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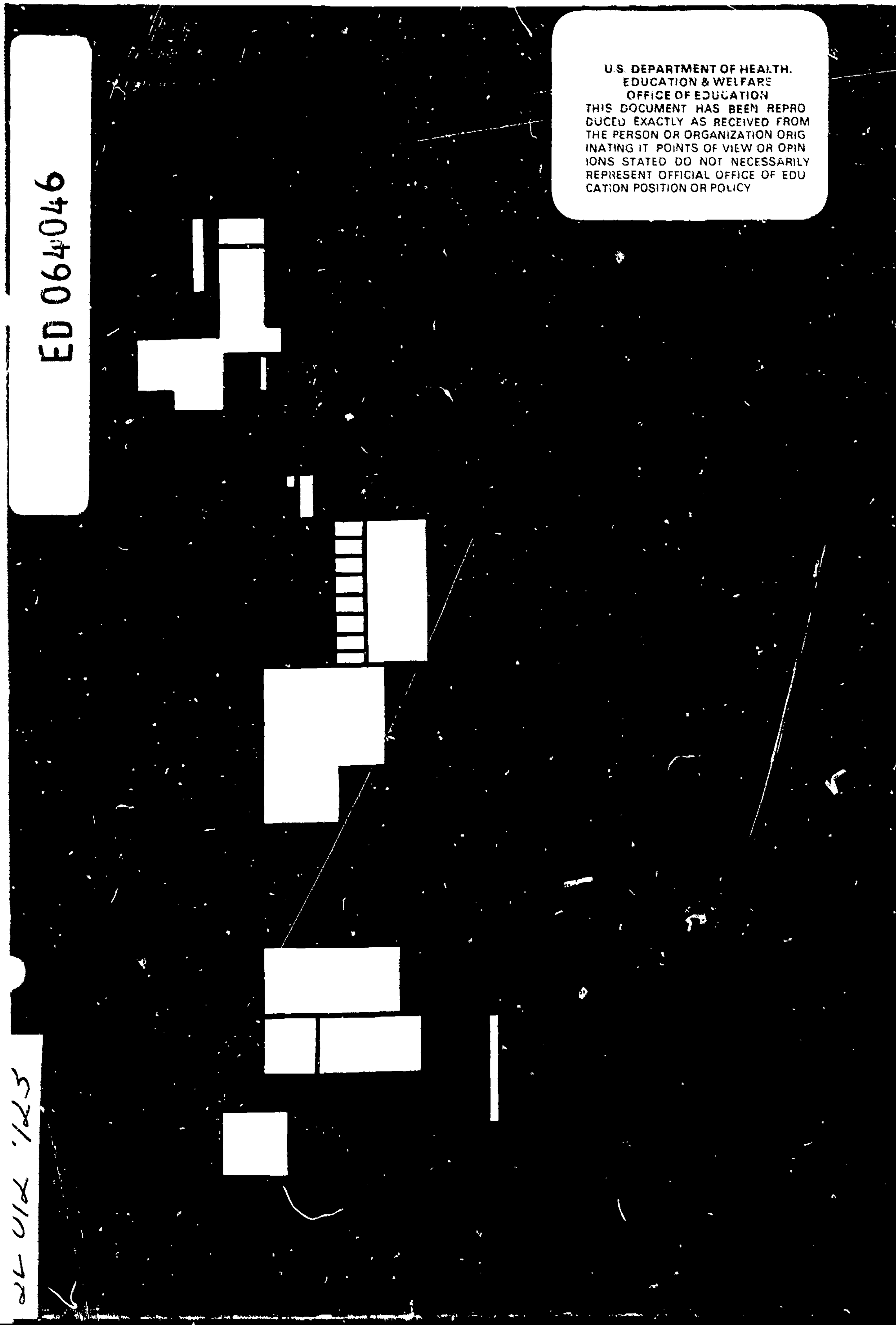
ABSTRACT

Included are three curriculum guides, Section A, Section B, and Section C, which are concerned with the development of number concepts in elementary school, through the use of Cuisenaire rods. Small group and individual instruction are encouraged, and many suitable learning activities are suggested. Section A is introductory and covers number names, one-to-one correspondence, ordinal and cardinal numbers and familiarization with the Cuisenaire materials. Section B develops the operations of addition, multiplication, subtraction and division in terms of the rods and their color. Equality of different combinations of rods, and manipulation of these combinations are stressed. Mastery of this section is emphasized as a prerequisite for later success in the arithmetic program. The aim of Section C is to translate the relations between rod lengths (found in Section B work) into relations between the numbers from one to ten. Included in Section C is work on recording, manipulating and solving equations, both with and without the support of the manipulative materials. Fractions are also introduced in this section. (MM)



# CURRICULUM GUIDE

## PURE NUMBER

## SECTION A



EDUCATION DEPARTMENT OF VICTORIA

10247/65

# CURRICULUM GUIDE
# PURE NUMBER COURSE
## Section A—Introductory
### *AIMS AND COMMENTS*

**AIMS**

This section has four main aims:

i The development of the ability to count.

ii The introduction and the development of the concept of ordinal number.

iii The introduction and the development of the concept of cardinal number.

iv The preparation of the child to use the Cuisenaire material.

**COMMENTS**

1. The development of the ability to count requires two skills:

KNOWLEDGE OF THE SEQUENCE OF NUMBER NAMES

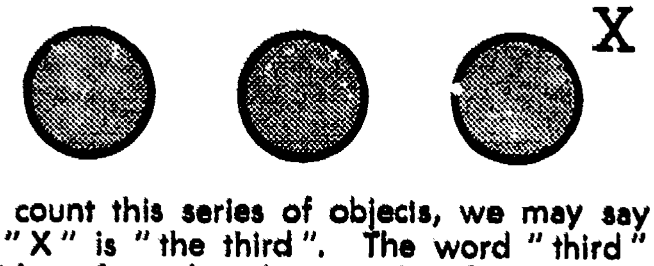
The child must be able to recite, in correct order, the number names from one to ten. When first learned these may simply be nonsense syllables that follow in a prescribed order.

APPRECIATION OF ONE-TO-ONE CORRESPONDENCE

The mere recital of number names, though necessary at first, is not sufficient. One-to-one correspondence must be developed, i.e. the child must be able to associate accurately the number names and the objects counted. The word "one", for example, must correspond with the first object indicated, the word "two", with the second, the word "ten", with the tenth in a series.

2. Once the child has mastered these two skills he has achieved the first aim of this section—the development of the ability to count. This ability may be used to develop the concepts of ordinal and cardinal number.

ORDINAL NUMBER



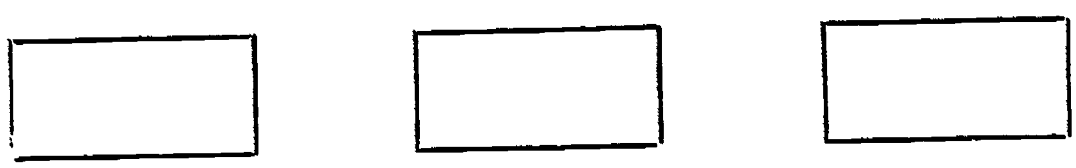

If we count this series of objects, we may say that the one marked "X" is "the third". The word "third" refers to the ordinal idea of number, because it refers to a particular object in a set position in a series.

CARDINAL NUMBER

We may count the objects and say that this is a group of three objects. This is a statement about cardinal number.

Thus with ordinal number we refer to a particular object within a group, while with cardinal number we refer to the group itself. The vocabulary used indicates the distinction—with ordinal number we say "the third" object, with cardinal number we say "three objects".

We must, however, refine the concept of cardinal number a little further. We may consider the group as a whole entity rather than a collection of parts. Instead of thinking of three as



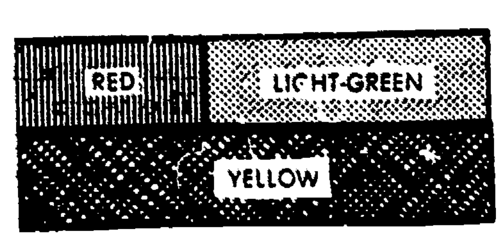
where we have a group of separate objects, we think of three as

where we have a "whole" three.

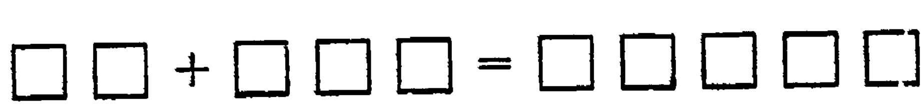
3. Unless the child develops this concept of cardinal number, the four operations cannot be understood—each of them is a manipulation of wholes. To say

"2 + 3 = 5"

is to say: A whole "two", plus a whole "three", equals a whole "five". This may be represented as



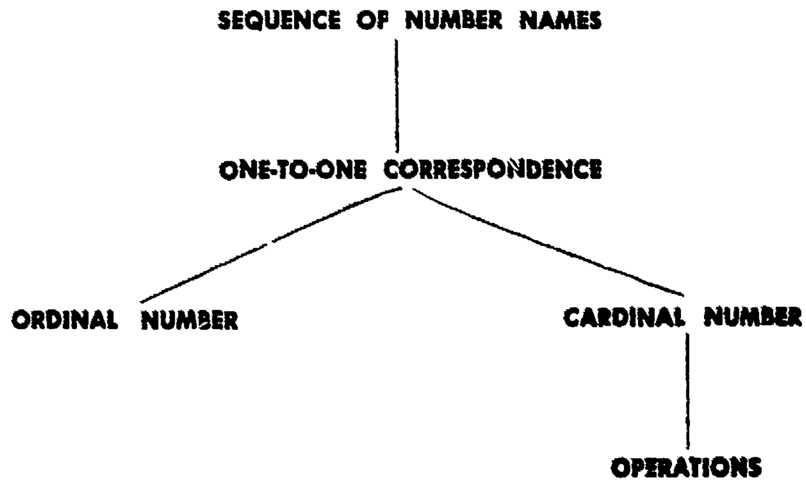

where it is clear that two wholes have been combined to equal a third whole. To represent this same example as



may result in the correct answer, but may also hide the fact that two wholes have been combined, and may even give the impression that addition is simply counting up a series of units.

Thus cardinal number, the concept of number as a whole (as distinct from a succession of units), must be understood if addition, or any of the operations, is to be grasped.

4. In the light of what is written above, the child's mathematical progress may be summarized by this diagram :




This shows the development from a parrot-like knowledge of the number names to an ability to count with accurate one-to-one correspondence. Once this ability is developed the child may use it to find a particular object in a group, e.g. the " sixth " boy in a line (ordinal number), or to develop the idea of the group as a whole (cardinal number). Upon the concept of cardinal number an understanding of the operations may be built.

5. This particular section does not, of course, take the child through all the above program. The program is outlined here simply to place the early sections in perspective. In this section the child learns the sequence of number names and develops one-to-one correspondence (Stage 1). He applies this counting ability to develop the concept of ordinal number (Stage 2), and to develop the simplest cardinal concept, number as a group (Stage 3).

The final concept of cardinal number, where number is seen as a whole, is to be taught using the Cuisenaire material. As this material is new to the child there is an obvious need for him to spend some time getting to know the material itself. This is the task of Stage 4.

The full concept of cardinal number and the understanding of the four operations are developed in later sections.

# STAGE 1

**AIM**

To develop the ability to count.

**NOTES ON AIM**

1. Because the ability to count precedes both ordinal and cardinal number it is the first skill to be taught.

2. As the COMMENTS to this section noted, two skills are required before the child can count—knowledge of the sequence of number names, and one-to-one correspondence. The development of these skills is the task of this stage.

**DEVELOPMENTAL STEPS**

**1. Sequence of Number Names**

a These are most readily learned by the repetition of counting rhymes, most of which are traditional, e.g.

"One, two, buckle my shoe.
Three, four, knock at the door.
Five, six, pick up sticks.
Seven, eight, lay them straight.
Nine, ten, the big fat hen."

b There are many games that are useful in developing this skill, e.g. "Buzz".

For this game children stand in a circle and count from one to ten. The child whose turn it is to say "ten" also says "buzz", and sits. The next child commences again at "one", and so the game continues until there is one child, the winner, left standing.

Although this game is usually played to ten, "buzz" may also be said on any other number, for example "seven", or in later stages "twenty".

This game can be adapted to provide many turns for the children in a short time when small groups play it as a contest under the teacher's supervision. The first group to finish correctly is the winner.

(For a list of activities to develop this skill, see Appendix 1 Part 1.)

**2. One-to-one Correspondence**

a It must be emphasized that children need to learn to **touch** the object as they say its number name. In later work with the rods, where it is necessary to count the white rods in close formation, success will be achieved only if children have formed accurate counting habits. Exercises such as the following may be used:

i **Counting Objects**

Objects such as chairs, tables, skittles, or toys are counted, the "number name" of each being spoken as the object is touched.

ii **Games such as "Old Tom"**

Mother Hen: Who goes round my house tonight?
Old Tom: Only poor Old Tom.
Mother Hen: Don't steal any of my chickens tonight.
Old Tom: Only this fat one.

The teacher chooses children to be Old Tom, Mother Hen, and a certain number of chickens.

In one corner, Old Tom sits in his den, while Mother Hen sits in the centre of her brood.

Tom circles the chickens for the first three lines, then, as the last line is spoken, he seizes one chicken and makes off with it to his den. The teacher then asks:

"How many chickens has Old Tom?"
"How many chickens has Mother Hen?"

The game continues thus, until all of the chickens are with Old Tom, who then goes off to gather wood for his fire. While he is away, Mother Hen rescues her brood and returns with them to her nest.

b Once simple one-to-one correspondence has been mastered the child learns the slightly more difficult task of stopping at a given number before the end of a series, e.g. the child may stop at "seven" in a series of ten. (No attempt is made, as yet, to group the seven objects.) This skill may be developed by—

i Counting of objects or activities, stopping at a particular number, e.g. "five jumps", "eight claps", "six skittles".

ii The game "Buzz" may be played, stopping at the various numbers.

(A list of activities to develop one-to-one correspondence is given in Appendix 1 Part 2.)

## NOTES ON METHOD

1. It is obvious that neither Stage 2 (the study of ordinal number) nor Stage 3 (the study of cardinal number) can be begun until success in this stage has been achieved. It is pointless to ask a child to bring the sixth object or to bring six objects until he can count to six. Thus the amount of time spent in this stage is determined by the child's ability to count. As soon as he has mastered this skill he leaves the stage; until he has mastered it he must remain.

2. There has been supplied a number of exercises designed to assist in the attainment of counting skill. It is assumed that teachers will supply exercises of their own to supplement those given. The introduction to this course has stressed that concepts do not "happen" in the child's mind—their growth must be fostered. A key factor in fostering them is a wide variety of experiences that reinforce each other and illustrate time and time again, though each time in a slightly different way, the concepts the child is striving to master.

## TESTING

SEQUENCE OF NUMBER NAMES

a The child counts to ten.

b The teacher begins a count and asks the child to complete it.

c Questions are asked, e.g.

"Which number is after six?"

"Which comes before ten?"

ONE-TO-ONE CORRESPONDENCE

a "How many pairs of scissors are on the table?"

"How many pieces of chalk are in the box?"

b Arrange a line of objects.

"How many objects are there in that row?"

Remove some of the objects.

"How many now?"

Spread the objects out without changing their total.

"How many now?"

c In order to test a child's ability to stop at a given number, ten objects of different colours may be placed on the table and the child told:

"Count and stop at seven."

"Which colour are you at when you stop at four? . . . . when you stop at five?"

TWO IMPORTANT MATTERS NEED TO BE NOTED IN CONNEXION WITH TESTING

a All the testing is oral. This enables a far wider range of questions to be put in much less time. The wider range of testing is vital. One of the surest indications of understanding is ability to use an idea in a wide range of situations —just as one of the surest indications of a lack of true understanding is a response that is correct only in a limited range of situations.

b In a sense some of the questions are "trick" questions, e.g. the questions asked on one-to-one correspondence. Unless the child is alert and fully master of what is being tested they will trap him. This policy is not adopted in a futile or perverse desire to catch the child out. It is a deliberate attempt to assess whether a child has sufficient confidence in his understanding to rise above, or ignore, factors which mislead him. It is because this course places so much stress on the need to develop strong, confident mastery of ideas that the policy is pursued of testing not only the child's mastery, but also the confidence and the quality of his mastery.

# STAGE 2

**AIM**

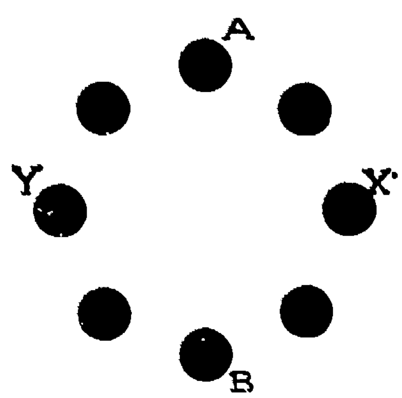
To introduce and develop the concept of ordinal number.

**NOTES ON AIM**

1. As soon as one-to-one correspondence is mastered, the child is able to begin his study of ordinal number, i.e. number that refers to an object in a set position in a series.

2. To master this concept, a child must be able to identify a particular object in a series, e.g. to pick up the fifth object in a group of five or more objects.

He must also realize that ordinal number assumes a point of reference. Obviously a child will not be able to point out the " third " thing in a group or circle unless some guidance is given as to the first object.




For example, if the counting is done in a clockwise direction and begins at A, the third object is X, whereas if the counting begins at B, the third object is Y.

**DEVELOPMENTAL STEPS**

These follow closely the pattern of exercises for Stage 1, and thus some of the activities suggested for that stage may be adapted for use in this one.

**1. Sequence of Number Names**

The terms " first, second, . . . . tenth " are introduced as a new way of counting. The child is shown that:

" one = first ",
" two = second " etc.

Later, after cardinal number is introduced through grouping, the child should realize that to say " two " may mean " second ", or may mean " a group of two ".

**2. One-to-one Correspondence**

This skill is taught in the same way as it was in the previous stage except that instead of saying, " One, two . . . ." the child says, " First, second . . . .".



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### 3. Ability to Select an Object from a Series

Using the same type of activity as was employed in the previous stage, the child learns to select particular objects from a series, e.g. " the fifth object from a line ", " the third pot-plant on the window-sill ".

### 4. Need for a Specific Point of Reference

When the child is able to identify an ordinal position in a straight series with complete accuracy, he begins a set of exercises to lead him to realize the need for a specific point of reference.

Previously, of course, the child will have read ordinal position, in his normal counting, from left to right because movement in this direction is the usual convention. He must now be made fully aware that ordinal position can be changed, if we change our starting point. (See **Notes on Aim.**)

Probably the easiest way to introduce this idea is to use a circular series where no conventional starting point is obvious. From there the child may return to a straight line series and see that ordinal values change depending on whether we start at the beginning of the line of objects or, for example, in the middle of the line. The teacher may ask :

" Which pot-plant is this? "

and expect the child to answer :

" The second from the back of the room "

or, " The third from the black-board ",

both answers being equally correct.

Not only should the child hear the teacher use the terms, but he should also be encouraged to use the terms fluently himself.

(The activities listed for the previous stage may be adapted to suit the needs of this stage.)

## NOTE ON METHOD

Once understanding of one-to-one correspondence is acquired the child may move to a study of either ordinal or cardinal number. In practice he studies both together, learning to apply his counting skill to pick out a particular object or to make a specific group. For purposes of convenience, the study of ordinal number has been treated first, but it must be stressed that, in the class-room, Stages 2 and 3 proceed simultaneously.

## TESTING

1. Arrange a group of eight objects and ask questions such as :

" What is the sixth object? "

" Is there a ninth object? "

2. Rearrange the objects to form a circle. Point to a particular object :

" If I start here, where's the third? "

Point to another object :

" If I start here, where's the third? "

# STAGE 3

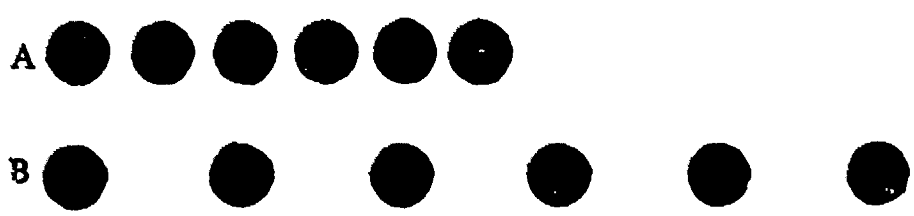
**AIM**

To introduce and develop the concept of cardinal number.

**NOTES ON AIM**

1. In this stage the child studies the first idea of cardinal number —the idea of number as a group. The more refined concept where number is seen as a whole is not tackled until later. (See **Comments on Aims of Section A.**)

2. Although the ability to count with one-to-one correspondence is a prerequisite for this stage, it is not sufficient of itself to lead to the full concept of number as a group. In fact it can co-exist with a very limited grasp of cardinal number. A child may, for example, be able to make a group of six objects but think the value of the group is changed if the group is rearranged. A typical response is to say that, in the pattern below, Row A contains fewer objects than Row B.

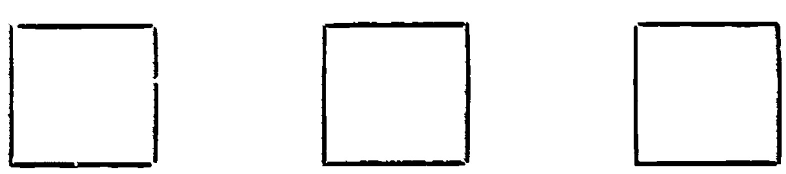



Until he realizes that the value of the group remains unaltered no matter how much its constituents are varied or rearranged, a child has not mastered the concept of number as a group. The aim of this stage is to bring about a realization of this concept.

**DEVELOPMENTAL STEPS**

**1. Grouping Objects**

To introduce the concept, practice is given in counting out groups of objects, e.g. three blocks :



Having put out many of these " threes ", the child is asked to point to " one three ", " another three . . . .". Extensive practice in making groups of all numbers to ten (and then studying the groups made) must be given.

**2. Rearranging Groups**

Once the child has developed the ability to construct a group of any given number, e.g. " five ", he studies differently arranged groups of " five ", e.g. five objects may be arranged in all these ways :



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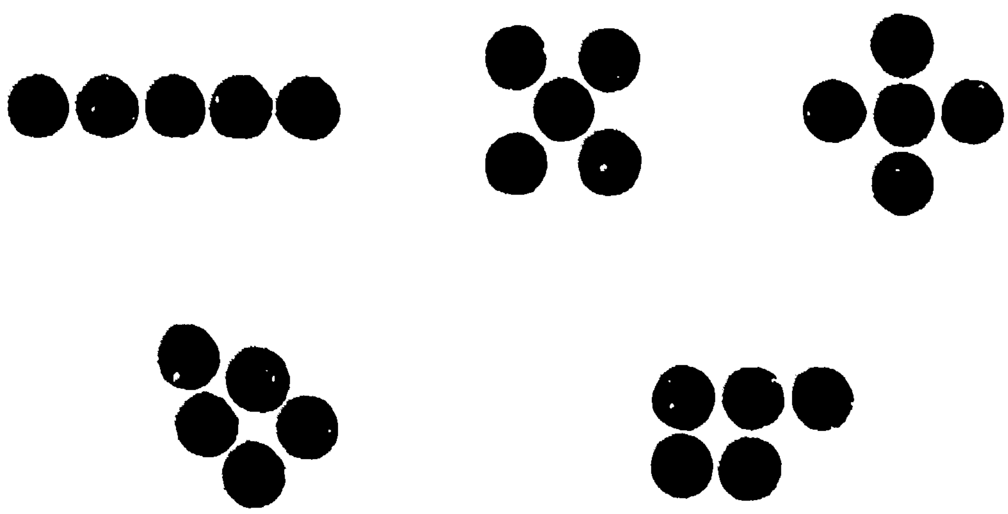
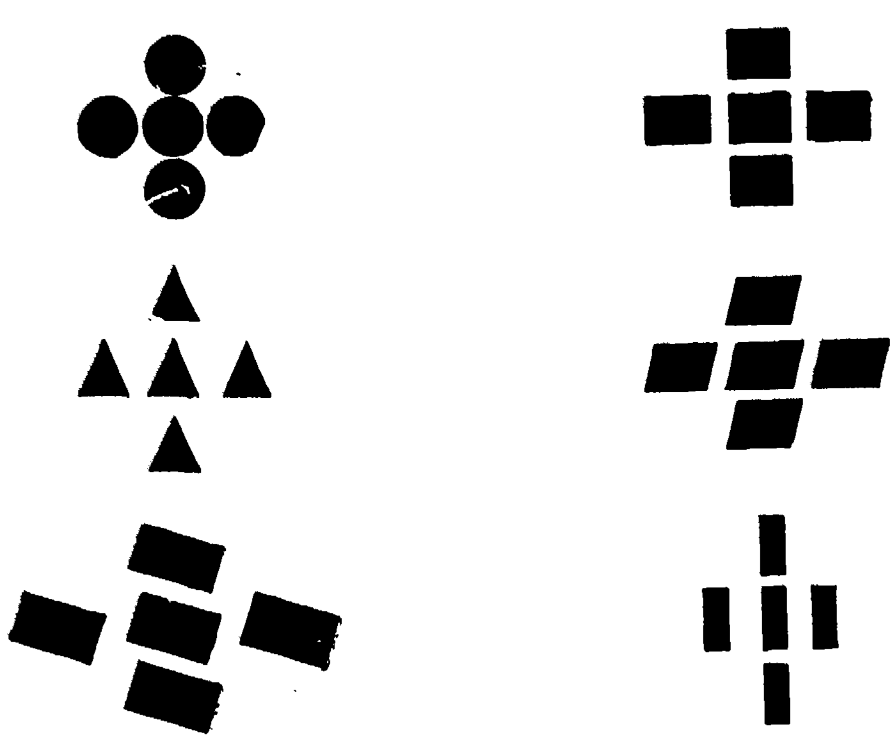


Thus the child is led to realize that the constant " fiveness " is unaffected by the arrangement of the group.

**3. Altering the Constituents of a Group**

This understanding is then extended so that he realizes that the " fiveness " remains even when the objects that make up the group are changed.



Substitution of different elements for those already in a group can assist the development of this idea. A group of three buttons could be replaced, one at a time, with cotton-reels. The group is still a group of three, though of objects different from those first used.

**4. Constructing a Specified Number of Groups with the Same Number of Elements in Each**

The child should be able to construct a number of groups of the same size, e.g. three groups of four counters; two groups of five counters.

**5. Constructing Groups of Un-related Objects**

This activity allows the child to abstract the cardinal idea. It could be approached thus:

"Jill, bring me three different things."
(Perhaps a toy, a flower, and a box would be chosen.)

"Now find me another three different things."

"What number is here?" (first group) "Three."

"Here?" (second group) "Three."

"What is the same about these two groups?"

"Both have three things." or "The number is the same."

(A list of activities suitable for developing this concept appears in Appendix 1 Part 3.)

**NOTES ON METHOD**

1. It was mentioned during the previous stage, and must be stressed again here, that Stage 2 and this stage, Stage 3, proceed simultaneously. Both stages are begun immediately, but not before, the skill of one-to-one correspondence is developed.

2. A full understanding of the concept of numbers as groups is not needed until the rods are given numerical value (Section C). Thus this stage may be completed at any time before the beginning of Section C. Stage 5 (Section B) is a continuation of this work.

3. It is necessary to mention again a point made earlier—the key to success in the formation of concepts is a wide range of directed activities.

The word "directed" is used because activity for its own sake may be of little value. The exercises must assist the child to develop the skill or the concept being studied.

The word "activities" is used to stress that the child must do the exercises himself. The degree of mastery attained by the child is directly related to the number of useful experiences he has had. It is not necesserily related to the number of times he has watched others perform the exercises.

**TESTING**

1. The child is asked questions such as:

"Show me six shells."

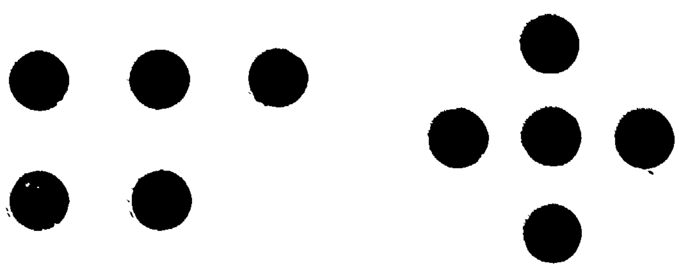
"Show me eight shells."

"Show me two lots of three shells; four lots of four shells."

2. A line of objects is arranged :

" How many counters in that group? "

They are then rearranged :



" How many now? "

3. Exercises of the type outlined in Step 5 (of Developmental Steps above) are given.

4. The child is not considered to have mastered the concept unless he is able to use the necessary vocabulary. He should be able to say, " This is a four . . . . Here are two fives . . . ." When testing, care should be taken to ensure that he has acquired this mastery.



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# STAGE 4

**AIM**

To develop skill in handling the rods and a knowledge of their properties.

**NOTES ON AIM**

1. This stage, as the **comments** for the section note, is purely a preparatory one. The material is soon to be used to give an understanding of mathematical ideas. Before this is done the material itself is studied, so that when the child uses it to work with mathematical ideas he will not be handicapped by an inability to handle the rods or an unfamiliarity with their properties.

2. This study of the material should develop in the child

a Skill in handling the rods, i.e. a purely physical dexterity, an ability to handle and pack the rods with ease and confidence.

b A knowledge of the colour names of the rods.

c An understanding of the vocabulary used when working with the rods. Apart from the colour names, the most commonly used words are :

| | | | | |
|---|---|---|---|---|
| **big,** | **bigger,** | **biggest** | **length** | **end** |
| **small,** | **smaller,** | **smallest** | **as long as** | **end to end** |
| **long,** | **longer,** | **longest** | **width** | **side** |
| **short,** | **shorter,** | **shortest** | | **side by side** |
| **wide,** | **wider,** | **widest** | | |
| | | | **size** | |
| | | | **shape** | **above** |
| **row of rods** | | | | **below** |
| **line of rods** | | | | |
| | | | **same** | |
| | | | **different** | |

N.B. " **difference** " is required later, but most children are probably not ready for it at this stage.

d A clear mental image of the colour and the size of each rod.

3. Although the aims of the stage are confined to those listed above, the actual benefits of the stage are wider and of sufficient importance to be noted.

Quite often, through the activities of this stage, the child begins to learn incidentally something of the mathematical qualities of the rods, e.g. he realizes that an orange rod is equal in length to two yellow rods, or to a black and a light-green rod placed end to end. While, at this time, no attempt is made to direct attention to such matters, the fact that the child discovers them is of considerable importance.

It is also remarkable how much can be learned about the child during this stage. Children approach building activities in very different ways. Some build the same thing, day after day, and display resentment when asked to change. Some build daringly, others with the utmost caution. Some share the rods with the rest of the group, others endeavour to monopolize the box. By observing the child's reaction and discussing it with him, the teacher may learn a great deal.

This stage is important in its own right—much of the child's future progress is determined by the extent of his familiarity with the properties of the material. But this importance is increased by the incidental benefits that can be shown to accrue.

## DEVELOPMENTAL STEPS

### 1. Introduction of the Material

The material may be introduced in one of two ways :

a The boxes are emptied on the table. In most of the work in the infant grades, one box is shared between four children. The rods are jumbled and the children are allowed to play with the material. At the end of the period, the rods are collected without any attempt being made to pack them correctly.

b The boxes are emptied on to the table and the children play with the material. At the end of the period the rods are packed correctly in their boxes.

The first method is the ideal. The child is not immediately shown that the rods are ordered according to colour and size, he is left to discover this for himself. Furthermore, he is not faced with the task of packing up until he has had a few days' unrestricted use of the material. The disadvantage of the method is the danger of loss—a danger that the second method, because it insists on immediate packing, obviates.

### 2. Sessions of Free Play

Irrespective of the method used (and the choice is, ultimately, for the individual teacher), the material is introduced through free play activities. The child is permitted to play with the material, to build whatever he wishes.

These first sessions of free play are of the utmost importance because it is through them that the child's attitude to the rods is formed. The rods are attractive to the child—so attractive that some children are inclined to "borrow" some of them. If introduced to the child through freely chosen activities, they capture his interest immediately. Later work in developing understanding of mathematical ideas can capitalize on this interest. It is this desire to capture interest, to introduce the rods in a pleasant way, that explains the sessions of free play.

Incidentally, however, this step contributes to the development of dexterity in handling the rods. The buildings made are often quite complicated and require considerable dexterity on the part of the child. In addition the child begins to get some idea of the relation between colour and size. He sees, and the very fact of packing emphasizes this, that all orange rods are of the same size, that they are always bigger than blue rods, which, in turn, are always bigger than brown rods.

Completely free play does not continue for very long. As soon as the child has had time to become used to the rods and to handle them with some semblance of skill, he moves to the next step—directed activities. It is most important to note, however, that although free play becomes less frequent, it does not cease. All through this stage free play continues, and in the later stages an occasional period of free play is most valuable.

**3. Directed Activities**

In this step attention is directed toward developing

a skill in handling the rods,

b a knowledge of the colour names,

c an understanding of the vocabulary used when working with the material,

d a clear mental image of the rods.

All of these skills may have received some treatment incidentally in the previous step. In this step, however, they are consciously and systematically developed. It is necessary to remember, too, that the formation of a clear mental image of the rods, while not consciously or systematically developed in this step, is considerably aided by the activities that are done. Thus the pattern of the activities in this stage is becoming clear—the four skills are developed side by side, though, in turn, each is concentrated on for a short period in order to ensure that it is acquired.

a **Skill in handling rods** To assist him to acquire this skill the child is given directed building. He is asked to build things that require considerable skill in handling the rods, e.g. " a tower "; " an aeroplane ". The type of building required is three-dimensional rather than flat, complicated rather than simple in design, using the smaller rods, which are more difficult to handle, as well as the bigger ones. It is in this particular step that the development of dexterity in handling the rods, begun in the free play step, is consolidated. It need not be fully present until the end of the stage, but it ought to be substantially achieved by the end of this step.

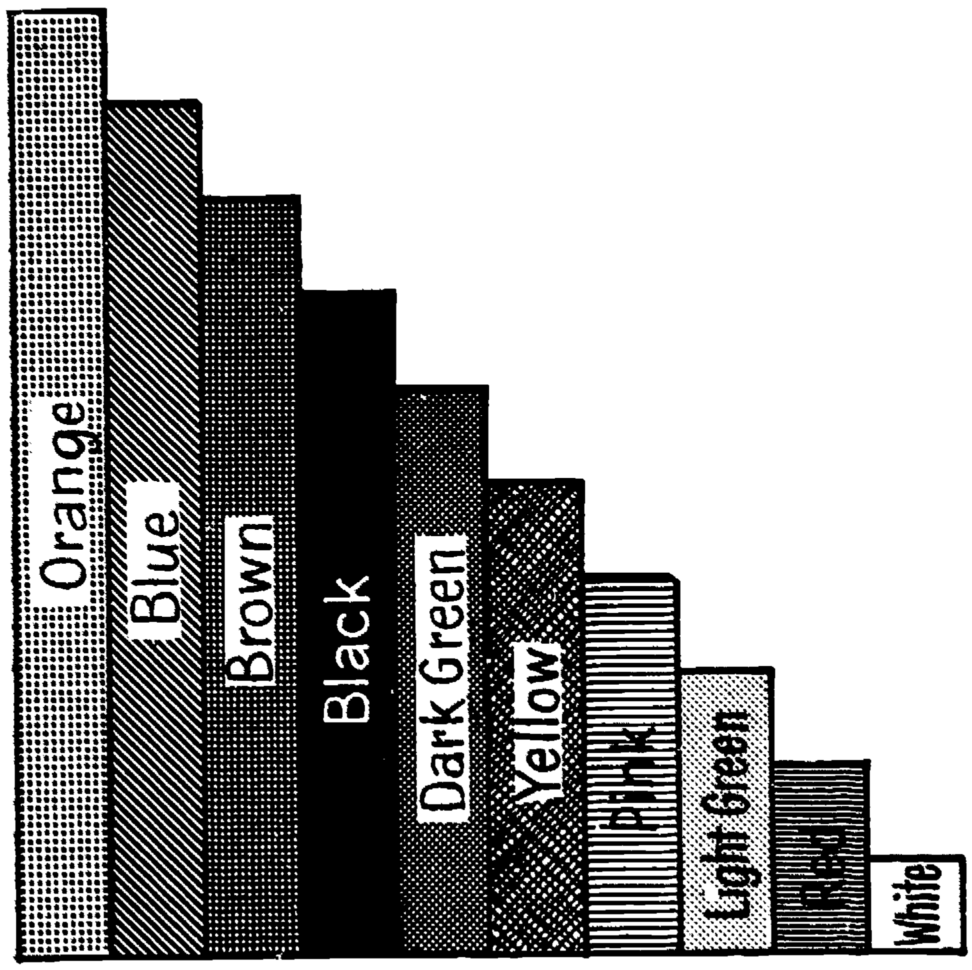
(For a list of suitable activities see Appendix 1 Part 4.)

b **Knowledge of colour names** At the same time as directed building is proceeding, the teacher introduces activities to ensure that the colour names are known. The names most commonly used are—white, red, light-green, pink, yellow, dark-green, black, brown, blue, orange. These names are, in most cases, learned incidentally in the previous steps. In any case it is necessary to ensure quite certainly that the names are known before any further work is done. Activities that assist this aim are :

i Question and answer games . . .

" What colour is this ? "

ii Staircases. The rods are arranged into a " staircase ", e.g.




The child moves up and down the "staircase" saying the colour names. (This game also helps to teach the relation of colour and size.)

c **Vocabulary** During these activities (and in fact, throughout the whole stage) the vocabulary that needs to be understood should be constantly used so that by the end of this stage the words listed in the **Notes on Aim** are completely familiar. It is not sufficient for the child to hear thr teacher use the words, he must also use them himself. Activities common in this step are :

i "Put a blue rod end to end with a black rod . . . . a white rod . . . . a brown rod."

ii "Put a blue rod side by side with a black rod . . . . a white rod . . . . a brown rod."

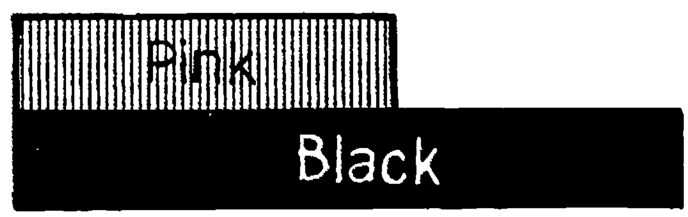
iii "Which of these rods is the biggest? . . . . smallest?"

iv "How have I placed these rods ?" "End-to-end."

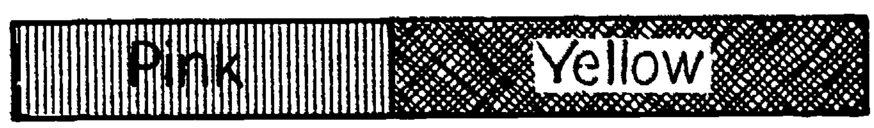
v "Tell me about these two rods." "They are of different lengths, but of the same width. The blue is longer than the yellow."

A child is asked to act as a teacher and to ask the children to put rods in various positions. He must then check to ensure that the class has done what he asked.

d **Mental image of rods** In this step, the last of the skills, that of securing a clear image of the rods, is emphasized. The achievement of this skill is of the utmost importance. Unless the child has a clear picture of the rods, his work in the later stages is cramped and hindered. Quite often, for example, the child is faced with a pattern such as




and asked to fill the gap, or




and asked to find a rod equal to these two rods together.

Unless he has a clear idea of the length of each rod these simple exercises are difficult. Instead of finding the correct rod the first time he may require two or three attempts. If this happens often, the work slows down and the study of the ideas becomes far more difficult and cumbersome. There is, therefore, an urgent practical need to give the child a clear image of the length of each rod.

This image is formed to some degree through the familiarity with the rods that the child acquires in the course of his normal work. Nevertheless in this step attention is concentrated upon it in order to ensure that it is fully achieved. The technique used is to lead the child through a series of games and activities to recognize the rods by touch and sight. This is not done because it is of intrinsic importance. It is done because in the process of learning this skill the child acquires a clear mental image of each rod.

The games (sometimes called the Touch Games) played to achieve this end are outlined below.

i Each child takes three specified rods, e.g. pink, yellow, and dark-green. He studies them, feels them, and discusses which is the biggest and which is the smallest.

