin^4 x \cos x \, dx$
3. $\int 2 \sin x \cos x \, dx$
4. $\int \frac{\cos 2x}{\sin^3 2x} \, dx$
5. $\int \cos^3 x \sin x \, dx$
6. $\int \sin x \cos x (\sin x + \cos x) \, dx$

ANSWERS:

1. $\frac{1}{3} \sin^3 x + C$
2. $\frac{1}{5} \sin^5 x + C$
3. $\sin^2 x + C$
4. $\frac{-1}{4 \sin^2 2x} + C$
5. $-\frac{1}{4} \cos^4 x + C$
6. $\frac{\sin^3 x - \cos^3 x}{3} + C$

CHAPTER 16

COMBINATIONS AND PERMUTATIONS

This chapter deals with concepts required for the study of probability and statistics. Statistics is a branch of science which is an outgrowth of the theory of probability. Combinations and permutations are used in both statistics and probability, and they, in turn, involve operations with factorial notation. Therefore, combinations, permutations, and factorial notation are discussed in this chapter.

DEFINITIONS

A combination is defined as a possible selection of a certain number of objects taken from a group with no regard given to order. For instance, suppose we were to choose two letters from a group of three letters. If the group of three letters were A, B, and C, we could choose the letters in combinations of two as follows:

AB, AC, BC

The order in which we wrote the letters is of no concern. That is, AB could be written BA but we would still have only one combination of the letters A and B.

If order were considered, we would refer to the letters as permutations and make a distinction between AB and BA. The permutations of two letters from the group of three letters would be as follows:

AB, AC, BC, BA, CA, CB

The symbol used to indicate the foregoing combination will be ${}_3C_2$, meaning a group of three objects taken two at a time. For the previous permutation we will use ${}_3P_2$, meaning a group of three objects taken two at a time and ordered.

An understanding of factorial notation is required prior to a detailed discussion of combinations and permutations. We define the product of the integers 1 through n as n factorial and use the symbol n! to denote this. That is,

$$3! = 1 \cdot 2 \cdot 3$$

$$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$$

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

EXAMPLE: Find the value of 5!

SOLUTION: Write

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$

$$= 120$$

EXAMPLE: Find the value of

$$\frac{5!}{3!}$$

SOLUTION: Write

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and

$$3! = 3 \cdot 2 \cdot 1$$

then

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$

and by simplification

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4$$

$$= 20$$

The previous example could have been solved by writing

$$\frac{5!}{3!} = \frac{3!\,4\cdot 5}{3!}$$
$$= 5\cdot 4$$

Notice that we wrote

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$$

and combined the factors

$$3\cdot 2\cdot 1$$

as

$$3!$$

then

$$5! = 3!\,4\cdot 5$$

EXAMPLE: Find the value of

$$\frac{6! - 4!}{4!}$$

SOLUTION: Write

$$6! = 4!\,5\cdot 6$$

and

$$4! = 4!\,1$$

then

$$\frac{6! - 4!}{4!} = \frac{4!\,(5\cdot 6 - 1)}{4!}$$
$$= (5\cdot 6 - 1)$$
$$= 29$$

Notice that 4! was factored from the expression

$$6! - 4!$$

THEOREM

If n and r are positive integers, with n greater than r, then

$$n! = r!\,(r + 1)(r + 2)\cdots n$$

This theorem allows us to simplify an expression as follows:

$$5! = 4!\,5$$
$$= 3!\,4\cdot 5$$
$$= 2!\,3\cdot 4\cdot 5$$
$$= 1\cdot 2\cdot 3\cdot 4\cdot 5$$

Another example is

$$(n + 2)! = (n + 1)!\,(n + 2)$$
$$= n!\,(n + 1)(n + 2)$$
$$= (n - 1)!\,n(n + 1)(n + 2)$$

EXAMPLE: Simplify

$$\frac{(n + 3)!}{n!}$$

SOLUTION: Write

$$(n + 3)! = n!\,(n + 1)(n + 2)(n + 3)$$

then

$$\frac{(n + 3)!}{n!} = \frac{n!\,(n + 1)(n + 2)(n + 3)}{n!}$$
$$= (n + 1)(n + 2)(n + 3)$$

PROBLEMS: Find the value of problems 1-4 and simplify problems 5 and 6.

1. $6!$
2. $3!\,4!$
3. $\frac{8!}{11!}$
4. $\frac{5! - 3!}{3!}$
5. $\frac{n!}{(n - 1)!}$
6. $\frac{(n + 2)!}{n!}$

ANSWERS:

1. 720
2. 144

3. $\frac{1}{990}$
4. 19
5. n
6. (n + 1) (n + 2)

COMBINATIONS

As indicated previously, a combination is the selection of a certain number of objects taken from a group of objects without regard to order. We use the symbol $_5C_3$ to indicate that we have five objects taken three at a time, without regard to order. Using the letters A, B, C, D, and E, to designate the five objects, we list the combinations as follows:

ABC ABD ABE ACD ACE

ADE BCD BCE BDE CDE

We find there are ten combinations of five objects taken three at a time. We made the selection of three objects, as shown, but we called these selections combinations. The word combinations infers that order is not considered.

EXAMPLE: Suppose we wish to know how many color combinations can be made from four different colored marbles, if we use only three marbles at a time. The marbles are colored red, green, white, and blue.

SOLUTION: We let the first letter in each word indicate the color, then we list the possible combinations as follows:

RGW RGY RWY GWY

If we rearrange the first group, RGW, to form GWR or RWG we still have the same color combination; therefore order is not important.

The previous examples are completely within our capabilities, but suppose we have 20 boys and wish to know how many different basket ball teams we could form, one at a time, from these boys. Our listing would be quite lengthy and we would have a difficult task to determine that we had all of the possible combinations. In fact, there would be over 15,000 combinations we would have to list. This indicates the need for a formula for combinations.

FORMULA

The general formula for possible combinations of r objects from a group of n objects is

$$_nC_r = \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdots r}$$

The denominator may be written as

$$1\cdot 2\cdot 3\ldots r = r!$$

and if we multiply both numerator and denominator by

$$(n-r)\cdots 2\cdot 1$$

which is

$$(n-r)!$$

we have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2.1}{r!\,(n-r)\cdots 2\cdot 1}$$

The numerator

$$n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1$$

is

$$n!$$

Then

$$_nC_r = \frac{n!}{r!\,(n-r)!}$$

This formula is read: The number of combinations of n objects taken r at a time is equal to n factorial divided by r factorial times n minus r factorial.

EXAMPLE: In the previous problem where 20 boys were available, how many different basketball teams could be formed?

SOLUTION: If the choice of which boy played center, guard, or forward is not considered, we find we desire the number of combinations of 20 boys taken five at a time and write

$$_nC_r = \frac{n!}{r!\,(n-r)!}$$

where

$$n = 20$$

and

$$r = 5$$

Then, by substitution we have

$$_nC_r = {}_{20}C_5 = \frac{20!}{5!\,(20-5)!}$$

$$= \frac{20!}{5!\,15!}$$

$$= \frac{15!\,16\cdot 17\cdot 18\cdot 19\cdot 20}{15!\,5!}$$

$$= \frac{16\cdot 17\cdot 18\cdot 19\cdot 20}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{16\cdot 17\cdot 3\cdot 19\cdot 1}{1}$$

$$= 15{,}504$$

EXAMPLE: A man has, in his pocket, a silver dollar, a half-dollar, a quarter, a dime, a nickel, and a penny. If he reaches into his pocket and pulls out three coins, how many different sums may he have?

SOLUTION: The order in not important, therefore the number of combinations of coins possible is

$$_6C_3 = \frac{6!}{3!\,(6-3)!}$$

$$= \frac{6!}{3!\,3!}$$

$$= \frac{3!\,4\cdot 5\cdot 6}{3!\,3!}$$

$$= \frac{4\cdot 5\cdot 6}{3\cdot 2\cdot 1}$$

$$= \frac{4\cdot 5}{1}$$

$$= 20$$

EXAMPLE: Find the value of

$$_3C_3$$

SOLUTION: We use the formula given and find that

$$_3C_3 = \frac{3!}{3!\,(3-3)!}$$

$$= \frac{3!}{3!\,0!}$$

This seems to violate the rule, "division by zero is not allowed," but we define 0! as equal 1. Then

$$\frac{3!}{3!\,0!} = \frac{3!}{3!} = 1$$

which is obvious if we list the combinations of three things taken three at a time.

PROBLEMS: Find the value of problems 1-6 and solve problems 7, 8, and 9.

1. $_6C_2$
2. $_6C_4$
3. $_{15}C_5$
4. $_7C_7$
5. $\dfrac{_6C_3 + {}_7C_3}{_{13}C_6}$
6. $\dfrac{(_7C_3)\,(_6C_3)}{_{14}C_4}$
7. We want to paint three rooms in a house, each a different color and we may choose from seven different colors of paint. How many color combinations are possible, for the three rooms?
8. If 20 boys go out for the football team, how many different teams may be formed, one at a time?
9. Two boys and their dates go to the drive-in and each wants a different flavor ice cream cone. The drive-in has 24 flavors of ice cream. How many combinations of flavors may they choose?

ANSWERS:

1. 15
2. 15
3. 3,003
4. 1
5. $\frac{5}{156}$
6. $\frac{100}{143}$

7. 35

8. 167,960

9. 10, 626

PRINCIPLE OF CHOICE

The principle of choice is discussed in relation to combinations although it is also, later in this chapter, discussed in relation to permutations. It is stated as follows:

If a selection can be made in n_1 ways, and after this selection is made, a second selection can be made in n_2 ways, and after this selection is made, a third selection can be made in n_3 ways, and so forth for r ways, then the r selections can be made together in

$$n_1 \cdot n_2 \cdot n_3 \cdots\cdot n_r \text{ ways}$$

EXAMPLE: In how many ways can a coach choose first a football team and then a basketball team if 18 boys go out for either team?

SOLUTION: First let the coach choose a football team. That is

$$\begin{aligned} {}_{18}C_{11} &= \frac{18!}{11!\,(18-11)!} \\ &= \frac{18!}{11!\,7!} \\ &= \frac{11!\ 12\cdot13\cdot14\cdot15\cdot16\cdot17\cdot18}{11!\ 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} \\ &= 31,824 \end{aligned}$$

The coach now must choose a basketball team from the remaining seven boys. That is

$$\begin{aligned} {}_7C_5 &= \frac{7!}{5!\,(7-5)!} \\ &= \frac{7!}{5!\,2!} \\ &= \frac{5!\ 6\cdot7}{5!\ 2!} \\ &= \frac{6\cdot7}{2} \\ &= 21 \end{aligned}$$

Then, together, the two teams can be chosen in

$$(31,824)(21) = 668,304 \text{ ways}$$

EXAMPLE: A man ordering dinner has a choice of one meat dish from four, a choice of three vegetables from seven, one salad from three, and one dessert from four. How many different menus are possible?

SOLUTION: The individual combinations are as follows:

meat. . . . . $_4C_1$

vegetable. . . . . . $_7C_4$

salad. . . . . $_3C_1$

dessert . . . . $_4C_1$

The value of

$$\begin{aligned} {}_4C_1 &= \frac{4!}{1!\,(4-1)!} \\ &= \frac{4!}{3!} \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} {}_7C_4 &= \frac{7!}{4!\,(7-4)!} \\ &= \frac{7!}{4!\,3!} \\ &= \frac{5\cdot6\cdot7}{2\cdot3} \\ &= 35 \end{aligned}$$

and

$$\begin{aligned} {}_3C_1 &= \frac{3!}{1!\,(3-1)!} \\ &= \frac{3!}{2!} \\ &= 3 \end{aligned}$$

therefore, there are

$$(4)(35)(3)(4) = 1680$$

different menus available to the man.

PROBLEMS: Solve the following problems.

1. A man has 12 different colored shirts and 20 different ties. How many shirt and tie combinations can he select to take on a trip, if he takes three shirts and five ties?

2. A petty officer, in charge of posting the watch, has in the duty section 12 men. He must post three different fire watches, then post four aircraft guards on different aircraft. How many different assignments of men can he make?

3. If there are 10 third class and 14 second class petty officers in a division which must furnish two second class and six third class petty officers for shore patrol, how many different shore patrol parties can be made?

ANSWERS:

1. 3,410,880
2. 27,720
3. 19,110

PERMUTATIONS

Permutations are similar to combinations but extend the requirements of combinations by considering order.

Suppose we have two letters, A and B, and wish to know how many arrangements of these letters can be made. It is obvious that the answer is two. That is

AB and BA

If we extend this to the three letters A, B, and C, we find the answer to be

ABC, ACB, BAC, BCA, CAB, CBA

We had three choices for the first letter, and after we chose the first letter, we had only two choices for the second letter, and after the second letter, we had only one choice. This is shown in the "tree" diagram in figure 16-1. Notice that there is a total of six different paths to the ends of the "branches" of the "tree" diagram.

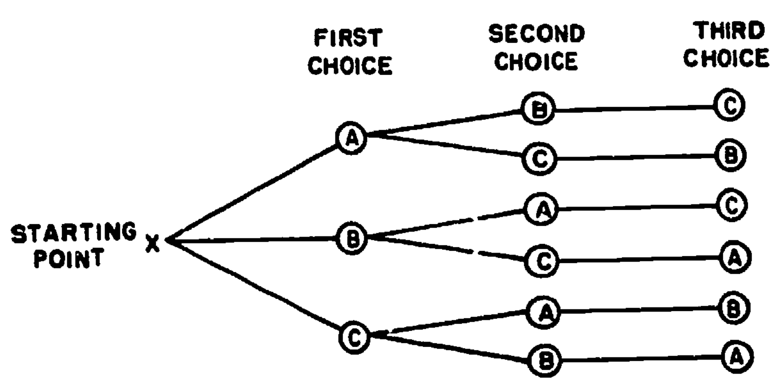


Figure 16-1.—"Tree" diagram.

If the number of objects is large, the tree would become very complicated; therefore, we approach the problem in another manner, using parentheses to show the possible choices. Suppose we were to arrange five objects in as many different orders as possible. We have for the first choice six objects.

$$(6)\ (\ )\ (\ )\ (\ )\ (\ )\ (\ )$$

For the second choice we have only five choices.

$$(6)\ (5)\ (\ )\ (\ )\ (\ )\ (\ )$$

For the third choice we have only four choices.

$$(6)\ (5)\ (4)\ (\ )\ (\ )\ (\ )$$

This may be continued as follows:

$$(6)\ (5)\ (4)\ (3)\ (2)\ (1)$$

By applying the principle of choice we find the total possible ways of arranging the objects to be the product of the individual choices. That is

$$6.5.4.3.2.1$$

and this may be written as

$$6!$$

This leads to the statement: The number of permutations of n objects, taken all together, is equal to n!.

EXAMPLE: How many permutations of seven different letters may be made?

SOLUTION: We could use the "tree" but this would become complicated. (Try it to see why.) We could use the parentheses as follows:

$$(7)\ (6)\ (5)\ (4)\ (3)\ (2)\ (1) = 5040$$

The easiest solution is to use the previous statement and write

$${}_7P_7 = 7!$$

That is, the number of possible arrangements of seven objects, taken seven at a time, is 7!. NOTE: Compare this with the number of COMBINATIONS of seven objects, taken seven at a time.

If we are faced with finding the number of permutations of seven objects taken three at a time, we use three parentheses as follows:

In the first position we have a choice of seven objects.

$$(7)(\ )(\ )$$

In the second position we have a choice of six objects

$$(7)(6)(\ )$$

In the last position we have a choice of five objects,

$$(7)(6)(5)$$

and by principle of choice, the solution is

$$7 \cdot 6 \cdot 5 = 210$$

FORMULA

At this point we will use our knowledge of combinations to develop a formula for the number of permutations of n objects taken r at a time.

