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ABSTRACT

This guide continues instruction of the three guides in SE 012 723 and of the two guides included in SE 012 724. Counting skills, appreciation of pattern and order, and understanding of place value are extended to numbers up to one million, and the associative, commutative and distributive laws are made explicit. Manipulation of equations is stressed throughout. This section contains a systematic approach to the development of automatic response in the basic number facts, and many activities are suggested for practicing these skills. Formal algorithms are discussed at length, but the teacher is warned not to develop these until the pupil is thoroughly familiar with place value, the laws of operation, the basic number facts, and manipulation of equations. There is also further work with fractions. The booklet concludes with a glossary and a summary of the contents of Sections A to F. (MM)

CURRICULUM GUIDE
PURE NUMBER COURSE

Section F—Extended Study of Basic Mathematical Ideas Using the Numbers to 144

Education Department of Victoria

1603/65.

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Section F—Extended Study of Basic Mathematical Ideas Using the Numbers to 144

INTRODUCTION

Section F sees the integration of the work done in the earlier sections. Experiences gained in carefully designed situations are drawn together.

As in previous sections, the various topics must be dealt with in parallel. However, teachers must realize that the activities leading towards formal processes demand, as prerequisites, work on the basic laws, rapid recall of number facts within the capacity of each individual child, and a knowledge of place value. Therefore, the systematic refinement of activities leading to formal processes will not be introduced until late in Section F.

A child will achieve the depth of understanding that allows him to move from Section E to Section F in the various topics at different times. This principle will also apply in the move from Section F to Section G. Teachers will realize that, as a child passes through Section F, his experience, his understanding, and his inquiry may logically take him to a position where he may usefully proceed with items classified as being in Section G.

The transition from Section F to Section G is not an easily defined one. There is no rigid border between the two. The answer to the question of how far a child should proceed depends on whether his understanding and his experiences are such that he can proceed to the next step, rather than on whether the work is classified as being in Section F or in Section G. Moreover, Section F should be viewed as a whole and the topics seen in relation one with the other. Care should be taken that one topic, such as formal process work, does not predominate. For instance, the child's creative and imaginative ideas should be continually encouraged. The development of an increased facility in calculation is an important aim, but only one of a number of equally important aims in this section.

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Oral work is still of the utmost importance. Planning should ensure that the time devoted to oral work is spread over all the topics in Section F. Teachers should continue to encourage children to observe and to discuss freely the ideas covered in written work. As far as possible some part of each period should be devoted to oral work.

During this course children have been encouraged to discover and use a variety of approaches when solving problems, therefore the teacher must be prepared to accept unorthodox but valid methods of solution.

In previous sections, emphasis has been placed on the use of the Cuisenaire material in order to establish mathematical ideas. Other concrete aids are also used during this section, for example, geometric shapes, abacus, applied number materials, equalizer, and arrays. Number lines will also prove a very useful aid in the treatment of most topics.

The question of the time necessary for a child of average ability to complete Section F needs careful consideration. The teacher, recalling that there is no clear line of demarcation between Section F and Section G, may find that most children require at least three school terms to complete the work in Section F. Some may take at least four terms. It is essential that sufficient time be given to permit the child to explore and discover for himself, instead of having the ideas formally presented to him by his teacher.

PART 1. PATTERN AND ORDER IN THE NUMBER SYSTEM

AIM

To continue the development of counting skills and to deepen the realization of pattern and order in the number system.

NOTES

1. Stage 23 (Section D) and Stage 27 (Section E) should be read in conjunction with this topic.

2. The work of this topic proceeds, throughout the entire section, parallel with and reinforcing work dealt with in other topics.

Some of the work outlined here is of value in other aspects of mathematics. For example, serial addition and subtraction will be used to speed computation, and some of the exercises in sequences are planned to assist in the activities leading to the formal process work. Their inclusion here, however, is to emphasize the logic of the number system.

3. Initially, most of the work will be oral because extensive use of written work would seriously limit the amount covered. As the child proceeds, more and more written activities will be undertaken. Throughout the topic, however, stress should still be placed on oral work.

DEVELOPMENT

Note : Four aspects of work are outlined below. Each continues and develops similar work from earlier sections, therefore the four should proceed side by side, not consecutively.

1. Counting Any Reasonable Sequence of Numbers within One Million

a **By Ones**—This activity serves mainly to make the child aware of large numbers. There is no point in counting extensively with the very large numbers. Appreciation of numbers beyond 1,000 is gained through the study of place value rather than through counting. The child should count forwards and backwards within specified limits.

e.g. By ones—from 36,327 to 36,356
—from 9,987 to 10,005
—from 65,090 to 65,110
—from 42,347 to 42,316
—from 16,011 to 15,996

Care should be taken to include examples that require alteration to the digit in places other than the units.

To show that in some counting with larger numbers only the last digit or digits will be affected, the following approach could be used :

Count by ones from	16	to	34
Count by ones from	216	to	234
Count by ones from	5,716	to	5,734
Count by ones from	48,916	to	48,934

This could be illustrated in the following way : A cloud (or a tree or a card) is hiding part of a number, leaving only the final digits visible. Behind the cloud are any digits desired, but these remain constant unless the count is extended considerably.

For example, to count forwards by ones from  879 :—

i If 1 is the hidden digit, then the starting number is 1,879.

ii If 176 are the hidden digits, then the starting number is 176,879.

Counting devices such as those found in duplicators and motor-cars (trip meters) help to reinforce the idea that, in counting by ones, the digits change in groups of ten, with the units changing the most frequently. Similarly, the tens digit changes more frequently than the hundreds digit, and any digit changes more frequently than any other digit to its left.

b **Group Counting from Nought (Zero)**—by twos, by threes, by fours, and by all the numbers to twelves. The work commenced in Section D and further developed in Section E must be extended. Abstract counting within the range 0—144 is required to assist with relevant multiplication tables. The usual upper limit of this counting will be 12 times the number being added, for example, 108 in the case of addition of nines.

Further extensions of group counting are possible through digit patterns.

c **Group Counting from Any Number**—by twos, by threes, by fours, and by all the numbers to twelves. This also is a continuation of similar work in Section E. Number charts and digit patterns probably provide the most effective means of carrying out this activity. (See Development 2 below.)

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2. Patterns in Sequences and Use of Number Charts

Note : The discovering of a pattern is more important than mere knowledge. However, where previously it was sufficient for a child to discover a pattern and accept it as a matter of interest, he should now be encouraged to seek reasons for the patterns he discovers. Most children are capable of seeing the reasons for simple patterns, though the ability to do this varies from one child to another.

a **Digit Patterns**—As well as the work outlined below, digit patterns form the basis of most of the number chart activities. (See b below.)

i The final digit patterns discovered in previous sections should be applied to larger numbers. The child who knows that the "final digit" pattern made when he counts by eights from zero is 8 6 4 2 0 can use the pattern to count by eights from, for example, 1,736.

ii Patterns also appear in places other than the units place.

Recognition of these patterns allows predictions to be made, as shown by digits in bold type.

The patterns can be checked by noting the difference between any two consecutive numerals.

8
16
24
32
40
48
56
64
72
80
88
96
104
112
120
128

b **Number Charts**—Exercises of the types suggested in Stage 23 and Stage 27 will form a satisfactory starting point. However, owing to the increasing maturity of the child and the length of time taken for Section F, teachers should ensure that the work increases in complexity according to the child's ability and is not a mere repetition of earlier experience. Quarter-inch squared paper will be found useful for children's individual work.

There is no need for very large numbers to be used on a number chart ; pattern when seen in the number range 0—144 can easily be applied, if appropriate, to large numbers.

The "nines" chart is reproduced here with a sample of possible uses. Similar ideas should be used with other charts, and other ideas will be found in the many books now dealing with pattern in number. (See lists of books published from time to time in the *Education Gazette*.)

1	2	3	4	5	**6**	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	**30**	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	**56**	57	58	59	60	61	62	63
64	65	**66**	67	68	69	70	71	72
73	74	75	**76**	77	78	79	80	81
82	83	84	85	**86**	87	88	89	90
91	92	93	94	95	**96**	97	98	99
100	101	102	103	104	105	**106**	107	108

i Counting by nines from any number reveals a digit pattern. Once seen, this can be applied to counting by, or adding on, nines at any level, for example, **3,264** ; **3,273** ; **3,282** ; **3,291** ; **3,300** ; **3,309** ; **3,318** . . .

or

2,653 + **9** = 2,662 ; **5,842** + **9** = 5,851 ;
10,039 + **9** = 10,048 .

ii For each of the numerals in the same vertical column of the chart, the sum or the extended sum of the digits is constant, for example, 6 ; 1 + 5 = 6 ; 2 + 4 = 6 ; . . . . 6 + 9 = 15 (1 + 5 = 6) ; . . . . 1 + 0 + 5 = 6. Why is this so ? Is it so on other charts ?

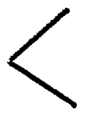
iii Why does a downward left-to-right diagonal always pass through numerals having the same final digit ?

iv What is the corresponding pattern of numerals in the downward right-to-left diagonal ? What is the reason for this pattern ?

(The last two questions would probably involve a study and discussion of the final digit pattern formed by moving once horizontally then once vertically on the chart.)

v Why does counting by threes on this chart show a vertical pattern ? Does any other count show a vertical pattern ? Look at vertical patterns on other charts to help find the reason.

vi Why are the numbers in each column alternately odd and even ?

vii Lines can be marked on the chart, and the resulting patterns examined. (This is particularly valuable where mixed counting techniques are involved.) Examples are a shape such as this  drawn from 6 to 46 then to 106 (shown on the chart by numbers in bold type), and zig-zag patterns (such as the one marked on the chart with a heavy black line).

viii Discover the pattern and continue the line (as indicated on the chart by the dotted line). Such an exercise enables the child to make predictions, which he can then check.

The questions included in the above exercises will encourage the child to seek reasons for the patterns.

It must be remembered that children's abilities cover a wide range. While some will be unable to cope with all the above activities, others will seek even more. For these advanced children, much enrichment material will be found in the books mentioned earlier (page 7).

c **Odd and Even Numbers**—Interesting and profitable discoveries can be made from studies of odd and even numbers. Problems such as the following could be investigated.

What sort of number results

i When two even numbers are added ? (e.g., 6 + 4)

ii When three even numbers are added ? (e.g., 2 + 4 + 6)

iii When four even numbers are added ? (e.g., 8 + 2 + 6 + 4)

iv When two odd numbers are added ? (e.g., 3 + 7)

v When three odd numbers are added ? (e.g., 1 + 5 + 3)

vi When five odd numbers are added ? (e.g., 7 + 5 + 3 + 1 + 9)

vii When an odd and an even number are added ? (e.g., 5 + 4) viii When two odd numbers and one even number are added ? (e.g., $3 + 5 + 2$)

ix When one odd number and two even numbers are added ? (e.g., $3 + 2 + 4$)

Many examples need to be given in each instance, and the order need not be as above. The conclusions drawn from such a study will open the way for further investigation and will assist the child in estimation and in checking later work in calculation.

d **Discovering and Continuing Pattern in a Sequence**—More complicated sequences than those formerly used should now be presented. The child should be asked to state the pattern discovered, and to continue it as far as required.

e.g.,	$\frac{1}{2}$	1	$1\frac{1}{2}$	—	—	3		
	28	27	25	24	22	—	—	—
	2	4	8	16	—	—	—	
	48	24	12	—	—	—		
	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	—	—	—	
	1	2	4	7	11	—	—	—
	0	1	3	6	10	—	—	—

Variations :

A3	B6	C9	—	—	—
A4Z	B8Y	C12X	—	—	—

3 dogs, 2 cats, 5 animals ; 4 dogs, 3 cats, 7 animals ; 5 dogs, 4 cats, 9 animals ; — — —

3. Doubling and Halving

a **From Any Number**—In Stage 27 (Section E) it was stressed that, in doubling and halving, the point of the exercise was to give the child an opportunity to think and to apply the understandings gained in that section and earlier sections, and that the teaching of formal rules should be very carefully avoided.

In Section F it is still vitally important that the child should continue to discover and develop his own techniques of doubling and of halving. But the child who has not yet achieved success may be given some assistance. The type of assistance is important. He should not be told to use one specific method, but rather, he should become aware of various other methods by hearing and by observing those used by other children. Activities directly connected with other topics, for example, place value and fractions, provide the child with opportunities for making discoveries which he can subsequently apply to doubling and halving.

It is now required that doubling and halving should commence from any number. Reference to Stage 27 may be found he[illegible]

b Using Multiplication Ideas—In Stage 27, it was mentioned that some children discover that they can double and halve with multiplication and apply their experience to creative work. It is now required that all children explore the ideas.

A method of approach is to take an equation, such as $2 \times 16 = 32$, double one factor and determine what the other must be to retain the same product.

e.g., $2 \times 16 = 32$
$4 \times \square = 32$
$8 \times \square = 32$
$16 \times \square = 32$
$32 \times \square = 32$

Examination of the factors supplied shows that they are being halved in each step. Experience with more examples leads the child to the conclusion that if one factor is doubled and the other halved the product remains the same. It is a simple matter then to continue the exercise using fractions.

Other examples, such as those listed below, are self-explanatory.

Double Second Factor	*Double Product*
$8 \times 3 = 24$	$2 \times 4 = 8$
$\square \times 6 = 24$	$2 \times \square = 16$
$\square \times 12 = 24$	$2 \times \square = 32$
$\square \times 24 = 24$	$2 \times \square = 64$

Double Both Factors

$2 \times 1 = 2$
$4 \times 2 = \square$
$8 \times 4 = \square$

(For detailed information on the use of frames see Use of Frames, page 104.)

To exercise the knowledge gained, a child could be given an equation, such as $4 \times 3 = 12$, and asked to develop other equations from it, using doubling and halving. Resulting equations could be—

$8 \times 3 = 24$	$8 \times 1\frac{1}{2} = 12$
$4 \times 6 = 24$	$2 \times 6 = 12$
$8 \times 6 = 48$	$2 \times 3 = 6$
	$4 \times 1\frac{1}{2} = 6$

and each of these could be further developed if required. The usefulness of such activities in developing and in exercising multiplication tables will be obvious.

Note : Doubling and halving exercises with division can be carried out by more advanced children.

4. Serial Addition and Subtraction

a The traditional activities of serial addition and subtraction are of use in this topic in emphasizing a particular pattern in the number system.

The child should be give.ı a number of activities such as :—

$$\begin{array}{rcl}
7+8= 15 & \quad & 15-9= 6 \\
17+8= 25 & & 25-9= 16 \\
27+8= 35 & & 35-9= 26 \\
37+8= 45 & & 45-9= 36 \\
\vdots & & \vdots \\
177+8=185 & & 215-9=206 \\
187+8=195 & & 225-9=216 \\
197+8=205 & & 235-9=226 \\
207+8=215 & & 245-9=236 \\
\vdots & & \vdots
\end{array}$$

The pattern should be applied to larger numbers, for example,

$$\begin{array}{r}
8+9= 17 \\
38+9= 47 \\
268+9= 277 \\
3{,}478+9= 3{,}487 \\
65{,}348+9=65{,}357
\end{array}$$

Once these exercises have served their primary purpose, that is, to illustrate this pattern in the number system, they can also be used to speed computation. (See Note 2, page 5.)

Extensions of the above exercises should include series such as :—

$$\begin{array}{rcl}
9+ 7= 16 & \quad & 8- 5=3 \\
9+ 17= 26 & & 18- 15=3 \\
9+ 27= 36 & & 28- 25=3 \\
\vdots & & \vdots \\
9 \quad 127=136 & & 128-125=3 \\
\vdots & & \vdots \\
9+627=636 & & 628-625=3
\end{array}$$

These patterns can also be seen as illustrations of the axioms : "If equals are added to equals the results are equal" and "If equals are added to unequals the difference remains constant"—thus showing that mathematics includes the study of topics that are gradually becoming integrated in the child's mind.

b Another type of sequence important to later work with formal processes can be illustrated by exercises such as those listed below :—

$$\begin{array}{rcl}
2+ 7= 9 & \quad & 9- 6= 3 \\
20+ 70= 90 & & 90- 60= 30 \\
200+ 700= 900 & & 900- 600= 300 \\
2{,}000+7{,}000=9{,}000 & & 9{,}000-6{,}000=3{,}000
\end{array}$$

$$
\begin{array}{rr}
2 \times 3 = 6 & 2 \times 3 = 6 \\
2 \times 30 = 60 & 20 \times 3 = 60 \\
2 \times 300 = 600 & 200 \times 3 = 600 \\
2 \times 3{,}000 = 6{,}000 & 2{,}000 \times 3 = 6{,}000
\end{array}
$$

When the idea has been established, other starting points should be taken, e.g., $8 + 7 = 15$, $13 - 6 = 7$, $3 \times 4 = 12$.

In division, it is sufficient at this stage to work with the divisor unchanged, e.g.,

$$
\begin{array}{rr}
6 \div 3 = 2 & 12 \div 4 = 3 \\
60 \div 3 = 20 & 120 \div 4 = 30 \\
600 \div 3 = 200 & 1{,}200 \div 4 = 300 \\
6{,}000 \div 3 = 2{,}000 & 12{,}000 \div 4 = 3{,}000
\end{array}
$$

When children have seen the pattern in these sequences, they could be given exercises to use the ideas, thus linking this topic with place value and activities leading towards formal processes, e.g.,

$$
\begin{array}{rr}
3 + 4 = 7 & 3 \times 3 = 9 \\
30 + 40 = \square & 30 \times 3 = \square \\
300 + \square = 700 & \square \times 3 = 900 \\
\square + 4{,}000 = 7{,}000 & \square \times 3 = \square
\end{array}
$$

Subtraction and division sequences would be included, and the position of the place holder varied.

TESTING

Testing procedures would differ little, if at all, from those used in daily work.

While it is possible to test counting formally (either orally or in writing), any such testing for realization of pattern and order would defeat the purpose of the exercises. A child's progress in this direction is best gauged by observation of his everyday work.

PART 2. NOTATION, NUMERATION, AND PLACE VALUE

AIM

To extend the study of notation, numeration, and place value.

NOTES

1. The child now learns to read and write numerals to 1,000,000 and words to one thousand. He is introduced to the decimal notation for tenths.

2. The study of place value, which is extended in this stage to thousands, is essential for—

a an understanding of our number system ; and

b the development of the formal processes of addition, subtraction, multiplication, and division.

3. The child should be developing an awareness of the dual value of a digit. In the numeral 476, the 7 has the absolute value 7, and, because of its position in the numeral, it is identified as 7 tens or 70, and it accounts for 70 of the total 476.

4. This work is a direct continuation of similar work in Section E, therefore no delay in its introduction is warranted. It should proceed throughout the section.

DEVELOPMENT

Note : The first three topics treated below should be taken side by side, and not in sequence.

1. Recognition and Writing of Numerals to 1,000,000

a The child is familiar with numerals to 1,000 and it is a simple step for him to read and to write numerals such as 2,000 ; 5,000 ; 9,000. The teacher now writes, and the child reads, examples such as the following :—

426	1,426	3,426	8,426
512	1,512	4,512	7,512
604	1,604	4,604	9,604

Special treatment should be given to examples such as :—

82	1,082	5,082	9,082
9	1,009	6,009	8,009

b Once the child is confident with the above, he can work with larger numerals. He can, if necessary, work first with whole thousands, e.g., 9,000 ; 10,000 ; 20,000 ; 67,000 ; 892,000 before proceeding to numerals such as 26,324; 47,802; 623,496; 504,912; 800,420 ; 500,016 ; 700,002.

c The final step is the introduction of one million. A suitable approach could be to ask the child to read 700,000; 800,000; 900,000; 1,000,000. The last numeral is likely to be read as " one thousand thousand ". If the child does not know another name for this, the teacher should introduce the word " million ".

Note: It is important to note that the comma is used to separate the units, the thousands, and the millions periods (or families). Recognition of this fact facilitates the reading and the writing of numerals.

2. Recognition and Writing of Words to One Thousand

As mentioned in other sections, this is not strictly a mathematical skill, and can be treated in reading and spelling periods.

3. Place Value to Thousands

a **The Thousands Place**—In Section E (see Stage 28) place value to 999 was studied. The child should readily discuss numerals in terms of the units place, the tens place, and the hundreds place.

Now he should be shown that in, for example, 6,429 there is another place, namely the thousands place. Much practice should be given in reading the values of digits according to their places, and studying place value in a variety of ways. A possible procedure is as follows :

2,356
Read this numeral.
How many digits are used ?
Why is the comma used ?
How many " places " are in this numeral ?
In which place is the 3 written ?
What number does the 5 represent ?

Numbers such as 2,222 and 5,555 are particularly useful in illustrating the relative values of places, especially when the notation is extended. (See c, Extended Notation, below.)

b **The Tens-of-thousands Place and the Hundreds-of-thousands Place**—When the child is competent in the work with the thousands place, he should extend the study to tens-of-thousands, and then to hundreds-of-thousands. A possible procedure is as follows :

28,479
Read this numeral.
How many places are used?
How does the comma help you to read the numeral ?
To what family does the 479 belong ?
To what family does the 28 belong ?
Which digit is in the thousands place ?
How many thousands are there altogether ?
What name is given to the place where the 2 is written ?
Read the value of each separate digit.

A similar procedure could be used for the hundreds-of-thousands place.

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Consolidation exercises, such as those listed in Section E, page 18, Step 9, extended to include the thousands places, should be used. The sign $>$ (is greater than) and the sign $<$ (is less than), if not used before, should be introduced in conjunction with these exercises.

e.g., Use the correct sign $>$ or $<$ in the following :—

5,050 □ 5,005
32,927 □ 32,972

(For detailed information on the use of frames see Use of Frames, page 104.)

It is important that numerals including zero should be studied, e.g., 7,208 ; 16,096 ; 324,080 ; 402,960; 260,050.

c **Extended Notation**—This exercise, commenced in Section E, is essential to Part 7, Activities Leading towards Formal Processes, dealt with later in this section. It also helps to show the values of separate digits and makes it clear that each digit has a greater value than any written to the right of it.

There should first be the simple extension of notation. For example :—

6,279 = 6,000 + 200 + 70 + 9
14,398 = 10,000 + 4,000 + 300 + 90 + 8
206,564 = 200,000 + 0 + 6,000 + 500 + 60 + 4

This is followed by exercises such as :—

2,846 = 2,000 + □ + 40 + 6
35,752 = □ + 5,000 + 700 + 50 + 2
40,000 + 8 + 900 + 200,000 + 3,000 + □ = 243,958
4,000 + 600 + 80 + 2 = □
□ = 30,000 + 7,000 + 400 + 80 + 6
200 + 7 + 600,000 + 40 + 3,000 + 90,000 = □

These are types only ; many variations are possible by altering the position of the frame.

It may be found useful to include words in the variations. For example :—

3,946 = 3 thousands + 9 hundreds + 4 tens + 6 units
8 units + 4 hundreds + 2 thousands + 5 tens = □
5,679 = 5 . . . + □ tens + 9 . . . + □ hundreds
46,387 = 4 tens-of-thousands + 6 . . . + 8 . . . + 7 . . . + □ hundreds.

Note : For slower pupils it will probably be unprofitable to attempt complicated exercises with six places.

d **Effects of Digits Written to the Left or to the Right of a Numeral**—One approach is suggested below. Questioning should be as varied as possible to allow for the deepest understanding, and answers in varying terms should be sought rather than stereotyped responses.

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i Consider 3. Read this numeral. (Three.) In what " place " is the 3 written ? (The units place.) What is its value ? (3 units.)

Now consider 23. Read this numeral. (Twenty-three.) Which digit was not used in the first number ? (2.) In what place is the 2 written ? (The tens place.) What is it worth (20.) In which place is the 3 ? (The units place.) What is it worth ? (3 units.) Has it changed in value ? (No.)

Now consider 523. Read this numeral. Question as before, leading the child to see that the 3 retains its value of 3 units.

A procedure such as the above should lead the child to see that, whereas each new digit occupies a new place, the original 3 retains its units place and therefore its original value. Extended notation reinforces the idea :

$$3 = 3$$
$$23 = 20 + 3$$
$$523 = 500 + 20 + 3$$

ii Consider 9. Read this numeral. (Nine.) In what place is the 9 written ? (The units place.) What is its value ? (9 units.)

Now consider 94. Read this numeral. (Ninety-four.) In what place is the 9 written ? (The tens place.) What number does it represent? (90.) Why has 9 now a different value ? (Because the 4 has replaced it in the units place.)

Now consider 947. Read this numeral. (Nine hundred and forty-seven.) In which place is the 9 written ? (The hundreds place.) What number does it represent now ? (900.) In 94, the 9 meant 90. In 947, it means 900. What makes the difference ? (The seven is now in the units place, and the other digits have changed their positions.)

Further questioning should lead the child to see that **each new digit written to the right has the effect of moving the original digit to a new place, thus increasing its value tenfold.** Extended notation reinforces the idea :

$$9 = 9$$
$$94 = 90 + 4$$
$$947 = 900 + 40 + 7$$

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e **Renaming Numbers**—Another exercise that helps the child to understand values of numbers is the renaming of numbers in various ways.

e.g.,
6 = 6 units
76 = 7 tens 6 units
76 = 76 units
876 = 8 hundreds 76 units
876 = 876 units
9,876 = 9 thousands 876 units
9,876 = 9,876 units

6 = 6 units
76 = 7 tens 6 units
876 = 87 tens 6 units
9,876 = 987 tens 6 units

6 = 6 units
76 = 76 units
876 = 8 hundreds 76 units
9,876 = 98 hundreds 76 units

Variations of the exercise :—
4 thousands 287 units = □
32 hundreds 92 units = □
596 = □ tens □ units
8,319 = □ thousands □ units.

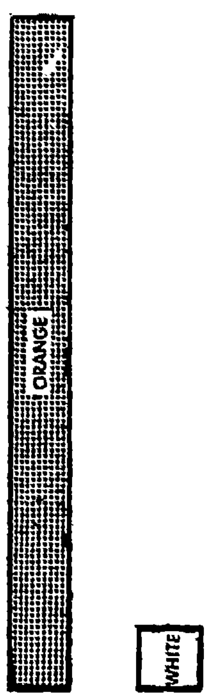
4. Extension of Decimal Notation to Tenths

The purpose is to master the convention of expressing tenths as part of the decimal notation. Prerequisites are :—

a Familiarity with tenths as vulgar fractions.

b Experience with value relations specifically in the ratio 1 to 10.

For example, using Cuisenaire rods :—



If orange has the value 10, what is the value of the white ? (1)

If orange has the value 100, what is the value of the white ? (10)

If orange has the value 1, what is the value of the white ? $(\frac{1}{10})$

It can be pointed out that, in all work so far, it has been understood that the right-hand digit of a numeral represents units, but it has not been labelled as such. The child can now be shown that numbers can be written with a special sign, thus, 111· 33· 462· 9· 320·. The special sign (·), called a decimal point, is a reminder that the digit just beside it to the left stands for a number of units.

A study of the relative values in 1,111 brings out the fact that each 1 is ten times the value of the 1 on its right, and one-tenth of the value of the one on its left.

Suggested Procedure for Introducing Tenths

Consider 11·	Establish the relative values of the digits.
Now consider 11·1	If each 1 is one-tenth of the value of the 1 to its left, what is the value of the right-hand 1 ? ($\frac{1}{10}$)
	What name could be given to this new place ? (Tenths.)
	The number can be read as " eleven and one tenth " or as " eleven point one ".
Now consider 2·2	Read this number. (Two and two tenths or two point two.) What is the name of the sign? (Decimal point.) Why is it used ? (To show the units place.) What does the right-hand 2 represent ? (Two tenths.) What is the name of the place to the right of the units place ? (Tenths.)

Other numbers should be similarly treated, e.g., 3·3, 2·6, 0·4, 0·6.

Notes : 1. Emphasis in verbalization should be strongly on " tenths " (e.g., " four and five tenths " rather than " four point five ") to ensure understanding of the place value involved.

2. The use of " 0 " in numbers consisting of tenths only (e.g., 0·4) helps the child to remember that the decimal point relates to the units place. The notation can later be refined to ·4, where the numeral in the units place is understood as being 0.

Relation to Vulgar Fraction Notation—The child should now see the alternative forms of notation for tenths, understanding that :—

·1 represents 1 in the tenths place, and can be written $\frac{1}{10}$

·7 represents 7 in the tenths place, and can be written $\frac{7}{10}$

2·3 represents 2 units and 3 tenths and can be written $2\frac{3}{10}$

Practice needs to be given in using the alternative forms.

PART 3. AUTOMATIC RESPONSE

AIM

To develop automatic response in the basic addition and subtraction facts to 20, and the multiplication and division facts to 144.

NOTES

1. Automatic response in the basic number facts is an essential tool for efficient calculation. Recall of these facts needs to be—

a accurate, and

b immediate.

2. The facts to which automatic response is required involve one step only. In addition and subtraction the upper limit is 20. The multiplication and division facts are those contained in the traditional tables to $12 \times 12 = 144$.

3. In each of the preceding sections, the aim has been primarily to establish and develop an understanding of certain basic mathematical ideas. At no time has there been a demand to develop automatic response to number combinations. However, in gaining this understanding, the child has had considerable incidental experience of number facts—

a from the reading of, and the creating of, equations, and

b by means of substitution work, group counting, and work with number patterns, often deliberately planned to increase the experience.

As a result, many children have a fairly rapid recall of some number facts. But automatic response is too important to be left to incidental teaching, and a systematic study needs to be undertaken.

4. It is now necessary—

a to **tabulate** facts, to ensure that all have been covered, and to limit the field in which a child works at a given time ;

b to **study** these facts systematically in order to ensure **correct** response ; and

c to **make frequent use** of the facts in varied and interesting ways to speed response and ensure retention.