Discussion of other properties is useful, and aids vocabulary extension. He then puts the rods behind his back and is asked to hold up a particular rod. If he makes a mistake he tries again If he is correct he puts the rods behind his back and is asked to show a different rod. The game may, of course, be played with more than three rods—the limit being the number a child can hold comfortably in his hands.

ii The second game develops from the skills achieved in the first. As before, the child places a group of rods before him, feels, studies, and discusses them. He then hides them with a card or similar object. He "stirs" them round with a finger. Using one hand, he selects the designated rod. **He must not feel two rods together.** He may pick one up, mentally assess it, put it down, and feel another before producing his choice. The child compares two mental images without having two rods in his hand at the one time. Thus the basis for the abstraction needed in the third game is established.

iii The third game is more difficult again because no comparison at all is possible. The child stands with his hands behind his back and a single rod is dropped into them. He must feel the rod and identify its colour. The child, to succeed in this game, must know the absolute length of the rod. He must have developed a clear mental picture of the rod.

(Activities suitable for use in this step appear in Appendix 1 Part 5.)

**NOTES ON METHOD**

1. The key to success in this stage is the realization that the four skills required are developed side by side, not separately—though, in order to ensure a full achievement, each of them receives specific attention at some time during the stage.

Accordingly, it must be realized that the steps outlined separately for convenience are not separate in practice. We do not begin Step 1, complete it, then drop all work connected with it and move to Step 2. If this rigid procedure is followed, an otherwise simple stage becomes very difficult.

The correct procedure is more gradual. After the rods are introduced, the child begins the sessions of free play and, until he gains some familiarity with the rods, does nothing else. Gradually free play, though it is never entirely cut out, becomes less frequent and the directed activities begin. Once these have ensured that the colour names and a certain basic vocabulary are known, the first of the Touch Games may be introduced. Thus three steps are going on together—free play, directed activities, and the first of the Touch Games. Once the first game is mastered the second begins; once that is mastered the third begins. Irrespective of what game is being played, the free play and the directed activities continue.

2. It is important to know when best to commence this stage. It is obvious that the three previous stages are in no way prerequisites for this, which is concerned solely with developing non-mathematical skills. Some experience in handling material is needed, but, as one purpose of the stage is to gain a highly developed skill, a brief experience would be sufficient.

However, the most important point to note is that this stage is aimed at making the child thoroughly familiar with the material in order that he may proceed efficiently with Section B, the study of the basic mathematical ideas of equality, addition, multiplication, subtraction, and division. The average child is ready for these ideas at the mental age of approximately six years. If the work of Stage 4 is taken too early, we find children unable to proceed to this mathematical study because of lack of maturity. An undesirable attitude can then be developed, due to the teacher's attempt to take children further than their mental development will allow.

It is found quite satisfactory to commence this stage about the middle of the beginners' year for children of average age and ability, with the necessary adjustments for those older or younger.

3. One of the first problems encountered is the problem of packing. In fact, at their first attempt it may take children twenty minutes to pack a box, though later the time is reduced to three or four minutes.

Definite practice in packing ought to be given very early in this stage. Speed in packing is greatly assisted if the teacher pastes in each compartment a strip of coloured paper which corresponds to the colour of the rods to be placed in it. Insistence on packing the rods in a uniform manner makes checking for loss a simple matter. The teacher can see at a glance if any rods are missing. Care taken in the early stages to ensure proper packing limits any likelihood of loss.

It must not be imagined that time spent in packing is time wasted. Quite apart from its importance in guarding against loss, packing emphasizes the fact that the rods are ordered according to colour and size. The counting and checking involved are important mathematical experiences. It is also an interesting group activity that helps children to work together.

4. The problem of when to leave the stage is quite simply solved. The child leaves the stage as soon as, but not before—

- he has developed skill in handling the rods,
- he has a knowledge of the colour names,
- he understands and uses the necessary vocabulary when working with the rods,
- he has a clear mental image of each rod.

The statement on individual differences in the introduction to the Revised Course in Mathematics makes it quite clear that skills such as these will be acquired by different children at different rates. Accordingly some children will be ready to leave this stage long before others. But, irrespective of the time taken, no child should move from this stage to the next until full mastery has been obtained. The reason for this insistence is quite clear—the skills developed in this stage are prerequisites for success in the later stages.

**TESTING**

1. The first skill to be obtained, dexterity in handling the rods, can be tested only by observation. The child builds using the rods, and the teacher watches the manner in which he handles them, particularly when the task the child is performing requires some

skill. When, from such observations, the teacher is satisfied that a child is handling the rods with skill, ease, and confidence, he is considered to have attained the aim of the stage.

2. Knowledge of colour names and vocabulary may be tested during the course of an everyday lesson. Questions asked of the children during these activities usually indicate whether the colours are known or not. Instructions such as " Put the pink rod beside (above, end to end with) the blue " give an opportunity to check the other words in the vocabulary. It is assumed, of course, that before children move from this stage they know all the colours and can use the vocabulary freely.

3. The tests for the final skill, the formation of a clear mental picture of the rods, demand a high degree of accuracy in the third Touch Game and complete competence in assessing lengths in terms of rods. A child who can recognize a single rod by touch alone, and who can quickly and efficiently match rods to appropriate outlines, has a clear mental picture of the rods.

## *SUMMARY OF WORK COVERED BY THE END OF SECTION A*

Let us consider what the child has accomplished by the end of this section.

COUNTING

He can count at least to 10, with efficient one-to-one correspondence, and with the ability to stop at a given number.

ORDINAL NUMBER

- a He understands the idea of number that refers to an object in a set position in a series.
- b He can confidently use the terms " first — — — tenth ".
- c He can select a designated article from a series.
- d He knows that he sometimes needs a specified point of reference and direction.

CARDINAL NUMBER

- a He can arrange objects in groups of up to 10.
- b He knows his group is truly a " four " or a " six " no matter what arrangement is used, or what constituents form his group.
- c He will have had some experience in seeing the cardinal nature of groups of unrelated objects, although this would probably be confined to small groups.
- d He is developing a concept of cardinal number which will, however, need to be extended during following sections.

FAMILIARITY WITH CUISENAIRE MATERIAL

- a He is thoroughly familiar with the material.
- b He handles it efficiently.

c He knows and uses freely i the colour names,
ii other necessary vocabulary.

d He has a clear mental image of the size of each rod.

GENERAL NOTES

a The child's interest in mathematics has been aroused and a desirable attitude developed.

b He has been given the opportunity for experiment and experience, and so has made discoveries and drawn conclusions for himself.

c His use of the rods should reveal that he has some awareness of the mathematical relationships that exist in the material.

d He is progressing at a rate suitable to his own powers. He has not been asked to master all the section in a specified time, or at the same rate as his class-mates. (In practice it is often found that a fair-sized group, perhaps a quarter or a third of a grade, will progress at much the same rate.)

# APPENDIX 1

This appendix contains activities that may be used to assist in the development of the concepts and skills that have been studied in Section A. The activities are listed under the skill or concept toward whose development they contribute most. In many cases, however, activities listed in one place may easily be adapted for use in another.

This appendix does not claim to include any more than a sample of possible activities. It is assumed that teachers will construct for themselves, or derive from other sources, many activities of a similar nature.

## PART 1 SEQUENCE OF NUMBER NAMES

COUNTING RHYMES

i One, two, three, four, five,
Once I caught a fish alive.
Six, seven, eight, nine, ten,
Then I let him go again.

ii One, two, three, four,
Mary at the cottage door;
Five, six, seven, eight,
Eating cherries off a plate.
Cherry nine and cherry ten,
Mary took inside again.

iii One, one,
Go for a run.
Two, two.
Touch your shoe.
Three, three,
Skip to me.
Four, four,
Run to the door.

Five, five,
    Bees in a hive.
Six, six,
    Do some tricks.
Seven, seven,
    Bend like seven.
Eight, eight,
    Learn to skate.
Nine, nine,
    Grow like a vine.
Ten, ten,
    Lions in a den.

iv One, two,
    Visit the zoo.
Three, four,
    Lions roar,
Five, six,
    Monkey tricks.
Seven, eight,
    Foxes wait.
Nine, ten,
    In their den.

v One, two, three, four,
    Sleepy giant starts to snore.
Five, six seven, eight,
    Clever hen is on the plate.
Nine, ten—it clucks, and then
    There's a golden egg again.

COMPLETING THE COUNT

In the simplest form of this activity the teacher begins a count and the child completes it, e.g. the teacher says, " One, two, three, four," and the child says, " Five, six, seven, eight, nine, ten."

Later this may be commenced by the teacher, who bounces a ball six times while the children count silently. A child is chosen to continue and say, " Seven, eight, nine, ten."

The game may be varied by having one child commence the counting then break off and leave another child to finish.

## PART 2 ONE-TO-ONE CORRESPONDENCE

ACTIVITY RHYMES

One little, two little, three little Indians,
Four little, five little, six little Indians,
Seven little, eight little, nine little Indians,
Ten little Indian boys.
Ten little, nine little, eight little Indians,
Seven little, six little, five little Indians,
Four little, three little, two little Indians,
One little Indian boy.

COUNTING OF INTERESTING OBJECTS

A beginners' room abounds in articles of interest to children. They can touch and count toys, balls, bean bags, skittles, flowers, chairs, tables, windows, rows of boys and girls, the children with school jumpers, the books on the library table, and so on.

USE OF BEAD FRAMES

The teacher at first (and later perhaps a child) moves each bead as the children count in correspondence. They may count the numbers of beads in a row, the red beads, the green beads . . . . Gradually the children learn to associate a bead with a separate number name.

GAMES

i **"Greedy Eagle"** This is a variation of the "Old Tom" game (see Stage 1) in which the children shout "Greedy Eagle" as each chicken is stolen.

ii **Bean Bags** This game may be integrated into the daily physical education period as an outdoor game or may be played in the normal number period. A child standing at a convenient distance aims a number of bean bags into a circle on the ground. At the conclusion of his turn the number of bags inside the circle is compared with the number outside the circle.

iii **"Shooting Ducks"** A pond is drawn in which a number of cardboard ducks are "swimming". The child bowls a ball at the ducks and counts the number he is successful in knocking over or "shooting" and the number he misses.

iv **Skittles** Coloured skittles may be used in the same way as the "ducks" were in the previous game.

v **"Apple-tree"** An "apple-tree" is made of felt or wood and hooks are placed in it. "Apples" cut out of coloured card are hung on the hooks while the following rhyme is spoken:

"Riddle me ree, riddle me ree,
How many apples are on the tree ?"

The apples are then counted.

The game may become pupil directed:

if the child counts correctly he may then rearrange the number of apples on the tree and choose the next child to count;

if the child counts incorrectly he tries again, with assistance if necessary. He then chooses a child to take his place, and returns to the group.

These activities may be used to teach simple one-to-one correspondence or may be adapted to teach the skill of using one-to-one correspondence and stopping on a particular number. It should be noted that all that is required of the child at this stage is the simple ability to stop at a designated number, e.g. seven. When he has stopped he is not asked to group the seven objects and say, "This is a seven", nor is he expected to say, "This is the seventh object." These adaptations can of course be used when he has the necessary skills.

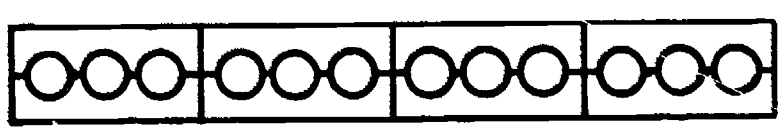
N.B. Care should be exercised with any activity to ensure that its purpose is not lost. If counting is the aim, most of the time should be spent in counting. Too much elaboration could make an exercise almost valueless.

## PART 3 CARDINAL NUMBER

All the activities listed in Part 2 of this appendix may be used so that the child groups the objects counted.

INDIVIDUAL ACTIVITIES

i **Bead Threading** Children thread beads according to a pattern shown on an individual card, e.g.



Another card could show a requirement of 2 red beads followed by 3 blue beads, then 2 red, 3 blue, and so on.

ii **Sorting and Matching** A child is given a box containing, say, 4 cardboard egg cups, 4 cardboard eggs, 3 flower-pots, 3 flowers. He sorts these and "pairs" or "matches" the eggs to the egg cups, the flowers to the flower-pots, and sees that he has groups of four and of three.

GAMES

**Musical Activity** Children move round the room freely or in a circle, to light running music, clapping or tapping (for rhythm). When the music stops, the teacher says, "Fours", and the children group themselves into fours. (Any children left over go into a corner of the room, or within the circle.) The music starts again and the children move round. When the music stops the teacher calls another number and the previous procedure is followed, the "left-over" children now joining in. The number of groups made can be counted as a further exercise.

## PART 4 PHYSICAL SKILL IN HANDLING RODS

It has been mentioned before that the activities chosen should incorporate three things—a complicated design, a three-dimensional effect, and practice in using the small rods. Suitable exercises are:

"Build your own house."



26

" Build the house where you would like to live."

" Build telegraph poles."

" Build a tower."

" Build a tower using only three rods at the bottom."

" Build a bus."

" Can you make a tent ? "

" Make a high fence with a smooth rail along the top."

" Build a circus tent."

" Build the great staircase where Cinderella lost her shoe."

" Make electric-light poles with more than one high rod."

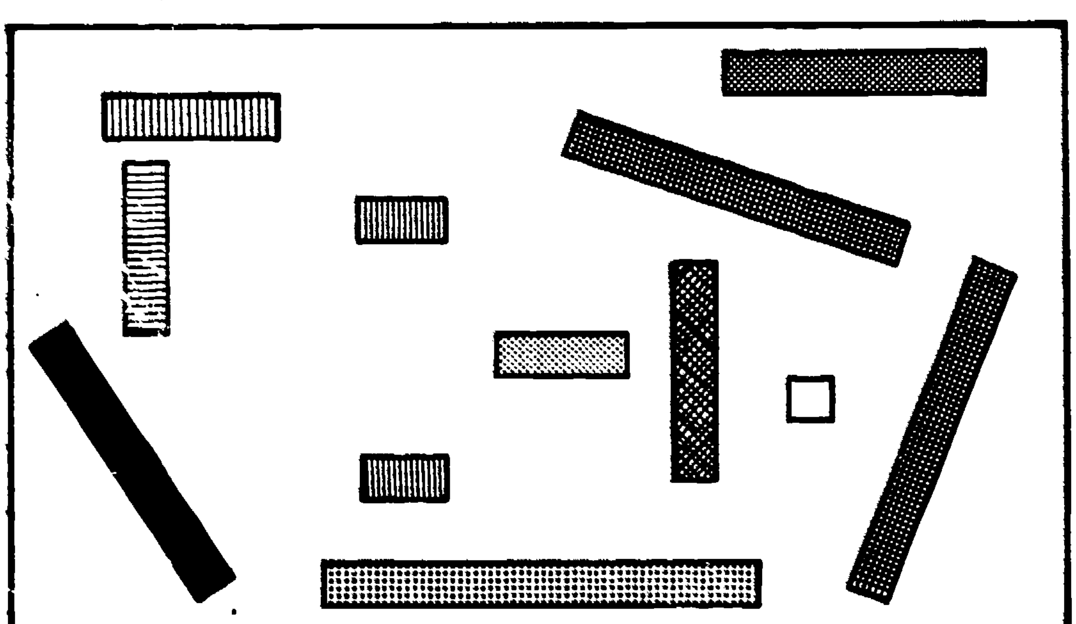
" Make a sheep-farm with pens and sheep."

" How many white rods can you hold in your hands ? "

" Make anything you like using white, red, and yellow rods."

## PART 5 MENTAL IMAGE OF THE RODS

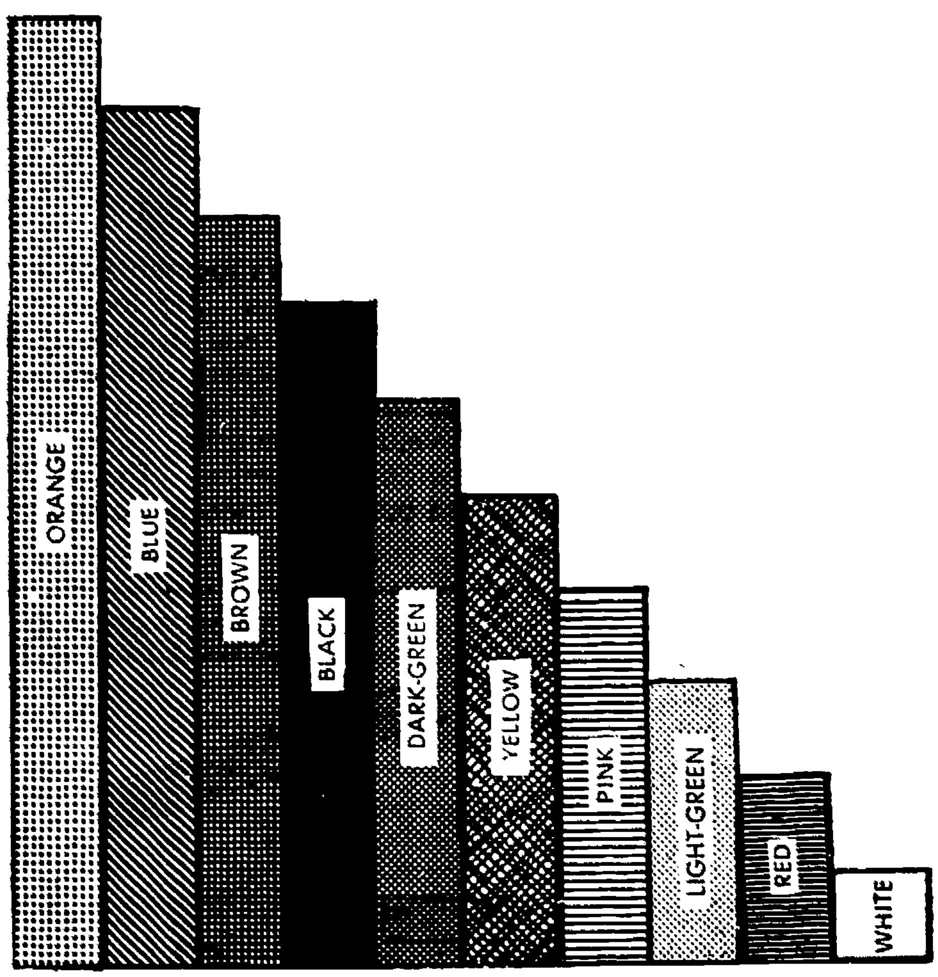
a **Individual Assignment Cards** Informal arrangements of accurate rod-outlines are marked on cards. The child looks at each outline, estimates the rod needed to cover it, checks, and if correct leaves the rod in place. He proceeds until all outlines are correctly covered. Because of the constant practice in estimation, this is a very valuable exercise. Careful planning of cards ensures a wide experience. The skill with which the exercise is carried out is the measure of the child's progress.



b **Estimation Exercises**

i " Hold your thumb and forefinger far enough apart so that a black rod will fit in exactly." A check is then made with an actual rod. This can be developed into a pairs game, where children test each other and keep the score of the number of times they were correct. (There is less likelihood of movement if finger and thumb are "anchored" on table edge, and the other hand is used for checking with a rod.)

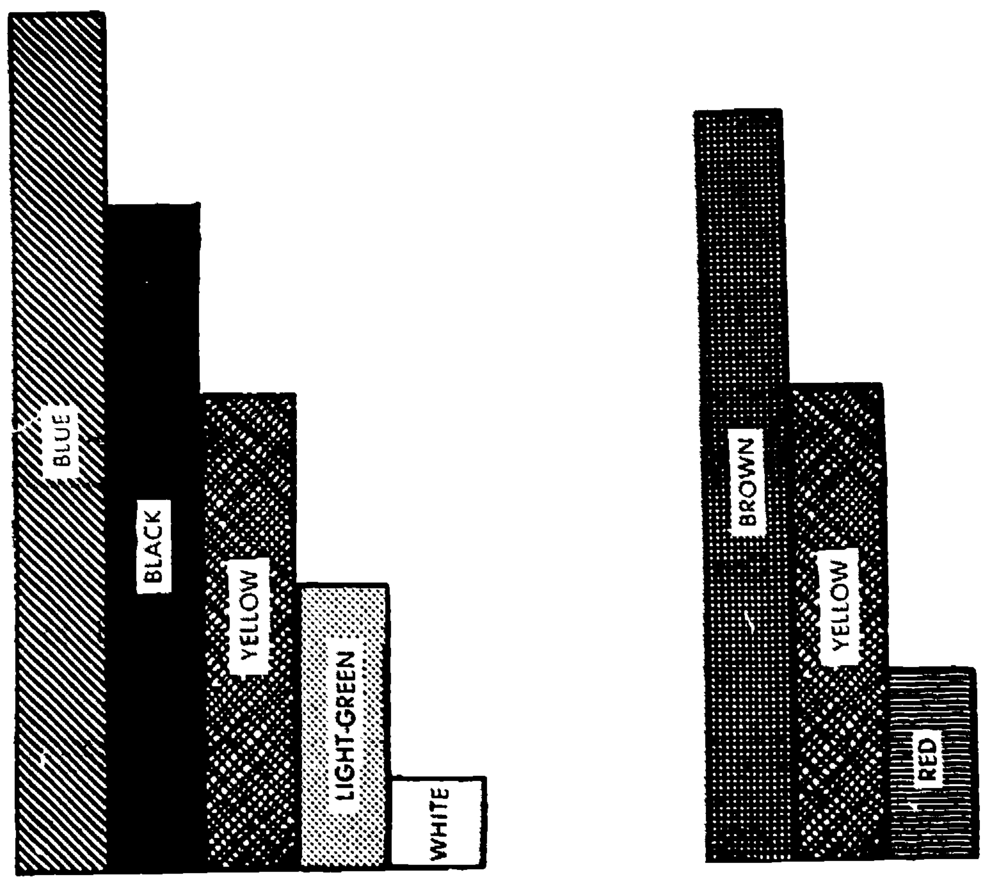
ii Children record the estimated length of, say, the light-green rod, by marking on their paper where they consider the rod should start and end. This attempt is checked against the actual rod, and the child tries to better his score each day.




c **Staircases** Mental images are aided by the study of the graded and complementary relationships of the rods, and these are well illustrated with various staircases.

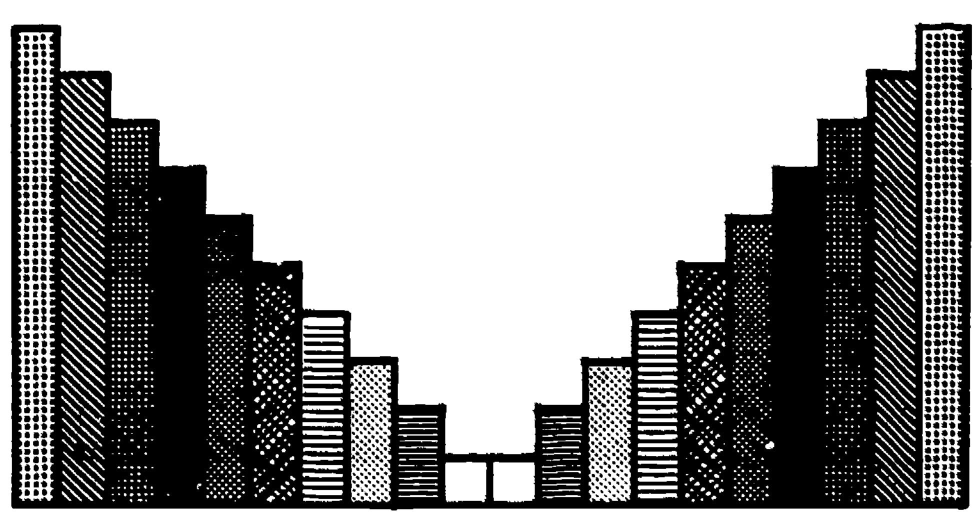
i The most simple one is that in which children commence with the longest rod, and finish with the smallest rod, or vice versa.

ii This may be varied by making the difference between the steps equal to: the white rod, the red rod, the light-green rod, etc.



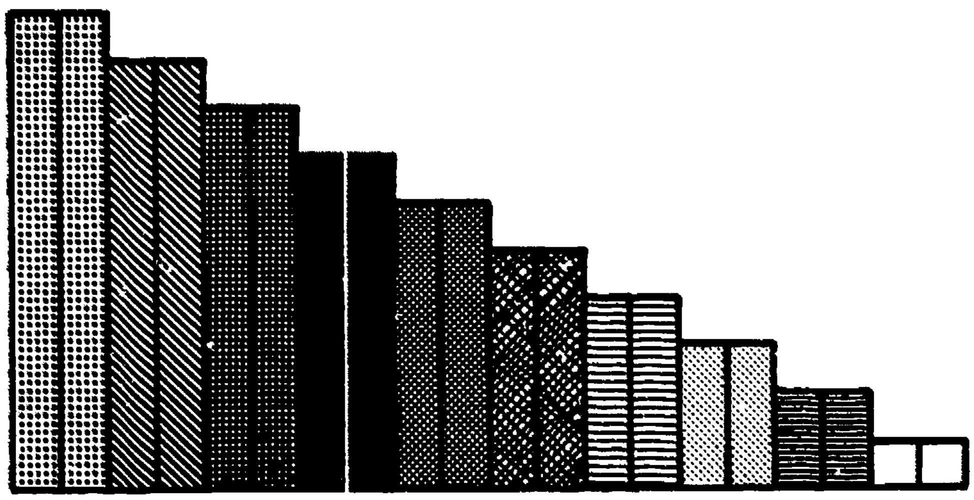

Children may emphasize the difference by making "a white carpet", "a red carpet", etc. on the stairs.

iii Children may join two staircases "standing up" to form a new pattern—large down to small, then up to large again.



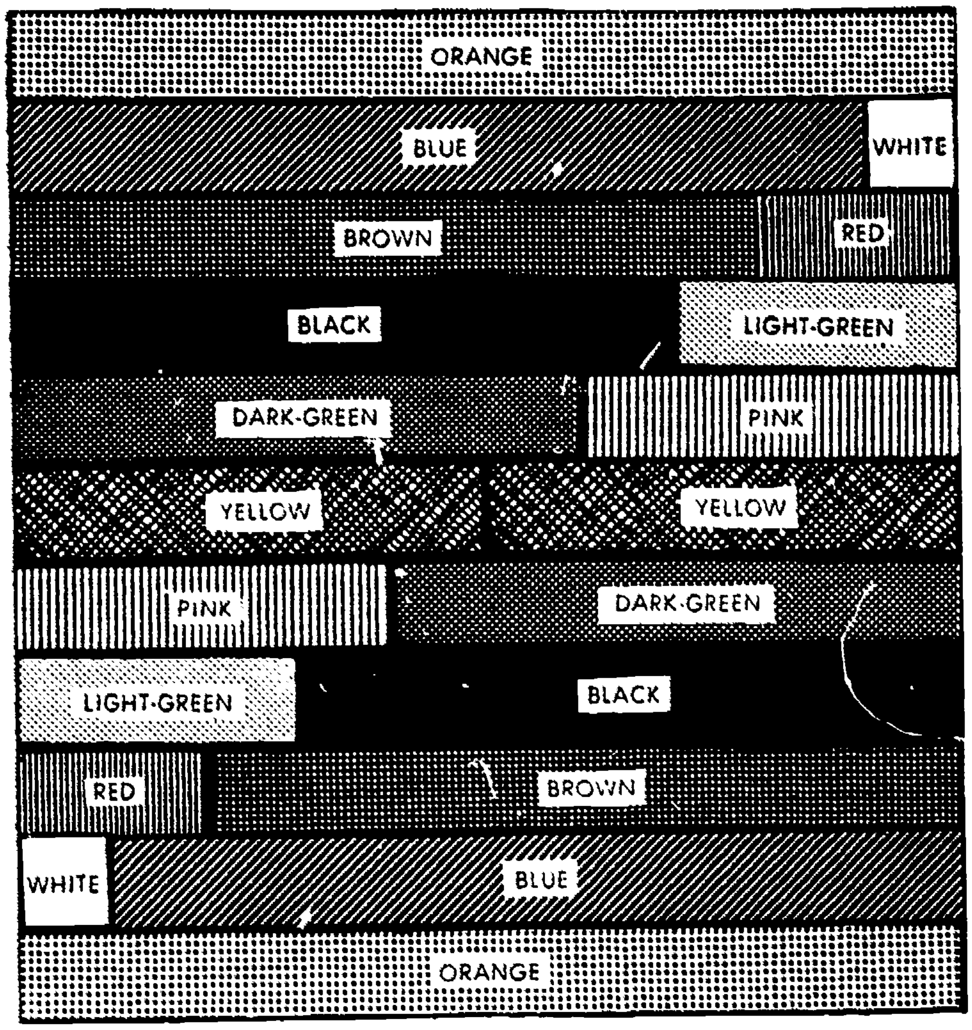
iv Double-rod staircases

These begin with two orange rods side by side and end with two white rods side by side.



v Complementary-paired staircases

One staircase is inverted and joined to the other to form a " mat ".




All these games are focusing attention on the relative sizes of the rods.

**d Touch Games**

These have been built upon the three games included in the Developmental Steps of Stage 4, and teachers will be a... to see even further adaptations to supply variety. The games chosen should be introduced by the teacher, and then allowed to become pupil directed.

i Children are chosen to stand facing the group, each with three specified rods held behind his back. While the group recites the rhyme, the children in front feel their rods without looking, and attempt to produce the rod that was named by the teacher or the child.

" Three little rods we can't see,
Would you show the (blue) to me ? "

If these children choose replacements for themselves, the game becomes pupil directed.

ii One or two of each rod are in a draw-string bag or an enclosed box, " the lucky dip ". While the rhyme is being said, the chosen child feels inside with one hand to select the colour named by the teacher or the child:

" Have a lucky dip to-day,

And see what luck will come your way.

Reds and yellows, greens and blues,

Here's the rod that you must choose." . . . . (Brown).

If the incorrect rod is produced, a new player is chosen by the group leader; if the correct rod is shown, the player chooses a replacement.

iii The child faces the group as the teacher or another child drops one rod into his hands, which are behind his back. The child has to identify the rod without seeing it, and without others for comparison, as the rhyme is spoken.

" Eeny, meeny, miny, mo,

Put your hands behind you so.

Feel carefully and tell us true,

The coloured rod I give to you."

If children are allowed to choose their own replacement, and a child is appointed to drop the rods into their hands, this game becomes self-directed also.

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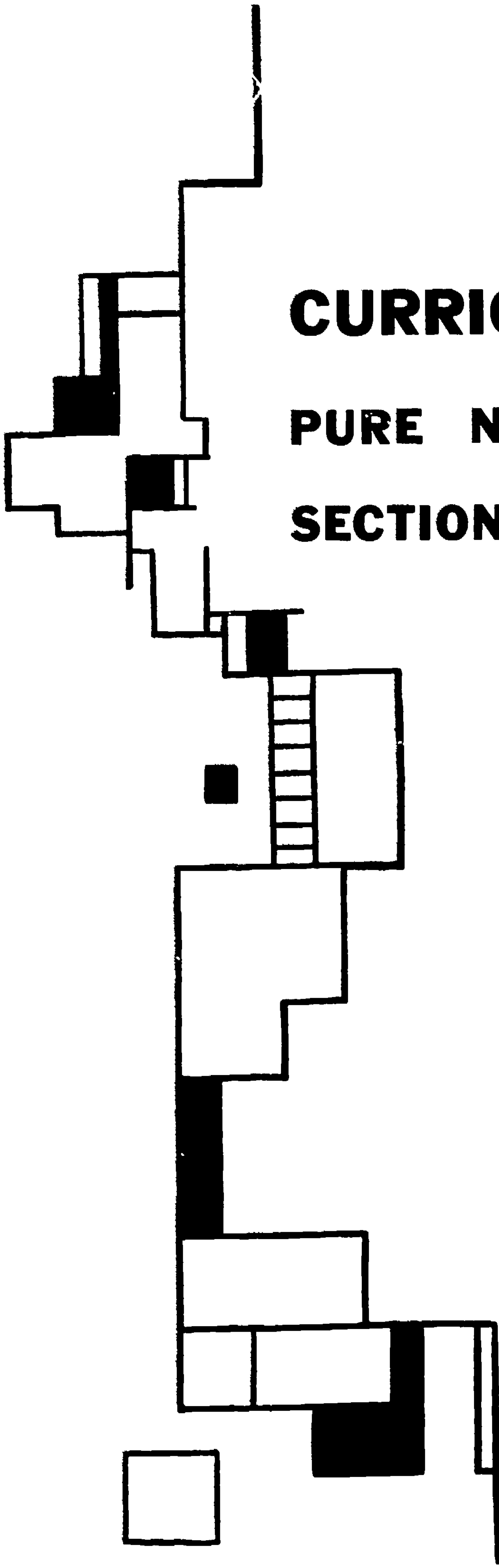
# CURRICULUM GUIDE

## PURE NUMBER

## SECTION B





EDUCATION DEPARTMENT OF VICTORIA

# CURRICULUM GUIDE
# PURE NUMBER COURSE

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## Section B—The Study of Basic Mathematical Ideas in Terms of Colour



**Education Department of Victoria**

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# CURRICULUM GUIDE
# PURE NUMBER COURSE

## Section B—The Study of Basic Mathematical Ideas in Terms of Colour

### AIM

This section aims to develop an understanding of the basic mathematical ideas of equality, cardinal number, and the nature of the four operations, and to express these ideas in the language of colour.

### COMMENTS

1. **Success in this section is vital. To a very large extent, the quality of the work done in this section will determine the quality of the work that will be done in the following sections.**

Hence for this section above all others it is essential to insist on the principle that no child be asked to tackle a new concept until he has mastered all the ideas necessary to understand it. This is a principle dictated, not by any fad of method, but by the very nature of mathematics.

2. The aim of this section has been given as the development of an understanding of certain basic ideas. Obviously it is not possible to teach the section effectively unless there is perfect clarity about the ideas to be studied. These ideas are :

CARDINAL NUMBER

Number considered as a whole, as an entity in itself.

EQUALITY

The fact that, as long as the totals on each side of an equation remain the same, the numbers of which the equation consists may be arranged or rearranged, increased or diminished, without disturbing the situation of equality.

ADDITION

The operation of combining or " putting together " numbers.

SUBTRACTION

The operation of finding the difference between numbers.

MULTIPLICATION

The operation of combining (i.e. adding) equal numbers.

DIVISION

The operation of dividing a number into a set of equal parts.

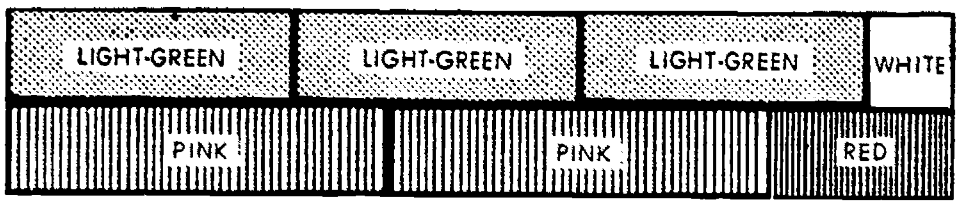
(In these definitions the word " number " has been used for the sake of convenience. In this particular section, of course, the rods are referred to by their colour names and are given no numerical values.)

A full treatment of these ideas is given in the stages that follow. What must be realized is that they, the ideas, are the course of study. **If they are understood, the section has been mastered; until they are understood, the section is incomplete.**

3. An understanding of these ideas should be on two levels. It is possible for a child to know what addition is and yet be very limited in his ability to add. He may know that addition is " putting numbers together " but, in fact, be unable to combine them. This child's concept is at the level of **definition**—he knows the meaning of the concept though he cannot use the concept.

There is obviously a need to go beyond this level. A child must not only know what the ideas imply, he must also be able to use them; he must not only know that addition is the combination of numbers, he must also be able to combine numbers. He must reach the level of **operation.**

A clear idea of what is involved in the term " operation " can be gained by looking at an example of a technique common in this section. The child is given a pattern such as




and is asked to read it in terms of addition, subtraction, and multiplication. A very small sample of readings would be:

Light-green + light-green + light-green + white = pink + pink + red.

White + light-green + light-green + light-green = red + pink + pink.

White + 3 light-greens = 2 pinks + red.

Red = 3 light-greens + white — 2 pinks.

Here the child is using his understanding of the operation to organize and manipulate equations. In each equation the relation between the rods is altered. If we look at the equations we find that

the rods have been combined by simple addition,

the rods are again added but the order of their addition is changed,

the rods have been combined by using multiplication and addition,

the rods have been combined by using addition, subtraction, and multiplication.

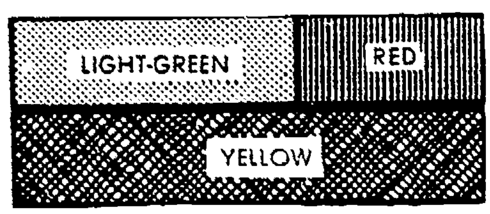
Thus the child has used his understandings of addition, subtraction, and multiplication, and, of course, equality to organize and rearrang equations. He is putting his understandings to use, he is **operating** with them.

Before leaving this section the child must have understanding at the levels of both definition and operation. Lack of understanding at either level renders the work incomplete. A child whose understanding is confined to definition, who knows what addition is but cannot add, is intellectually hobbled. A child whose understanding is at the level of operation but not at the level of definition (who has, for example, been taught to perform the process of multiplication without realizing that he is adding equal numbers) is being treated as a performing bear who has learned a meaningless trick.

4. It is a considered and deliberate assumption that a child who, fully aware of what he is doing and without prompting from a teacher, can create and manipulate equations in the manner shown has a thorough mastery of the ideas being studied. It is the development of this mastery that is the guiding principle of the whole section, that determines what is taught, the method of teaching it, and the techniques used to test it.

5. No numerical value is attached to the rods during this section, the rods are referred to solely by colour names. (For this reason the section is often referred to as the **colour section.**)

In order to make clear the reason for retaining the colour names (and it is of the utmost importance that they are retained) it is necessary to study this pattern :




In reading the pattern in this way the child sees four ways in which unit, the pattern may be read, in terms of addition, as :

$$3 + 2 = 5$$
$$2 + 3 = 5$$
$$5 = 3 + 2$$
$$5 = 2 + 3.$$

In reading the pattern in this way the child sees four ways in which the combination of the "three" and the "two" can be expressed.

It is obvious that if mathematical statements such as the four above are to be fully understood, the child must have mastered two types of idea :

a **The Mathematical Idea :** the signs + and = are symbols that represent the concepts of addition and equality. Unless a child understands these concepts he cannot understand what the statements mean.

b **The Numerical Idea :** the child must also understand what it means to say "two" and "three" and "five", i.e. his concept of cardinal number must be fully developed.

Clearly, if reading of the type illustrated is to be done, both mathematical and numerical ideas must be fully understood.

It is possible, however, to read the same pattern in a different way:

Light-green + red = yellow.
Red + light-green = yellow.
Yellow = light-green + red.
Yellow = red + light-green.

If this is done the child is able to see, just as he was with the first reading, the four ways in which the red and the light-green rods can be combined to equal the yellow rod—the mathematical ideas of addition and equality are not altered.

On the other hand the cardinal number ideas have been completely eliminated and the colour names substituted for them. This substitution is vital. It enables the child to concentrate solely on the mathematical ideas. Any difficulty that he had with number is removed, and the child is no longer trying to master two ideas, the mathematical and the numerical, at the one time.

Thus the function of colour in this section is to provide a substitute vocabulary. It does not, as is sometimes alleged, constitute a bondage —it creates a freedom. It enables the child to master one thing at a time instead of struggling with two things at once.

6. Before discussing the section in detail it is necessary to stress again the importance of its contribution. Because of the use of the colour vocabulary the child is able to concentrate his whole attention on basic mathematical ideas. The aim of the stage is to give the child a complete understanding of these ideas, an understanding at the level of definition and of operation. It should not be assumed that an apparent facility with the use of mathematical terms such as " equals ", " plus ", and " minus " necessarily indicates a full understanding of the ideas. Therefore at every stage in this section, and also throughout all the sections that follow, there should be frequent reference to the meanings of these terms. **Because understanding is the basis for all future work, success in this section is vital. Because success is vital no child should leave the section until he has mastered all the ideas studied in it.**

## STAGE 5

### AIM

To continue the study of ordinal and cardinal number.

### ORDINAL NUMBER

#### COUNTING

1. By the end of this section, Section B, the child needs the ability to count from one to ten. He needs this ability because at the beginning of the next section, Section C, he is to apply these numerical values to the rods.

2. It will be remembered that the ability to count orally (though, of course, not to write down the figures) from one to ten was developed by the end of Stage 1 in Section A. Thus the child begins Section B with as much counting ability as he needs to have at the end of the section.

Clearly, therefore, the main aim of this section is to give the child sufficient practice in counting to ensure that he does not lose his skill. There is, strictly speaking, no need to extend the child's counting ability—though, of course, there can be no objection to extending it further if the child is able and the teacher desires.

3. However, irrespective of whether it is decided to consolidate the child's skill or extend it further, counting activities must continue side by side with the work using the Cuisenaire material. (The activities for counting given in Appendix 1 Part.1 may be of assistance to the teacher.)

ORDINAL NUMBER

The work of Stage 2, developing the idea of ordinal number as a number in a series, should be extended throughout this section in conjunction with the counting work.