Suppose we wish to find the number of permutations of five things taken three at a time. We first determine the number of groups of three, as follows:

$$_5C_3 = \frac{5!}{3!\,(5-3)!}$$

$$= \frac{5!}{3!\,2!}$$

$$= 10$$

Thus, there are 10 groups of three objects each.

We are now asked to arrange each of these ten groups in as many orders as possible. We know that the number of permutations of three objects, taken together, is 3!. We may arrange each of the 10 groups in 3! or six ways. The total number of possible permutations of $_5C_3$ then is

$$_5C_3 \cdot 3! = 10 \cdot 6$$

which is written

$$_5C_3 \cdot 3! = {_5P_3}$$

Put into the general form, then

$$_nC_r \cdot r! = {_nP_r}$$

and knowing that

$$_nC_r = \frac{n!}{r!\,(n-r)!}$$

then

$$_nC_r \cdot r! = \frac{n!}{r!\,(n-r)!} \cdot r!$$

$$= \frac{n!}{(n-r)!}$$

but

$$_nC_r \cdot r! = {_nP_r}$$

therefore

$$_nP_r = \frac{n!}{(n-r)!}$$

EXAMPLE: How many permutations of six objects, taken two at a time, can be made?

SOLUTION: The number of permutations of six objects, taken two at a time, is written

$$_6P_2 = \frac{6!}{(6-2)!}$$

$$= \frac{6!}{4!}$$

$$= \frac{4!\,5 \cdot 6}{4!}$$

$$= 5 \cdot 6$$

$$= 30$$

EXAMPLE: In how many ways can eight people be arranged in a row?

SOLUTION: All eight people must be in the row; therefore, we want the number of permutations of eight people, taken eight at a time, which is

$$_8P_8 = \frac{8!}{(8-8)!}$$

$$= \frac{8!}{0!}$$

(Remember that 0! was defined as equal to 1) then

$$\frac{8!}{0!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1}$$

$$= 40,320$$

Problems dealing with combinations and permutations often require the use of both formulas to solve one problem.

EXAMPLE: There are eight first class and six second class petty officers on the board of the fifty-six club. In how many ways can they elect, from the board, a president, a vice-president, a secretary, and a treasurer if the president and secretary must be first class petty officers and the vice-president and treasurer must be second class petty officers?

SOLUTION: Since two of the eight first class petty officers are to fill two different offices, we write

$$\begin{aligned} {}_8P_2 &= \frac{8!}{(8-2)!} \\ &= \frac{8!}{6!} \\ &= 7 \cdot 8 \\ &= 56 \end{aligned}$$

Then, two of the six second class petty officers are to fill two different offices; thus we write

$$\begin{aligned} {}_6P_2 &= \frac{6!}{(6-2)!} \\ &= \frac{6!}{4!} \\ &= 5 \cdot 6 \\ &= 30 \end{aligned}$$

The principle of choice holds in this case; therefore, there are

$$56 \cdot 30 = 1680$$

ways to select the required office holders. The problem, thus far, is a permutation problem, but suppose we are asked the following: In how many ways can they elect the office holders from the board, if two of the office holders must be first class petty officers and two of the office holders must be second class petty officers?

SOLUTION: We have already determined how many ways eight things may be taken two at a time and how many ways six may be taken two at a time, and also, how many ways they may be taken together. That is

$${}_8P_2 = 50$$

and

$${}_6P_2 = 30$$

then

$${}_8P_2 \cdot {}_6P_2 = 1680$$

Our problem now is to find how many ways we can combine the four offices, two at a time. Therefore, we write

$$\begin{aligned} {}_4C_2 &= \frac{4!}{2!\,(4-2)!} \\ &= \frac{4!}{2!\,2!} \\ &= \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2} \\ &= 6 \end{aligned}$$

Then, in answer to the problem, we write

$${}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2 = 10,080$$

In words, if there are ${}_4C_2$ ways of combining the four offices, and then, for every one of these ways there are ${}_8P_2 \cdot {}_6P_2$ ways of arranging the office holders, then there are

$${}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2$$

ways of electing the petty officers.

PROBLEMS: Find the answers to the following.

1. ${}_6P_3$
2. ${}_4P_3$
3. ${}_7P_2 \cdot {}_5P_2$
4. In how many ways can six people be seated in a row?
5. There are seven boys and nine girls in a club. In how many ways can they elect four different officers designated by A, B, C, and D if:
   (a) A and B must be boys and C and D must be girls?
   (b) two of the officers must be boys and two of the officers must be girls?

ANSWERS:

1. 120
2. 24
3. 840
4. 720
5. (a) 3,024
   (b) 18,144

In the question "How many different arrangements of the letters in the word STOP can be made?" were asked, we would write

$$_4P_4 = \frac{4!}{(4-4)!}$$

$$= \frac{4!}{0!}$$

$$= 24$$

We would be correct since all letters are different. If some of the letters were the same, we would reason as given in the following problem.

EXAMPLE: How many different arrangements of the letters in the word ROOM can be made?

SOLUTION: We have two letters alike. If we list the possible arrangements, using subscripts to make a distinction between the O's, we have

$R\,O_1O_2M$	$O_1O_2M\,R$	$O_1M\,O_2R$	$M\,O_1O_2R$
$R\,O_2O_1M$	$O_2O_1M\,R$	$O_2M\,O_1R$	$M\,O_2O_1R$
$R\,O_1M\,O_2$	$O_1O_2R\,M$	$O_1R\,M\,O_2$	$M\,O_1R\,O_2$
$R\,O_2M\,O_1$	$O_2O_1R\,M$	$O_2R\,M\,O_1$	$M\,O_2R\,O_1$
$R\,M\,O_1O_2$	$O_1M\,R\,O_2$	$O_1R\,O_2M$	$M\,R\,O_1O_2$
$R\,M\,O_2O_1$	$O_2M\,R\,O_1$	$O_2R\,O_1M$	$M\,R\,O_2O_1$

but we cannot distinguish between the O's and $R\,O_1O_2M$ and $R\,O_2O_1M$ would be the same arrangement without the subscript. Notice in the list that there are only half as many arrangements without the use of subscripts or a total of twelve arrangements. This leads to the statement: The number of arrangements of n items, where r_1, r_2, and r_k are alike, is given by

$$\frac{n!}{r_1!\,r_2!\cdots r_k!}$$

In the previous example n was equal to four and there were two letters alike; therefore, we would write

$$\frac{4!}{2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1}$$

$$= 12$$

EXAMPLE: How many arrangements can be made using the letters in the word ADAPTATION?

SOLUTION: We use

$$\frac{n!}{r_1!\,r_2!\cdots r_k!}$$

where

$$n = 10$$

and

$$r_1 = 2 \text{ (two T's)}$$

and

$$r_2 = 3 \text{ (three A's)}$$

Then

$$\frac{n!}{r_1!\,r_2!\cdots r_k!} = \frac{10!}{2!\,3!}$$

$$= \frac{2\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{1}$$

$$= 302,400$$

PROBLEMS: Find the number of possible arrangements of the letters in the following words.

1. DOWN
2. STRUCTURE
3. BOOK
4. MILLIAMPERE
5. TENNESSEE

ANSWERS:

1. 24
2. 45,360
3. 12
4. 2,494,800
5. 3,780

Although the previous discussions have been associated with formulas, problems dealing with combinations and permutations may be analyzed and solved in a more meaningful way without complete reliance upon the formulas.

EXAMPLE: How many four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, and 7

(a) without repetitions?

(b) with repetitions?

SOLUTION: The (a) part of the question is a straight forward permutation problem and we reason that we want the number of permutations of six items taken four at a time.

Therefore

$$_6P_4 = \frac{6!}{(6-4)!}$$

$$= \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1}$$

$$= 360$$

The (b) part of the question would become quite complicated if we tried to use the formulas; therefore, we reason as follows:

We desire a four digit number and find we have six choices for the first digit. That is, we may use any of the digits 2, 3, 4, 5, 6, or 7 in the thousands column which gives us six choices for the digit to be placed in the thousands column. If we select the digit 4 for the thousands column we still have a choice of any of the digits 2, 3, 4, 5, 6, or 7 for the hundreds column. This is because we are allowed repetition and may select the digit 4 for the hundreds column as we did for the thousands column. This gives us six choices for the hundreds column.

Continuing this reasoning, we could write the number of choices for each place value column as shown in table 16-1.

Table 16-1.—Place value choices.

thousands column	hundreds column	tens column	units column
six choices	six choices	six choices	six choices

In table 16-1, observe that the total number of choices for the four digit number, by the principle of choice, is

$$6\cdot 6\cdot 6\cdot 6 = 1{,}296$$

Suppose, in the previous problem, we were to find how many three-digit odd numbers could be formed from the given digits, without repetition. We would be required to start in the units column because an odd number is determined by the units column digit. Therefore, we have only three choices. That is, either the 3, 5, or 7. For the tens column we have five choices and for the hundreds column we have four choices. This is shown in table 16-2.

Table 16-2.—Place value choices.

hundreds column	tens column	units column
four choices	five choices	three choices

In table 16-2, observe that there are

$$4\cdot 5\cdot 3 = 60$$

three-digit odd numbers that can be formed from the digits 2, 3, 4, 5, 6, and 7, without repetition.

PROBLEMS: Solve the following problems.

1. Using the digits 4, 5, 6, and 7, how many two-digit numbers can be formed:
   (a) without repetitions?
   (b) with repetitions?

2. Using the digits 4, 5, 6, 7, 8, and 9, how many five-digit numbers can be formed:
   (a) without repetitions?
   (b) with repetitions?

3. Using the digits of problem 2, how many four-digit odd numbers can be formed, without repetitions?

ANSWERS

1. (a) 12
   (b) 16
2. (a) 720
   (b) 7,776
3. 180

CHAPTER 17

PROBABILITY

The history of probability theory dates back to the 17th century and at that time was related to games of chance. In the 18th century it was seen that probability theory had applications beyond the scope of games of chance. Some of the applications in which probability theory is applied are situations with outcomes such as life or death and boy or girl. In the present century, statistics and probability are applied to insurance, annuities, biology, and social investigations.

The treatment of probability in this chapter is limited to simple applications. These applications will be, to a large extent, based on games of chance which lend themselves to an understanding of basic ideas of probability.

BASIC CONCEPTS

If a coin were tossed, the chance that it would land heads up is just as likely as the chance it would land tails up. That is, the coin has no more reason to land heads up than it has to land tails up. Every toss of the coin is called a trial.

We define probability as the ratio of the different number of ways a trial can succeed (or fail) to the total numbers of ways in which it may result. We will let p represent the probability of success and q represent the probability of failure.

One commonly misunderstood concept of probability is the effect prior trials have on a single trial. That is, after a coin has been tossed many times and every trial resulted in the coin falling heads up, will the next toss of the coin result in tails up? The answer is "not necessarily," and will be explained later in this chapter.

PROBABILITY OF SUCCESS

If a trial must result in any of n equally likely ways, and if s is the number of successful ways and f is the number of failing ways, then the probability of success is

$$p = \frac{s}{s + f}$$

where

$$s + f = n$$

EXAMPLE: What is the probability that a coin will land heads up?

SOLUTION: There is only one way the coin can land heads up, therefore s equals one. There is also only one way the coin can land other than heads up; therefore, f equals one. Then

$$s = 1$$

and

$$f = 1$$

Thus the probability of success is

$$\begin{aligned} p &= \frac{s}{s + f} \\ &= \frac{1}{1 + 1} \\ &= \frac{1}{2} \end{aligned}$$

This, then, is the ratio of successful ways in which the trial can succeed to the total number of ways the trial can result.

EXAMPLE: What is the probability that a die (singular of dice) will land with a three showing on the upper face.

SOLUTION: There is only one favorable way the die may land and there are a total of five ways it can land without the three face up.

$$s = 1$$

and

$$f = 5$$

and

$$p = \frac{s}{s+f}$$

$$= \frac{1}{1+5}$$

$$= \frac{1}{6}$$

EXAMPLE: What is the probability of drawing a black marble from a box of marbles if all six of the marbles in the box are white?

SOLUTION: There are no favorable ways of success and there are six total ways, therefore,

$$s = 0$$

and

$$f = 6$$

then

$$p = \frac{0}{0+6}$$

$$= \frac{0}{6}$$

$$= 0$$

EXAMPLE: What is the probability of drawing a black marble from a box of six black marbles?

SOLUTION: There are six successful ways and no unsuccessful ways of drawing the marble, therefore

$$s = 6$$

and

$$f = 0$$

then

$$p = \frac{6}{6+0}$$

$$= \frac{6}{6}$$

$$= 1$$

The previous two examples are the extremes of probabilities and intuitively demonstrate that the probability of an event ranges from zero to one inclusive.

EXAMPLE: A box contains six hard lead pencils and twelve soft lead pencils. What is the probability of drawing a soft lead pencil from the box?

SOLUTION: We are given

$$s = 12$$

and

$$f = 6$$

then

$$p = \frac{12}{12+6}$$

$$= \frac{12}{18}$$

$$= \frac{2}{3}$$

PROBLEMS:

1. What is the probability of drawing an ace from a standard deck of fifty-two playing cards?
2. What is the probability of drawing a black ace from a standard deck of playing cards?
3. If a die is rolled, what is the probability of an odd number showing on the upper face?
4. A man has three nickels, two dimes, and four quarters in his pocket. If he draws a single coin from his pocket, what is the probability that:
   (a) he draws a nickel?
   (b) he draws a half-dollar?
   (c) he draws a quarter?

ANSWERS:

1. $\frac{1}{13}$
2. $\frac{1}{26}$
3. $\frac{1}{2}$
4. (a) $\frac{1}{3}$
   (b) 0
   (c) $\frac{4}{9}$

PROBABILITY OF FAILURE

If a trial results in any of n equally likely ways, and s is the number of successful ways and f is the number of failures then, as before,

$$s + f = n$$

or

$$n - s = f$$

The probability of failure is given by

$$q = \frac{f}{s + f}$$

$$= \frac{n - s}{n}$$

A trial must result in either success or failure. If success is certain then p equals one and q equals zero. If success is impossible then p equals zero and q equals one. By combining both events—that is, in either case—the probability of success plus the probability of failure is equal to one as shown by

$$p = \frac{s}{s + f}$$

and

$$q = \frac{f}{s + f}$$

then

$$p + q = \frac{s}{s + f} + \frac{f}{s + f}$$

$$= 1$$

If, in any event

$$p + q = 1$$

then

$$q = 1 - p$$

In the case of tossing a coin, the probability of success is

$$p = \frac{s}{s + f}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

and the probability of failure is

$$q = 1 - p$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

EXAMPLE: What is the probability of not drawing a black marble from a box containing six white, three red, and two black marbles from a box containing six white, three red, and two black marbles?

SOLUTION: The probability of drawing a black marble from the box is

$$p = \frac{s}{s + f}$$

$$= \frac{2}{2 + 9}$$

$$= \frac{2}{11}$$

Since the probability of drawing a marble is one, then the probability of not drawing a black marble is

$$q = 1 - p$$

$$= 1 - \frac{2}{11}$$

$$= \frac{9}{11}$$

PROBLEMS: Compare the following problems and answers with the preceding problems dealing with the probability of success.

1. What is the probability of not drawing an ace from a standard deck of fifty-two playing cards?
2. What is the probability of not drawing a black ace from a standard deck of playing cards?
3. If a die is rolled, what is the probability of an odd number not showing on the upper face?
4. A man has three nickels, two dimes, and four quarters in his pocket. If he draws a single coin from his pocket, what is the probability that:
   - (a) he does not draw a nickel?
   - (b) he does not draw a half-dollar?
   - (c) he does not draw a quarter?

ANSWERS:

1. $\frac{12}{13}$

2. $\frac{25}{26}$

3. $\frac{1}{2}$

4. (a) $\frac{2}{3}$

   (b) 1

   (c) $\frac{5}{9}$

EXPECTATION

In this discussion of expectation we will consider two types. One is a numerical expectation and the other is value expectation.

