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Five models are introduced for the factor analytic treatment of a set of measures obtained for the same sample of persons on two different occasions. The models differ in terms of the assumptions made regarding the constancy of the (1) factor (actually component) score and (2) factor pattern matrices from occasion 1 to 2. Least-squares procedures are developed for the estimation of the component scores and patterns under four of the models; canonical correlation procedures are developed for the fifth. Illustrative examples using these procedures are presented, and the research implications of the hypotheses and procedures underlying each model are discussed.

SOME NOTES ON THE FACTOR ANALYTIC TREATMENT
OF MEASURES OBTAINED ON TWO DIFFERENT OCCASIONS

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The purpose of this paper is not to develop a single, in some sense preferred, procedure for the factor analysis of data matrices obtained on two occasions, but rather to bring into clearer focus the parameters of such a situation and to suggest possible analysis strategies depending upon the assumptions the experimenter wishes to make about his data and the particular questions he wishes to answer. In a strict sense, what is to follow is only generically "factor analysis," as the estimation of communality implied by the common-factor model is absent. Instead, considerable use is made of the Eckart-Young theorem (see Eckart & Young, 1936; Johnson, 1963), and the component model is generally implied. We begin with a discussion of some important characteristics and assumptions associated with longitudinal data. Following this, certain of the possible alternatives are selected and developed into clearly operational analysis strategies. Empirical examples of some of these strategies follow, and we conclude with a word concerning practical implications.

1. Parameters of Longitudinal Data

One way to analyze longitudinal data is to treat occasion either as the observational unit--on which scores on either variables or persons are recorded--or as the attribute being measured, in conjunction with either a single variable or a single person. Two dimensional designs involving occasions by persons or variables constitute four of Cattell's [1952] six factor analytic designs--O-, P-, S-, and T-techniques. For these designs to lend stability to the obtained correlations and factors, however, a large number of occasions is generally required, analogously to the requirement of a reasonably sizeable person sample in standard, or R-technique, factor analysis.

The experimenter may not wish to obtain factors either representing occasion groupings or based on inter-occasion variation, but, instead may require standard factors--representing variable groupings and based on inter-person variation--that are, in some sense, stable or present over two or more occasions. Given the latter requirement, several alternatives exist. Historically, longitudinal studies involving the same battery of tests administered to the same sample of persons on two or more occasions have reflected the assumption of unchanging factor scores, with the changing factor pattern matrices the locus of interest [e.g., Evans, 1967; Fleishman, 1957, 1960, 1966; Harris, 1963; Meyer & Bendig, 1961]. Tucker's [1963] application of three-mode factor analysis to the measurement of change reflected the converse assumption, that of unchanging pattern coefficients but changing factor scores. The procedures derived by Corballis and Traub [1970] are based on both changing factor scores and changing factor patterns, and an excellent discussion of the conceptual merits of the differing assumptions can be found in this latter paper.

A feature of the model proposed by Corballis and Traub [1970] is that although factor scores may change, from occasion 1 to 2, these factor scores are orthogonal both within and between occasions. It is the opinion of the present author that although the restriction to within-occasion factor orthogonality may not be excessive, that to between-occasion orthogonality is unwarranted. The effect of this latter constraint is to specify uniquely the factor pattern matrices for the two occasions, thus precluding a simple structure rotation if the initially obtained factor patterns are difficult to interpret.

The purpose of the remainder of the present paper, then, is the presentation of procedures for analyzing longitudinal data under several of the assumptions, in turn, mentioned earlier. We are interested, exclusively, in the experimental situation in which the same sample of persons is measured on the same set of variables on two different occasions. We thus are concerned with the pretest-posttest or change paradigm so useful in psychological and educational research, where the aim may be to isolate time-stable constructs, to study correlates of the constructs at the two different times, to note changes over time in constructs of interest, or perhaps, to observe the effects on these constructs of intervening experimental treatments.

2. Derivation of Analysis Procedures

In this section, our aim is to consider, in turn, various experimental situations which differ in the assumptions made about the constancy of the factor score and factor pattern matrices over the two occasions. As has been mentioned, the common-factor model is not implied as it was in the paper by Corballis and Traub [1970]; that is to say we are not concerned with the estimation of communalities as well as common-factor loadings. Instead, the component model is implied, and extensive use is made of the Eckart-Young theorem and its implications (see Eckart & Young, 1936; Johnson, 1963). In most cases, however, although the component model is implied, the task is to find a least-squares fit to either two sets of components or two matrices of pattern coefficients, or perhaps both.

In the various situations for which procedures are derived, we begin with two data matrices, $\underline{Z}_1$ and $\underline{Z}_2$, both of order $\underline{N}$ persons x $\underline{n}$ variables--the latter of which are standardized to have zero mean and unit variance--representing, respectively, occasions 1 and 2. The particular $\underline{N}$ persons and $\underline{n}$ variables are, of course, common to both occasions. Standard scores on new, derived components are contained in the matrices $\underline{X}_1$ and $\underline{X}_2$--if scores are separately derived for each occasion--or $\underline{X}$--if a matrix common to both occasions is hypothesized. These matrices are $\underline{N}$ x $\underline{r}$, where $\underline{r} < \underline{n}$ components are considered adequate. We return to the issue of determing $\underline{r}$ later in the paper, but will assume $\underline{r}$ is known in the procedures that follow. The component pattern matrices, relating the $\underline{X}$'s to the $\underline{Z}$'s, are designated $\underline{F}_1$ and $\underline{F}_2$, or simply $\underline{F}$, if a common pattern is hypothesized; all are $\underline{n}$ x $\underline{r}$. In all cases, the $\underline{F}$ matrices may be transformed to a simple structure pattern, $\underline{P}$, $\underline{n}$ x $\underline{r}$, by a transformation matrix $\underline{T}$, $\underline{r}$ x $\underline{r}$, either orthonormal or in some cases, oblique. Given the two data matrices, $\underline{Z}_1$ and $\underline{Z}_2$, a standard principal-components analysis of each is embodied in the following well-known equations:

$$(1)\qquad \begin{aligned} Z_1 &= X_1F_1' + E_1 = X_1T_1P_1' + E_1, \text{ and} \\ Z_2 &= X_2F_2' + E_2 = X_2T_2P_2' + E_2, \end{aligned}$$

where $\underline{E}_1$ and $\underline{E}_2$, both $\underline{N}$ x $\underline{n}$, are matrices of "errors" of fit of the $\underline{XF}'$ or $\underline{XTP}'$ matrices to the $\underline{Z}$'s. Given $\underline{X}_j\underline{F}_j'$ or $\underline{X}_j\underline{T}_j\underline{P}_j'$ of rank $\underline{r} < \underline{n}$, then by the Eckart-Young [1936] theorem, $\mathrm{tr}[\underline{E}_j'\underline{E}_j]$ is minimal. In the specific instances that follow, (1) is modified according to the specific hypothesis made.

Case I

We begin with the most restrictive situation, that in which, for descriptive and conceptual reasons, we seek a single component pattern matrix that relates a single matrix of component scores, in a least-squares sense, to the two observed data matrices, $\underline{Z}_1$ and $\underline{Z}_2$. Alternatively, we hypothesize a single set of occasion-stable components which are related by a single pattern matrix--again hypothesized to be constant over occasions--to the observed data matrices. The Case I model, in place of (1), is

$$Z_1 = XF' + E_1, \text{ and}$$
$$Z_2 = XF' + E_2. \tag{2}$$

Our criterion is

$$\phi_I = tr[E_1'E_1] + tr[E_2'E_2] = \text{minimum}, \tag{3}$$

subject to the constraints

$$X'X/N = I. \tag{4}$$

Combining (3) and (4) yields

$$\phi_I^* = tr[Z_1'Z_1 + Z_2'Z_2 - 2FX'Z_1 - 2FX'Z_2 + 2FX'XF'] + tr[\Lambda(X'X/N - I)], \tag{5}$$

where Λ, $r \times r$, is a matrix of Lagrange multipliers. Differentiating ϕ_I^* (see Schönemann, 1965) with respect to the matrices of interest, and setting the matrix of partial derivatives, in each case, to the null matrix yields

$$\partial\phi_I^*/\partial F = -2Z_1'X - 2Z_2'X + 4FX'X = 0, \text{ and} \tag{6}$$

$$\partial\phi_I^*/\partial X = -2Z_1F - 2Z_2F + 4XF'F + (1/N)X(\Lambda + \Lambda') = 0. \tag{7}$$

Combining (4) and (6) yields

$$F = (1/2N)(Z_1'X + Z_2'X), \text{ or}$$

$$F = (1/2N)Z^{*\prime}X, \tag{8}$$

where Z^*, $N \times n$, is given by $Z_1 + Z_2$. Substituting (8) in (7), we have

$$(1/N)Z_1Z^{*\prime}X + (1/N)Z_2Z^{*\prime}X - (1/N^2)XX'Z^*Z^{*\prime}X = XQ, \tag{9}$$

where the symmetric matrix $Q = (\Lambda + \Lambda')/N$. Combining the first two terms of (9) and premultiplying by $(1/N)X'$, we have

$$(1/N^2)X'Z^*Z^{*\prime}X - (1/N^2)X'Z^*Z^{*\prime}X = Q = 0, \tag{10}$$

so that (9) may be rewritten

(11) $$(1/N)SX = (1/N^2)XX'SX,$$

where $S = Z^*Z^{*\prime}$. If we denote the canonical decomposition of S by

(12) $$S = WL^2W',$$

choosing for X the matrix $N^{1/2}W_r$--where W_r, $N \times r$, is composed of the first r columns of W--satisfies both (11) and (4). The initial pattern matrix, F, is given by (8).

It is unlikely, however, that the F matrix so obtained will be of great interpretive value, and transformation to a simple structure will be generally required at this point. Thus, we seek the transformation to a simple structure, T, $r \times r$, and our new pattern P, $n \times r$, and matrix of transformed components, X^*, $N \times r$, are given by

(13) $$P = F(T')^{-1}, \text{ and}$$
$$X^* = XT.$$

Both Z_1 and Z_2 are approximated by X^*P', leaving the criterion, ϕ_I in (3) unaffected. If transformed components that are still orthogonal are desired, T is, of course, orthonormal and $P = FT$.

Computationally, the process of finding the latent vectors, W, in (12) can be hastened by obtaining the canonical decomposition of $Z^{*\prime}Z^*$ as

(14) $$Z^{*\prime}Z^* = VL^2V',$$

and then obtaining W by

(15) $$W = Z^*VL^{-1}.$$

Such a procedure is considerably faster than that indicated in (12), since $Z^{*\prime}Z^*$, $n \times n$, will generally be of much smaller order than will $Z^*Z^{*\prime}$, $N \times N$.