RECOGNITION AND WRITING OF FIGURES

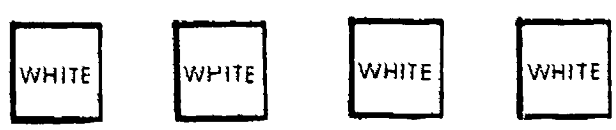
During Section A the child will have seen most of the figures from 1 to 10 and undoubtedly will be able to recognize many of them, and perhaps relate them to the work in grouping. With the writing of figures, current practice as outlined in the handwriting course tends to be to postpone the more difficult mechanical skill of writing until the child has reached an age approaching six years. Therefore it is suggested that the majority of children might well postpone this task until some time during this section. The essential point, however, is that during Sections A and B the simple ability to count is sufficient for the child to proceed with the work. The recognition and the writing of figures must, however, be taught during these stages in order to have the skill ready for use in Section C.

## CARDINAL NUMBER

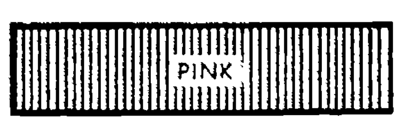
1. In the previous section the study of cardinal number was proceeding in two ways. In the first place there was the work in Stage 3 designed to give the idea of number as a group. It was mentioned then, and should be noted now, that this work continued side by side with, though separate from, the main work of the present section.

2. The concept was also approached from a different direction. The nature of this approach can be seen by an examination of the "touch games". It was pointed out that these games were designed to give a child a clear image of the rods so that he could work with them more efficiently. They have another value also—they assist in the development of ideas of cardinal number.

It will be remembered that the final concept of cardinal number is that a number is thought of, not as a group of separate objects, but as a whole, e.g. four is not thought of as




but as—




The games are valuable because they focus the child's attention on the fact that he has to distinguish between wholes of different sizes. They accustom him to working with wholes, thus preparing him for cardinal number.

3. Therefore it may be said that the Introductory Section approached the study of cardinal number in two ways :

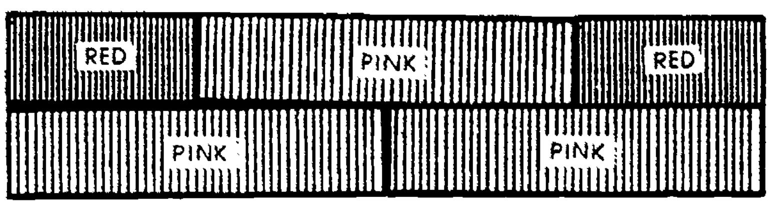
a Through the work in Stage 3 it began to develop the concept of number as a group.

b Through the work in Stage 4 it prepared for the view of number as a whole by getting the child used to working with wholes.

4. It remains now to consider the contribution of this section. Like the previous section it takes both approaches :

a As mentioned before, the work begun in Stage 3 continues side by side with the work outlined for this section. Thus the concept of number as a group is fostered.

b This section continues also the unconscious preparation for cardinal number begun in the last section. Whether a child reads this pattern




in terms of addition, subtraction, multiplication, or division he is manipulating wholes. Thus when, later, he begins to read it using numbers, he sees each number as a separate whole. To think of addition, for example, as a counting on of units would be contrary to his whole training.

5. It is clear that, if the contribution of this section to the development of cardinal number is of the kind outlined, there is no need for a separate list of activities. The concept of number as a group is developed by continuing the activities outlined in the previous section. The concept of number as a whole is prepared for by the reading of patterns and by other work done during the course of the section.

6. Though there are no activities specific to this stage, it has been given this separate treatment to draw attention to the importance of the " touch games " and of the conditioning effect of the activities of this section. It is, as it were, a teacher's stage not a child's. As always, it is essential that the teacher be aware of the aims and the effects of the activities the children perform.

# STAGE 6

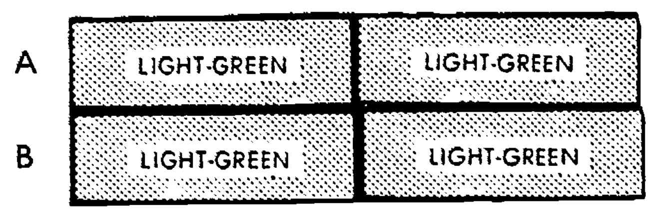
**AIM**

To develop an understanding of the concept of equality.

**NOTES ON AIM**

1. The concept of equality is basic to all mathematical work. For example, $2 + 2 = 4$ is a mathematical sentence which means that a whole, two, combined with a whole, two, equals a whole, four. It is not possible to understand the meaning of this sentence unless we have thoroughly grasped what is meant by the word " equal ". The same knowledge is needed to understand what is meant by $2 \times 2 = 4$, or $4 - 2 = 2$, or $4 \div 2 = 2$. An understanding of equality is a prerequisite for an understanding of the four operations. In fact, no matter how complicated our computation gets, no matter how abstruse the ideas we are working with, we must ultimately return to a situation of equality.

2. It is important, then, to know what is meant by the word " equal ". In order to make this clear let us look at this pattern :

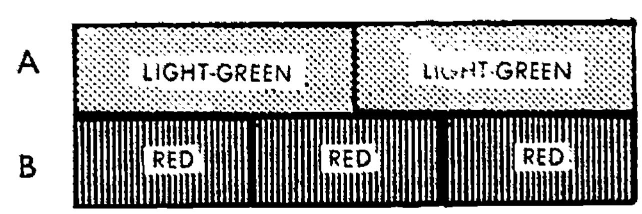



Two statements may be made about it :

a The rows are similar, i.e. Row A and Row B consist of two light-green rods placed end to end.

b The rows are equal, i.e. the length of each row is the same.

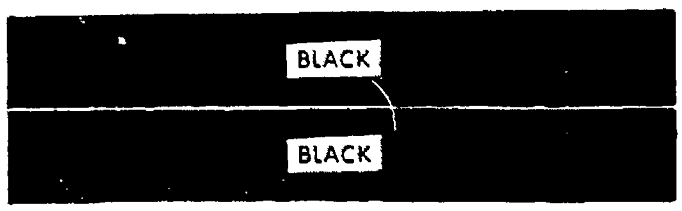
If, however, we look at this pattern



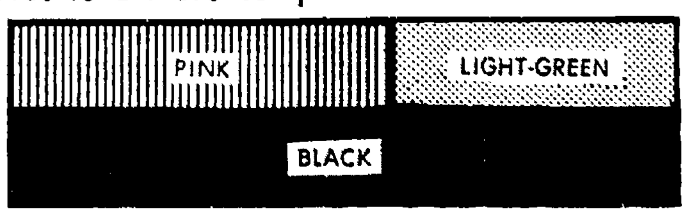

we are not able to say that Row A is the same as Row B, for one is composed of two light-green rods, the other of three red rods. Yet we may still say that Row A = Row B, i.e. each row has the same length.

Thus all that is required for equality is that the totals are the same. When we say that $2 \times 3 = 3 \times 2$ we are saying that, although $3 \times 2$ is not the same arrangement of numbers as $2 \times 3$, the two expressions have the same total. Because this is what is meant by " equal " we may say that $3 \times 2 = 2 \times 3 = 1 + 4 + 1 = 1 \times 5 + 1$, for all these expressions have the same total, although the numbers used in them are arranged in different ways.

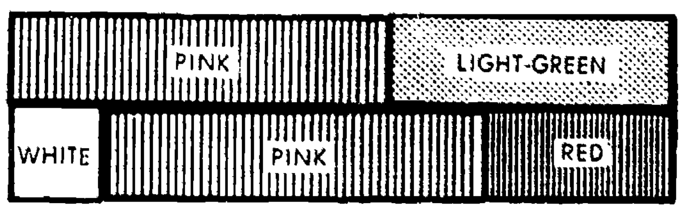
3. This idea is presented to the child through a series of exercises that present increasingly complex examples of the basic idea. From a simple situation such as



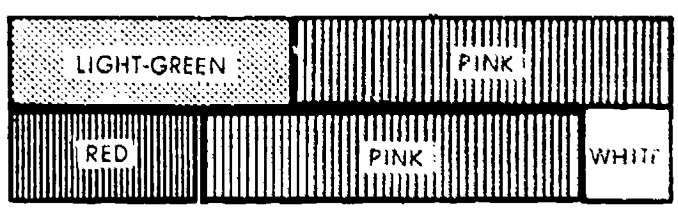

the child moves to a more complex one:




The idea is shown in another way when the bottom row is broken up.




The child can then be led to realize that the rods in this last pattern may be retained and combined differently without disturbing equality.

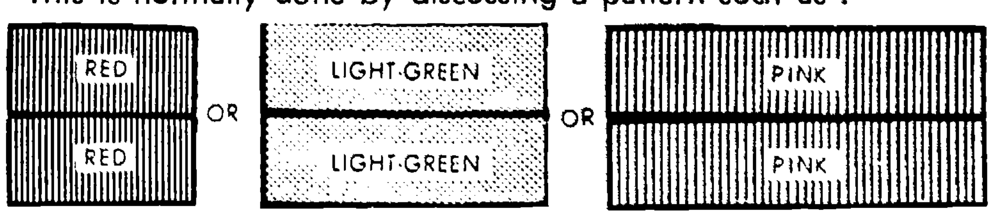



**DEVELOPMENTAL STEPS**

The steps outlined below illustrate a variety of types of equality. It is not important that the child should cover the steps in the order listed. Very often, in fact, this is not possible. But it is important that he should cover **all** the steps, and meet many examples in each step, because the aim of the stage is to give the child a wide experience in the various types of equality.

1. **Introduction of the Word " Equal "**

This is normally done by discussing a pattern such as :




The child realizes that, in each of these patterns, the rods have the same colour, are of the same length and width, are, in fact, identical in all respects. The word " equal " is introduced by the teacher to describe this situation.

Concentrated attention should then be focused on the word to ensure that the child knows what it means, e.g.

"Show me a rod equal to this rod . . . to this rod . . . to this rod . . ."

"Is this rod equal to that one? . . . to that one?"

"Show me any two rods that are equal."

"Show me two rods that are not equal."

"I am holding up five rods. Only two of them are equal. Which are they?"

"Does this rod equal that one?"

"What do you mean by 'equal'?"

It is most important that the child should be able to answer this last question. The word "equal", as introduced, means similar in length, in width, in colour, in all respects. Thus the full concept requires from the child an understanding of three dimensions.

If the child's familiarity with the material indicates that he realizes the constancy of its cross-section, then length or the term "as long as" could be accepted as a reasonable indication of understanding of equality.

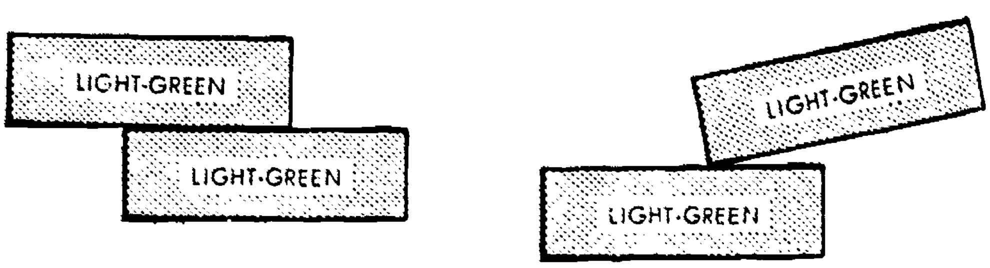
The concept, however, can be extended by discussion of the materials to be found around the class-room. The understanding of "equal in weight", through use of a balance-scale, "equal in size", through observation of window-panes and cupboard doors, "equal in capacity", through use of cups and tins, and "equal in length", through observation of pencils and sticks of chalk, can further the idea.

Finally the introduction of the term "unequal" and the use of the material to illustrate this can develop the concept. (Refer to Step 11.)

Before any further work is done the child should be completely master of the word "equal", he should be able to give a meaning for it, to understand it when used by others, and to use it correctly himself.

2. **a = a**

When establishing the meaning of the word "equal" the idea of equality used was of the type "a = a", i.e. a rod is always equal to itself. In that step, however, the rods were always placed in the position illustrated above—not in positions such as:




It is surprising to find that, as soon as the position of the rods is altered so that their equality is not immediately obvious, the child begins to doubt the fact of their equality. He has not, as yet, a grasp of the concept confident enough to retain the realization of equality unless the equality is apparent.

Therefore this step aims to show the child that if two rods are equal, a change in position cannot affect their equality.

It has been found useful here to take two equal rods and work as follows:


"What can you say about these rods?"

"They are equal."

"Why?"

"They are the same size." (Or other interpretations as given.)

"Can I change their size?"

"No."

"Do I change the size of the rod if I put it here?" (Moving one to a position on a table, chalk ledge, or elsewhere.)

"No." (If doubt exists, a check should be made.)

"If they are still the same size, can I say they are equal?"

"Yes." (Or "What is the word we use to say they are the same size?" "Equal.")

Proceed by putting rods in varying positions, high, low—asking "Are they still equal?" and gradually returning to positions such as those shown at the beginning of the step.

3. **a = b + c**

The child now moves to a more complicated idea.

"Put end to end a pink rod and a light-green rod."

"Can you find a rod equal to these two together?"

"What is it?"

"Black?"

Read your pattern. "Black equals a pink and a light-green."

"What do you mean by equal?"

"They are the same length."

"What are the same length?"

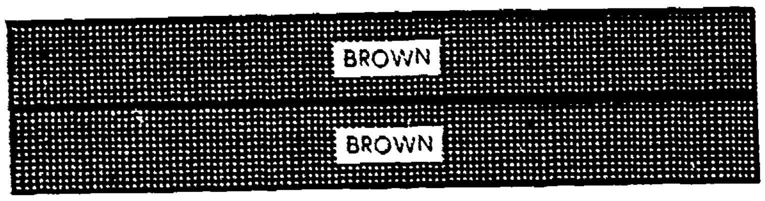
"The two rows."

"Are the rows the same colour?"

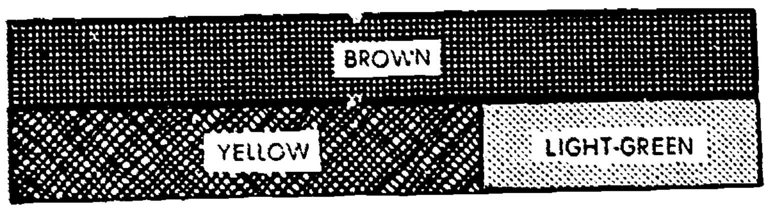
"No."

Put out a blue rod. "Could you find two rods that end to end are equal to the blue? Tell me the two you found, and the two **you** found." The reading should be complete, i.e. "A red and a black equal a blue." The probing question to test the child's understanding of equality should be used continually.

Another useful technique involves the substitution of two rods for one. "Show me two rods that are equal."




"Can you remove one of them and put two other rods in its place? Read your pattern."



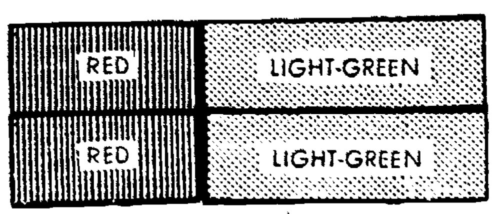

Continue the probing for understanding.

4. **a = b + c + d**

Using the techniques employed in the previous step, the teacher leads the child to realize that it is possible to have one rod equal to three rods, to four rods, to five rods . . . As before, all the rods are used to illustrate these examples, which are shown in as many different ways as possible. The child reads his patterns and the teacher constantly asks, "What do you mean?" "Why are they equal?"

5. **a + b = a + b**

Looking at a pattern such as:




the teacher may ask,

"Does the top row equal the bottom row?"

"Read what the rods say." (The child would read, "Red and light-green equal red and light-green.")

"Can you make me another pattern like this one?"

"Does the top row equal the bottom row?"

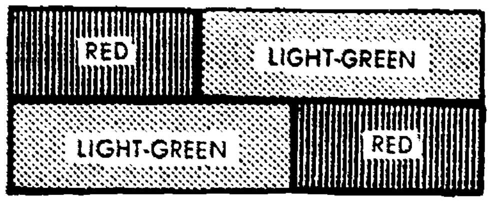
"What do you mean by 'equal'?"

And, as before, example after example of this particular type of equality is built up and studied.

6. **a + b = b + a**

"Read your pattern." (The child reads, for example, the pattern used above.)

" Change the places of the rods on the bottom row."




" Are the rows still equal? "

" Why are they equal? "

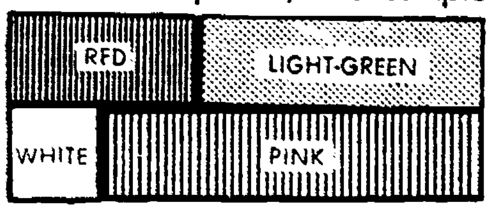
" Are these equal? " (Showing another example of this type.) " Read your pattern."

" Why are the rows equal? "

The step is continued with a large number of similar examples in which the child makes patterns, reads patterns, and the teacher asks, " Why? "

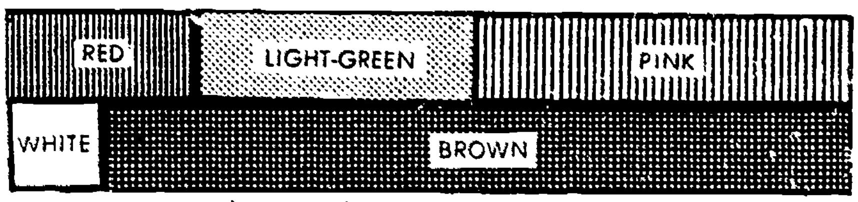
7. $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$

This is an extension of the previous step. Instead of the bottom row being the reverse of the top row, two completely different rods are substituted, e.g.




8. $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} + \mathbf{e}$

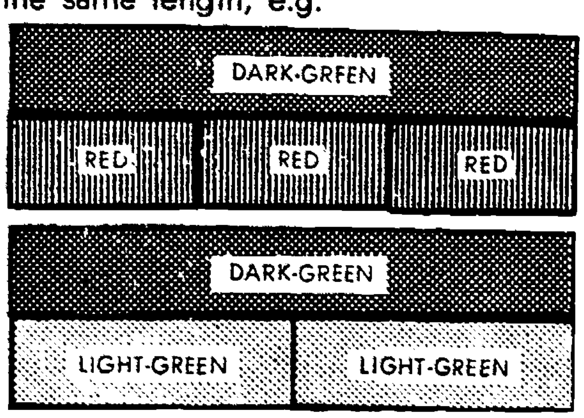
Here the child sees that two rods may equal three totally different rods :




This step is extended to show the child a wide range of examples where equality is preserved even though there is a different number of rods in each row. He sees, for example, that two rods may equal five rods, that four rods may equal six rods, or that nine rods may equal three rods. He is being led to see the vast number of possible combinations that can be enclosed in the equation form.

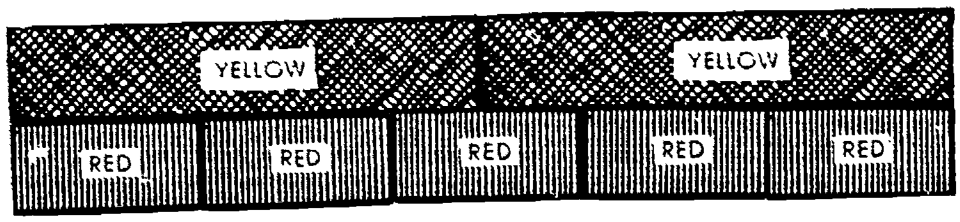
9. $\mathbf{a} = \mathbf{yb}$

Here the child realizes that one rod may equal two or three (or more) rods of the same length, e.g.



10. **xa = yb**

In this step the child sees, for example, two rods of equal length equalling five rods, each of the same length, e.g.

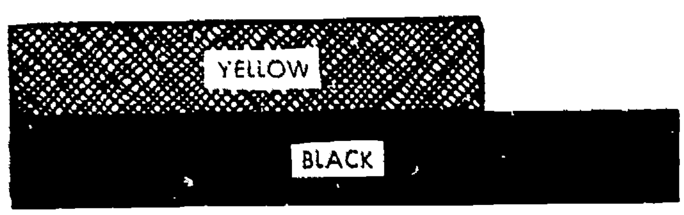



11. **Inequality**

As mentioned in Step 1 some experience of inequality is important in completing the understanding of equality. Through the understanding of difference shown in inequality we are laying the foundation for the work in complementary addition and subtraction.

" Show me two rods that are not equal . . . Why are they not equal? . . . We say these rods are unequal . . . Find two other rods that are unequal . . . Tell me about them."

When the term " unequal " is used confidently proceed thus:




" Are these rods equal ? " " No."

" Can we make the rows equal? "

" How ? " " By putting a red rod with the yellow rod."

" Do it."

**Note** The term " difference " should, if not already used, be introduced during this step.

12. **If Equals are Added to Equals the Results are Equal**

The teacher may introduce this by saying :

" Pick up a pink rod. Put below it another pink rod. Are they equal? "

" Put a red rod with the top pink rod and another red rod with the bottom pink rod. Are the rows still equal? "

" Why do you say they are equal? "

As always, this idea is then illustrated in a large variety of ways.

There is no need for a generalization in the form of the heading of this step, but through using many examples the child should understand the principle involved.

**13. If Equals are Added to Unequals, the Difference Remains Constant**

" Put a yellow rod below a brown rod . . . Are they equal?"

" No."

" Show me the difference . . . Which rod equals the difference? "

" Light-green."

(Although the rod may be used to check the difference, it should not be left in the pattern.)

" Place a red rod in the top row and another red rod in the bottom row." (Experience should be given in placing the rods at either end of the pattern.)

" Does the top row equal the bottom row? "

" No."

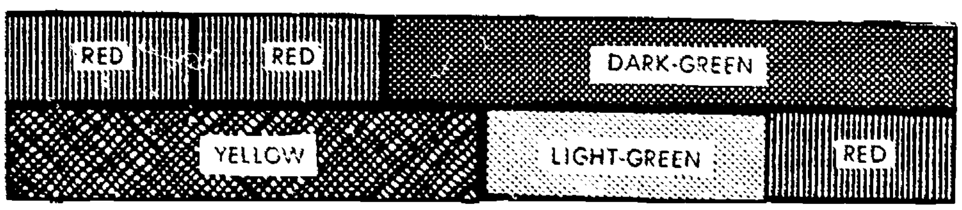
" Which rod equals the difference? "

" The light-green rod."

As in Step 12, no formal generalization is necessary, but understanding of the principle must be shown.

**14. Rearranging an Equation**

Frequently during the stage, the rods in a pattern are picked up and rearranged, and the child is asked whether the rows are still equal. From a pattern such as :

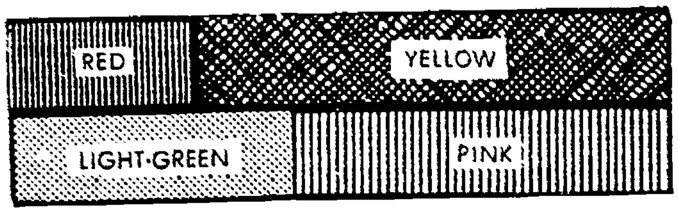



we may derive a pattern such as this :


This type of work is not confined to any particular step. It should be done frequently during the stage because, more perhaps than any other exercise, it emphasizes the fact that the rods may be changed about and rearranged without disturbing the situation of equality.

## 15. Replacing Part of an Equation

This exercise should be done as often as the previous one. A pattern such as the following is made :




"If I put, instead of the yellow rod, a pink rod and a white rod, are the rows still equal? . . . Show me."

This type of example shows that not only can rods be rearranged, they may also be replaced completely and, provided the replacement does not affect the total length, the equation remains intact.

**NOTES ON METHOD**

1. It has been mentioned previously, and because of its importance must be mentioned again, that one of the key factors determining success in this stage is the number of examples of each type of equality that the child is given. Any technique that seriously limits the amount of work the child covers defeats the purpose of this stage.

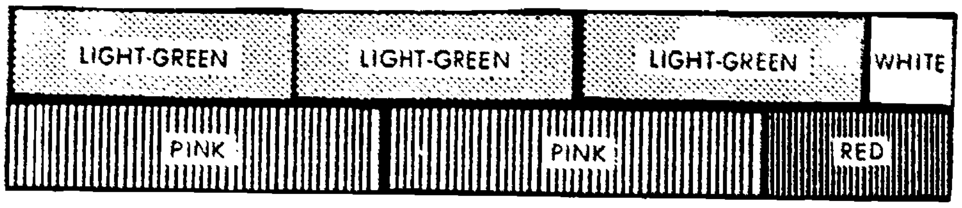
2. It is important in this stage that a large number of examples should be given ; to achieve this the work is mainly oral. There are, obviously, very real difficulties in requiring children to write using colour names or abbreviations of them. But the basic reason for oral work is not so much the difficulty of writing but the limitations it imposes—the number of examples covered is so small compared with the number that can be covered orally. Oral work, too, is far more flexible, allowing the teacher to change from example to example at will.

3. The teacher's role is that of a problem setter, not a problem solver. The child is faced with a pattern that illustrates the idea being studied, he is questioned, his attention is directed to the idea, but he is never given a pat answer and asked to memorize it. At every step he must be led, and if necessary, forced, to think for himself.

In many ways this role is not easy for a teacher. There is a temptation, especially if a child is slow, to help him through his difficulty by giving an answer. Once this has been done, however, there is always the problem—has the child really understood the idea, or does he just remember what he has been told?

4. When the three principles stated above are taken together—the need for a great number of examples, for oral work, and for the problem-setting approach, the difficulty of grade organization becomes obvious. This matter is so important that it has been treated fully in Appendix 2. It would be most advisable to consult this appendix before beginning work on this stage.

5. Ability to read the same pattern in a large number of different ways, i.e., the skill of organizing and rearranging equations, is not required in this stage. As long as the child can read a pattern well enough to tell the teacher what he has made, work can proceed satisfactorily. For example, the pattern




may be read in a large number of ways (see page 3). It is **sufficient** for this stage if the child reads it as :

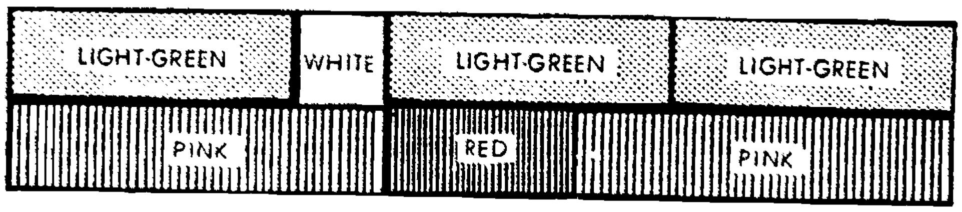
" Light-green and light-green and light-green and white equals pink and pink and red."

because such a reading conveys the idea of equality. Ability to rearrange this **mentally,** e.g.

" White and light-green and light-green and light-green equals red and pink and pink."

requires a skill of addition that has not as yet been taught.

This does not gainsay the fact that a child must realize that a **physical** rearrangement of the rods does not alter equality. If, for example, the above pattern is changed to




the child must realize that the rows are still equal and should read the pattern as

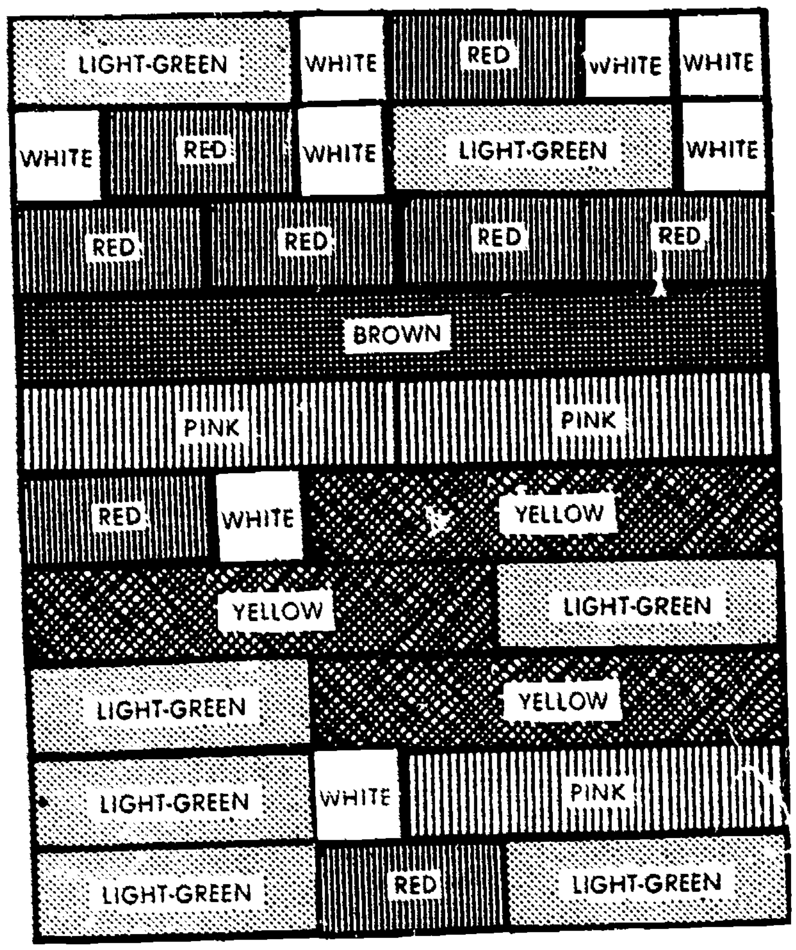
" Light-green and white and light-green and light-green equals pink and red and pink."

6. It is noticeable that in this pattern reading the word " and " is used. The word " plus " is, of course, introduced later. It is, however, not necessary for an understanding of equality at this level. Hence it is not used. There is, in fact, quite a strong argument for avoiding it. The word " plus " is strange to a child. When it is introduced stress must be laid upon it and care taken to ensure that its meaning is fully understood. If this attention were given it here, the child could be distracted from the concept of equality. If it were introduced without sufficient attention, it could be given a wrong meaning. It would, therefore, seem preferable to avoid it during this stage.

7. In this stage more than a study of equality has been achieved. In the patterns studied, situations have arisen that will later be used to introduce the ideas of addition, multiplication, subtraction, and

division. The techniques of manipulation and substitution, so widely used in the work of later sections, have been commenced. The presentation of equality through carefully graded steps sets a pattern of development from simple to complex equations which will be followed throughout the Guide.

8. The concept of equality is in the main taught from a pattern, not a "mat". The pattern, which shows two rows side by side, each equal to the other, illustrates clearly the normal equation where the left-hand side equals the right-hand side. A restricted amount of work with mats is, however, necessary. This mat, for example, shows the child something of a number of combinations that are equal.

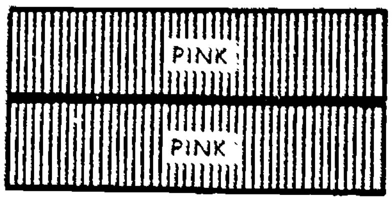



While it is advisable to introduce each step through a pattern and to do the bulk of the work in each step with patterns, it is also useful, before leaving the step, to allow the child some work with mats. This helps to emphasize the large number of ways in which each type of equality can be expressed with the rods.

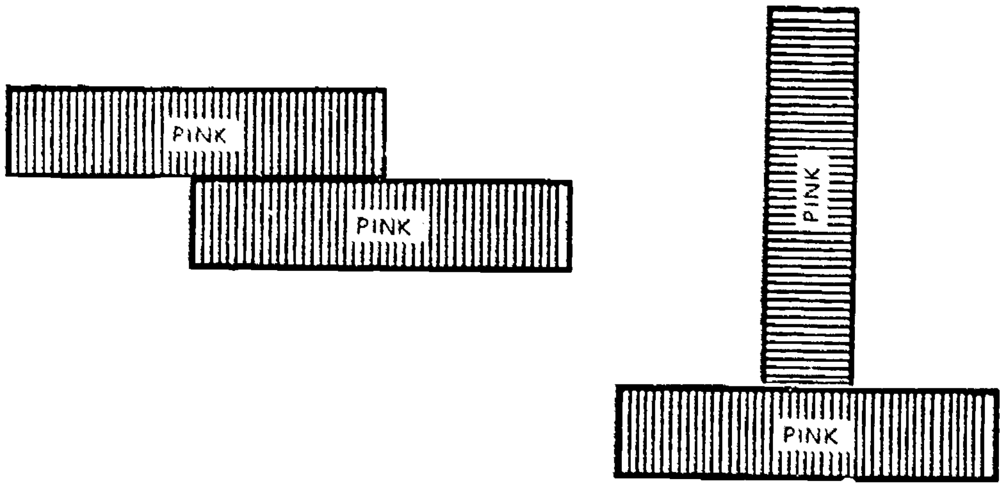
## TESTING

There is no need to hold a special test. The child's progress can be gauged by watching the way he moves through the **Developmental Steps.** If a quick test is thought necessary, however, a series of exercises such as those listed below may be used.

1. The child makes this pattern :




and is asked, " Are the rods equal? . . . What do you mean by ' equal ' ? " (At this stage the child would mean that the lengths were the same.) " If the rods are arranged in these ways are they still equal? "




This ensures that the child realizes that the situation of equality is preserved in all positions.

This step is repeated with different rods to ensure that the child realizes the universality of the idea.

2. Questions such as :

" Can two rods equal one rod? . . . Show me with your rods."

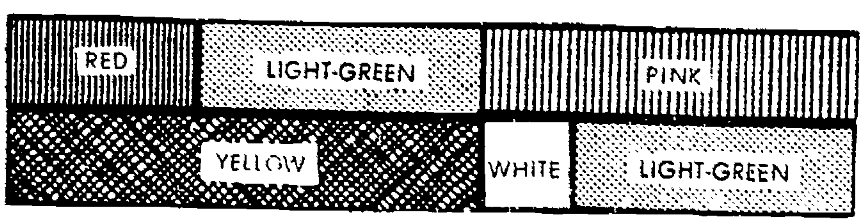
" Can two rods equal three others? . . . Show me ! "

" Can you find three rods equal to these four? "

" What do you mean by ' equal ' ? "

" Show me two rods that are unequal . . . Why are they unequal? "

3. Make this pattern :




" Does it matter if we put the red where the pink is and put the pink where the red is? " (Rearranging an equation.)

" If I were to remove the yellow rod and put in its place a red and a light-green rod would the rows still be equal? " (Replacing part of an equation.)

# STAGE 7

## AIM

To develop an understanding of the operation of addition.

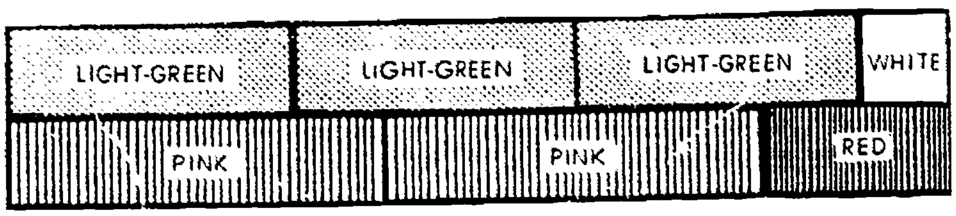
## NOTES ON AIM

1. Addition is the operation of combining wholes. If a child says,

"Black + red = blue",

he means that a black rod combined with a red rod is equal to a blue rod. The word "plus" is the operative word; and the child who realizes that it means "combined with" (or "put together with" or "added to" or "joined with") has achieved understanding at the level of definition. He knows what addition is.

2. Understanding at the level of operation is also essential. As well as knowing what addition is the child must know **how** to add, how to combine wholes. A clear idea of what is meant by the combination of wholes can be gained by studying this pattern :




In terms of addition this pattern may be read, among many other ways, as :

Light-green + light-green + light-green + **white** = pink + pink + **red.**

Light-green + **white** + light-green + light-green = pink + **red** + pink.

**White** + light-green + light-green + light-green = **red** + pink + pink.

Pink + **red** + pink = **white** + light-green + light-green + light-green.

**Red** + pink + pink = light-green + light-green + **white** + light-green.

A child who is able to manipulate rods in this way is able to add, i.e. he can combine groups in a large variety of ways. Thus, although he cannot do "an addition sum", although he cannot calculate, he has mastered the mathematical operation of addition.

3. The aim of this stage is, therefore, twofold. The child is required to know that addition is the combination of groups. He is required, also, to have developed the ability to combine groups in a large variety of ways. The stage is completed when both these demands are satisfied.

**DEVELOPMENTAL STEPS**

**1. Introduction of the Word "Plus"**

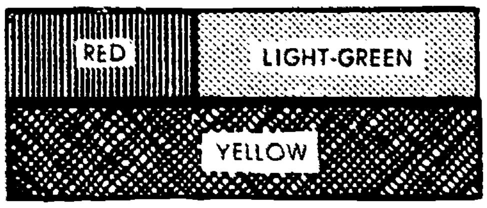
This step aims to ensure that the child has a thorough grasp of the meaning of the word "plus". A series of exercises to achieve this aim follows :—

"Show me black put together with red."

"Show me blue put together with white."

"What do pink and yellow equal when put together?"

"Read this pattern."




The child reads, "Red put together with light-green equals yellow." This type of exercise continues until the child has become used to the idea of combining rods. Such pattern reading is simply concerned with impressing this idea. The child is not expected to read the pattern in a number of different ways.

When the child performs the above exercises easily and confidently the teacher introduces the accurate and simple word "plus" to replace the cumbersome expression "put together with".

Once the word "plus" has been introduced a large number of exercises should be done to ensure that its meaning is fully mastered.

"Show me black plus white . . . What does this mean?"

"Show me black plus white plus blue plus red. . . ."

"Read for me what this pattern says . . . what this pattern says . . . what this pattern says . . ." (The child is given practice using the term "plus".)

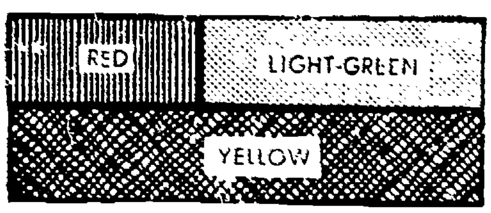
"What does 'plus' mean?"

"Make me a pattern with 'plus'." (The child may say "Black + white = brown.") "Is he right? Check with your rods."

The purpose of this step is quite obvious. Through the exercises and the questions given above, the child gains an understanding of the word "plus". The key to success in this stage, as always, is the number of examples covered by the child.

**2. Simple Pattern Reading**

Now that the basic idea of addition as the combination of groups has been established, the child takes the next step—he learns to combine groups, i.e. he learns to add. He begins with a simple pattern such as :



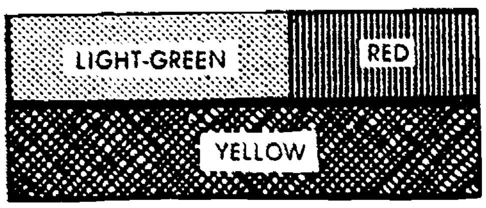
This can be read, in terms of addition, in four ways:

Red + light-green = yellow,
Light-green + red = yellow,
Yellow = red + light-green,
Yellow = light-green + red.

When presented with this pattern and asked to read it, the child may be unable to read it immediately in all four ways. His ability to combine groups has not yet been fully developed. He is not, however, told the four ways and asked to memorize them. He is questioned, he is asked to read it another way, but he is left to work out this "other way" himself. If a child, after many attempts, can manage, for example, only two ways, we may rearrange the rods to make things clearer, e.g. he may read only:

Red + light-green = yellow,
Yellow = red + light-green.

If the rods are rearranged:




the other readings are obvious,

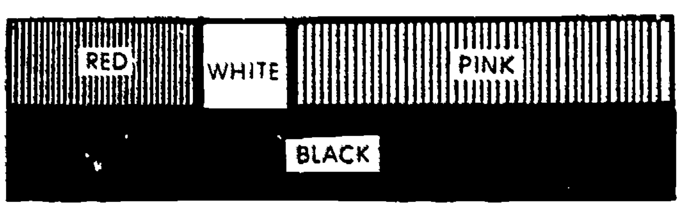
Light-green + red = yellow,
Yellow = light-green + red.

After practice with this type of work the child develops the ability to give the four readings from a fixed pattern, i.e. he is able to manipulate the rods mentally without the need for a physical rearrangement.

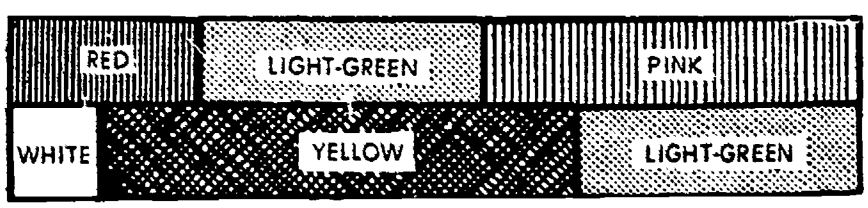
### 3. More Complex Pattern Reading

In the last step the ability to manipulate the rods using the process of addition was developed. This step aims to improve this ability by introducing patterns that increase gradually in complexity.