Once a child is accurate in a certain set of facts, he should begin working on them for speed, while at the same time working on another set for accuracy.

5. It was suggested (Section E, pages 4, 27) that, towards the end of Section E, the child who had sufficient experience of numbers to 10 could classify the relevant facts and commence working on them for automatic response. Not all children would have done this, and no treatment was given in Section E. The work outlined below covers all the required facts and tables.

Work for automatic response must commence at the beginning of Section F and continue in one form or another throughout. In fact, consolidation exercises will be necessary even in future sections.

6. The child should realize that knowledge of one fact implies knowledge of others. By applying his experience of the inter-relation of operations, axioms, and the commutative property of addition and multiplication (informally experienced before and formally treated in this section) the child can see the following :—

$7 + 7 = 14$	$2 \times 7 = 14$	$14 \div 7 = 2$	$\frac{1}{2}$ of $14 = 7$
$14 - 7 = 7$	$7 \times 2 = 14$	$14 \div 2 = 7$	$\frac{1}{7}$ of $14 = 2$

$5 + 3 = 8$	$15 + 3 = 18$	$5 + 13 = 18$
$3 + 5 = 8$	$3 + 15 = 18$	$13 + 5 = 18$
$8 - 3 = 5$	$18 - 3 = 15$	$18 - 13 = 5$
$8 - 5 = 3$	$18 - 15 = 3$	$18 - 5 = 13$

7. Teachers may find it useful to have the child keep his own book for the recording of work covered.

8. Attention is drawn to the statements in the Introduction to the Course of Study for Primary Schools, Mathematics, 1964, regarding individual differences in children's attainments.

9. Because the field to be covered is so wide, it is obvious that a limit must be set to the area in which a child works at a given time. This involves the grouping of facts according to some plan, and this can be done in various ways. Some plans and groupings of facts within plans are suggested below.

SUGGESTED PLANS FOR ORDER OF WORKING

A. 1. Addition, subtraction, multiplication, and division facts to 10.

2. Addition, subtraction, multiplication. and division facts to 20.

3. Multiplication and division facts to 144 (allied to group counting) by—

   a twos, fours, eights ;

   b fives, tens ;

   c threes, sixes, twelves ;

   d nines ;

   e elevens ;

   f sevens.

B. 1. Addition and subtraction facts to 10.
   2. Addition and subtraction facts to 20.
   3. Multiplication and division facts to 20.
   4. Multiplication and division facts to 40.
   5. Multiplication and division facts to 80.
   6. Multiplication and division facts to 144.
   (For 4 and 5, any appropriate limits could be substituted.)

C. 1. Addition and subtraction facts to 10.
   2. Addition and subtraction facts to 20.
   3. Multiplication and division facts by doubling factors and products.

Within any plan, grouping of facts is necessary—

a to limit the facts being studied at a given time, and

b because an orderly arrangement assists memorization and rapid recall.

SUGGESTED TABULATIONS OF FACTS

Note : While the groupings of facts shown below are intended initially to ensure that all facts are covered, it should be noted that the building up of such lists is in itself a step towards gaining accurate response.

No grouping should be used exclusively, and it is profitable for a child to see facts in various settings.

A. ADDITION AND SUBTRACTION

1. Facts about a Given Number

e.g., 7

7 + 0 = 7	7 − 0 = 7
0 + 7 = 7	7 − 7 = 0
6 + 1 = 7	7 − 1 = 6
1 + 6 = 7	7 − 6 = 1
5 + 2 = 7	7 − 2 = 5
2 + 5 = 7	7 − 5 = 2
4 + 3 = 7	7 − 3 = 4
3 + 4 = 7	7 − 4 = 3

12

6 + 6 = 12	12 − 6 = 6
7 + 5 = 12	12 − 5 = 7
5 + 7 = 12	12 − 7 = 5
8 + 4 = 12	12 − 4 = 8
4 + 8 = 12	12 − 8 = 4
9 + 3 = 12	12 − 3 = 9
3 + 9 = 12	12 − 9 = 3
10 + 2 = 12	12 − 2 = 10
2 + 10 = 12	12 − 10 = 2
11 + 1 = 12	12 − 1 = 11
1 + 11 = 12	12 − 11 = 1
12 + 0 = 12	12 − 0 = 12
0 + 12 = 12	12 − 12 = 0

2. Addition of or to a Constant Number and Subtraction of or from a Constant Number

0 + 2 = 2	6 + 0 = 6	10 + 3 = 13	12 + 0 = 12
1 + 2 = 3	6 + 1 = 7	11 + 3 = 14	12 + 1 = 13
2 + 2 = 4	6 + 2 = 8	12 + 3 = 15	12 + 2 = 14
3 + 2 = 5	6 + 3 = 9	13 + 3 = 16	12 + 3 = 15
4 + 2 = 6	6 + 4 = 10	14 + 3 = 17	12 + 4 = 16
5 + 2 = 7		15 + 3 = 18	12 + 5 = 17
6 + 2 = 8		16 + 3 = 19	12 + 6 = 18
7 + 2 = 9		17 + 3 = 20	12 + 7 = 19
8 + 2 = 10			12 + 8 = 20

10 − 1 = 9	10 − 0 = 10	20 − 3 = 17	17 − 0 = 17
9 − 1 = 8	10 − 1 = 9	19 − 3 = 16	17 − 1 = 16
8 − 1 = 7	10 − 2 = 8	18 − 3 = 15	17 − 2 = 15
7 − 1 = 6	10 − 3 = 7	17 − 3 = 14	17 − 3 = 14
6 − 1 = 5	10 − 4 = 6	16 − 3 = 13	17 − 4 = 13
5 − 1 = 4	10 − 5 = 5	15 − 3 = 12	17 − 5 = 12
4 − 1 = 3	10 − 6 = 4	14 − 3 = 11	17 − 6 = 11
3 − 1 = 2	10 − 7 = 3	13 − 3 = 10	.
2 − 1 = 1	10 − 8 = 2	12 − 3 = 9	.
1 − 1 = 0	10 − 9 = 1	11 − 3 = 8	.
	10 − 10 = 0		.
			.
			.
			17 − 17 = 0

B. MULTIPLICATION AND DIVISION

Note : Only the multiplication facts are listed. The division tables should be treated in conjunction with these.

1. Traditional Tables

e.g.,	0 × 5 = 0	8 × 0 = 0
	1 × 5 = 5	8 × 1 = 8
	2 × 5 = 10	8 × 2 = 16
	.	.
	.	.
	.	.
	.	.
	.	.
	12 × 5 = 60	8 × 12 = 96

2. A Tabulation Involving Doubling

This presupposes ability to double and to recognize that doubling one factor results in doubling the product. (See Section F, Part 1, Pattern and Order in the Number System.) In addition, children should realize the commutative property of multiplication (if $3 \times 2 = 6$, then $2 \times 3 = 6$).

In each of the following sets, the bracketed facts are those which would already have been covered by applying the commutative law to a previous set, and those underlined would have been treated by doubling the second factor in a previous set. Only the unmarked facts remain to be treated in any particular set, a total of 54 separate facts.

1 × 0 = 0 2 × 0 = 0 4 × 0 = 0 8 × 0 = 0	3 × 0 = 0 6 × 0 = 0 12 × 0 = 0	5 × 0 = 0 10 × 0 = 0	7 × 0 = 0	9 × 0 = 0	11 × 0 = 0
1 × 1 = 1 2 × 1 = 2 4 × 1 = 4 8 × 1 = 8	3 × 1 = 3 6 × 1 = 6 12 × 1 = 12	5 × 1 = 5 10 × 1 = 10	7 × 1 = 7	9 × 1 = 9	11 × 1 = 11
(1 × 2 = 2) 2 × 2 = 4 4 × 2 = 8 8 × 2 = 16	3 × 2 = 6 6 × 2 = 12 12 × 2 = 24	5 × 2 = 10 10 × 2 = 20	7 × 2 = 14	9 × 2 = 18	11 × 2 = 22
(1 × 3 = 3) (2 × 3 = 6) 4 × 3 = 12 8 × 3 = 24	3 × 3 = 9 6 × 3 = 18 12 × 3 = 36	5 × 3 = 15 10 × 3 = 30	7 × 3 = 21	9 × 3 = 27	11 × 3 = 33
(1 × 4 = 4) (2 × 4 = 8) 4 × 4 = 16 8 × 4 = 32	(3 × 4 = 12) 6 × 4 = 24 12 × 4 = 48	5 × 4 = 20 10 × 4 = 40	7 × 4 = 28	9 × 4 = 36	11 × 4 = 44
(1 × 5 = 5) (2 × 5 = 10) (4 × 5 = 20) 8 × 5 = 40	(3 × 5 = 15) 6 × 5 = 30 12 × 5 = 60	5 × 5 = 25 10 × 5 = 50	7 × 5 = 35	9 × 5 = 45	11 × 5 = 55
(1 × 6 = 6) (2 × 6 = 12) (4 × 6 = 24) 8 × 6 = 48	(3 × 6 = 18) 6 × 6 = 36 12 × 6 = 72	(5 × 6 = 30) 10 × 6 = 60	7 × 6 = 42	9 × 6 = 54	11 × 6 = 66

(1 × 7 = 7) (2 × 7 = 14) (4 × 7 = 28) 8 × 7 = 56	(3 × 7 = 21) (6 × 7 = 42) 12 × 7 = 84	(5 × 7 = 35) 10 × 7 = 70	7 × 7 = 49	9 × 7 = 63	11 × 7 = 77
(1 × 8 = 8) (2 × 8 = 16) (4 × 8 = 32) 8 × 8 = 64	(3 × 8 = 24) (6 × 8 = 48) 12 × 8 = 96	(5 × 8 = 40) 10 × 8 = 80	(7 × 8 = 56)	9 × 8 = 72	11 × 8 = 88
(1 × 9 = 9) (2 × 9 = 18) (4 × 9 = 36) (8 × 9 = 72)	(3 × 9 = 27) (6 × 9 = 54) 12 × 9 = 108	(5 × 9 = 45) 10 × 9 = 90	(7 × 9 = 63)	9 × 9 = 81	11 × 9 = 99
(1 × 10 = 10) (2 × 10 = 20) (4 × 10 = 40) (8 × 10 = 80)	(3 × 10 = 30) (6 × 10 = 60) 12 × 10 = 120	(5 × 10 = 50) 10 × 10 = 100	(7 × 10 = 70)	(9 × 10 = 90)	11 × 10 = 110
(1 × 11 = 11) (2 × 11 = 22) (4 × 11 = 44) (8 × 11 = 88)	(3 × 11 = 33) (6 × 11 = 66) 12 × 11 = 132	(5 × 11 = 55) (10 × 11 = 110)	(7 × 11 = 77)	(9 × 11 = 99)	11 × 11 = 121
(1 × 12 = 12) (2 × 12 = 24) (4 × 12 = 48) (8 × 12 = 96)	(3 × 12 = 36) (6 × 12 = 72) 12 × 12 = 144	(5 × 12 = 60) (10 × 12 = 120)	(7 × 12 = 84)	(9 × 12 = 108)	(11 × 12 = 132)

3. An Alternative Tabulation Involving Doubling

The activity outlined below can provide a grouping to limit the immediate field of work, a means of working for accuracy, and a form of consolidation.

a Start with a number, say 9.

b Children supply multiplication facts, which the teacher records on the black-board.

c Relevant division facts are listed.

d The original number is doubled, and the multiplication facts for this new number are recorded.

e Doubling continues as far as is relevant. For example :—

$9 = 3 \times 3$	1×9	$\longrightarrow$	$9 \div 3 = 3$	$9 \div 9 = 1$
	9×1			$9 \div 1 = 9$
$18 = 6 \times 3$	2×9	$\longrightarrow$	$18 \div 3 = 6$	$18 \div 9 = 2$
3×6	9×2		$18 \div 6 = 3$	$18 \div 2 = 9$
$36 = 12 \times 3$	4×9	$\longrightarrow$	$36 \div 3 = 12$	$36 \div 9 = 4$
3×12	9×4		$36 \div 12 = 3$	$36 \div 4 = 9$
6×6			$36 \div 6 = 6$	
$72 = 12 \times 6$	8×9	$\longrightarrow$	$72 \div 6 = 12$	$72 \div 9 = 8$
6×12	9×8		$72 \div 12 = 6$	$72 \div 8 = 9$
$144 = 12 \times 12$		$\longrightarrow$	$144 \div 12 = 12$	

f The original number (9) can be trebled to introduce the series 27, 54, 108.

$27 = 9 \times 3$	$\longrightarrow$	$27 \div 3 = 9$
3×9		$27 \div 9 = 3$
$54 = 9 \times 6$	$\longrightarrow$	$54 \div 6 = 9$
6×9		$54 \div 9 = 6$
$108 = 9 \times 12$	$\longrightarrow$	$108 \div 12 = 9$
12×9		$108 \div 9 = 12$

Further development of the activity could include, for example :—

If $4 \times 9 = 36$, what is $\frac{1}{4}$ of 36 ? $\frac{2}{4}$ of 36 ? $\frac{4}{4}$ of 36 ? $\frac{3}{4}$ of 36 ? $\frac{5}{4}$ of 36 ?

If $9 \times 4 = 36$, what is $\frac{1}{9}$ of 36 ? $\frac{5}{9}$ of 36 ? $\frac{7}{9}$ of 36 ?

4. Matrices

A matrix can provide a concise form of tabulation. (It can also be used in studying number facts, and consolidation and speed exercises can be developed from it.)

The matrix **should not be presented in complete form, but built up by the child** in conjunction with his experience in various fields, such as group counting, use of commutative law, and pattern in number.

Below is an example of a partly developed matrix.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1									10		
2	0		4								20		
3	0			9							30		
4	0				16						40		
5	0					25					50		
6	0						36				60		
7	0										70		
8	0										80		
9	0										90		
10	0	10	20	30	40	50	60	70	80	90	100		
11	0												
12	0												

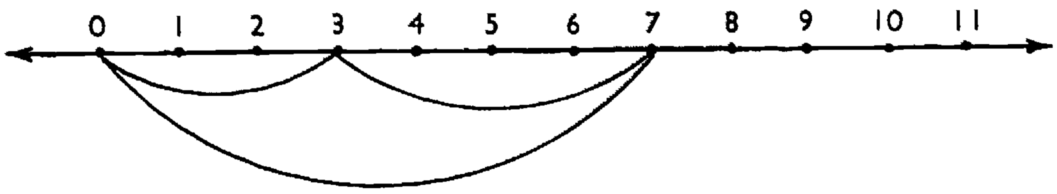
Matrices can also be used for addition and subtraction.

REPETITION

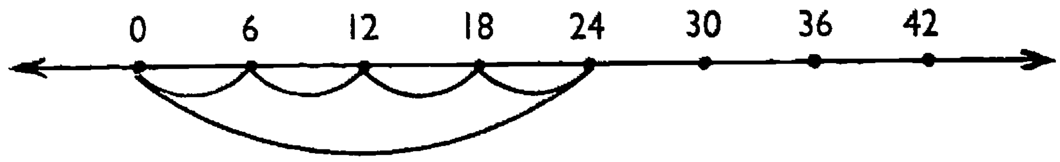
The facts must be used frequently in all possible ways to establish them in the child's mind. Meaningful repetition in a variety of ways is essential. Oral recitation, making use as it does of the sense of hearing, has some value, though this is limited.

ACTIVITIES TO HELP ESTABLISH NUMBER FACTS

1. Number Lines



3 + 4 = 7 7 − 4 = 3



4 × 6 = 24 24 ÷ 6 = 4

2. Number Charts

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

1 × 4 = 4
2 × 4 = 8
3 × 4 = 12
4 × 4 = 16
5 × 4 = 20

8, 12, 16, and 20 are multiples of 4.

Vertical movement shows addition and subtraction of fours.

7 + 4 = 11	14 + 4 = 18	9 + 4 = 13
7 − 4 = 3	14 − 4 = 10	9 − 4 = 5

Diagonal movement shows addition and subtraction of threes and fives.

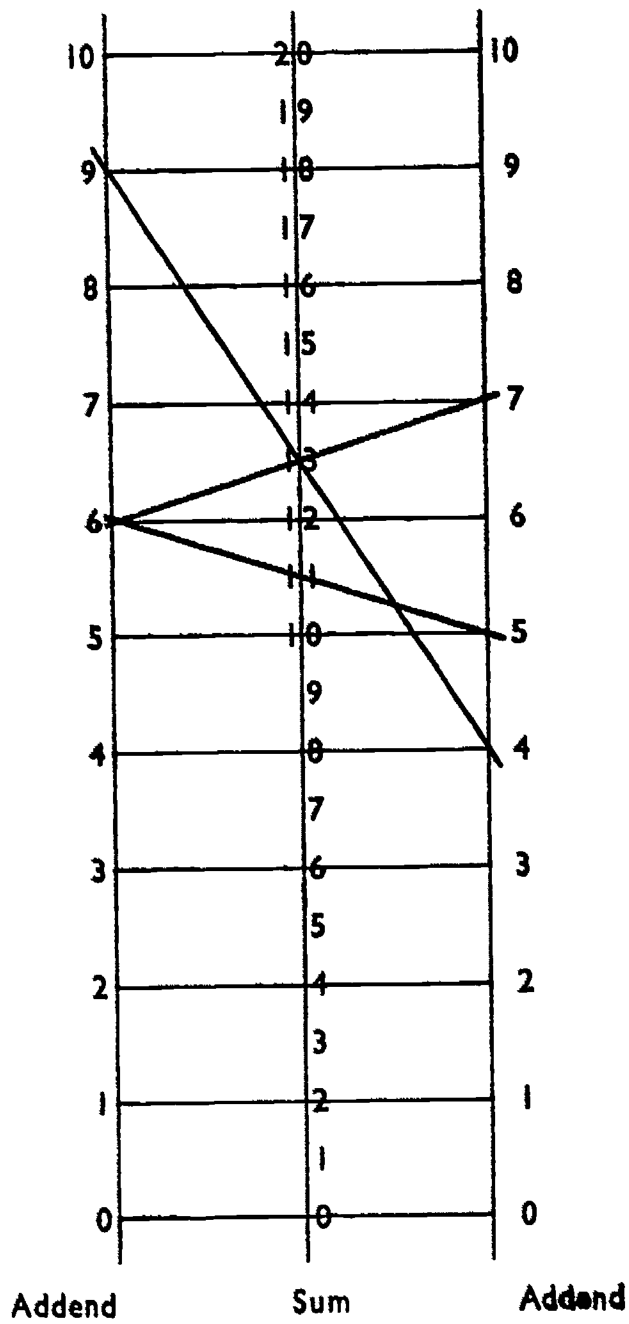
10 + 3 = 13	15 + 3 = 18	14 + 5 = 19	6 + 5 = 11
10 − 3 = 7	15 − 3 = 12	14 − 5 = 9	6 − 5 = 1

From a given number, movement can be made in eight directions.

11 + 1 = 12	11 + 4 = 15	11 + 5 = 16	11 + 3 = 14
11 − 1 = 10	11 − 4 = 7	11 − 5 = 6	11 − 3 = 8

Note: As required, larger charts can be used. They are helpful for consolidation exercises beyond the range of automatic response.

3. A Nomogram



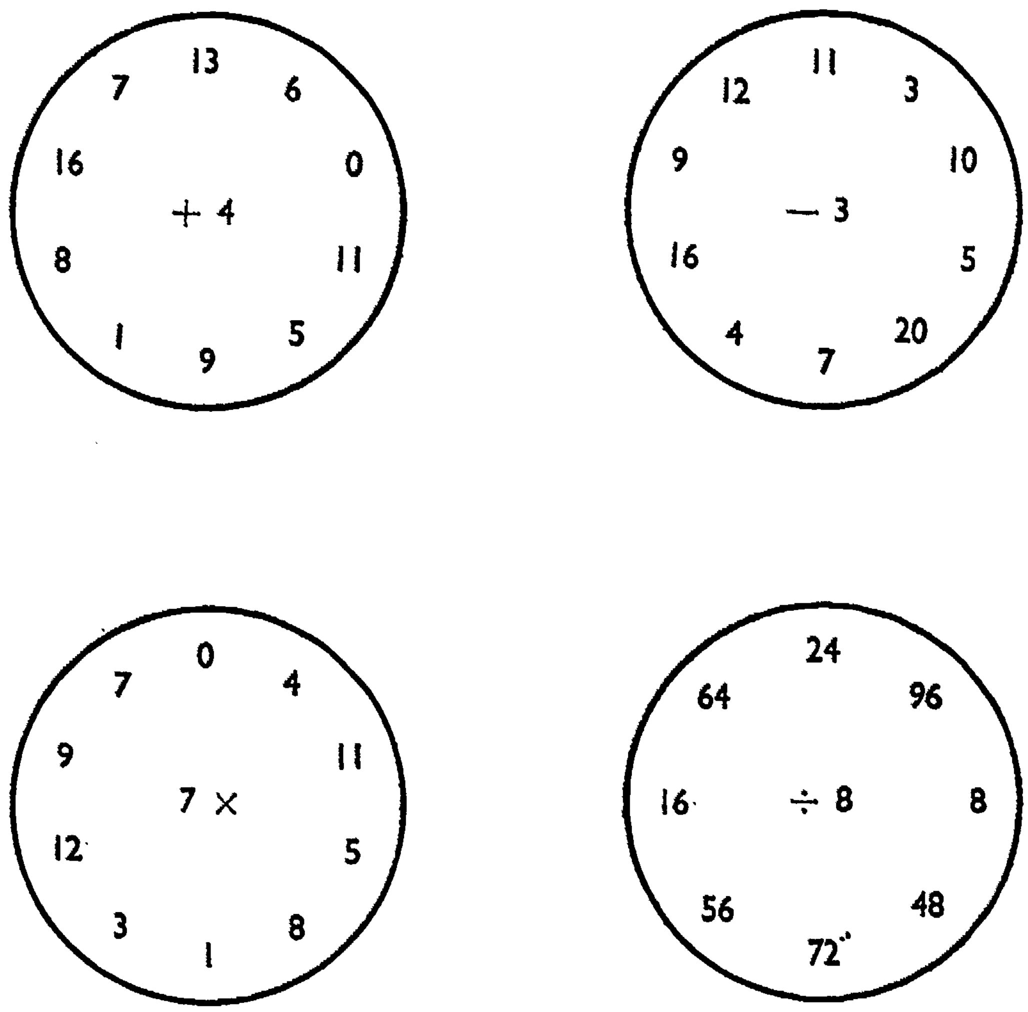
A line taken from one addend to another passes through their sum. This device is especially useful in establishing bridging facts, e.g., $9 + 4 = 13$. Subtraction can also be read from it.

Knowledge of known facts, such as $6 + 6 = 12$, can lead to knowledge of related facts such as $6 + 5 = 11$ and $6 + 7 = 13$.

ACTIVITIES FOR CONSOLIDATION

Note : The aim of these activities is to present the facts in a wide variety of situations in order to consolidate them and to speed recall. Only sample exercises are included here ; teachers will adapt them to their own needs, and find many others.

1. Number Rings



As the outside number is indicated, the child carries out the operation indicated in the centre of the ring.

2. Flash Cards

(For detailed information on the use of frames see Use of Frames, page 104.)

5 + 4 = □	16 − 12 = □
9 × 8 = □	42 ÷ 6 = □

These could be used :—

a For rapid response only.

b With the child's being asked to supply related facts.

The cards could be arranged :—

a In sets that include only one type of operation.

b In sets that include mixed types of operations.

3. Complementary Addends or Factors

Sum is 20	*Product is* 12
8 + □	4 × □
6 + □	6 × □
12 + □	12 × □
14 + □	2 × □
9 + □	3 × □

For effective use, rapid oral presentation is necessary. For example, the sum is stated, the teacher says one addend, and the child supplies the complementary one.

4. Number Grids

The teacher dictates numbers, which the child writes horizontally along the top line. The teacher then gives numbers as shown in the left-hand column, one at a time, allowing limited time for answers to be recorded.

+	**6**	**3**	**4**	**7**	**9**	. . .
8	14	11	12	15	17	
4	10	7	8	11	13	
7	13					
.						
.						
.						

×	**7**	**5**	**8**	**0**	**9**	. . .
3	21	15	24	0	27	
9	63	45				
12						
.						
.						
.						

Variety can be added by giving different instructions verbally for each line, for example,

	8	4	3	6	9
Add 3 ..	11	7	6	9	12
Subtract 5 ..	6	2	1	4	7
Double ..	12	4	2	8	14
Add 6 ..	18	10	8	14	20
Halve ..	9	5	4	7	10

5. Multiples

The teacher dictates a series of numbers which the child writes, for example, 24, 18, 13, 32, 42, 37.

The child is then instructed to mark multiples, for example, " Circle the multiples of 8 and underline the multiples of 6."

6. Number Squares

a

2	7	6
9	5	1
4	3	8

" Magic " square. Add horizontally, vertically, and diagonally. Totals will be the same.

As the child gains experience, the amount of information he is given to solve the puzzle may gradually be decreased.

b

	4	5	
	7	6	

Add vertically, horizontally, and diagonally. Look for a pattern in the border numbers.

As a variation, some of the given numbers can be in the border, and spaces left in the inner square.

EXTENSION

Application of serial addition and subtraction and of doubling and halving in multiplication and division will enable the child to work more efficiently with numbers in cases where automatic response is not required.

The extended use of number charts and larger numbers in some of the suggested activities will give practice and help to reinforce automatic response as the child proceeds to Section G.

TESTING

Testing is carried out—

a for accuracy, and

b for speed.

For b, strict timing is essential. The measure of a child's success should be improvement on his previous performance rather than comparison with that of other children.

PART 4. THE BASIC LAWS OF MATHEMATICS

AIM

To develop in the child—

a an understanding of the basic mathematical laws, and

b an ability to use them freely.

IMPLICATIONS FOR TEACHING

An understanding of these laws is important for the following reasons :—

1. They help the child to understand new, unfamiliar mathematical situations, to make correct judgments, and to reach sound conclusions.
2. They help him to choose from different possible courses of action those that are appropriate and those that are not.
3. They allow the child to operate with confidence when dealing with abstract operations and relations.
4. They assist the child in committing to memory sets of number combinations.
5. The vast number of techniques for dealing with different number situations can be reduced to a small set of fundamental laws.

BACKGROUND INFORMATION FOR TEACHERS

1. The Nature of the Laws

a **The Commutative Law of Addition (Property of Order)—** The order in which two numbers are added does not affect the sum.

e.g., $3 + 4 = 7$ and $4 + 3 = 7$

$3 + 4 = 4 + 3$

In general,

$a + b = b + a$

b **The Commutative Law of Multiplication (Property of Order)**—The order in which two factors are multiplied does not affect the product.

e.g., $3 \times 2 = 6$ and $2 \times 3 = 6$

$3 \times 2 = 2 \times 3$

In general,

$a \times b = b \times a$

c **The Associative Law of Addition (Property of Grouping)—** When adding three or more numbers the manner of grouping does not change the sum of the numbers.

e.g., $2 + 3 + 4$

$(2 + 3) + 4 = 2 + (3 + 4)$

$5 \quad + 4 = 2 + \quad 7$

$9 = 9$

In general,

$(a + b) + c = a + (b + c)$

d **The Associative Law of Multiplication (Property of Grouping)**—When multiplying three or more numbers, the manner of grouping does not change the product of the numbers.

e.g., $2 \times 3 \times 4$

$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$
$$6 \times 4 = 2 \times 12$$
$$24 = 24$$

In general,

$$(a \times b) \times c = a \times (b \times c)$$

e **The Distributive Law**—The distributive property of multiplication over addition (or subtraction) and the distributive property of division over addition (or subtraction) allow us to link the two operations involved in each instance.

e.g., $6 \times 8 = 6 \times (5 + 3)$

$$= (6 \times 5) + (6 \times 3)$$
$$= 30 + 18$$
$$= 48$$

In general,

$$a \times (b + c) = a \times b + a \times c$$

Furthermore,

$$a \times (b - c) = a \times b - a \times c$$

$$(a + b) \div c = a \div c + b \div c$$
$$(a - b) \div c = a \div c - b \div c$$

Note : Division may be distributed over the dividend only, and not over the divisor.

e.g., $36 \div 6 = (30 + 6) \div 6$

$$36 \div 6 = (30 \div 6) + (6 \div 6)$$

$$36 \div 6 = 36 \div (4 + 2)$$

but $36 \div 6 \neq (36 \div 4) + (36 \div 2)$

f **Inverse Operations**—Consider the relationship between the three numbers 3, 4, and 7. This can be written :—

$$3 + 4 = 7 \quad \text{or} \quad 4 + 3 = 7$$

However, it can also be written :—

$$7 - 4 = 3 \quad \text{or} \quad 7 - 3 = 4$$

This example illustrates what is meant by saying that subtraction and addition are inverse operations of each other.

Similarly, the relationship between 2, 5, and 10 can be written :—

$$2 \times 5 = 10 \quad \text{or} \quad 5 \times 2 = 10$$

However, it can also be written :—

$$10 \div 5 = 2 \quad \text{or} \quad 10 \div 2 = 5$$

This example illustrates what is meant by saying that division and multiplication are inverse operations of each other.

In general,

$$a + b = c \Leftrightarrow b + a = c$$
$$\Leftrightarrow c - b = a \Leftrightarrow c - a = b$$

and

$$p \times q = r \Leftrightarrow q \times p = r$$
$$\Leftrightarrow r \div q = p \Leftrightarrow r \div p = q$$

The sign $\Leftrightarrow$ means "is equivalent to".

g **Properties of Equations**—See pages 54–57.

h **Properties of Difference**—See pages 57–58.