Case II

In this case, which is somewhat less restrictive than is Case I, we hypothesize a constant pattern matrix over occasions, but permit the component scores to change, presumably as a function of maturation, experimental treatment, or other possible causes, in much the same fashion that we would expect the scores on the variables themselves to change over time. The Case II model is

(16)
$$Z_1 = X_1F' + E_1, \text{ and}$$
$$Z_2 = X_2F' + E_2.$$

As before, our criterion, ϕ_{II} is

(17)
$$\phi_{II} = tr[E_1'E_1] + tr[E_2'E_2] = \text{minimum},$$

subject to the constraints

(18)
$$X_1'X_1/N = I = X_2'X_2/N.$$

Combining (17) and (18) yields

(19)
$$\phi^*_{II} = tr[Z_1'Z_1 + Z_2'Z_2 - 2FX_1'Z_1 - 2FX_2'Z_2 + FX_1'X_1F' + FX_2'X_2F'] + tr[\Lambda_1(X_1'X_1/N - I)] + tr[\Lambda_2(X_2'X_2/N - I)],$$

where Λ_1 and Λ_2, both $r \times r$, are matrices of Lagrange multipliers. Differentiating ϕ^*_{II} with respect to F, X_1, and X_2, and setting the resulting matrices of partial derivatives to zero, we have

(20)
$$\partial\phi^*_{II}/\partial F = -2Z_1'X_1 - 2Z_2'X_2 + 2FX_1'X_1 + 2FX_2'X_2 = 0;$$

(21)
$$\partial\phi^*_{II}/\partial X_1 = -2Z_1F + 2X_1F'F + (1/N)X_1(\Lambda_1 + \Lambda_1') = 0;$$

(22)
$$\partial\phi^*_{II}/\partial X_2 = -2Z_2F + 2X_2F'F + (1/N)X_2(\Lambda_2 + \Lambda_2') = 0.$$

Noting the constraints (18), we have, for (20)

(23)
$$F = (1/2N)(Z_1'X_1 + Z_2'X_2),$$

which, when substituted into (21), yields

(24)
$$(-1/N)(Z_1Z_1'X_1 + Z_1Z_2'X_2) + (1/2N^2)(X_1X_1'Z_1Z_1'X_1 + X_1X_2'Z_2Z_1'X_1 + X_1X_1'Z_1Z_2'X_2 + X_1X_2'Z_2Z_2'X_2) + X_1Q = 0,$$

where the symmetric matrix, $Q = (\Lambda_1 + \Lambda_1')/N$. Premultiplying (24) by $(1/N)X_1'$, we have

(25) $$(1/N^2)(X_1'Z_1Z_1'X_1 + X_1'Z_1Z_2'X_2) - (1/2N^2)(X_1'Z_1Z_1'X_1 + X_2'Z_2Z_1'X_1 + X_1'Z_1Z_2'X_2 + X_2'Z_2Z_2'X_2) = Q.$$

Thus, the left side of (25) is symmetric, since $\underline{Q}$ is, and inspection of the left side of (25) reveals that for this symmetry to exist, the matrix $\underline{X}_1'\underline{Z}_1\underline{Z}_2'\underline{X}_2$ must also be symmetric. If we represent $\underline{Z}_1\underline{Z}_2'$ by the Eckart-Young factorization,

(26) $$Z_1Z_2' = WLV',$$

where we have the canonical decompositions

(27) $$Z_2Z_1'Z_1Z_2' = VL^2V', \text{ and}$$
$$Z_1Z_2'Z_2Z_1' = WL^2W',$$

then choosing for $\underline{X}_1$ the matrix $\underline{N}^{\frac{1}{2}}\underline{W}_r$ and for $\underline{X}_2$, $\underline{N}^{\frac{1}{2}}\underline{V}_r$, where $\underline{W}_r$ and $\underline{V}_r$ contain the first $\underline{r}$ columns of $\underline{W}$ and $\underline{V}$, respectively, renders $\underline{X}_1'\underline{Z}_1\underline{Z}_2'\underline{X}_2$ symmetric and satisfies (18). It should be noted that substituting (23) into (22), rather than into (21), leads to exactly the same result. We then obtain $\underline{F}$ by (23). We return to optimally efficient procedures for obtaining latent roots and vectors of the rather large ($\underline{N}$ x $\underline{N}$) matrices in (27) later in the paper.

As in the Case I situation, we may transform the obtained $\underline{F}$ to a simple structure solution, $\underline{P}$, in terms of either orthogonal or oblique components, $\underline{X}$*--by the formulas in (13). Such a transformation does not alter the minimal value obtained for $\underline{\phi}_{II}$ in (17).

Case III

This case differs from Case II in that instead of the component scores changing while being related to the observed data by a stable, unchanging pattern matrix, the scores are hypothesized to be constant over occasions, but related to the changing observed data variables by changing pattern matrices. Thus, instead of interest centering upon changes over time of scores on variables that themselves are defined by unchanging linear composites of known variables, as with Case II, the locus of interest with Case III is the change over time in composition--in terms of the changing observed variables--of the components, which themselves are hypothesized to be constant over occasions. As was pointed out earlier, longitudinal factor analytic studies have usually been based on this assumption.