Numerical Expectation

If you tossed a coin fifty times you would expect the coin to fall heads about twenty-five times. Your assumption is explained by the following definition.

If the probability of success in one trial is p, and k is the total number of trials, then pk is the expected number of successes in the k trials.

In the above example of tossing the coin fifty times the expected number of heads (successes) is

$$E_n = pk$$

where

E_n = expected number

p = probability of heads (successes)

k = number of tosses

Substituting values in the equation, we find that

$$E_n = (\frac{1}{2})\,50$$
$$= 25$$

EXAMPLE: A die is rolled by a player. What is the expectation of rolling a six in 30 trials?

SOLUTION: The probability of rolling a six in one trial is

$$p = \frac{1}{6}$$

and the number of rolls is

$$k = 30$$

therefore

$$E_n = pk$$
$$= \frac{1}{6}(30)$$
$$= 5$$

In words, the player would expect to roll a six five times in thirty rolls.

EXAMPLE: If a box contained seven numbered slips of paper, each numbered differently, how many times would a man expect to draw a single selected number slip, if he returned the drawn slip after each draw and he made a total of seventy draws?

SOLUTION: The probability of drawing the selected number slip in one drawing is

$$p = \frac{1}{7}$$

and the number of draws is

$$k = 70$$

therefore

$$E_n = pk$$
$$= (\frac{1}{7})\,70$$
$$= 10$$

Note: When the product of pk is not an integer, we will use the nearest integer to pk.

Value Expectation

We will define value expectation as follows: If, in the event of a successful result, a person is to receive m value and p is the probability of success of that event, then mp is his value expectation.

If you attended a house party where a door prize of $5.00 was given and ten people attended the party, what would be your expectation? In this case, instead of using k for expected number we use m for expected value. That is

$$E_v = pm$$

where

$$p = \text{probability of success}$$

$$m = \text{value of prize}$$

and

$$E_v = \text{expected value}$$

Then, by substitution

$$E_v = pm$$

$$= \left(\frac{1}{10}\right) \$5.00$$

$$= \$.50$$

EXAMPLE: In a game, a wheel is spun and when the wheel stops a pointer indicates one of the digits 1, 2, 3, 4, 5, 6, 7, or 8. The prize for winning is $16.00. If a person needed a 6 to win, calculate the following:

(a) What is his probability of winning?
(b) What is his value expectation?

SOLUTION

(a) $p = \frac{1}{8}$

(b) $p = \frac{1}{8}$ and $m = \$16.00$

therefore

$$E_v = pm = \left(\frac{1}{8}\right) \$16.00$$

$$= \$2.00$$

PROBLEMS:

1. When a store opened, each person who made a purchase was given one ticket on a chance for a door prize of $400. At the close of the day 2,000 people had registered.
   (a) If you made one purchase, what is your expectation?
   (b) If you made 5 purchases, what is your expectation?
2. Each person at a Bingo game purchased a fifty-cent chance for the jackpot of twenty dollars. If fifty people purchased chances, what is each person's
   (a) probability of winning?
   (b) probability of not winning?
   (c) expectation?

ANSWERS:

1. (a) $.20
   (b) $1.00
2. (a) $\frac{1}{50}$
   (b) $\frac{49}{50}$
   (c) $.40

COMPOUND PROBABILITIES

The probabilities to this point have been single events. In the discussion on compound probabilities, events which may affect others will be covered. The word "may" is used because included with dependent events and mutually exclusive events is the independent event.

INDEPENDENT EVENTS

Two events are said to be independent if the occurrence of one has no effect on the occurrence of the other.

When two coins are tossed at the same time or one after the other, whether one falls heads or tails has no effect on the way the second coin falls. Suppose we call the coins A and B. There are four ways in which the coins may fall, as follows:

1. A and B fall heads.
2. A and B fall tails.
3. A falls heads and B falls tails.
4. A falls tails and B falls heads.

The probability of any one way for the coins to fall is calculated as follows:

$$s = 1$$

and

$$n = 4$$

therefore

$$p = \frac{1}{4}$$

This probability may be determined by considering the product of the separate probabilities; that is

p that A falls heads is $\frac{1}{2}$

p that B falls heads is $\frac{1}{2}$

and the probability that both fall heads is

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

In other words, when two events are independent, the probability that one and then the other will occur is the product of their separate probabilities.

EXAMPLE: A box contains three red marbles and seven green marbles. If a marble is drawn, then replaced and another marble is drawn, what is the probability that both marbles are red?

SOLUTION: Two solutions are offered. First, there are, by the principle of choice, $10 \cdot 10$ ways in which two marbles can be selected. There are three ways the red marble may be selected on the first draw and three ways on the second draw and by the principle of choice there are $3 \cdot 3$ ways in which a red marble may be drawn on both trials. Then the required probability is

$$p = \frac{9}{100}$$

The second solution, using the product of independent events, follows: The probability of drawing a red ball on the first draw is $\frac{3}{10}$ and the probability of drawing a red ball on the second draw is $\frac{3}{10}$. Therefore, the probability of drawing a red ball on both draws is the product of the separate probabilities

$$p = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100}$$

PROBLEMS:

1. If a die is tossed twice, what is the probability of a two up followed by a three up?
2. A box contains two white, three red, and four blue marbles. If after each selection the marble is replaced, what is the probability of drawing, in order:
   (a) a white then a blue marble?
   (b) a blue then a red marble?
   (c) a white, a red, then a blue marble?

ANSWERS:

1. $\frac{1}{36}$
2. (a) $\frac{8}{81}$
   (b) $\frac{4}{27}$
   (c) $\frac{8}{243}$

DEPENDENT EVENTS

In some cases one event is dependent on another. That is, two or more events are said to be dependent if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others.

Consider that two or more events are dependent. If p_1 is the probability of a first event, p_2 the probability that after the first happens the second will occur, p_3 the probability that after the first and second have happened the third will occur, etc., then the probability that all events will happen in the given order is the product $p_1 \cdot p_2 \cdot p_3 \cdots$

EXAMPLE: A box contains three white marbles and two black marbles. What is the probability that in two draws both marbles drawn will be black. The first marble drawn is not replaced.

SOLUTION: On the first draw the probability of drawing a black marble is

$$p_1 = \frac{2}{5}$$

and on the second draw the probability of drawing a black marble is

$$p_2 = \frac{1}{4}$$

then the probability of drawing both black marbles is

$$\begin{aligned} p &= p_1 \cdot p_2 \\ &= \frac{2}{5} \cdot \frac{1}{4} \\ &= \frac{1}{10} \end{aligned}$$

EXAMPLE: Slips numbered one through nine are placed in a box. If two slips are drawn, what is the probability that

(a) both are odd?
(b) both are even?

SOLUTION:

(a) The probability that the first is odd is

$$p_1 = \frac{5}{9}$$

and the probability that the second is odd is

$$p_2 = \frac{4}{8}$$

therefore, the probability that both are odd is

$$p = p_1 \cdot p_2$$

$$= \frac{5}{9} \cdot \frac{4}{8}$$

$$= \frac{5}{18}$$

(b) The probability that the first is even is

$$p_1 = \frac{4}{9}$$

and the probability that the second is even is

$$p_2 = \frac{3}{8}$$

therefore, the probability that both are even is

$$p = p_1 \cdot p_2$$

$$= \frac{4}{9} \cdot \frac{3}{8}$$

$$= \frac{1}{6}$$

A second method of solution involves the use of combinations.

(a) There are a total of nine slips taken two at a time and there are five odd slips taken two at a time, therefore

$$p = \frac{{}_5C_2}{{}_9C_2}$$

$$= \frac{\frac{5!}{2!\,(3!)}}{\frac{9!}{2!\,(7!)}}$$

$$= \frac{5}{18}$$

(b) There are a total of ${}_9C_2$ choices and four even slips taken two at a time, therefore

$$p = \frac{{}_4C_2}{{}_9C_2}$$

$$= \frac{1}{6}$$

PROBLEMS: In the following problems assume that no replacement is made after each selection.

1. A box contains five white and six red marbles. What is the probability of successfully drawing, in order, a red marble then a white marble?

2. A bag contains three red, two white, and six blue marbles. What is the probability of drawing, in order, two red, one blue and two white marbles?

3. There are fifteen airmen in the line crew. They must take care of the coffee mess and line shack cleanup. They put slips numbered 1 through 15 in a hat and decide that any who draws a number divisible by 5 will be assigned the coffee mess and any who draws a number divisible by 4 will be assigned cleanup. The first person draws a 4, the second a 3, and the third an 11. As fourth person to draw, what is the probability that you will:

(a) be assigned the coffee mess?

(b) be assigned the cleanup?

ANSWERS:

1. $\frac{3}{11}$

2. $\frac{1}{770}$

3. (a) $\frac{1}{4}$

(b) $\frac{1}{6}$

MUTUALLY EXCLUSIVE EVENTS

Two or more events are called mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. The probability of occurrence of some one of two or more mutually exclusive events is the sum of the probabilities of the individual events.

It sometimes happens that when one event has occurred, the probability of another event is excluded, it being understood that we are referring to the same given occasion or trial.

For example, throwing a die once can yield 5 or 6, but not both, in the same toss. The probability that either a 5 or a 6 occurs is the sum of their individual probabilities.

$$p = p_1 + p_2$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{3}$$

EXAMPLE: From a bag containing five white balls, two black balls, and eleven red balls, one ball is drawn. What is the probability that it is either black or red?

SOLUTION: There are eighteen ways in which the draw can be made. There are two black ball choices and eleven red ball choices which are favorable, or a total of thirteen favorable choices. Then, the probability of success is

$$p = \frac{13}{18}$$

Since drawing a red ball excludes the drawing of a black ball, and vice versa, the two events are mutually exclusive. Then, the probability of drawing a black ball is

$$p_1 = \frac{2}{18}$$

and the probability of drawing a red ball is

$$p_2 = \frac{11}{18}$$

Therefore the probability of success is

$$p = p_1 + p_2$$
$$= \frac{2}{18} + \frac{11}{18} = \frac{13}{18}$$

EXAMPLE: What is the probability of drawing either a king, a queen, or a jack from a deck of playing cards?

SOLUTION: The individual probabilities are

$$\text{king} = \frac{4}{52}$$

$$\text{queen} = \frac{4}{52}$$

$$\text{jack} = \frac{4}{52}$$

Therefore the probability of success is

$$p = \frac{4}{52} + \frac{4}{52} + \frac{4}{52}$$
$$= \frac{12}{52}$$
$$= \frac{3}{13}$$

EXAMPLE: What is the probability of rolling a die twice and having a 5 and then a 3 show or having a 2 and then a 4 show?

SOLUTION: The probability of having a 5 and then a 3 show is

$$p_1 = \frac{1}{6} \cdot \frac{1}{6}$$
$$= \frac{1}{36}$$

and the probability of having a 2 and then a 4 show is

$$p_2 = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$$
$$= \frac{1}{36}$$

Then, the probability of either p_1 or p_2 is

$$p = p_1 + p_2$$
$$= \frac{1}{36} + \frac{1}{36}$$
$$= \frac{1}{18}$$

PROBLEMS:

1. When tossing a coin, what is the probability of getting either a head or a tail?
2. A bag contains twelve blue, three red, and four white marbles. What is the probability of drawing:
   (a) in one draw, either a red or a white marble?
   (b) in one draw, either a red, white, or blue marble?
   (c) in two draws, either a red marble followed by a blue marble or a red marble followed by a red marble?
3. What is the probability of getting a total of at least 10 points in rolling two dice?

(HINT: You want either a total of 10, 11, or 12.)

ANSWERS:

1. 1
2. (a) $\frac{7}{19}$

   (b) 1

   (c) $\frac{7}{57}$
3. $\frac{1}{6}$

EMPIRICAL PROBABILITIES

Among the most important applications of probability are those in situations where we cannot list all possible outcomes. To this point we have considered problems in which the probabilities could be obtained from situations in terms of equally likely results.

Because some problems are so complicated for analysis we can only estimate probabilities from experience and observation. This is empirical probability.

In modern industry probability now plays an important role in many activities. Quality control and reliability of a manufactured article have become extremely important considerations in which probability is used.

RELATIVE FREQUENCY OF SUCCESS

We define relative frequency of success as follows. After N trials of an event have been made, of which S trials are success, then the relative frequency of success is

$$P = \frac{S}{N}$$

Experience has shown that empirical probabilities, if carefully determined on the basis of adequate statistical samples, can be applied to large groups with the result that probability and relative frequency are approximately equal. By adequate samples we mean a large enough sample so that accidental runs of "luck," both good and bad, cancel each other. With enough trials, predicted results and actual results agree quite closely. On the other hand, applying a probability ratio to a single individual event is virtually meaningless.

For example, table 17-1 shows a small number of weather forecasts from April 1 to April 10. The actual weather on the dates is also given.

Observe that the forecasts on April 1, 3, 4, 6, 7, 8, and 10 were correct. We have observed ten outcomes. The event of a correct forecast has occurred seven times. Based on this information we might say that the probability for future forecasts being true is $\frac{7}{10}$. This number is the best estimate that we can make from the given information. In this case, since we have observed such a small number of outcomes, it would not be correct to say that the estimate of P is dependable. A great many more cases should be used if we expect to make a good estimate of the probability that a weather forecast will be accurate. There are a great many factors which affect the accuracy of a weather forecast. This example merely indicates something about how successful a particular weather office has been in making weather forecasts.

Another example may be drawn from industry. Many thousands of articles of a certain type are manufactured. The company selects 100 of these articles at random and subjects them to very careful tests. In these tests it is found that 98 of the articles meet all measurement requirements and perform satisfactorily. This suggests that $\frac{98}{100}$ is a measure of the reliability of the article.

One might expect that about 98% of all of the articles manufactured by this process will be satisfactory. The probability (measure of chance) that one of these articles will be satisfactory might be said to be 0.98.

This second example of empirical probability is different from the first example in one very important respect. In the first example all of the possibilities could be listed and in the second example we could not do so. The selection of a sample and its size is a problem of statistics.

Considered from another point of view, statistical probability can be regarded as relative frequency.

EXAMPLE: In a dart game, a player hit the bull's eye 3 times out of 25 trials. What is the statistical probability that he will hit the bull's eye on the next throw?

SOLUTION:

$$N = 25$$

and

$$S = 3$$

hence

$$P = \frac{3}{25}$$

Table 17-1. —Weather forecast.

Date	Forecast	Actual weather	Did the actual forecasted event occur?
1	Rain	Rain	yes
2	Light showers	Sunny	No
3	Cloudy	Cloudy	Yes
4	Clear	Clear	Yes
5	Scattered showers	Warm and sunny	No
6	Scattered showers	Scattered showers	Yes
7	Windy and cloudy	Overcast and windy	Yes
8	Thundershowers	Thundershowers	Yes
9	Clear	Cloudy and rain	No
10	Clear	Clear	Yes

EXAMPLE: Using table 17-2, what is the probability that a person 20 years old will live to be 50 years old?

SOLUTION: Of 95,148 persons at age 20, 81,090 survived to age 50. Hence

$$P = \frac{81,090}{95,148}$$

$$= 0.852$$

EXAMPLE: How many times would a die be expected to land with 5 or 6 showing in 20 trials?

SOLUTION: The probability of a 5 or 6 showing is

$$p = \frac{1}{3}$$

The relative frequency is approximately equal to the probability

$$P \approx p$$

therefore

$$P = \frac{S}{N}$$

where

$$P = \frac{1}{3}$$

$$S = ?$$

and

$$N = 20$$

Rearranging and substituting, find that

$$P = \frac{S}{N}$$

$$\frac{1}{3} = \frac{S}{20}$$

$$S = \frac{20}{3}$$

$$= 7$$

Table 17-2.—Mortality table (based on 100,000 individuals 1 year of age).