2. The Binary Nature of the Operations

An operation that links only two numbers is called a binary operation. Addition, subtraction, multiplication, and division are all binary operations.

e.g., $3 + 5$ Two numbers, one operation.
$7 - 3$ Two numbers, one operation.
7×4 Two numbers, one operation.
$50 \div 5$ Two numbers, one operation.

However, we are often concerned with combinations of these binary operations. This is clear from such examples as the following :—

$7 + 3 + 4$ Three numbers, two binary operations.
$9 + 7 - 2$ Three numbers, two binary operations.
$5 \times 4 \times 2$ Three numbers, two binary operations.
$3 \times (2 + 4)$ Three numbers, two binary operations.
$3 \times 2 + 4$ Three numbers, two binary operations.

Since only one of these binary operations will be carried out at a time, it is important to be able to decide the sequence in which these separate operations will take place. The associative and the distributive laws help us to make correct judgments when deciding the sequence or sequences of these binary operations.

DEVELOPMENT

Notes :

i It should be remembered that, from and inclusive of Section B, the child has begun to build up a body of knowledge of relationships from which an understanding of the laws can develop. At the level of Section F, it is intended that the child will begin a study of the laws as such.

ii **The child will need to experience personally a large variety of concrete situations before being asked to give general statements of the laws.**

iii At any one time, individual children will have different degrees of understanding. Some children may already have reached a generalization and will need only a little practice to reinforce it. (The term "generalization" is used here to indicate a general understanding rather than a formal definition.)

iv Formal definitions and algebraic expressions are not required and are, in fact, generally undesirable at this level. Naming of the laws may be left to the discretion of the class teacher. (For example, Order Law or Commutative Law.)

v The laws must be regarded as an integrated whole, not just a series of isolated topics. Although the laws are listed, for convenience, in the order below, the actual sequence of treatment should be determined by the class teacher according to the needs of the individual child, keeping in mind that some laws can be dealt with incidentally. **However, the teacher must check that all the laws are studied and remain available for use by the child, even after the formal processes are refined. Furthermore, the teacher should realize that the understanding, the knowledge, and the confident use of the basic laws constitute a prerequisite for the successful handling, with understanding, of the activities planned to lead to formal processes, and of the processes themselves.**

vi Initially the form of setting out of certain equations should be such that one side of the equation is repeated in its original form. The other side shows the recording of a series of manipulations, each one equal to the opposite side.

e.g., $4 \times 21 = 4 \times (20 + 1)$

$$\begin{aligned} 4 \times 21 &= 4 \times 20 + 4 \times 1 \\ 4 \times 21 &= 80 + 4 \\ 4 \times 21 &= 84 \end{aligned}$$

By using this form of setting out, a child can compare each new step on one side of the equation with the original (4×21).

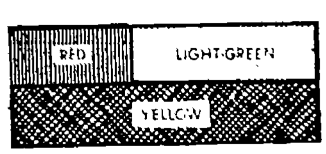
Only after experience and understanding have developed with the child's gradually increasing mathematical maturity as he proceeds through Section F will the setting out be shortened to—

$$\begin{aligned} 4 \times 21 &= 4 \times (20 + 1) \\ &= 4 \times 20 + 4 \times 1 \\ &= 80 + 4 \\ &= 84 \end{aligned}$$

A. COMMUTATIVE LAW OF ADDITION

I. Activities

a **The Rods**—The teacher asks the child to select a rod and then to find a different one to add to the first. The rod that equals these two rods is then selected. Such a situation may be :—



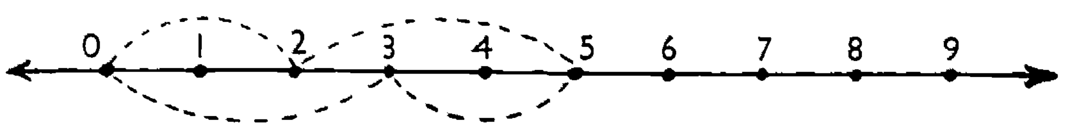
If the child placed the red rod first and then the light-green rod the teacher would ask the child, " If we had put the light-green rod down first and then added the red rod, would the total length of the two rods still equal yellow ? "

The child should be encouraged to use different sets of rods to see if reordering will ever result in a different total.

b **Concrete Objects**—The objects used may be counters, beads, strips of paper, pencils, match-boxes, or any other pieces of concrete material that are readily available.

The child may be asked to take two match-boxes and three bottle-tops from the scrap-box. "How many things have you?" "What did you take out first?" "Will it make any difference to the total if you take out the bottle-tops first?" "Try it."

c **The Number Line**—The number line can be used to discover the sum of two numbers.



In adding 2 and 3, we start at 0, move first 2 units to the right, and then move three more units to the right. This gives us the sum of 2 and 3. 3 plus 2 can then be tried. "How do the answers compare?" Work would then proceed using other number combinations.

2. Generalization of the Law

Sufficient varied situations should be attempted by the child for him to establish the understanding that by changing the order of the two addends the sum is not affected.

During the initial stages it is sufficient to speak of the "order" in which numbers are added or of "order in addition". Later, when the child has firmly grasped this idea, the teacher may wish to introduce the child to the term "commutative".

The child should be encouraged to describe orally what he is doing. For example, "If two numbers are added together it does not matter which is used first."

3. Non-numerical Situations

Once the child has reached a generalization about the commutative property, he may be given situations that are non-numerical and asked if they, too, are commutative.

e.g., a In mixing a glass of cordial, is the result the same if the water is put in first as if the cordial is put in first? (Yes.)

b Is the result the same if a boat is in the water as if the water is in the boat? (No.)

4. Activities with Numerals

(For detailed information on the use of frames see Use of Frames, page 104.)

a Complete these equations:

$$3 + 5 = \square + 3$$
$$\square + 4 = 4 + 7$$

b State whether these equations show the commutative (or order) property of addition.

$$7 + 5 = 5 + 7 \quad \text{(Yes.)}$$
$$75 + 26 = 26 + 75 \quad \text{(Yes.)}$$
$$9 + 8 = 7 + 10 \quad \text{(No.)}$$

Note : Just as the child has discovered that the commutative property applies to addition, he should be encouraged to experiment to establish for himself that it cannot be applied to subtraction.

$7 - 5$ is not equal to $5 - 7$, i.e., $7 - 5 \neq 5 - 7$.

$7 - 5 = 2$, but $5 - 7$ is meaningless if counting numbers are being dealt with.

B. ASSOCIATIVE LAW OF ADDITION

Before commencing work on this law, the child should be made fully aware of the binary nature of addition. The following procedure could be adopted:—

$$3 + 7 + 8$$

Find the sum of these numbers. How did you get this ? How many numbers did you add together at any one time ? Can you add more than two numbers together at the one time ?

If the child feels that more than two numbers can be added in the one operation, he should be asked to work the problem aloud.

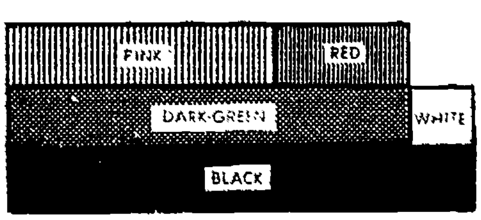
The child must now realize that when combining three or more numbers he must make a decision as to which addition he will carry out first ; that is to say, he must " group " two of his addends. How can this grouping take place ? If the manner of grouping changes does the answer also change ?

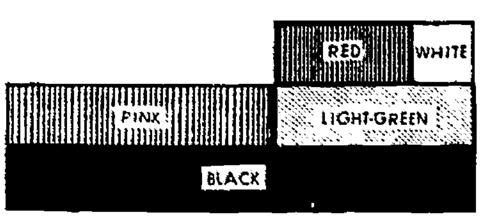
I. Activities

a **The Rods**—The child could be presented with the following set of rods and asked :—

' How many ways can you discover to group the rods in pairs without changing the order ? "

" Does the change in the manner of joining these rods alter the sum ? "





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The child should be able to see that :—

Pink plus red equals dark-green ; and dark-green plus white equals black.

He should also see that :—

White plus red equals light-green ; and light-green plus pink equals black.

In terms of numerals this could be :—

$$(4 + 2) + 1 = 6 + 1 = 7 \qquad 4 + (2 + 1) = 4 + 3 = 7$$

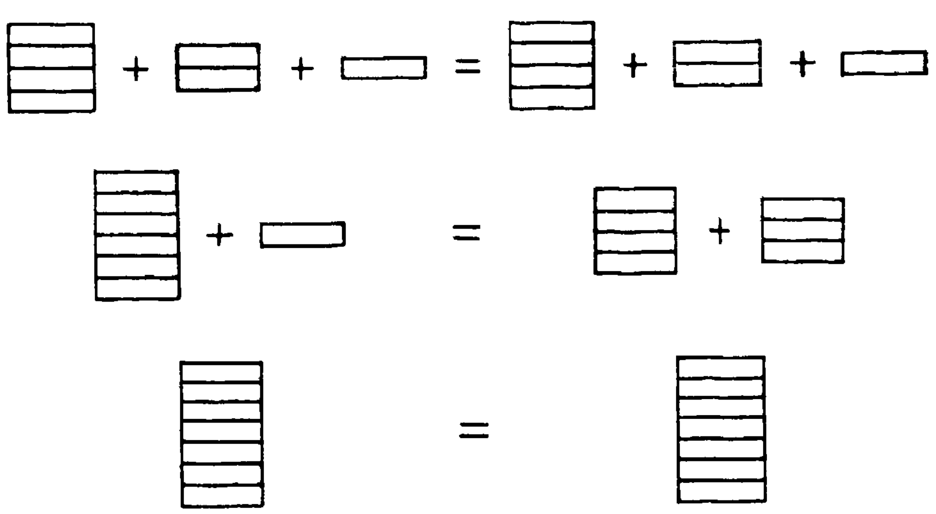
b **Meccano**—This is an excellent aid because, through the use of the material, the binary nature of addition is made obvious to the child. When the child is asked to join three strips of different lengths together using the nuts and the bolts provided, he can operate with only two strips at any one moment.

The following is one procedure which may be used :—

The child is asked to select three strips of different lengths. He then arranges them in any order.

He is then asked to find out in how many different ways the strips can be joined together, using the nuts and the bolts, without changing the order. Once he has discovered the two ways of joining the strips, his attention should be drawn to the fact that the method of grouping them does not affect the final length of the combined strips.

c **Building Blocks**—A procedure similar to that used with Meccano strips could be used with building blocks. This material is joined together very simply and presents a visual pattern that varies from that achieved with the rods.



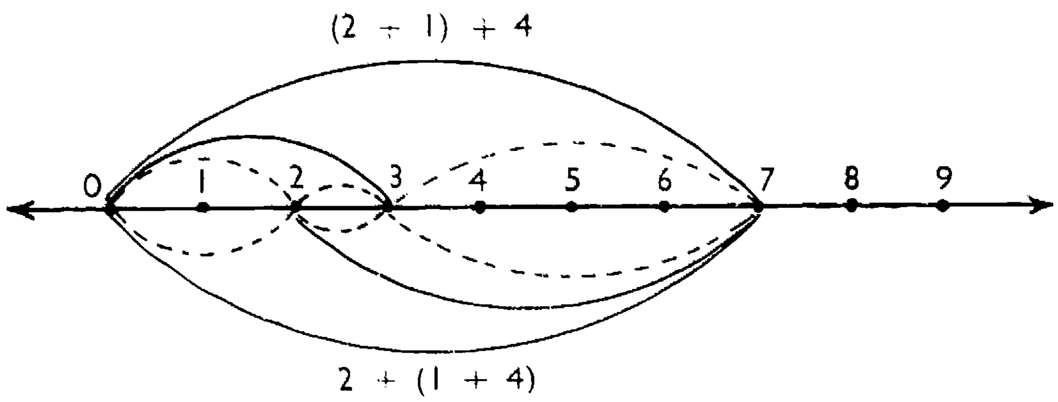
d **Scrap Materials**—Any material available in the class-room may be used here. Situations similar to the following may be presented :—

Three sets of objects are placed before the child, for example, pencils, counters, rubbers. He is told that new sets are to be made by combining these sets in different ways, and reminded that only two sets may be grouped at any one time. He is then asked to find ways of combining the sets.

i He may combine the set of pencils with the set of counters ; then combine the resulting set with the set of rubbers. He should find the cardinal number of this resulting set.

ii On the other hand, he may combine the sets of counters and rubbers ; then combine the resulting set with the set of pencils. He should find that the cardinal number of the resulting set is the same as in situation i.

e **The Number Line**—The number line can be used to illustrate this law.



Starting from 0, we mark the numbers to be added. The child should discover that, no matter which ways the numbers are grouped, the total is the same.

2. Generalization of the Law

Sufficient varied situations should be used by the child for him to establish the understanding that the order in which additions take place does not affect the sum. It would be expected that the child, after a variety of experiences, would be able to put into words his understanding of this property of addition.

e.g., "In 3 + 5 + 7, I can either add the 3 and the 5 first, then the 8 and the 7, or, I can add the 5 and the 7, then the 12 and the 3. The answer is still 15."

"It doesn't matter which of the two additions I do first, the answer will be the same."

"When we add, we can change the grouping and still get the same answer."

During the child's initial experiences with the ideas of associativity, it would be sufficient for him to use the term "grouping" rather than the term "associative property". At a later date the teacher may wish to introduce the child to the more technical term.

3. Activities with Numerals

Once the child has had sufficient concrete experiences he may be introduced to the abstract situation of numerals. The teacher may wish to link this with the concrete work discussed above. As the child manipulates the concrete materials he may be encouraged to record the operations in terms of numerals.

a □ □ □ ○ ○ □ □ □ □

$3 + 2 + 4$	$3 + 2 + 4$
$(3 + 2) + 4$	$3 + (2 + 4)$
$5 \quad + 4$	$3 + \quad 6$
9	9

Before using numerals it may be necessary to review the use of brackets as a means of indicating which pair of numbers is added first.

b Complete the equation :

$$(7 + 3) + 4 = \square + (3 + 4)$$
$$(3 + \square) + 5 = 3 + (2 + 5)$$

c Complete each equation :

$$\begin{aligned} 12 + 3 + 4 &= (12 + 3) + 4 \\ &= \square + 4 \\ &= \square \end{aligned}$$

$$\begin{aligned} 12 + 3 + 4 &= 12 + (3 + 4) \\ &= 12 + \square \\ &= \square \end{aligned}$$

$$\begin{aligned} 20 + 7 + 12 &= 20 + (7 + 12) \\ &= 20 + \square \\ &= \square \end{aligned}$$

$$\begin{aligned} 20 + 7 + 12 &= (20 + 7) + 12 \\ &= \square + 12 \\ &= \square \end{aligned}$$

d Fill the place holders. Find the sums :

$$\begin{aligned} 13 + 4 &= (10 + 3) + \square \\ &= 10 + (\square + 4) \\ &= \square + 7 \\ &= \square \end{aligned}$$

$$\begin{aligned} 15 + 3 &= (\square + 5) + 3 \\ &= 10 + (5 + \square) \\ &= 10 + \square \\ &= \square \end{aligned}$$

$$
\begin{aligned}
34 + 2 &= (\square + 4) + \square \\
&= 30 + (\square + \square) \\
&= \square + \square \\
&= \square
\end{aligned}
$$

e Using brackets, show the easiest way of obtaining the sum.

e.g., $6 + 4 + 9 = (6 + 4) + 9$

i $54 + 1 + 9$

ii $15 + 9 + 11$

C. COMMUTATIVE LAW WITH ASSOCIATIVE LAW IN ADDITION

It is important that the child has sufficient experience and understanding of the individual properties of addition before any attempt is made to combine them. If this understanding is not achieved, the following activities become merely the means by which the child develops facility in manipulation.

The child should be led to realize that, by using the commutative and the associative properties of addition in combination, an easier way of finding the sum of a set of numbers is sometimes gained.

Activities

a Reorder and regroup to make addition easier.

$$
\begin{aligned}
&5 + 7 + 2 + 5 + 3 + 8 + 6 \\
&= (5 + 5) + (7 + 3) + (8 + 2) + 6 \\
&= 10 + 10 + 10 + 6 \\
&= 36
\end{aligned}
$$

b Complete these equations:

$(50 + 4) + (20 + 2) = (50 + 20) + (4 + \square) = \square + 6 = \square$

$(10 + 6) + (50 + 2) = (10 + \square) + (6 + \square) = 60 + \square = \square$

$(20 + 3) + (30 + 4) = (\square + \square) + (\square + \square) = \square + \square = \square$

c Using extended notation and the associative and the commutative laws of addition, find the sum of 36 and 42, and of 243 and 315.

$$
\begin{aligned}
36 + 42 &= (30 + 6) + (40 + 2) && \text{(Extended notation)} \\
&= (30 + 40) + (6 + 2) && \text{(Grouping and ordering)} \\
&= 70 + 8 \\
&= 78
\end{aligned}
$$

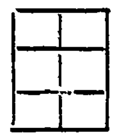
$$
\begin{aligned}
243 + 315 &= (200 + 40 + 3) + (300 + 10 + 5) \\
&= (200 + 300) + (40 + 10) + (3 + 5) \\
&= 500 + 50 + 8 \\
&= 558
\end{aligned}
$$

Note : These are in fact preliminary activities leading to the formal process of addition. The ability to reorder and regroup will assist the child when he comes later to the addition of vertical columns of figures.

D. COMMUTATIVE LAW OF MULTIPLICATION

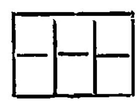
I. Activities

a **Cards**—The child is presented with a rectangular card divided into squares. He should be encouraged to think of this array in terms of rows and the number of squares in each row.



$3 \times 2 = 6$

The card is then turned on its side and the child is asked to record the number of rows and squares.

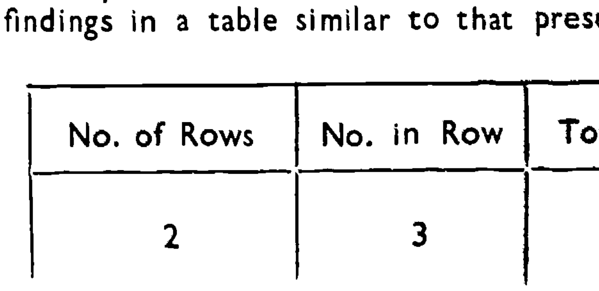


$2 \times 3 = 6$

Other arrays are then presented to the child so that the common mathematical idea can be recognized.

b **Peg-board**—The peg-board is a slight variation from the cards presented above. However, it has certain advantages in the class-room setting. A procedure that might be adopted could be as follows :—

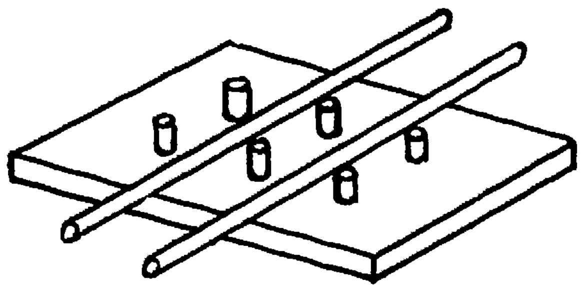
The child is asked to mark a rectangle on the board with chalk. The rectangle is then filled with pegs. Using straws, the child is directed to divide the pegs into rows. The child's observation is then directed to the number of rows and the number of pegs in each row.



At this point the teacher may wish the child to record his findings in a table similar to that presented below.

No. of Rows	No. in Row	Total No. of Pegs
2	3	6

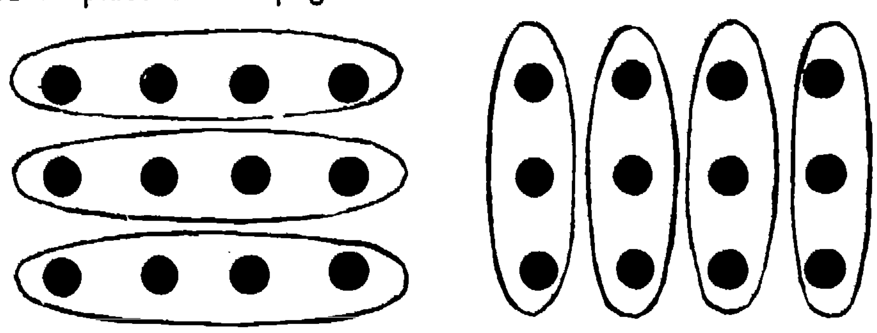
Without changing the pegs the child is asked to divide the rectangle into rows in a different way.



The observations of the child are again recorded in the table so that it would now appear as :—

No. of Rows	No. in Row	Total No. of Pegs
2	3	6
3	2	6

The child should be encouraged to experiment further to see if there are cases where this feature does not hold. If the teacher wishes, counters or an array of dots could be used in place of the peg-board.

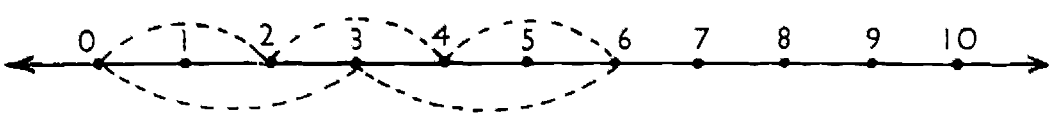


3 · 4 · 12   4 3 12

c **Number Line**—The number line can be an effective means of illustrating the commutative property of multiplication.

Three sets of two and two sets of three each give us one set of six. For example :—

$3 \times 2 = 6$



$2 \times 3 = 6$

d **Equalizer**—Another useful piece of equipment, which can be used to illustrate the commutative law of multiplication, is the equalizer. One form of equalizer is to be found in the Dienes equipment. (As with all structured aids, the child must have a period of familiarization, which is often no more than free play.)

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2. Generalization of the Law

At this stage it is sufficient to speak of the order of multiplication. When the idea is firmly grasped, the term "commutative" may be introduced. The child should be encouraged to talk freely about the law. For example :—

> "When we change the order of the factors we get the same product."
>
> "When two numbers are multiplied it does not matter which is used first."

3. Activities with Numerals

a Complete each sentence with $=$ or $\neq$:

$3 \times 4 \triangle 4 \times 3$

$2 \times 9 \triangle 9 \times 5$

$87 \times 34 \triangle 34 \times 87$

b Solve each equation :

$4 \times 8 = \square$ $\quad$ $8 \times 4 = \square$

$9 \times 8 = \square$ $\quad$ $8 \times 9 = \square$

c Complete each equation :

$4 \times 8 = 8 \times \square$

$\square \times 28 = 28 \times 56$

4. Problem Situations Involving the Commutative Property

e.g., Bill takes 3 steps, each of which is 2 feet long. John takes 2 steps, each of which is 3 feet long. Did Bill walk further than John ?

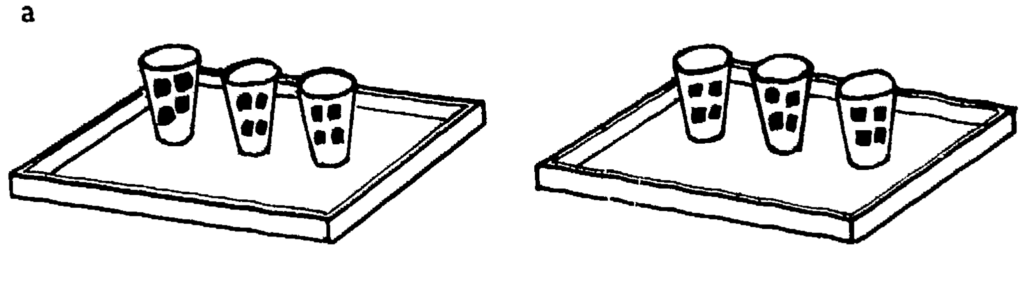
E. ASSOCIATIVE LAW OF MULTIPLICATION

As with the associative law of addition, the child needs to be aware of the binary nature of the operation of multiplication before he studies the associative law. A technique similar to that used for addition (see page 38) will enable him to see that when faced with the problem $2 \times 3 \times 5$ he can carry out only one multiplication at a time. He should now be brought to realize that he has a choice as to which multiplication he will do first, that is, he must decide how to group his factors.

1. Activities

It is difficult to show this property adequately using a variety of concrete materials. However, if the teacher feels that it is essential to present the embodiment of this law in a concrete situation, variations of the following examples may suffice.

a



The presentation of this problem to the child may be best in story form. As the story is told, the child may construct the situation using infant chalk-boards, paper cups, and counters.

It is a hot day and you have five pals playing with you. Mother has provided a drink for you and for each of your friends. Two trays have been left in the kitchen and on each tray there are three glasses. In each glass there are four cubes of ice.

How many sets of glasses ? (2)

How many glasses in each set ? (3)

How many glasses altogether ? (6)

How many cubes of ice in each glass ? (4)

How many cubes of ice are there in all the glasses ? (24)

The child now has the situation : $(2 \times 3) \times 4 = 24$.

He may also be guided into discovering the situation of $2 \times (3 \times 4) = 24$.

How many glasses on a tray ? (3)

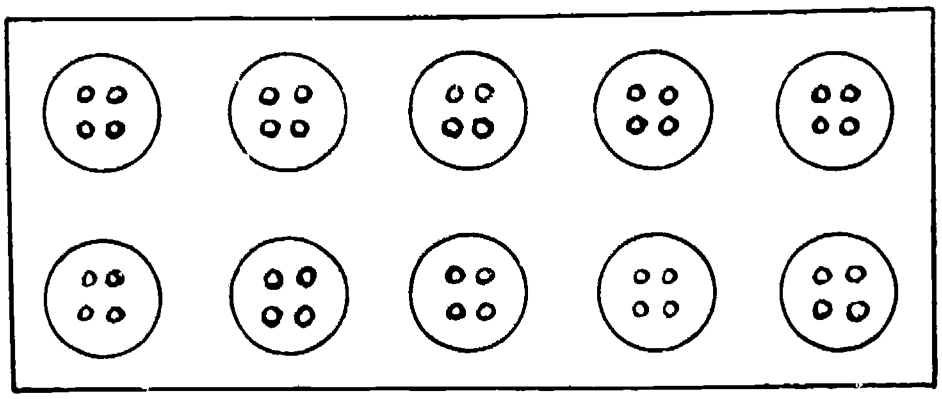
How many cubes of ice in a glass ? (4)

How many cubes of ice in these three glasses ? (12)

How many trays of glasses do we have ? (2)

How many cubes of ice altogether ? (24)

b



In finding the total number of holes in the set of buttons, the children can calculate in several ways.

e.g., Two rows of five buttons give ten buttons ; each has four holes, giving a total of forty holes.

i.e., $(2 \times 5) \times 4 = 40$

Take one row of buttons, each of which has four holes. In one row there are twenty holes. Two rows would give forty holes.

i.e., $2 \times (5 \times 4) =$

2. Generalization of the Law

At this stage it is sufficient to speak of "grouping" in multiplication. The more technical term may be introduced later. The child should be encouraged to talk freely about the law. For example :—

"When we multiply we can change the grouping and still get the same product."

"When we do multiplications only we can please ourselves which we do first."

3. Activities with Numerals

e.g.,

$$4 \times 2 \times 3 = (4 \times 2) \times 3 = 8 \times 3 = 24$$

$$4 \times 2 \times 3 = 4 \times (2 \times 3) = 4 \times 6 = 24$$

Complete each equation :

a $\quad 3 \times 2 \times 5 = (3 \times 2) \times 5 = \square \times \square = \square$

b $\quad 3 \times 2 \times 5 = 3 \times (2 \times 5) = \square \times \square = \square$

c $\quad 3 \times 60 = 3 \times (\square \times 10) = (\square \times 6) \times 10 = \square \times \square = \square$

d $\quad 4 \times 15 = 4 \times (\square \times \square) = (4 \times \square) \times \square = \square \times \square = \square$

F. THE COMMUTATIVE LAW WITH THE ASSOCIATIVE LAW IN MULTIPLICATION

Activities involving both the associative and the commutative properties of multiplication should be presented to the child once he has understanding of each of these properties.

Use the commutative and the associative properties of multiplication to make the equations easier to solve. (The first example is worked.)

i $\quad 5 \times (19 \times 2) = 5 \times (2 \times 19) = (5 \times 2) \times 19 = 10 \times 19 = 190$

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ii $(2 \times 27) \times 5 = (27 \times \square) \times 5$
$$= \square \times (2 \times \square)$$
$$= \square \times \square$$
$$= \square$$

iii $(3 \times 11) \times 4 = \square \times (11 \times 4)$
$$= 3 \times (4 \times \square)$$
$$= (\square \times 4) \times 11$$
$$= \square \times \square$$
$$= \square$$

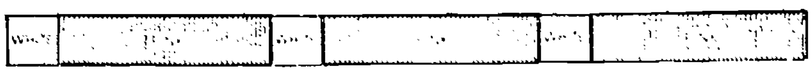
G. THE DISTRIBUTIVE LAW

The distributive property of multiplication over addition (or subtraction) and the distributive property of division over addition (or subtraction) allow us to link the two operations involved in each instance.

If the child does not know the product of two numbers he may find it by using the distributive law, that is, through the use of both addition and multiplication. It is probably the most important law in extending the child's ability to manipulate numbers beyond his range of automatic response.

I. Activities

a **The rods**—As a prerequisite, the child should have the experience of repeating a pattern of rods, such as white plus pink, so that he can see this pattern—



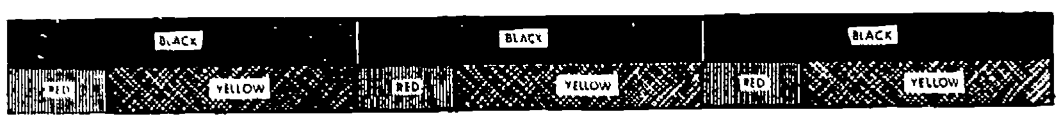
as 3 times (white plus pink).