The Case III model is

(28) $$Z_1 = XF_1' + E_1, \text{ and} \qquad Z_2 = XF_2' + E_2.$$

Our criterion is

(29) $$\phi_{III} = \mathrm{tr}[E_1'E_1] + \mathrm{tr}[E_2'E_2] = \text{minimum},$$

subject to the constraints

(30) $$X'X/N = I.$$

By combining (29) and (30), we have

(31) $$\phi^*_{III} = \mathrm{tr}[Z_1'Z_1 + Z_2'Z_2 - 2Z_1'XF_1' - 2Z_2'XF_2' + F_1X'XF_1' + F_2X'XF_2'] + \mathrm{tr}[\Lambda(X'X/N - I)],$$

where, as before, $\underline{\Lambda}$, $\underline{r}$ x $\underline{r}$, is a matrix of Lagrange multipliers. Differentiating $\underline{\phi}^*_{III}$ with respect to $\underline{X}$, $\underline{F}_1$, and $\underline{F}_2$, and setting the resulting matrices to the null matrix, we have

(32) $$\partial\phi^*_{III}/\partial X = -2Z_1F_1 - 2Z_2F_2 + 2XF_1'F_1 + 2XF_2'F_2 + (1/N)X(\Lambda + \Lambda') = 0;$$

(33) $$\partial\phi^*_{III}/\partial F_1 = -2Z_1'X + 2F_1X'X = 0;$$

(34) $$\partial\phi^*_{III}/\partial F_2 = -2Z_2'X + 2F_2X'X = 0.$$

Noting the constraints, (30), we may write for (33) and (34)

(35) $$F_1 = Z_1'X/N, \text{ and} \qquad F_2 = Z_2'X/N.$$

Substituting (35) into (32) yields

$$(36) \qquad (2/N)(Z_1Z_1' + Z_2Z_2')X - (2/N^2)XX'(Z_1Z_1' + Z_2Z_2')X = XQ,$$

where the symmetric matrix $\underline{Q} = (\underline{\Lambda} + \underline{\Lambda}')/\underline{N}$. By reasoning similar to that employed in the Case I solution, we note that by premultiplying (36) by $(1/\underline{N})\underline{X}'$, we have

$$(37) \qquad (2/N^2)[X'(Z_1Z_1' + Z_2Z_2')X] - (2/N^2)[X'(Z_1Z_1' + Z_2Z_2')X] = Q = 0,$$

so that (36) and (37) together imply that

$$(38) \qquad (1/N)UX = (1/N^2)XX'UX,$$

where $\underline{U} = \underline{Z}_1\underline{Z}_1' + \underline{Z}_2\underline{Z}_2'$. If we denote the canonical decomposition of $\underline{U}$ as

$$(39) \qquad U = VL^2V',$$

then choosing for $\underline{X}$ the matrix $\underline{N}^{\frac{1}{2}}\underline{V}_r$--where $\underline{V}_r$, $\underline{N} \times \underline{r}$, is composed of the first $\underline{r}$ columns of $\underline{V}$--satisfies both (39) and (30). We then obtain $\underline{F}_1$ and $\underline{F}_2$ by (35).

The matter of transforming the obtained $\underline{F}_1$ and $\underline{F}_2$ matrices to a simple structure is not as straightforward as with Cases I and II, since unlike these applications, Case III involves two pattern matrices, but a single matrix of component scores. For these scores to remain identical over occasions after transformation they must be transformed by the same matrix, which implies that a single matrix, $\underline{T}$, must be found that results simultaneously in optimal simple structures for both the occasion 1 and occasion 2 pattern matrices. Although the problem would best be solved by optimizing a two-matrix analytic criterion function--for example, a simultaneous two-matrix varimax function--we offer the following rather simplistic alternative. Form

$$(40) \qquad F^* = (1/2)(F_1 + F_2),$$

that is, average the unrotated patterns. Next, we would derive $\underline{T}$ as maximizing the criterion function of choice when applied to $\underline{F}^*$. Then

$$X^* = XT;$$

$$P_1 = F_1(T')^{-1};$$

$$P_2 = F_2(T')^{-1},$$

where, as before, if an orthogonal transformed solution is desired, $\underline{P}_j = \underline{F}_j\underline{T}$.

It is undoubtedly clear to the reader at this point that we have assumed in the first three cases that a basic and enlightening structure common to both occasions exists--either in the form of component scores or pattern matrices relating these scores to the observed data. Needless to say, if there is such a substantial change from the $\underline{Z}_1$ to the $\underline{Z}_2$ matrix that no such common structure exists, then attempting to fit matrices of scores or pattern coefficients to the two sets of data will be unsuccessful.

Obtaining Latent Roots and Vectors for Cases II and III

The prospect of having to obtain the latent roots and vectors of the $\underline{N} \times \underline{N}$ matrices implied by Cases II and III, where the number of subjects can be expected to be fairly large, say 100 or more, may seem, at first, prohibitive, in terms of computing time. Certainly, if one were to employ the well-known Jacobi procedure, which yields all non-zero roots and vectors simultaneously, the computation time involved would be prohibitive. In such instances, Hotelling's [1936] procedure is much more efficient than the Jacobi method and certainly accurate enough.