Age	Number of people
5	98,382
10	97,180
15	96,227
20	95,148
25	93,920
30	92,461
35	90,655
40	88,334
45	85,255
50	81,090
55	75,419
60	67,777
65	57,778
70	45,455
75	31,598
80	18,177
85	7,822
90	2,158

(NOTE: The number observed in an experiment may differ from that predicted; therefore, the results may be taken to the nearest integer.)

PROBLEMS:

1. In a construction crew there are six electricians and 38 other workers. How many electricians would you expect to choose if you chose one man each day of a week for your helper?
2. How many times would a tossed die be expected to turn up 3 or less in thirty tosses?
3. Using table 17-2, find the probability that a person whose age is 30 will live to age 60.

ANSWERS:

1. 1
2. 15
3. 0.733

DATA PROCESSING

Data processing is an extremely large and complex field and applications are usually made for individual situations. For a general understanding of how data processing can be related to probability and statistics, the functional operation of the computer is needed.

COMPUTER OPERATIONS

High speed computers are used in data processing because they are able to solve problems in seconds where humans may require months and even years to solve the same problem.

A problem which arises, when using a computer, is how the human can communicate with the computer. This communication is a function of mathematics. We refer to this form of mathematics as computer-oriented mathematics.

Digital computers are high speed adding machines. To perform multiplication they make repeated additions. To perform subtraction the addition sequence may be reversed, and division is the process of repeated subtraction.

An example of an operation of a digital computer is finding the square root of 36. We know that the sum of the first n odd integers is equal to n^2. The computer is programed to subtract successive odd integers from 36 until zero is reached. That is,

$$36 - 1 - 3 - 5 - 7 - 9 - 11 = 0$$

When zero is reached, the computer then counts the number of odd integers it has subtracted and this sum is the square root of 36. For a further discussion on computers and number systems refer to Mathematics, Volume 3, NavPers 10073.

APPLICATION TO PROBABILITY

Many problems may be solved on computers with the use of the proper mathematical model. The mathematical model may include any of the variables and the probability with which they occur. The collection of statistical data plays an important part in building a mathematical model for the computer.

The mathematical model may be used to determine probabilities by the use of computers. That is, the computer will "play a game" many times and give a result comparable to many trials for determining probabilities.

To understand how a game of chance may be used to produce a useful result, consider the problem of determining the product of 3/8 and 2/3. Place eight ping-pong balls of which three are coated with a conducting material in one container. Place three other ping-pong balls of which two are coated with the conducting material in a second container. A trial consists of a detecting head from the computer touching a ball in each container. If both balls are coated a point is registered; if not, a zero is registered. The number of points registered divided by the total trials, if the number of trials is large, will closely approximate the fraction 6/24.

The reasoning for the result is that the probability of touching a coated ball in the first container is 3/8 and the probability of touching a coated ball in the second container is 2/3. The events of touching a coated ball in the first container and one in the second container are independent, and the probability of both balls being coated is the product of the individual probabilities. This is exactly what we set out to determine.

The preceding example is extremely simplified, in comparison with the complexity of the actual statistical problems solved by computers. However, it does serve to indicate some of the possibilities of computer-oriented mathematics.

USAGE OF STATISTICAL DATA

Suppose a squadron, through years of operation, has accumulated statistical data on the operation of an aircraft. By using a computer, the probability of the failure of an engine can be determined from the many bits of information regarding individual parts of the engine. A related problem is to determine how the engine failure probability can be decreased.

While changing all of the components of an aircraft engine to try to improve efficiency is unsound, the mathematical model, from a selected group of experiments, may be used to predict the change in efficiency for any combination of component changes.

One solution is by trial and error, but this would take years of time. Another solution is to make a few flights using various configurations of the engine and then use the computer to simulate years of operation. Researchers would then determine the probability of failure, based on the new statistical data obtained from the few flights. By this high speed determination of the failure of the engine with different combinations of new parts, the optimum design of the modified engine may be determined.

The field of data processing with computers is just beginning and many new techniques are being developed. It is hoped that this brief introduction will stimulate the reader to search further for a better understanding of this new field.

APPENDIX I

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

38° → ↓ '	sin	Diff. 1'	csc	tan	Diff. 1'	cot	sec	Diff. 1'	cos	← 141° ↓ '
0	9.78934	16	10.21066	9.89281	26	10.10719	10.10347	10	9.89653	60
1	.78950	17	.21050	.89307	26	.10693	.10357	10	.89643	59
2	.78967	16	.21033	.89333	26	.10667	.10367	9	.89633	58
3	.78983	16	.21017	.89359	26	.10641	.10376	10	.89624	57
4	.78999	16	.21001	.89385	26	.10615	.10386	10	.89614	56
5	9.79015	16	10.20985	9.89411	26	10.10589	10.10396	10	9.89604	55
6	.79031	16	.20969	.89437	26	.10563	.10406	10	.89594	54
7	.79047	16	.20953	.89463	26	.10537	.10416	10	.89584	53
8	.79063	16	.20937	.89489	26	.10511	.10426	10	.89574	52
9	.79079	16	.20921	.89515	26	.10485	.10436	10	.89564	51
10	9.79095	16	10.20905	9.89541	26	10.10459	10.10446	10	9.89554	50
11	.79111	17	.20889	.89567	26	.10433	.10456	10	.89544	49
12	.79128	16	.20872	.89593	26	.10407	.10466	10	.89534	48
13	.79144	16	.20856	.89619	26	.10381	.10476	10	.89524	47
14	.79160	16	.20840	.89645	26	.10355	.10486	10	.89514	46
15	9.79176	16	10.20824	9.89671	26	10.10329	10.10496	9	9.89504	45
16	.79192	16	.20808	.89697	26	.10303	.10505	10	.89495	44
17	.79208	16	.20792	.89723	26	.10277	.10515	10	.89485	43
18	.79224	16	.20776	.89749	26	.10251	.10525	10	.89475	42
19	.79240	16	.20760	.89775	26	.10225	.10535	10	.89465	41
20	9.79256	16	10.20744	9.89801	26	10.10199	10.10545	10	9.89455	40
21	.79272	16	.20728	.89827	26	.10173	.10555	10	.89445	39
22	.79288	16	.20712	.89853	26	.10147	.10565	10	.89435	38
23	.79304	15	.20696	.89879	26	.10121	.10575	10	.89425	37
24	.79319	16	.20681	.89905	26	.10095	.10585	10	.89415	36
25	9.79335	16	10.20665	9.89931	26	10.10069	10.10595	10	9.89405	35
26	.79351	16	.20649	.89957	26	.10043	.10605	10	.89395	34
27	.79367	16	.20633	.89983	26	.10017	.10615	10	.89385	33
28	.79383	16	.20617	.90009	26	.09991	.10625	11	.89375	32
29	.79399	16	.20601	.90035	26	.09965	.10636	10	.89364	31
30	9.79415	16	10.20585	9.90061	25	10.09939	10.10646	10	9.89354	30
31	.79431	16	.20569	.90086	26	.09914	.10656	10	.89344	29
32	.79447	16	.20553	.90112	26	.09888	.10666	10	.89334	28
33	.79463	15	.20537	.90138	26	.09862	.10676	10	.89324	27
34	.79478	16	.20522	.90164	26	.09836	.10686	10	.89314	26
35	9.79494	16	10.20506	9.90190	26	10.09810	10.10696	10	9.89304	25
36	.79510	16	.20490	.90216	26	.09784	.10706	10	.89294	24
37	.79526	16	.20474	.90242	26	.09758	.10716	10	.89284	23
38	.79542	16	.20458	.90268	26	.09732	.10726	10	.89274	22
39	.79558	15	.20442	.90294	26	.09706	.10736	10	.89264	21
40	9.79573	16	10.20427	9.90320	26	10.09680	10.10746	10	9.89254	20
41	.79589	16	.20411	.90346	25	.09654	.10756	11	.89244	19
42	.79605	16	.20395	.90371	26	.09629	.10767	10	.89233	18
43	.79621	15	.20379	.90397	26	.09603	.10777	10	.89223	17
44	.79636	16	.20364	.90423	26	.09577	.10787	10	.89213	16
45	9.79652	16	10.20348	9.90449	26	10.09551	10.10797	10	9.89203	15
46	.79668	16	.20332	.90475	26	.09525	.10807	10	.89193	14
47	.79684	15	.20316	.90501	26	.09499	.10817	10	.89183	13
48	.79699	16	.20301	.90527	26	.09473	.10827	11	.89173	12
49	.79715	16	.20285	.90553	25	.09447	.10838	10	.89162	11
50	9.79731	15	10.20269	9.90578	26	10.09422	10.10848	10	9.89152	10
51	.79746	16	.20254	.90604	26	.09396	.10858	10	.89142	9
52	.79762	16	.20238	.90630	26	.09370	.10868	10	.89132	8
53	.79778	15	.20222	.90656	26	.09344	.10878	10	.89122	7
54	.79793	16	.20207	.90682	26	.09318	.10888	11	.89112	6
55	9.79809	16	10.20191	9.90708	26	10.09292	10.10899	10	9.89101	5
56	.79825	15	.20175	.90734	25	.09266	.10909	10	.89091	4
57	.79840	16	.20160	.90759	26	.09241	.10919	10	.89081	3
58	.79856	16	.20144	.90785	26	.09215	.10929	11	.89071	2
59	.79872	15	.20128	.90811	26	.09189	.10940	10	.89060	1
60	9.79887		10.20113	9.90837		10.09163	10.10950		9.89050	0
↑ 128° →	cos	Diff. 1'	sec	cot	Diff. 1'	tan	csc	Diff. 1'	sin	← 51° ↑

APPENDIX II

COMMON LOGARITHMS OF NUMBERS

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

APPENDIX III

NATURAL SINES AND COSINES

M.	0° Sin.	0° Cos.	1° Sin.	1° Cos.	2° Sin.	2° Cos.	3° Sin.	3° Cos.	4° Sin.	4° Cos.	
0	0.00000	1.0000	0.01745	0.99985	0.03490	0.99939	0.05234	0.99863	0.06976	0.99756	60
1	029	000	774	984	519	938	263	861	.07005	754	59
2	058	000	803	984	548	937	292	860	034	752	58
3	087	000	832	983	577	936	321	858	063	750	57
4	116	000	862	983	606	935	350	857	092	748	56
5	.00145	1.0000	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99740	55
6	175	000	920	982	664	933	408	854	150	744	54
7	204	000	949	981	693	932	437	852	179	742	53
8	233	000	978	980	723	931	466	851	208	740	52
9	262	000	.02007	980	752	930	495	849	237	738	51
10	.00291	1.0000	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	320	.99999	065	979	810	927	553	846	295	734	49
12	349	999	094	978	839	926	582	844	324	731	48
13	378	999	123	977	868	925	611	842	353	729	47
14	407	999	152	977	897	924	640	841	382	727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	465	999	211	976	955	922	698	838	440	723	44
17	495	999	240	975	984	921	727	836	469	721	43
18	524	999	269	974	.04013	919	756	834	498	719	42
19	553	998	298	974	042	918	785	833	527	716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	611	998	356	972	100	916	844	829	585	712	39
22	640	998	385	972	129	915	873	827	614	710	38
23	669	998	414	971	159	913	902	826	643	708	37
24	698	998	443	970	188	912	931	824	672	705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	756	997	501	969	246	910	989	821	730	701	34
27	785	997	530	968	275	909	.06018	819	759	699	33
28	814	997	560	967	304	907	047	817	788	696	32
29	844	996	589	966	333	906	076	815	817	694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	902	996	647	965	391	904	134	812	875	689	29
32	931	996	676	964	420	902	163	810	904	687	28
33	960	995	705	963	449	901	192	808	933	685	27
34	989	995	734	963	478	900	221	806	962	683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	047	995	792	961	536	897	279	803	.08020	678	24
37	076	994	821	960	565	896	308	801	049	676	23
38	105	994	850	959	594	894	337	799	078	673	22
39	134	994	879	959	623	893	366	797	107	671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	193	993	938	957	682	890	424	793	165	666	19
42	222	993	967	956	711	889	453	792	194	664	18
43	251	992	996	955	740	888	482	790	223	661	17
44	280	992	.03025	954	769	886	511	788	252	659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	338	991	083	952	827	883	569	784	310	654	14
47	367	991	112	952	856	882	598	782	339	652	13
48	396	990	141	951	885	881	627	780	368	649	12
49	425	990	170	950	914	879	656	778	397	647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	483	989	228	948	972	876	714	774	455	642	9
52	513	989	257	947	.05001	875	743	772	484	639	8
53	542	988	286	946	030	873	773	770	513	637	7
54	571	988	316	945	059	872	802	768	542	635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	629	987	374	943	117	869	860	764	600	630	4
57	658	986	403	942	146	867	889	762	629	627	3
58	687	986	432	941	175	866	918	760	658	625	2
59	716	985	461	940	205	864	947	758	687	622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
	Cos. 89°	Sin. 89°	Cos. 88°	Sin. 88°	Cos. 87°	Sin. 87°	Cos. 86°	Sin. 86°	Cos. 85°	Sin. 85°	M.

APPENDIX III.—Natural sines and cosines—Continued

M.	5°		6°		7°		8°		9°		
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	
0	0.08716	0.99619	0.10453	0.99452	0.12187	0.99255	0.13917	0.99027	0.15643	0.98769	60
1	745	617	482	449	216	251	946	023	672	764	59
2	774	614	511	446	245	248	975	019	701	760	58
3	803	612	540	443	274	244	.14004	015	730	755	57
4	831	609	569	440	302	240	033	011	758	751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	889	604	626	434	360	233	090	002	816	741	54
7	918	602	655	431	389	230	119	.98998	845	737	53
8	947	599	684	428	418	226	148	994	873	732	52
9	976	596	713	424	447	222	177	990	902	728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	034	591	771	418	504	215	234	982	959	718	49
12	063	588	800	415	533	211	263	978	988	714	48
13	092	586	829	412	562	208	292	973	.16017	709	47
14	121	583	858	409	591	204	320	969	046	704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	179	578	916	402	649	197	378	961	103	695	44
17	208	575	945	399	678	193	407	957	132	690	43
18	237	572	973	396	706	189	436	953	160	686	42
19	266	570	.11002	393	735	186	464	948	189	681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	98676	40
21	324	564	060	386	793	178	522	940	246	671	39
22	353	562	089	383	822	175	551	936	275	667	38
23	382	559	118	380	851	171	580	931	304	662	37
24	411	556	147	377	880	167	608	927	333	657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	469	551	205	370	937	160	666	919	390	648	34
27	498	548	234	367	966	156	695	914	419	643	33
28	527	545	263	364	995	152	723	910	447	638	32
29	556	542	291	360	.13024	148	752	906	476	633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	614	537	349	354	081	141	810	897	533	624	29
32	642	534	378	351	110	137	838	893	562	619	28
33	671	531	407	347	139	133	867	889	591	614	27
34	700	528	436	344	168	129	896	884	620	609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	758	523	494	337	226	122	954	876	677	600	24
37	787	520	523	334	254	118	982	871	706	595	23
38	816	517	552	331	283	114	.15011	867	734	590	22
39	845	514	580	327	312	110	040	863	763	585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	903	508	638	320	370	102	097	854	820	575	19
42	932	506	667	317	399	098	126	849	849	570	18
43	961	503	696	314	427	094	155	845	878	565	17
44	990	500	725	310	456	091	184	841	906	561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	048	494	783	303	514	083	241	832	964	551	14
47	077	491	812	300	543	079	270	827	992	546	13
48	106	488	840	297	572	075	299	823	.17021	541	12
49	135	485	869	293	600	071	327	818	050	536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	192	479	927	286	658	063	385	809	107	526	9
52	221	476	956	283	687	059	414	805	136	521	8
53	250	473	985	279	716	055	442	800	164	516	7
54	279	470	.12014	276	744	051	471	796	193	511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	337	464	071	269	802	043	529	787	250	501	4
57	366	461	100	265	831	039	557	782	279	496	3
58	395	458	129	262	860	035	586	778	308	491	2
59	424	455	158	258	889	031	615	773	336	486	1
60	.10453	.99452	.12187	99255	.13917	.99027	.15643	.98769	.17365	.98481	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	84°		83°		82°		81°		80°		