Rearranging the rods,

he should recognize that 3 times (white plus pink) = 3 times white + 3 times pink.

The child is directed to take three black rods and place them end to end.

He is then asked to find two rods to equal black and to place them, end to end, under the first black rod. He repeats this procedure, using similar rods for each black rod, thus :—



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The child's attention is then drawn to the fact that he has—

i a line of three black rods, and

ii a second line consisting of three sets of red and yellow rods,

and that these two lines have the same length.

Numerically this could be expressed :—

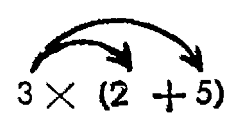
$$3 \times 7 = 3 \times (2 + 5)$$

By rearranging the rods to group the reds and the yellows—

he can see that the equation could be expressed :—

$$3 \times 7 = 3 \times (2 + 5)$$
$$= (3 \times 2) + (3 \times 5)$$

He should be led to see that the number outside the brackets operates on both the numbers inside the brackets, that the three is " distributed " (or " spread ") over both the two and the five.



b **Arrays**—The child is asked to construct a rectangular array. This can be done with counters, peg-board, bead frame, or with dots on paper. Such an array could be :—

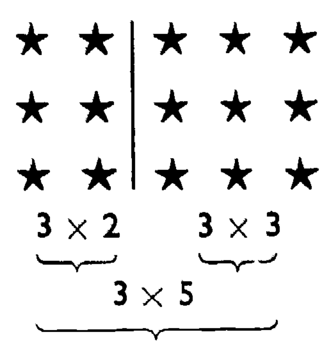
★ ★ ★ ★ ★

★ ★ ★ ★ ★

★ ★ ★ ★ ★

The array is then expressed, by the child, in the form of an equation, for example, $3 \times 5 = 15$.

A ruler or pencil is used to divide the array into two sets, and the new situation is recorded.

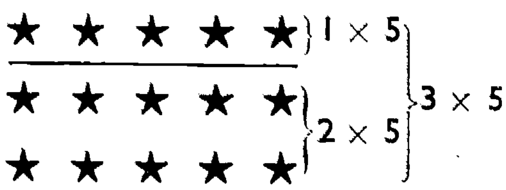


In numerals :—

$$3 \times 5 = 3 \times (2 + 3)$$
$$= (3 \times 2) + (3 \times 3)$$
$$= 6 + 9$$
$$= 15$$

The child should become aware that the sum of the products for these two sets remains equal to the number of elements in the array.

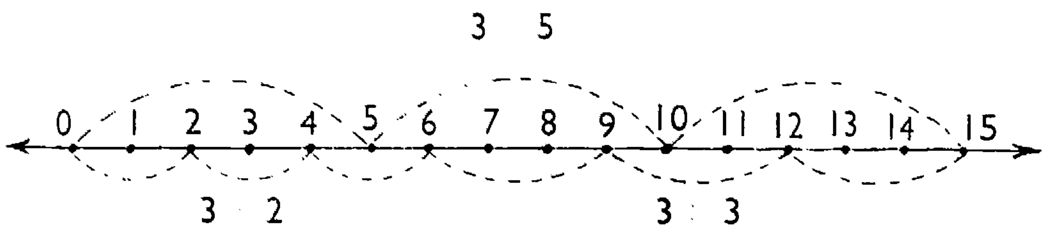
In the example used above, it is the second factor in the equation that is being renamed. By splitting the array into rows, it is the first factor that is being renamed.



In numerals :—

$$
\begin{aligned}
3 \times 5 &= (1 + 2) \times 5 \\
&= (1 \times 5) + (2 \times 5) \\
&= \quad 5 \quad + \quad 10 \\
&= \quad 15
\end{aligned}
$$

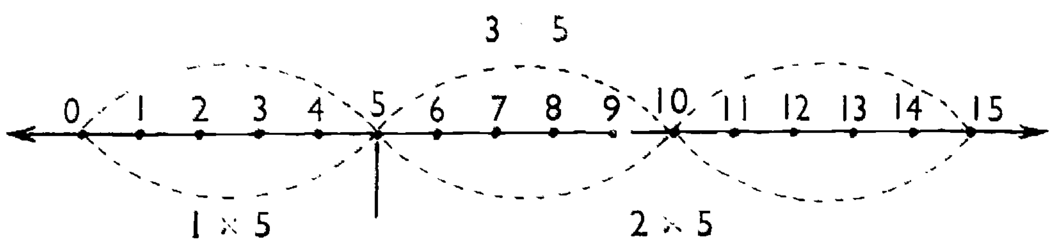
c **Number Line**—The number line can also be used to examine the distributive property. In the example 3 × 5, the five can be renamed as two plus three. In such a case there would be three sets of two plus three sets of three. For example :—



The first factor in the statement 3 × 5 can be renamed as one plus two.

$$(1 + 2) \times 5 = 3 \times 5$$

This relation can also be shown on the number line.



2. Activities with Numerals

a Rename the second factor as the sum of two addends:

i $5 \times 7 = 5 \times (\square + 2)$

ii $8 \times 6 = 8 \times (\square + \square)$

b Complete each equation:

i $5 \times (3 + 6) = 5 \times \square = \square$

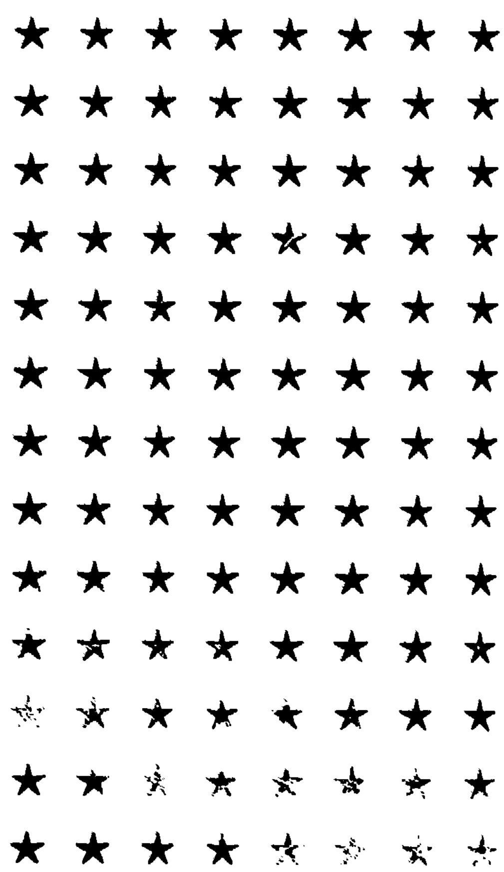
ii $(5 \times 3) + (5 \times 6) = \square + \square = \square$

c Complete each equation:

i $8 \times 9 = 8 \times (\square + \square)$
$= (8 \times \square) + (8 \times \square)$
$= \square + \square$
$= \square$

ii $14 \times 6 = (\square + \square) \times 6$
$= (\square \times 6) + (\square \times 6)$
$= \square + \square$
$= \square$

d



$13 \times 8 = \square$

How many different ways can you find to solve this equation by renaming the number of rows ?

e.g., $(2 + 11) \times 8$
$(5 + 8) \times 8$

How many different ways can you find to solve this equation by renaming the number of columns ?

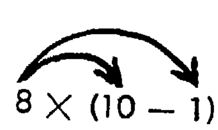
e.g., $13 \times (2 + 6)$
$13 \times (5 + 3)$

3. Multiplication Is Distributive over Subtraction

The child who has understood how the distributive property links multiplication and addition may now be asked to see if the distributive property can also link multiplication and subtraction. Instead of a factor being renamed in terms of two addends, it can be renamed in terms of subtraction.

e.g., $8 \times 9 = 8 \times (5 + 4)$ or $8 \times 9 = 8 \times (10 - 1)$

If this is done, will the multiplication spread over the subtraction statement ?



The child should reach the conclusion that multiplication is distributive over subtraction.

e.g., $8 \times 9 = 8 \times (5 + 4)$
$= (8 \times 5) + (8 \times 4)$
$= 40 + 32$
$= 72$

$8 \times 9 = 8 \times (10 - 1)$
$= (8 \times 10) - (8 \times 1)$
$= 80 - 8$
$= 72$

4. The Distributive Law and Division

The child has been encouraged to generalize about the properties of number after he has had suitable experiences concerning them. As he progresses in mathematics, these generalizations are subject to refinement. For example, the child has probably discovered that the commutative property applies to addition but not to subtraction.

Once the child has an understanding of the distributive law, he should be encouraged to experiment with its application to division.

e.g., $15 \div 5 = (10 + 5) \div 5$
$= (10 \div 5) + (5 \div 5)$
$= 2 + 1$
$= 3$

For children who are unable to attain understanding at this time, the systematic exploration of the distributive law as applied to division should be postponed.

Situations used for discovering the distributive property of multiplication over addition are suitable here.

i.e., i Rods.
ii Arrays.
iii Number lines. (See pages 48–50)

Frequent practice should be given in renaming larger numbers in terms of multiples of smaller numbers.

e.g., Rename 48 as multiples of 4.
$48 = 40 + 8$
$48 = 32 + 16$
$48 = 20 + 20 + 8$, etc.

Only after the child can confidently do this should he be allowed to proceed to examples of division involving the distributive law.

e.g., Solve $48 \div 4$ in as many ways as possible using the distributive law.

$48 \div 4 = (40 + 8) \div 4$
$= (40 \div 4) + (8 \div 4)$
$= 10 + 2$
$= 12$

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$$
\begin{aligned}
48 \div 4 &= (20 + 20 + 8) \div 4 \\
&= (20 \div 4) + (20 \div 4) + (8 \div 4) \\
&= \quad 5 \quad + \quad 5 \quad + \quad 2 \\
&= \quad 12
\end{aligned}
$$

No examples involving remainders should be given at this level.

H. INVERSE OPERATIONS

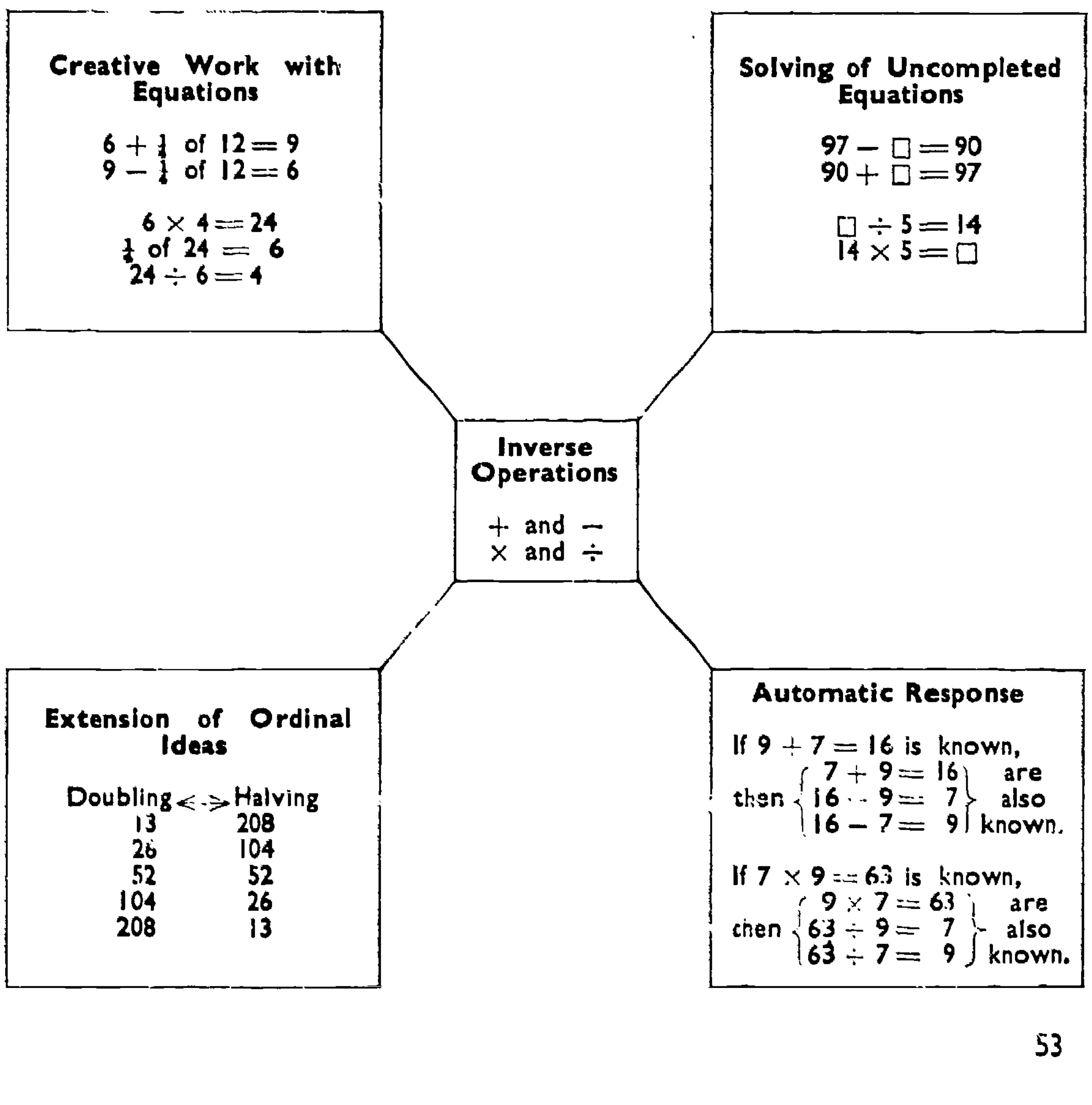
The four operations can be seen as pairs of inverse operations. Addition is linked with subtraction ; multiplication with division. They are so called because the use of the other operation in a pair causes the first operation to be " undone ".

e.g., $6 + 5 - 5 = 6$

or, if $6 + 5 = 11$, then $11 - 5 = 6$.

Experience with the inverse relationships between the operations has been continuing since Section B, while a special study was made in Section E, Stage 30.

In Section F the child should be encouraged to recognize and use the inverse operations in various aspects of his work.



I. PROPERTIES OF EQUATIONS

Through the ability to use the following properties, the child will have greater power over situations involving equations. Zero is to be understood as something other than mere nothing, that is, the absence of something. In mathematics zero is a number that has unique properties.

1. Law for Addition of Zero (Identity Law of Addition)

If zero is added to any number the total is the same as the original number.

e.g., $7 + 0 = 7$.

The child's attention might best be drawn to this law through a variety of numerical examples. A pattern may be built up by using examples similar to :—

$7 = 7 + \square$	$5 = \square + 5$
$8 + 0 = \square$	$6 + \square = 6$
$0 + 3 = \square$	$\square + 0 = 9$

Further activities using numerals :—

$\square = 0 + \square$	$6 - 0 = \square$
$5 = 5 + \square - \square$	$9 - \square = 0$

2. Law for Multiplication by One (Identity Law of Multiplication)

If any number is multiplied by one, the product gained is the same as the original number, that is, the identity of the original number remains unchanged.

e.g., $3 \times 1 = 3$.

This idea is basic for work in later sections, for example, the comparison of fractions, the ordering of fractions, and the four processes using fractions.

e.g., $\frac{3}{7} + \frac{2}{5} = (\frac{3}{7} \times \frac{1}{1}) + (\frac{2}{5} \times \frac{1}{1})$ (Identity Law of Multiplication)

$= (\frac{3}{7} \times \frac{5}{5}) + (\frac{2}{5} \times \frac{7}{7})$ ($\frac{1}{1}$ has been renamed as $\frac{5}{5}$ and $\frac{7}{7}$)

$= \frac{15}{35} + \frac{14}{35}$

$= \frac{29}{35}$

The teacher could direct the child to this property through a variety of numerical examples.

a	$1 \times 3 = \square$	$\square = 6 \times 1$
	$3 \times \square = 3$	$17 = 17 \times \square$
	$\square \times 1 = 57$	$34 = \square \times 1$
b	$\square \times 1 = \square$	
	$\square = 1 \times \square$	

3. Law for Multiplication by Zero

If any number is multiplied by zero the product is always zero.

e.g., $3 \times 0 = 0$.

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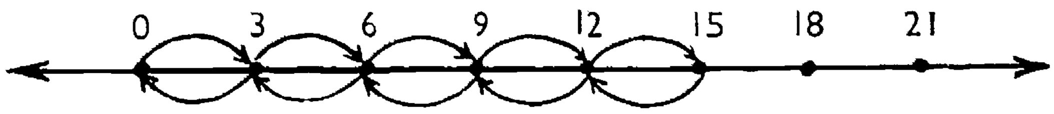
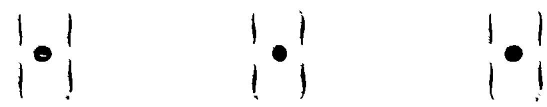
This property may be approached through the number line or through number patterns via serial counting. For example :—



$1 \times 3 = 3$	$5 \times 3 = 15$
$2 \times 3 = 6$	$4 \times 3 = 12$
$3 \times 3 = 9$	$3 \times 3 = 9$
$4 \times 3 = 12$	$2 \times 3 = 6$
$5 \times 3 = 15$	$1 \times 3 = 3$
	$0 \times 3 = 0$

Another approach makes use of counters or scrap material. The child is asked to show three sets of one. The total number of the elements is then established as being three.



This can be followed by the direction to show three empty sets.

So he concludes that three times zero is zero.

Numerical activities can be used to reinforce the idea.

a	$3 \times 0 = \square$	b	$0 = \square \times 0$
	$3 \times \square = 0$		$0 = 0 \times \square$
	$\square = 3 \times 0$		
	$0 = 3 \times \square$		

4. If Equals Are Added to Equals the Sums Are Equal

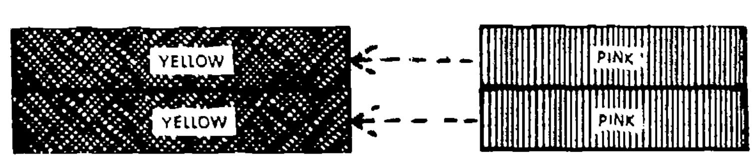
e.g., $5 = 5$

$5 + 3 = 5 + 3$

This property can be readily illustrated by rods, by balance, by peg-board, or by scrap materials.

a **With the Rods**

i



Yellow equals yellow.

Yellow plus pink equals yellow plus pink.

(The child first had experience of this situation in Section B, Stage 6.)

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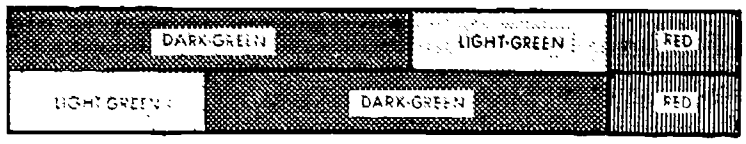
ii

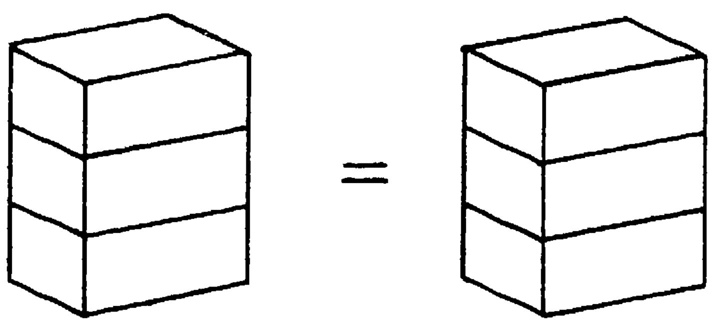


Dark-green plus light-green equals light-green plus dark-green.

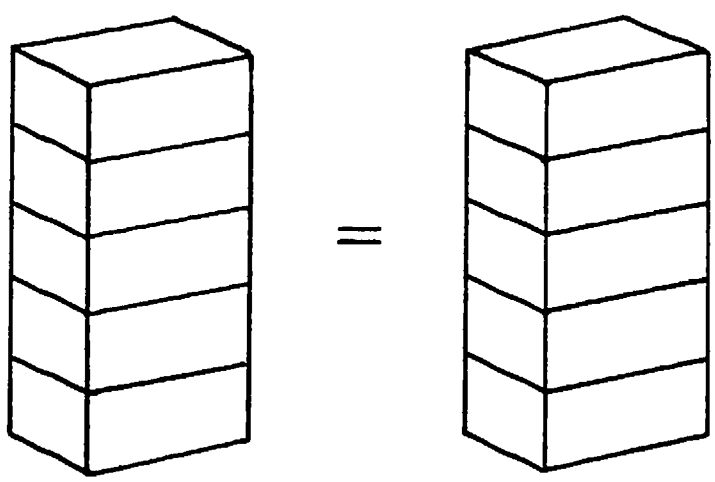
Dark-green plus light-green plus red equals light-green plus dark-green plus red.

b With Towers Constructed from Wooden Blocks

Three blocks equals three blocks.



Add two blocks to each pile and the situation of equality is maintained.



c Activities with Numerals

$$5 + \square = 5 + \square$$

$$64 + \square = 64 + \square$$

$$\square + 21 = \square + 21$$

The child should be encouraged to develop equations of his own to show this property.

Note : Some children will have incidental experience of other properties of equations :—

i If equals are subtracted from equals, the differences remain equal.

ii If equals are multiplied by equals, the products are equal.

iii If equals are divided by equals, the quotients are equal, provided the divisor is not zero.

These other properties will be treated in Section G.

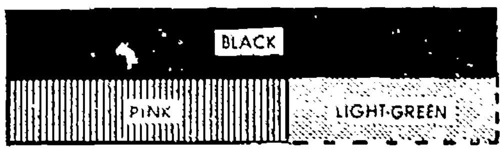
J. PROPERTIES OF DIFFERENCE

1. If Equals Are Added to Unequals, the Original Difference Remains Unchanged

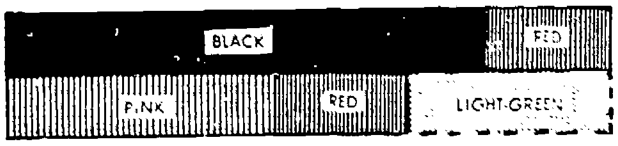
e.g., $7 - 4 = (7 + 5) - (4 + 5)$.

This property is the basis of formal subtraction using the method of equal additions. It can be shown by varying both situations and materials.

a **The Rods**



In this example, the difference between the two rods is equal to a light-green rod. If red is added to both lines, difference in length between the two remains unchanged.



(The child first experienced this situation in Section B, Stage 6.)

b **Activities with Numerals**—The child can solve equations similar to the following through an application of the property under discussion.

e.g., $36 - 24 = (36 + 6) - (24 + 6)$
$= 42 - 30$
$= 12$

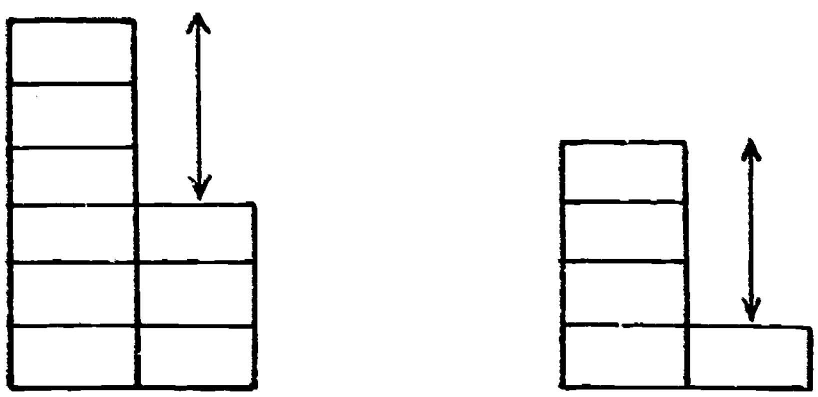
(The subtrahend has been rounded off to 30 by the addition of 6.)

2. If Equals Are Subtracted from Unequals the Original Difference Remains Unchanged

e.g., $7 - 4 = (7 - 3) - (4 - 3)$

This can be shown with :—

a **Towers**



Subtract three blocks from six and the difference is three. If before obtaining the difference, an equal number of blocks were taken from each pile, then the difference between the two piles would remain unchanged. That is, after removing two blocks from each pile the difference remains three.

b **Activities with Numerals**—This property of difference can be used in solving uncompleted equations.

e.g., $36 - 24 = \square$

$$\begin{aligned} 36 - 24 &= (36 - 4) - (24 - 4) \\ &= 32 - 20 \\ &= 12 \end{aligned}$$

Note : The decision as to whether the child will add or subtract equal amounts will depend upon the child's opinion as to which is the easier method in a given situation.

$$\begin{aligned} 85 - 27 &= (85 + 3) - (27 + 3) \\ &= 88 - 30 \\ &= 58 \end{aligned}$$

Here the subtrahend has been rounded off through the addition of three units.

$$\begin{aligned} 89 - 27 &= (89 - 7) - (27 - 7) \\ &= 82 - 20 \\ &= 62 \end{aligned}$$

Here the subtrahend has been rounded off by the subtraction of seven units.

Some children may prefer to round off the minuend rather than the subtrahend. They should be encouraged to use the method that is easier for them as individuals.

$$\begin{aligned} 89 - 27 &= (89 + 1) - (27 + 1) \\ &= 90 - 28 \\ &= 62 \end{aligned}$$

The child should be encouraged to use a variety of approaches when solving equations. The teacher must be prepared to accept unorthodox but valid methods of solution.

PART 5. EQUATIONS

REVIEW OF EQUATION WORK IN PREVIOUS SECTIONS

In **Section A**, the child had incidental experience that formed a background for a study of equality (possibly the most fundamental of all basic mathematical ideas), for example, similarities of colour, of shape, and of size in a wide variety of situations not limited to mathematics.

In **Section B**, a deliberate study was commenced. The first type of equivalence recognized by the child was one in which two rods of the same size and colour were placed side by side. The child learned to relate this situation to a statement such as "Yellow equals yellow". His understanding was then refined through a series of graded exercises including rearrangement and substitution of rods. (See Section B, Stage 6.)

Understanding of the operations of addition, multiplication, subtraction, and division was developed against this background of equivalence.

In this section, the coloured rods provided the means whereby the child could perceive and state an equivalence. The statement so made was important only as a means of expressing some mathematical understanding. The language used was that of colour, uncomplicated by numerical ideas.

In **Section C**, the change of language to number opened the way to recording and the beginning of a deliberate move towards abstract work. Manipulation of equations, both oral and written, provided a background of experience from which there would later develop a gradual awareness of the relations between operations.

Section D, a short section, marked a consolidation period. The child's awareness of the relations between the operations grew and an understanding began to develop. (Written equations were no more important in themselves than the oral reading of rod patterns.)

In **Section E**, the number field widened considerably. From a simple understanding of the operations, there was a deliberate treatment of their relationships to one another. Interpretation of his own and of the teacher's equations helped to reveal the extent of the child's understanding. The solving of equations was important mainly in so far as it utilized the understanding of operations.

Teachers will realize then, that the evidence of maturity or development of mathematical thought is not to be found merely in the use of larger numbers, but also in the fact that more complex combinations of operations are used, and more complex numerals for numbers.

e.g., ... 6 . . . . . .

. . . . 2×3 . . . .

. . $2 \times \frac{1}{2}$ of 6 . . .

In renaming 6 as 2×3, the child has used a more complicated name for the number. In the next renaming (or substitution) he has introduced a further operation.

It is essential that teachers should be familiar with Stages 29 and 30 (Section E) so that they understand the varying types of experience the child has already gained. Such experience as the child has had in Section E (and this will vary greatly) must of necessity be his starting point in Section F.

DEVELOPMENT

By the time Section F is reached, the child is thoroughly familiar with the equation as a means of expressing his mathematical ideas. He can create, manipulate, interpret, and solve equations. When a discovery is made in some particular aspect of his work (for example, fractions, place value, basic mathematical laws) it is by means of the equation that he expresses and utilizes this discovery. The child begins to consider, informally at first, the various means by which equality can be achieved.

Because of this teachers should not regard equations as an isolated topic for separate treatment in Section F. They should rather think of equations as being a method of approach to most other topics and a means of developing logical thought. Therefore, this part of the Guide (Part 5, Equations) must be read by teachers in conjunction with other parts of Section F.

In previous sections, creating, manipulating, interpreting, and solving of equations were developed separately. In Section F it is virtually impossible to separate these four divisions. To find an unknown, the child must :—

a Interpret the equation, thus drawing on his experience gained in creating equations.

b Exercise where necessary his understanding of the relations between operations, thus drawing on his experience of creating and manipulating equations.

c Use his skill with number facts to arrive at the correct answer.

For example, in solving $\frac{5}{6}$ of $60 - \square \times \square = 30$, a child would need to use the following steps, though not necessarily in this order :—

> He interprets. (e.g., "The difference between $\frac{5}{6}$ of 60 and what I must find equals 30.")
>
> He renames $\frac{5}{6}$ of 60 as 50, using his understanding of the fraction as an operator, together with his knowledge of number facts.
>
> He relates addition and subtraction. ("What must be added to 30 to equal 50 ?")
>
> He interprets $\square \times \square$ as "factors of twenty", and decides which he will use.

(For detailed information on the use of frames see Use of Frames, page 104.)

I. Creative Work

Opportunity should still be given for the child to create equations freely, in order to maintain an imaginative approach to mathematics. A smaller proportion of time than formerly will be spent in this way.