Steps 4 to 8 of the previous stage give an example of a series of patterns graded in order of complexity. An examination of these shows that the child begins with a pattern such as:




and concludes with one such as this:



Whatever the pattern used, however, the technique is the same, and the child is encouraged to read the rods in terms of addition in as many ways as possible. A child who has mastered the process of addition should be able to read the last pattern in ways such as:

Red + light-green + pink = white + yellow + light-green,

Red + pink + light-green = white + light-green + yellow,

Yellow + white + light-green = red + light-green + pink,

Light-green + white + yellow = pink + red + light-green,

Pink + red + light-green = white + yellow + light-green,

Pink + light-green + red = light-green + yellow + white.

Once the child is able to combine rods with the freedom and the flexibility shown in these examples the aim of this stage has been achieved.

## NOTES ON METHOD

1. This stage is begun immediately the previous stage, which was devoted to a study of equality, has been completed. As an understanding of equality is basic to an understanding of addition, this stage obviously cannot be begun until the concept of equality is mastered.

2. Three matters stressed in the Notes on Method for Stage 6 need to be stressed again. The amount of success the child achieves is in direct proportion to the number of examples he covers. Because the quantity of experience is so vital all the work needs to be oral. Because understanding is the aim, the teacher's task is to set the problems, not to solve them. He endeavours to lead the child to discover for himself the number of ways in which a pattern may be read; he does not tell him.

3. The making of mats is a useful teaching technique during this stage. It enables the child to build and read a large number of examples at the one time. It is not, however, the staple technique, and can even be dangerous. Too much attention to reading long mats can develop the habit of skimming quickly from row to row. The vitally important step of reading two rows (compare the examples given above) in as many ways as possible can be neglected. As this stage aims at developing the child's ability to manipulate the rods, pattern reading is the staple technique. It should not be ignored for a facile skipping from row to row of a big mat.

4. It is sometimes felt that during this stage it is necessary to devote a certain amount of time to each rod, that, for example, as much time should be spent reading patterns that include the blue rod as is spent reading patterns that include the black rod. In fact, however, as long as ability to manipulate an equation is developed, the actual rods used are not important. What is important is that

the patterns made and read are of gradually increasing complexity so that the child, who is beginning to become skilful in manipulating equations, is extended according to an organized and coherent plan. Thus the exercises the child does are determined not by the particular rods used but by the simplicity or the complexity of the patterns studied.

## TESTING

1. In order to test whether a child understands the meaning of the word " plus " a ser es of questions such as the following may be used :

> " Show me black plus yellow plus red."
>
> " What does pink plus red equal? "
>
> " What do you mean by ' plus '? " (The child should know that it means " joined with " or some similar term.)

2. The second aim of this stage is the ability to perform the process of addition, i.e. the combining of wholes. The best test of this is to ask the child to read a pattern. In his reading three characteristics must be present :

### QUANTITY

The child must be able to read a pattern in a great number of ways.

### QUALITY

Too much stress, however, should not be given to the number of readings. The quality is equally important. It is possible, for example, to read the pattern shown in Step 3 in many more ways than are indicated. If, however, a child supplied quickly and easily the readings given, he would have done sufficient to show that he understood what he was doing. If the readings are examined it will be seen that the white rod in the top row and the red rod in the bottom row occur in different places—the child is consciously altering the combinations each time. It is this ability to take an equation and manipulate it at will that is meant by the word " quality ".

### EASE

A vital matter to watch is the ease and speed with which the readings are given. If a child's reading is hesitant and laboured he is not fully master of the process.

The method of testing stages based on pattern reading is the assessment of the examples given by the child in terms of quantity, quality, and ease of reading.

# STAGE 8

**AIM**

To develop an understanding of the operation of multiplication.

**NOTES ON AIM**

1. Multiplication is the process of adding equal wholes. We may, for example, read this pattern


as :

Red + red + red = dark-green,

in which case we are using the addition process. We may, however, read it as :

3 reds = dark-green,

or 3 times red = dark-green,

in which cases we are using the operation of multiplication, i.e. we have combined equal groups.

2. At a stage much later than this, when the rods have numerical value, a multiplication table may be constructed. We may, for example, collect all the " twos " into the table of twos, e.g.

$1 \times 2 = 2$

$2 \times 2 = 4$

$3 \times 2 = 6$

$4 \times 2 = 8$, etc.

This organization of the facts into tables is done for the purpose of memorization. They can be learned and applied in calculations much more easily than the corresponding group of addition facts would have been. For this reason multiplication is often called " quick addition ".

3. During this stage, tables, as such. are not used. In the first place, of course, no numerical value is attached to the rods. Thus the formal table is not possible. It would be possible, however, to construct the colour equivalent of the numerical table, e.g.

One red = red

Two reds = pink

Three reds = dark-green

Four reds = brown, etc.

This, however, is not done. In the first place it is not necessary. If understanding is the sole aim (and at this stage it is) the order in which facts are introduced is utterly irrelevant. In any case, the teaching of multiplication through the formal tables prevents the child from obtaining a clear view of the function of the tables. The child who has first studied multiplication facts in a haphazard order sees, when he comes to memorize them, the purpose of the formal tables.

Their simple and logical organization makes this task easier. The child who first met multiplication in the form of the neatly arranged tables is much less likely to see their true purpose.

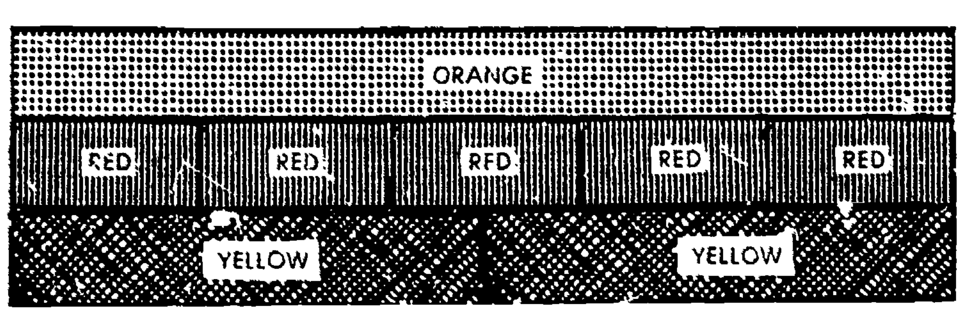
4. Throughout this stage the expression 4 × 3 will be read as "four times three". In this expression the 3 is the cardinal number and the 4 is the operator. This appears to be a natural development in the colour stage where the child puts out light-green rods, counts them, and says "four light-green rods", or, later "four times light-green". This does not deny that the same expression 4 × 3 is often read as "four multiplied by three", where 4 is the cardinal number and 3 is the operator, or, to be consistent with our example using four light-green rods, the expression is read as "three multiplied by four" or in the colour stage as "light-green multiplied by 4". The use of "times" throughout this and the following stages appears appropriate with this material, but no objection can be offered if the alternative expression is used. The essential point is that the child realizes that in multiplication one number has a cardinal value, while the other, the operator, has an ordinal value. Provided this understanding is present, the language used is not important. In addition, the child must discover the reversibility of these factors (commutative law of multiplication), and this is completed through the work on mathematical laws outlined in Section F.

**DEVELOPMENTAL STEPS**

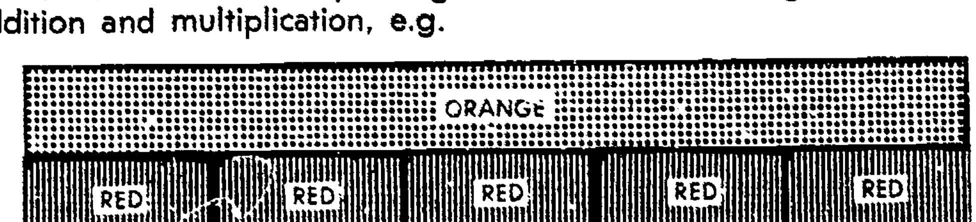
As was pointed out above, a completely haphazard study of multiplication facts would, provided an adequate number of examples was covered, be sufficient to enable an understanding of multiplication to be gained. The method of introducing multiplication that is set out below is not haphazard, but it is based on a systematic coverage of the main multiplication ideas, not the multiplication tables. It is offered as a convenient method of treating this operation.

1. **Factors**

A mat such as that shown below :




contains what may be called the "factors" of the orange rod. A common method of introducing multiplication is to work through a mat such as that shown, taking each row and reading it in terms of addition and multiplication, e.g.



Orange = red + red + red + red + red
Orange = 5 reds.

Similarly a pattern may be made with orange and two yellow rods and read as :

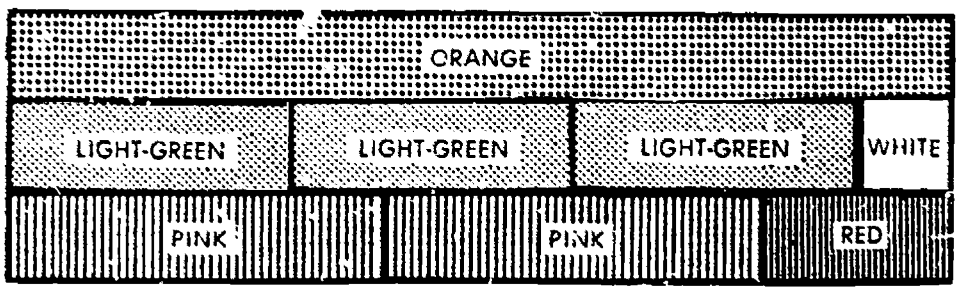
Orange = yellow + yellow
Orange = 2 yellows.

The aim here is to stress the fact that multiplication is a specialized form of addition.

This step is repeated with the " factors " of the other rods, so that, through a large number of examples, the basic concept is mastered.

2. **Mats without Factors**

The following mat of orange shows a sample of rows where factors are not used :




This step can be handled in the same manner as the previous step, i.e. the child takes two rows and reads them as :

Orange = light-green + light-green + light-green + white
Orange = 3 light-greens + white.

The aim remains the same—to stress the relation between multiplication and addition. The step is, of course, repeated using a wide variety of examples.

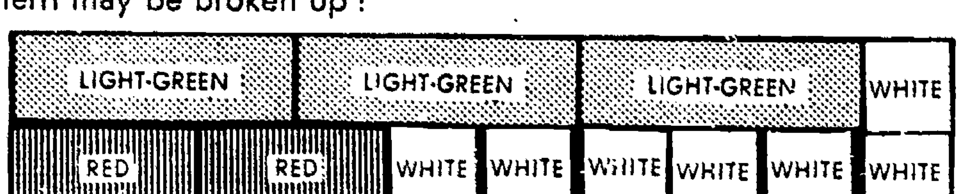
3. **Simple Pattern Reading**

In the previous steps, the child has been directed to readings that draw attention to the relation between multiplication and addition. No real pressure was put on the child to read each pattern in a large variety of ways. Thus this step consists of a repetition of the two steps, except that the child is asked to read each pattern in all ways that involve multiplication using the term " times ". The last pattern shown may, for example, be read as :

Orange = 3 times light-green + white
Orange = white + 3 times light-green
White + 3 times light-green = orange
3 times light-green + white = orange.

4. **Complex Pattern Reading**

In order to increase the complexity of readings, both rows of a pattern may be broken up :



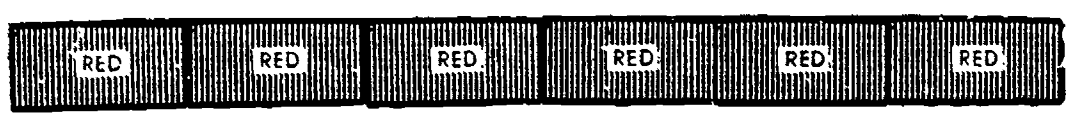
This may be read as :

3 times light-green + white = 2 times red + 6 times white,
white + 3 times light-green = 2 times red + 6 times white,
3 times light-green + white = 6 times white + 2 times red,
6 times white + 2 times red = 3 times light-green + white,
2 times red + 6 times white = white + 3 times light-green.

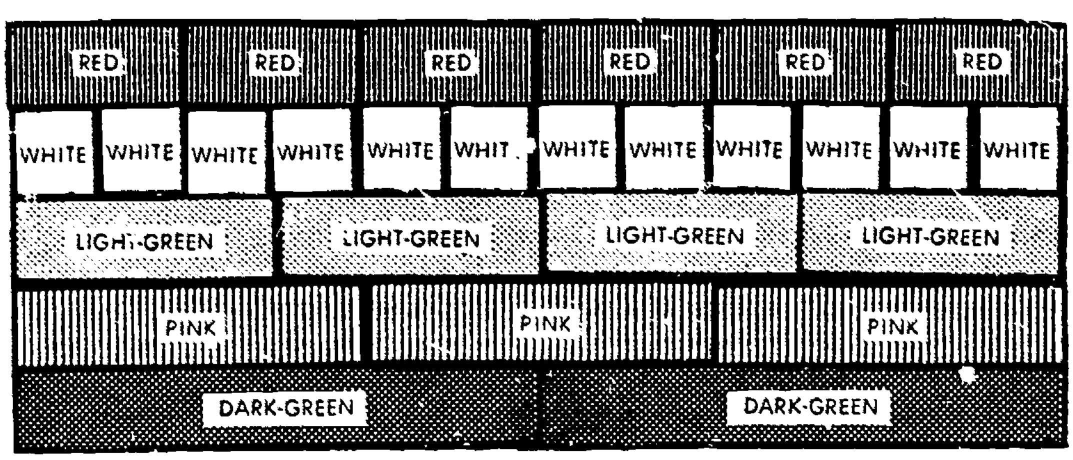
Thus by the end of this step, the child is able to manipulate the rods in terms of multiplication as freely and easily as he can in terms of addition.

5. **" Trains "**

A useful exercise that may be done at any time during this stage after the basic multiplication concept has been grasped is the making of " trains ". To start this exercise a child makes a row (or " train ") of rods of the same colour, e.g.




He then makes " trains " using other rods and finds which of the " trains " will finish exactly at the end of the first row, e.g.




The child then reads the mat his " trains " have made. The value of a wide experience in this type of work for the later teaching of factors is obvious.

**NOTES ON METHOD**

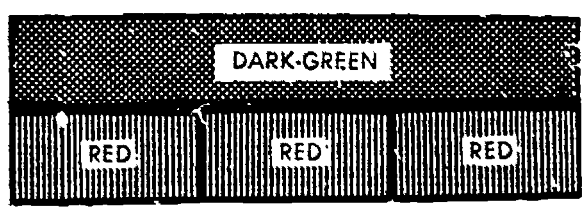
1. Multiplication is introduced after addition because it is the operation toward which the child moves most naturally. Without prompting from the teacher, he tends to say " 3 reds " instead of the more long-winded " red + red + red ". There is, of course, no mathematical necessity to introduce multiplication here. Subtraction, for example, could just as easily be introduced.

2. Because the child moves naturally to multiplication, the teacher will often find that long before the child is fully master of addition he has begun the first steps of multiplication. Thus addition and multiplication cannot, in the class-room, be separated as arbitrarily as they are in this Guide. It is important, however, that, even though both are proceeding side by side, deliberate and separate treatment of each operation should occur.

3. Though a child may build a mat and from time to time read from a mat, the basic teaching device is (as is the case throughout the whole section) the pattern of two rows. The child develops fullest mastery of a concept when he is given a pattern and asked to read it in the largest possible number of ways.

4. For purposes of convenience, it was suggested in Step 1 that the "factors" of the orange rod should be studied first, then the factors of the other rods introduced. It must be stressed, of course, that this choice of rods is quite arbitrary. As long as the child has a wide variety of readings the actual rods used (or the order in which the rods are used) are of no great importance.

5. The words used when reading a pattern in terms of multiplication vary. This pattern




is sometimes read as :

3 reds = dark-green

3 times red = dark-green.

The most obvious (and usually the first an unprompted child uses) is "3 reds". Because, later, this gives no **obvious** meaning to **the** multiplication sign in an expression such as $3 \times 2 = 6$, the term "times" should be introduced.

**TESTING**

1. Read this pattern :


The child reads this as (for example) :

3 times red = dark-green.

"Show me 2 times red."

"Show me 4 times blue."

"Show me 3 times yellow."

"Show me 5 times red."

The aim of this testing is to ensure that the child understands what is meant by "times".

2. Using the same pattern, the child is asked to read as many equations as he can. The teacher watches his reading to ensure that he can move easily from addition to multiplication, thus assessing his mastery of the relation between these two operations.

3. The final test is to ascertain the child's ability to read a pattern in terms of multiplication with the quantity, the quality, and the ease of reading outlined previously. The pattern read in Step 4 of the **Developmental Steps** shows the type of reading expected.

# STAGE 9

**AIM**

To consolidate the ability to read a pattern in terms of addition and multiplication.

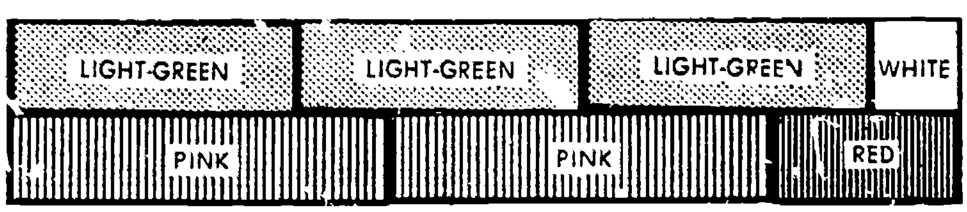
**NOTES ON AIM**

1. While it is true to say that the main aim of Stage 7 was the mastery of addition and the main aim of Stage 8 was the mastery of multiplication, it has already been remarked that, in practice, the two stages cannot be separated. A child reading patterns in the manner illustrated in Step 4 of the **Developmental Steps** in the previous stage, is, in fact, reading in terms of multiplication and addition.

2. The linking of the operations is so important, however, that it is given separate treatment in this stage to ensure that it is properly made. The stage is thus more a teacher's stage than a child's. All that is required is that the type of work the child is doing in the latter part of the previous stage be continued and developed.

**DEVELOPMENTAL STEPS**

Patterns such as the following are taken and read in terms of addition and multiplication :



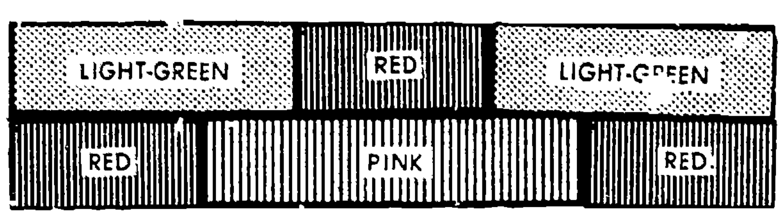

Light-green + light-green + light-green + white = pink + pink + red,

3 times light-green + white = 2 times pink + red,

White + 2 times light-green + light-green = red + 2 times pink,

Pink + pink + red = light-green + 2 times light-green + white,

2 times pink + red = 3 times light-green + white.




Light-green + red + light-green = red + pink + red,

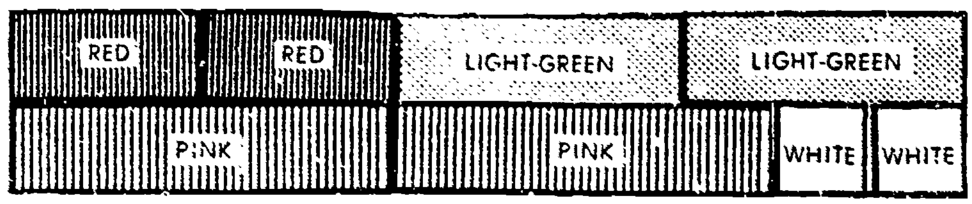
2 times light-green + red = 2 times red + pink,

Red + 2 times light-green = pink + 2 times red,

Pink + red + red = 2 times light-green + red,

2 times red + pink = red + light-green + light-green,

Pink + 2 times red = red + 2 times light-green.




Red + red + light-green + light-green = pink + pink + white + white,

2 times red + 2 times light-green = 2 times pink + 2 times white,

Light-green + light-green + red + red = white + white + pink + pink,

2 times light-green + 2 times red = 2 times white + 2 times pink,

Pink + pink + white + white = red + red + light-green + light-green,

2 times pink + 2 times white = 2 times red + 2 times light-green,

White + white + pink + pink = red + red + light-green + light-green,

2 times white + 2 times pink = 2 times red + 2 times light-green.

**TESTING**

A child who reads patterns such as those shown above with requisite quantity, quality, and ease of reading is ready to leave this stage.

# STAGE 10

**AIM**

To develop an understanding of the operation of subtraction.

**NOTES ON AIM**

1. Before introducing the operation of subtraction it is important to study the different approaches that are used to explain it. An expression such as

$$5 - 3 = 2$$

may be read as :

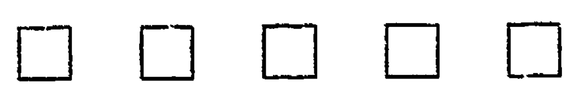
**Take Away** Five take away three. Answer : Two.

**Difference** What is the difference between five and three ? Answer : Two.

**Complementary Addition** What do I need to add to three to equal five? Answer : Two.

2. Each of these approaches may be illustrated with the Cuisenaire material.

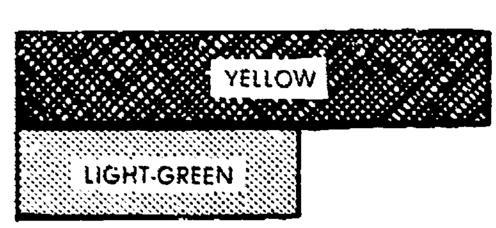
**Take Away** To illustrate this approach we put out five rods



If we "take away" three it is obvious that two remain.

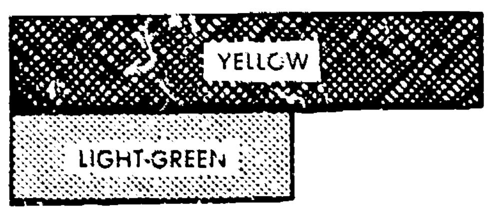
Here the rods are illustrating a process that occurs quite commonly in everyday life, e.g. with a boy who has sixteen marbles and gives away five.

**Difference** To find the difference between three and five the child places the appropriate rods side by side.

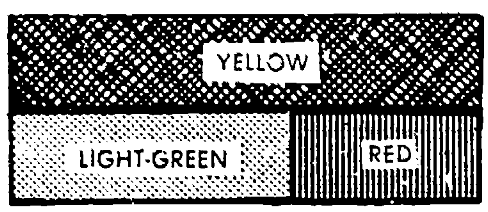



By observation, or by trial and error, the child discovers that the difference between the rods is equal to two. Thus the final pattern is

**Complementary Addition** The pattern for complementary addition is the same as for the previous approach. To find what is added to three to equal five, the rods are placed side by side.

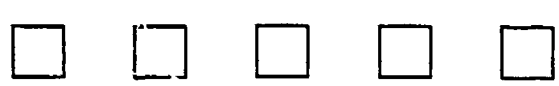



Again, by observaion or trial and error, the child sees that it is necessary to add two :

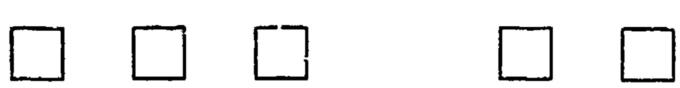



3. The question now arises—which approach ought to be used? The answer to this question depends on the aim in teaching. Throughout this section the aim has been constant—to develop an understanding of basic mathematical ideas. Thus the approach chosen must be the one that illustrates most accurately the ideas of subtraction and equality involved in the statement $5 - 3 = 2$. In the light of this statement there is a strong case for preferring the approaches that use difference or complementary addition.

In the equation $5 - 3 = 2$ there is present a five, a three, and a two, related in a particular way. With the "take away" approach we begin with a five

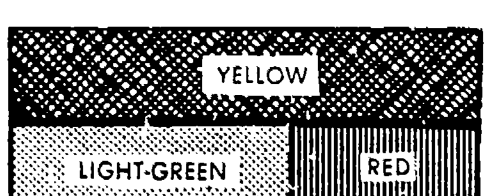


and split it into a three and a two :



As soon as this is done the five is gone. The five, the three, and the two remain present in the equation, but the five disappears when the rods are used to explain the equation. Thus, instead of illustrating the equation, this approach destroys it and puts a barrier in the way of understanding.

If the other two methods are used we are able to represent the equation exactly. In these the equation $5 - 3 = 2$ is represented as



The five in the equation is represented by a yellow rod, the three is represented by a light-green rod, and the two by a red rod. The arrangement of the rods corresponds with the arrangement of the figures in the equation.

4. The operation of subtraction is the inverse of the operation of addition, therefore we need to work from the ideas already established, and from the understandings of equality and inequality. These are best brought together through a clear understanding of difference. Therefore in the first four developmental steps we start with a state of inequality or difference between two rods. Complementary addition is used as a procedure for restoring the original state of equality. These understandings, although listed in this stage, have their beginnings in preceding stages concerned with equality and addition.

The essential new work in this stage is the concentration on the idea of difference, and this must be explored in depth, and thorough understanding must be achieved in this step before proceeding. We then move to the operation of subtraction involving the use of the term " minus ", which is used in these notes to describe that situation of difference where the larger number precedes the smaller. The remaining parts of the stage then exercise this basic operation in a variety of situations involving the procedure of substitution and the use of brackets.

5. Thus the operation of subtraction is defined as a specially directed form of difference. The terms " take away " and " complementary addition " and " the difference between " are the varying vocabularies used in describing the process. The " take away " approach has no place in the expression of abstract ideas and is usually mastered by the child in out-of-school activities. The choice between " what do I need to add to " and " what is the difference between " may need to be made when the formal subtraction operation is introduced later on.

6. Two further ideas concerning subtraction will need to be considered, once the main concept of subtraction as " the difference between " has been mastered. The first is met through substitution, where (using numbers to illustrate the point) in the equation $6 - 5 = 1$ we wish to substitute $3 + 2$ for the 5. To hold to our original intention, $3 + 2$ must be contained in a bracket, or the same result can be achieved by omitting the bracket and writing $6 - 3 - 2$. The use of brackets is discussed in this stage. While repeated subtraction may also be studied in this, the colour stage, it is probably advisable to defer it until Section C, by which time the child's understanding will be deeper and the use of numbers provides greater scope to illustrate the point. (Refer to Section C.)

The second idea relates to a special form of repeated subtraction, where the quantity subtracted each time is the same, e.g. $6 - 2 - 2 - 2$. This may be considered first as three consecutive equations, i.e. $6 - 2 = 4$; $4 - 2 = 2$; $2 - 2 = 0$, or may be looked at from the multiplication point of view as $6 - 3$ twos $= 0$, or finally as the division idea, " How many times can 2 be subtracted from 6? " or $6 \div 2 = 3$. Although it is not essential to study all these ideas in this stage concerning subtraction, it is important to realize first that they *may* be studied here, and second that subtraction lays the foundation for more advanced ideas.

## DEVELOPMENTAL STEPS

### 1. Establishing the Idea and the Term "Difference"

"Show me two rods that are the same."

"Tell me about these rods." (Two brown rods.)

"They are the same length, colour, size. They are equal."

"Now show me two rods that are different."

"Tell me about these rods." (A blue and a pink.)

"They are different in colour, different in length, and of different sizes. They are unequal."

Sufficient work of this type should be done to ensure that children are quite at home with the terms and use them confidently.

"Look at these rods." (Orange and black.)

"Are they the same or different?"

"Show me how much different they are . . . We call this the difference."

(This term, "difference", was introduced during the study of inequality and should be well known. It is repeated here to emphasize that it is an essential foundation for subtraction.)

"Measure the difference with your finger. Can you measure it with a rod? Which rod do you need?"

Much attention should now be given to exercises such as this, to consolidate the idea of difference.

"Show me two rods with a little difference." (E.g. yellow and pink, brown and orange.)

"Show me two rods with a big difference." (E.g. white and brown, orange and red.)

"Show me two rods with no difference." (Two yellow, two blue.)

"Measure the difference between a yellow rod and a blue rod. Which rod is equal to the difference?"

"Take out any two unequal rods. Show me the difference. Measure it."

### 2. Consolidating the Idea through Oral Reading of Patterns

"Can you find unequal rods with the difference equal to a red rod? . . . Tell me about them."

"The difference between a yellow rod and a black rod equals a red rod."

"A red rod is equal to the difference between a black rod and a yellow rod."

"Tell me about yours."

"The difference between a pink rod and a dark-green rod is equal to a red rod."

"A red rod equals the difference between a dark-green rod and a pink rod."

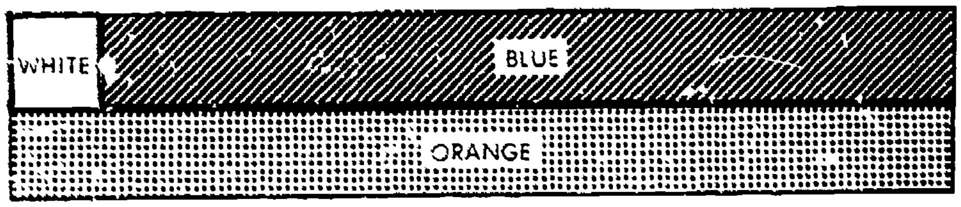
Repeat with many examples.

"Find unequal rods with the difference equal to a pink rod."

Repeat questions as above, and encourage the widest variety of ways of expressing the difference.

Continue, finding rods with a specified difference, and stating this difference.

Children can now make their own patterns and read them. Any pattern of two rods equal to one ($a + b = c$) can be used, e.g. a child making this pattern




would read,

"The difference between orange and blue equals white."

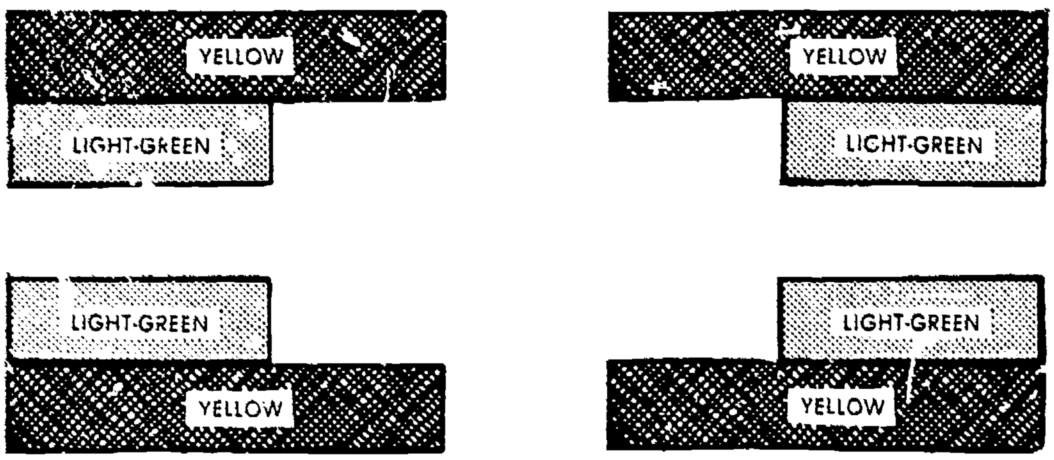
"White equals the difference between blue and orange."

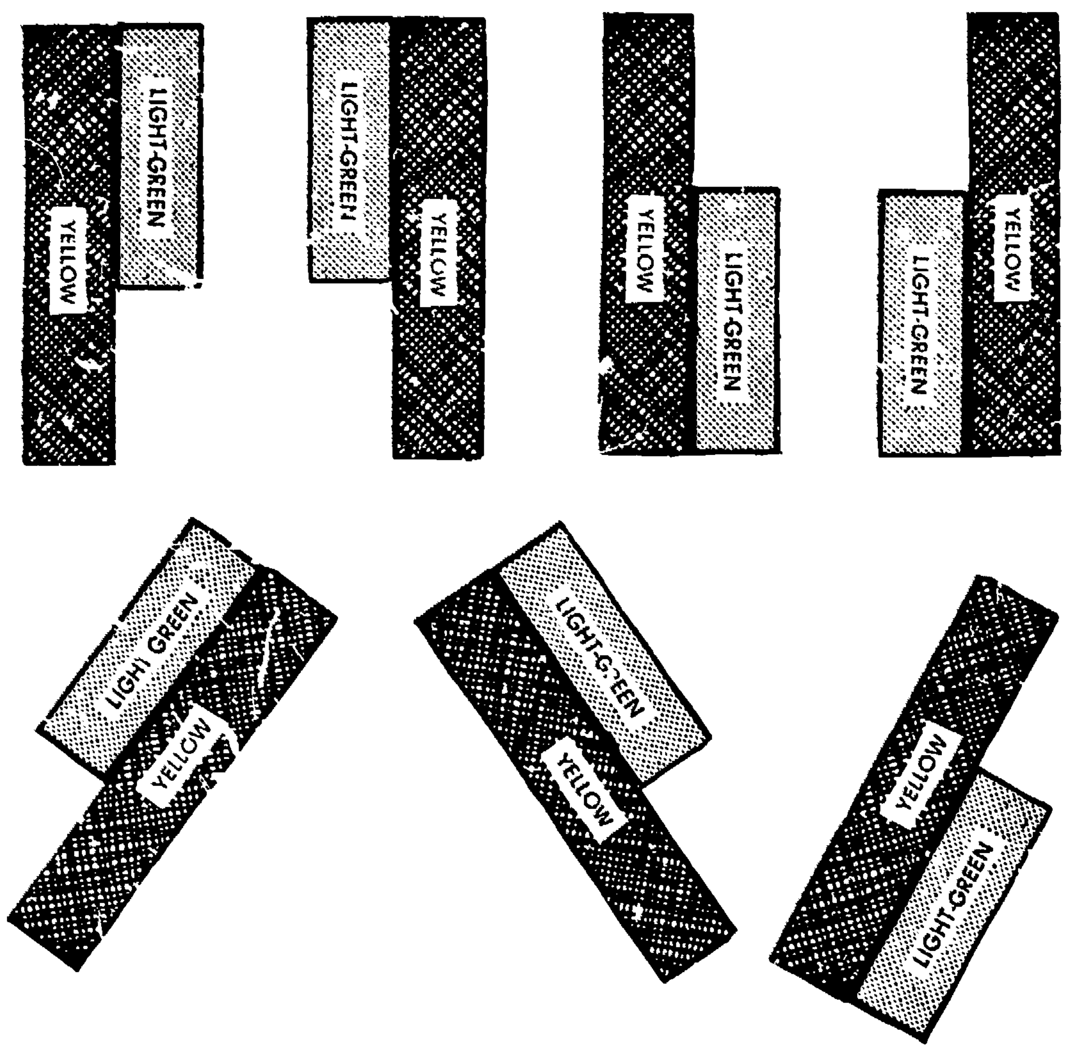
"The difference between blue and orange is equal to white."

and whatever other interpretation he can give.

**Note** i Until "minus" is introduced as a term which gives special direction to the difference, it is immaterial whether the larger or the smaller rod is named first.

ii It is important that "difference" be seen with the rods in a number of positions.






3. **Compleme ːiary Addition**

The understanding of difference can now be reinforced by expressing it in terms of complementary addition. Although essentially thi: is how we arrived at difference in the previous step, the re-phrasing to "What do I need to add to . . ." emphasizes the inverse relationship between addition and subtraction.

> "What must I add to yellow to equal brown? . . . Show me."
>
> "Light-green."
>
> "What must I add to light-green to equal brown? . . . Show me."
>
> "Yellow."

(The emphasis here is on the addition of a rod to the smaller one to establish equality, not simply finding a rod to fill a gap—this is merely a matter of estimating size, and was thoroughly dealt with in Stage 4 of Section A.) Now the child should see that in establishing equality he is measuring the difference shown by a comparison of the rods.

Much of this type of work needs to be done, with the child verbalizing as well as handling rods. Understanding is thus shown at the levels of definition and operation.

**Note** i The "impossible" type of problem must be included, i.e. "What must I add to yellow to equal red?" to show that we add to the smaller rod to establish equality. It is impossible to add anything to a longer rod to make it equal to a shorter rod.

ii Another type of problem is that encountered when equal rods are compared. "What must I add to this red rod to make it equal to this red rod?"

"Nothing, they are already equal."

This paves the way for the later use of "nought" and "zero".

Full attention to i and ii is essential for a complete understanding.

**4. Combining Steps 2 and 3**

Steps 2 and 3 should now be brought together. Patterns should be read in terms of difference and of complementary addition with equal facility, each being used to explain the other, e.g. in this pattern


the child should see

that the red rod is the one equal to the difference between dark-green and pink and

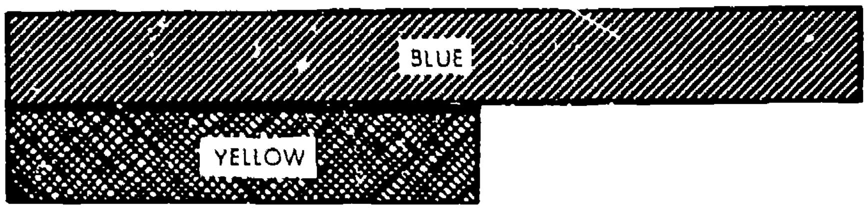
that the red rod is the one added to the pink to establish equality.

**5. Change to Minus**

The previous steps have been aimed at giving a wide background of experience and understanding. Now the formal term should be introduced. There should be no undue haste in moving to the use of "minus". Complete confidence in the above steps is necessary before proceeding, and it could well be that weeks of this preparatory work will be needed.

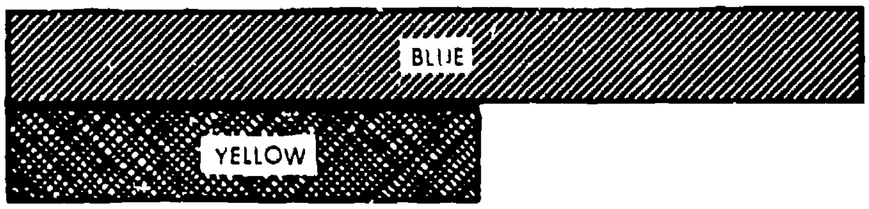
This could be a suitable approach:

a Show pattern

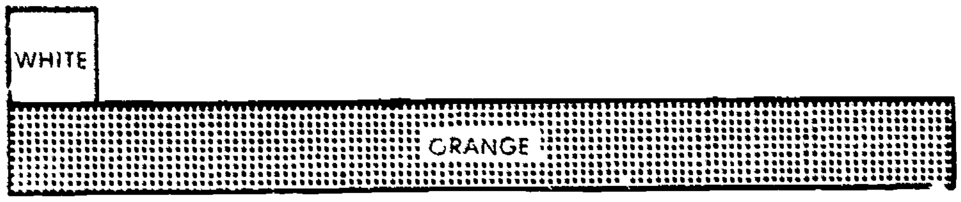



"Instead of saying 'The difference between blue and yellow equals pink' we have a shorter way. The word we use is 'minus'. Say it."

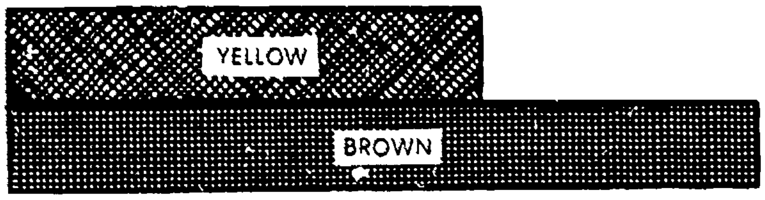
"Now look at these patterns and listen while I read them."



"Blue minus yellow equals (placing the pink rod) pink."



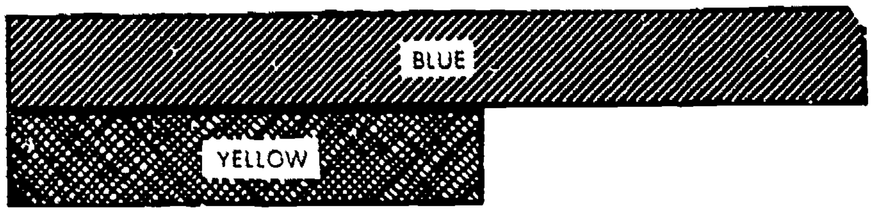

"Orange minus white equals (placing the blue rod) blue."




"Brown minus yellow equals (placing the light-green rod) light-green."

A number of patterns should be treated thus.

b Read this pattern in the old way:




"The difference between blue and yellow equals pink."

"Now read it in the new way."

"Blue minus yellow equals pink."

Treat each pattern in this way.

"Do you notice which rod we name first when we use the word 'minus'?"

"Yes, the longer rod."

(If this is not yet recognized, the teacher should read the patterns again.)