As a check on the speed of the Hotelling procedure, a symmetric matrix of order 98 x 98 was factored into latent roots and vectors, with the process stopped after the first $\underline{r} = 5$ vectors had been obtained. The CPU time on the University of Alberta System 360/67 computer for this problem was slightly over 30 seconds, a not excessive time requirement relative to that of most factor analytic computations.

Case IV

The researcher may wish to permit both the component scores and the pattern matrices to vary between occasions. The procedures developed by Corballis and Traub [1970] would be appropriate in such an instance--the model developed involving components as a special case--but the fact that no simple structure solution is generally possible must be considered a shortcoming. In any case, if no restrictions to equality are placed on either the component scores or pattern matrix, various criteria may be considered to identify an optimal pair of solutions. One such criterion involves obtaining true components for each occasion--that is linear composites of the observed data matrices (unlike in Cases I, II, and III, where the observed data are only approximated in a least-squares sense)--that are maximally related in a pairwise sense. This criterion--clearly expressible as a canonical correlation problem--constitutes our Case IV.

The Case IV model is as in (1), which we repeat here

$$Z_1 = X_1F_1' + E_1, \text{ and} \qquad (1)$$
$$Z_2 = X_2F_2' + E_2.$$

The criterion is

$$\phi_{IV} = tr[X_1'X_2/N] = \text{maximum}, \qquad (42)$$

subject to the constraints, expressed earlier,

$$X_1'X_1/N = I = X_2'X_2/N. \qquad (18)$$

It should be clear that in (1) above, if all $\underline{n}$ components are obtained for both occasion 1 and occasion 2, the $\underline{E}_1$ and $\underline{E}_2$ matrices will be null.

As was noted above, $\underline{X}_1$ and $\underline{X}_2$ are $\underline{N}$ x $\underline{r}$ linear composites of, respectively, $\underline{Z}_1$ and $\underline{Z}_2$. We seek the matrices $\underline{K}_1$ and $\underline{K}_2$, both $\underline{n}$ x $\underline{r}$, such that

$$Z_1K_1 = X_1, \text{ and} \qquad (43)$$
$$Z_2K_2 = X_2,$$

and the conditions (42) and (18) are met. Canonical correlation analysis yields matrices $\underline{K}_1^*$, $\underline{X}_1^*$, $\underline{K}_2^*$, and $\underline{X}_2^*$, where the respective columns of $\underline{X}_1^*$ and $\underline{X}_2^*$ are, indeed, maximally correlated, and columns <u>within</u> $\underline{X}_1^*$ and $\underline{X}_2^*$ are mutually uncorrelated, but, in general, the new variates, while certainly having zero mean, do not have unit variance. Since

(44)
$$Z_1K_1^* = X_1^*, \text{ and}$$
$$Z_2K_2^* = X_2^*,$$

the diagonal matrices of reciprocal standard deviations of the $\underline{X}_1^*$ and $\underline{X}_2^*$ variables are obtained by

(45)
$$D_1 = [\text{diag}(K_1^{*\prime}R_1K_1^*)]^{-\frac{1}{2}}, \text{ and}$$
$$D_2 = [\text{diag}(K_2^{*\prime}R_2K_2^*)]^{-\frac{1}{2}},$$

and we have, for the matrices of standardized component scores, $\underline{X}_1$ and $\underline{X}_2$,

(46)
$$X_1 = Z_1K_1^*D_1, \text{ and}$$
$$X_2 = Z_2K_2^*D_2.$$

From (46), initial pattern matrices, $\underline{F}_1$ and $\underline{F}_2$, are obtained by

(47)
$$F_1 = Z_1'X_1/N = R_1K_1^*D_1, \text{ and}$$
$$F_2 = Z_2'X_2/N = R_2K_2^*D_2.$$

In the interests of preserving the orthogonal characteristics of the linear composites, and also ensuring the constancy of the value of $\underline{\phi}_{IV}$ in (42), we may seek, at this point, an orthonormal transformation matrix, $\underline{T}$, which when applied to $\underline{F}_1$ and $\underline{F}_2$, renders a reasonably interpretable simple structure. Since we are seeking a single transformation matrix to be applied to both pattern matrices, we are faced with the same situation as in Case III. Again, it is suggested that the $\underline{F}$* matrix in (40) be that for which the $\underline{T}$ matrix is established. We then have

(48)
$$P_1 = F_1T; \quad Y_1 = X_1T, \text{ and}$$
$$P_2 = F_2T; \quad Y_2 = X_2T,$$

so that

(49) $$\text{tr}[Y_1'Y_2/N] = \text{tr}[TT'X_1'X_2/N] = \text{tr}[X_1'X_2/N].$$

We note that although an orthonormal transformation does not alter the value of the criterion, ϕ_{IV}, in (42), the individual diagonal elements of $\underline{Y}_1'\underline{Y}_2/\underline{N}$ will not, in general, be the same as those of $\underline{X}_1'\underline{X}_2/\underline{N}$, the latter diagonal elements being, of course, canonical correlations between the component scores for occasion 1 and those for occasion 2. It is noted, finally, that a Case IV solution could be obtained by using the Orthogonal Procrustes procedures outlined in detail by Schönemann [1966]. Using such procedures, we would seek orthonormal transformations to be applied to the $\underline{X}_1$ and $\underline{X}_2$ matrices obtained by standard component analyses of $\underline{R}_1$ and $\underline{R}_2$--transformations that would satisfy (42). We judge the procedure outlined in this section as more direct.