2. Manipulation

The manipulation of equations (that is, rearrangement and substitution, sometimes separately and sometimes in combination) is now carried out with a purpose more definitely stated than in previous sections. Some aspects of this work, with typical examples, are as follows :—

a **Automatic Response**—As an activity to assist in reinforcing rapid recall of number facts, the child could be given an equation, such as $11 + 36 = 47$, and asked to substitute for 11 using addition only, and for 36 using multiplication only.

e.g.,

$$11 + 36 = 47$$
$$8 + 3 + 4 \times 9 = 47$$
$$9 + 2 + 6 \times 6 = 47$$
$$6 + 5 + 12 \times 3 = 47$$

b **Basic Laws of Mathematics**—As an illustration of the distributive law the child may use :—

$$3 \times 21 = 3 \times (20 + 1)$$
$$3 \times 21 = 3 \times 20 + 3 \times 1$$
$$3 \times 21 = 60 + 3$$
$$3 \times 21 = 63$$

The child's ability to manipulate the original equation in terms of his understanding allows him to arrive at the solution.

c **Interrelation of Operations**—In Section D, it was pointed out that "the relationships existing between operations should be constantly in the teacher's mind, and informally brought to the child's notice." (Section D, page 11.)

In Section E, the child was required to make a definite study of these relationships in simple examples. (Section E, pages 29 and 30.)

All the relationships that needed to be studied were outlined in Section E, but not all children will have had sufficient experience. Such experience must continue in order to gain full understanding. In any case, all children need consolidation exercises.

ACTIVITIES FOR CONSOLIDATION MAY INCLUDE :—

i *Number Triples*

Given, for example, 3, 2, 6, the child is asked to write as many equations as he can, using these numbers and any symbols,

e.g.,

$$3 \times 2 = 6$$
$$2 \times 3 = 6$$
$$6 - 2 - 2 - 2 = 0$$
$$6 - 3 \times 2 = 0$$
$$6 \div 2 = 3$$
$$6 \div 3 = 2$$
$$\tfrac{1}{3} \text{ of } 6 = 2$$
$$\tfrac{1}{2} \text{ of } 6 = 3$$

Similarly, using the triple 5, 4, 9, he could write:—

$5+4=9$	$9-4=5$	$9-(4+5)=0$	$9-5-4=0$
$4+5=9$	$9-5=4$	$9-(5+4)=0$	$9-4-5=0$
$9=4+5$	$4=9-5$	$0=9-(4+5)$	$0=9-5-4$
$9=5+4$	$5=9-4$	$0=9-(5+4)$	$0=9-4-5$

ii *Manipulation of Equations of the Type* $n \times a = b$

e.g., $4 \times 5 = 20$

$5+5+5+5=20$

$20 \div 5 = 4$

$\frac{1}{4}$ of $20 = 5$

$20-5-5-5-5=0$

Given any one of the above equations, the child could be asked to supply the others.

REPEATED SUBTRACTION

The formal process of division will later be introduced through repeated subtraction, and the child needs preparation for this. In Section E, page 30, exercises such as the following were suggested :—

$24-6-6-6-6=0$	$15 \div 5 = 3$
$24-4\times 6=0$	$15-3\times 5=0$
$24 \div 6 = 4$	$15-5-5-5=0$

No remainders will be required in early formal division work, but exercises, as under, will provide useful background.

$15-6-6=3$	$17-3\times 5=2$
$15-2\times 6=3$	$17-5-5-5=2$

d **Order of Operations**—In Section E, page 33 (Solving Uncompleted Equations), it was stated : "In the absence (as yet) of a formalized rule for the order of working, the teacher must depend on the child's experience in creating, manipulating, and interpreting equations." **The formalizing of this convention will certainly not be required before Section G.** However, the child can be given experience that will result in his more readily accepting the convention at the appropriate time. Exercises should be limited to those in which the initial equation involves only addition and/or subtraction. Examples of this experience are :—

i Direction of the type of substitution to be used can assist the child to see the equation as a set of terms.

e.g.,

Substitute using division :

$5-2=3$

$10 \div 2 - 4 \div 2 = 6 \div 2$

Substitute using multiplication :

$24 = 16 + 8$

$4 \times 6 = 4 \times 4 \quad + \quad 4 \times 2$

Substitute using multiplication or division :—

$12 - 10 + 2 = 4$

$6 \times 2 \quad - \quad 20 \div 2 \quad + \quad 4 \times \frac{1}{2} = 12 \div 3$

ii Rearranging of equations in conjunction with substitution.

e.g., $18 - 3 = 15$

Rename the 3 in this equation, using multiplication or division :

$18 - \frac{1}{4}$ of $12 = 15.$

Rewrite the first equation using addition instead of subtraction :

$15 + 3 = 18.$

Rewrite the second equation using addition instead of subtraction :

$15 + \frac{1}{4}$ of $12 = 18.$

Following experience such as this, the child could be asked to rewrite, for example,

$17 - 3 \times 4 = 5,$

using addition.

Examination of other topics treated in Section F (Notation, Numeration and Place Value, Automatic Response, Basic Laws of Mathematics, Fractions, and Activities Leading towards Formal Processes) will suggest other specific purposes for which equations may be manipulated.

3. Verbal Interpretation

Throughout earlier sections of the course, the child is required to interpret his own and his teacher's equations in order to reveal his understanding of mathematical ideas. This verbal interpretation continues to be extremely important in Section F and should be carried right through the section.

Interpretation will provide the best opportunity for a child to show his increasing awareness of, for example, the basic laws of mathematics, and to discuss within the range of his vocabulary their meaning to him and how he sees them at work.

e.g., $6 \times 7 = 7 \times 6.$

What does this equation tell you ?
(Six times seven equals seven times six.)

What do you mean by "equals" ?
(Has the same value as.)

If I changed the sevens for fives and the sixes for twos, what would the equation tell you ?

It is from discussions such as the above that the child will clarify his ideas and eventually be able to put his understanding of the laws into words.

$$6 + \frac{1}{2} \text{ of } 18 = 15$$

$$15 - \frac{1}{2} \text{ of } 18 = 6$$

In explaining this manipulation, a child may interpret in terms somewhat as follows : "The first equation tells me that if 6 and $\frac{1}{2}$ of 18 are added, the result is 15. I know then that the difference between 15 and $\frac{1}{2}$ of 18 equals 6."

As the child proceeds through Section F, he will tend to use more sophisticated terms and to express his understanding more easily.

4. Solving Uncompleted Equations

It is certainly true that in solving an uncompleted equation the child is concerned with getting the correct solution, but this is not the only reason for studying such equations. It is important that the child should **search for, recognize, and apply some principle** in order to achieve his aim. If the child works through an understanding and a use of principles rather than some formal rote process, there is a much better prospect of his both solving the equation correctly and understanding what is involved in so doing.

e.g., a $\frac{\square}{7} = \frac{14}{14}$; $\frac{2}{7} = \frac{\square}{14}$; $\frac{5}{7} = \frac{10}{\square}$; $\frac{6}{\square} = \frac{12}{\square}$

To solve these equations, the child needs to understand that there are many different names for the same fraction. He must also recall a way of finding the required equivalent. (See Part 6, Fractions, pages 75–80.)

b $23 + (14 + 11) = (\square + 14) + 11$

To solve this equation, the child recognizes and puts to use the associative law of addition.

Reference should be made to Section E, pages 33 to 35, and similar exercises should be continued. The comments made there are also valid for Section F. In the main, equations need not be any more complex than those shown in Section E.

FREE APPLICATION OF TECHNIQUES

As the child's knowledge of number and the basic laws extends and deepens, so will he find more ways of solving equations. In fact, the child should be encouraged to use a variety of approaches when solving an equation. However, the increase in the variety of methods of solution is not an end in itself. The ultimate aim for each child is the ability to select from the methods of solution he has at his command the one that best suits him for any particular problem. The teacher must encourage originality of thought and be prepared to accept unorthodox but valid methods of solution.

If a child does not express all steps of a solution, the teacher can ask for oral interpretation rather than demand a large amount of written work.

The following examples are not intended as prescribed methods of solving equations. They include some (but certainly not all) of the solutions children are likely to use. Further variations are possible, and some children will enjoy finding complicated ways of solving simple equations.

Note : For a commentary on methods of the setting out of equations, see Section F, Part 4, Basic Laws of Mathematics, Development, Note vi, page 36.

a $24 + 28 = \square$

i

$$\begin{aligned} 24 + 28 &= 24 + (24 + 4) && \text{(Renaming)} \\ &= (24 + 24) + 4 && \text{(Regrouping)} \\ &= 2 \times 24 + 4 \\ &= 48 + 4 && \text{(Doubling)} \\ &= 52 \end{aligned}$$

ii

$$\begin{aligned} 24 + 28 &= 20 + 4 + 20 + 8 && \text{(Extended notation)} \\ &= (20 + 20) + (4 + 8) \\ &= 40 + 12 \\ &= 52 \end{aligned}$$

Note : This method is closely linked with work in the Basic Laws (see page 42) and is an essential prerequisite for the process of addition (see Activities Leading towards Formal Processes, page 82).

iii

$$\begin{aligned} 24 + 28 &= 22 + 2 + 30 - 2 && \text{(Renaming)} \\ &= 22 + 30 \\ &= 52 \end{aligned}$$

iv

$$\begin{aligned} 24 + 28 &= 24 - 4 + 28 + 4 \\ &= 20 + 32 \\ &= 52 \end{aligned}$$

v

$$\begin{aligned} 24 + 28 &= 6 \times 4 + 7 \times 4 \\ &= 13 \times 4 \\ &= 26 \times 2 && \text{(Doubling one factor, halving the other)} \\ &= 52 \end{aligned}$$

b $5 \times 14 = \square$

i

$$\begin{aligned} 5 \times 14 &= 10 \times 7 && \text{(Doubling one factor, halving the other)} \\ &= 70 \end{aligned}$$

ii

$$\begin{aligned} 5 \times 14 &= 5 \times (10 + 4) && \text{(Renaming)} \\ &= 5 \times 10 + 5 \times 4 && \text{(Distributive law)} \\ &= 50 + 20 \\ &= 70 \end{aligned}$$

65

iii $5 \times 14 = 5 \times (7 \times 2)$ (Renaming)
$= (5 \times 7) \times 2$ (Regrouping)
$= 35 \times 2$
$= 70$

iv $5 \times 14 = 14 + 14 + 14 + 14 + 14$
$= 10 + 4 + 10 + 4 + 10 + 4 + 10 + 4 + 10 + 4$ (Renaming)
$= 10 + 10 + 10 + 10 + 10 + 4 + 4 + 4 + 4 + 4$ (Reordering)
$= 50 + 20$ (Counting)
$= 70$

v $5 \times 14 = 5 \times (20 - 6)$ (Renaming)
$= 5 \times 20 - 5 \times 6$ (Distributive law)
$= 100 - 30$
$= 70$

c $36 - 19 = \square$

i $36 - 19 = 37 - 20$ (Equals added to unequals, the difference remains constant)
$= 17$

ii $36 - 19 = 40 - 23$ (Equals added to unequals, the difference remains constant)
$= 17$

iii $36 - 19 = 36 - 10 - 9$
$= 26 - 9$
$= 17$

iv If $36 - 19 = \square$, then $19 + \square = 36$ (Complementary addition)
$19 + (1 + 10 + 6) = 36$
$19 + 17 = 36$
$36 - 19 = 17$

v $36 - 19 = 2 \times 18 - (1 \times 18 + 1)$ (Renaming)
$= 2 \times 18 - 1 \times 18 - 1$
$= 1 \times 18 - 1$
$= 17$

vi $36 - 19 = 30 + 6 - 10 - 9$
$= 30 - 10 - 9 + 6$
$= 20 - 9 + 6$
$= 11 + 6$
$= 17$

66

vii $36 - 19 = 30 + 6 - (10 + 9)$
$= 20 + 16 - 10 - 9$
$= 20 - 10 + 16 - 9$
$= 10 + 7$
$= 17$

d $120 \div 8 = \square$

i $120 \div 8 = \frac{1}{8}$ of 120
$= \frac{1}{4}$ of 60
$= \frac{1}{2}$ of 30
$= 15$

ii $120 \div 8 = (80 + 40) \div 8$ (Renaming)
$= 80 \div 8 + 40 \div 8$ (Distributive law)
$= 10 + 5$
$= 15$

iii $120 \div 8 = 60 \div 4$ (Halving the dividend, halving the divisor)
$= 30 \div 2$
$= 15$

iv $120 = 40 + 40 + 40$ (Renaming through counting)
$= 5 \times 8 + 5 \times 8 + 5 \times 8$ (Renaming)
$= 15 \times 8$
$120 \div 8 = 15$

v $120 - 40 - 40 - 40 = 0$ (Division as repeated subtraction)
$120 - 5 \times 8 - 5 \times 8 - 5 \times 8 = 0$
$120 - 15 \times 8 = 0$
$120 \div 8 = 15$

PART 6. FRACTIONS

AIM

To continue the study of fractions.

NOTES

1. This study involves the fraction—

   a as a relation between two numbers ;

   b as an operator on another number ;

   c as a number ;

   d as an equivalent of other fractions.

2. These ideas have been presented to the child both in concrete and in abstract situations during the preceding sections. The purpose of this stage is to consolidate the ideas, using a greater variety of concrete materials and abstract situations.

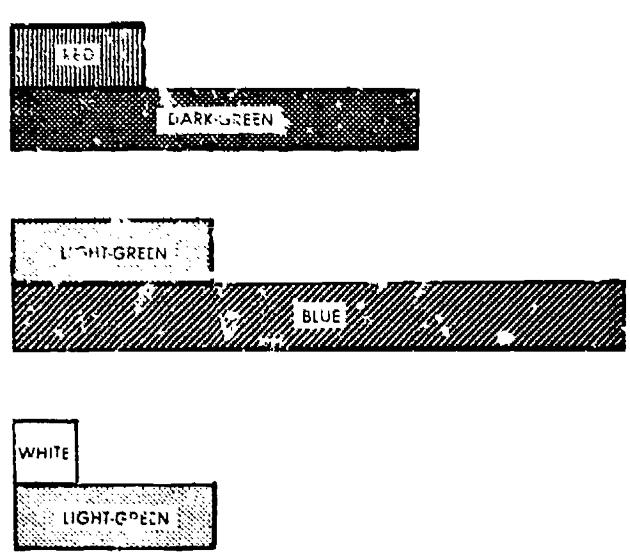
3. A sound knowledge of equivalence of fractions must precede addition and subtraction of fractions, which is commenced in Section G.

DEVELOPMENT

1. The Fraction as a Relation between Two Numbers

In previous sections the child discovered :—

i That the same relationship exists in different pairs of rods, e.g.,



each of which expresses $\frac{1}{3}$.

ii That the relationship between two rods can be expressed in two ways, according to which rod is the " measurer ".

e.g., Yellow is $\frac{5}{7}$ of the black rod.

Black is $\frac{7}{5}$ of the yellow rod.

This experience should now be consolidated and extended through activities such as those listed below.

a **Using Rods**

i Exercises in which the child is asked to show the relationship in a number of different ways with rods could be extended to include examples such as $\frac{5}{3}$, $\frac{9}{4}$, $\frac{10}{5}$. The child should be able to indicate which rod is being used as the " measurer ".

ii Exercises involving value relations :—

If black is 1, what is white ? ($\frac{1}{7}$)

If black is 1, what is yellow ? ($\frac{5}{7}$)

If black is 1, which rod is $\frac{3}{7}$? (Light-green)

If black is 1, what is blue ? ($\frac{9}{7}$ or $1\frac{2}{7}$)

If orange is 1, what is white ? ($\frac{1}{10}$)

If orange is 1, which rod is $\frac{3}{5}$? $\frac{7}{10}$? $\frac{1}{2}$?

If brown is 1, what is white ? ($\frac{1}{8}$)

If brown is 2, what is white ? ($\frac{2}{8}$)

If brown is 3, what is white ? ($\frac{3}{8}$)

If brown is 3, what is light-green ? ($\frac{9}{8}$)

If light-green is $\frac{3}{7}$, which rod is 1 ? (Black)

If light-green is $\frac{3}{7}$, which rod is $\frac{5}{7}$? (Yellow)

If dark-green is 1, which rod is $\frac{1}{3}$? (Red)

If dark-green is 1, what is brown ?

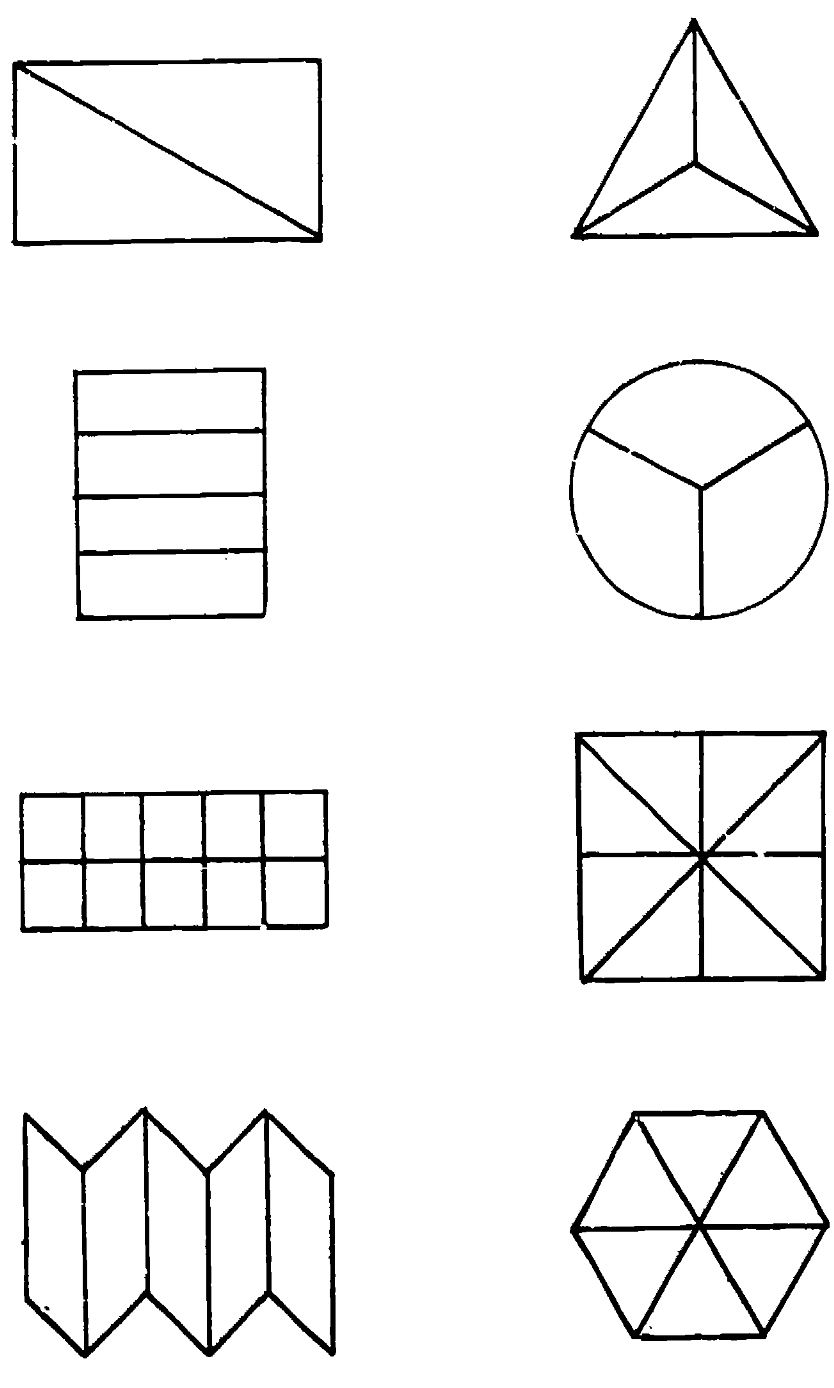
($\frac{4}{3}$ or $1\frac{1}{3}$, i.e., $1 + \frac{1}{3}$)

b Using Geometric Shapes

Note : Shapes made of materials such as wood, pressed board, cardboard, or plastic are invaluable for initial experience.

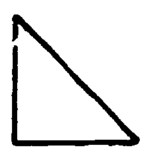
i Colour one part of each shape and write this as a fraction of the whole.

Colour two parts of each shape and write this as a fraction of the whole.

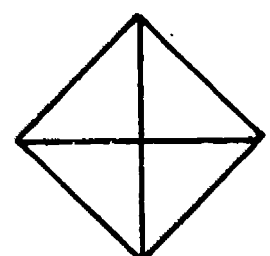


ii Express as a fraction the relation of A to B
A to C
A to D
A to E

Express as a fraction the relation of E to B
E to C
E to D



A



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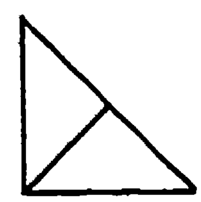

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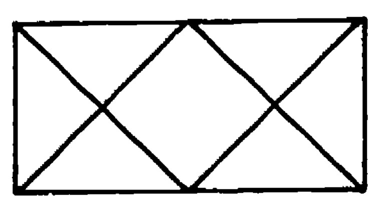

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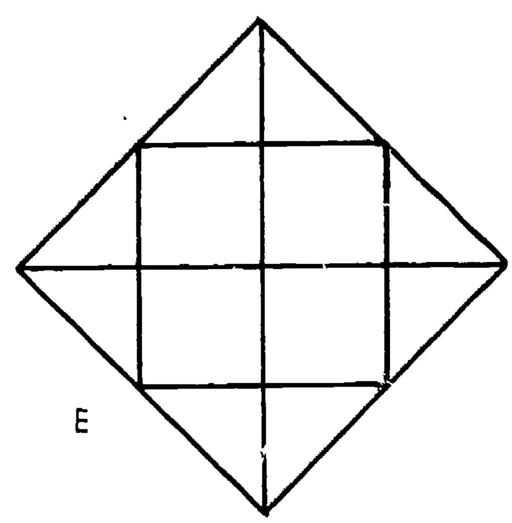
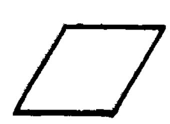


E

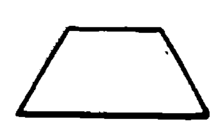
iii Applying value relations to geometric shapes.

Once the idea of relationship has been established, a value can be given to any one shape in a set of related shapes, thus fixing the values of all the shapes in that set. For example :—

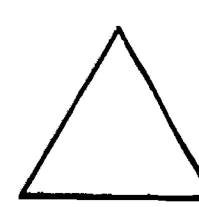
A



B



C



D

If A has the value 1, what is the value of B ? C ? D ?

If C has the value 1, what is the value of A ? B ? D ?

If D has the value 1, what is the value of A ? B ? C ?

If B has the value $\frac{1}{2}$, what is the value of A ? C ? D ?

c **Using Applied Number Materials**—The child should be able to apply his knowledge of relationship in many different situations. Applied number materials present many opportunities for developing this aspect of the work. For example, asked to find pairs of objects that show a relationship of $\frac{1}{2}$, the child could select :—

i Pieces of string 6 inches and 1 foot long.

ii Strips of paper 3 inches and 6 inches long.

iii Packets of rice weighing 8 ounces and 1 pound.

iv Tins of gravel weighing 4 ounces and 8 ounces.

v Vessels of different shapes holding 1 half-pint and 1 pint.

2. The Fraction as an Operator on Another Number

In previous sections the child has had experience in using the fraction as an operator on numbers to 144. (See Section E, Stage 31.) Rods have been used for multiplication mats in Sections D and E, and this work could now be extended. The emphasis is still on numbers to 144, although the child may extend beyond this range, particularly in his creative work.

The understanding of the fraction as an operator is developed through various aspects of work, some of which are outlined below. The order of listing is not necessarily the order of handling.

a **Pattern Work**—The child often follows a pattern in creative work with fractions. For example :—

i Doubling and halving may show :—

$\frac{1}{48}$ of $48 = 1$	$\frac{1}{2}$ of $8 = 4$	$\frac{2}{3}$ of $9 = 6$
$\frac{1}{24}$ of $48 = 2$	$\frac{1}{4}$ of $16 = 4$	$\frac{2}{3}$ of $18 = 12$
$\frac{1}{12}$ of $48 = 4$	$\frac{1}{8}$ of $32 = 4$	$\frac{2}{3}$ of $36 = 24$
$\frac{1}{6}$ of $48 = 8$	$\frac{1}{16}$ of $64 = 4$	$\frac{2}{3}$ of $72 = 48$

ii When working with the three numbers 3, 4, 12 (a number triple), to demonstrate the relationships between operations, the child may write :—

$3 \times 4 = 12$	$12 \div 4 = 3$	$\frac{1}{3}$ of $12 = 4$
$\frac{1}{4}$ of $12 = 3$	$4 \times 3 = 12$	$12 \div 3 = 4$

(Note the usefulness of these exercises in assisting rapid recall of number facts.)

b **Substitution Involving Fractions**—Many substitutions used in creative work involve the use of the fraction as an operator.

e.g., $46 - 24 + 9 = 31$

$\frac{7}{7}$ of $46 - \frac{2}{3}$ of $36 + \frac{9}{5}$ of $5 = \frac{1}{3}$ of 93.

c **Uncompleted Equations**

(For detailed information on the use of frames see Use of Frames, page 104.)

$\frac{4}{9}$ of $72 = \square$ $\qquad$ $\frac{\square}{\square}$ of $36 = 24$

$\frac{\square}{9}$ of $90 = 10$ $\qquad$ $8 + \frac{3}{4}$ of $16 - \frac{1}{3}$ of $3 = \square$

$\frac{\square}{\square}$ of $16 = 16$

Late in the section, the most advanced children could probably handle more difficult examples :—

$$\text{e.g., } 24 - \frac{\square}{\square} \text{ of } \square = 2$$

$$7 + \frac{\square}{9} \text{ of } 18 = 9$$

$$\frac{\square}{3} \text{ of } 12 - \frac{\square}{8} \text{ of } 16 = 2$$

It should not be demanded that the simplest form of the fraction should always be used.

d **Applied Number**—In some applied number situations it will be found that fractions are being used as operators. For example, in ruling a line $\frac{1}{3}$ of a foot long, the child will probably find $\frac{1}{3}$ of 12 inches before ruling the line.

3. The Fraction as a Number

The child has become aware of fractions as numbers in addition to their roles as operators or relationships (see Section E, Stage 31). The study is now extended to include the fraction as a number—

less than 1, e.g., $\frac{3}{5}$

equal to 1, e.g., $\frac{5}{5} = 1$

greater than 1, e.g., $\frac{7}{5}$, i.e., $1\frac{2}{5}$

Note : Any whole number can be regarded as a fraction with 1 as its denominator. For example, 3 can be written as $\frac{3}{1}$.

Whole numbers can be expressed as fractions with denominators other than 1. For example, 5 can be expressed as $\frac{10}{2}$.

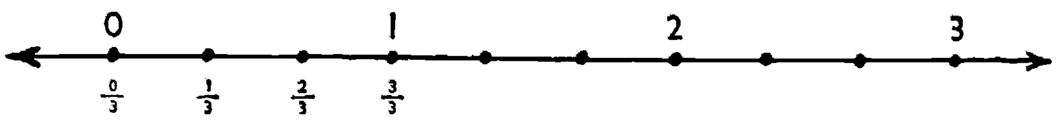
These points can be covered through the activities listed below, which are not necessarily sequential. It should be recalled that some experience with equivalence of fractions was gained in Section E. Such experience is a prerequisite for this unit of work.

Activities

a Rod Staircases—These activities, commenced in section E, Stage 31, should be continued and extended.

b Number Lines

i Continue the labelling of this number line.



ii Mark the following points on this number line :—

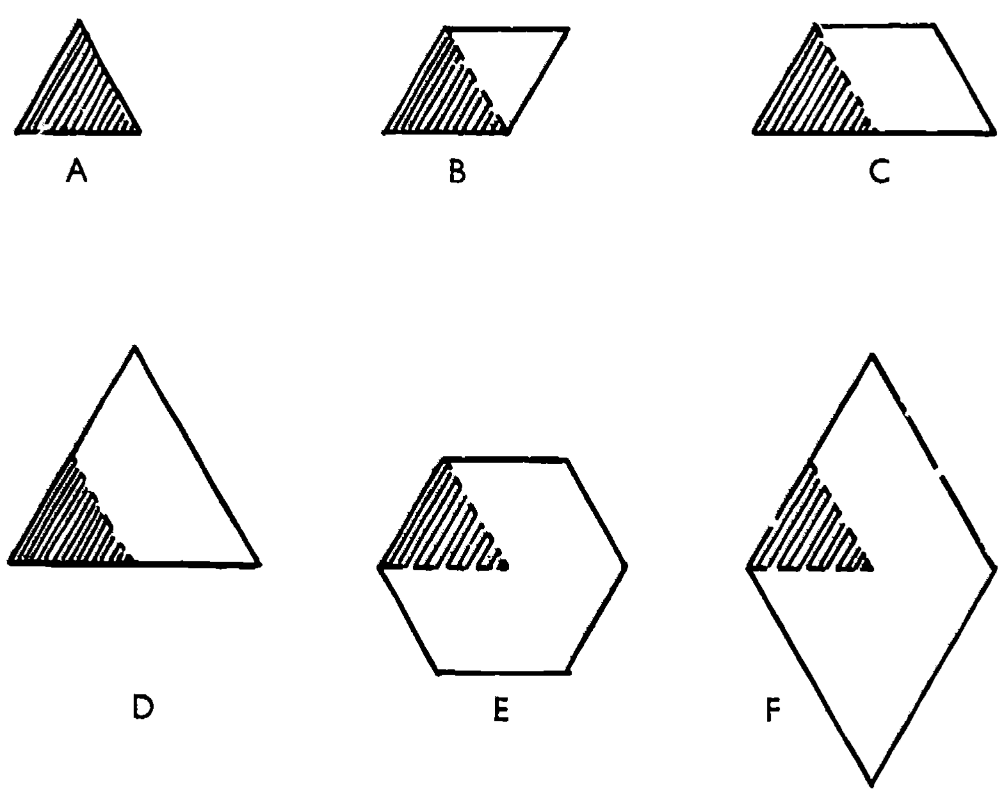
$\frac{4}{4}$, $\frac{8}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{9}{4}$, $\frac{6}{4}$

iii Estimate and show the positions of these numbers on this number line :—

$\frac{7}{8}$, $1\frac{4}{6}$, $2\frac{1}{8}$, $\frac{16}{10}$, $\frac{5}{2}$

(The use of quarter-inch square graph paper could assist the child in making his estimations.)