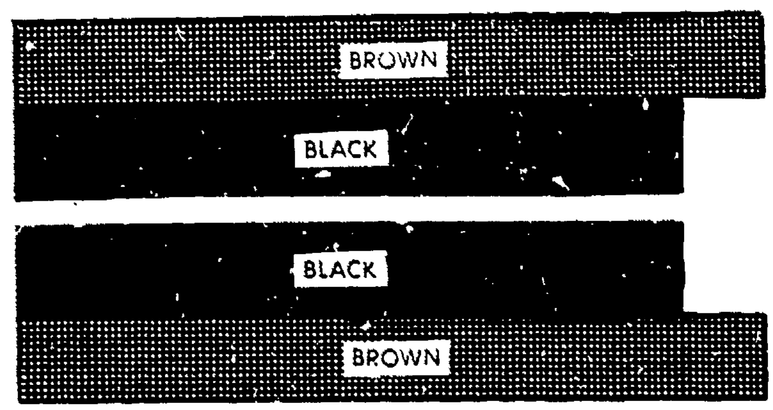
It is essential that the child should go no further till he sees clearly that when using the term "minus" we must name the longer rod first, i.e. the word "minus" directs the order in which the rods are named.

c Position of rods. Exercises should be given to show that the child understands the convention of placing the rods end to end for addition, and side by side for subtraction, e.g.

"Show me yellow plus red
yellow minus red
brown minus light-green
brown plus light-green."

### 6 Simple Manipulation and Pattern Reading

" With rods show me brown minus black."




" What does it mean ? "

a " What is the difference between brown and black ? "

b " What must be added to black to equal brown ? "

Hold the pattern in one hand.

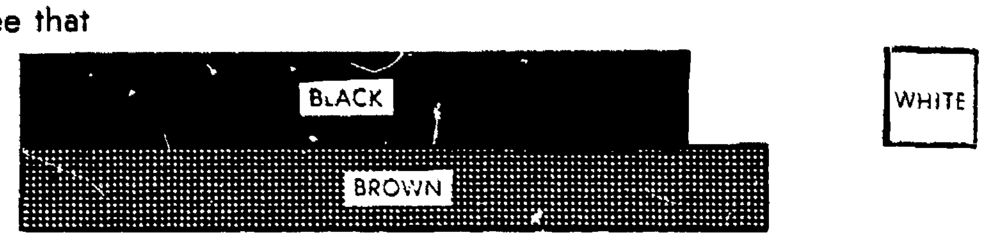
" What is the important part of the pattern ? "

" The difference (the space)."

" Which rod is equal to the difference ? "

" The white rod."

Hold it in the other hand. Here the child should be brought to see that

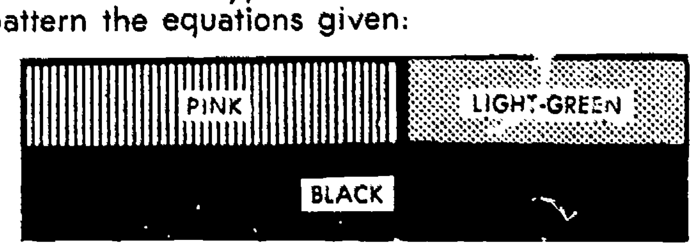



the most significant part is the difference, not the rods themselves. Therefore (b) equals (a) and the pattern should be read both as

" White equals brown minus black " and as

" Brown minus black equals white."

Much practice in this type of work will lead to the ability to read from this pattern the equations given:



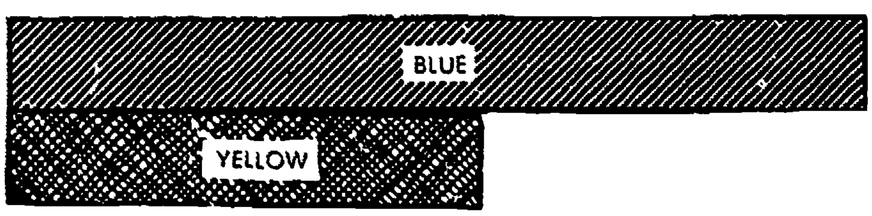

Black minus pink equals light-green,
black minus light-green equals pink,
pink equals black minus light-green,
light-green equals black minus pink.

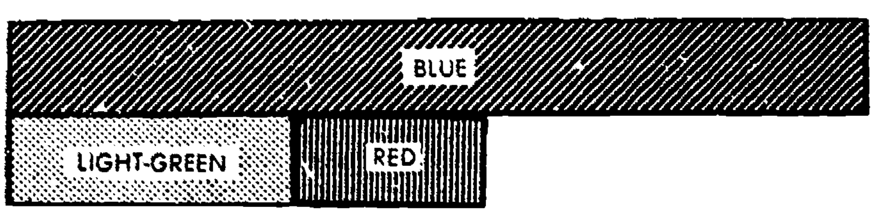
Teachers should note that the two rods between which we are finding the difference (i.e. one complete side of the equation) always remain present, side by side. If it is necessary to remove any rod it will be the rod that equals the difference, i.e. the other side of the equation.

7. **More Complex Manipulation**

The need for brackets.

Once the idea of " minus " is firmly established, we will need to work with more complicated examples. Probably the best approach is to use the technique of substitution already studied in Stage 6. However, difficulties occur when we wish to substitute two or more rods for the smaller of the two rods in our subtraction pair, e.g.

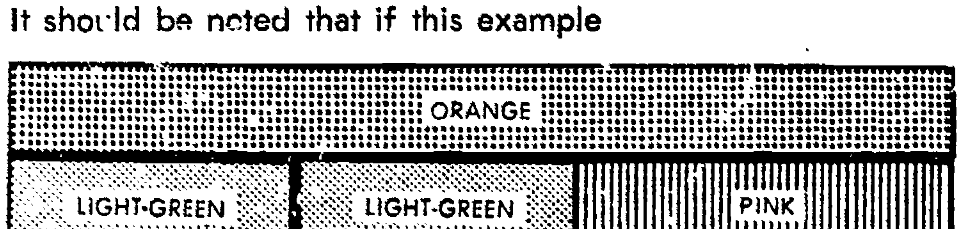



Here we are looking for the difference between the whole blue rod and the whole yellow rod. If we substitute for the yellow rod we now wish to find the difference between the whole blue rod and the whole group of rods (light-green and red). Therefore we need some means by which the child can indicate this. When we work with numbers, this is done by enclosing the " 3 " and the " 2 " (light-green and red) in brackets. In this stage, as we are not writing the equation, we need some oral or physical expression to convey our meaning. This can be done in several ways. The child may raise his left hand (or a finger) as the first bracket, and his right hand (or a finger) as the final bracket. Alternatively, he may indicate his meaning by careful phrasing, e.g. " Blue minus (pause) light-green plus red (pause) equals pink." Another approach is to use the word " together ", e.g. " Blue minus light-green plus red together equals pink."

Possibly the best approach is to use careful phrasing and the word " together ", e.g. " Blue minus (pause) light-green and red together (pause) equals pink."

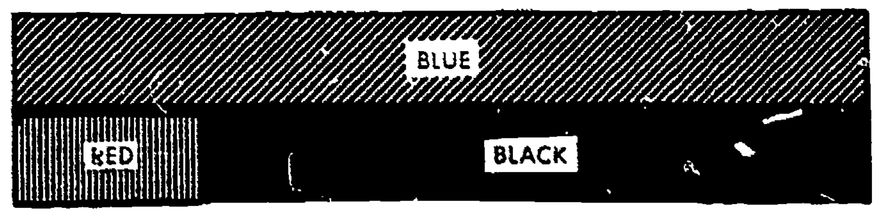
It should be noted that if this example



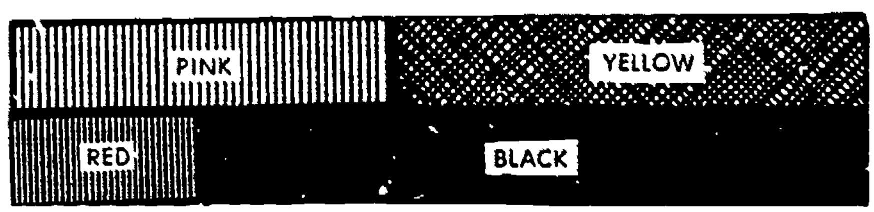

is read as " Orange minus light-green plus light-green together equals pink ", brackets are required. However, if the two light-green rods are interpreted as multiplication, where an ordinal number (2) operates on the cardinal idea contained in the light-green rods, then, because of the nature of multiplication, which involves the idea 2 times light-green as one whole, no brackets are required.

Further treatment of the use of brackets will be found in Section C.

## 8. Further Examples of Substitution

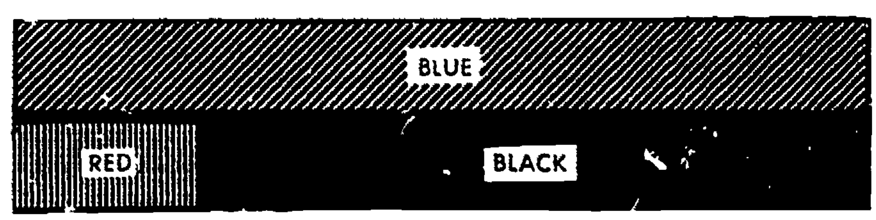



Examples such as this can be developed to give more complicated readings, although at this stage it is sufficient if the child realizes that he can substitute for the first-named rod.

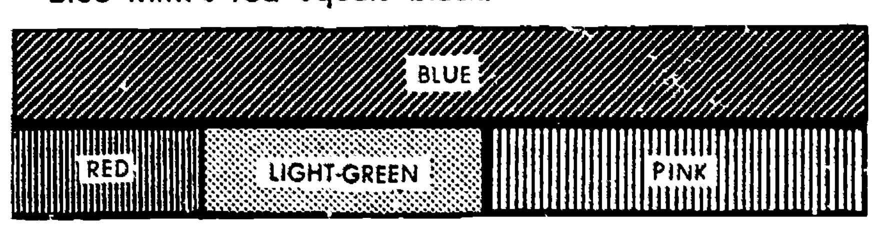



" Pink plus yellow minus red equals black " (and the many possible variations).

Examples can then be introduced, substituting for the " answer ".




" Blue minus red equals black."




" Blue minus red equals light-green plus pink."

**NOTE**

In this step it is sufficient for the child to realize that he can substitute two or more rods for each element in the equation. The degree of complexity of the equation will vary according to the ability of the child, and the work of this step will merge into the work of Stage 11.

Occasionally, bright children will present the " answer " to a subtraction statement in terms of subtraction, e.g. " Blue minus red equals orange minus light-green." It is not necessary for teachers to force children to this idea, but if it appears naturally they should accept it thankfully and understand just what it is the child is doing.

**NOTES ON METHOD**

1. At some opportune time during the stage, the word " subtraction " as a description of the operation should be introduced, e.g. " Read this pattern as subtraction." From this will develop words such as " subtract " and " subtracting ".

2. " Minus " in terms such as " Minus sums ", or " What did I minus? " should not be used.

3. A very common source of error is a failure to give a thorough understanding of the word " minus ". Accordingly, the utmost attention must be given to the early developmental steps (1 — 5) which aim to clinch the idea.

4. Throughout the steps outlined three basic procedures have been stressed. As they are so vital to success they must be mentioned again :

a The child must have the widest possible range of experiences. He must handle the rods as often as the teacher can permit.

b There must be constant probing by the teacher to ensure that terms, e.g. " minus ", are understood. Hence the importance of switching back and forth from " minus " to " difference " and to complementary addition.

c The child must use the word himself. Ability to understand what is meant when the teacher says " minus " indicates one level of understanding. Ability to use the word correctly indicates an even higher level.

**TESTING**

A In order to test whether a child understands the process of subtraction, questions such as the following may be asked :

1 a The difference between black and yellow equals . . . .?
  b What does the red rod equal?

2 a Show me with your rods blue minus pink.
  b What does " minus " mean ?
  c Which rod must I add to pink to equal blue?

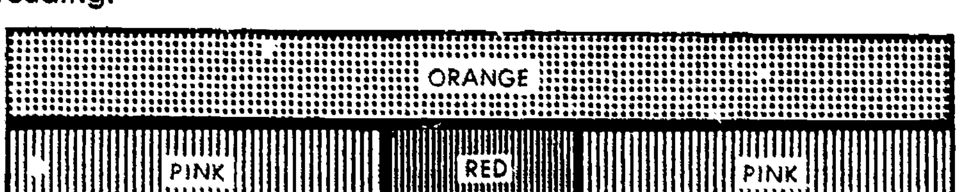
3 What do you mean when you say " orange minus dark-green equals pink "?

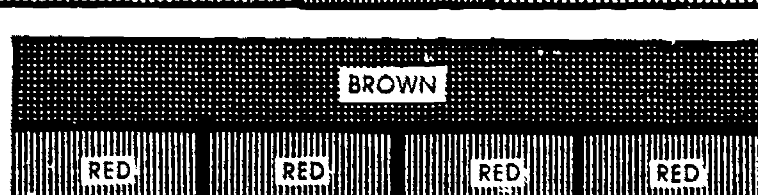
4 What is the difference between pink and pink?

5 What do I need to add to light-green to equal black?

6 Show me any two rods where the difference equals red, light-green, etc.

B Patterns similar to those shown below are read and the child's reading is assessed in terms of the quantity, the quality, and the ease of reading.



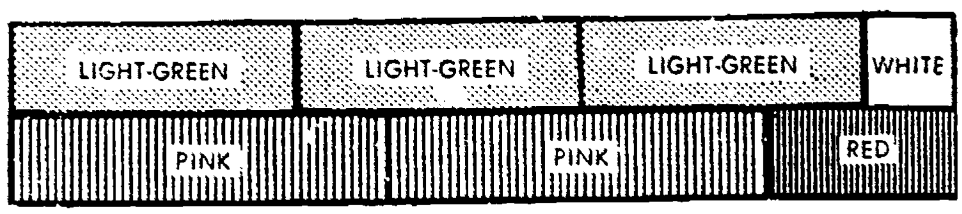
# STAGE 11

**AIM**

To develop the ability to manipulate equations using the operations of addition, multiplication, and subtraction.

**NOTES ON AIM**

1. A typical procedure is to take a pattern such as




and to ask the child to read it in as many ways as possible. A sample of readings would be :

Light-green + light-green + light-green + white = pink + pink + red.

Light-green + white + light-green + light-green = red + pink + pink.

Pink + red + pink = white + light-green + light-green + light-green.

3 times light-green + white = 2 times pink + red.

Red + 2 times pink = white + 3 times light-green.

Pink + pink + red = white + 2 times light-green + light-green.

White = 2 times pink + red — 3 times light-green.

Red = 3 times light-green + white — 2 times pink.

3 times light-green + white — red = 2 times pink.

2 times pink = 3 times light-green + white — red.

2 times pink + red — 2 times light-green = light-green + white

Pink = 3 times light-green + white (pause) — pink + red together.

Here the child is manipulating the rods using his understanding of the three operations. He is seeing that, if he understands what he is doing, he is able to create equations at will, to rearrange them and reverse them, to use first one operation then another, then a combination of operations. A child who is able to do this with facility and confidence has achieved the aim of this stage.

2. It must not be imagined that this stage is completely new to the child. He has been doing this type of work on a more limited scale in the other stages (cf. Stage 9 and Step 7 of Stage 10). In fact the quality of the work done in previous stages will determine the ease with which this stage is accomplished.

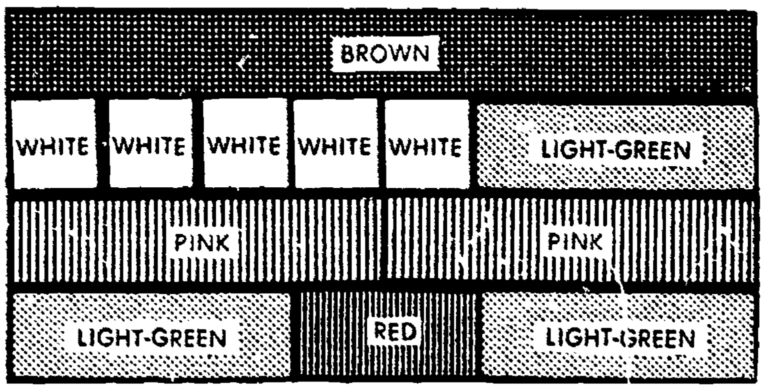
**DEVELOPMENTAL STEPS**

**1. Pattern Reading**

The basic technique in this stage is to take a pattern such as that illustrated on page 45 and let children read it in as many ways as they can. As previously mentioned, the rods used are not important, though a graded series of examples that leads the child from simple arrangements of the patterns to more complex ones is advisable. The steps outlined for the treatment of equality would serve this purpose.

**2. Use of Mat**

The larger mat




has a place in this stage since it stresses the large number of combinations that may be constructed using one rod as the base. The previous stages have been primarily concerned with introducing the operations, and this larger mat has had a limited value, because the flexibility of reading required could be developed more effectively with the pattern. Once this flexibility has been developed, however, it is important for the child to see at a glance the large number of possible combinations. Thus a mat is of assistance. Despite the increased value of the mat, it is true to say that the pattern remains the principal avenue of approach.

**NOTES ON METHOD**

1. It is obvious that beginning this stage before the previous stages are thoroughly mastered is largely self-defeating.

2. The degree of flexibility achieved by the child in this stage will determine his later performance with numbers. The degree of flexibility is related to the amount of time he has spent working in the stage and the number of opportunities he has had to manipulate the rods. Hence it is vitally important to give the child as much practice in manipulation as possible.

**NOTES ON TESTING**

Quite clearly this stage may be tested only by hearing a child read patterns or mats and assessing his performance in terms of the quantity, the quality, and the ease of his reading.

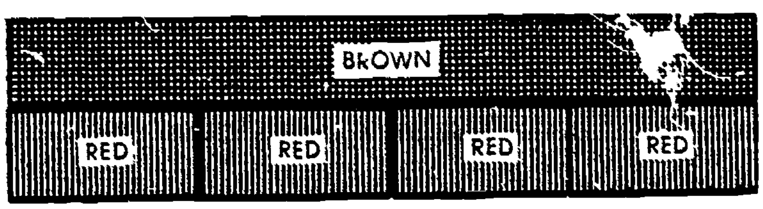
# STAGE 12

**AIM**

To develop an understanding of the operation of division.

**NOTES ON AIM**

1. There are two concepts of division that a child needs to understand. Assuming (for the sake of this point) that the rods are given their commonest numerical values, this pattern




suggests two questions:

a How many twos equal eight? i.e. $8 \div 2 = ?$

Here it is necessary to find the number of equal groups into which eight has been divided. To find the answer, red rods (i.e. "twos") are counted. If he does this the child finds the answer—four.

This is **quotition division.**

b What does one-quarter of eight equal? i.e. $\frac{1}{4}$ of $8 = ?$

Here the number of groups into which eight has been divided (i.e. four) is known and the value of each group is to be found. To find the answer, the child must discover the numerical value of each rod (i.e. two). This is **partition division.**

Thus when division is being considered two questions may be asked :

i How many parts are there ?

ii What is the value of each part ?

2. When these two approaches to the operation of division are realized it is clear that this stage has three main aims :

a To ensure that the child realizes that both these questions may be asked.

b To ensure that the quotition aspect of division is thoroughly understood so that answers may confidently be expected to questions such as, "How many light-greens equal blue ?" and "How many pinks equal orange?"

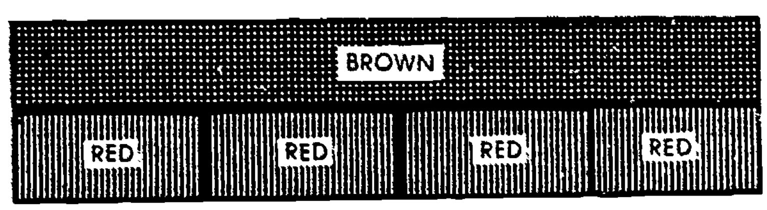
c To introduce, though not to complete, the partition aspect of division. For example, a child may be asked questions such as : "Pick up an orange rod. Can you find two rods of the same length which together equal the orange rod? What colour are they?"

When he carries out this operation the child's attention is directed to the fact that instead of asking for the number of equal rods equal to another rod, it is possible to begin with a knowledge of the number of rods and be asked to find the colour (later, the value) of them. This process will be fully developed when numbers are introduced and the child is able to give an actual numerical value to the rods.

## DEVELOPMENTAL STEPS

### 1. Quotition Division

a **Factors** The child is asked, for example, " How many reds equal brown? " He puts out a brown rod, and by placing red rods beside it and counting them he discovers the answer.




In a similar way he finds the answer to :

" How many whites equal brown? "

" How many pinks equal brown? "

" How many browns equal brown? "

Having studied the " factors " of brown, he uses the same procedure to study the " factors " of the other rods.

b **Remainders** Once the child has mastered the first step thoroughly he may be asked questions such as:

" How many light-greens equal brown ? "

Using the same technique as he used in the previous step, he discovers that " There are two light-greens and something equal to a red left over."

He may then be asked:

" How many dark-greens equal brown ? "

" How many yellows equal brown ? "

" How many blacks equal brown ? " and so on.

Using all the other rods, he studies quotition division with remainders.

c **Mixed Examples** The child is now asked questions chosen indiscriminately from the two steps.

d **Introduction of Term " Remainder "** As soon as the child is thoroughly master of the first three steps the term " remainder " is introduced to replace the term " left over ". As in all cases where a vocabulary change is made, the child must be given a great deal of practice in using it to ensure that he has grasped its meaning.

### 2. Partition Division

In this stage, partition division is approached in the manner outlined in the **Notes on Aim**. The child is asked :

" Put out an orange rod. Find five rods of the same length that together are equal to it. What colour rod have you used ? "

This type of exercise must be done many times using the widest possible range of rods. It is obvious that in this stage the exercise would be done using " factors " of rods only.

**NOTES ON METHOD**

1. This stage may be begun at any time after the completion of multiplication, for once that process is understood all the ideas necessary for an understanding of division have been tackled.

2. The need to provide a wide range of activities to ensure that the child has a thorough mastery of the matter he is studying has been stressed many times. It is as necessary in this stage as in any of the others.

3. The pattern reading of the previous stage continues side by side with the work of this stage.

**TESTING**

**Quotition Division** This may be tested by asking questions such as:

" How many light-greens equal brown ? "

" How many blues equal orange ? "

" How many reds equal pink ? "

The child must use his rods to give the answer.

He should also be asked,

" What do you mean by ' remainder ' ? "

and be able to say that it is what is left over.

**Partition Division** This is best tested with the type of exercise given in Step 2 above.

# STAGE 13

**AIM**

To develop ability to manipulate equations using the operations of addition, subtraction, multiplication, and division.

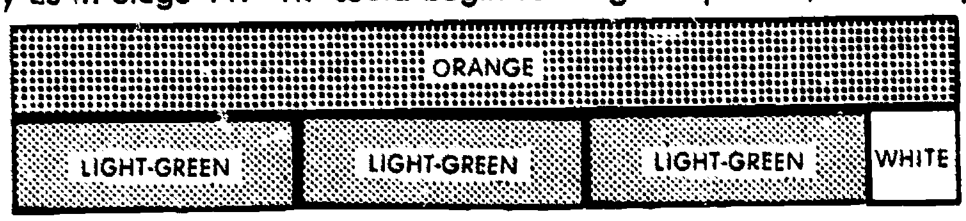
**NOTES ON AIM**

This stage is a repetition of Stage 11, except that the operation of division is added to the operations of addition, subtraction, and multiplication which were used in that stage. The aim of this stage is the same as that of Stage 11—to lead the child to use his understanding of the operations to construct and rearrange equations (see **Notes on Aim,** Stage 11).

**DEVELOPMENTAL STEPS**

1. **Pattern Reading (Quotition Division)**

In this stage a child would be asked to read a pattern in the same way as in Stage 11. He could begin reading this pattern, for example,




in ways such as :

" Light-green + light-green + light-green + white = orange."

" Orange − 2 times light-green = light-green + white."

" 3 times light-green + white = orange."

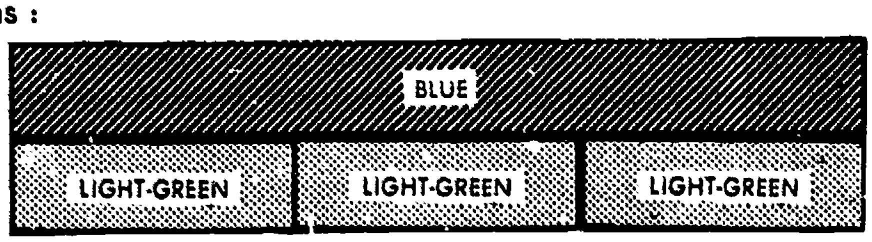
The teacher would then interrupt and ask,

" How many light-greens equal orange ? "

A constant linking of division with the other operations in this manner achieves the aim of the stage.

2. **Pattern Reading (Partition Division)**

Partition division may be introduced by working from a pattern such as :




After a child has read this in ways that involve the use of other operations the teacher may ask :

" How many rods are equal to blue ? "

" Three."

" What is the colour of each of them ? "

(See Notes on Aim, Stage 12.)

It must be realized that, at this time in his development, the child would be able to approach division only as an answer to a question. It is not until he is able to write a division sign that he is able to construct examples of his own volition.

**3. Link with Multiplication**

It will be noted that in the example given above the teacher asked the division question immediately following an example of multiplication. It is not, of course, necessary or desirable that this order should be followed every time. It is important, however, that it should occur frequently so that the relation between the two operations may be stressed.

**NOTES ON METHOD**

1. As this stage is dependent on skills developed in the previous two stages it cannot be begun until they are completed.

2. The ability to use the four operations to make equations is, in many ways, the climax to this section. If a child has developed this level of understanding he has achieved the aim of the section. Until he has achieved it, his work is not complete. He is not ready to leave the stage until he is able to manipulate the rods using the four operations.

**TESTING**

This stage is tested in the same way as Stage 11, except that, in addition to reading patterns and mats with the characteristics of quantity, quality, and ease, the operation of division is added; the child must be able to answer, fluently, any division question the teacher puts to him.

**SUMMARY OF WORK COVERED BY THE END OF SECTION B**

Let us consider what the child has accomplished by the end of this section.

1. **Ordinal Number**
   - a Although this section has not required an advance in counting and the idea of ordinal number as a number in a series, most children will almost certainly have extended this area to 20 (see Stage 5).
   - b Figures to 10 must be recognized and written.

2. **Cardinal Number**
   - a Number as a Group. Exercises in grouping objects (related and unrelated), counting groups, making a limited number of groups of a specified size, and seeing the "number" in a group as distinct from the objects of which it is composed, having refined the concept of cardinal number. Faced with groups of three totally different objects, e.g. (i) a book, a doll, and a box, (ii) a bottle of milk, a chair, and a school bag, seeing that the number "3" is common to both.
   - b Number as a Whole. Reading from patterns of rods has accustomed the child to thinking in terms of wholes, e.g. "A brown rod plus a red rod equals an orange rod." Three wholes of different sizes are represented in this statement.

3. **Basic Mathematical Ideas**
   - a **Equality** The child understands equality, can explain it in his own terms, and knows it holds under many conditions (Stage 6).

b **Addition** He understands the operation of addition as a putting together or combination of wholes; he can define this operation, and can add, using the material.

c **Multiplication** This operation is seen as a way of combining equal wholes. To show his understanding, the child can read a specified pattern in terms of both addition and multiplication.

d **Subtraction** The operation of subtraction is understood as finding the difference. Experience has shown the child that to measure the difference he establishes equality by using complementary addition. When the term "minus" is used, he knows this requires a specified order, namely, the bigger rod (or group of rods) is named before the smaller.

e **Division** This operation has been understood as:

i Finding the number of wholes of a given size which equal a larger specified whole (Quotition Division).

ii Finding the size (expressed in colour) of the rod that must be used a given number of times to equal a larger specified rod (Partition Division).

**Note** The operations have not only been used separately, but in relation to each other (as in Stages 9, 11, 13). More and more complex work has been required, though the degree of complexity varies with children of different ability levels.

**4. Understanding and Experience**

Understanding has been the keynote, and repeated experience is required to develop this. The child freely uses the terms "equals", "plus", "times", "minus", "how many", but this does not necessarily indicate understanding. He must be able to define them, however simply. Unless he can do this, he will be unable to apply his knowledge to the more complex work of later sections.

**5. General Notes**

a The nature of the work has maintained and extended the child's interest.

b He has continued to experiment, and to draw his own conclusions. The teacher's probing attitude has led him to justify his statements, and he is developing confidence in so doing.

c He is still progressing at a rate suitable to his own abilities. The difference between most advanced and least advanced pupils in a grade is more marked than at the beginning of the section.

d His readiness to proceed is shown by his understanding, and by the quantity, the quality, and the fluency of readings evident in his interpretations of patterns of rods.

e Although it is recognized that the slower child will never achieve the depth of understanding and the complexity of expression shown by the brighter one, it would be pointless for him to proceed till he understands the simple, basic idea behind equality and behind each of the four operations.

# APPENDIX 2

Acceptance of the fact that children progress at different rates is basic to the methodology of this course. Because of this fact the course is written as a series of sections through which each child moves at his own pace.

Thus it may well happen (it is, in fact, expected) that groups of children will be working at different levels within the Colour Section. The use of ability groups at any time and in any subject raises certain problems of grade organization. Their use in the Colour Section of the mathematics course highlights these problems.

The reason why the problems are highlighted can be seen very easily if certain basic points, stressed when discussing Section B, are considered:

1 Success in mastering a concept is, to a very large extent, dependent upon the number of experiences (involving the concept) that the child has had.

2 Because the number of experiences is so important all work needs to be oral.

3 Even if written work were desirable it would not be done effectively, since the child is still learning to write.

4 The most typical and important activity is pattern reading.

Thus the problem of organization in this section can be summarized by saying:

1 There is a need for children to work in groups at different levels.

2 The teacher can work (particularly if pattern reading is involved) with only one group at a time.

3 If he is to do this, the groups with which he is not working must have purposeful, self-directed activities to occupy them.

4 Such activities are not easy to find in a section where written work with the rods is not done.

This appendix attempts to solve the problem by giving a list of activities that meet the requirements of the section and help to free the teacher to work with one group at a time.

The activities are divided into four main groups:

1 Activities to assist in recognition of figures.

2 Applied number activities.

3 Card activities.

4 Games.

The first two groups are concerned with aspects of the mathematics course that are covered at the same time as, but separate from, the work with the Cuisenaire material. Obviously when a teacher is directing these activities, the whole grade is **not** using the Cuisenaire material at the one time.

The last two groups are based on the Colour Section itself as taught using the Cuisenaire material.

It must always be remembered that, while many of these activities are pupil-directed and leave the teacher free to work with other groups, they must still be supervised and corrected at some time during the period.

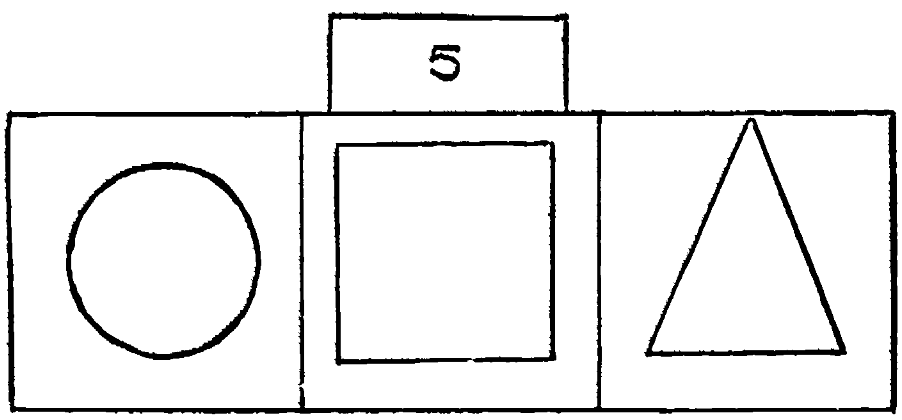
As stated in Appendix 1, the activities are no more than samples. Many teachers will evolve others more suited to their particular needs.

## PART 1 ACTIVITIES TO ASSIST IN RECOGNITION OF FIGURES

The activities listed below are designed primarily to assist in the development of the ability to recognize figures but may be adapted to assist the writing of figures or the recognition and writing of words. In each case added value is given to the exercise if the figure is written alongside each group (if the child is writing figures) or a small card bearing the appropriate figure is selected from a box and placed with the group.

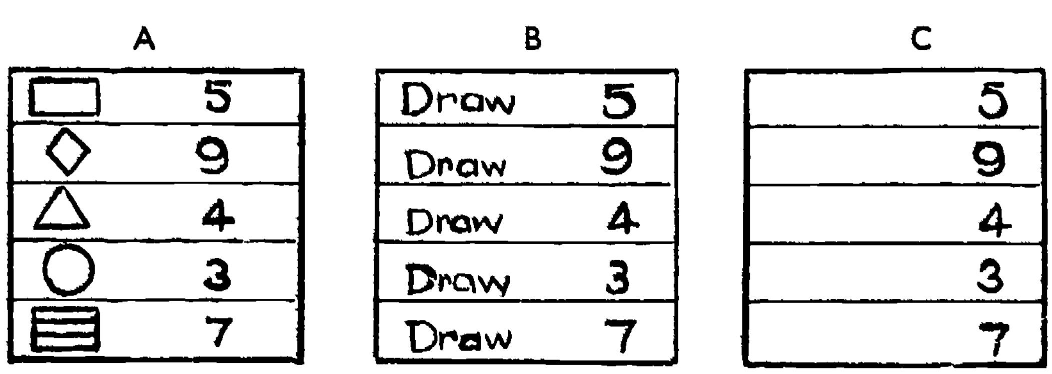
a **Assignments**

i



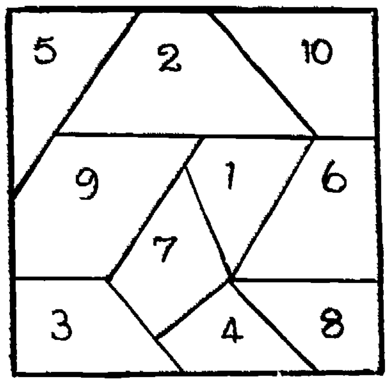

A figure (written on a card) is attached to a larger card as illustrated. The child draws the required number of outlines.

ii This assignment may be arranged so that children meet groups of numerals. They may receive cards such as these, and draw the required number of shapes for Card A, and whatever they choose for Cards B and C.



b **Games**

i **Bean Bags**

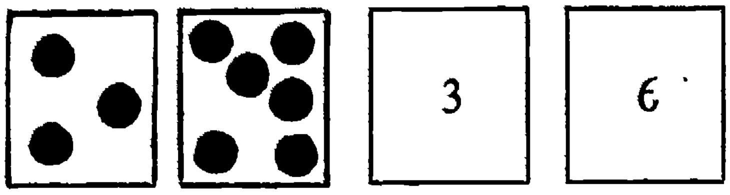



The child throws a bean bag into a chart and names the figure in the area on which the bag falls.

ii " **Grab** " Two similar packs of miniature cards are made, each card with a different number:


One pack is given to each of two children. The children murmur the name as they turn up each of the cards. When the two children turn up the same number the first child to say the number loudly wins all the cards that have been previously called.

iii " **Find Your Partner** " Children have cards from matching packs, e.g.




They skip around to music and, when the music stops, the children holding the figure cards run and stand beside the children holding the appropriate quantity card. Cards are then changed and the game starts again.

iv " **Postman** "

Group: " Postman, will you bring letters to me,
Letters to me from over the sea? "

Postman: " Yes, I'll bring letters to your door,
Sometimes one, and sometimes more."

The postman delivers figure cards round the small circle, as children chant the rhyme. When the child receives the card he either names the figure and returns the card, or is told to "jump" the number of times indicated by his card. The game may be varied by having the teacher call the numbers in turn; the child, when his number is called, "posts" it in a box.

v **"Last Man Standing"** The children stand in a line holding the figure cards. They say the rhyme:

"All the little figure cards
Standing in a row,
John says — (Seven)
So — (seven) must go."

The child holding the card with the specified number must sit.

vi **The "Numbers with Actions"** Children collect figure cards and spread out over a movement area. When a child's particular number is mentioned he must do the activity prescribed by his rhyme, which all the children say:

"Figure one, go for a run!
Figure two, touch your shoe!
Figure three, skip to me!
Figure four, run to the door!
Figure five, bees in a hive!
Figure six, do some tricks!
Figure seven, bend like seven!
Figure eight, shut the gate!
Figure nine, grow like a vine!
Figure ten, lions in a den!"

vii **"Dice"** Dice marked with figures are a source of many games. Marked spinning tops, bobs, and skittles may also be used in a variety of ways.

viii **Matchbox Train** A train of the trays from matchboxes is made and a number placed in each tray. The child then places the appropriate number of buttons (or some suitable counting material) in the trays.

ix **Community Revision** The teacher holds up a figure card, e.g.

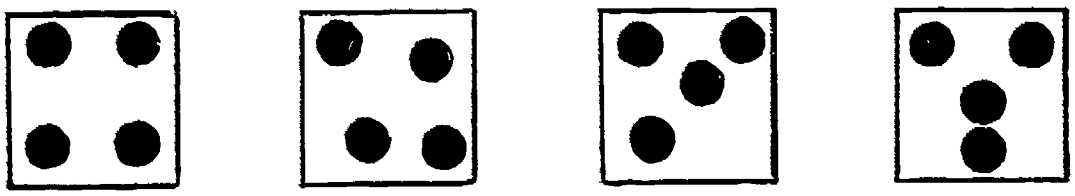
| 4 |
|---|

The children are then asked to

"Bring this number of balls"
"Clap this number of times"
"Find this number of beads."

The teacher may vary this community activity by holding two figure cards behind her back. The children are given turns to choose " which hand " they would like and then to name the figure on the card held in that hand.

**Note** It is important to ensure that cards which show a series of objects should show the objects arranged differently, e.g. " four " could be shown as :



The use of a fixed pattern to represent any number may lead the child to confuse value with position. He should realize that " fourness " is determined by number not by position.

**PART 2 CARDS**

Effective use of this type of activity presupposes that children :

1 Know how to complete the card being used.

2 Have available sufficient cards for each member of the group, plus extra cards so that a finished card may be exchanged for a spare one at any time.

3 Know how to indicate silently (by standing or by " sitting tall ") when they have finished a card.

4 Know that to " prove " a pattern is to put below the pattern the single rod they have been attempting to " equal ".

It must be stressed that the activities outlined in this appendix are intended to supplement, not displace, oral work. They free the teacher to give him more opportunity for oral work. It would be to misuse the activities grossly if they were allowed to reduce the amount of oral work done.

Finally four points connected with their use in the class-room ought to be noted :

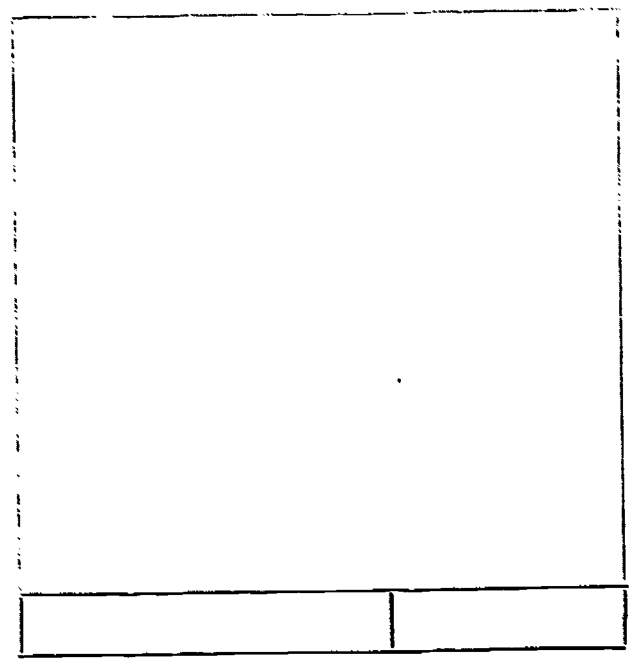
1 The illustrations on the cards that follow are not the same size as the actual rods. This is to save space. Any illustration that a child worked with would, however, be the same size as the rod illustrated, e.g. an orange rod would be 1 cm. wide and 10 cm. long.

2 Colouring the spaces, generally speaking, reduces their value. If they are uncoloured, the child must exercise his powers of assessment.

3 In most instances added value is given to an exercise if children " read " from their patterns. This should be done wherever possible.

4 The cards that follow are intended to suggest the type of useful activity that children can undertake. They do not pretend to cover the whole range of possible activities. It is assumed that teachers wishing to use this type of activity will construct other cards of their own design.

CARD 1



**Purpose of Card**

This type of card is of particular value during the study of equality.