Case V

As with Case IV, with Case V we hypothesize that both the component scores and pattern matrices vary between occasions. With Case V, however, we attempt to bring both pattern matrices, rather than matrices of component scores, to as similar a position as possible. As with previous cases, a simple structure resolution is possible.

The Case V model is, again, as in (1)

(1) $$Z_1 = X_1F_1' + E_1, \text{ and}$$ $$Z_2 = X_2F_2' + E_2.$$

Our criterion, however, is

(50) $$\phi_V = tr[E_{12}'E_{12}] = \text{minimum},$$

where $\underline{E}_{12} = \underline{F}_1 - \underline{F}_2$. Our constraints are as in (18),

(18) $$X_1'X_1/N = I = X_2'X_2/N.$$

If we let $\underline{F}_1 = \underline{Z}_1'\underline{X}_1/\underline{N}$, and $\underline{F}_2 = \underline{Z}_2'\underline{X}_2/\underline{N}$, the criterion, (50), may be written subject to the constraints, (18), as

(51) $$\phi_V^* = (1/N^2)tr[X_1'Z_1Z_1'X_1 + X_2'Z_2Z_2'X_2 - 2X_1'Z_1Z_2'X_2] + tr[\Lambda_1(X_1'X_1/N - I)] + tr[\Lambda_2(X_2'X_2/N - I)],$$

where Λ_1 and Λ_2, $r \times r$, are matrices of Lagrange multipliers. Differentiating ϕ_V^* with respect to X_1 and X_2 and setting the partial derivatives to the null matrix, we have

(52) $$\partial\phi_V^*/\partial X_1 = (2/N^2)(Z_1Z_1'X_1 - Z_1Z_2'X_2) + (1/N)X_1(\Lambda_1 + \Lambda_1') = 0, \text{ and}$$

(53) $$\partial\phi_V^*/\partial X_2 = (2/N^2)(Z_2Z_2'X_2 - Z_2Z_1'X_1) + (1/N)X_2(\Lambda_2 + \Lambda_2') = 0.$$

Letting the symmetric matrix $(\Lambda_1 + \Lambda_1')/2 = Q_1$, and premultiplying (52) by X_1', we have, for (52),

(54) $$X_1'Z_1Z_2'X_2 - X_1'Z_1Z_1'X_1 = N^2Q_1.$$

Equation (54), in addition to similar manipulation of (53), reveals that, by a symmetry argument like that used in the somewhat similar Case II, the Case V criterion is satisfied, subject to the constraints noted, by taking for X_1 and X_2, the matrices, respectively, $N^{1/2}W_r$ and $N^{1/2}V_r$, where W_r and V_r are composed of the first r columns of W and V, obtained by the Eckart-Young factorization given in (26),

(26) $$Z_1Z_2' = WLV'.$$

The F_1 and F_2 matrices are subsequently given by

(55) $$F_1 = Z_1'X_1/N, \text{ and}$$
$$F_2 = Z_2'X_2/N.$$

At this point we would likely seek simple structure resolutions for F_1 and F_2. If a single <u>orthonormal</u> transformation, T, is applied to F_1 and F_2, yielding P_1 and P_2, the criterion (50) is unaffected (although the individual diagonal elements of $E_{12}'E_{12}$ certainly are affected), since if $E_{12}^* = F_1T - F_2T = (F_1 - F_2)T$, then

(56) $$tr[E_{12}^{*\prime}E_{12}^*] = tr[(E_{12}T)'(E_{12}T)] = tr[T'E_{12}'E_{12}T] = tr[E_{12}'E_{12}].$$

We thus recommend that the same simplistic procedure outlined with Cases III and IV solutions be used, in which we form $\underline{F}^* = (1/2)(\underline{F}_1 + \underline{F}_2)$, and then rotate $\underline{F}^*$ to a position satisfying some analytic orthogonal simple structure criterion. We then have, for interpretive purposes, $\underline{P}_1 = \underline{F}_1\underline{T}$ and $\underline{P}_2 = \underline{F}_2\underline{T}$, where $\underline{T}$ is the orthonormal transformation applied to $\underline{F}^*$.

As with Case IV, a Case V solution could, alternatively, have been obtained by applying Orthogonal Procrustes procedures [Schönemann, 1966]. Such procedures would be applied to $\underline{F}_1$ and $\underline{F}_2$ matrices obtained by standard component analyses of $\underline{R}_1$ and $\underline{R}_2$. Again, we judge the technique outlined in this section to be more direct.

<u>The Problem of the Number of Components to Retain</u>

The decision regarding the appropriate value of $\underline{r}$ in the analysis procedures just described may be seen as either possibly the result of the analysis procedures themselves or the result of an assessment of the congruence between components derived by standard component analyses on the two occasions. At this time not enough experience has been gained with data to suggest rules of thumb for deciding the correct number of components when the various cases described are adopted and the analysis proceeds exactly as outlined. It is likely, furthermore, that different rules will apply in the different cases. In short, few suggestions can be made at this time for deciding on the appropriate number of Case II components, for example, <u>as a result of a Case II analysis</u>. As more experience is gained involving real data and the analysis procedures described, it is hoped that some suggestions regarding the number of components will come out of these procedures themselves. As is noted later, an exception to the above situation may be Case IV.