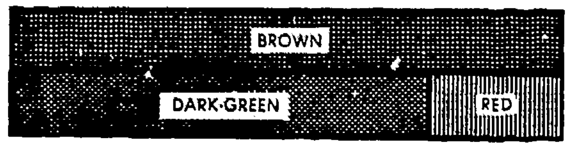
c Geometric Shapes



Name the value of the shaded triangle in each shape if the whole shape has the value 1.

d Reading Equations from Rod Patterns

In this exercise, rods other than white are named as " one " so that fractional numbers are used. The activity is possible only after the child has had a wide range of value relationship experiences. It is not directed primarily to operations with fractions but is merely an activity which helps the child to discover that whole numbers and fractions have common properties.



In the above pattern, the brown rod may be called " one " and the child is asked the values of the dark-green and the red rods. ($\frac{3}{4}$ or $\frac{6}{8}$; and $\frac{1}{4}$ or $\frac{2}{8}$.)

The pattern can then be read :—

$$1 = \tfrac{3}{4} + \tfrac{2}{8} \qquad \tfrac{6}{8} = 1 - \tfrac{1}{4}$$

$$1 - \tfrac{1}{4} = \tfrac{3}{4} \qquad 1 = \tfrac{1}{4} + \tfrac{3}{4}$$

$$\tfrac{1}{4} = 1 - \tfrac{3}{4} \qquad 1 - \tfrac{3}{4} = \tfrac{2}{8}$$

If the orange rod is called " one ", the values change. The value of the brown rod becomes $\frac{8}{10}$ or $\frac{4}{5}$; the red rod $\frac{2}{10}$ or $\frac{1}{5}$; and the dark-green rod $\frac{6}{10}$ or $\frac{3}{5}$. Readings would then include :—

$$\tfrac{6}{10} + \tfrac{2}{10} = \tfrac{8}{10} \qquad \tfrac{4}{5} - \tfrac{1}{5} = \tfrac{3}{5}$$

$$\tfrac{2}{10} + \tfrac{3}{5} = \tfrac{8}{10} \qquad \tfrac{}{10} - \tfrac{1}{5} = \tfrac{6}{10}$$

This activity could be extended to include more complicated patterns and other operations.

e Substitution of Fractions for Whole Numbers in Equations

Substitute for the 1 in the following examples :—

i $7 + 1 = 8$	ii $8 - 1 = 7$
$7 + \frac{1}{2} + \frac{1}{2} = 8$	$8 - \frac{7}{7} = 7$
$7 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 8$	$8 - \frac{6}{6} = 7$
$7 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 8$	$8 - (\frac{3}{6} + \frac{3}{6}) = 7$
$7 + \frac{2}{8} + 3 \times \frac{1}{4} = 8$	$8 - 2 \times \frac{3}{6} = 7$
$7 + \frac{2}{8} + 3 \times \frac{2}{8} = 8$	$8 - 2 \times \frac{1}{2} = 7$

4. The Fraction as an Equivalent

By the end of Section E, the child had sufficient understanding to be able to take a fraction such as $\frac{1}{3}$ and discover, with the help of rods, fractions equivalent to it, for example, $\frac{1}{3} = \frac{4}{12} = \frac{2}{6} = \frac{3}{9}$.

In this section the child should have a wide range of experiences with exercises, such as $\frac{2}{3} = \frac{\square}{6}$, using structured aids. During these experiences he may discover a "pattern" or generalization for finding equivalent fractions, that is, when numerator and denominator are multiplied by the same number, an equivalent fraction is formed. Discussion with the teacher of a sequence such as $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}$ may help some children to form their generalization. Once this generalization has been established, many children will discover, through extensions of the basic activities, that when numerator and denominator are divided by the same number, an equivalent fraction will result, e.g., $\frac{4}{8} = \frac{1}{2}$, $\frac{2}{10} = \frac{1}{5}$.

Note: The order of listing the activities below is not intended to indicate an order of presentation.

Activities with Structured Aids

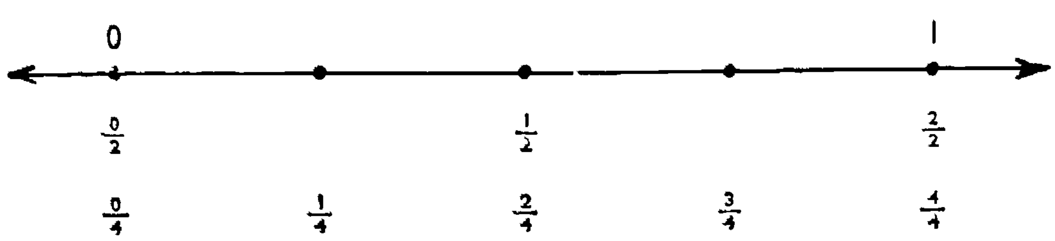
a Using Rods

There should be an extension of the work of Section E, Stage 31, Equivalence of Fractions.

b Using Number Lines

Typical activities, using the number line, are discussed below.

i

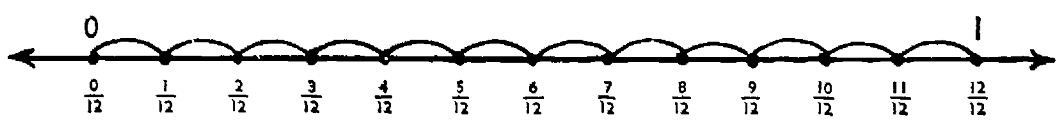


The segment of the number line to be used should first be established and marked (0 1).

Counting by halves, then by quarters, the child can discover that any point on a number line may have different names, for example, 1, $\frac{2}{2}$, $\frac{4}{4}$.

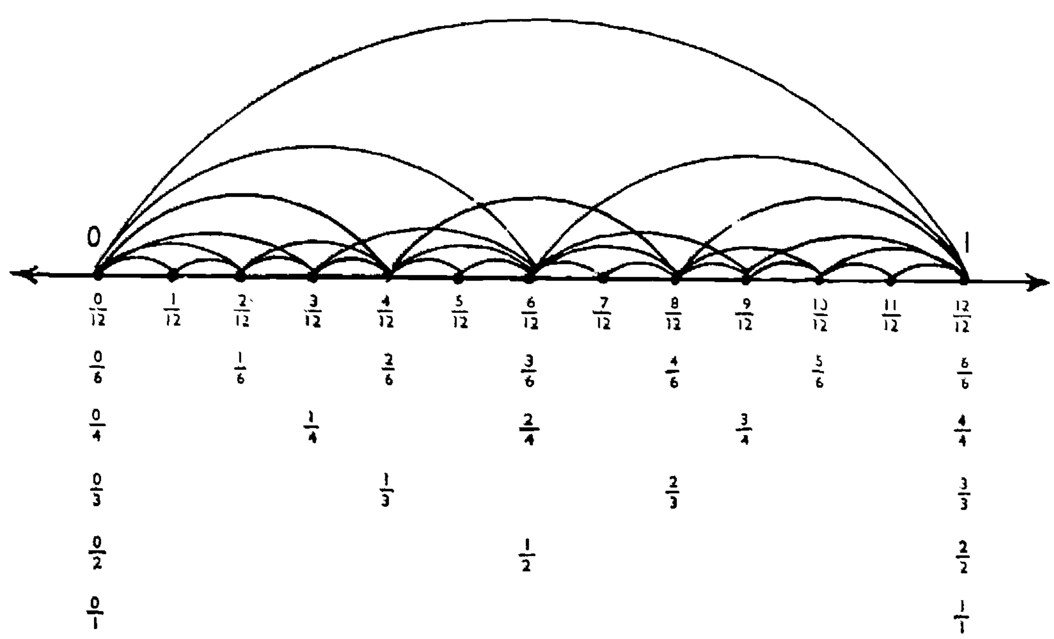
ii

After establishing the segment of the number line to be used, the teacher should lead the child to see that twelve "one-step jumps" are needed to move from 0 to 1. Each of these is one of a set of twelve equal jumps and so may be called a twelfth. The line should be marked accordingly.



The same procedure can be followed with two-step jumps (each of which is one of a set of six equal jumps and so may be called a sixth), three-step jumps (fourths or quarters), four-step jumps (thirds), six-step jumps (halves), and a twelve-step jump (one large jump).

These could be shown on separate number lines, but if all are marked on one number line the following diagram will result.



Equivalence can be seen clearly at many points on the line.

For example, $\frac{2}{12} = \frac{1}{6}$; $\frac{4}{12} = \frac{2}{6} = \frac{1}{3}$; $\frac{6}{12} = \frac{3}{6} = \frac{2}{4} = \frac{1}{2}$;

$\frac{12}{12} = \frac{6}{6} = \frac{4}{4} = \frac{3}{3} = \frac{2}{2} = \frac{1}{1}$.

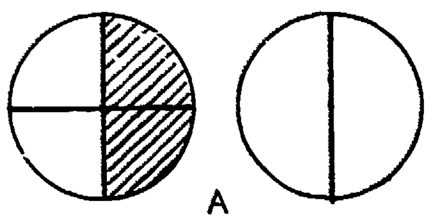
When this diagram is complete, discussion and oral questioning should follow, and a record could be made of some of the equivalents discovered.

Note : Quarter-inch graph paper could be used by a child in making his own number lines and discovering equivalent fractions.

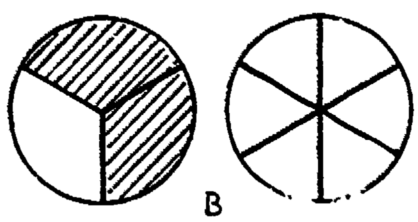
77

c Using Geometric Shapes

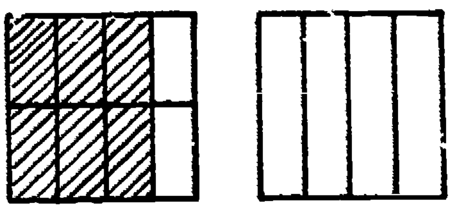
i Write the name of the fraction represented by the shaded portion in the first figure, colour an equivalent space in the second figure, and write the name of the fraction.



A

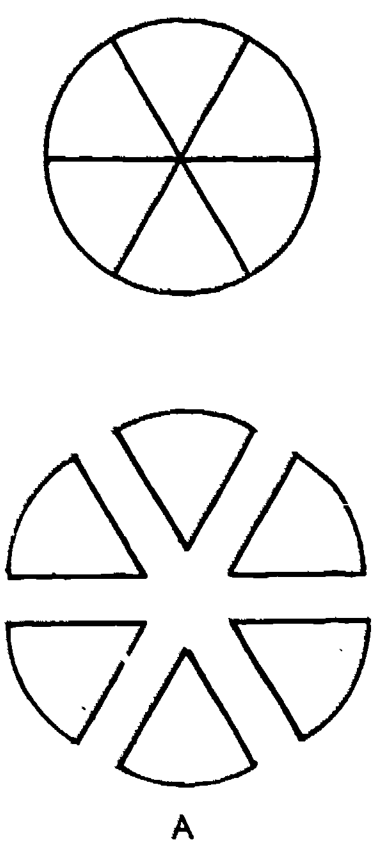


B

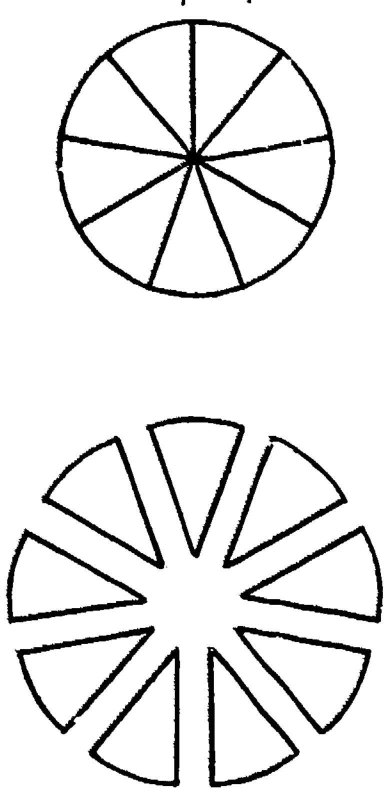


C

ii This exercise requires four cardboard circles of the same diameter. One is marked in sixths, the second is cut into six equal parts, the third is marked in ninths, and the last is cut into nine equal parts.



A



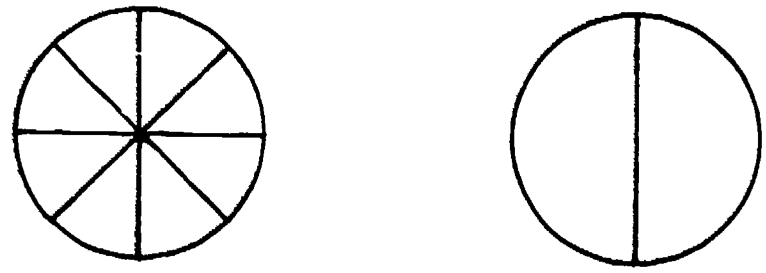
B

By placing the cut-out segments on the uncut circles, a child could discover that $\frac{2}{6} = \frac{3}{9}$.

This idea could also be used with other fractions.

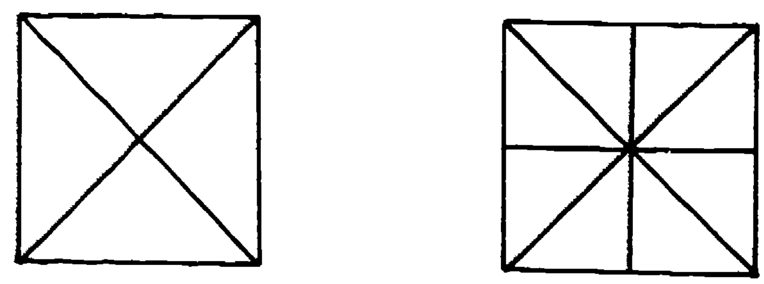
iii Colour the required part o. each figure, and supply the missing sign $>$, $<$, or $=$.

A



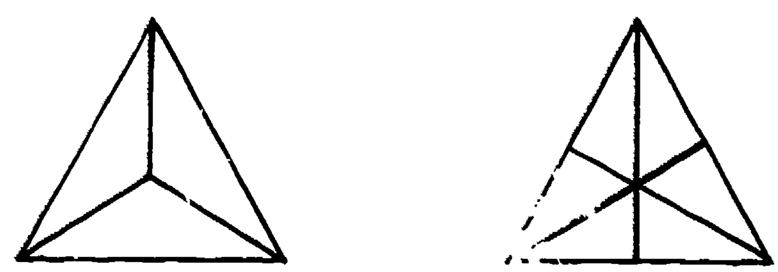
$\frac{4}{8}$ ☐ $\frac{1}{2}$

B



$\frac{3}{4}$ ☐ $\frac{6}{8}$

C



$\frac{2}{3}$ ☐ $\frac{5}{6}$

Activities with Numerals

i Write several equivalents for each of these numbers.

$\frac{1}{1}$ . . . . . .

$\frac{1}{2}$ . . . .

$\frac{3}{1}$ . . . .

$\frac{1}{5}$ . . . .

$\frac{2}{5}$ . . . .

$\frac{6}{5}$ . . . .

$\frac{0}{2}$ . . . .

Imaginative thinking on the part of the pupil should be encouraged, for example, $\frac{2}{3} = \frac{100}{150} = \frac{20}{30} = \frac{2000}{3000} = \frac{3}{4\frac{1}{2}}$.

ii Circle the numerals that represent the number $\frac{1}{2}$.

$\frac{2}{4} \quad \frac{2}{1} \quad \frac{4}{2} \quad \frac{6}{18} \quad \frac{10}{20} \quad \frac{6}{3} \quad \frac{50}{100}$.

Circle the numerals that represent the number 3.

$\frac{3}{1} \quad \frac{1}{3} \quad \frac{9}{3} \quad \frac{3}{9} \quad \frac{27}{3} \quad \frac{300}{100} \quad \frac{100}{3} \quad \frac{3}{3} \quad \frac{3}{0}$.

iii Complete the following statements.

$\frac{1}{5} = \frac{\square}{10}$

$\frac{4}{8} = \frac{1}{\square}$

$\frac{\square}{3} = \frac{3}{9}$

$\frac{6}{9} = \frac{\square}{3}$

$\frac{3}{4} = \frac{\square}{8} = \frac{9}{\square} = \frac{\square}{16}$

$\frac{21}{28} = \frac{\square}{4}$

$\frac{35}{45} = \frac{7}{\square}$

iv Supply the missing sign =, >, or <.

$\frac{2}{5} \square \frac{4}{10}$

$\frac{10}{15} \square \frac{1}{3}$

$\frac{2}{12} \square \frac{2}{24}$

PART 7. ACTIVITIES LEADING TOWARDS FORMAL PROCESSES

AIM

To bring together—

a the child's understanding of the four basic operations,

b his experience with the basic laws of mathematics,

c his knowledge of place value, and

d all other relevant experiences,

in order to develop a logical approach to the formal processes of addition, subtraction, multiplication, and division.

NOTES

1. It is not the intention in Section F to take each process to its final refinement, except in certain simple cases. However, as stated in the introduction to this section (see Section F, Introduction), there is no rigid border between Section F and Section G. Care should be taken that one process is not developed unduly at the expense of another.

2. The stages of refinement are of necessity sequential. The child should not leave a particular stage till he has thoroughly mastered it, though some stages may need only brief treatment for some children.

Having mastered a stage, however, there is no need for the child to continue to use it. The stage has served its purpose.

e.g., $24 + 37 = \square$

Having mastered Stage 1,

i.e.,
$$\begin{array}{l} 20 + 4 \\ 30 + 7 \\ \hline 50 + 11 = 50 + 10 + 1 \\ \quad\;\; ,, \;\; = (50 + 10) + 1 \\ \quad\;\; ,, \;\; = 60 + 1 \\ \quad\;\; ,, \;\; = 61 \end{array}$$

the child will begin Stage 2,

i.e.,
$$\begin{array}{r} 24 \\ 37 \\ \hline 11 \\ 50 \\ \hline 61 \\ \hline \end{array}$$

and discontinue Stage 1.

ADDITION

AIM

To prepare the child to perform the traditional process of addition with understanding of the basic mathematical principles involved. To achieve this aim, the child will proceed through a series of activities and stages of refinement leading finally to the traditional process.

LIMITS FOR SECTION F

1. Two addends only.

2. Each addend to comprise no more than three digits.

3. The final refinement, within the above limits, is reached **where there is no regrouping of the units, the tens, or the hundreds.**

4. Where there is regrouping of units or tens, or both, the steps of refinement stop **one step short of the final refinement.**

PREREQUISITE SKILLS AND UNDERSTANDINGS

1. The nature of the operation of addition.

2. Place value ; extended notation (renaming).

3. Commutative and associative laws (reordering and regrouping).

4. Knowledge of basic addition facts.

5. Experience in solving equations involving addition by means of extended notation, reordering, and regrouping. (See Section F, Part 4, Basic Laws of Mathematics, page 42 ; and Part 5, Equations, page 65.) For example :—

a	$26 + 32 = 20 + 6 + 30 + 2$	(Renaming 26 and 32)
	$26 + 32 = 20 + 30 + 6 + 2$	(Reordering)
	$26 + 32 = (20 + 30) + (6 + 2)$	(Regrouping)
	$26 + 32 = 50 + 8$	(Renaming)
	$26 + 32 = 58$	(Renaming)

b $325 + 146 = 300 + 20 + 5 + 100 + 40 + 6$

„ $= 300 + 100 + 20 + 40 + 5 + 6$

„ $= 300 + 100 + 20 + 40 + 11$

„ $= (300 + 100) + (20 + 40 + 10) + 1$

„ $= 400 + 70 + 1$

„ $= 471$

Work on the above should be commenced at the beginning of Section F and thoroughly established before pupils are started on the systematic refinement leading to the formal process. This in effect means that work on the stages of refinement will not be commenced until late in Section F.

VOCABULARY

The following terms should be understood and used by the child :—

Addition, addend, sum, total.

DEVELOPMENT

1. Vertical Recording : Basic Addition Facts

e.g., i $6 + 3 = 9 \rightarrow \begin{array}{r} 6 \\ +3 \\ \hline 9 \\ \hline \end{array}$ ii $13 + 5 = 18 \rightarrow \begin{array}{r} 13 \\ +\ 5 \\ \hline 18 \\ \hline \end{array}$

iii $7 + 4 = 11 \rightarrow \begin{array}{r} 7 \\ +\ 4 \\ \hline 11 \\ \hline \end{array}$

2. Vertical Recording : Addition of Multiples of 10

(See Section F, Part 1, Pattern and Order in the Number System, pages 12 and 13.)

e.g., i $60 + 30 = 90 \rightarrow \begin{array}{r} 60 \\ +30 \\ \hline 90 \\ \hline \end{array}$ This may be verbalized as :— "6 tens plus 3 tens equals 9 tens, which is 90."

ii $70 + 40 = 110 \rightarrow \begin{array}{r} 70 \\ +40 \\ \hline 110 \\ \hline \end{array}$ "7 tens plus 4 tens equals 11 tens. This equals 1 hundred and 1 ten, which is 110."

3. Vertical Recording : Addition of Hundreds

e.g., $300 + 600 = 900 \rightarrow \begin{array}{r} 300 \\ +600 \\ \hline 900 \\ \hline \end{array}$

4. Graded Examples, with Stages of Refinement

A. Examples Involving No Regrouping

a *TWO ADDENDS, EACH OF TWO DIGITS*

e.g., 35 + 24 = □

i *Vertical Arrangement and Extended Notation*

$$\begin{array}{r} 35 \\ +24 \\ \hline \\ \hline \end{array} \rightarrow \begin{array}{r} 30 + 5 \\ +20 + 4 \\ \hline 50 + 9 \\ \hline \end{array} = 59$$

ii *Vertical Arrangement without Extended Notation*

```
  35
+ 24
 ---
   9
  50
 ---
  59
 ---
```

No regrouping being involved in this type of example, the child may proceed to iii.

iii *The Final Refinement*

```
  35        Verbalization could be :—
+ 24          "4 plus 5 equals 9 ;
 ---           2 tens plus 3 tens equals 5 tens."
  59
 ---
```

b TWO ADDENDS, EACH OF THREE DIGITS

e.g. $361 + 234 = \square$

i *Vertical Arrangement and Extended Notation*

```
  361 →    300 + 60 + 1
+ 234    + 200 + 30 + 4
 ----      ------------
           500 + 90 + 5 = 595
 ----      ------------
```

ii *Vertical Arrangement without Extended Notation*

```
  361
+ 234
 ----
    5
   90
  500
 ----
  595
 ----
```

No regrouping being involved in this type of example, the child may proceed to iii.

iii *The Final Refinement*

```
  361       Verbalization could be :—
+ 234         "4 plus 1 equals 5 ;
 ----          3 tens plus 6 tens equals 9 tens ;
  595          2 hundreds plus 3 hundreds equals
 ----             5 hundreds."
```

c *TWO ADDENDS, DIFFERENT NUMBERS OF DIGITS*

e.g., 823 + 56 = □

i *Vertical Arrangement and Extended Notation*

```
 823 →   800 + 20 + 3
+ 56   +       50 + 6
----     -------------
         800 + 70 + 9 = 879
----     -------------
```

ii *Vertical Arrangement without Extended Notation*

```
 823
+ 56
----
   9
  70
 800
----
 879
----
```

No regrouping being required, the child may proceed to iii.

iii *The Final Refinement*

```
 823      Verbalization could be :—
+ 56        "6 plus 3 equals 9 ;
----        5 tens plus 2 tens equals 7 tens ;
 879        8 hundreds."
----
```

d *AN EXAMPLE INVOLVING ZERO*

e.g., 304 + 123 = □

i *Vertical Arrangement and Extended Notation*

```
 304 →   300 +  0 + 4
+123   + 100 + 20 + 3
----     -------------
         400 + 20 + 7 = 427
----     -------------
```

ii *Vertical Arrangement without Extended Notation*

```
 304
+123
----
   7
  20
 400
----
 427
----
```

No regrouping being required, the child may proceed to iii.

iii *The Final Refinement*

304 + 123 — 427 —	Verbalization could be :— "3 plus 4 equals 7 ; 2 tens plus zero tens equals 2 tens ; 1 hundred plus 3 hundreds equals 4 hundreds."

Note : Examples such as 242 + 302 = □, 275 + 410 = □, 50 + 234 = □ should receive similar treatment.

B. Examples Involving Regrouping

a *TWO ADDENDS, EACH OF TWO DIGITS, WITH RENAMING AND REGROUPING OF THE UNITS*

e.g., 36 + 28 = □

i *Vertical Arrangement and Extended Notation*

```
 36 →   30 + 6
+28   + 20 + 8
 —     ———
        50 + 14 = 50 + 10 + 4
 —                      (Renaming 14)
             ,,  = (50 + 10) + 4
                        (Regrouping)
             ,,  =    60    + 4 (Renaming)
             ,,  =    64        (Renaming)
```

ii *Vertical Arrangement without Extended Notation*

36 + 28 — 14 50 — 64 —	The final refinement of this type of example will be treated in Section G.

b *TWO ADDENDS, EACH OF TWO DIGITS, WITH RENAMING AND REGROUPING OF THE TENS*

e.g., 87 + 41 = □

i *Vertical Arrangement and Extended Notation*

87 → 80 + 7 + 41 + 40 + 1 — ——— 120 + 8 = 128 — ———	Verbalization could be :— " 1 plus 7 equals 8 ; 4 tens plus 8 tens equals 12 tens ; 12 tens are 1 hundred and twenty."

ii *Vertical Arrangement without Extended Notation*

```
   87
+  41
  ---
    8
  120
  ---
  128
  ---
```

The final refinement of this type of example will be treated in Section G.

c *TWO ADDENDS, EACH OF TWO DIGITS, WITH RENAMING AND REGROUPING OF THE TENS AND THE UNITS*

e.g., 84 + 37 = □

i *Vertical Arrangement and Extended Notation*

```
   84 →    80 + 4
+  37   +  30 + 7
  ---     --------
  ---     110 + 11 = 100 + 10 + 10 + 1
                                (Renaming 110
                                 and 11)
               ,,  = 100 + (10 + 10) + 1
                                (Regrouping)
               ,,  = 100 + 20 + 1
               ,,  = 121
```

ii *Vertical Arrangement without Extended Notation*

```
   84
+  37
  ---
   11
  110
  ---
  121
  ---
```

The final refinement of this type of example will be treated in Section G.

d *TWO ADDENDS, EACH OF THREE DIGITS, WITH RENAMING AND REGROUPING OF THE TENS AND THE UNITS*

e.g., 146 + 389 = □

i *Vertical Arrangement and Extended Notation*

```
  146 →    100 +  40 + 6
+ 389    + 300 +  80 + 9
  ---      --------------
  ---      400 + 120 + 15 = 400 + 100 + 20 + 10 + 5
                 ,,       = (400 + 100) + (20 + 10) + 5
                 ,,       =     500     +    30     + 5
                 ,,       =     535
```

ii *Vertical Arrangement without Extended Notation*

```
  146
+ 389
-----
   15
  120
  400
-----
  535
-----
```

The final refinement of this type of example will be treated in Section G.

e *EXAMPLES INVOLVING ZERO*

Types of examples are :

70 + 45 = □
304 + 159 = □
603 + 209 = □
63 + 27 = □
46 + 54 = □

Stages of refinement will be the same as for examples above.

*   *   *   *   *   *   *

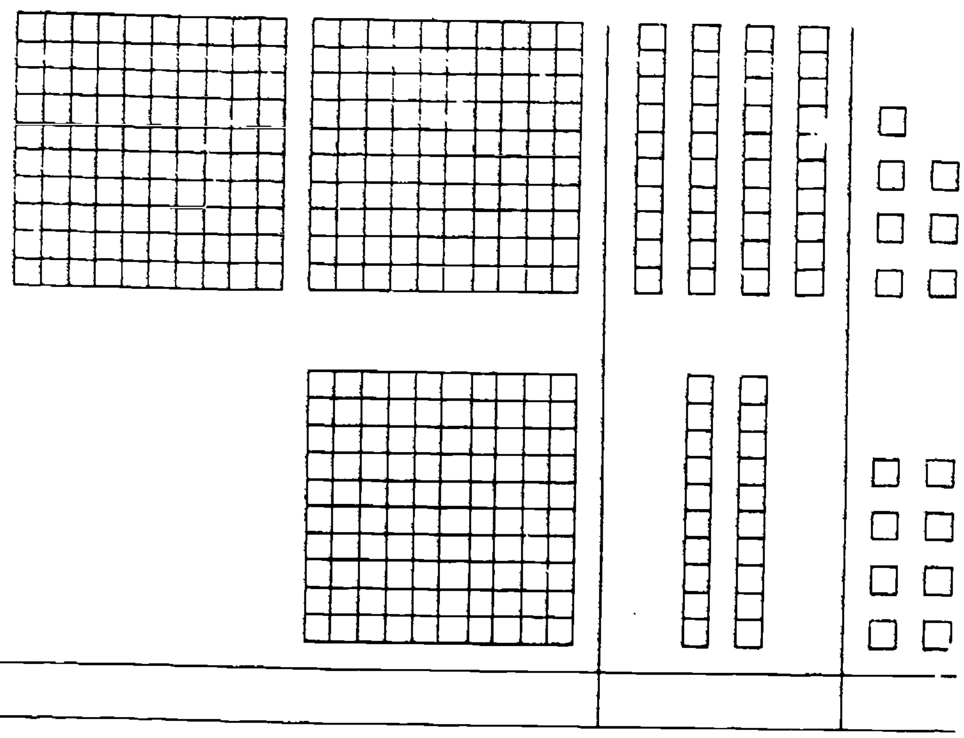
Note: If activities involving the use of structured material are required for this part of the work, then the Multibase Arithmetic Blocks designed by Dienes could be utilized.