**Method of Use**

1 The child finds how many other combinations of two rods will equal the two rods given.

2 The child may also be asked to find combinations of three or four (or more) rods that equal the two rods given.

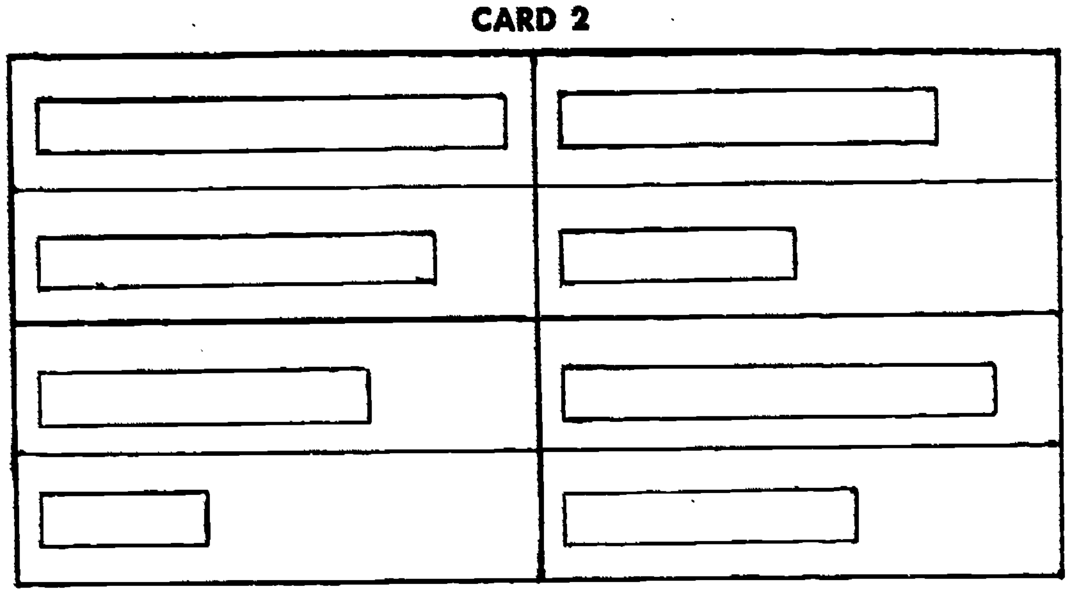
3 He should " read " his patterns.

**Adaptations**

1 The card shown belongs to a set which has, on the bottom line, different combinations of two rods equalling orange. Similar sets may be made using as base the rods from blue to yellow.

2 Sets may be made in which the bottom line is comprised of three rods (or more than three rods) that equal orange—or any other rod from blue to yellow.

CARD 2



**Purpose of Card**

This activity assists in developing further the child's images of the rods and thus provides a valuable follow-up of the touch games from Section A, Stage 4.

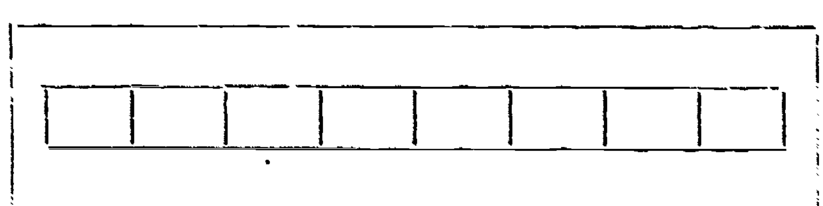
**Method of Use**

A group of children is given a set of cards such as that shown. The leader of the group holds up a chosen rod, e.g. brown. Each child must then construct below his card a combination of two rods that equal brown. The first child correct is named by the leader. He " reads " his combination, then covers the brown rod on his card with an actual brown rod. The first child to cover all the rods is the winner.

**Adaptations**

The children may be required to find different numbers of rods equal to the named one, e.g. instead of two rods equal to brown they may be required to find three or four.

**CARD 3**



**Purpose of Card**

This card is useful when studying " trains " (see Stage 8, Developmental Step 5).

**Method of Use**

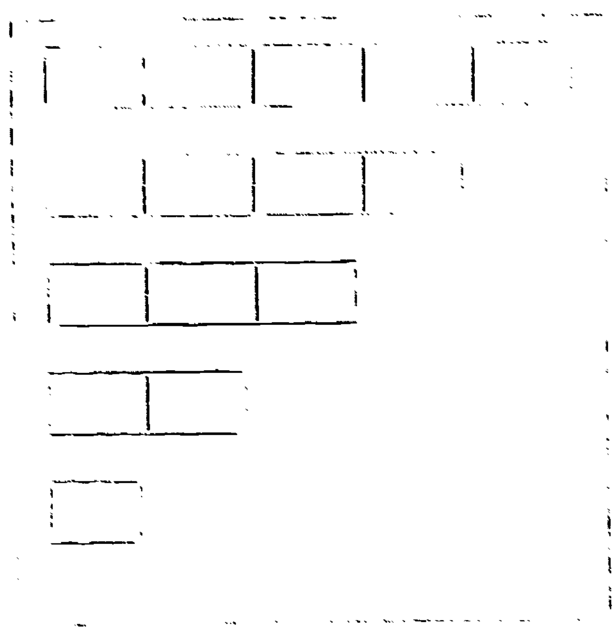
The child is asked to find a number of different " trains " which begin and end at the same points as the row on the card. The " trains " which the child makes may be left on the desk below the card.

(Space is not provided on the card for this activity since the cards would be too bulky, e.g. a card which has a " train " of five brown rods.)

**Adaptations**

The number of rods that constitute the " train " on the card, and, of course, the sizes of the rods used, may be varied.

**CARD 4**



**Purpose of Card**

This activity provides valuable background for the study of concepts of multiplication, division, and fractions.

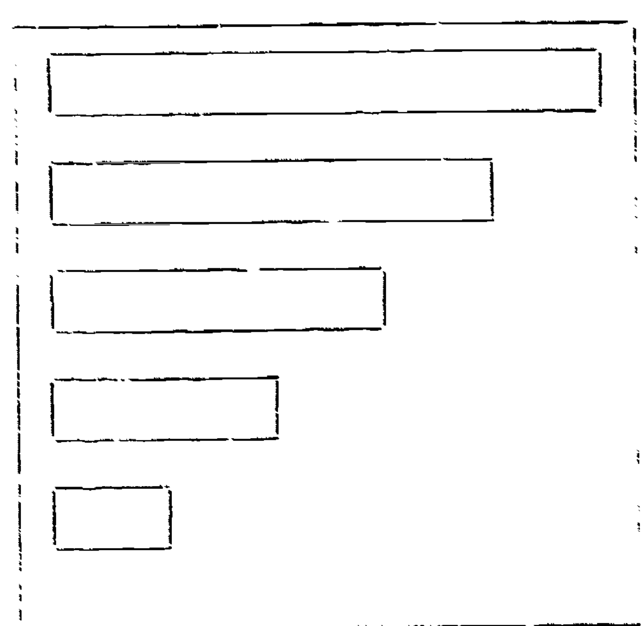
**Method of Use**

The child must match rods to the spaces, then find a single rod to equal each line of red rods, e.g. orange to equal five reds.

**Adaptations**

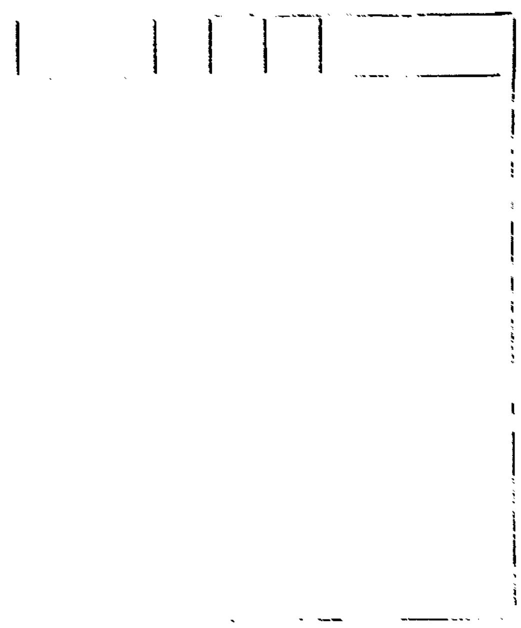
1 Similar cards using other rods, e.g. white, light-green, or pink rods.

2 The card may be reversed in principle, e.g.



The child must now discover the number of red rods that equal each of the single rods shown.

**CARD 5**



**Purpose of Card**

This card affords an excellent preparation for mental manipulation of equations as well as emphasizing that location does not affect equality.

**Method of Use**

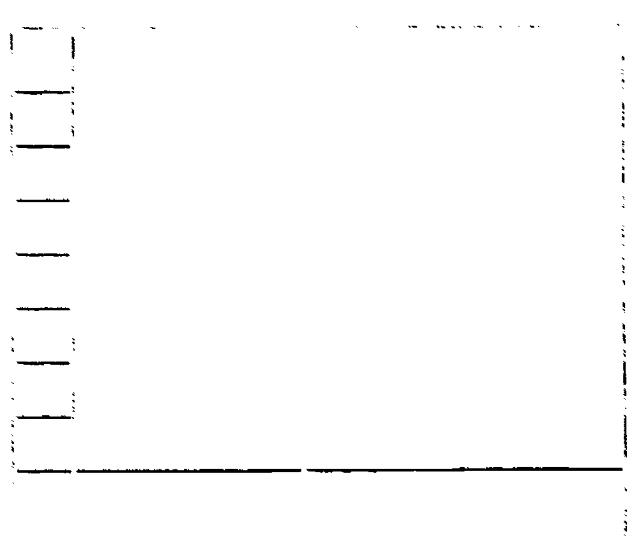
The child uses the same group of rods as shown on the top of the card and rearranges them differently each time, e.g. the second row may read:

Red, white, red, light-green, white, white.

**Adaptations**

Cards with different rods on the top line may be made.

**CARD 6**



**Purpose of Card**

This card may be used to assist in developing the concept of complementary addition, and to give experience in substitution.

**Method of Use**

The child adds to each white rod a stipulated number of rods to make each row equal to orange.

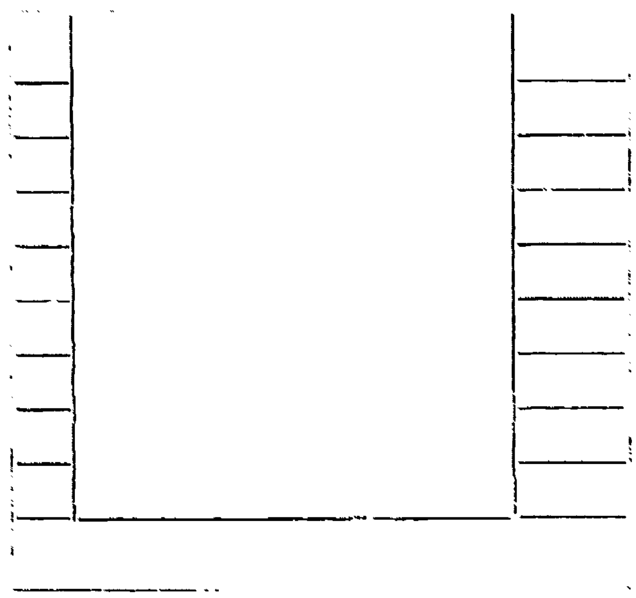
**Adaptations**

1 This activity may be varied by asking for different numbers of rods to be added.

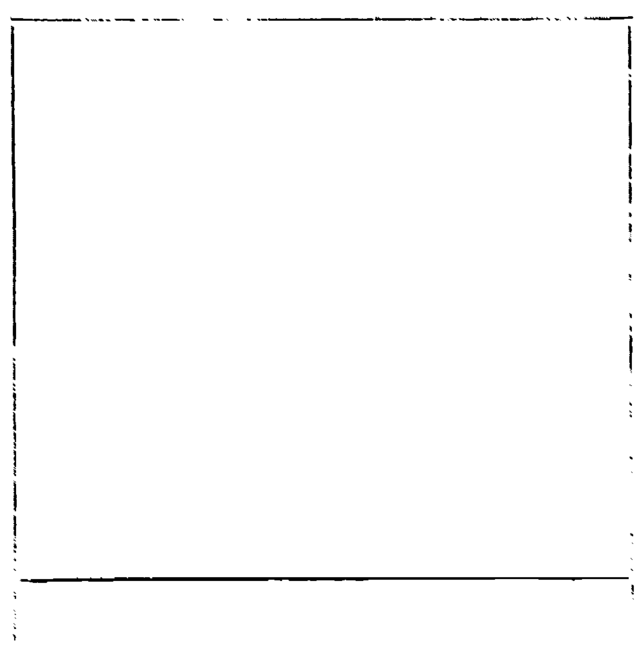
2 The base rod may be varied.

3 The rods on the side of the card may be varied.

4 Rods may be put on both sides of the card and children asked to fill the gaps, e.g.



**CARD 7**



**Purpose of Card**

This type of card fulfils a limited need and may be used if the teacher wishes to save a few minutes of time when starting a group to work. Rather than naming each child and the mat he is to build, the teacher directs the children to collect a card, knowing that they will obtain a variety of bases.

**Method of Use**

1 The child may build a mat upon the base shown.

2 The child may be told to build a mat using—

two rods in each line

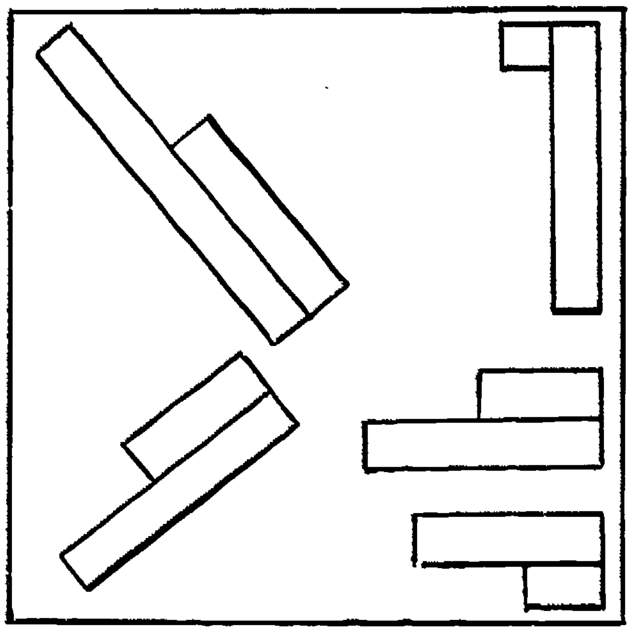
three (or more) rods in each line.

**Adaptations**

1 In addition to a set of cards with orange as the base rod, separate sets may be made using as the base the rods from blue to yellow.

2 Another set may be constructed using a mixed group of bases from orange to yellow.

**CARD 8**



**Purpose of Card**

This card is useful at several stages, namely, for mental image, equality, addition, complementary addition, and subtraction.

**Methods of Use**

1 The child matches rods to spaces, and "fills the gap." (Mental image.)

2 At the stage of equality, he can "read" his patterns in terms of equality.

3 Later he can "read" in terms of addition.

4 By comparing the unequal rods, the child sees and measures difference.

5 After the term " minus " has been introduced, subtraction readings may be given.

**Adaptations**

By using more than one rod to cover an outline, situations where two rods equal two others, three rods equal two, and so on, can be obtained. This opens up wide possibilities with reading.

## PART 3 GAMES

It is assumed that, before playing these games under their own direction, the children will learn the mechanics thoroughly under the teacher's supervision.

A matter stressed when discussing the card games must be stressed again. The games that follow are intended to supplement, not displace, the oral work so essential if success is to be achieved.

Some of these games require the child to remember certain rod combinations which equal others. While this is not a specific requirement of the section, it is desirable that children should be encouraged to develop memory along these lines.

It is realized that some of the games may require more space and organization than some teachers will consider worth while for the value they contain. However, they are included as ideas which may constitute " starting points " from which individual teachers will work out their own ideas.

GAME—" Guess the Mat ".

**Aim**

To develop in children the ability to visualize equations.

**Preparation**

A small group of children sit in a semi-circle around a leader, who holds a card upon which is coloured a typical mat.

<table>
<tr><td>W.</td><td colspan="4">Pink</td><td colspan="5">Yellow</td></tr>
<tr><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td><td>W.</td></tr>
<tr><td colspan="2">Red</td><td colspan="4">Pink</td><td colspan="4">Pink</td></tr>
<tr><td colspan="3">Light-green</td><td colspan="4">Pink</td><td colspan="3">Light-green</td></tr>
<tr><td>W.</td><td>W.</td><td colspan="3">Light-green</td><td>W.</td><td>W.</td><td colspan="3">Light-green</td></tr>
<tr><td colspan="3">Light-green</td><td>W.</td><td>W.</td><td colspan="5">Yellow</td></tr>
<tr><td>W.</td><td colspan="4">Pink</td><td colspan="5">Yellow</td></tr>
<tr><td colspan="5">Yellow</td><td colspan="5">Yellow</td></tr>
<tr><td colspan="2">Red</td><td colspan="8">Brown</td></tr>
<tr><td colspan="9">Blue</td><td>W.</td></tr>
</table>

**Development**

Children listen as the leader reads the mat, line by line, until one of the children realizes that the colour base of the mat being read is orange. The first child to name this correctly becomes the new leader, and chooses a new card with a different base for the next reading.

**Note** The mat is coloured on a card for two reasons:

1 For speed of playing. To construct a real mat each turn would occupy too much time.

2 For practical reasons. The children are not able to see the combinations (as they would be able to if a real mat were used).

**Adaptation**

The children play the game in pairs taking turns. One selects and reads a card, while the other guesses.

GAME—" Musical Rods ".

**Aim**

This game assists a child to develop a clear mental image of the rods and to realize the large number of combinations that are equal to any one rod.

**Preparation**

Children stand in a circle on the floor. A box of rods is emptied on the floor within the circle. The teacher has one of each rod beside her.

Before each round of activity the teacher announces, " This time I am talking about the brown rod." Children then know that brown is the length which they will try to equal for the next set of turns.

**Development**

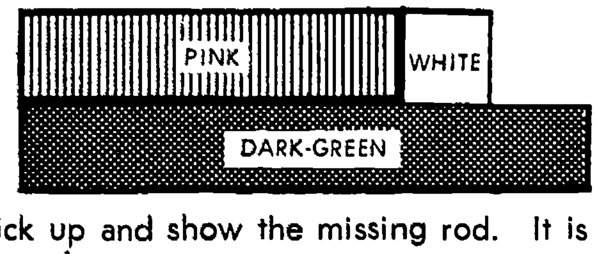
The children move around the rods as the teacher plays or taps a rhythm. When the music stops, the teacher holds up one rod, e.g. pink. The children know that the rod they choose must, when added to this, equal brown, and so they each select and hold up the one that they think will complete the line, i.e. pink. A child checks this with the teacher's rods to " prove " that it was the correct rod.

**Note** This game would be best played with a small group of children in a setting that allows for movement between the " turns ".

**Adaptations**

1 The teacher holds up one rod, while children bend and construct on the floor a combination to equal the rod displayed.

2 The teacher shows an unfinished equation on her display board, e.g.




Children pick up and show the missing rod. It is checked against the teacher's equation.

GAME—" Postman ".

**Aim**

This game assists in the development of the idea of equality.

**Preparation**

a The children are seated in a circle or hollow square, with an individual display board at one point of the circle.

b The child selected to be " Postman " carries a bag containing rods, and may wear a postman's cap.

c The children need to know the following rhyme:

1 " Postman, will you bring letters to me?
2 Letters to me from over the sea? "
3 " Yes, I'll bring letters to your door,
4 Sometimes one, and sometimes more."

**Development**

As the children say lines 1 and 2, the postman circles the group. When the postman says lines 3 and 4, he quickly delivers one rod to each child.

On the display board, the postman then constructs a row, e.g.


The first child to hold up the rod that equals that row is allowed to prove whether he is correct. He then places the rod on the floor in front of him, before he collects a new rod from the postman.

The first child with three rods on the floor in front of him becomes the new postman.

**Note** The children should be encouraged to have many turns in a short time.

**Adaptations**

1 The postman makes an incomplete pattern, and the children hold up the missing rod. One is selected to check.

2 The postman may show a rod and the children create a pattern.

3 The children may be asked for two or more rods to equal the displayed pattern.

GAME—" Remembering the Mat ".

**Aim**

This activity assists in the development of equations.

**Preparation**

The children build a large mat, e.g. orange. An individual display board is ready on a low chair in front of an open space. A child, chosen to work on the display board from a box of rods on the chair, puts the orange rod out as base.

**Development**

The children who are mat building at their tables are asked to look carefully at their mat, and to remember " the line they like best ".

They then move out and sit in the space before the display board, where the monitor chooses the child to have first turn.

This child must dictate the **complete** line as he remembers it, **before** the monitor places it upon the orange base. If the first child is correct, he may choose the next child. If incorrect, he must correct his row and the monitor chooses the child for the next turn.

The game continues until all children in the small gro ,› have had a turn.

**Note** Training in working quickly and giving clear " dictation " is necessary to make this game effective.

**Adaptations**

1 The monitor may commence with a certain pattern, and children orally " twist " it to create new patterns using the same rods.

2 Children may play the game in pairs, taking turns to dictate a line.

GAME—" Password ".

**Aim**

This game assists the development of ideas in addition.

**Preparation**

When the teacher has chosen the form which the game is to take, e.g. Robin Hood, children should have practice in the sequence of the game, so that the situation in which the password is used creates no confusion during the playing of the game. E.g. Robin Hood goes inside a circle of " trees ", while Friar Tuck challenges passers-by to give the password before they are allowed to proceed.

**Development**

Each child is challenged by a sentry thus :

" Finish the password for orange !
Dark-green plus . . . . . . . ? "

The person challenged is expected to answer,

" Dark-green plus pink equals orange."

If he does, he is allowed to pass. If the child is incorrect he must be " imprisoned " and redeem himself by answering a similar question later.

**Adaptations**

Themes may be varied, e.g. Soldiers, Spacemen, Cowboys and Indians.

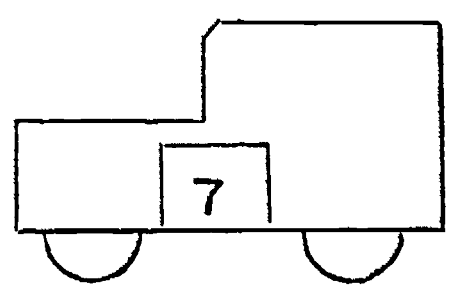
GAME—" D.24 ".

**Aim**

This game assists the development of flexibility in mat reading.

**Preparation**

Enough cardboard cut-outs of cars are made for each member of the group. A monitor is chosen to broadcast as " D.24 ". To each cut-out of a car, a number card is attached with a paper clip.




The children build a mat for dark-green.

**Development**

When the children have had sufficient time for their mat building, the game begins :

1 The monitor, " D.24 ", distributes the car cut-outs, so that each child has before him his mat and a numbered car, i.e. his " call-number ".

2 " D.24 " then proceeds :

" D.24 calling car number seven. Report please, car number seven."

3 The child whose car is numbered seven then reads out a line of his mat as he answers,

" Car seven reporting . . . . red plus pink plus white equals dark-green."

4 This is repeated until each child has had a turn to read his mat (or part of his mat).

**Adaptations**

1 Mats may be different for each child.

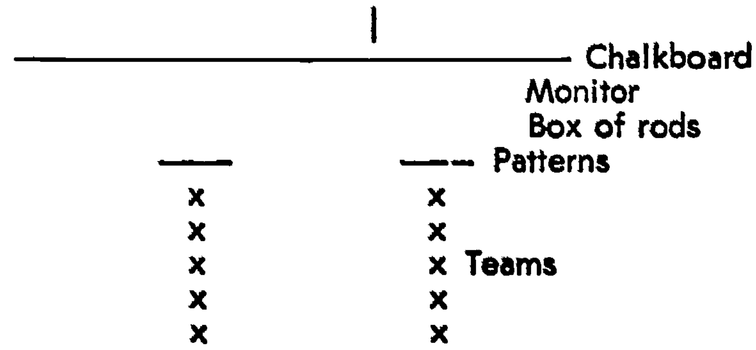
2 " D.24 " may read one pattern, and call upon the " cars " to twist it. He quickly constructs each one as it has been dictated, and each car must report a different version.

GAME—" Race ".

**Aim**

This activity develops speed in combining the rods.

**Preparation**




Lay-out for the game would be as above, with the score recorded as follows :—

| Team 1 | Team 2 |
|---|---|
| III | II |

1 A monitor is chosen to call the problem and to keep the score.

2 The children stand or sit in two teams, with a box of rods in front of each leader.

**Development**

1 The monitor calls " Black ! "

2 The two leaders vie to—

i compose a pattern for the black rod, and

ii prove it.

3 The first child to do so scores a point for his team. The two leaders then put the rods back and go to the rear of their teams.

4 The children then leave the others to continue the game, and the highest scoring team at the end is the winner.

**Adaptations**

The monitor may call out a combination and the teams be asked to vary it or find another combination equal to it.

GAME—" The Old Woman from Botany Bay ".

**Aim**

This game assists in developing skill in pattern building and the manipulation of rods.

**Preparation**

The children are seated in a circle, with one box of rods between two people. The " Old Woman " is chosen, and given a basket containing tokens, e.g. shells.

The children learn the following rhyme :—

" Here comes the old woman
From Botany Bay,
And what does she want us
To give her today? "

**Development**

The children speak the rhyme as the Old Woman circles the group. She then answers by saying what she " wants ", e.g. " A pattern to equal the brown rod."

Each child first constructs a pattern, then " proves it " with the brown rod. If wrong, he must leave it until it has been seen, then correct it.

The Old Woman quickly distributes shells to those correct, and the game is repeated until one child has five shells, when he or she becomes the Old Man or the Old Woman.

**Adaptations**

1 The Old Woman may ask for the missing rod from a pattern.

2 The Old Woman may show the pattern, while the children show the rod to equal it.

3 The children may be required to manipulate a given pattern.

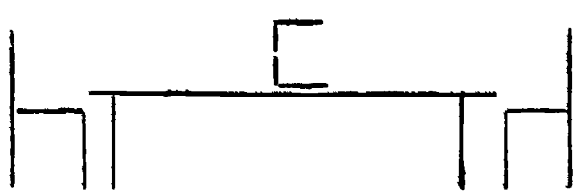
GAME—" Partner Post ".

**Aim**

To assist in addition (or subtraction, through complementary addition).

**Preparation**

Each pair of children collects a chalk-box lid. Into the top of this lid has been cut a square hole, large enough to accommodate a rod, end-on. The chalk-box is tipped on its side on the table so that the former top surface containing the hole is now a " wall " between the two children.



Each child has access to rods.

**Development**

1 The first child " posts " a rod through from his side, e.g. black.

2 The second child posts back two rods (as exchange) which equal black, e.g. light-green and pink.

3 The first child " proves " it. If correct, the second child has first turn in the next round of the game.

**Adaptations**

If the mat being used is " blue ", this rod may be put on top of the box to remind the children. The game would then proceed :

1 The first child posts a rod smaller than blue, e.g. yellow.

2 The second child posts back the rod which, when added to the first, equals blue, i.e. pink.

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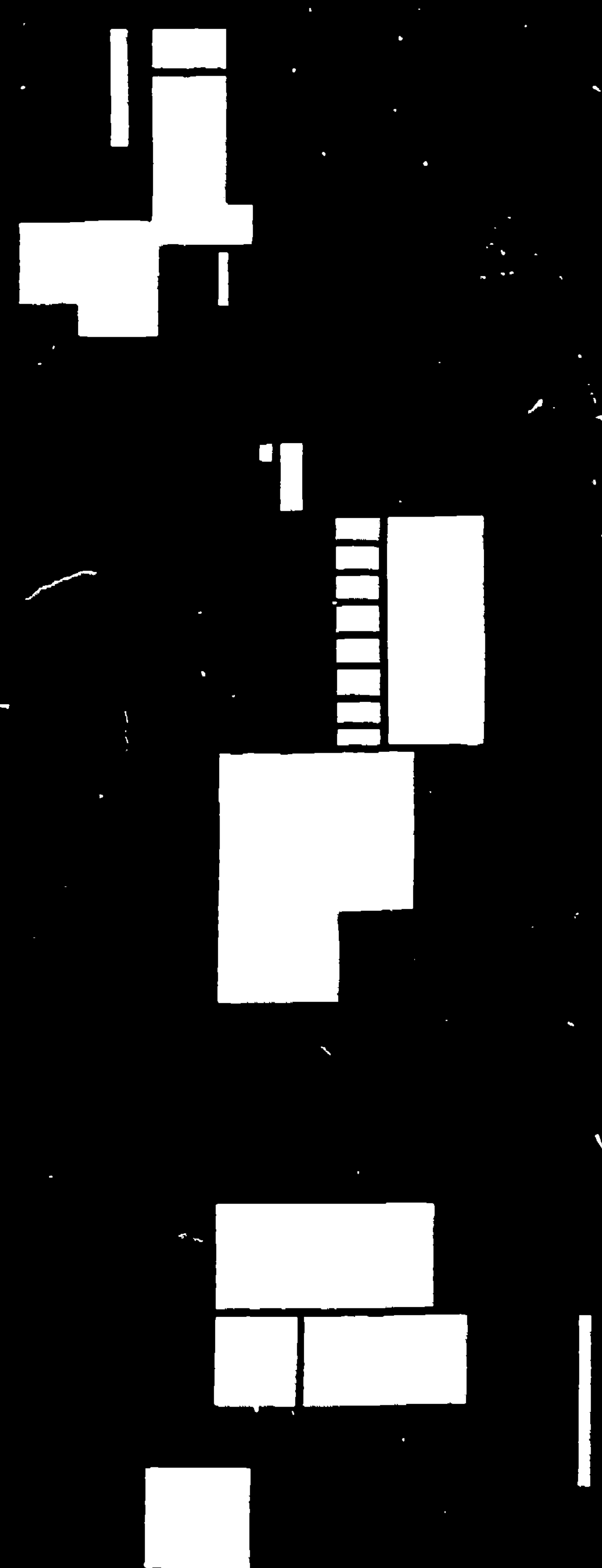


# CURRICULUM GUIDE
# PURE NUMBER COURSE

## Section C—The Study of Basic Mathematical Ideas Using the Numbers from One to Ten



**Education Department of Victoria**

6715/67.

# CURRICULUM GUIDE

# PURE NUMBER COURSE

## Section C—The Study of Basic Mathematical Ideas Using the Numbers from One to Ten

### AIMS

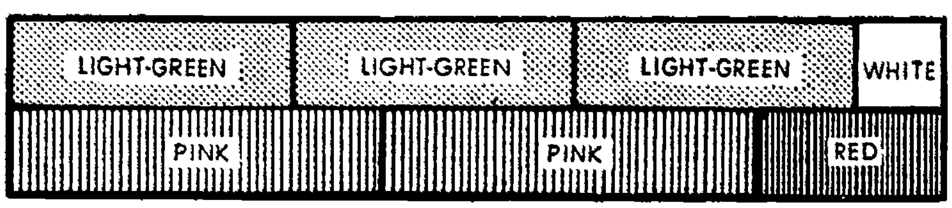
The aims of this section are :

1. To extend the child's understanding of the ideas introduced in Section B.
2. To enable him to express these ideas using the numbers from one to ten.
3. To introduce elementary ideas of fractions.

### COMMENTS

1. The aim of the last section was to develop an understanding of certain basic mathematical ideas—ordinal number, cardinal number, equality, and the operations of addition, subtraction, multiplication, and division. The aim in this section is the same, but numbers, not colours, are used to express the ideas. In the last section the final indication of the understanding required was the ability to organize and create equations using the ideas studied. This ability is the final aim of this section also.

In the Comments on Section B it was stressed that a child should read a pattern such as this :




in many ways. A small selection of readings would be :

Light-green + light-green + light-green + white = pink + pink + red.

3 times light-green + white = 2 times pink + red.

Red = 3 times light-green + white — 2 times pink.

White = 2 times pink + red — 3 times light-green.

By the end of this section, Section C, he should be reading this pattern, assuming that white is the unit, in ways such as:

$$3+3+3+1=4+4+2$$
$$3\times3+1=2\times4+2$$
$$2=3\times3+1-2\times4$$
$$1=2\times4+2-3\times3$$

The main purpose of this section is to develop the child's understanding of the ideas being studied to the stage where he is able to use them to create and re-arrange equations.

2. Because of this it is accurate to describe this section as being mainly concerned with a change of vocabulary. The type of exercise done is the same as that done in the last section. The aim is the same—understanding of basic ideas. In Section B, however, the vocabulary of colour was used, whereas in this section the vocabulary of number is used.

3. It would be a misunderstanding of the section to refer to its purpose as the "analysis of number". The child is not analysing numbers, he is using an understanding of certain ideas to build, to put together, equations. Numbers, in themselves, are of little more importance in this section than colours were in the previous one. Both colours and numbers are a means to an end—they provide a vocabulary through which ideas are discussed and studied. It is the ideas which are important—not the numbers that express them.

4. Automatic response to number facts or tables is of no concern in this section. In the first place, automatic response is not necessary for the achievement of its main aim. The ability to say, "Five" when the teacher says, "Two plus three", gives no indication of the child's ability to create and manipulate equations—and it is this ability that the section is endeavouring to develop. Concentration on automatic response can be more than the cultivation of an unnecessary skill: it may seriously hinder the main aim of the section. To gain the level of understanding needed for the successful completion of this section, a considerable amount of time is required. If time is limited by work designed to secure automatic response (a skill which also requires regular attention if it is to be developed efficiently), the degree of understanding will be correspondingly less. Thus, not only is automatic response unnecessary for the achievement of the main aim of this section, but concentration on it may seriously limit the level of understanding obtained.

5. Though the ideas studied in the last section constitute the main bulk of the work in this, they are not the only matters needing attention. Introduction of numerical values for the rods is the first task. The method of doing it is outlined in Stages 15 to 17. Once this problem has been overcome, the child is free to concentrate on the main task of this section—the understanding of basic mathematical ideas and their expression in terms of number. This occupies Stages 18–21. When this is reasonably well established, the child's range of ideas is extended to include a new concept, fractions. **This is the task of Stage 22.**

# STAGE 14

**AIM**

To continue the study of ordinal number.

**NOTES ON AIM**

1. The work of counting continues parallel with, and not separate from, the other stages of this section.

2. The threefold purpose of counting should now be considered.

a In the first place, counting is an essential skill in its own right. Unless a child can count, he is of necessity limited. He cannot understand ordinal or cardinal number; he cannot even read a pattern, for the ability to say "three light-green rods" assumes the ability to count to three. Previous sections have established this ability.

b Secondly, counting introduces the child to numbers which, later, he will study intensively. It extends his horizon so that he explores at a superficial level the ground he will later cover in detail. In this section, it is therefore necessary to count to 20 in preparation for the detailed work of Section D. This is, however, a minimum requirement; most children are quite capable of counting further and should be encouraged to do so. Upward limits are governed by children's ability and interest and the time available.

c Thirdly, counting can lead the child to discover the pattern and order that exist in the number system. The importance of this will further be stressed in later sections.

**DEVELOPMENTAL STEPS**

**1. Counting by Ones to Twenty**

Counting by ones to twenty, forwards and backwards, should be treated as a preparation for the work of Section D. Counting within this range (e.g. from 9 to 14, from 20 to 15, etc.) should also be done.

**2. Group Counting**

Children at this stage should begin group counting, but only where a pattern can be noted, either orally or visually.

Counting by tens, by fives, and by twos provides scope for the earliest work, and should be taken only as far as a child can proceed comfortably.

Counting by threes and by fours, possibly within a small range, could follow, but counting by sixes, sevens, eights, and nines would serve little purpose at this stage, because the child's counting skills are insufficient for the pattern to appear.

a First group counting will probably emerge from oral, rhythmic counting, e.g.

i One **two** three **four** five **six** seven **eight** nine **ten.**

ii Ten twenty thirty forty fifty sixty ......

iii One two **three** four five **six** seven eight **nine.**

b When written work is possible, exercises such as filling the gap 1 2 — 4 5 — 7 8 — could be used. A pattern can then be seen as well as heard, e.g. 5 10 15 20 25 30 (alternating 5 and 0).

c Black-board recording of group counting can lead to the building up of simple number charts, revealing patterns such as the following:

| | |
|---|---|
| 1 | 2 |
| 3 | 4 |
| 5 | 6 |
| 7 | 8 |
| 9 | 10 |
| 11 | 12 |

(counting by twos)

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |
| 10 | 11 | 12 |

(counting by threes)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |

(counting by fours)

(The construction of number charts is an introduction to the multiplication tables which will be studied in later sections.)

### 3. The Study of Ordinal Number

Work with the aspect of number dealing with an object in a set position in a series (16th, 18th, 19th) as outlined in Stage 2 must be extended as counting develops.

### 4. The Writing of Figures

Just as in Section B the child learned to recognize and write figures to 10 in preparation for the work of this section, so now he must extend this skill to 20 in order to be equipped for the next.

### 5. The Recognition of Words to Ten

The recognition and writing of "number words" is not strictly a mathematical skill, and may well be handled during reading and spelling periods, but the matching of figures and words should not be neglected.

N.B.—These five steps are not sequential. Steps 3, 4, and 5 should proceed along with 1 and 2.

## NOTES ON METHOD

1. Counting work is basically oral, associated at first with concrete materials. When the spoken number names have been related to symbols, written work should be done. (The children can write a certain amount, and the teacher can do more on the black-board.)

2. A pattern is heard in oral work before it is seen from written symbols (see Developmental Step 2, Group Counting).

3. The familiar counting from 1 to 9 finds further application in 21-29, 31-39, etc.

4. It is important that counting ideas should be presented in as many ways as possible, for one may be more significant to a particular child than to another. Concrete materials, oral work, and study of various arrangements of numbers should all be used. Counting frames are useful. The work in applied number involves counting. For example, in measuring we count 12 steps, 7 spoonfuls, 3 cupfuls.

**TESTING**

**1. Counting by Ones to 20**

a Count from 7 to 13.

b Go on counting when I stop. (Teacher begins a "count", child continues.)

c If I am counting by ones, what number comes after twelve ?

d Fill in the missing number :

i 11 12 13 14 — 16

ii 20 19 18 — 16

e Which is greatest : 2, 9, or 7 ? 13, 11, or 9 ?

**2. Group Counting**

a Count by twos to 10.

b 5 10 15 20. "What am I counting by ?" "What is the pattern ?"

c Counting by threes, what number comes after 6 ?

d Fill in the missing number :

i 2 4 6 8 —

ii 3 6 9 — 15

**3. Study of Ordinal Number**

If children correctly used the terms "first . . . . tenth" in Sections A and B, and were able to perform the activities outlined there, specific tests would hardly be needed now. The ability to use the terms "eleventh . . . . twentieth" in respect of days of the month, positions of children in line, and wherever appropriate is in itself a demonstration of understanding.

e.g. i "Today is Tuesday the sixteenth of August. What will tomorrow be ?"

ii "Who is twelfth in line ?"

**4. Writing of Figures**

Write the figures 19, 17, 13.

Write the number that comes after 18.

**5. Number Words**

a Write the words for these figures : 2, 4, 8.

b Write the figures for these words : three, nine, seven.

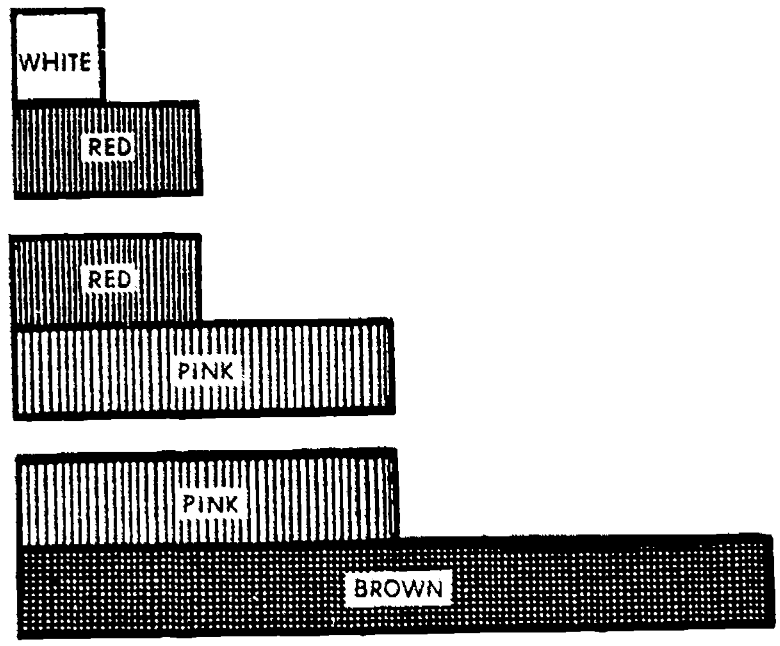
# STAGE 15

**AIM**

To lead the child to realize that the rods have no fixed value, but vary in value according to the rod used as the unit.

**NOTES ON AIM**

1. It is important that, from the moment when number is introduced, the child should realize that no rod has a fixed, pre-ordained value. The value of any rod is determined solely by its relation to the rod nominated as the unit or measure. In the patterns below, for example, the number "two" is represented by three different rods; in the first case by a red rod (where white is the unit), in the next by a pink rod (where red is the unit), and finally by a brown rod (where pink is the unit).




2. There are important reasons why the child should understand this idea.

a It is fundamental to the development of future mathematical ideas.