APPENDIX III.—Natural sines and cosines—Continued

M	10°		11°		12°		13°		14°		
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	
0	0.17365	0.98481	0.19081	0.98163	0.20791	0.97815	0.22495	0.97437	0.24192	0.97030	60
1	393	476	109	157	820	809	523	430	220	023	59
2	422	471	138	152	848	803	552	424	249	015	58
3	451	466	167	146	877	797	580	417	277	008	57
4	479	461	195	140	905	791	608	411	305	001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	537	450	252	129	962	778	665	398	362	987	54
7	565	445	281	124	990	772	693	391	390	980	53
8	594	440	309	118	.21019	766	722	384	418	973	52
9	623	435	338	112	047	760	750	378	446	966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	680	425	395	101	104	748	807	365	503	952	49
12	708	420	423	096	132	742	835	358	531	945	48
13	737	414	452	090	161	735	863	351	559	937	47
14	766	409	481	084	189	729	892	345	587	930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	823	399	538	073	246	717	948	331	644	916	44
17	852	394	566	067	275	711	977	325	672	909	43
18	880	389	595	061	303	705	.23005	318	700	902	42
19	909	383	623	056	331	698	033	311	728	894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	966	373	680	044	388	686	090	298	784	880	39
22	995	368	709	039	417	680	118	291	813	873	38
23	.18023	362	737	033	445	673	146	284	841	866	37
24	052	357	766	027	474	667	175	278	869	858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	109	347	823	016	530	655	231	264	925	844	34
27	138	341	851	010	559	648	260	257	954	837	33
28	166	336	880	004	587	642	288	251	982	829	32
29	195	331	908	.97998	616	636	316	244	.25010	822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	252	320	965	987	672	623	373	230	066	807	29
32	281	315	994	981	701	617	401	223	094	800	28
33	309	310	.20022	975	729	611	429	217	122	793	27
34	338	304	051	969	758	604	458	210	151	786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	395	294	108	958	814	592	514	196	207	771	24
37	424	288	136	952	843	585	542	189	235	764	23
38	452	283	165	946	871	579	571	182	263	756	22
39	481	277	193	940	899	573	599	176	291	749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	538	267	250	928	956	560	656	162	348	734	19
42	567	261	279	922	985	553	684	155	376	727	18
43	595	256	307	916	.22013	547	712	148	404	719	17
44	624	250	336	910	041	541	740	141	432	712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	681	240	393	899	098	528	797	127	488	697	14
47	710	234	421	893	126	521	825	120	516	690	13
48	738	229	450	887	155	515	853	113	545	682	12
49	767	223	478	881	183	508	882	106	573	675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	824	212	535	869	240	496	938	093	629	660	9
52	852	207	563	863	268	489	966	086	657	653	8
53	881	201	592	857	297	483	995	079	685	645	7
54	910	196	620	851	325	476	.24023	072	713	638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	967	185	677	839	382	463	079	.058	769	623	4
57	995	179	706	833	410	457	108	051	798	615	3
58	.19024	174	734	827	438	450	136	044	826	608	2
59	052	168	763	821	467	444	164	037	854	600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	79°		78°		77°		76°		75°		

APPENDIX III.—Natural sines and cosines—Continued

M.	15° Sin.	15° Cos.	16° Sin.	16° Cos.	17° Sin.	17° Cos.	18° Sin.	18° Cos.	19° Sin.	19° Cos.	
0	0.25882	0.96593	0.27564	0.96126	0.29237	0.95630	0.30902	0.95106	0.32557	0.94552	60
1	910	585	592	118	265	622	929	097	584	542	59
2	938	578	620	110	293	613	957	088	612	533	58
3	966	570	648	102	321	605	985	079	639	523	57
4	994	562	676	094	348	596	.31012	070	667	514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	050	547	731	078	404	579	068	052	722	495	54
7	079	540	759	070	432	571	095	043	749	485	53
8	107	532	787	062	460	562	123	033	777	476	52
9	135	524	815	054	487	554	151	024	804	466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	191	509	871	037	543	536	206	006	859	447	49
12	219	502	899	029	571	528	233	.94997	887	438	48
13	247	494	927	021	599	519	261	988	914	428	47
14	275	486	955	013	626	511	289	979	942	418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	331	471	.28011	.95997	682	493	344	961	997	399	44
17	359	463	039	989	710	485	372	952	.33024	390	43
18	387	456	067	981	737	476	399	943	051	380	42
19	415	448	095	972	765	467	427	933	079	370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	471	433	150	956	821	450	482	915	134	351	39
22	500	425	178	948	849	441	510	906	161	342	38
23	528	417	206	940	876	433	537	897	189	332	37
24	556	410	234	931	904	424	565	888	216	322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	612	394	290	915	960	407	620	869	271	303	34
27	640	386	318	907	987	398	648	860	298	293	33
28	668	379	346	898	.30015	389	675	851	326	284	32
29	696	371	374	890	043	380	703	842	353	274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	752	355	429	874	098	363	758	823	408	254	29
32	780	347	457	865	126	354	786	814	436	245	28
33	808	340	485	857	154	345	813	805	463	235	27
34	836	332	513	849	182	337	841	795	490	225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	892	316	569	832	237	319	896	777	545	206	24
37	920	308	597	824	265	310	923	768	573	196	23
38	948	301	625	816	292	301	951	758	600	186	22
39	976	293	652	807	320	293	979	749	627	176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	032	277	708	791	376	275	034	730	682	157	19
42	060	269	736	782	403	266	061	721	710	147	18
43	088	261	764	774	431	257	089	712	737	137	17
44	116	253	792	766	459	248	116	702	764	127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	172	238	847	749	514	231	171	684	819	108	14
47	200	230	875	740	542	222	199	674	846	098	13
48	228	222	903	732	570	213	227	665	874	088	12
49	256	214	931	724	597	204	254	656	901	078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	312	198	987	707	653	186	309	637	956	058	9
52	340	190	.29015	698	680	177	337	627	983	049	8
53	368	182	042	690	708	168	364	618	.34011	039	7
54	396	174	070	681	736	159	392	609	038	029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	452	158	126	664	791	142	447	590	093	009	4
57	480	150	154	656	819	133	474	580	120	.93999	3
58	508	142	182	647	846	124	502	571	147	989	2
59	536	134	209	639	874	115	529	561	175	979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	74°		73°		72°		71°		70°		

APPENDIX III.—Natural sines and cosines—Continued

M.	20° Sin.	20° Cos.	21° Sin.	21° Cos.	22° Sin.	22° Cos.	23° Sin.	23° Cos.	24° Sin.	24° Cos.	
0	0.34202	0.93969	0.35837	0.93358	0.37461	0.92718	0.39073	0.92050	0.40674	0.91355	60
1	229	959	864	348	488	707	100	039	700	343	59
2	257	949	891	337	515	697	127	028	727	331	58
3	284	939	918	327	542	686	153	016	753	319	57
4	311	929	945	316	569	675	180	005	780	307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	366	909	.36000	295	622	653	234	982	833	283	54
7	393	899	027	285	649	642	260	971	860	272	53
8	421	889	054	274	676	631	287	959	886	260	52
9	448	879	081	264	703	620	314	948	913	248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	503	859	135	243	757	598	367	925	966	224	49
12	530	849	162	232	784	587	394	914	992	212	48
13	557	839	190	222	811	576	421	902	.41019	200	47
14	584	829	217	211	838	565	448	891	045	188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	639	809	271	190	892	543	501	868	098	164	44
17	666	799	298	180	919	532	528	856	125	152	43
18	694	789	325	169	946	521	555	845	151	140	42
19	721	779	352	159	973	510	581	833	178	128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	775	759	406	137	.38026	488	635	810	231	104	39
22	803	748	434	127	053	477	661	799	257	092	38
23	830	738	461	116	080	466	688	787	284	080	37
24	857	728	488	106	107	455	715	775	310	068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	912	708	542	084	161	432	768	752	363	044	34
27	939	698	569	074	188	421	795	741	390	032	33
28	966	688	596	063	215	410	822	729	416	020	32
29	993	677	623	052	241	399	848	718	443	008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	048	657	677	031	295	377	902	694	496	984	29
32	075	647	704	020	322	366	928	683	522	972	28
33	102	637	731	010	349	355	955	671	549	960	27
34	130	626	758	.92999	376	343	982	660	575	948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	184	606	812	978	430	321	035	636	628	924	24
37	211	596	839	967	456	310	062	625	655	911	23
38	239	585	867	956	483	299	088	613	681	899	22
39	266	575	894	945	510	287	115	601	707	887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	320	555	948	924	564	265	168	578	760	863	19
42	347	544	975	913	591	254	195	566	787	851	18
43	375	534	.37002	902	617	243	221	555	813	839	17
44	402	524	029	892	644	231	248	543	840	826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	456	503	083	870	698	209	301	519	892	802	14
47	484	493	110	859	725	198	328	508	919	790	13
48	511	483	137	849	752	186	355	496	945	778	12
49	538	472	164	838	778	175	381	484	972	766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	592	452	218	816	832	152	434	461	.42024	741	9
52	619	441	245	805	859	141	461	449	051	729	8
53	647	431	272	794	886	130	488	437	077	717	7
54	674	420	299	784	912	119	514	425	104	704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	728	400	353	762	966	096	567	402	156	680	4
57	755	389	380	751	993	085	594	390	183	668	3
58	782	379	407	740	.39020	073	621	378	209	655	2
59	810	368	434	729	046	062	647	366	235	643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	69°		68°		67°		66°		65°		

APPENDIX III.—Natural sines and cosines—Continued

M.	25° Sin.	25° Cos.	26° Sin.	26° Cos.	27° Sin.	27° Cos.	28° Sin.	28° Cos.	29° Sin.	29° Cos.	
0	0.42262	0.90631	0.43837	0.89879	0.45399	0.89101	0.46947	0.88295	0.48481	0.87462	60
1	288	618	863	867	425	087	973	281	506	448	59
2	315	606	889	854	451	074	999	267	532	434	58
3	341	594	916	841	477	061	.47024	254	557	420	57
4	367	582	942	828	503	048	050	240	583	406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	420	557	994	803	554	021	101	213	634	377	54
7	446	545	.44020	790	580	008	127	199	659	363	53
8	473	532	046	777	606	.88995	153	185	684	349	52
9	499	520	072	764	632	981	178	172	710	335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	552	495	124	739	684	955	229	144	761	306	49
12	578	483	151	726	710	942	255	130	786	292	48
13	604	470	177	713	736	928	281	117	811	278	47
14	631	458	203	700	762	915	306	103	837	264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	683	433	255	674	813	888	358	075	888	235	44
17	709	421	281	662	839	875	383	062	913	221	43
18	736	408	307	649	865	862	409	048	938	207	42
19	762	396	333	636	891	848	434	034	964	193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	815	371	385	610	942	822	486	006	.49014	164	39
22	841	358	411	597	968	808	511	.87993	040	150	38
23	867	346	437	584	994	795	537	979	065	136	37
24	894	334	464	571	.46020	782	562	965	090	121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	946	309	516	545	072	755	614	937	141	093	34
27	972	296	542	532	097	741	639	923	166	079	33
28	999	284	568	519	123	728	665	909	192	064	32
29	.43025	271	594	506	149	715	690	896	217	050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	077	246	646	480	201	688	741	868	268	021	29
32	104	233	672	467	226	674	767	854	293	007	28
33	130	221	698	454	252	661	793	840	318	.86993	27
34	156	208	724	441	278	647	818	826	344	978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	209	183	776	415	330	620	869	798	394	949	24
37	235	171	802	402	355	607	895	784	419	935	23
38	261	158	828	389	381	593	920	770	445	921	22
39	287	146	854	376	407	580	946	756	470	906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	340	120	906	350	458	553	997	729	521	878	19
42	366	108	932	337	484	539	.48022	715	546	863	18
43	392	095	958	324	510	526	048	701	571	849	17
44	418	082	984	311	536	512	073	687	596	834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	471	057	036	285	587	485	124	659	647	805	14
47	497	045	062	272	613	472	150	645	672	791	13
48	523	032	088	259	639	458	175	631	697	777	12
49	549	019	114	245	664	445	201	617	723	762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	602	.89994	166	219	716	417	252	589	773	733	9
52	628	981	192	206	742	404	277	575	798	719	8
53	654	968	218	193	767	390	303	561	824	704	7
54	680	956	243	180	793	377	328	546	849	690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	733	930	295	153	844	349	379	518	899	661	4
57	759	918	321	140	870	336	405	504	924	646	3
58	785	905	347	127	896	322	430	490	950	632	2
59	811	892	373	114	921	308	456	476	975	617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	64°		63°		62°		61°		60°		

APPENDIX III.—Natural sines and cosines—Continued

M.	30° Sin.	30° Cos.	31° Sin.	31° Cos.	32° Sin.	32° Cos.	33° Sin.	33° Cos.	34° Sin.	34° Cos.	
0	0.50000	0.86603	0.51504	0.85717	0.52992	0.84805	0.54464	0.83867	0.55919	0.82904	60
1	025	588	529	702	.53017	789	488	851	943	887	59
2	050	573	554	687	041	774	513	835	968	871	58
3	076	559	579	672	066	759	537	819	992	855	57
4	101	544	604	657	091	743	561	804	.56016	839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	151	515	653	627	140	712	610	772	064	806	54
7	176	501	678	612	164	697	635	756	088	790	53
8	201	486	703	597	189	681	659	740	112	773	52
9	227	471	728	582	214	666	683	724	136	757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	277	442	778	551	263	635	732	692	184	724	49
12	302	427	803	536	288	619	756	676	208	708	48
13	327	413	828	521	312	604	781	660	232	692	47
14	352	398	852	506	337	588	805	645	256	675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	403	369	902	476	386	557	854	613	305	643	44
17	428	354	927	461	411	542	878	597	329	626	43
18	453	340	952	446	435	526	902	581	353	610	42
19	478	325	977	431	460	511	927	565	377	593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	528	295	026	401	509	480	975	533	425	561	39
22	553	281	051	385	534	464	999	517	449	544	38
23	578	266	076	370	558	448	.55024	501	473	528	37
24	603	251	101	355	583	433	048	485	497	511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	654	222	151	325	632	402	097	453	545	478	34
27	679	207	175	310	656	386	121	437	569	462	33
28	704	192	200	294	681	370	145	421	593	446	32
29	729	178	225	279	705	355	169	405	617	429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	779	148	275	249	754	324	218	373	665	396	29
32	804	133	299	234	779	308	242	356	689	380	28
33	829	119	324	218	804	292	266	340	713	363	27
34	854	104	349	203	828	277	291	324	736	347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	904	074	399	173	877	245	339	292	784	314	24
37	929	059	423	157	902	230	363	276	808	297	23
38	954	045	448	142	926	214	388	260	832	281	22
39	979	030	473	127	951	198	412	244	856	264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	029	000	522	096	.54000	167	460	212	904	231	19
42	054	.85985	547	081	024	151	484	195	928	214	18
43	079	970	572	066	049	135	509	179	952	198	17
44	104	956	597	051	073	120	533	163	976	181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	154	926	646	020	122	088	581	131	024	148	14
47	179	911	671	005	146	072	605	115	047	132	13
48	204	896	696	.84989	171	057	630	098	071	115	12
49	229	881	720	974	195	041	654	082	095	098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	279	851	770	943	244	009	702	050	143	065	9
52	304	836	794	928	269	.83994	726	034	167	048	8
53	329	821	819	913	293	978	750	017	191	032	7
54	354	806	844	897	317	962	775	001	215	015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	404	777	893	866	366	930	823	969	262	982	4
57	429	762	918	851	391	915	847	953	286	965	3
58	454	747	943	836	415	899	871	936	310	949	2
59	479	732	967	820	440	883	895	920	334	932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	59°		58°		57°		56°		55°		