An advantage of this material is that when using base ten the child can see instantly a unit, a ten, and a hundred. With these blocks the child can present the addition problem in concrete form, and can then physically combine the sets, making any regrouping that is necessary. For example :—

247 + 128 =

247

128



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MULTIPLICATION

AIM

To prepare the child to perform the traditional process of multiplication with understanding of the basic mathematical principles involved. To achieve this aim, the child will proceed through a series of activities and stages of refinement leading finally to the traditional process.

LIMITS FOR SECTION F

1. Two-digit multiplicand, one-digit multiplier.

2. The final refinement, within the above limits, is reached **where there is no regrouping of the units or the tens, or in the final addition.**

3. Where there is regrouping of the units and the tens but not in the final addition, the steps of refinement stop **one step short of the final refinement.**

4. Examples involving regrouping of the units and the tens and in the final addition will be treated in Section G.

PREREQUISITE SKILLS AND UNDERSTANDINGS

1. Understanding of the multiplication operation as repeated addition.

2. Knowledge of basic multiplication and addition facts.

3. Place value :—
   - i Multiples of 10.
   - ii Extended notation (renaming).

4. Associative law, that is, regrouping.

5. Distributive law of multiplication over addition.

" MULTIPLIED BY "

If in previous sections the child's ability to interpret the sign " $\times$ " has been limited to " times " (2×3 read as " two times three ") he can now be introduced to the alternative interpretation, that is, " multiplied by " (2×3 read as " two multiplied by three ").

A possible method of introduction follows :

1. The child is asked to make 2 sets of 3, using actual objects,

e.g.,

This is read (2 times 3) and recorded (2×3) in the familiar way. (Attention may be drawn to the fact that the 2 refers to the number of sets.)

The child can now be told that there is another way of reading this, starting with the number in each set (3). This other way is " **3 multiplied by** 2 ". The same sign is used, and the recording is therefore " 3×2 ".

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2. Now the child is asked to make 4 sets of 5.

e.g.,

He reads and records :—

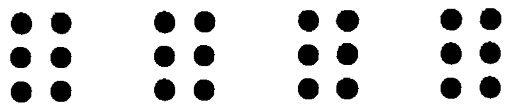
a In the familiar way : 4 times 5 ; 4×5.

b In the new way : 5 **multiplied by** 4 ; 5×4.

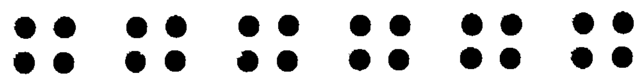
Many other examples are given.

3. An expression such as 4×6 is written on the chalkboard. The child is asked to show this in different ways.

e.g., a



b



The ability to do this shows that the child can interpret 4×6 either as :—

a 4 times 6, i.e., $6 + 6 + 6 + 6$, or

b 4 multiplied by 6, i.e., $4 + 4 + 4 + 4 + 4 + 4$.

VOCABULARY

The following terms should be understood and used by the child :—

Multiplication, factor, product, multiplier, multiplied by multiple.

DEVELOPMENT

I. Vertical Recording of Basic Multiplication Facts

e.g., i $3 \times 2 = 6 \rightarrow \begin{array}{r} 3 \\ \times\ 2 \\ \hline 6 \\ \hline \end{array}$ or $\begin{array}{r} 2 \\ \times\ 3 \\ \hline 6 \\ \hline \end{array}$

ii $4 \times 9 = 36 \rightarrow \begin{array}{r} 4 \\ \times\ 9 \\ \hline 36 \\ \hline \end{array}$ or $\begin{array}{r} 9 \\ \times\ 4 \\ \hline 36 \\ \hline \end{array}$

2. Vertical Recording of Basic Multiplication Facts Applied to Multiples of 10

(See Section F, Part I, Pattern and Order in the Number System, page 13.)

e.g., i $3 \times 20 = 60 \rightarrow \begin{array}{r} 20 \\ \times\ 3 \\ \hline 60 \\ \hline \end{array}$

This may be verbalized as : "3 times 2 tens equals 6 tens."

90

ii $60 \times 3 = 180 \rightarrow$

$$\begin{array}{r} 60 \\ \times \; 3 \\ \hline 180 \\ \hline \end{array}$$

Verbalization could be :—
"6 tens multiplied by 3 equals 18 tens. This can be renamed as 1 hundred and 8 tens, or 180."

3. Graded Examples with Stages of Refinement

Note : See Section F, Part 4, Basic Laws of Mathematics, page 36, for comment on the setting out of equations.

a No Regrouping

i *Extended Notation and the Distributive Law*

$2 \times 31 = 2 \times (30 + 1)$	(Renaming 31)
$2 \times 31 = 2 \times 30 + 2 \times 1$	(Distributive Law)
$2 \times 31 = 60 + 2$	(Renaming)
$2 \times 31 = 62$	(Renaming)

or

$31 \times 2 = (30 + 1) \times 2$	(Renaming 31)
$31 \times 2 = 30 \times 2 + 1 \times 2$	(Distributive Law)
$31 \times 2 = 60 + 2$	(Renaming)
$31 \times 2 = 62$	(Renaming)

ii *Vertical Arrangement and Extended Notation*

$$\begin{array}{r} 31 \\ \times \; 2 \\ \hline \\ \hline \end{array} \rightarrow \begin{array}{r} 30 + 1 \\ \times \quad\; 2 \\ \hline 60 + 2 = 62 \\ \hline \end{array}$$

iii *Vertical Arrangement without Extended Notation*

$$\begin{array}{r} 31 \\ \times \; 2 \\ \hline 2 \\ 60 \\ \hline 62 \end{array}$$

As the final addition in this example requires no regrouping, the child may proceed to iv.

iv *The Final Refinement*

$$\begin{array}{r} 31 \\ \times \; 2 \\ \hline 62 \\ \hline \end{array}$$

Verbalization could be :—
"2 times 1 equals 2 ;
2 times 3 tens equals 6 tens."

Note : It is vital that all digits are correctly positioned according to their place value.

b Regrouping of the Units, but Not in the Final Addition

i *Extended Notation and the Distributive Law*

$$\begin{aligned}
4 \times 14 &= 4 \times (10 + 4) && \text{(Renaming 14)}\\
&= 4 \times 10 + 4 \times 4 && \text{(Distributive Law)}\\
&= 40 + 16 && \text{(Renaming)}\\
&= 40 + 10 + 6 && \text{(Renaming)}\\
&= (40 + 10) + 6 && \text{(Regrouping)}\\
&= 50 + 6 && \text{(Renaming)}\\
&= 56 && \text{(Renaming)}
\end{aligned}$$

or

$$\begin{aligned}
14 \times 4 &= (10 + 4) \times 4 && \text{(Renaming 14)}\\
&= 10 \times 4 + 4 \times 4 && \text{(Distributive Law)}\\
&= 40 + 16 && \text{(Renaming)}\\
&= 40 + 10 + 6 && \text{(Renaming)}\\
&= (40 + 10) + 6 && \text{(Regrouping)}\\
&= 50 + 6 && \text{(Renaming)}\\
&= 56 && \text{(Renaming)}
\end{aligned}$$

ii *Vertical Arrangement and Extended Notation*

$$\begin{array}{rl}
14 \rightarrow & 10 + 4\\
\times\ 4 & \times\ \ \ \ 4\\
\hline
 & 40 + 16 = (40 + 10) + 6\\
 & \quad = 50 + 6\\
 & \quad = 56
\end{array}$$

iii *Vertical Arrangement without Extended Notation*

$$\begin{array}{r}
14\\
\times\ 4\\
\hline
16\\
40\\
\hline
56\\
\hline
\end{array}$$

Verbalization could be :—
"4 times 4 equals 16 ;
4 times 1 ten equals 4 tens ;
zero plus 6 equals 6, 4 tens
plus 1 ten equals 5 tens."

(No regrouping in the final addition.)

Note : The final refinement of the above example will be treated in Section G.

c. Regrouping of Units and Tens, but Not in the Final Addition

i *Extended Notation and the Distributive Law*

$$\begin{aligned}
4 \times 34 &= 4 \times (30 + 4) && \text{(Renaming 34)}\\
&= 4 \times 30 + 4 \times 4 && \text{(Distributive Law)}\\
&= 120 + 16 && \text{(Renaming)}\\
&= 100 + 20 + 10 + 6 && \text{(Renaming)}\\
&= 100 + (20 + 10) + 6 && \text{(Regrouping)}\\
&= 100 + 30 + 6 && \text{(Renaming)}\\
&= 136 && \text{(Renaming)}
\end{aligned}$$

or

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$$
\begin{aligned}
34 \times 4 &= (30 + 4) \times 4 && \text{(Renaming 34)}\\
\text{,,} \quad &= 30 \times 4 \quad + \quad 4 \times 4 && \text{(Distributive Law)}\\
\text{,,} \quad &= 120 \quad + \quad 16 && \text{(Renaming)}\\
\text{,,} \quad &= 100 + 20 \quad + \quad 10 + 6 && \text{(Renaming)}\\
\text{,,} \quad &= 100 + \quad (20 + 10) \quad + 6 && \text{(Regrouping)}\\
\text{,,} \quad &= 100 + \quad 30 \quad + 6 && \text{(Renaming)}\\
\text{,,} \quad &= 136 && \text{(Renaming)}
\end{aligned}
$$

ii *Vertical Arrangement and Extended Notation*

$$
\begin{array}{rl}
34 \rightarrow & 30 + 4 \\
\times\ 4 & \times \quad 4 \\
\hline
 & 120 + 16 = 100 + (20 + 10) + 6 \\
 & \quad \text{,,} \quad = 100 + \quad 30 \quad + 6 \\
 & \quad \text{,,} \quad = 136
\end{array}
$$

iii *Vertical Arrangement without Extended Notation*

$$
\begin{array}{r}
34 \\
\times\ 4 \\
\hline
16 \\
120 \\
\hline
136 \\
\hline
\end{array}
$$

Verbalization could be :—
"4 times 4 equals 16 ;
4 times 3 tens equals 12 tens ;
12 tens are 120 ;
zero plus 6 equals 6 ;
2 tens plus 1 ten equals 3 tens ;
1 hundred."

(No regrouping in the final addition.)

Note : The final refinement of the above example will be treated in Section G.

SUBTRACTION

AIM

To prepare the child to perform the formal process of subtraction with understanding of the basic mathematical principles involved. To achieve this aim the child will proceed through a series of activities and stages of refinement leading finally to the formal process.

LIMITS FOR SECTION F

1. **Two-digit Minuend**

a Where the **units in the subtrahend do not exceed those in the minuend** (e.g., $39 - 8 = \square$; $47 - 23 = \square$) the child can proceed to the final refinement.

b Where the units in the subtrahend **do** exceed those in the minuend (e.g., $43 - 7 = \square$; $52 - 28 = \square$) **the final refinement will be treated in Section G.**

2. **Three-digit Minuend**

The child who is competent with 1.a above can proceed to the final refinement in examples involving three digits, where no digit in the subtrahend exceeds the corresponding digit in the minuend (e.g., $975 - 234 = \square$; $758 - 32 = \square$).

THREE ASPECTS OF SUBTRACTION

(See also Section B, pages 33 to 35.)

The operation of subtraction may be viewed as :—

a The comparison of two numbers, that is, the difference.

b Complementary addition, for example, $9 - 5 = \square$. What must be added to five to equal nine ? $(5 + \square = 9)$. That is, the operation is the inverse of addition.

c "Taking" a quantity or number from an equal or larger quantity or number.

The difference and the complementary addition approaches were used to introduce subtraction in Section B (see Section B, page 33) and continued in Sections C, D, and E. The "taking from" aspect, which the child experiences freely with concrete materials in everyday life, needs little teaching and has been delayed until the other two aspects are firmly established.

It should be noted that the "difference" approach leads to more applications in our number system than does the idea of "taking away". For example, minimum temperature, -27°, maximum, $+3^\circ$. What was the rise in temperature ? This is best explained as : "The difference between -27° and $+3^\circ$ equals 30°." Therefore, it is most important that the difference and the complementary addition aspects of subtraction be continued in order to lay a solid foundation for later work.

However, the "taking from" approach should be considered in Section F for three reasons :—

a It is a valid aspect of subtraction.

b It provides a form of verbalization in the formal process of subtraction.

c. It assists with an understanding of the division process.

PREREQUISITE SKILLS AND UNDERSTANDINGS

1. Understanding of the operation of subtraction from the three aspects outlined above.
2. Place value ; extended notation (renaming).
3. Associative law (regrouping).
4. Knowledge of basic addition and subtraction facts.
5. Properties of difference : If equals are added to (or subtracted from) unequals, the difference remains unchanged. (See Section F, Part 4, Basic Laws of Mathematics, pages 57–58.)

VOCABULARY

The following terms should be understood and used by the child :—

Subtract, subtraction.

DEVELOPMENT

1. Vertical Recording of Basic Subtraction Facts

e.g., i $9 - 5 = 4 \rightarrow \begin{array}{r} 9 \\ -5 \\ \hline 4 \\ \hline \end{array}$ ii $16 - 7 = 9 \rightarrow \begin{array}{r} 16 \\ -7 \\ \hline 9 \\ \hline \end{array}$

iii $14 - 12 = 2 \rightarrow \begin{array}{r} 14 \\ -12 \\ \hline 2 \\ \hline \end{array}$

2. Activities

a "**Taking From**"—First use actual objects, and lead on to problem situations.

e.g., i "8 pencils are in this box. Take out 3 of them. How many are left ?" (5)

Record as $8 - 3 = 5$ or $\begin{array}{r} 8 \\ -3 \\ \hline 5 \\ \hline \end{array}$ or both.

ii "20 bottles of milk were in the crate. I gave out 12 of them. How many were still in the crate?" (8)
Record as $20 - 12 = 8$ or $\begin{array}{r} 20 \\ -12 \\ \hline 8 \\ \hline \end{array}$ or both.

b **Complementary Addition**

i $9 - 4 = \square$
$4 + \square = 9$
$4 + \mathbf{5} = 9$
$9 - 4 = \mathbf{5}$

ii $17 - 9 = \square$
$9 + \square = 17$
$9 + \mathbf{8} = 17$
$17 - 9 = \mathbf{8}$

iii $25 - 19 = \square$
$19 + \square = 25$
$19 + 1 + 5 = 25$
$19 + \mathbf{6} = 25$
$25 - 19 = \mathbf{6}$

iv $34 - 6 = \square$
$6 + \square = 34$
$6 + 4 + 20 + 4 = 34$
$6 + \mathbf{28} = 34$
$34 - 6 = \mathbf{28}$

In examples such as iii and iv children will use various combinations in the addition, according to their own experience. Examples should be graded in difficulty, leading to, for example, $85 - 23 = \square$.

c **Equals Added to or Subtracted from Unequals**—The addition of equals to unequals was first introduced in Section B (page 16). The child had incidental experience following this, and definite treatment was suggested in Section E (page 32).

In preparation for the more formal work of subtraction, the child must now be made fully aware of this principle. A full treatment was given in Section F, Part 4, Basic Laws of Mathematics, pages 57–58, which must be considered in conjunction with this. Exercises will include examples such as :—

i $14 - 8 = \square$
$14 - 8 = 16 - 10$
$14 - 8 = 6$

ii $31 - 17 = \square$
$31 - 17 = 34 - 20$
$31 - 17 = 14$

iii $49 - 23 = \square$
$49 - 23 = 50 - 24$
$49 - 23 = 26$

or

$49 - 23 = \square$
$49 - 23 = 56 - 30$
$49 - 23 = 26$

In developing efficiency with this technique, a child will sometimes add ones until he reaches a familiar or simple situation, for example,

$31 - 17 =$ $(31 - 17,\ 32 - 18,\ 33 - 19,\ 34 - 20 = 14)$.

With practice he becomes more expert, and selects the number to be added at the first attempt. Experience leads him to see that "rounding off" to a 10 is often useful. For most children it is easier to round off the subtrahend than the minuend.

Some experience should be given in subtracting equals from unequals. In some examples this is more useful than adding equals.

e.g., $78 - 31 = \square$
$78 - 31 = 77 - 30$
$78 - 31 = 47$

Note : These activities should precede the work outlined in 3 below, but should not necessarily cease when that work is commenced.

3. Graded Examples, with Stages of Refinement

a Where Digits in the Subtrahend Do Not Exceed the Corresponding Digits in the Minuend

e.g., $57 - 25 = \square$

i *Vertical Arrangement with Extended Notation*

$$\begin{array}{rcl} 57 & \rightarrow & 50 + 7 \\ -25 & & -(20 + 5) \\ \hline & & 30 + 2 = 32 \end{array}$$

ii *Final Refinement*

$$\begin{array}{r} 57 \\ -25 \\ \hline 32 \end{array}$$

Other examples such as $49 - 6 = \square$ and $86 - 60 = \square$ should receive similar treatment.

b Where the Units in the Subtrahend Exceed Those in the Minuend

e.g., $64 - 17 = \square$

i *Extended Notation and Equal Additions*

$$\begin{array}{rl} 64 \rightarrow & 60 + 4 \\ -\ 17 & -\ (10 + 7) \\ \hline \\ \hline \end{array}$$

Verbalization (as outlined below) leads the child to see that he cannot proceed as in **a**. Of the three activities he used to assist in subtraction in equation form, two are not helpful. He is reminded of the third, namely, the addition of equals to unequals.

It is important that the child should be allowed to experiment with the addition of various numbers. For instance, he may decide to add 6.

$$\begin{array}{rl} 60 + 4 + 6 \rightarrow & 60 + 10 \\ -\ (10 + 7 + 6) & -\ (10 + 13) \\ \hline \\ \hline \end{array}$$

He still has difficulty in proceeding.

He may next try reordering and regrouping,

$$\begin{array}{rl} 60 + 4 + 6 \rightarrow & 60 + 10 \\ -\ (10 + 6 + 7) & -\ (16 + \ 7) \\ \hline \\ \hline \end{array}$$

and find that $60 - 16$ presents a difficulty.

Experiment should continue until the child sees that in all situations the addition of 10 units in the minuend and 1 ten in the subtrahend leads to a ready solution.

$$\begin{array}{rlll} 64 \rightarrow & 60 + 4 \rightarrow & 60 + 10 + 4 \rightarrow & 60 + 14 \\ -\ 17 & -\ (10 + 7) & -\ (10 + 10 + 7) & -\ (20 + \ 7) \\ \hline & & & 40 + \ 7 = 47 \\ \hline \end{array}$$

ii *Final Refinement*

The final refinement will be treated in Section G.

VERBALIZATION

1. Verbalization in the subtraction process can follow various forms. It is important to remember these points :—

a Whatever form is used must have meaning for the child, and not be a formal " patter ".

b No one verbalization should be a standard requirement. A child who sees the situation in some unorthodox but valid way, and explains it accordingly, need not be required to conform to a pattern.

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c The **complete** subtrahend is to be subtracted from the **complete** minuend. Any partial subtraction (units from units, or tens from tens) is attempted merely in order to solve the problem more easily.

2. The term "minus" means to the child :—

a the difference between . . . . . .

b what must be added . . . . . .

c taking . . . from . . .

For the purpose of carrying out the formal process of subtraction, "the difference between" presents difficulties of verbalization and is not useful.

3. Types of verbalization suggested for use in Section F are :—

a "from"

b "What must I add ?"

c "minus", (where "from" or "What must I add ?" is understood).

4. Examples of verbalization :

```
 72    a  "9 from 2", or
−39       "2 minus 9", or
 —        "What must I add to 9 to equal 2 ?"
 —          None of these proving possible, the child adds ten
          to the 2 units and to the 3 tens (see b i, page 97).

       b  "9 from 12", or
          "12 minus 9", or
          "What must I add to 9 to equal 12 ?"

       c  "4 tens from 7 tens", or
          "7 tens minus 4 tens", or
          "What must I add to 4 tens to equal 7 tens ?"
```

Note : The child who sees "2 minus 9" as "negative 7" could use this to help him find an answer to the problem. When he realizes that this leads to an unwieldy method of solution, he can be led to use one of the customary forms.

DIVISION

AIM

To present activities that will prepare the child for an understanding of the formal process of division.

FOUR ASPECTS OF DIVISION

1. **Quotition** : How many sets of 3 may be made from 12 ? (4) ($12 \div 3 = 4$).

2. **Partition** : 12 is divided into 3 equal sets. How many in each set ? (4) ($\frac{1}{3}$ of $12 = 4$).

3. **Repeated subtraction** : How many subtractions of 3 can I make starting with 12 ? (4) ($12 - 3 - 3 - 3 - 3 = 0$, i.e., $12 \div 3 = 4$.)

4. **Inverse of multiplication** : If 4 is multiplied by 3, the result is 12. If 12 is divided by 3, the result is 4.

(See Section F, Part 4, Basic Laws of Mathematics, pages 34–35 and page 53.)

PREVIOUS EXPERIENCE

Quotition and partition division were introduced in Section B. (See Stage 12.)

Repeated subtraction was probably not introduced until Section C (page 25). It was related to division in Section E (page 30), though some children may have detected the relationship earlier.

Division as the inverse of multiplication was specifically treated in a previous part of this section. Multiplication and division were seen as related operations very early, possibly in Sections B and C, and certainly in Sections D and E.

PREREQUISITE SKILLS AND UNDERSTANDINGS

1. Understanding of the operation of division.
2. Place value ; extended notation.
3. Some experience with the distributive law.
4. Knowledge of basic number facts, especially subtraction, and ability to name multiples of a given number.
5. Ability to rearrange equations to show interrelation of operations.

VOCABULARY

The child should understand and use the following terms :—

Division, divide, divisor, divided by.

DEVELOPMENT

In this course, "short" division will be deferred until Section H. The distributive law provides background experience for this method, and to this end exercises as shown in Section F, Part 4, Basic Laws of Mathematics, pages 52–53, should be continued. There is no need for complicated examples. Examples should be limited to divisors of 12 or less, and dividends that the child can rename in multiples of the divisor, for example, $36 \div 3 = \square$; $48 \div 3 = \square$.

Because it follows more logically from the child's experience, "long" division through continued estimation will be used to introduce the formal process. The development of the necessary understanding and skills is dependent on a wide range of experience of various activities (including the final refinement of the subtraction process). It is therefore not proposed to formalize the division process until Section G.

1. Relation of Division to Repeated Subtraction

a **Through Rearranging of Equations**—Exercises as outlined in Section E (page 30) should be continued and extended.

e.g., $27 - 9 - 9 - 9 = 0$	$28 \div 7 = 4$
$27 - 3 \times 9 = 0$	$28 - 4 \times 7 = 0$
$27 \div 9 = 3$	$28 - 7 - 7 - 7 - 7 = 0$

b **Vertical Arrangement of Repeated Subtraction**—Instead of always using the equation, the child can set out his subtraction vertically.

e.g., $18 - 6 - 6 - 6 = 0$

$$\begin{array}{rll} 18 & & \\ -\ 6 & 1 \times 6 & \\ \hline 12 & & \\ -\ 6 & 1 \times 6 & \\ \hline 6 & & \\ -\ 6 & 1 \times 6 & \\ \hline 0 & 3 \times 6 & 18 - 3 \times 6 = 0 \end{array}$$

He can see clearly that he has made three subtractions of six. (It will be noted that a certain facility with subtraction is necessary, and examples should be selected accordingly. Counting backwards on a number chart will be found helpful.)

c **Through the Use of Concrete Materials**—Children should perform these and similar activities themselves, and see that in order to perform the division, they are making a number of subtractions.

e.g., i "Put 35 match-sticks and a number of empty boxes on the table. Take 5 match-sticks at a time from the heap, and put each set of five in a box." — — — — — —

"How many boxes did you need ?" "How many fives did you take from the heap ?"

ii "From a bag of 54 plastic sticks, make bundles of 9 sticks." — — — — — —

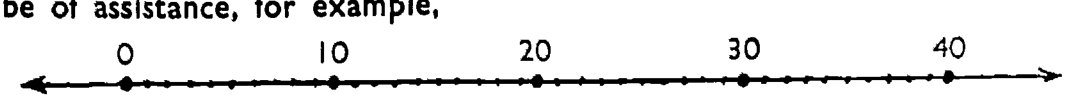
"How many bundles did you make ?" or "How many boys can each take 9 sticks from the bag ?"

iii "Obtain a piece of string 32 inches long. Measure and cut it into 8-inch lengths." — — — — — —

"How many short lengths can you make ?"

2. Estimation

It will be necessary in Section G for children to estimate partial quotients. To prepare for this, exercises using a number line would be of assistance, for example,



Use the number line to find :—

a Which multiple of 10 is nearest to 19, 13, 28, 35.

b Which multiple of 10 is nearest **below** 27, 34, 19, 21, 35.

Another exercise useful in estimation is division into multiples of 10 and of 100. This work was outlined in Part I of this section, Pattern and Order in the Number System (see page 13).

e.g.,

$$\begin{array}{rcl} 8 \div 2 = 4 & \qquad & 12 \div 4 = 3 \\ 80 \div 2 = 40 & & 120 \div 4 = 30 \\ 800 \div 2 = 400 & & 1200 \div 4 = 300 \end{array}$$

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3. Cases Where the Dividend Is Not a Multiple of the Divisor

When the division process is formalized in Section G, it will first be developed with "exact" division. However, it is desirable that children should have some experience of situations where the dividend is not a multiple of the divisor.

a **Practical Examples as Suggested in 1.c Above May Be Adapted**—For example :—

i "This piece of string is 22 inches long. Measure and cut off 10-inch lengths." - - - - - -

"Did you use all the string ?" "How many 10-inch pieces did you make ?" "How much string is left ?"

ii "Here are 40 stamps. Put them in sets of 12." - - - - -

"How many sets did you make ?" "Did you use all the stamps ?" "How many are left ?"

b **Rearranging of Equations, and Vertical Recording**

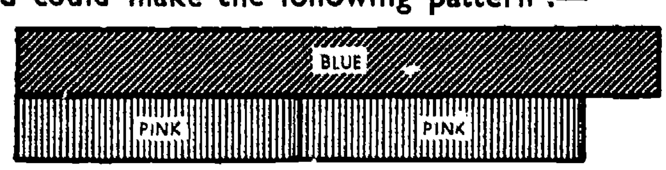
Exercises as in **1.a** and **2** above can be extended to include examples such as :—

$15 - 6 - 6 = 3$
$15 - 2 \times 6 = 3$
(For division, see c below)

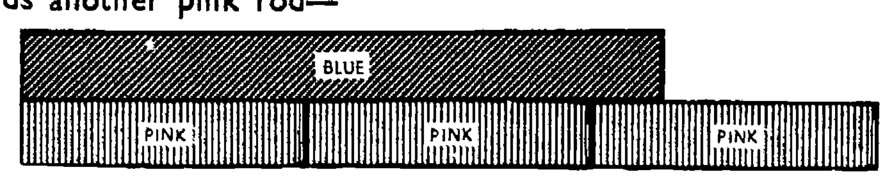
$$\begin{array}{r} 15 \\ -\ 6 \\ \hline 9 \\ -\ 6 \\ \hline 3 \end{array}$$

c **Quotients Written as Fractions**

A child asked to show how many pink rods equal a blue rod could make the following pattern :—

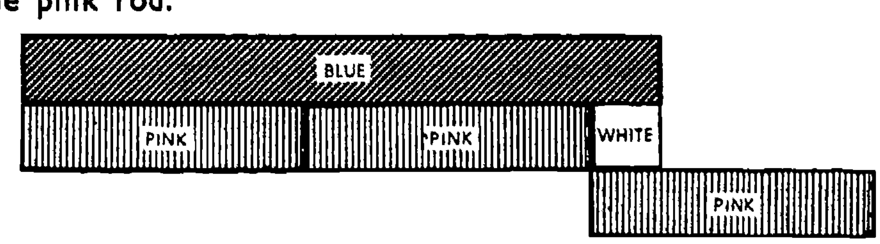


He has two pink rods, but they do not equal the blue. He adds another pink rod—



and sees that three pink rods do not equal the blue, but two pink rods and part of another pink rod are equal to the blue.

This part can be represented by a white rod, which is $\frac{1}{4}$ of the pink rod.



Therefore $2\frac{1}{4}$ pink rods equal the blue.
Numerically, this can be recorded as $9 \div 4 = 2\frac{1}{4}$.