The other alternative is to perform principal component analyses on the $\underline{Z}_1$ and $\underline{Z}_2$ matrices, and apply current rules of thumb to ascertain $\underline{r}$; subsequently, with a knowledge of $\underline{r}$, the experimenter would apply the appropriate procedures as previously described. An advantage exists with longitudinal data in deciding on $\underline{r}$, since the notion of stability over time can be added to the other various rationales. As one possibility, the experimenter could factor $\underline{R}_1$ and $\underline{R}_2$ into latent roots and vectors and apply the Kaiser-Guttman rule of accepting for $\underline{r}$, the number of latent roots greater than one. If this rule led to the same value of $\underline{r}$ for the two occasions, some faith could be placed in this particular value. On the other hand, if the number of latent roots greater than one differed on the two occasions, other tests would be indicated.

Cross-correlating the components obtained on the two occasions, $\underline{X}_1$ and $\underline{X}_2$, is another possibility that could be employed either with or without the aforementioned inspection of latent roots. If the canonical decompositions of $\underline{R}_1$ and $\underline{R}_2$ are designated, respectively, $\underline{Q}_1\underline{M}_1^2\underline{Q}_1'$, and $\underline{Q}_2\underline{M}_2^2\underline{Q}_2'$, and the matrix of correlations between variables on occasion 1 and 2 is designated $\underline{R}_{12}$, then the matrix

$$X_1'X_2/N = M_1^{-1}Q_1'R_{12}Q_2M_2^{-1} \tag{57}$$

contains the cross-occasion correlations of components. If we arrange the rows and columns of this matrix to maximize $tr[\underline{X}_1'\underline{X}_2/\underline{N}]$, then inspection of the diagonal elements would indicate at what point the components at occasion 1 cease to have clearly identifiable pairmates at occasion 2. We might regard these between-occasion component correlations as component stability coefficients and take for $\underline{r}$ the number of components for which the stability coefficient exceeds some minimal level of reliability, for example, .7.

Perhaps a better approach than that just described would be to determine the maximum congruence that can be obtained between columns of $\underline{X}_1$ and $\underline{X}_2$. The canonical correlation procedures outlined in Case IV would provide a way of applying this rationale. We would simply obtain the matrices $\underline{X}_1$ and $\underline{X}_2$ as given in (46), with the one difference being that we would continue to obtain components until the canonical correlation dropped below some predetermined value, $\underline{k}$, again, perhaps, .7. The value of $\underline{r}$ would be the number of canonical correlations greater than $\underline{k}$.

Summary of the Procedures Proposed

The preceding analysis procedures are brought together in Table 1, in the interests of presenting a unified view of these techniques. The $\underline{E}_1$ and $\underline{E}_2$ matrices given in the column headed "Criterion" are, in general, matrices of discrepancies between the observed data matrices, $\underline{Z}_1$ and $\underline{Z}_2$, and their reproduction from respective products of component scores and patterns. The "solutions" given will, for the most part, require augmentation from material in the text. It may be worth noting, finally, the great similarity between Case II and Case V.

3. Empirical Examples

In this section results of solutions based on three of the five cases developed earlier are presented. Each is discussed in turn.

A Case I Solution

Data for the application of Case I procedures were constructed by computer simulation techniques. This alternative was found necessary since very few longitudinal studies were found in the literature, and those data that were available did not include the original score matrices, needed for a Case I solution. The results of the Case I analysis appear in Table 2.

First, a 50 x 3 matrix was constructed using random number generation. Next, by rescaling of the columns and a roots and vectors decomposition, a matrix, $\underline{X}$, of "true" component scores, 50 x 3, was obtained such that each column had exactly zero mean, unit variance, and zero intercorrelation with each other column. A "true" rotated pattern matrix, $\underline{P}$, displayed in Table 2, was introduced, and the product $\underline{XP}'$ formed, yielding a 50 x 9 matrix. Finally,

TABLE 1

Summary of the Analysis Strategies Proposed
under Varying Assumptions Regarding the Component Scores and Patterns

Case	Hypothesized Stability Over Time: Scores	Hypothesized Stability Over Time: Patterns	Criterion	Solutions (Relevant Formulas in Text)	Transformation to Simple Structure
I	Unchanging	Unchanging	$tr[E_1'E_1] + tr[E_2'E_2] = min.$	$X=N^{\frac{1}{2}}W_r$, where $(Z_1+Z_2)(Z_1+Z_2)' =WL^2W'$ (12); $F=(1/2N)(Z_1+Z_2)'X$ (8)	Orthogonal or oblique
II	Changing	Unchanging	$tr[E_1'E_1] + tr[E_2'E_2] = min.$	$\left.\begin{matrix}X_1=N^{\frac{1}{2}}W_r\\X_2=N^{\frac{1}{2}}V_r\end{matrix}\right\}$, where $Z_1Z_2'=WLV'$ (26) $F=(1/2N)(Z_1'X_1+Z_2'X_2)$ (23)	Orthogonal or oblique
III	Unchanging	Changing	$tr[E_1'E_1] + tr[E_2'E_2] = min.$	$X=N^{\frac{1}{2}}V_r$, where $Z_1Z_1'+Z_2Z_2'=VL^2V'$ (39) $F_1=Z_1'X/N$; $F_2=Z_2'X/N$ (35)	Orthogonal or oblique
IV	Changing	Changing	$tr[X_1'X_2/N] = max.$	X_1 and X_2 obtained by canonical correlation analysis (43)-(46) $F_1=Z_1'X_1/N$; $F_2=Z_2'X_2/N$ (47)	Orthogonal
V	Changing	Changing	$tr[(F_1-F_2)'(F_1-F_2)] = min.$	$\left.\begin{matrix}X_1=N^{\frac{1}{2}}W_r\\X_2=N^{\frac{1}{2}}V_r\end{matrix}\right\}$, where $Z_1Z_2'=WLV'$ (26) $F_1=Z_1'X_1/N$; $F_2=Z_2'X_2/N$ (55)	Orthogonal