APPENDIX III.—Natural sines and cosines—Continued

M.	35°		36°		37°		38°		39°		
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	
0	0.57358	0.81915	0.58779	0.80902	0.60182	0.79864	0.61566	0.78801	0.62932	0.77715	60
1	381	899	802	885	205	846	589	783	955	696	59
2	405	882	826	867	228	829	612	765	977	678	58
3	429	865	849	850	251	811	635	747	.63000	660	57
4	453	848	873	833	274	793	658	729	022	641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	501	815	920	799	321	758	704	694	068	605	54
7	524	798	943	782	344	741	726	676	090	586	53
8	548	782	967	765	367	723	749	658	113	568	52
9	572	765	990	748	390	706	772	640	135	550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	619	731	037	713	437	671	818	604	180	513	49
12	643	714	061	696	460	653	841	586	203	494	48
13	667	698	084	679	483	635	864	568	225	476	47
14	691	681	108	662	506	618	887	550	248	458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	738	647	154	627	553	583	932	514	293	421	44
17	762	631	178	610	576	565	955	496	316	402	43
18	786	614	201	593	599	547	978	478	338	384	42
19	810	597	225	576	622	530	.62001	460	361	366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	857	563	272	541	668	494	046	424	406	329	39
22	881	546	295	524	691	477	069	405	428	310	38
23	904	530	318	507	714	459	092	387	451	292	37
24	928	513	342	489	738	441	115	369	473	273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	976	479	389	455	784	406	160	333	518	236	34
27	999	462	412	438	807	388	183	315	540	218	33
28	.58023	445	436	420	830	371	206	297	563	199	32
29	047	428	459	403	853	353	229	279	585	181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	094	395	506	368	899	318	274	243	630	144	29
32	118	378	529	351	922	300	297	225	653	125	28
33	141	361	552	334	945	282	320	206	675	107	27
34	165	344	576	316	968	264	342	188	698	088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	212	310	622	282	.61015	229	388	152	742	051	24
37	236	293	646	264	038	211	411	134	765	033	23
38	260	276	669	247	061	193	433	116	787	014	22
39	283	259	693	230	084	176	456	098	810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	330	225	739	195	130	140	502	061	854	959	19
42	354	208	763	178	153	122	524	043	877	940	18
43	378	191	786	160	176	105	547	025	899	921	17
44	401	174	809	143	199	087	570	007	922	903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	449	140	856	108	245	051	615	970	966	866	14
47	472	123	879	091	268	033	638	952	989	847	13
48	496	106	902	073	291	016	660	934	.64011	828	12
49	519	089	926	056	314	.78998	683	916	033	810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	567	055	972	021	360	962	728	879	078	772	9
52	590	038	995	003	383	944	751	861	100	754	8
53	614	021	.60019	.79986	406	926	774	843	123	735	7
54	637	004	042	968	429	908	796	824	145	717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	684	970	089	934	474	873	842	788	190	679	4
57	708	953	112	916	497	855	864	769	212	661	3
58	731	936	135	899	520	837	887	751	234	642	2
59	755	919	158	881	543	819	909	733	256	623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	54°		53°		52°		51°		50°		

APPENDIX III.—Natural sines and cosines—Continued

M.	40°		41°		42°		43°		44°		
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	
0	0.64279	0.76604	0.65606	0.75471	0.66913	0.74314	0.68200	0.73135	0.69466	0.71934	60
1	301	586	628	452	935	295	221	116	487	914	59
2	323	567	650	433	956	276	242	096	508	894	58
3	346	548	672	414	978	256	264	076	529	873	57
4	368	530	694	395	999	237	285	056	549	853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	412	492	738	356	043	198	327	016	591	813	54
7	435	473	759	337	064	178	349	.72996	612	792	53
8	457	455	781	318	086	159	370	976	633	772	52
9	479	436	803	299	107	139	391	957	654	752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	524	398	847	261	151	100	434	917	696	711	49
12	546	380	869	241	172	080	455	897	717	691	48
13	568	361	891	222	194	061	476	877	737	671	47
14	590	342	913	203	215	041	497	857	758	650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	635	304	956	165	258	002	539	817	800	610	44
17	657	286	978	146	280	.73983	561	797	821	590	43
18	679	267	.66000	126	301	963	582	777	842	569	42
19	701	248	022	107	323	944	603	757	862	549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	746	210	066	069	366	904	645	717	904	508	39
22	768	192	088	050	387	885	666	697	925	488	38
23	790	173	109	030	409	865	688	677	946	468	37
24	812	154	131	011	430	846	709	657	966	447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	856	116	175	973	473	806	751	617	.70008	407	34
27	878	097	197	953	495	787	772	597	029	386	33
28	901	078	218	934	516	767	793	577	049	366	32
29	923	059	240	915	538	747	814	557	070	345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	967	022	284	876	580	708	857	517	112	305	29
32	989	003	306	857	602	688	878	497	132	284	28
33	.65011	.75984	327	838	623	669	899	477	153	264	27
34	033	965	349	818	645	649	920	457	174	243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	077	927	393	780	688	610	962	417	215	203	24
37	100	908	414	760	709	590	983	397	236	182	23
38	122	889	436	741	730	570	.69004	377	257	162	22
39	144	870	458	722	752	551	025	357	277	141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	188	832	501	683	795	511	067	317	319	100	19
42	210	813	523	664	816	491	088	297	339	080	18
43	232	794	545	644	837	472	109	277	360	059	17
44	254	775	566	625	859	452	130	257	381	039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	298	738	610	586	901	413	172	216	422	.70998	14
47	320	719	632	567	923	393	193	196	443	978	13
48	342	700	653	548	944	373	214	176	463	957	12
49	364	680	675	528	965	353	235	156	484	937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	408	642	718	489	.68008	314	277	116	525	896	9
52	430	623	740	470	029	294	298	095	546	875	8
53	452	604	762	451	051	274	319	075	567	855	7
54	474	585	783	431	072	254	340	055	587	834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	518	547	827	392	115	215	382	015	628	793	4
57	540	528	848	373	136	195	403	.71995	649	772	3
58	562	509	870	353	157	175	424	974	670	752	2
59	584	490	891	334	179	155	445	954	690	731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	M.
	49°		48°		47°		46°		45°		

APPENDIX IV.—NATURAL TANGENTS AND COTANGENTS

M	0° Tan.	0° Cot.	1° Tan.	1° Cot.	2° Tan.	2° Cot.	3° Tan.	3° Cot.	
0	0.00000	0	0.01746	57.2900	0.03492	28.6363	0.05241	19.0811	60
1	029	3437.75	775	56.3506	521	.3994	270	18.9755	59
2	058	1718.87	804	55.4415	550	.1664	299	.8711	58
3	087	1145.92	833	54.5613	579	27.9372	328	.7678	57
4	116	859.436	862	53.7086	609	.7117	357	.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	175	572.957	920	52.0807	667	.2715	416	.4645	54
7	204	491.106	949	51.3032	696	.0566	445	.3655	53
8	233	429.718	978	50.5485	725	26.8450	474	.2677	52
9	262	381.971	.02007	49.8157	754	.6367	503	.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	320	312.521	066	48.4121	812	.2296	562	17.9602	49
12	349	286.478	095	47.7395	842	.0307	591	.8863	48
13	378	264.441	124	47.0853	871	25.8348	620	.7934	47
14	407	245.552	153	46.4489	900	.6418	549	.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	465	214.858	211	45.2261	958	.2644	708	.5205	44
17	495	202.219	240	44.6386	987	.0798	737	.4314	43
18	524	190.984	269	44.0661	.04016	24.8978	766	.3432	42
19	553	180.932	298	43.5081	046	.7185	795	.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	611	163.700	357	42.4335	104	.3675	854	.0837	39
22	640	156.259	386	41.9158	133	.1957	883	16.9990	38
23	669	149.465	415	41.4106	162	.0263	912	.9150	37
24	698	143.237	444	40.9174	191	23.8593	941	.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	756	132.219	502	39.9655	250	.5321	999	.6681	34
27	785	127.321	531	39.5059	279	.3718	.06029	.5874	33
28	815	122.774	560	39.0568	308	.2137	058	.5075	32
29	844	118.540	589	38.6177	337	.0577	087	.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	902	110.892	648	37.7686	395	.7519	145	.2722	29
32	931	107.426	677	37.3579	424	.6020	175	.1952	28
33	960	104.171	706	36.9560	454	.4541	204	.1190	27
34	989	101.107	735	36.5627	483	.3081	233	.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	047	95.4895	793	35.8006	541	.0217	291	.8945	24
37	076	92.9085	822	35.4313	570	21.8813	321	.8211	23
38	105	90.4633	851	35.0695	599	.7426	350	.7483	22
39	135	88.1436	881	34.7151	628	.6056	379	.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	193	83.8435	939	34.0273	687	.3369	438	.5340	19
42	222	81.8470	968	33.6935	716	.2049	467	.4638	18
43	251	79.9434	997	33.3662	745	.0747	496	.3943	17
44	280	73.1263	.03026	33.0452	774	20.9460	525	.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	338	74.7292	084	32.4213	833	.6932	584	.1893	14
47	367	73.1390	114	32.1181	862	.5691	613	.1222	13
48	396	71.6151	143	31.8205	891	.4465	642	.0557	12
49	425	70.1533	172	31.5284	920	.3253	671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	484	67.4019	230	30.9599	978	.0872	730	.8596	9
52	513	66.1055	259	30.6833	.05007	19.9702	759	.7954	8
53	542	64.8580	288	30.4116	037	.8546	788	.7317	7
54	571	63.6567	317	30.1446	066	.7403	817	.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	629	61.3829	376	29.6245	124	.5156	876	.5438	4
57	658	60.3058	405	29.3711	153	.4051	905	.4823	3
58	687	59.2659	434	29.1220	182	.2959	934	.4212	2
59	716	58.2612	463	28.8771	212	.1879	963	.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
	Cot. 89°	Tan. 89°	Cot. 88°	Tan. 88°	Cot. 87°	Tan. 87°	Cot. 86°	Tan. 86°	M.

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	4° Tan.	4° Cot.	5° Tan.	5° Cot.	6° Tan.	6° Cot.	7° Tan.	7° Cot.	
0	0.06993	14.3007	0.08749	11.4301	0.10510	9.51436	0.12278	8.14435	60
1	.07022	.2411	778	.3919	540	.48781	308	.12481	59
2	051	.1821	807	.3540	569	.46141	338	.10536	58
3	080	.1235	837	.3163	599	.43515	367	.08600	57
4	110	.0655	866	.2789	628	.40904	397	.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	168	13.9507	925	.2048	687	.35724	456	.02848	54
7	197	.8940	954	.1681	716	.33155	485	.00948	53
8	227	.8378	983	.1316	746	.30599	515	7.99058	52
9	256	.7821	.09013	.0954	775	.28058	544	.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	314	.6719	071	.0237	834	.23016	603	.93438	49
12	344	.6174	101	10.9882	863	.20516	633	.91582	48
13	373	.5634	130	.9529	893	.18028	662	.89734	47
14	402	.5098	159	.9178	922	.15554	692	.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	461	.4039	218	.8483	981	.10646	751	.84242	44
17	490	.3515	247	.8139	.11011	.08211	781	.82428	43
18	519	.2996	277	.7797	040	.05789	810	.80622	42
19	548	.2480	306	.7457	070	.03379	840	.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	607	.1461	365	.6783	128	8.98598	899	.75254	39
22	636	.0958	394	.6450	158	.96227	929	.73480	38
23	665	.0458	423	.6118	187	.93867	958	.71715	37
24	695	12.9962	453	.5789	217	.91520	988	.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	753	.8981	511	.5136	276	.86862	047	.66466	34
27	782	.8496	541	.4813	305	.84551	076	.64732	33
28	812	.8014	570	.4491	335	.82252	106	.63005	32
29	841	.7536	600	.4172	364	.79964	136	.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	899	.6591	658	.3538	423	.75425	195	.57872	29
32	929	.6124	688	.3224	452	.73172	224	.56176	28
33	958	.5660	717	.2913	482	.70931	254	.54487	27
34	987	.5199	746	.2602	511	.68701	284	.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	046	.4288	805	.1988	570	.64275	343	.49465	24
37	075	.3838	834	.1683	600	.62078	372	.47806	23
38	104	.3390	864	.1381	629	.59893	402	.46154	22
39	134	.2946	893	.1080	659	.57718	432	.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	192	.2067	952	.0483	718	.53402	491	.41240	19
42	221	.1632	981	.0187	747	.51259	521	.39616	18
43	251	.1201	.10011	9.98931	777	.49128	550	.37999	17
44	280	.0772	040	.96007	806	.47007	580	.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	339	11.9923	099	.90211	865	.42795	639	.33190	14
47	368	.9504	128	.87338	895	.40705	669	.31600	13
48	397	.9087	158	.84482	924	.38625	698	.30018	12
49	427	.8673	187	.81641	954	.36555	728	.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	485	.7853	246	.76009	.12013	.32446	787	.25310	9
52	514	.7448	275	.73217	042	.30406	817	.23754	8
53	544	.7045	305	.70441	072	.28376	846	.22204	7
54	573	.6645	334	.67680	101	.26355	876	.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	632	.5853	393	.62205	160	.22344	935	.17594	4
57	661	.5461	422	.59490	190	.20352	965	.16071	3
58	690	.5072	452	.56791	219	.18370	995	.14553	2
59	720	.4685	481	.54106	249	.16398	.14024	.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
	Cot. 85°	Tan. 85°	Cot. 84°	Tan. 84°	Cot. 83°	Tan. 83°	Cot. 82°	Tan. 82°	M.

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	8° Tan.	8° Cot.	9° Tan.	9° Cot.	10° Tan.	10° Cot.	11° Tan.	11° Cot.	
0	0.14054	7.11537	0.15838	6.31373	0.17633	5.67128	0.19438	5.14455	60
1	084	.10038	868	.30189	663	.66165	468	.13658	59
2	113	.08546	898	.29007	693	.65205	498	.12862	58
3	143	.07059	928	.27829	723	.64248	529	.12069	57
4	173	.05579	958	.26655	753	.63295	559	.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	232	.02637	.16017	.24321	813	.61397	619	.09704	54
7	262	.01174	047	.23160	843	.60452	649	.08921	53
8	291	6.99718	077	.22003	873	.59511	680	.08139	52
9	321	.98268	107	.20851	903	.58573	710	.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	381	.95385	167	.18559	963	.56706	770	.05809	49
12	410	.93952	196	.17419	993	.55777	801	.05037	48
13	440	.92525	226	.16283	.18023	.54851	831	.04267	47
14	470	.91104	256	.15151	053	.53927	861	.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	529	.88278	316	.12899	113	.52090	921	.01971	44
17	559	.86874	346	.11779	143	.51176	952	.01210	43
18	588	.85475	376	.10664	173	.50264	982	.00451	42
19	618	.84082	405	.09552	203	.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	678	.81312	465	.07340	263	.47548	073	.98188	39
22	707	.79936	495	.06240	293	.46648	103	.97438	38
23	737	.78564	525	.05143	323	.45751	133	.96690	37
24	767	.77199	555	.04051	353	.44857	164	.95945	36
25	.14796	6.75838	.16585	6.02962	.18384	5.43966	.20194	4.95201	35
26	826	.74483	615	.01878	414	.43077	224	.94460	34
27	856	.73133	645	.00797	444	.42192	254	.93721	33
28	886	.71789	674	5.99720	474	.41309	285	.92984	32
29	915	.70450	704	.98646	504	.40429	315	.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	975	.67787	764	.96510	564	.38677	376	.90785	29
32	.15005	.66463	794	.95448	594	.37805	406	.90056	28
33	034	.65144	824	.94390	624	.36936	436	.89330	27
34	064	.63831	854	.93335	654	.36070	466	.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87892	25
36	124	.61219	914	.91236	714	.34345	527	.87162	24
37	153	.59921	944	.90191	745	.33487	557	.86444	23
38	183	.58627	974	.89151	775	.32631	588	.85727	22
39	213	.57339	.17004	.88114	805	.31778	618	.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	272	.54777	063	.86051	865	.30080	679	.83590	19
42	302	.53503	093	.85024	895	.29235	709	.82882	18
43	332	.52234	123	.84001	925	.28398	739	.82175	17
44	362	.50970	153	.82982	955	.27553	770	.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	421	.48456	213	.80953	.19016	.25880	830	.80068	14
47	451	.47206	243	.79944	046	.25048	861	.79370	13
48	481	.45961	273	.78938	076	.24218	891	.78673	12
49	511	.44720	303	.77936	106	.23391	921	.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	570	.42253	363	.75941	166	.21744	982	.76595	9
52	600	.41026	393	.74949	197	.20925	.21013	.75906	8
53	630	.39804	423	.73960	227	.20107	043	.75219	7
54	660	.38587	453	.72974	257	.19293	073	.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	719	.36165	513	.71013	317	.17671	134	.73170	4
57	749	.34961	543	.70037	347	.16863	164	.72490	3
58	779	.33761	573	.69064	378	.16058	195	.71813	2
59	809	.32566	603	.68094	408	.15256	225	.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	81°		80°		79°		78°		

APPENDIX [illegible]—Natural tangents and cotangents—Con.