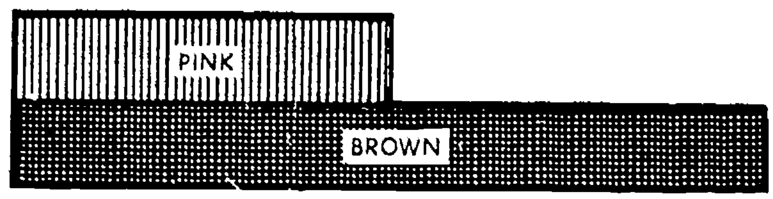
b Unless the child realizes that the value of the rods is dependent on the relations between them he does not understand the material. The belief that the rods have necessarily fixed values is mathematically false, and there can be no ju:ification for allowing the child to form an incorrect idea.

c Quite apart from the mathematical principle involved, an inability to use different rods as units severely limits the later applications of the material. The understanding of many ideas is dependent upon an ability to use different units. In a later section, for example, the concept of equivalence of fractions can very simply be demonstrated

If white is called one, this pattern could be used to represent $\frac{2}{6}$:


But if red is called one, and dark-green three, the pattern represents $\frac{1}{3}$. So we can see that $\frac{1}{3} = \frac{2}{6}$.

Similarly the pattern:




can represent $\frac{4}{8}$, $\frac{2}{4}$, or $\frac{1}{2}$ according to whether white, red, or pink is the unit, and we can show that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. This would not have been possible had a fixed value been given to each rod.

d The fourth reason for the study of this concept is its contribution to the treatment of cardinal number. This matter is discussed fully in Stage 17.

3. If the child is introduced to the fact that the rods have no fixed value on his first use of number with the material, he accepts the idea without any difficulty. If, however, he is allowed to think, or, worse still, positively taught, that a certain rod is always "one", the child is later confronted with the problem of unlearning. This difficulty is avoided if the changing of units is introduced immediately number values are given to the rods. Delay in introducing the idea is a spurious economy. It does not assist; it complicates.

## DEVELOPMENTAL STEPS

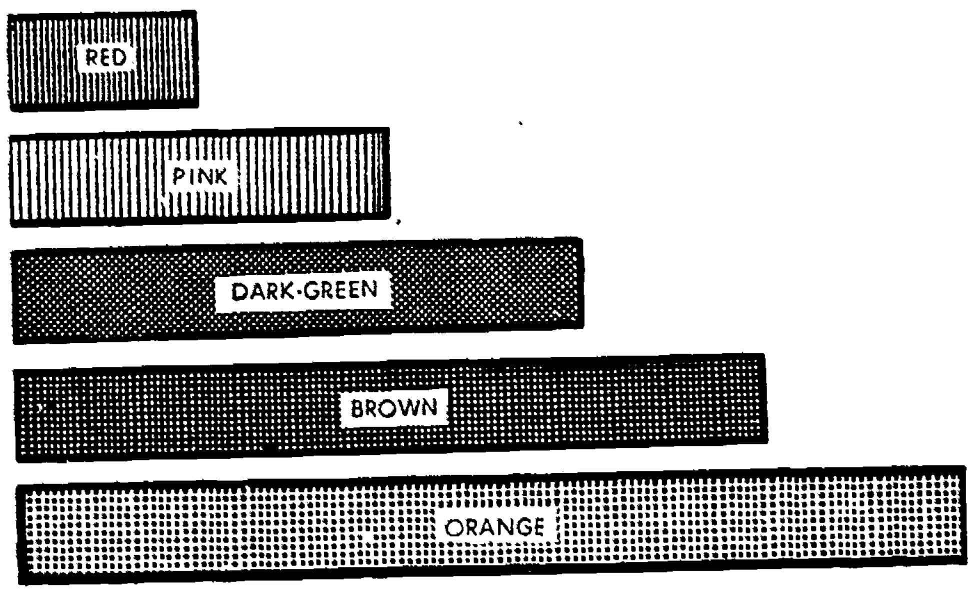
### 1. Introduction of the Concept

a Exercises should be given to emphasize that one quantity can be *measured* by another to prove equality.

e.g. The child shows 3 red rods. He then finds a rod to equal these three. When asked how he knows they are equal, he should be encouraged to use the term "measure".

b Children are asked to show one rod. They show various rods, and are then asked if theirs could be called "one". If there is any difficulty, further discussion will be necessary. This is to emphasize that any rod can be called "one".

c A rod (say red) is chosen as the unit, and called "one". Two of these are placed out and measured with one rod (pink). It is seen that the pink rod can be called "two" because it is equal to two of the rods called "one". This activity is repeated; the child, using the red rod as unit, measures to find rods with values of 3, 4, 5. A staircase is built from these rods:




d The same procedure is then followed using other rods as units and the appropriate staircases are built.

The child must be given a considerable amount of practice in this type of work. If this is given there is little need for the teacher to emphasize that the class is using different rods as the unit. The child simply uses different rods without considering it in the least strange and, at the same time, is learning that "two", for example, is not a fixed and definite length but varies according to the rod used as the unit. He sees also that, for example, the dark-green rod has been called "two", "three", or "six" according to the unit selected.

By the end of this step the teacher should be able to ask questions and receive answers such as:

"If red is one, what is four?"

(The child should immediately choose the brown rod.)

"Why is that four?"

"Because it is equal to four red rods."

The child able to do a wide range of such exercises easily and confidently, using different rods as units, has mastered the concept.

## 2. Application of the Concept in More Difficult Examples

This step introduces more difficult exercises involving the use of different rods as units. However, to proceed to the next stage, it is necessary to have mastered the previous step only, because mastery of that step gives sufficient guarantee that the child has realized that the rods have no fixed value. The more difficult examples of this step are not prerequisites for a move to the next stage, but should be developed throughout the whole of this and the following sections. They are included here to ensure that later ideas are not limited by a partial grasp of the concept.

In the next stage, Stage 16, the child learns that, although he can use any rod as the unit, it saves time and trouble if the same rod is used on most occasions. The rod chosen is, for purposes of convenience, the white rod. This step is most necessary but there is a danger. If, for most practical purposes, the child uses the white rod as the unit, he may, if steps are not taken to prevent it, forget that it is possible to use other rods.

In order to keep this point clearly in mind it is imperative that every time the material is used practice should be given in exercises similar to those outlined above. As his experience widens, the exercises should become more difficult, in order both to extend the child and to keep this aspect of his work in line with others.

The type of exercise listed below shows the path to be followed during **this and later sections.** Some of the exercises rely on a knowledge of fractions, which may, if desired, be studied before the end of Section C. No exercise should be attempted before the prerequisite understandings have been mastered.

a *Name Unit or Measure, Find Rods of Specified Values*

If red is one, what is brown?

If light-green is one, what is blue?

b *Give Rod a Specified Value, Find Unit*

If orange is five, which rod is one?

If dark-green is two, what is one?

c *A Combination of a and b*

If orange is five, which rod must equal three?

If brown is four, what is five?

d *Name Unit, Find Value of a Group of Rods*

If light-green is one, find value of these rods.

(Show, for example, dark-green and blue.)

e *Use Exercises of the Above Types, Involving Fractions*

If brown is one, what is pink?

If light-green is one, what is white?

If pink is one, what is orange?

If light-green is one, what are these worth together?—

i Orange and blue.

ii Yellow and white.

iii Brown and pink.

## NOTES ON METHOD

1. This stage cannot be started until the work of the previous section is completed. The main purpose of the present section is to change the vocabulary used to express the ideas the child has studied in the previous section.

If he has not mastered them thoroughly (i.e., if he has not completed the work of Section B), any attempt to introduce numbers will result in confusion.

2. It is important that the child should be able to explain any statement he may make. If, for example, he says that a certain rod equals "five" he must be able to give a reason for the statement, i.e., he must know that the rod designated equals five of the unit rods.

3. As always, one of the key factors in the acquiring of a new concept is the number and variety of experiences that can be provided for the child.

## TESTING

It has been pointed out that the child is ready to move to Stage 16 when he has mastered the work outlined in Step 1. Thus a fair test of this stage would be the last activity in Step 1.

# STAGE 16

## AIM

To lead the child to realize that, though the rods have no fixed value, it is desirable for most practical purposes to use "white" as the unit.

## NOTES ON AIM

This aim is accomplished in two stages:

a The child must be led to realize that if a different unit is being used by everyone a great deal of confusion is unavoidable. When there is ignorance of the unit being used, all that may safely be said is that a group of rods is arranged in a pattern. The value of the rods can only be guessed; it cannot be worked out independently.

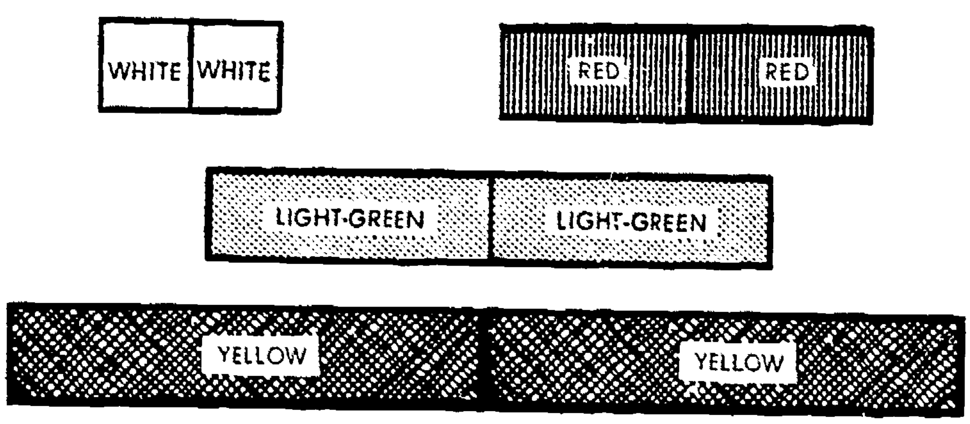
b When the need for an agreed unit has been recognized, the problem of selecting the unit must be faced. The child must realize that the white rod is chosen, not because of a whim or fancy, but because it is the most convenient rod to use as the unit.

## DEVELOPMENTAL STEPS

### 1. Establishing the Need for a Common Unit

A typical method of bringing home to the child the need for a common unit begins with the children being asked to show: "one plus one" using the rods.

Because the direction is vague a wide range of patterns may be given, e.g.




The children are asked to read out the rods they have used, and it becomes evident that a large number of different patterns have been supplied to represent the same expression. The teacher is then in a position to point out the difficulties and confusion inherent in this type of situation and to show the child that the problem would be solved if the same rod were used as a unit by everyone.

## 2. Choice of a Unit

Having decided that a common unit is necessary, the child now faces the task of deciding which rod should be chosen. At first, when asked this question he tends to choose as a unit a big rod which is easy to handle (e.g., an orange rod), or a rod whose colour he particularly likes. As soon as he begins to work with these rods he can be made to see their limitations: if a big rod is used as a unit the supply of rods runs out quickly; a line of rods spreads right across the desk, or, because it is long, is difficult to see at one glance.

After experiences such as these the child can be led to realize that the most convenient rod to use as the unit is the smallest—the white rod.

### NOTES ON METHOD

1. This stage is begun immediately the previous stage is completed—to begin before then is pointless.

2. This stage is one of the shortest in terms of teaching time. Quite often it can be completed in one period. It is nevertheless important. It is an endeavour to show a child the reasons for making a particular decision.

3. Because this stage is a prerequisite for efficient performance in the next, a child must continue with it until he has mastered it.

### TESTING

The test of a child's understanding is his ability to say, in language appropriate to the level of his mathematical development, why the white rod is the most convenient one to be the unit. That is, when white is the unit there are sufficient rods to represent each number from 1 to 10.

# STAGE 17

### AIMS

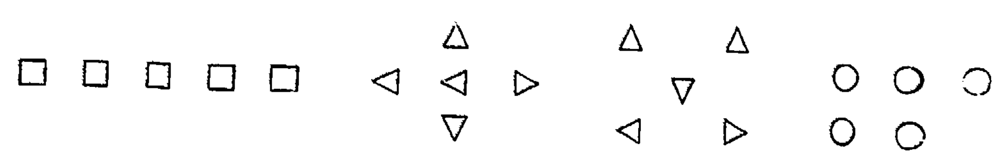
1. To obtain automatic recognition of the numerical value of the rods when " white " is the unit.

2. To develop further the concept of cardinal number.

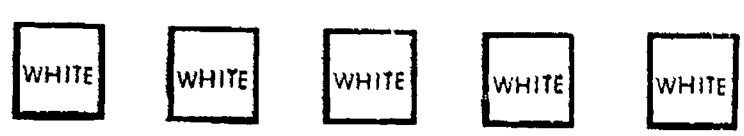
### NOTES ON AIM

1. Before the child can work efficiently with the rods using the vocabulary of number he must be able to recognize, without the least hesitation, the numerical values of the rods when white is the unit. If, for example, he is not certain whether the pink rod has the value of " three ", " four ", or " five ", any attempt to read a pattern will be hampered while he decides the value of the rod. It is then an urgent practical necessity that the child should be able to recognize immediately the value of each rod.

2. During this stage the child's concept of cardinal number is extended. It will be remembered that the development of this concept is gradual. The Comments on Section A show that the child's first appreciation of cardinal number is as a group. He realizes, for example, that despite differences in arrangements and the objects used, each of the groups illustrated is a group of " five ".



He has not, at this stage, moved to the more refined concept, where instead of thinking of " five " as a group of separate objects, e.g.




he thinks of it as a whole, a " five ".


The notes on Stage 5 point out that the child is being prepared for this concept. In this stage he should master it fully.

All through this and the following stages he sees

a two


a three


a ten

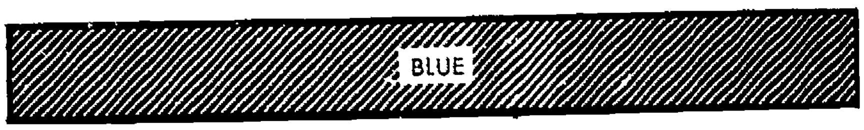

as distinct wholes. The rest of the work throughout the section emphasizes the concept: whether a child is asked to produce "a two", "a seven", "a nine" . . . . . he produces a single rod, not a collection of separate rods.

3. It must be emphasized, however, that although the child thinks of numbers as *wholes* he does not think that a given number must always be represented by a rod of the same length. Because of the work done in Stage 16 he knows that a "three", for example, may be represented by

this rod


or this


oi this




depending on the unit chosen. Thus his concept of three is not that of a fixed or rigid pattern, but of a variety of patterns, each with the common characteristic of a unit repeated three times. "Three" is not an arrangement of objects into an unvarying group, but an abstract relation.

4. His thinking may therefore be summarized by saying that he thinks of " three " as a *whole*, not as a group of separate objects; that he knows that the size of this whole may be different depending on the unit used; that usually he uses white as the unit; and that when using white as the unit he instantly recognizes the value of the other rods.

## DEVELOPMENTAL STEPS

### 1. Automatic Recognition of the Numerical Values of the Rods When White Is the Unit

Any activity that helps the child to see and remember how many white rods are needed to " measure " any other rod will be useful. Work with staircases, using number names instead of colour, will give practice in the use of the new vocabulary.

### 2. Cardinal Number

There is no need for separate activities to develop the child's concept of number as a *whole*. All the activities in the previous step are developing and reinforcing this concept at the same time as they are developing automatic recognition of the rods.

## NOTES ON METHOD

1. This stage is begun immediately the previous stage is completed. It cannot be begun earlier because the child is not in a position to give numerical values to the rods until he has decided upon a common unit.

2. It is most important to continue the work of Stage 15.

3. Before leaving this stage, the child's ability to recognize the numerical values of the rods, when white is the unit, should be sufficiently well developed to ensure that later work can be done without being hindered by an inability to recognize values quickly.

## TESTING

The oral work of Step 1 is the test for this stage.

# STAGE 18

## AIM

To master the vocabulary of number, and to develop the understandings of equality and the four operations through manipulation of equations.

## NOTES ON AIM

1. The first task is to achieve a simple change of vocabulary, and to regain a fluency that may have been lost.

2. Once fluency has been regained, the ability to manipulate equations is re-established and developed through more complex patterns.

3. This stage is concerned only with oral work. Competence is required here before written work is commenced. (See Stage 19.) Once recorded work does begin, however, oral and written work must develop side by side.

## DEVELOPMENTAL STEPS

### 1. Simple Readings (involving only equality and addition)

Patterns of the type used in the early steps of Stage 6, where only equality and addition are involved, will provide sufficient scope to ensure that the number vocabulary is used confidently.

### 2. More Complex Readings (to involve all operations)

Once the vocabulary is established, this should be used in patterns involving the four operations. Alternative approaches are suggested:

i Use a simple pattern. Read it in terms of all possible operations. Gradually increase the complexity of the patterns used;

or

ii introduce each operation separately, working from simple to complex patterns within each operation. Finally, combine operations (as in colour in Stages 9, 11, and 13 of Section B).

## NOTES ON METHOD

1. This stage is begun as soon as the prerequisite language skill has been achieved in Stage 17.

2. Irrespective of the approach taken in Developmental Step 2, the result should be that the child will be as proficient with the ideas using number as he was when using colour. The bright child will probably regain this facility almost immediately, whereas the slower child may need to be led gradually through the selected steps.

3. This stage marks the beginning of a greater depth of understanding. Firstly, the child whose work in Section B was somewhat limited in quality, should, through both maturity and experience, reach a deeper understanding. Secondly, the move to number extends the cardinal ideas of the previous section. Thirdly, an extension of the ideas studied in Section B and greater facility in their use will be possible now that the child is working with numbers, and should be evident as he progresses through the following stages.

4. It is imperative that the work in "value relations", as outlined in Step 2 of Stage 15, should continue, according to the child's capability.

**TESTING**

The testing for this stage is the same as for Stage 13, except that numbers, not colours, are used. The child should not move from this stage until his work in number has at least the same quality as it showed in colour.

# STAGE 19

**AIMS**

1. To develop ability to record an equation illustrated by the rods.
2. To develop the ability to interpret a written equation.

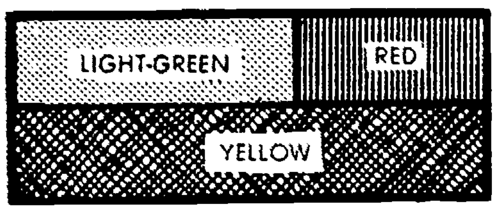
**NOTES ON AIM**

1. Until this stage the child has been concerned with developing ability to manipulate equations orally using, first, colours, and then by the end of the last stage, numbers. He has not yet learned to record equations using numbers. He has, of course, learned to write the numbers during his work with counting. (See Stage 5.)

The task of this stage, therefore, is not to teach the child how to write figures, but to show him how to use his ability to write figures in order to record equations that he can express orally.

2. There are two skills the child must acquire:

a He must learn to record in figures an expression previously made by the rods. He must know that this pattern



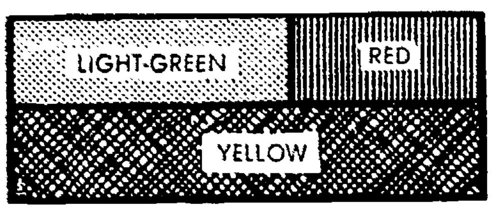

may be written as

$$3 + 2 = 5.$$

b He must be able to interpret a written equation with the rods. i.e., to move from

$$3 + 2 = 5$$

to the pattern




thus showing his understanding of the signs and their meanings.

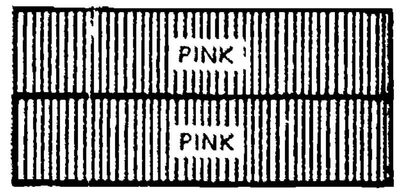
3. The two skills, recording in writing a pattern made with the rods, and interpreting with the rods a written equation, constitute the aim of this stage. While developing these two skills the child is concerned with the mechanics of writing equations.

At a later stage, he must learn to solve uncompleted equations without using the rods at all. He is not, however, concerned with this during the present stage.

**DEVELOPMENTAL STEPS**

**1. Recording an Equation Illustrated by the Rods**

**a Equality** In order to introduce the child to written work, the teacher may ask him to make a simple pattern such as:




and to read it (" Four equals four ").

Faced with the problem of writing this, the child finds he knows how to write " 4 ", but not " equals ". He can be shown the word and the sign; it is then a simple matter to record 4 = 4. Sufficient practice is given to make him familiar with this sign.

**b Addition, Multiplication, Subtraction, Division** A similar procedure to that above can be followed to introduce the four signs + × — ÷. (It should be noted that the division sign (÷) is taken to indicate quotition division. The expression 8 ÷ 4 is read as " How many fours equal eight? " or " In eight, how many fours? " Partition division is not recorded until Stage 22, when fractional notation can be introduced.)

Each sign should be used long enough for the child to be quite at home with its use before another is introduced.

**c Brackets** In Stage 10 (Subtraction in Colour) the idea of brackets was used, even though the term may not have been introduced. It was suggested that phrasing and the word " together " be used when a group of rods was compared with another rod to find the difference, e.g., blue minus (pause) light-green plus red together (pause) equals pink. This technique would have been carried over to oral work with number (9 minus 3 + 2 together equals 4). Brackets must now be shown as the means of holding the written symbols together, resulting in the recording 9 — (3+2) = 4.

NOTE: In this earliest recording it is advisable that the child should:

a construct an equation with the rods;

b read the equation;

c write the symbols to form the recorded equation.

**2. Interpreting a Written Equation**

a The teacher writes on the black-board an equation such as

$$3 + 2 = 5$$

and asks the child to construct this with his rods. (The equation could also be written by the child, but more work in interpretation can be covered in a given time if most of the work is oral. As the work progresses, recording and interpretation will go hand in hand.)

b Simple equations involving multiplication, subtraction (without and with brackets), and division are introduced, each operation being treated separately.

The emphasis is on showing an understanding of the symbols, and being able to do what they direct, e.g., in $8 - 3 = 5$ it should be clearly seen that it is required to show that the difference between the eight and the three is equal to five.

## NOTES ON METHOD

1. It is most important that the child should not begin written work until he has the ability to read patterns orally with fluency and ease. Thus this stage should not begin until a high degree of skill has been achieved in the previous stage.

2. The basic method used in this stage is the matching of figures and rods. Either the child begins with a pattern of rods and writes in figures the equation it represents or he begins with a written equation and forms a pattern to represent it with the rods. The child is not asked to manipulate numbers and record results. His task is simply to learn how to use figures to record mathematical ideas.

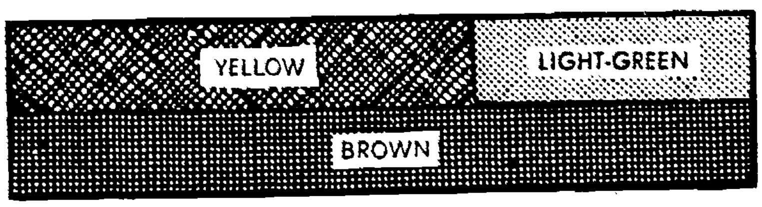
3. The next stage will require a complexity not needed here, e.g. $4 + 3 + 2 = 6 + 3$; $2 \times 5 - 3 = 7$. However, it should be noted that the stages merge. There is no sharp division, and some children will be able to move fairly rapidly through the simple work of Stage 19.

4. Work involving the use of units other than white is continued as outlined in Stage 15.

5. During this stage the oral manipulation of equations begun in the previous stage is continued.

## TESTING

1. In order to test ability to record an expression illustrated by the rods the teacher gives a pattern such as this:




and asks the child to write down an equation this represents. The child may write ;

$$5 + 3 = 8 \text{ or } 8 = 3 + 5$$

He is then given other suitable patterns and asked to record subtraction, multiplication, and division examples.

2. To test the child's ability to interpret a written equation, a set of examples such as those shown may be written down :

| | |
|---|---|
| $4 + 2 = 6$ | $4 \times 2 = 8$ |
| $4 - 2 = 2$ | $4 \div 2 = 2$ |

The child is asked to write each one and illustrate it with the rods.

# STAGE 20

## AIMS

1. To give experience in more complex manipulation, both oral and written.

2. To extend the work in interpretation of equations.

3. To introduce the solving of simple equations.

## NOTES ON AIM

1. This stage has an aim similar to that of Stage 18—the development of the ability to manipulate numbers using the four operations. In this stage, however, the child is led to express his results in writing as well as orally. Thus the difference between the first part of this stage and Stage 18 lies in the manner in which the child expresses his ideas. There is no difference in his activities.

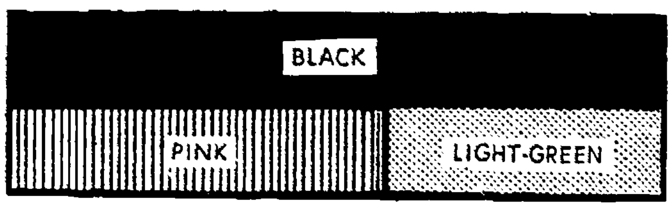
2. Added to the oral and recorded manipulation is the extension of interpretation of written equations, including specific types not previously encountered, and this leads to the solving of uncompleted equations.

3. Once introduced, all aspects of the stage (reading, recording, interpreting, and solving equations) should proceed side by side, with much greater emphasis on the first three than on solving equations.

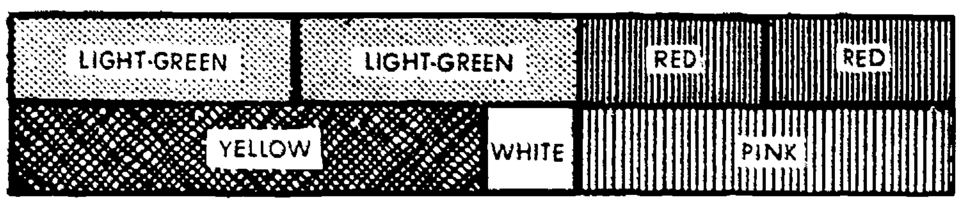
## DEVELOPMENTAL STEPS

### 1. Oral Pattern Reading

This type of activity, begun in Stage 18, is continued in this stage. The child is given a pattern and asked to read it in many different ways. In doing such manipulative work the particular rods used and the order in which they are used are of no great importance. What is necessary is that the complexity of the patterns increases gradually from the simple type :




to the more complex pattern where both rows in the pattern are broken up :

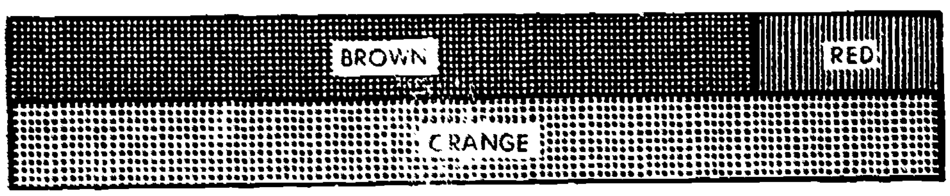



(For a graded series of patterns that increase in complexity see Developmental Steps for Stage 6.)

It should be noted that the value of the readings is not in terms of the great number of addition and subtraction readings possible, but in the variety of operations used.

**2. Recording of Manipulations**

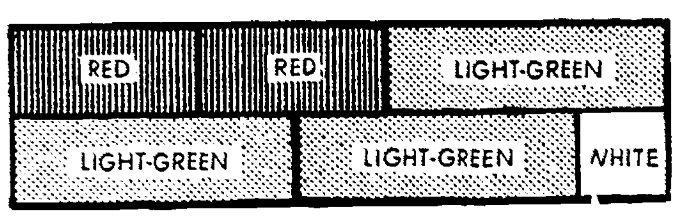
Side by side with the work of the previous step goes the work of recording in writing the results of manipulative work. The child again begins with a simple pattern:




and writes down the ways in which he can manipulate it, e.g.

| | |
|---|---|
| $8 + 2 = 10$ | $10 - 2 = 8$ |
| $2 + 8 = 10$ | $10 - 8 = 2$ |
| $10 = 2 + 8$ | $8 = 10 - 2$ |
| $10 = 8 + 2$ | $2 = 10 - 8$ |

In Stage 19, any one of the above equations would be acceptable; now the child should be capable of all the variations. Gradually (as in the previous step) the complexity of the pattern is increased. From this pattern:




equations such as the following could be written:

$$2 + 2 + 3 = 3 + 3 + 1$$
$$2 \times 3 + 1 = 2 \times 2 + 3$$
$$3 + 2 \times 2 - 1 = 3 + 3$$
$$2 \times 3 + 1 - (3 + 2) = 2$$
$$1 = 2 + 2 + 3 - 2 \times 3$$

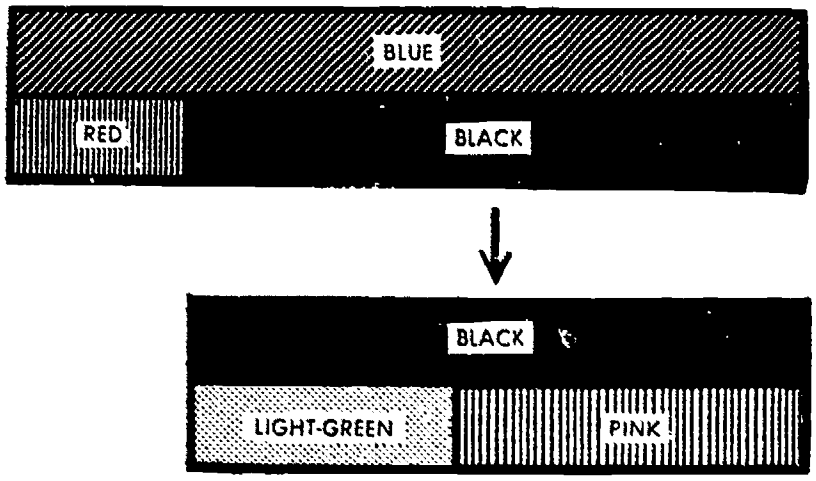
NOTE: It is most important to stress that these two steps, the oral and the written manipulation of numbers, proceed side by side, gradually increasing in complexity. It is here that the child begins to see what happens when he records many equations from one pattern. He needs a great deal of experience in this. It will be noted that, as time goes on, he manipulates the equation he has written, without always referring to the rod pattern. This is the skill that will enable him in the next stage to move towards abstraction (see Stage 21, Step 2).

### 3. Repeated Subtraction

a In Section B, Stage 10, Notes on Aim, reference was made to repeated subtraction. If it has not been treated before, it should be during this stage.

The idea involved is as follows: 9 — 2 — 3 = 4

In detail this means: 9 — 2 = 7; 7 — 3 = 4 and it can be illustrated thus:

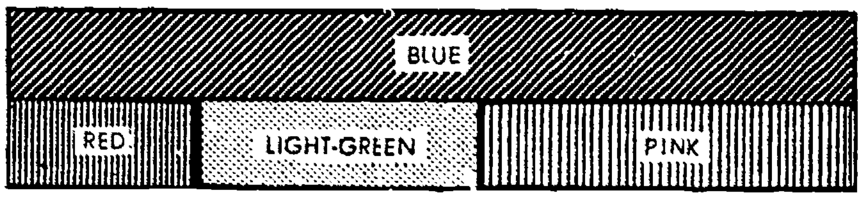



9 — 2 = (placing the black rod) 7

Bring the black rod to new position.

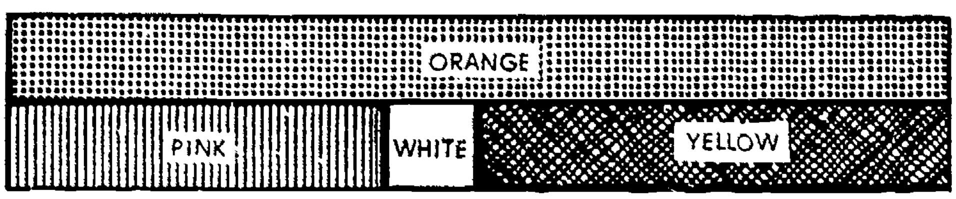
7 — 3 = (placing the pink rod) 4

After extensive practice in such examples, through which the child sees the repetition of the subtraction operation, he can place the rods thus:




and read 9 — 2 — 3 = 4. He should understand that he is taking a short cut, i.e., he places rods to show 9 — 2, and before placing an answer, proceeds to the next subtraction, — 3.

Once this idea is mastered, the child should read and write from a pattern such as this:

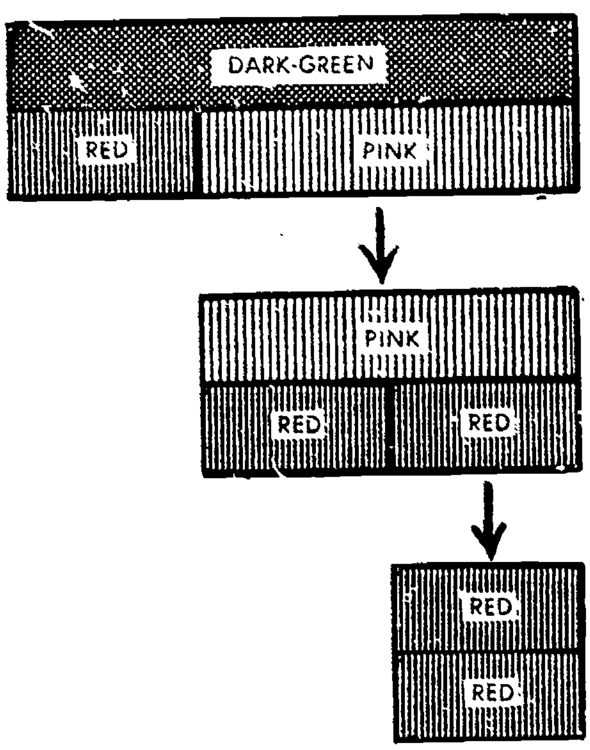



10 — (4 + 1) = 5 and

10 — 4 — 1 = 5

b Another idea which needs to be understood is that of the special form of repeated subtraction where the same number is subtracted each time (e.g. $6 - 2 - 2 - 2 = 0$), leading to division.

A child illustrating this with his rods would first see :




$6 - 2 = 4$

$4 - 2 = 2$

$2 - 2 = 0$

Then, after sufficient practice in similar examples :


$6 - 2 - 2 - 2 = 0$

This pattern, in isolation, has previously been read as addition $(2 + 2 + 2 = 6)$ and multiplication $(3 \times 2 = 6)$. After the above exercises, it should be seen also as $6 - 3$ twos $= 0$, and $6 \div 2 = 3$ (i.e. " How many twos equal six ? "—" Three ". Two is being used three times to measure six.)

This idea is important in that it lays the foundation for relating division and subtraction.

**4. Interpretation of Equations**

Development here is in two directions :

a More complicated examples of the use of rods to illustrate a written equation.

b A verbal explanation of the meaning of a written equation.

In paragraph a, examples should include two operations, e.g.

$4 + 2 \times 3 = 10$ $\quad$ $9 + 1 - 4 = 6$

$6 + 3 = 3 \times 3$.

NOTE: It should be brought to the child's notice that in illustrating $4 \times 2 = 8$ he does not have a rod to represent the 4. It is the number of twos that equal eight. Similarly, in illustrating $10 \div 5 = 2$, no rod is used to represent the 2. It is the number of fives needed to measure the ten. For this reason, care needs to be taken in selection of examples, e.g. $8 \div 2 + 3 = 7$ involves a complication for which most children are not yet ready.

In paragraph b, children may be asked to read equations and explain what they understand.

e.g. i $4 + 3 = 7$ is written, then read aloud.

Questions such as these could be asked:

"What do you mean by 'plus'?" (Joined to, added to . . .)

"What are added (joined, put together)?" (The 4 and the 3.)

"Which is bigger, the 4 and 3 together, or the 7?" (Neither. Both are of the same size.)

"How do you know?" or "What tells you?" (The equals sign.)

NOTE: The idea of equality needs now to be developed from "same length" or "same size", which was adequate for rods, to "worth the same" or "of the same value", when applied to abstract numbers.

e.g. ii $2 \times 3 + 1 = 7$

Questions here should lead to the explanation that the two threes and the one are the "parts" joined by the "plus".

e.g. iii $9 - 2 \times 4 = 1$

"What do you mean by this?" (The minus sign.)

"Find the difference between what?" (The 9 and the 2 fours.)

"How much difference is there?"

e.g. iv $4 \times 2 = 8$ should continue to be related to the addition $2 + 2 + 2 + 2 = 8$.

e.g. v $9 - 9 = 0$ can be explained thus: "There is no difference (in size or value) between the two nines" or "Nothing needs to be added to the second nine to equal the first."

Many other examples will suggest themselves to the teacher. The study of equations written by the children themselves will prove profitable.

## 5. Solving Uncompleted Equations

It is in this step that the child, for the first time, begins to study uncompleted equations. He is given examples such as :

$3 + 4 = \square$ $\qquad$ $3 \times 3 = \square$

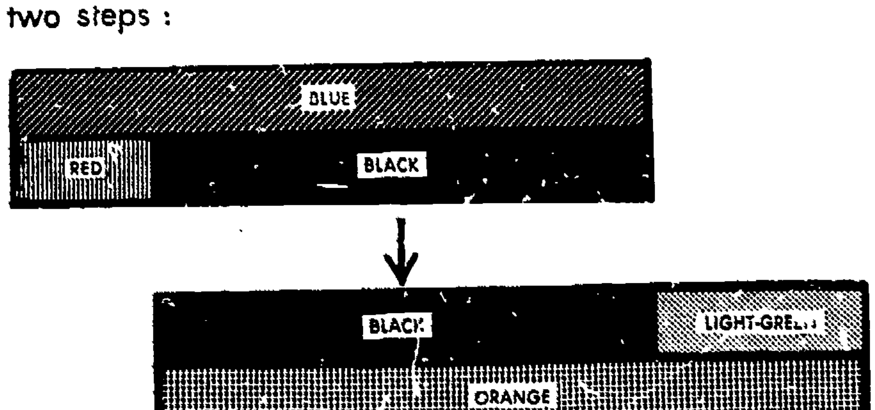
$9 - 2 = \square$ $\qquad$ $10 \div 2 = \square$

and asked to use his rods to find an answer. It is through his work in interpreting completed equations that the child can be led to do this. When he is confronted with $3 + 4 = \square$ he understands that he must add together a 3 and a 4 and find the number which is equal to these. In $9 - 2 = \square$ he knows 9 and 2 must be compared in such a way as to find their difference. Thus he carries his understanding and experience to a new situation.

Ability to solve equations is an essential skill, and, although it is much less important in this stage than the creation of equations, it is nevertheless necessary to ensure that the child has developed ability to handle this type of work. This is an activity, however, that should occupy a much smaller proportion of the child's time than his creative and interpretative work.

The examples of the uncompleted equations must be graded so that, like the work in the previous steps, they move from simple to complex examples. At first it is sufficient to solve equations involving one operation only, with the unknown in different positions. Gradually the difficulty is increased in terms of the number of operations used. It is no more difficult to solve $6 + 4 = \square$ than $2 + 1 = \square$, but $2 \times 3 - (2 + 1) = \square$ is very much more complicated than $8 - 4 = \square$, although it involves smaller numbers. The child should not be asked to solve an equation of a type he cannot interpret. From the point of view of complexity, creative work (dealt with in detail in Stage 21) should be furthest ahead, then should come interpretation, and finally solving uncompleted equations.

NOTE : A type of equation needing special attention is $9 - 2 + 3 = \square$. This has not been read by the child in his oral work. He needs to be shown that this is a problem that must be attacked in two steps :




$9 - 2 = 7$

$7 + 3 = 10$

(There is here a foreshadowing of the convention of working from left to right when only addition and subtraction occur, and more will be said of this later.)

It will be found helpful to associate :

$$9 - (2 + 3) = \square$$

$$\text{and } 9 - 2 + 3 = \square$$

through many examples, so that children become familiar with the difference made by using the brackets.

## NOTES ON METHOD

1. This stage is a continuation of the previous one, and develops naturally out of it. The techniques established here (oral manipulation reading from patterns of rods, recording these readings, interpretation and solving of equations) must all be used from now on; none should be neglected at any time.

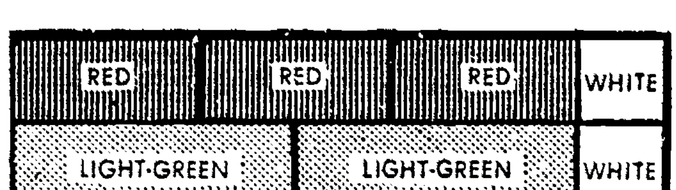
2. A danger in this stage is the neglect of oral work in order to concentrate on written work. It must always be remembered that oral work, because it allows a large number of examples to be covered, is of the utmost importance, and although less time is devoted to it than in previous stages, it must not be omitted or skimped of time.

3. A noticeable feature of the work of some children in this stage is a reluctance to use the rods at times. They find that their mastery of the skills and concepts required is so great that the rods tend to slow them down. This matter should be carefully watched. In the next stage the child is asked to work without using the rods. In that stage, for the first time, the ability to work without the rods is deliberately developed by the teacher. Before then, however, some children begin to work independently. This ability does not suddenly *happen*, it develops gradually. Often it is in this stage that the first indications of it are noticed.