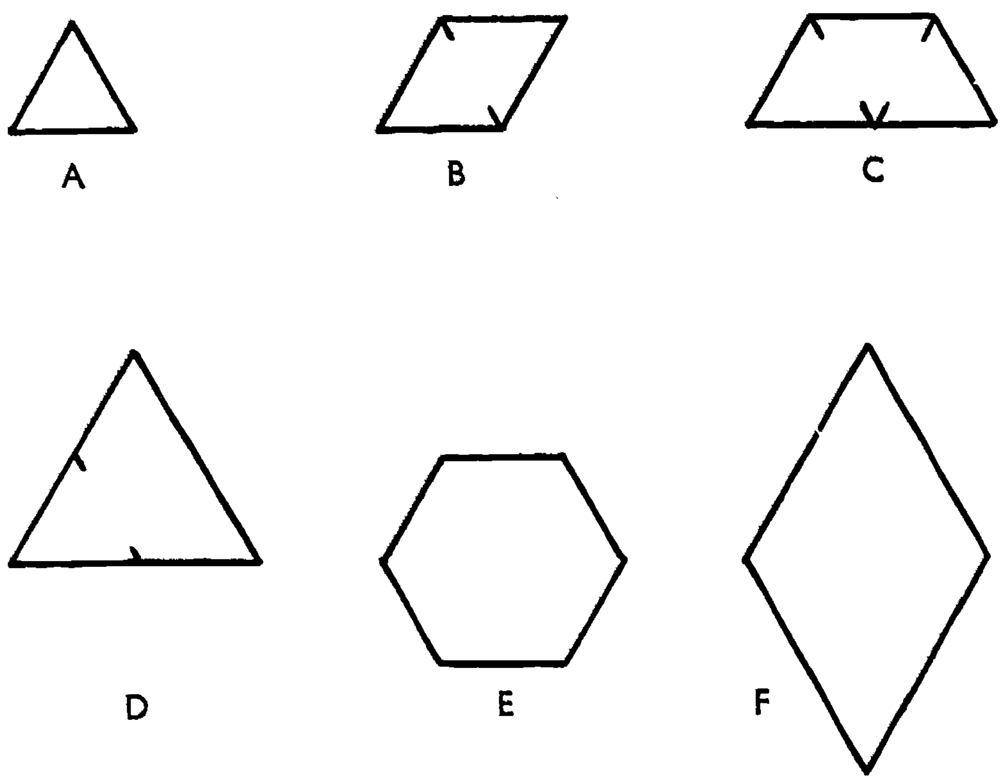
PART 8. VALUE RELATIONS

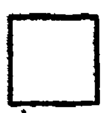
Since Section C, the child has been doing exercises in value relations, using the rods. (See Section C, page 11 ; Section E, pages 25 to 27.) In addition, he may have been finding examples of value relations in other situations such as the windows in a class-room. If one pane of glass were given the value 3, what would be the value of the whole window ?

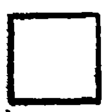
In Section F, the work in value relations is continued and extended. The child should be encouraged to see relationships in a wide variety of situations as well as with the rods. Exercises using geometric shapes, as illustrated in Fractions (Section F, pages 71 and 74), provide such an opportunity.

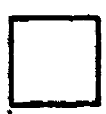
Other exercises could include the following :—

1. If A has the value 3, what is the value of B, C, D, E, F ?
   If C has the value 15, what is the value of A, B, D, E, F ?
   If D has the value 1, what is the value of A, B, C, E, F ?



2. If the basic shape  has the value 1, what is the value of each of the letters below ?

   If the basic shape  has the value 6, what is the value of each of the letters below ?

   If the basic shape  has the value $\frac{1}{6}$, what is the value of each of the letters below ?

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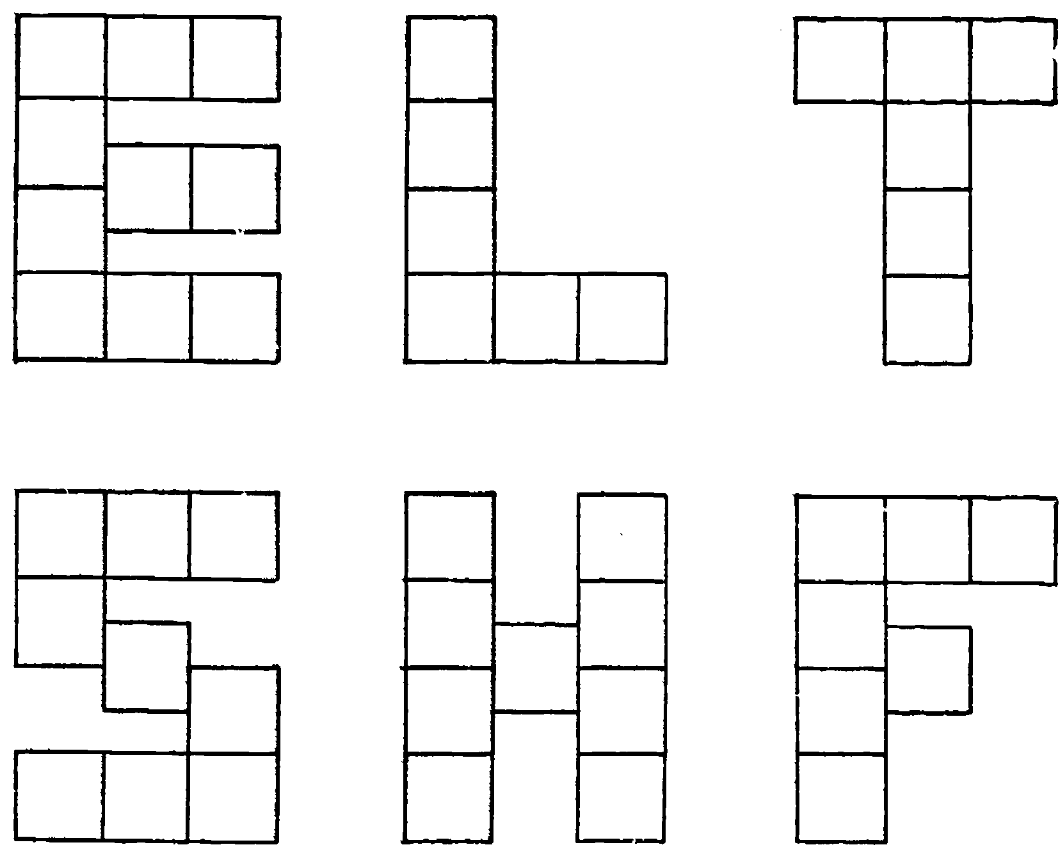

THE USE OF FRAMES

WHY FRAMES ?

In the new mathematics course increased emphasis is placed on work with equations, and some of these will be open sentences. An open sentence has one or more gaps or openings in which numerals, relation signs, or operation signs are to be placed. There is a need for some clear way of indicating where these openings or gaps occur in mathematical sentences. One way of writing an open sentence is $5 + \quad = 7$. However, it is preferable to make clear the fact that a certain part of the sentence is yet to be named, and where this part is to occur. Thus $5 + \square = 7$ is preferred.

The missing part of a sentence will not always be a numeral. For example :—

a $5 \square 2 = 7$ (Here, $\square$ is to be replaced by one of the operation signs $\times \div + -$.)

b $2 + 5 \square 8$ (Here, $\square$ is to be replaced by one of the relation signs $= \neq < >$.)

CONVENTIONS WITH FRAMES

There are different conventions in the use of frames. In the writing of this section of the guide, the convention followed is that the use of a frame merely indicates that an element of the sentence has yet to be named. This is not to be confused with the use of symbols such as *x* or *n*, which always represent the same number if used more than once in an equation.

It should be understood, however, that other conventions regarding the use of frames are likely to be met in various textbooks and tests.

Another widely used and accepted convention is the consistent use of frames of the same shape where the same numeral is intended in any one example. For example :—

a $20 - \square \times \square = 16$ ($\square$ is a place holder for 2)

b $7 \times 8 = 7 \times (\triangle + \triangle)$ ($\triangle$ is a place holder for 4)

$$= (7 \times \triangle) + (7 \times \triangle)$$
$$= \square + \square$$
$$= \nabla$$

However, frames of different shapes need not necessarily imply different numerals. For example :—

$$30 - \triangle \times \Diamond = 21$$

In this equation, some of the acceptable replacements for $\triangle \times \Diamond$ are 9×1, 3×3, $\frac{1}{2} \times 18$.

SHAPES OF FRAMES

Some commonly used shapes are—

$$\square \quad \triangle \quad \nabla \quad \odot \quad \bigcirc$$

The teacher should feel free to use any other shape that suits his purpose. The child may see symbols such as * or ?. The teacher may wish to use such symbols instead of frames.

TESTING

WHY IS TESTING NECESSARY?

The primary purpose of testing is to gain information on which action may be taken if necessary. The teacher needs to be informed continuously of the child's achievements in various areas of the work, the nature of any errors made, and to what extent the goals outlined in the introduction to the Course of Study are being attained. With this information, he can decide whether or not the child is ready to proceed with new work and what steps he will take to overcome difficulties.

WHAT IS TO BE TESTED?

1. **Understanding** of basic ideas concerned with such topics as operations, basic laws, place value, fractions, formal processes, and measurement.

2. **Computational skills** involving accuracy and speed associated with, for example, recall of basic number combinations, formal processes, substitution in equations, as well as exercises including fractions and compound quantities.

3. **Application** of the child's understanding and computational skills to quantitative situations. Consideration should be given to creativeness, flexibility, pattern work, and the ability of the child not only to think broadly about a new situation, but also to be able to deal with it confidently and effectively.

HOW IS TESTING CARRIED OUT?

Although formal tests may be given from time to time, most of the testing program will be carried out during the normal teaching period. Techniques such as oral tests, written tests, and observation of the child at work may be profitably employed. To ensure that reliable judgments will be made, it will be found necessary at times to supplement written tests with oral questioning. It must be realized too that testing carried out during normal teaching practice could reveal adequate information so that no formal test in certain areas of the work will be required.

The following are examples of test items that could be suitable for use. It should be remembered that, once a child has had experience of a particular form of test item, a later test using the same item may well be testing a different aspect of the topic. For example, early in Section F, a child would need an understanding of extended notation and the distributive law to answer the following question with understanding: $23 \times 3 = ?$ Later, however, the same question is likely to become a test simply of computational skill.

Topic	Aspect to be Tested	Test Item
Identity Law of 1	Understanding	a $7 \times \frac{\square}{\square} = 7$ (Includes numerals for 1.)
		b $5 \div \Diamond = \Diamond \div 5$. If $\Diamond$ represents the same number each time, what is this number ?
Interrelationship of Operations	Understanding	a Using only the numbers 6, 4, and 24, write as many equations as you can.
		b $6 + 6 + 6 = 18$. Rewrite this equation using both 18 and 6, and replacing addition by division.
		c $92 - 87 = ?$ Rewrite this sentence using addition.
		d $15 - \Diamond - \Diamond - \Diamond = 0$. If $\Diamond$ represents the same number each time, what is this number ?
Equivalence of Fractions	Understanding	a  Use this number line to help you write an equation about quarters and eighths.
		b Show on a number line that $\frac{2}{3} = \frac{6}{9}$.
		c Draw a circle around the fraction that does not belong to the set : $\frac{10}{15}$, $\frac{6}{9}$, $\frac{8}{12}$, $\frac{14}{18}$, $\frac{12}{18}$.
Addition	Understanding	a When I see the numbers 12 and 4 and I think of 16, what operation have I used ?
		b $32 + 45 = ?$ Solve in as many ways as you can.
		c Continue a pattern using the numbers as shown : 123, 127, 131, $\square$, $\square$
	Computation	$143 + 253 = \square$.
	Application	Tom has as many marbles as Bill and John have together. If Bill has 36 and John has 23, how many marbles has Tom ?
Place Value	Understanding	4,382
		a What number does the 8 represent ?
		b How many tens does the 3 represent ?

When open-ended situations are used, information beyond that sought may be revealed. For example, a child asked to write equations using the numbers 9, 3, and 27 may write $9 \times 3 = 27$. He has satisfied the requirement. Another child may include $27 \div \frac{3}{3}$ of $3 = 9$. He has satisfied the requirement and given the additional information that he understands special properties of the number 1, and furthermore that there are many different ways of naming this number. This does not mean that the first child does not have this understanding. It is simply an observation that contributes to the teacher's overall knowledge of the child.

RECORDING EVALUATIONS

When adequate information on any topic has been gained, the teacher is able to make an evaluation and decide the course of action to be taken. This evaluation, when recorded, can serve as a guide not only for planning future work and assessing individual progress, but also for compiling reports when requested. Records can be kept on a progress chart similar to the charts used for earlier sections of the course. Alternatively, information can be entered in a special record book. Individual progress made can be shown by series of ticks (√) or by other symbols.

If a record showing degrees of achievement at various stages in the development of each topic is required, a scheme such as the following may be used.

Top Group:
- Very Good
- Good
- Satisfactory
- Below Standard

Middle Group:
- Very Good
- Good
- Satisfactory
- Below Standard

Lowest Group:
- Very Good
- Good
- Satisfactory
- Below Standard

A general rating of " Below Standard " in the top and middle groups would indicate that the child has been wrongly placed, whereas a " Very Good " rating in the middle and lowest groups **may** mean that the child is ready for promotion to the next group.

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GLOSSARY

Abacus—A calculating device consisting of a frame with movable beads on parallel rods.

Addend—The number to be added. In $3 + 4 = 7$, the numbers 3 and 4 are addends.

See **Addition, Sum.**

Addition—A binary operation on two numbers, called addends, to obtain a third number, called the sum.

See **Operation, Sum.**

Algorithm (Algorism)—A formal procedure for calculation, such as in long division.

See **Process.**

Associative Law of Addition (Property of Grouping)—When adding three or more numbers, the manner of grouping does not change the sum of the numbers.

For example :—

$$(2 + 3) + 4 = 2 + (3 + 4)$$
$$(a + b) + c = a + (b + c)$$

Associative Law of Multiplication (Property of Grouping)—When multiplying three or more numbers the manner of grouping does not change the product of the numbers.

For example :—

$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$
$$(a \times b) \times c = a \times (b \times c)$$

Automatic Response—Immediate and accurate recall of basic number facts.

Base—The number used in the fundamental grouping procedure. The base of our decimal system of numeration is ten.

Thus $1{,}111 = 1 \times 1{,}000 + 1 \times 100 + 1 \times 10 + 1$
$= 1 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 1$
and $3{,}572 = 3 \times 10^3 + 5 \times 10^2 + 7 \times 10^1 + 2$

Basic Laws (or Properties)—General principles or rules to which all cases must conform.

For examples see **Associative Law, Commutative Law, Distributive Law, Identity Law.**

Binary Operation—An operation applied to a pair of numbers. All the basic operations are binary because they involve working with two numbers at a time.

See **Operation.**

Braces—1. See **Parenthesis.**

2. It also has a special meaning with reference to sets. (Not important at this stage.)

Brackets—1. Often used as a general term to include parenthesis ( ), braces { }, and brackets [ ].

2. Can also be used as referring only to [ ].

See **Parenthesis.**

Cardinal Number (Syn. Whole Number)—A number that tells **how many** things there are in a set or group. The cardinal numbers are $\{0, 1, 2, 3, 4, 5, 6 \ldots\ldots\}$.

Commutative Law of Addition—The order of two numbers in an addition does not affect their sum. For all numbers a and b, $a + b = b + a$.

For example :—

$$7 + 2 = 2 + 7.$$

Commutative Law of Multiplication—The order of two numbers in a multiplication does not affect their product. For all numbers a and b, $a \times b = b \times a$.

For example :—

$$4 \times 8 = 8 \times 4.$$

Composite Number—A number that has whole number factors besides itself and one.

For example, 4, 6, and 10 are composite numbers as distinct from prime numbers like 3, 5, and 7.

Correspondence (One-to-one)—See **One-to-one Correspondence.**

Counting Number (Syn. Natural Number)—Any member of the set $\{1, 2, 3, 4, 5 \ldots\}$.

Note that zero is not a counting number.

Decimal Numeration—A place-value system with ten as the base for grouping.

Denominator—In a fraction written in the form $\frac{a}{b}$, the denominator is the number below the bar, in this case, b.

See **Fraction.**

Difference—The result of subtracting one quantity from another.

See **Subtraction.**

Digit—One of the basic symbols used to write numerals in a numeration system. The digits of our customary system, called the Hindu-Arabic system of notation, are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

See **Numeral.**

Distributive Law of Division over Addition—Division can be "distributed" over addition. For all numbers a, b, and c,

$$(a + b) \div c = (a \div c) \quad + \quad (b \div c)$$

For example :—

$$(12 + 4) \div 4 = 12 \div 4 \quad + \quad 4 \div 4.$$

Distributive Law of Division over Subtraction—Division can be "distributed" over subtraction. For all numbers a, b, and c,

$$(a - b) \div c = (a \div c) \quad - \quad (b \div c)$$

For example :—

$$(12 - 4) \div 4 = 12 \div 4 \quad - \quad 4 \div 4.$$

Distributive Law of Multiplication over Addition—Multiplication can be "distributed" over addition. For all numbers a, b, and c,

$$a \times (b + c) = (a \times b) \quad + \quad (a \times c)$$

For example :—

$$3 \times (9 + 2) = 3 \times 9 \quad + \quad 3 \times 2.$$

Distributive Law of Multiplication over Subtraction—Multiplication can be "distributed" over subtraction. For all numbers a, b, and c,

$$a \times (b - c) = (a \times b) \quad - \quad (a \times c)$$

For example :—

$$3 \times (9 - 2) = 3 \times 9 \quad - \quad 3 \times 2.$$

Dividend—In the operation of division, the dividend is the number to be divided. It corresponds to the product in multiplication.

Division—A binary operation performed on two numbers to give a third number called the quotient. Division is the inverse operation of multiplication ; that is, division "undoes" what multiplication "does". If a number is first multiplied and then divided by the same number, the original number is left unchanged.

For example :—

$$7 \times 4 = 28 \qquad 28 \div 4 = 7$$

or

$$7 \times 4 \div 4 = 7.$$

Divisor—A number that divides another number. In the example $48 \div 6 = 8$, 6 is the divisor, 48 is the dividend, and 8 is the quotient.

Equal Fractions—Fractions that name the same number. The fraction $\frac{3}{4}$ names the same number as $\frac{9}{12}$, that is, $\frac{3}{4} = \frac{9}{12}$. (Note that $3 \times 12 = 4 \times 9$.)

Equality—The relation expressing sameness, and symbolized by the sign $=$. For example, in the statement $3 + 2 = 5$, $3 + 2$ and 5 name the same number.

In a less formal way, we may say that two objects are equal in a certain respect. For example, two children are equal in height. This does not mean that the two children are the same, but merely that the height of the first child is the same as the height of the second child.

Equation—A mathematical sentence or statement to the effect that two expressions name the same thing. The sentence or equation $8 - 3 = 4 + 1$ is a true one because both $8 - 3$ and $4 + 1$ are different names for the same number, whose simplest name is 5.

Equivalent Fractions—See **Equal Fractions**.

Estimation—Judgment based upon very general considerations, as contrasted to finding the quantity by exact mathematical procedure. One might estimate the product of 49 and 31 as 50 × 30, giving 1,500, but one would compute it systematically to obtain the exact value of 1,519.

Even Number—An integer that is divisible by 2. The sum of, the product of, or the difference between, even numbers is an even number.

Extended Notation—A particular way of rewriting a number as a sum of simpler numbers, each with only one significant digit.

379 can be expressed in extended notation as $300 + 70 + 9$, or $(3 \times 100) + (7 \times 10) + (9 \times 1)$.

Factor—When multiplying two or more numbers, each number is called a factor of the product. In the equation $5 \times 6 = 30$, 5 and 6 are factors of 30. A prime factor is also a prime number. The prime factors of 12 are 2 and 3.

See **Prime Number.**

Fraction—A ratio of whole numbers. A number that can be expressed in the form $\frac{a}{b}$. Here, a is the numerator and b the denominator. (If the fraction is to have meaning as a number then b cannot be zero.) A fraction can be written as an ordered pair, for example, (1, 2) instead of $\frac{1}{2}$ or 1/2.

Fractional Number—A number that is the result of dividing a whole number by a counting number.

Examples :—

$$2 \div 3 = \frac{2}{3}$$
$$4 \div 3 = \frac{4}{3}$$

Note : Every whole number is a fractional number, since it can be considered as the result of dividing a whole number by a counting number.

Examples :—

$$0 \div 6 = 0$$
$$2 \div 1 = 2$$
$$1 \div 1 = 1$$
$$6 \div 2 = 3$$

Group—See **Set.**

Grouping—See **Quotition.**

Identity Law for Addition—There is an identity element for addition, namely, zero. Zero added to any number does not change that number, for example, $6 + 0 = 6$.

Identity Law for Multiplication—There is an identity element for multiplication, namely, one. Any number multiplied by 1 remains unchanged, for example, $6 \times 1 = 6$.

Improper Fraction—A fraction in which the numerator is greater than (or equal to) the denominator. $\frac{4}{3}$ is an improper fraction.

Inequality—A statement that one number is less than or greater than another number. The statement that five is less than seven is symbolized $5 < 7$; the statement that seven is greater than five is symbolized $7 > 5$. The statement that something is not equal to another thing is less informative. For example, $5 \neq 7$, point A $\neq$ point B. Inequalities as statements may be either true or false. For example, $5 < 5$ is a false inequality.

Integer—Any element of the set $\{\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \ldots\}$

Inverse (of Operation)—The inverse of an operation is another operation which "undoes" the first. Thus, subtraction and addition are inverse operations, as are multiplication and division.

An inverse of $3 + 2 = 5$ is $5 - 3 = 2$
An inverse of $4 \times 5 = 20$ is $20 \div 5 = 4$

Mathematical Sentence—A mathematical sentence may be either an equation or an inequality.

See **Equation, Inequality.**

Minuend—In a — b, a is called the minuend. For example, in 35 — 18, 35 is the minuend.

Mixed Number—An integer plus a proper fraction. For example, $2 + \frac{3}{8}$ (abbreviated $2\frac{3}{8}$) is a mixed number. An improper fraction can always be renamed as a mixed number.

See **Improper Fraction, Proper Fraction.**

Multiple—When two or more numbers are multiplied, their product is a multiple of each.

Multiplicand—When two numbers are multiplied, one can be called the multiplicand and the other the multiplier, for example, $12 = 3 \times 4$. We may call 3 the multiplicand and 4 the multiplier.

Multiplication—A binary operation on two numbers, called factors, to obtain a third number, called the product.

Multiplier—See **Multiplicand.**

Natural Number—See **Counting Number.**

Nought (Naught)—See **Zero.**

Number—See **Cardinal Number, Composite Number, Counting Number, Even Number, Fractional Number, Integer, Mixed Number, Numeral, Odd Number, Ordinal Number, Prime Number, Whole Number, Zero.**

Number Line—A visual model of numbers by means of a line. Each point on the line is associated with exactly one number. The number line is used as a visible means for showing operations on numbers and relations between them.

Number Period—A number period is the means of grouping places in a number to facilitate ease in reading the number represented. Beginning at the right of a whole number, the first three places to the left are designated as units, the next three places are thousands, the next three places are millions, and so on. An illustration of number periods is 2,345,000.

Number Sentence—A statement of mathematical relationship involving numbers.

Examples:—

$$7 + 3 > 4 ; \tfrac{3}{8} = \tfrac{1}{3}.$$

Numeral—A name or symbol for a number. For example, 36 is a numeral. Other numerals for 36 can be 12×3 ; $72 \div 2$; $\frac{1}{4}$ of 144 ; XXXVI.

Numeration System—A system for naming numbers in figures.

Numerator—In a fraction written in the form $\frac{a}{b}$, the numerator is the number named above the bar, in this case, a.

See **Fraction.**

Odd Number—An integer that is not divisible by 2 ; not an even number ; the set of integers $\{\ldots -7, -5, -3, -1, +1, +3 \ldots\}$

One-to-one Correspondence—Two sets, or collections of things, A and B are said to be in one-to-one correspondence if—

i each member of A is associated with exactly one member of B ; and

ii each member of B is matched exactly with one member of A.

The members of A and B are paired off exactly. For example, when a hall is filled, a one-to-one correspondence exists between the chairs and the people occupying them. The two sets have the same cardinal number.

Open Sentence—A mathematical sentence containing one or more pronumerals.

Examples :—

$$9 - \square = 7$$
$$5 + x = 9$$

Operation—In elementary mathematics, operation refers to one of the following :—

Addition, Subtraction, Multiplication, Division, Squaring . . .

Ordinal Number—A counting number related to a place in an ordered sequence. The following are the early numbers in this sequence :—

First, second, third, ...

Parenthesis—i The marks ( ) used to indicate that an enclosed expression is to be regarded as an individual part of a larger expression, for example, $12 - (2 + 3) = 7$.

Varieties of parentheses are braces $\{ \}$ and brackets [ ] used to avoid ambiguity in larger expressions, for example, $\{16 - [12 - (5 - 3)] + 4\} \div 2 = 5$.

ii It also has a special meaning for indicating ordered pairs. (Not important at this stage.)

Partition (Syn. Sharing)—Finding the result of a division into parts where the number of parts is known and the number in each part is to be found. For example, divide 10 into 5 equal sets. Each is a set of two.

Place Holder—In a mathematical expression, a symbol that holds a place for a numeral, an operation sign (+, −, ×, ÷), or a relation symbol (>, <, =). Place holders used include □, △, ⬡.

Place Value—The property of a numeration system that gives a digit a different value depending upon the position that the digit holds in the numeral. For example, the digit 5 in the decimal numeral 65 represents five ones, but in 157 the digit 5 represents five tens or fifty.

Prime Number—An integer that has no factors except unity and itself, as 2, 3, 5, 7, ... ; 1 is usually excluded. A whole number greater than 1 that cannot be expressed as the product of two smaller whole numbers (each greater than 1). A prime number has only two factors, itself and 1.

Process (Syn. Algorithm)—Frequently used in reference to any one of the four fundamental operations and combinations of such operations. Actually "algorithm" is the more meaningful and correct term.

See **Algorithm.**

Product—The result of the operation of multiplication on two or more numbers (called factors).

Pronumeral—A place holder for a numeral. The place holder is usually shown as either a letter of the alphabet, e.g., x, y, or as a frame, e.g., □.

Proper Fraction—A fraction in which the numerator is smaller than the denominator. $\frac{2}{3}$ is a proper fraction.

Quotient—The result of the operation of division.

Quotition (Syn. Grouping)—This is division into parts, where the number of each part is known and the number of parts is to be found, for example, division of 20 by 4. Divide 20 into equal sets, each containing four. There are 5 such sets.

Ratio—A comparison between two numbers. The ratio of two to five is written 2 : 5 or $\frac{2}{5}$.

See **Fraction.**

Rearranging—Changing the order of numerals. $4 + 5 + 3 = 12$ may be rearranged as $3 + 4 + 5 = 12$ or $12 - (4 + 5) = 3$.

Reciprocal of Number (Syn. Inverse of Number)—The number 1 divided by the given number. The reciprocal of 2 is $\frac{1}{2}$. The product of any number and its reciprocal is 1. There is no reciprocal for zero.

The reciprocal of any fractional number is the number that is obtained by interchanging the numerator and the denominator. The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

Relation—A property that holds between two mathematical objects in a specified order. The simplest examples are the numerical relations of "is equal to", "is less than", and "is greater than" ($=$, $<$, $>$).

Renaming—See **Substitution.**

Sequence—A collection of numbers given in a definite order, usually according to some rule or pattern. There may or may not be a last number in a sequence.

Examples :—

3, 5, 7, 9, 11, 13 . . . .
5, 10, 15, 20, 25, 30 . . . .
18, 15, 12, 9, 6, 3, 0.

Set—A collection of things that can be distinguished from one another.

The individual objects of a set are the elements, or members, of the set ; the element is said to belong to the set.

Examples :—

1. The children in a class form a set. Any one child in the class is a member of that set.
2. Any point on a line, considered as a set of points, belongs to the line.
3. The number 5 belongs to the set of prime numbers.

Subset—If a particular set A forms part of a larger set B, then A is a subset of B.

Examples :—

1. The boys in a mixed class form a subset of the set of all children in the class.
2. The set of all numbers that are multiples of 3 is a subset of the set of all counting numbers.

Substitution (Syn. Renaming)—1. The replacement of a numeral by an equivalent numeral to name the same number. The numeral 256 names a particular number ; other ways of naming this number are $250 + 6$ and $200 + 50 + 6$. Likewise 16 names a particular number ; other names for this number include $9 + 7$; 4×4 ; $\frac{1}{2}$ of 32 ; $48 \div 3$; and $20 - 4$.

2. The replacement of rods in the colour stage. A dark-green rod may be substituted for three red rods.

Subtraction—The binary operation on two numbers that results in a third number called the difference. One of the four fundamental operations of arithmetic. The inverse operation of addition.

Subtrahend—A quantity to be subtracted from another. For example, in the equation $7 - 3 = 4$, 3 is the subtrahend.

Sum—The result of addition. 13 is the sum of 6 and 7, since $6 + 7 = 13$. See **Addition.**

Symbols—$+$ (plus), $-$ (minus), $\times$ (times, or multiplied by), $\div$ (divided by), $=$ (equals), $\neq$ (is not equal to), $>$ (is greater than), $<$ (is less than), $\Longleftrightarrow$ (is equivalent to).

Verbalization—Expressing in words (mainly oral).

Vulgar Fraction—A fraction that is a ratio of whole numbers, such as $\frac{2}{3}$ or $\frac{8}{5}$.

Whole Numbers—The set of numbers $\{0, 1, 2, 3, 4, \ldots\ldots\}$

Zero—1. The cardinal number indicating the absence of elements or things.

2. The ordinal number denoting the initial point, or origin. The product of any number and zero is equal to zero.

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