M.	12° Tan.	12° Cot.	13° Tan.	13° Cot.	14° Tan.	14° Cot.	15° Tan.	15° Cot.	
0	0.21256	4.70463	0.23087	4.33148	0.24933	4.01078	0.26795	3.73205	60
1	286	.69791	117	.32573	964	.00582	826	.72771	59
2	316	.69121	148	.32001	995	.00086	857	.72338	58
3	347	.68452	179	.31430	.25026	3.99592	888	.71907	57
4	377	.67786	209	.30860	056	.99099	920	.71476	56
5	.21408	4.67121	[illegible]	4.30291	.25087	3.98607	.26951	3.71046	55
6	438	.66458	[illegible]	.29724	118	.98117	982	.70616	54
7	469	.65797	301	.29159	149	.97627	.27013	.70188	53
8	499	.65138	332	.28595	180	.97139	044	.69761	52
9	529	.64480	363	.28032	211	.96651	076	.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	590	.63171	424	.26911	273	.95680	138	.68485	49
12	621	.62518	455	.26352	304	.95196	169	.68061	48
13	651	.61868	485	.25795	335	.94713	201	.67638	47
14	682	.61219	516	.25239	366	.94232	232	.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	743	.59927	578	.24132	428	.93721	294	.66376	44
17	773	.59283	608	.23580	459	.92793	326	.65957	43
18	804	.58641	639	.23030	490	.92316	357	.65538	42
19	834	.58001	670	.22841	521	.91839	388	.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	895	.56728	731	.21387	583	.90890	451	.64289	39
22	925	.56091	762	.20842	614	.90417	482	.63874	38
23	956	.55458	793	.20298	645	.89945	513	.63461	37
24	986	.54826	823	.19756	676	.89474	545	.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	047	.53568	885	.18675	738	.88536	607	.62224	34
27	078	.52941	916	.18137	769	.88068	638	.61814	33
28	108	.52316	946	.17600	800	.87601	670	.61405	32
29	139	.51693	977	.17064	831	.87136	701	.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	200	.50451	039	.15997	893	.86208	764	.60181	29
32	231	.49832	069	.15465	924	.85745	795	.59775	28
33	261	.49215	100	.14934	955	.85284	826	.59370	27
34	292	.48600	131	.14405	986	.84824	858	.58966	26
35	.22322	[illegible]	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	353	[illegible]	193	.13350	048	.83906	921	.58160	24
37	383	.46764	223	.12825	079	.83449	952	.57758	23
38	414	.46155	254	.12301	110	.82992	983	.57357	22
39	444	.45548	285	.11778	141	.82537	.28015	.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	505	.44338	347	.10736	203	.81630	077	.56159	19
42	536	.43735	377	.10216	235	.81177	109	.55761	18
43	567	.43134	408	.09699	266	.80726	140	.55364	17
44	597	.42534	439	.09182	297	.80276	172	.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	658	.41340	501	.08152	359	.79378	234	.54179	14
47	689	.40745	532	.07639	390	.78931	266	.53785	13
48	719	.40152	562	.07127	421	.78485	297	.53393	12
49	750	.39560	593	.06616	452	.78040	329	.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	811	.38381	655	.05599	515	.77152	391	.52219	9
52	842	.37793	686	.05092	546	.76709	423	.51829	8
53	872	.37207	717	.04586	577	.76268	454	.51441	7
54	903	.36623	747	.04081	608	.75828	486	.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	964	.35459	809	.03076	670	.74950	549	.50279	4
57	995	.34879	840	.02574	701	.74512	580	.49894	3
58	.23026	.34300	871	.02074	733	.74075	612	.49509	2
59	056	.33723	902	.01576	764	.73640	643	.49126	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	77°		76°		[illegible]		74°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	16° Tan.	16° Cot.	17° Tan.	17° Cot.	18° Tan.	18° Cot.	19° Tan.	19° Cot.	
0	0.28675	3.48741	0.30573	3.27085	0.32492	3.07768	0.34433	2.90421	60
1	706	.48359	605	.26745	524	.07464	465	.90147	59
2	738	.47977	637	.26406	556	.07160	498	.89873	58
3	769	.47596	669	.26067	588	.06857	530	.89600	57
4	801	.47216	700	.25729	621	.06554	563	.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	864	.46458	764	.25055	685	.05950	628	.88783	54
7	895	.46080	796	.24719	717	.05649	661	.88511	53
8	927	.45703	828	.24383	749	.05349	693	.88240	52
9	958	.45327	860	.24049	782	.05049	726	.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	.44576	923	.23381	846	.04450	791	.87430	49
12	053	.44202	955	.23048	878	.04152	824	.87161	48
13	084	.43829	987	.22715	911	.03854	856	.86892	47
14	116	.43456	.31019	.22384	943	.03556	889	.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	179	.42713	083	.21722	.33007	.02963	954	.86089	44
17	210	.42343	115	.21392	040	.02667	987	.85822	43
18	242	.41973	147	.21063	072	.02372	.35020	.85555	42
19	274	.41604	178	.20734	104	.02077	052	.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	337	.40869	242	.20079	169	.01489	118	.84758	39
22	368	.40502	274	.19752	201	.01196	150	.84494	38
23	400	.40136	306	.19426	233	.00903	183	.84229	37
24	432	.39771	338	.19100	266	.00611	216	.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	495	.39042	402	.18451	330	.00028	281	.83439	34
27	526	.38679	434	.18127	363	2.99738	314	.83176	33
28	558	.38317	466	.17804	395	.99447	346	.82914	32
29	590	.37955	498	.17481	427	.99158	379	.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	653	.37234	562	.16838	492	.98580	445	.82130	29
32	685	.36875	594	.16517	524	.98292	477	.81870	28
33	716	.36516	626	.16197	557	.98004	510	.81610	27
34	748	.36158	658	.15877	589	.97717	543	.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	811	.35443	722	.15240	654	.97144	608	.80833	24
37	843	.35087	754	.14922	686	.96858	641	.80574	23
38	875	.34732	786	.14605	718	.96573	674	.80316	22
39	906	.34377	818	.14288	751	.96288	707	.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	970	.33670	882	.13656	816	.95721	772	.79545	19
42	.30001	.33317	914	.13341	848	.95437	805	.79289	18
43	033	.32965	946	.13027	881	.95155	838	.79033	17
44	065	.32614	978	.12713	913	.94872	871	.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	128	.31914	042	.12087	978	.94309	937	.78269	14
47	160	.31565	074	.11775	.34010	.94028	969	.78014	13
48	192	.31216	106	.11464	043	.93748	.36002	.77761	12
49	224	.30868	139	.11153	075	.93468	035	.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	287	.30174	203	.10532	140	.92910	101	.77002	9
52	319	.29829	235	.10223	173	.92632	134	.76750	8
53	351	.29483	267	.09914	205	.92354	167	.76498	7
54	382	.29139	299	.09606	238	.92076	199	.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	446	.28452	363	.08991	303	.91523	265	.75746	4
57	478	.28109	396	.08685	335	.91246	298	.75496	3
58	509	.27767	428	.08379	368	.90971	331	.75246	2
59	541	.27426	460	.08073	400	.90696	364	.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	73°		72°		71°		70°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	20° Tan.	20° Cot.	21° Tan.	21° Cot.	22° Tan.	22° Cot.	23° Tan.	23° Cot.	
0	0.36397	2.74748	0.38386	2.60509	0.40403	2.47509	0.42447	2.35585	60
1	430	.74499	420	.60283	436	.47302	482	.35395	59
2	364	.74251	453	.60057	470	.47095	516	.35205	58
3	496	.74004	487	.59831	504	.46888	551	.35015	57
4	529	.73756	520	.59606	538	.46682	585	.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	595	.73263	587	.59156	606	.46270	654	.34447	54
7	628	.73017	620	.58932	640	.46065	688	.34258	53
8	661	.72771	654	.58708	674	.45860	722	.34069	52
9	694	.72526	687	.58484	707	.45655	757	.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	760	.72036	754	.58038	775	.45246	826	.33505	49
12	793	.71792	787	.57815	809	.45043	860	.33317	48
13	826	.71548	821	.57593	843	.44839	894	.33130	47
14	859	.71305	854	.57371	877	.44636	929	.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	925	.70819	921	.56928	945	.44230	998	.32570	44
17	958	.70577	955	.56707	979	.44027	.43032	.32383	43
18	991	.70335	988	.56487	.41013	.43825	067	.32197	42
19	.37024	.70094	.39022	.56266	047	.43623	101	.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	090	.69612	089	.55827	115	.43220	170	.31641	39
22	123	.69371	122	.55608	149	.43019	205	.31456	38
23	157	.69131	156	.55389	183	.42819	239	.31271	37
24	190	.68892	190	.55170	217	.42618	274	.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	256	.68414	257	.54734	285	.42218	343	.30718	34
27	289	.68175	290	.54516	319	.42019	378	.30534	33
28	322	.67937	324	.54299	353	.41819	412	.30351	32
29	355	.67700	357	.54082	387	.41620	447	.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	422	.67225	425	.53648	455	.41223	516	.29801	29
32	455	.66989	458	.53432	490	.41025	550	.29619	28
33	488	.66752	492	.53217	524	.40827	585	.29437	27
34	521	.66516	526	.53001	558	.40629	620	.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	588	.66046	593	.52571	626	.40235	689	.28891	24
37	621	.65811	626	.52357	660	.40038	724	.28710	23
38	654	.65576	660	.52142	694	.39641	758	.28528	22
39	637	.65342	694	.51929	728	.39645	793	.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	754	.64875	761	.51502	797	.39253	862	.27987	19
42	787	.64642	795	.51289	831	.39058	897	.27806	18
43	820	.64410	829	.51076	865	.38863	932	.27626	17
44	853	.64177	862	.50864	899	.38668	966	.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	920	.63714	930	.50440	968	.38279	036	.27088	14
47	953	.63483	963	.50229	.42002	.38084	071	.26909	13
48	986	.63252	997	.50018	036	.37891	105	.26730	12
49	.38020	.63021	.40031	.49807	070	.37697	140	.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	086	.62561	098	.49386	139	.37311	210	.26196	9
52	120	.62332	132	.49177	173	.37118	244	.26018	8
53	153	.62103	166	.48967	207	.36925	279	.25840	7
54	186	.61874	200	.48758	242	.36733	314	.25663	6
55	.38220	2.61646	.40234	2.48549	42276	2.36541	.44349	2.25486	5
56	253	.61418	267	.48340	310	.36349	384	.25309	4
57	286	.61190	301	.48132	345	.36158	418	.25132	3
58	320	.60963	335	.47924	379	.35967	453	.24956	2
59	353	.60736	369	.47716	413	.35776	488	.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	69°		68°		67°		66°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	24° Tan.	24° Cot.	25° Tan.	25° Cot.	26° Tan.	26° Cot.	27° Tan.	27° Cot.	
0	0.44523	2.24604	0.46631	2.14451	0.48773	2.05030	0.50953	1.96261	60
1	558	.24428	666	.14288	809	.04879	989	.96120	59
2	593	.24252	702	.14125	845	.04728	.51026	.95979	58
3	627	.24077	737	.13963	881	.04577	063	.95838	57
4	662	.23902	772	.13801	917	.04426	099	.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	732	.23553	843	.13477	989	.04125	173	.95417	54
7	767	.23378	879	.13316	.49026	.03975	209	.95277	53
8	802	.23204	914	.13154	062	.03825	246	.95137	52
9	837	.23030	950	.12993	098	.03675	283	.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	907	.22683	.47021	.12671	170	.03376	356	.94718	49
12	942	.22510	056	.12511	206	.03227	393	.94579	48
13	977	.22337	092	.12350	242	.03078	430	.94440	47
14	.45012	.22164	128	.12190	278	.02929	467	.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	082	.21819	199	.11871	351	.02631	540	.94023	44
17	117	.21647	234	.11711	387	.02483	577	.93885	43
18	152	.21475	270	.11552	423	.02335	614	.93746	42
19	187	.21304	305	.11392	459	.02187	651	.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	257	.20961	377	.11075	532	.01891	724	.93332	39
22	292	.20790	412	.10916	568	.01743	761	.93195	38
23	327	.20619	448	.10758	604	.01596	798	.93057	37
24	362	.20449	483	.10600	640	.01449	835	.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	432	.20108	555	.10284	713	.01155	909	.92645	34
27	467	.19938	590	.10126	749	.01008	946	.92508	33
28	502	.19769	626	.09969	786	.00862	983	.92371	32
29	538	.19599	662	.09811	822	.00715	.52020	.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	608	.19261	733	.09498	894	.00423	094	.91962	29
32	643	.19092	769	.09341	931	.00277	131	.91826	28
33	678	.18923	805	.09184	967	.00131	168	.91690	27
34	713	.18755	840	.09028	.50004	1.99986	205	.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	784	.18419	912	.08716	076	.99695	279	.91282	24
37	819	.18251	948	.08560	113	.99550	316	.91147	23
38	854	.18084	984	.08405	149	.99406	353	.91012	22
39	889	.17916	.48019	.08250	185	.99261	390	.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	960	.17582	091	.07939	258	.98972	464	.90607	19
42	995	.17416	127	.07785	295	.98828	501	.90472	18
43	.46030	.17249	163	.07630	331	.98684	538	.90337	17
44	065	.17083	198	.07476	368	.98540	575	.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	136	.16751	270	.07167	441	.98253	650	.89935	14
47	171	.16585	306	.07014	477	.98110	687	.89801	13
48	206	.16420	342	.06860	514	.97966	724	.89667	12
49	242	.16255	378	.06706	550	.97823	761	.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97681	.52798	1.89400	10
51	312	.15925	450	.06400	623	.97538	836	.89266	9
52	348	.15760	486	.06247	660	.97395	873	.89133	8
53	383	.15596	521	.06094	696	.97253	910	.89000	7
54	418	.15432	557	.05942	733	.97111	947	.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52985	1.88734	5
56	489	.15104	629	.05637	806	.96827	.53022	.88602	4
57	525	.14940	665	.05485	843	.96685	059	.88469	3
58	560	.14777	701	.05333	879	.96544	096	.88337	2
59	595	.14614	737	.05182	916	.96402	134	.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	65°		64°		63°		62°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	28° Tan.	28° Cot.	29° Tan.	29° Cot.	30° Tan.	30° Cot.	31° Tan.	31° Cot.	
0	0.53171	1.88073	0.55431	1.80405	0.57735	1.73205	0.60086	1.66428	60
1	208	.87941	469	.80281	774	.73089	126	.66318	59
2	246	.87809	507	.80158	813	.72973	165	.66209	58
3	283	.87677	545	.80034	851	.72857	205	.66099	57
4	320	.87546	583	.79911	890	.72741	245	.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	395	.87283	659	.79665	968	.72509	324	.65772	54
7	432	.87152	697	.79542	.58007	.72393	364	.65663	53
8	470	.87021	736	.79419	046	.72278	403	.65554	52
9	507	.86891	774	.79296	085	.72163	443	.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	582	.86630	850	.79051	162	.71932	522	.65228	49
12	620	.86499	888	.78929	201	.71817	562	.65120	48
13	657	.86369	926	.78807	240	.71702	602	.65011	47
14	694	.86239	964	.78685	279	.71588	642	.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	769	.85979	041	.78441	357	.71358	721	.64687	44
17	807	.85850	079	.78319	396	.71244	761	.64579	43
18	844	.85720	117	.78198	435	.71129	801	.64471	42
19	882	.85591	156	.78077	474	.71015	841	.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	957	.85333	232	.77834	552	.70787	921	.64148	39
22	995	.85204	270	.77713	591	.70673	960	.64041	38
23	.54032	.85075	309	.77592	631	.70560	.61000	.63934	37
24	070	.84946	347	.77471	670	.70446	040	.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	145	.84689	424	.77230	748	.70219	120	.63612	34
27	183	.84561	462	.77110	787	.70106	160	.63505	33
28	220	.84433	501	.76990	826	.69992	200	.63398	32
29	258	.84305	539	.76869	865	.69879	240	.63292	31
30	.54296	1.84177	.56577	1.76749	.58905	1.69766	.61280	1.63185	30
31	333	.84049	616	.76629	944	.69653	320	.63079	29
32	371	.83922	654	.76510	983	.69541	360	.62972	28
33	409	.83794	693	.76390	.59022	.69428	400	.62866	27
34	446	.83667	731	.76271	061	.69316	440	.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	522	.83413	808	.76032	140	.69091	520	.62548	24
37	560	.83286	846	.75913	179	.68979	561	.62442	23
38	597	.83159	885	.75794	218	.68866	601	.62336	22
39	635	.83033	923	.75675	258	.68754	641	.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	711	.82780	.57000	.75437	336	.68531	721	.62019	19
42	748	.82654	039	.75319	376	.68419	761	.61914	18
43	786	.82528	078	.75200	415	.68308	801	.61808	17
44	824	.82402	116	.75082	454	.68196	842	.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	900	.82150	193	.74846	533	.67974	922	.61493	14
47	938	.82025	232	.74728	573	.67863	962	.61388	13
48	975	.81899	271	.74610	612	.67752	.62003	.61283	12
49	.55013	.81774	309	.74492	651	.67641	043	.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	089	.81524	386	.74257	730	.67419	124	.60970	9
52	127	.81399	425	.74140	770	.67309	164	.60865	8
53	165	.81274	464	.74022	809	.67198	204	.60761	7
54	203	.81150	503	.73905	849	.67088	245	.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	279	.80901	580	.73671	928	.66867	325	.60449	4
57	317	.80777	619	.73555	967	.66757	366	.60345	3
58	355	.80653	657	.73438	.60007	.66647	406	.60241	2
59	393	.80529	696	.73321	046	.66538	446	.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	61°		60°		59°		58°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	32°		33°		34°		35°		
	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	
0	0.62487	1.60033	0.64941	1.53986	0.67451	1.48256	0.70021	1.42815	60
1	527	.59930	982	.53888	493	.48163	064	.42726	59
2	568	.59826	.65024	.53791	536	.48070	107	.42638	58
3	608	.59723	065	.53693	578	.47977	151	.42550	57
4	649	.59620	106	.53595	620	.47885	194	.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	730	.59414	189	.53400	705	.47699	281	.42286	54
7	770	.59311	231	.53302	748	.47607	325	.42198	53
8	811	.59208	272	.53205	790	.47514	368	.42110	52
9	.852	.59105	314	.53107	832	.47422	412	.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	933	.58900	397	.52913	917	.47238	499	.41847	49
12	973	.58797	438	.52816	960	.47146	542	.41759	48
13	.63014	.58695	480	.52719	.68002	.47053	586	.41672	47
14	055	.58593	521	.52622	045	.46962	629	.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	136	.58388	604	.52429	130	.46778	717	.41409	44
17	177	.58286	646	.52332	173	.46686	760	.41322	43
18	217	.58184	688	.52235	215	.46595	804	.41235	42
19	258	.58083	729	.52139	258	.46503	848	.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	340	.57879	813	.51946	343	.46320	935	.40974	39
22	380	.57778	854	.51850	386	.46229	979	.40887	38
23	421	.57676	896	.51754	429	.46137	.71023	.40800	37
24	462	.57575	938	.51658	471	.46046	066	.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	544	.57372	.66021	.51466	557	.45864	154	.40540	34
27	584	.57271	063	.51370	600	.45773	198	.40454	33
28	625	.57170	105	.51275	642	.45682	242	.40367	32
29	666	.57069	147	.51179	685	.45592	285	.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	748	.56868	230	.50988	771	.45410	373	.40109	29
32	789	.56767	272	.50893	814	.45320	417	.40022	28
33	830	.56667	314	.50797	857	.45229	461	.39936	27
34	871	.56566	356	.50702	900	.45139	505	.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	953	.56366	440	.50512	985	.44958	593	.39679	24
37	994	.56265	482	.50417	.69028	.44868	637	.39593	23
38	.64035	.56165	524	.50322	071	.44778	681	.39507	22
39	076	.56065	566	.50228	114	.44688	725	.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	158	.55866	650	.50038	200	.44508	813	.39250	19
42	199	.55766	692	.49944	243	.44418	857	.39165	18
43	240	.55666	734	.49849	286	.44329	901	.39079	17
44	281	.55567	776	.49755	329	.44239	946	.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	363	.55368	860	.49566	416	.44060	.72034	.38824	14
47	404	.55269	902	.49472	459	.43970	078	.38738	13
48	446	.55170	944	.49378	502	.43881	122	.38653	12
49	487	.55071	986	.49284	545	.43792	167	.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	569	.54873	071	.49097	631	.43614	255	.38399	9
52	610	.54774	113	.49003	675	.43525	299	.38314	8
53	652	.54675	155	.48909	718	.43436	344	.38229	7
54	693	.54576	197	.48816	761	.43347	388	.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	775	.54379	282	.48629	847	.43169	477	.37976	4
57	817	.54281	324	.48536	891	.43080	521	.37891	3
58	858	.54183	366	.48442	934	.42992	565	.37807	2
59	899	.54085	409	.48349	977	.42903	610	.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	57°		56°		55°		54°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	36°		37°		38°		39°		
	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	
0	0.72654	1.37638	0.75355	1.32704	0.78129	1.2794	0.80978	1.23490	60
1	699	.37554	401	.32624	175	.27917	.81027	.23416	59
2	743	.37470	447	.32544	222	.27841	075	.23343	58
3	788	.37386	492	.32464	269	.27764	123	.23270	57
4	832	.37302	538	.32384	316	.27688	171	.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	921	.37134	629	.32224	410	.27535	268	.23050	54
7	966	.37050	675	.32144	457	.27458	316	.22977	53
8	.73010	.36967	721	.32064	504	.27382	364	.22904	52
9	055	.36883	767	.31984	551	.27306	413	.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	144	.36716	858	.31825	645	.27153	510	.22685	49
12	189	.36633	904	.31745	692	.27077	558	.22612	48
13	234	.36549	950	.31666	739	.27001	606	.22539	47
14	278	.36466	996	.31586	786	.26925	655	.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	368	.36300	088	.31427	881	.26774	752	.22321	44
17	413	.36217	134	.31348	928	.26698	800	.22249	43
18	457	.36134	180	.31269	975	.26622	849	.22176	42
19	502	.36051	226	.31190	.79022	.26546	898	.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	592	.35885	318	.31031	117	.26395	995	.21959	39
22	637	.35802	364	.30952	164	.26319	.82044	.21886	38
23	681	.35719	410	.30873	212	.26244	092	.21814	37
24	726	.35637	456	.30795	259	.26169	141	.21742	36
25	.73771	1.3554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	816	.35472	548	.30637	354	.26018	238	.21598	34
27	861	.35389	594	.30558	401	.25943	287	.21526	33
28	906	.35307	640	.30480	449	.25867	336	.21454	32
29	951	.35244	686	.30401	496	.25792	385	.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	.35060	779	.30244	591	.25642	483	.21238	29
32	086	.34978	825	.30166	639	.25567	531	.21166	28
33	131	.34896	871	.30087	686	.25192	580	.21094	27
34	176	.34814	918	.30009	734	.25417	629	.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	267	.34650	.77010	.29853	829	.25268	727	.20879	24
37	312	.34568	057	.29775	877	.25193	776	.20808	23
38	357	.34487	103	.29696	924	.25118	825	.20736	22
39	402	.34405	149	.29618	972	.25044	874	.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	492	.34242	242	.29463	067	.24895	972	.20522	19
42	538	.34160	289	.29385	115	.21820	.83022	.20451	18
43	583	.34079	335	.29307	163	.24746	071	.20379	17
44	628	.33998	382	.29229	211	.24672	120	.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	719	.33835	475	.29074	306	.24523	218	.20166	14
47	764	.33754	521	.28997	354	.24449	268	.20095	13
48	810	.33673	568	.28919	402	.24375	317	.20024	12
49	855	.33592	615	.28842	450	.24301	366	.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	946	.33430	708	.28687	546	.24153	465	.19811	9
52	991	.33349	754	.28610	594	.24079	514	.19740	8
53	.75037	.33268	801	.28533	642	.24405	564	.19669	7
54	082	.33187	848	.28456	690	.23931	613	.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	173	.33026	941	.28302	786	.23784	712	.19457	4
57	219	.32946	988	.28225	834	.23710	761	.19387	3
58	264	.32865	.78035	.28148	882	.23637	811	.19316	2
59	310	.32785	082	.28071	930	.23563	860	.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19176	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	53°		52°		51°		50°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	40° Tan.	40° Cot.	41° Tan.	41° Cot.	42° Tan.	42° Cot.	43° Tan.	43° Cot.	
0	0.83910	1.19175	0.86929	1.15037	0.90040	1.11061	0.93252	1.07237	60
1	960	.19105	980	.14969	093	.10996	306	.07174	59
2	.84009	.19035	.87031	.14902	146	.10931	360	.07112	58
3	059	.18964	.082	.14834	199	.10867	415	.07049	57
4	108	.18894	133	.14767	251	.10802	469	.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	208	.18754	236	.14632	357	.10672	578	.06862	54
7	258	.18684	287	.14565	410	.10607	633	.06800	53
8	307	.18614	338	.14498	463	.10543	688	.06738	52
9	357	.18544	389	.14430	516	.10478	742	.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	457	.18404	492	.14296	621	.10349	852	.06551	49
12	507	.18334	543	.14229	674	.10285	906	.06489	48
13	556	.18264	595	.14162	727	.10220	961	.06427	47
14	606	.18194	646	.14095	781	.10156	.94016	.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	706	.18055	749	.13961	887	.10027	125	.06241	44
17	756	.17986	801	.13894	940	.09963	180	.06179	43
18	806	.17916	852	.13828	993	.09899	235	.06117	42
19	856	.17846	904	.13761	.91046	.09834	290	.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	956	.17708	.88007	.13627	153	.09706	400	.05932	39
22	.85006	.17638	059	.13561	206	.09642	455	.05870	38
23	057	.17569	110	.13494	259	.09578	510	.05809	37
24	107	.17500	162	.13428	313	.09514	565	.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	207	.17361	265	.13295	419	.09386	676	.05624	34
27	257	.17292	317	.13228	473	.09322	731	.05562	33
28	308	.17223	369	.13162	526	.09258	786	.05501	32
29	358	.17154	421	.13096	580	.09195	841	.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	458	.17016	524	.12963	687	.09067	952	.05317	29
32	509	.16947	576	.12897	740	.09003	.95007	.05255	28
33	559	.16878	628	.12831	794	.08940	062	.05194	27
34	609	.16809	680	.12765	847	.08876	118	.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	710	.16672	784	.12633	955	.08749	229	.05010	24
37	761	.16603	836	.12567	.92008	.08686	284	.04949	23
38	811	.16535	888	.12501	062	.08622	340	.04888	22
39	862	.16466	940	.12435	116	.08559	395	.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	963	.16329	.89045	.12303	224	.08432	506	.04705	19
42	.86014	.16261	097	.12238	277	.08369	562	.04644	18
43	064	.16192	149	.12172	331	.08306	618	.04583	17
44	115	.16124	201	.12106	385	.08243	673	.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	216	.15987	306	.11975	493	.08116	785	.04401	14
47	267	.15919	358	.11909	547	.08053	841	.04340	13
48	318	.15851	410	.11844	601	.07990	897	.04279	12
49	368	.15783	463	.11778	655	.07927	952	.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	470	.15647	567	.11648	763	.07801	064	.04097	9
52	521	.15579	620	.11582	817	.07738	120	.04036	8
53	572	.15511	672	.11517	872	.07676	176	.03976	7
54	623	.15443	725	.11452	926	.07613	232	.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	725	.15308	830	.11321	.93034	.07487	344	.03794	4
57	776	.15240	883	.11256	088	.07425	400	.03734	3
58	827	.15172	935	.11191	143	.07362	457	.03674	2
59	878	.15104	988	.11126	197	.07299	513	.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	Cot.	Tan.	M.
	49°		48°		47°		46°		