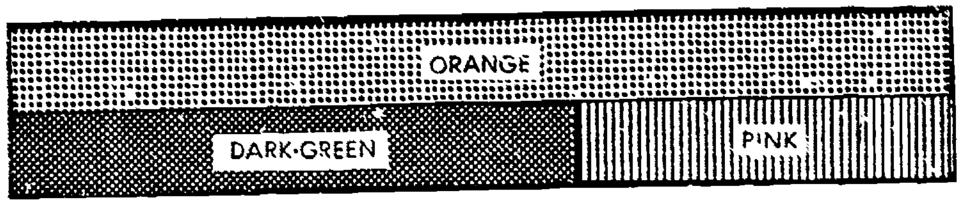
4. Probably the most common source of error in this stage is the desire to organize the child's work in terms of the numbers studied. Sometimes this desire takes the form of an attempt to devote an equal amount of time to the study of each number. At other times it can be seen in an attempt to grade the child's work so that he moves from the smaller to the bigger numbers.

It cannot be stressed too strongly that the numbers used are not important. An aim of the stage is to develop facility in the manipulation of equations. Accordingly the stage is organized so that the child begins working with simple equations and moves to more complicated ones.

A failure to organize the stage on this basis can confuse the child and cause unnecessary difficulties. If, for example, the stage is organized so that the child moves from smaller to bigger numbers he may be presented with a pattern such as is common when using number seven :



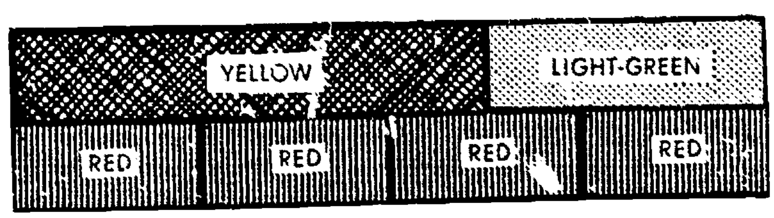
before he meets a pattern such as :




which is met when using ten. Yet the first pattern suggests far more complicated equations (e.g. $2 \times 3 + 1 - 3 \times 2 = 1$) than the simpler type of equation suggested by the second pattern (e.g. $10 - 6 = 4$).

If thought is given to the type of equation (or the idea to be studied), and this is applied to rods of varying sizes, the desired results should be obtained.

5. The child's creative work sometimes reveals a need for brackets apart from that already dealt with. It is most undesirable to force children into such situations, but the teacher must be alert to the possibility of their occurrence, and ready to deal with them if they occur. For example, some children express the equation :




as $5 + 3 \div 2 = 4$, seeing the division idea involved. While this expresses the child's meaning it could be dangerous to allow repeated recording without brackets. The query " What do you mean ? " wil probably bring the response " How many twos equal the five and three together ? " The " togetherness " of the $5 + 3$ is the justification for using the brackets to record correctly $(5 + 3) \div 2 = 4$.

6. The use of mats, as distinct from patterns, can be valuable in this stage. They are a useful device for providing at the one time a large number of possible equations. They should not be over-used, however, since they may result in the child's skipping from row to row without really exhausting the possibilities of each equation in terms of all possible operations. (See " Notes on Method ", Stage 7.)

7. Considerable skill in manipulating equations with the rods is necessary before the child attempts Stage 21, which aims to develop his ability to work without the material. Once Stage 21 has been commenced, the two stages proceed side by side.

8. Work in " value relationships " must continue side by side with all the other work of this stage. (Refer to Stage 15 for exercises.) Children should, by now, be thoroughly at home with :

a finding rods of given values or giving values of rods, when the unit is specified, and

b finding the unit, when given value of a rod.

Most will also be able to cope with a combination of these, e.g.

" If dark-green is called ' two ', what must I call blue ? "

" If orange is called ' five ', which rod will we call ' three ' ? "

## TESTING

### 1. Oral Manipulation

This is tested in the same way as it was tested at earlier levels. The child must read patterns orally and the teacher must assess his reading in terms of the three characteristics of quantity, quality, and ease. The quality is shown by the ability to use combinations of operations.

### 2. Written Manipulation

This is tested in the same way as the above step except that the three characteristics are assessed from the written work.

### 3. Interpreting Equations

Success in this step is measured :

a by the child's ability to use rods to illustrate a written equation, with emphasis on his understanding of what the signs and symbols tell him to do, and

b by his ability to explain what is meant by a written equation.

### 4. Uncompleted Equations

The child should be able to solve equations with reasonably complicated use of signs, and the unknown element in varying positions. It should be noted that the solving of equations will go on alongside Stage 21 and all future work. Therefore, the degree of difficulty must gradually increase. From simple one-operation examples such as were listed in Developmental Step 5, progress will be made to more complex examples.

e.g.

| | |
|---|---|
| $10 - 2 \times 4 = \square$ | $3 + 4 - 6 = \square$ |
| $8 - (3 + 2) = \square$ | $9 - 3 - 2 = \square$ |
| $\square \times 4 = 8$ | $2 \times 5 - (3 + 1) = \square$ |
| $8 \div 2 = \square$ | $\square = 2 \times 3 + 2 \times 1$ |
| $9 - 4 + 2 = \square$ | $9 - (2 \times 2 + 1) = \square$ |
| $3 + \square + 2 = 10$ | |

# STAGE 21

**AIM**

To develop ability to create equations without using the material.

**NOTES ON AIM**

1. The aim in using the Cuisenaire material is not to teach the child to manipulate coloured rods but to develop an understanding of basic mathematical ideas. The rods are used to introduce these ideas because they illustrate them in a simple and concrete way and, by providing the child with a wide variety of experiences, enable him to develop a thorough understanding of what he is studying. Once the child has mastered the ideas, the rods have served their purpose and the child is able to use the concepts he has acquired to organize and manipulate numbers. Like a mathematician he works with abstract concepts.

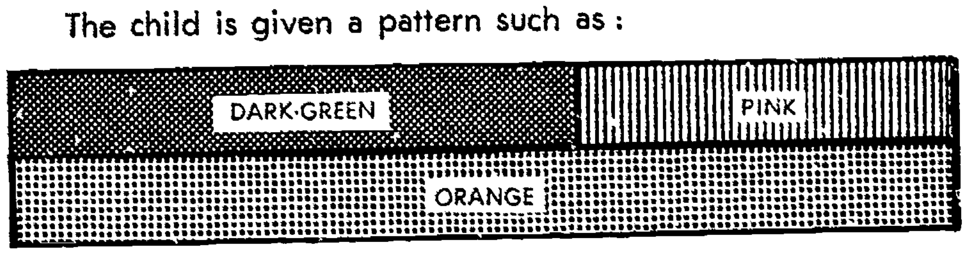
2. This ability to work abstractly has been developing during the preceding stages (see " Notes on Method ", Stage 20). Gradually, as the child gains confidence in his understanding and experience with various rod combinations, he begins to show less dependence on the rods. This stage aims to develop his ability to work confidently with abstract ideas, independent of rods. The achievement of this ability is the culmination of the work with the rods in the previous stages. The rods, like any good aid, have achieved their purpose when the child is independent of them.

3. It should not be supposed that the child no longer uses rods at all. He may work without rods for parts of this section, but he will still need them as he encounters bigger numbers and more advanced ideas, and for experience which is to lead to automatic response.

**DEVELOPMENTAL STEPS**

**1. Mental Substitution**

The child is given a pattern such as :




This may be read as $6 + 4 = 10$ and recorded on the black-board. The child may then be asked to think of rods that could replace the dark-green, and to say the " six " another way.

e.g.

$$6 + 4 = 10$$
$$3 + 3 + 4 = 10$$
$$3 + 3 \times 1 + 4 = 10$$
$$2 \times 3 + 4 = 10$$

He should do the same thing with other numbers in the equation.

In the example above, he could substitute for the 4 as :

$$6 + 2 \times 2 = 10$$

or the 10, as :

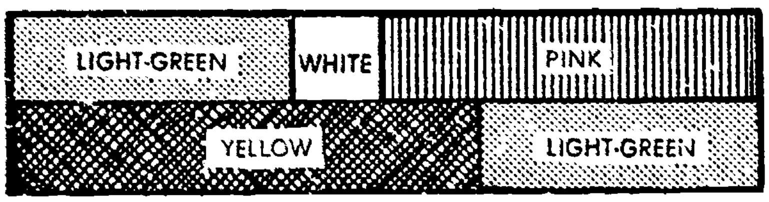
$$6 + 4 = 8 + 2 \times 1$$

Gradually the position is reached where substitution can be made for all parts of the equation, producing a result such as :

$$4 + 2 + 2 \times 2 = 7 + 3$$

$$\text{or } 2 \times 3 + 1 + 1 + 1 + 1 = 5 \times 2$$

Gradually, too, the complexity of the given equation should be increased, and substitution required for varying elements. From this example :




a wide variety of substitution could be expected.

Children who do not find it easy to recall rod combinations for the purposes of substitution will be helped by making mats and noting the results.

In this step the child is not fully free of the rods—they act as a guide to his substitutions. It is nevertheless a useful " bridge " on the way to work which is completely independent of the rods.

The child may make these substitutions either orally or in writing. It is highly desirable, however, that before he does much written work he should be given extensive oral work. (The teacher should record this oral work on the black-board.) Skill and flexibility in this type of exercise is dependent upon the amount of experience given—and so much more can be done orally.

The teacher should note that to substitute " $7 - 2$ " or " $10 \div 2$ " for 5 requires an abstraction and a number experience impossible with the majority of children at this level. It can be expected during Section D, and would certainly be acceptable here, but should not be forced.

**2. Manipulating a Written Expression**

The child is given an expression such as :

$$3 + 3 + 1 = 4 + 3$$

He is then asked to say this equation another way, without using substitution. The ability to do this is a direct result of the experience of Stage 20, Developmental Step 2.

e.g.
$$1 + 3 + 3 = 3 + 4$$
$$2 \times 3 + 1 = 4 + 3$$
$$2 \times 3 + 1 - 4 = 3$$
$$4 + 3 - 3 = 3 + 1$$
$$4 + 3 - 2 \times 3 = 1$$
$$3 + 4 - (3 + 1) = 3$$
$$3 + 4 - 3 - 1 = 3$$
$$2 \times 3 + 1 - (4 + 3) = 0$$

The child manipulating equations in this way is freer of the rods than he was in the previous step. Basically what he is doing is "pattern reading without a pattern".

As with the previous step the work can be done orally or in writing, though, for the reasons given above, a large amount of oral work is advisable before much written work is done. Recording the oral readings on the black-board is of considerable help.

### 3. Creating Equations

The previous step, though an important one, is limited in one aspect. The child is given an equation and told to reorganize it. The final test of ability to manipulate equations is made when the child creates his own equations without assistance from the teacher, and without using the rods. He is asked, for example, to make up an equation about eight. A child with little knowledge of number facts but a sound understanding of the basic operations and their relation to one another may write :

$$4 + 4 = 8$$

and manipulate his equation thus :

$$2 \times 4 = 8$$
$$8 = 2 \times 4$$
$$8 - 4 = 4$$
$$8 - (4 + 4) = 0$$
$$8 - 2 \times 4 = 0$$
$$8 - 4 - 4 = 0$$

and by using substitution :

$$2 \times 2 + 2 \times 2 = 8$$
$$2 \times 4 - 4 = 4$$
$$3 + 1 + 2 \times 2 = 6 + 2$$

As his experience widens, the child may offer, about 10 :

$$7 + 3 = 2 \times 5$$
$$100 - 90 = 5 + 5$$
$$10 - 2 \times 3 = 4 \times 1$$
$$15 - \tfrac{1}{2} \text{ of } 10 = \tfrac{1}{2} \text{ of } 20$$

A child able to create and manipulate equations in this way has achieved the aim of this stage.

Three points need to be noted about this step :

a The main emphasis in this whole stage is on using the understandings gained with the numbers below ten. Quite often, however, the child uses numbers above ten in his examples. He has become acquainted with these through his counting. (For details see Stage 14.)

b The ability to create equations without reference to rods is the requirement of this stage. Some children will do quite complicated work, as above. Less gifted children will do only very simple work. But all children are expected to do *some* creative work without rods, however simple it may be.

c The examples given above include fractions. The method for introducing these is not discussed until the next stage. It is, of course, not until that stage is completed that children would use fractions in abstract work.

**NOTES ON METHOD**

1. As mentioned in the Notes on Aim the child has been moving gradually to the stage where he is able to work without using the rods. Thus this stage is, to some extent, running simultaneously with the preceding stages. It begins, not at any definite time, but as soon as the child begins to manifest an ability to work without rods. As this ability develops, the type of work outlined above becomes more and more important until by the end of the section it occupies much of the child's time. Section C is not complete until the child is able to work abstractly.

2. It was stressed under Developmental Steps that, because the amount of experience the child has in this type of exercise is vital, oral work must not be neglected. A good rule is to introduce each of the steps orally and gradually allow more written work to be done as the child becomes skilful and confident in the abstract manipulation of equations. Remember that "oral work" can include black-board recording.

3. The work with units other than white continues during this stage. Refer to Stage 15 for suggested exercises. If the child is competent with the first three of these, he should attempt work with groups of rods—exercise d (page 11).

4. Work with uncompleted equations was introduced in Stage 20, and continues during this stage. By the end of Section C the child ought to be able to solve simple equations without rods, and to do without rods for parts of more complicated equations.

5. It is most important to realize that, although this stage requires the child to know the answers to various number facts without recourse to the rods, there is no insistence on automatic response. (By "automatic response" is meant an immediate and correct answer to a number fact or table.) This skill is not required for work at this level. All that is needed is a knowledge of the answer. Further experience in number bonds to 10 will be given in substitution exercises in Section D.

**TESTING**

The aim of this stage has been achieved when the child is able to create equations without reference to the rods or help from the teacher. The simplest way to assess whether a child has completed this stage is to give the types of exercise shown in each of the three steps and to assess the child's equations in terms of the three characteristics of quality, quantity, and ease. The work should show a thorough understanding of all operations, and ability to manipulate and substitute.

# STAGE 22

**AIM**

To introduce the study of fractions and to complete the study of partition division.

**NOTES ON AIM**

1. At this stage the study of fractions embraces two concepts :

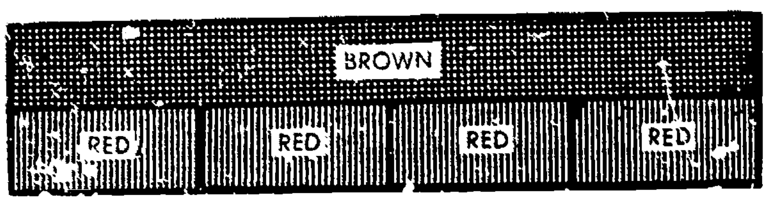
a the fraction as the relation between two whole numbers ; and

b the fraction as an operator.

Development of a mastery of these concepts is essential for an understanding of fractions. They are introduced because they are important in themselves (unless a child understands them there is a serious gap in his mathematical knowledge) and because they are a prerequisite for an understanding of other mathematical ideas, e.g., ratio, proportion, and partition division.

All the ideas and skills needed for the understanding of the concepts have been mastered—thus the child's present mathematical knowledge has prepared him to work with fractions.

2. The concept of the fraction as an operator is very important because, through it, the concept of partition division is completed. In Section B, Stage 12, two aspects of division (quotition and partition) were stressed. For example, a pattern such as :




suggests two questions :

a " How many twos equal eight ? " i.e., $8 \div 2 = \square$

Here we are seeking the number of groups into which eight has been divided. This is quotition division and has been studied in Stages 18 and 19 of this section.

b " What does one-quarter of eight equal ? "

i.e., $\frac{1}{4}$ of $8 = \square$

In this question we know the number of groups into which eight has been divided (four) but we wish to know the value of each group. To discover the answer the child finds the numerical value of the rod. This is partition division.

The importance of a study of fractions in the development of mathematical concepts is illustrated by this link with division. Until a child is able to understand fractions he has not completely mastered division.

## DEVELOPMENTAL STEPS

### 1. Introduction and Vocabulary

a From a set of five equal rods, of any colour, or of five identical objects used in applied number, the child should be led to recognize one of a set of five. When he can confidently show " one of five ", he should be given the term for this, " one-fifth ".

The fractional notation $\frac{1}{5}$ can be shown and recognized, but need not be written by the child.

b When the child has a thorough grasp of the term " one-fifth ", similar exercises should be used to extend the idea to $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$. The fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ can be left till last because of the slight vocabulary difficulty. Apart from this, the order of presentation is unimportant.

c After much practice has been given in Steps a and b the teacher should extend the idea to $\frac{2}{5}$ (2 of a set of 5), $\frac{3}{5}$ (3 of a set of 5), and $\frac{4}{5}$ (4 of a set of 5). Similar treatment will make the child familiar with $\frac{2}{6}$ ...... $\frac{5}{6}$, $\frac{2}{7}$ ...... $\frac{6}{7}$, and so on.

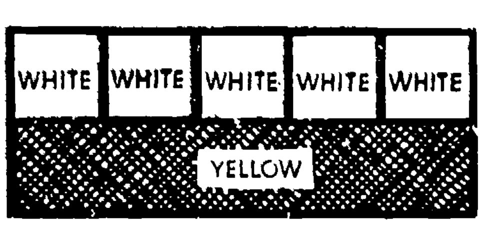
### 2. The Fraction as the Relation between Two Numbers

a The previous work has been devoted to introducing the idea of a fraction to the child. It has prepared for the concept of the fraction as the relation between two numbers. This concept can be introduced in the following way. The child makes a pattern : e.g.


" How many rods in this row ? " (Five.)

" Show me one-fifth of the row."

The child indicates one rod. A yellow rod is then added :



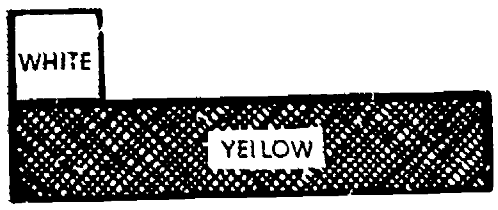

" What can you tell me about these rows ? " (They are equal.)

" What is one-fifth of the top row ? "

The child indicates one rod.

" Would that be equal to one-fifth of the bottom row ? " (Yes.)

Four rods are then removed so that the pattern becomes

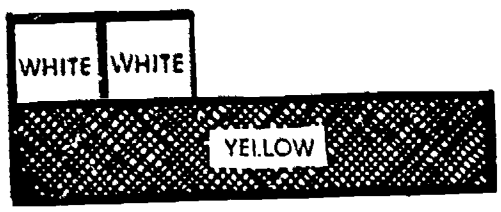



and the child is told that this is another way of showing one-fifth (i.e., one of the five needed to equal the yellow rod).

b Next the teacher says:

"Show me two-fifths."

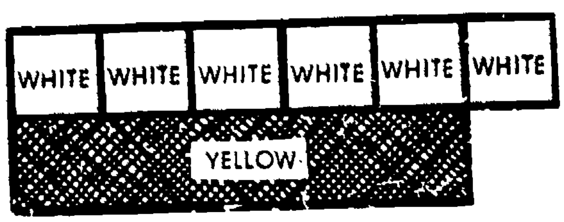
This is usually shown by the pattern:



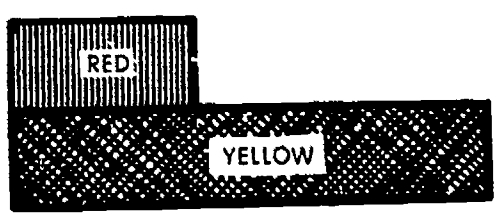

Then:

"Show me three-fifths .... four-fifths .... five-fifths .... six-fifths .... seven-fifths ...."

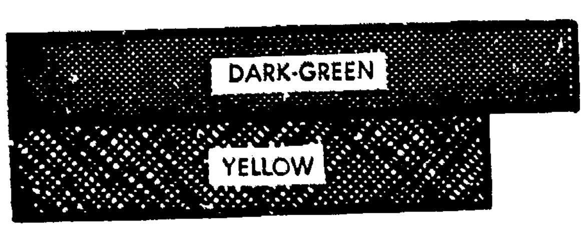
Six-fifths is shown as:




c Once the child has mastered the concept he can be shown that, in the fraction two-fifths, one red rod may replace the two white rods, in which case the fraction $\frac{2}{5}$ would appear as:




and the fraction $\frac{6}{5}$ would appear as:

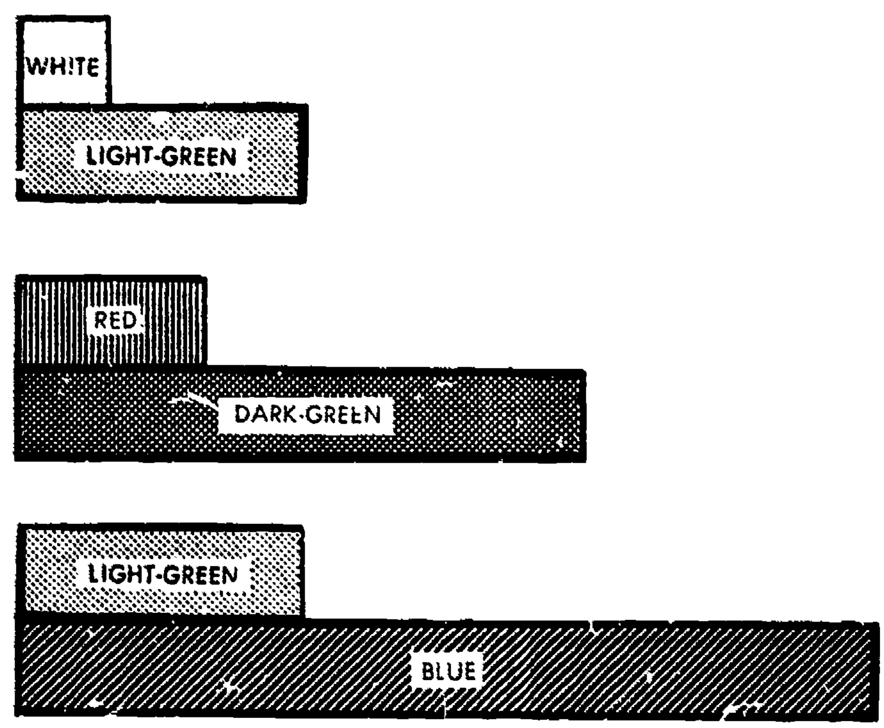


Use of the rods in this way stresses the fact that two whole numbers are being compared, or related.

d In the same way the fractions introduced previously are studied. The child is given exercises such as :

"Show me with your rods two-thirds .... three-quarters .... five-sixths .... seven-eighths."

e Great care should be taken to ensure that, when working with these fractions, the child uses units other than white. He should, for example, see $\frac{1}{3}$ as :




Much practice in this type of exercise is necessary to help develop the idea of the relationship. In each case, the smaller rod is one of the three needed to equal the larger rod.

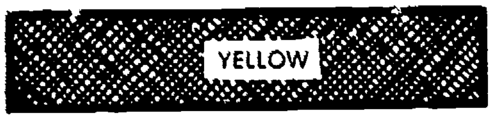
### 3. The Fraction as an Operator

An equation such as $\frac{1}{5}$ of $5 = 1$ introduces a new concept of a fraction. In the previous work the child has studied the fraction in isolation. He has been asked to show $\frac{1}{3}$ or $\frac{3}{8}$ or $\frac{5}{8}$. In all cases he is simply looking at the relation between two numbers.

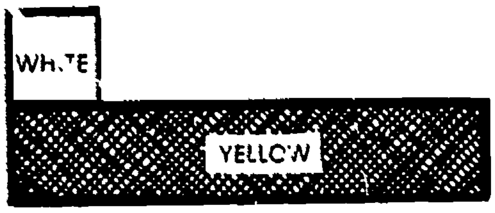
When the child works with an equation such as $\frac{1}{5}$ of $5 = 1$, however, the fraction is not simply an isolated statement about the relation between two numbers. It is part of an equation and like any other part of an equation it has an effect on the other numbers in the equation. When the fraction takes its place in an equation it becomes known as an operator— and has to be distinguished from the fraction in isolation. When we speak of $\frac{1}{5}$ we refer to the relation between two numbers. When we say $\frac{1}{5}$ of $5 = 1$ we have placed the fraction in an equation where it is operating on other numbers.

The fraction as an operator is introduced in the following way:

a "Show me a five with your rods."




"Show me one-fifth of that five."




"What does one-fifth of five equal?" (One.)

"Show me two-fifths of five."

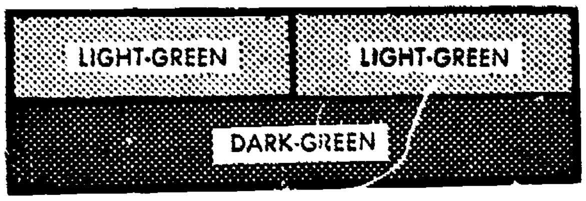
"What does two-fifths of five equal?"

This step is repeated using fractions introduced previously. In order to control the difficulty of the work, however, the fractions used should be chosen so that the denominator is the same as the number upon which the fraction is operating, e.g. $\frac{1}{5}$ of $5 = \square$, $\frac{3}{8}$ of $8 = \square$, $\frac{2}{7}$ of $7 = \square$, before $\frac{1}{2}$ of $4 = \square$, $\frac{1}{3}$ of $9 = \square$

As always, a great deal of oral work should be done before written work on any large scale is introduced.

b The child is now given exercises of a more complex nature, e.g. $\frac{1}{2}$ of $6 = \square$, $\frac{1}{4}$ of $8 = \square$, $\frac{1}{3}$ of $9 = \square$

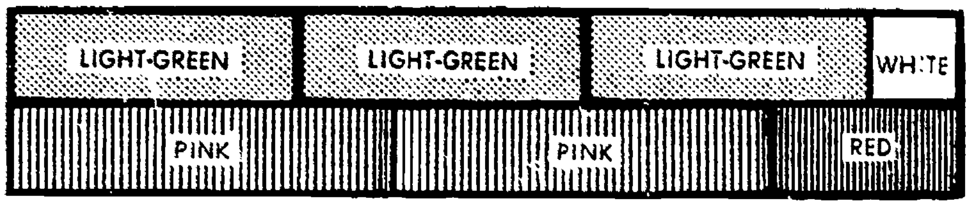
When finding the pattern for these the child may work at first by trial and error. In attempting to solve $\frac{1}{2}$ of $6 = \square$ he knows that he must find two equal rods which, when placed end to end, equal six. He may then have to experiment before he finds the correct rods:




With practice, and increasing mastery of the concept, he discovers the answer more quickly.

A child following this procedure begins his work with the knowledge of the number of groups into which six is divided (two) and discovers the value of each group (three). Although perhaps not aware of it, he is doing partition division.

c When the concept of the fraction as an operator is mastered, the child begins to use it in other work. In pattern reading, for example, the child should be encouraged to use fractions, together with the basic operations. From this pattern :




he may read :

$2 \times 4 + 2 = 3 \times 3 + 1$

$2 \times 4 + \frac{1}{2}$ of $4 = \frac{1}{9}$ of $9 + 3 \times 3$

$4 + \frac{4}{4}$ of $4 + \frac{1}{4}$ of $8 = 3 \times 3 + \frac{1}{10}$ of $10$

$3 \times 3 + \frac{1}{6}$ of $6 = 2 \times 4 + \frac{2}{7}$ of $7$

The fraction should also be used when the child is working with uncompleted equations (Stage 20) or doing manipulative work without the rods (Stage 21).

### 4. Distinction between Quotition and Partition Division

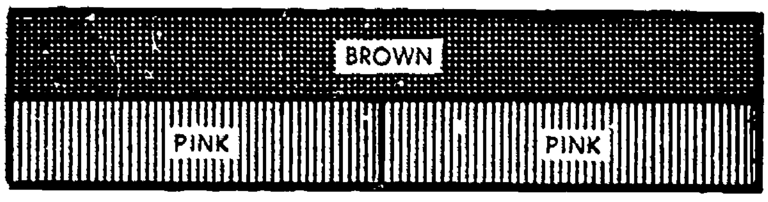
Until this step the child has concentrated on the fraction as an isolated relation between two numbers or as a working part of an equation. It has already been mentioned (see "Notes on Aim" and Step 3 b) that, when the child is using the fraction as an operator, he is completing the work on partition division—even if he is not fully aware of it. The purpose of this step is to consolidate this concept. It may be done through exercises such as the following :

**Quotition** The teacher writes on the board :

$8 \div 4 = \square$

The child reads, "How many fours equal eight ?" and is asked to find the answer with his rods.

The child makes the pattern :




"What is the answer ?" (Two.)

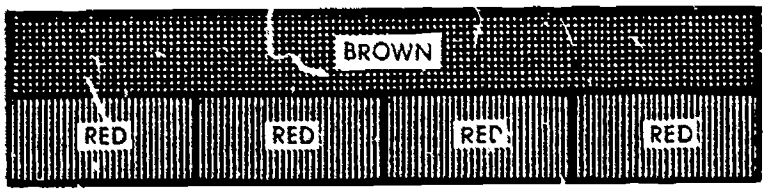
"How did you find the answer ?" (By counting the pink rods.)

**Partition** The teacher then writes :

$\frac{1}{4}$ of 8 = □

This is read as " One-quarter of eight equals .... ? " and the answer is found with rods.

The child makes the pattern :




" What is the answer ? " (Two.)

" How did you find the answer ? " (I know what the red rod equals.)

The purpose of this approach is to make the child realize that, although the answer is " two " in both cases, it has been reached in different ways. In one case the " two " referred to the number of rods used, in the other to the value of the rod.

There is no need for the child to have advanced beyond this level. If he is able to answer questions about the method he used to reach his answer he has understood the distinction between the two equations. He need not, at this stage, be able to express this distinction unaided by the teacher's questions, nor does he need to hear the terms " quotition " and " partition ".

The child needs a large amount of practice in this type of work to ensure that the distinction is fully mastered.

## NOTES ON METHOD

1. The study of fractions may begin when the child has a thorough mastery of the ideas and skills necessary for its understanding. The ideas and skills concerned are :

a ability to count ;

b the concept of cardinal number ;

c the concept of equality.

These prerequisites are present early in this section. Exactly when the study of fractions is introduced is a matter for individual teachers to decide. It should be remembered that, although the child may have the skills, he may lack the maturity necessary to understand the fraction ideas. When fraction work is commenced, it should proceed parallel to the other aspects of the section.

2. As in every other stage a great deal of oral work needs to be done before any extensive written work is introduced. The success of this stage depends more on the extent and quality of the oral work than on any other factor.

3. The correct use of vocabulary is of the utmost importance. It will be noticed that, when working with the fraction considered solely as a relation between two numbers, the teacher sets such tasks as:

"Show me two-fifths."

"Show me three-ninths."

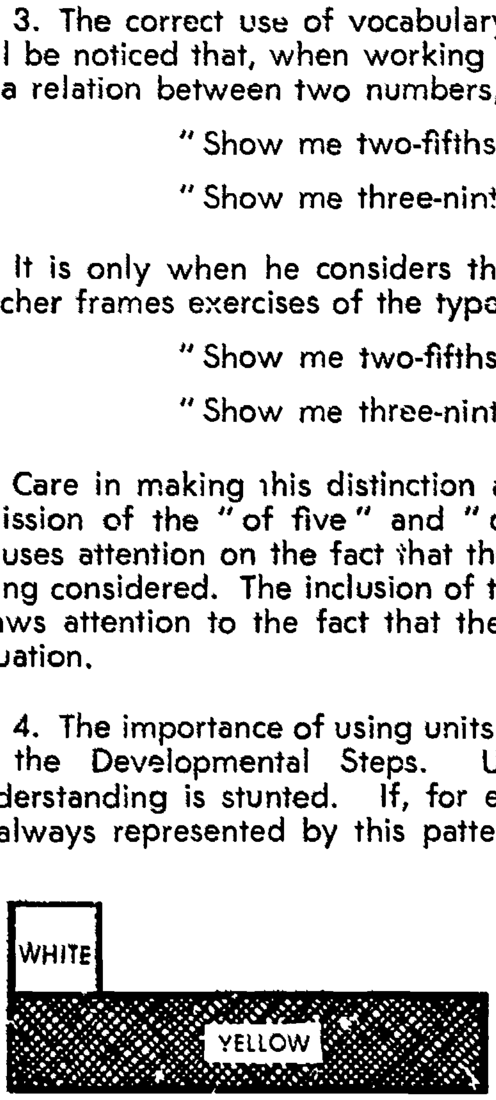
It is only when he considers the fraction as an operator that the teacher frames exercises of the type:

"Show me two-fifths of five."

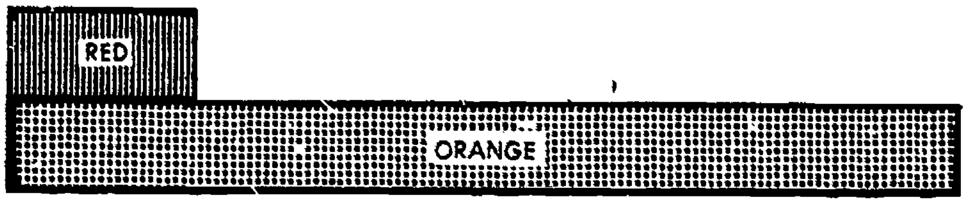
"Show me three-ninths of nine."

Care in making this distinction assists the child considerably. The omission of the "of five" and "of nine" from the first examples focuses attention on the fact that the relation between the numbers is being considered. The inclusion of the phrases in the second examples draws attention to the fact that the fraction is a working part in an equation.

4. The importance of using units other than white has been stressed in the Developmental Steps. Unless this is done the child's understanding is stunted. If, for example, he believes that one-fifth is always represented by this pattern:



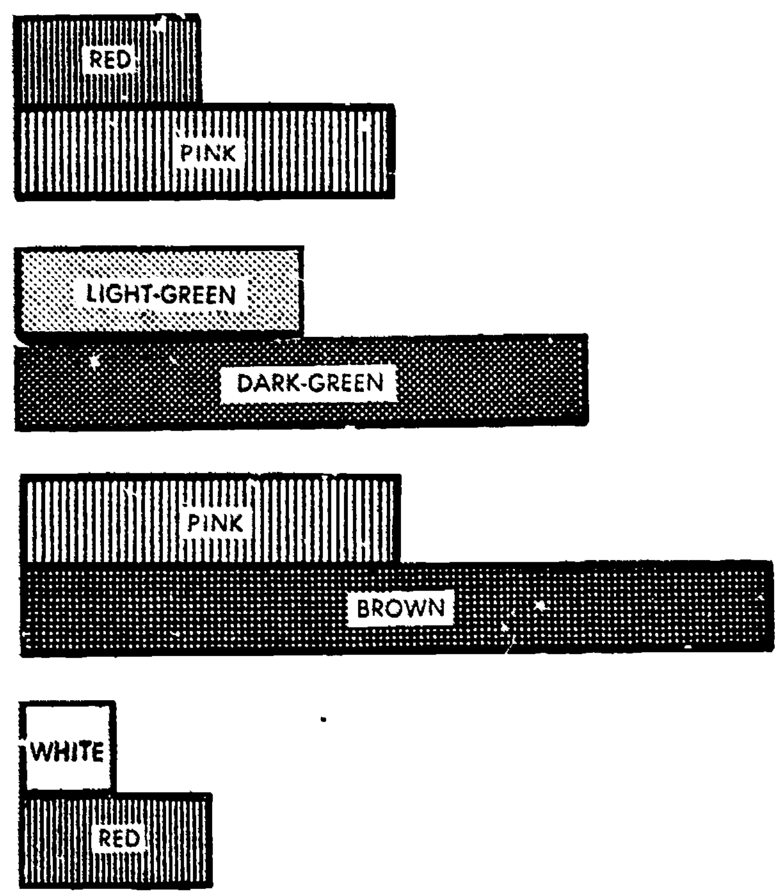

and never by this:




he has failed to understand that "$\frac{1}{5}$" expresses a relationship between two numbers, a relationship as fully present in the second pattern as in the first.

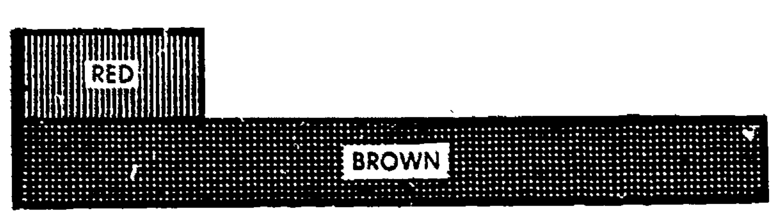
5. During this stage some children begin to realize that fractions may be equivalent. For example, they may realize, when working with $\frac{2}{4}$ or $\frac{3}{6}$ or $\frac{4}{8}$, that these fractions are equal to $\frac{1}{2}$. The rods themselves

thrust the idea before them, as shown by a study of the patterns for the four fractions illustrated :




The rods differ, but the relation of $\frac{1}{2}$ remains. While no attempt need be made to teach this fact directly, it is important to be aware that the rods are already beginning to develop this understanding and to be prepared to assist any child who is starting to explore the idea.

6. Exercises in value relations must continue. As fraction work develops from this early stage through Section D, the teacher should begin to include fractions in these exercises. Straightforward examples are suggested in Stage 15, Exercise e (page 11). In addition, the child will be required to recognize that a comparison of these rods :




could make him think of $\frac{1}{4}$ or 4, according to which rod is considered the unit.

7. To begin further work in fractions without a thorough mastery of this stage would be self-defeating. The ideas treated in later work with fractions depend upon ideas studied in this stage. Hence a thorough mastery of the work outlined is essential.

**TESTING**

1. In order to ensure that the child understands the concept of the fraction as a relation between two numbers he may be told :

"Show me two-fifths."
"Show me three-eighths."
"Show me nine-sevenths."
"Show me five-quarters."

The child must produce a correct pattern quickly and confidently.

2. Two matters need to be watched when testing the concept of the fraction as an operator. In the first place the child should be able to solve equations involving fractions,

e.g. $\frac{1}{2}$ of $4 = \square$, $\frac{3}{4}$ of $8 = \square$,
$\frac{9}{4}$ of $8 = \square$, $\frac{1}{6}$ of $6 + 2 = \square$.

Secondly, the most important indication of mastery is the use of fractions in the pattern reading and manipulative work. The appearance of the fraction in this type of exercise is a sign that the concept of the fraction as an operator has been mastered.

3. The work outlined in Step 4 is an effective method of testing whether a child has a clear idea of the distinction between quotition and partition division.

## *SUMMARY OF WORK COVERED BY THE END OF SECTION C*

**1. Ordinal Number**

i Counting has been continued throughout the section—
  a by ones at least to 20 ;
  b in groups (by tens, fives, twos), to open the way to discovery of the pattern and order of the number system.

ii The idea of a number in a set position in a series (12th, 15th, . . . .) has been developed.

iii Figures to 20 can be recognized and written.

iv Words to "ten" can be recognized and written.

**2. Cardinal Number**

Children are now working with numbers as wholes, e.g., in $4 + 3 = 7$, the wholes 4 and 3, are added together to equal the whole, 7.

**3. Value Relations**

While number values have been given to the rods, children realize that these values are not fixed ; they vary according to the rod chosen as the unit. Exercises are continually given to extend experience.

**4. Basic Mathematical Ideas**

Extension of the understandings gained in Section B has been held to be of primary importance.

**5. Oral and Written Work**

Although the child now has the skill to record any equation he can read, it is important that a great deal of oral work should still be done.

**6. Interpretation**

Written equations can be interpreted :

a with rods, and

b orally

to show understanding of symbols.

**7. Solving Equations**

Through his understanding of operations and his ability to interpret symbols, the child can now solve equations, generally with the help of rods, but to a limited extent without concrete aid.

**8. Abstract Work**

Continued experience in making patterns, reading equations, and recording them is giving the child the ability to work abstractly, and this ability is deliberately developed by the teacher (see Stage 21). Every child should be able to do some creative work without rods, however limited it may be. Understanding of the basic operations and their inter-relationship is being extended in this abstract work.

**9. Fractions**

Elementary ideas of fractions have been introduced and the child has some awareness of the fraction as :

a the relation between two numbers, and

b an operator.

Some children are able to use fractions in their creative work.

**10. General Notes**

i Confidence is increased because the child's knowledge continues to be in terms of his own experience.

ii The change to the language of number has allowed the child to record his ideas, and opened the way to abstraction. He is beginning to think in terms of numbers as distinct from rods and other concrete materials.

iii The numbers mainly used (1-10) have provided a means of expressing ideas and establishing certain techniques which will be developed later. (More intensive experience with the numbers themselves will follow in Section D.) At the same time counting activities have made children aware of a wider area which they will shortly explore in more detail.

iv A marked difference in quality of work is evident. All children must be able to cover the basic work of each stage, but many will do work of a much greater complexity than others.

By Authority: A. C. Brooks, Government Printer, Melbourne.