APPENDIX IV.—Natural tangents and cotangents—Con.

M.	44° Tan.	44° Cot.		M.	44° Tan.	44° Cot.		M.	44° Tan.	44° Cot.	
0	0.96569	1.03553	60	20	0.97700	1.02355	40	40	0.98843	1.01170	20
1	625	.03493	59	21	756	.02295	39	41	901	.01112	19
2	681	.03433	58	22	813	.02236	38	42	958	.01053	18
3	738	.03372	57	23	870	.02176	37	43	.99016	.00994	17
4	794	.03312	56	24	927	.02117	36	44	073	.00935	16
5	.96850	1.03252	55	25	.97984	1.02057	35	45	.99131	1.00876	15
6	907	.03192	54	26	.98041	.01998	34	46	189	.00818	14
7	963	.03132	53	27	098	.01939	33	47	247	.00759	13
8	.97020	.03072	52	28	155	.01879	32	48	304	.00701	12
9	076	.03012	51	29	213	.01820	31	49	362	.00642	11
10	.97133	1.02952	50	30	.98270	1.01761	30	50	.99420	1.00583	10
11	189	.02892	49	31	327	.01702	29	51	478	.00525	9
12	246	.02832	48	32	384	.01642	28	52	536	.00467	8
13	302	.02772	47	33	441	.01583	27	53	594	.00408	7
14	359	.02713	46	34	499	.01524	26	54	652	.00350	6
15	.97416	1.02653	45	35	.98556	1.01465	25	55	.99710	1.00291	5
16	472	.02593	44	36	613	.01406	24	56	768	.00233	4
17	529	.02533	43	37	671	.01347	23	57	826	.00175	3
18	586	.02474	42	38	728	.01288	22	58	884	.00116	2
19	643	.02414	41	39	786	.01229	21	59	942	.00058	1
20	.97700	1 02355	40	40	.98843	1.01170	20	60	1.00000	1.00000	0
	Cot.	Tan.	M.		Cot.	Tan.	M.		Cot.	Tan.	M.
	45°				45°				45°		

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