TABLE 2

Correlations within and between Occasions, Correlations between True and Estimated Component Scores, and True and Estimated Rotated Pattern Matrices for a Case I Solution Using Artificial Data (Decimal Points Omitted)

Correlations of Variables

Occasion 1 ($\underline{R}_1$)

	1	2	3	4	5	6	7	8	9
1	100								
2	75	100							
3	78	74	100						
4	-09	-09	-01	100					
5	-04	-04	-07	67	100				
6	-12	-14	-07	67	69	100			
7	-11	-01	02	01	03	-22	100		
8	-07	06	03	07	05	-16	68	100	
9	-14	-14	-04	07	08	00	73	60	100

Occasion 2 ($\underline{R}_2$)

	1	2	3	4	5	6	7	8	9
1	100								
2	73	100							
3	74	59	100						
4	-11	-07	11	100					
5	-25	-17	-04	72	100				
6	-11	-02	13	72	72	100			
7	10	20	12	02	-03	14	100		
8	-13	06	-11	-06	04	11	67	100	
9	-19	08	-08	-03	-05	14	68	68	100

Correlations between Variables on Occasion 1 (Rows) and Occasion 2 (Columns) ($\underline{R}_{12}$)

	1	2	3	4	5	6	7	8	9
1	82	78	63	-09	-15	-08	-01	-21	-25
2	80	74	68	-09	-19	-07	-02	-18	-21
3	73	75	65	-04	-17	02	07	-12	-10
4	-09	-02	21	67	71	80	08	08	05
5	-03	-06	20	64	66	62	07	18	-04
6	-13	-09	06	69	73	65	-05	-08	-14
7	-02	17	09	-06	-07	03	72	73	71
8	-04	16	05	05	-03	13	65	65	64
9	-05	16	03	02	-07	08	76	71	72

Correlations between True and Estimated Rotated Component Scores

True Component	Estimated Component I	II	III
I	967	-031	022
II	026	967	-006
III	-021	016	961

Rotated Pattern Matrices

	True ($\underline{P}$) I	II	III	Estimated ($\hat{\underline{P}}$) I	II	III
1	80	00	00	90	-09	-10
2	75	00	00	88	-08	05
3	70	00	00	85	07	01
4	00	80	00	00	87	02
5	00	70	00	-06	85	01
6	00	75	00	-03	86	-01
7	00	00	75	08	-01	88
8	00	00	65	-03	03	84
9	00	00	70	-08	00	87

two matrices of "errors"--$\underline{E}_1$ and $\underline{E}_2$, both 50 x 9, of (2) were constructed using random numbers. These matrices were then deviated and rescaled so that each column had zero mean and variance .25. The matrices $\underline{XP}' + \underline{E}_1$ and $\underline{XP}' + \underline{E}_2$ were then formed and column rescaled to have unit variance, yielding, respectively, $\underline{Z}_1$ and $\underline{Z}_2$.

The resulting $\underline{R}_1$, $\underline{R}_2$, and $\underline{R}_{12}$ matrices appear in Table 2. Using formulas (8), (14), and (15), matrices of, respectively, estimated pattern coefficients and component scores were obtained. A normal varimax rotation was applied to the obtained pattern, yielding the $\underline{\hat{P}}$ matrix in Table 2. The obtained orthonormal transformation was then applied to the estimated component scores. These transformed component scores were then correlated with the "true" component scores constructed earlier, with the resulting correlations presented in Table 2. Separate component analyses of $\underline{R}_1$ and $\underline{R}_2$ ensured that a decision of three components would be reached, and this value for $\underline{r}$ was retained throughout the analyses.

It is probably true that the great congruence between the "true" and estimated elements of this example are, in large part, due to the simplicity and artificiality of the data, even though an attempt was made to simulate real conditions by generating variables with reliability only approximately .70 or so, as can be seen from the $\underline{R}_{12}$ matrix. In any case, it appears true that if all that changes from occasion 1 to occasion 2 is error, and that time-stable variables and components can be reasonably hypothesized, the Case I solution is useful in delineating this stable configuration.

A Case III Solution

Data for the application of Case III procedures were constructed in somewhat the same manner as for the Case I example. It will be recalled that the component scores are hypothesized to be constant in Case III, with possibly changing patterns. The same component score matrix, $\underline{X}$, used with

Case I was postmultiplied by the transposes of two different "true" rotated patterns, $\underline{P}_1$ (identical to the $\underline{P}$ matrix of the previous example) and $\underline{P}_2$, for the two occasions. These patterns, as well as the other matrices of interest for this example, are displayed in Table 3. It will be noted that $\underline{P}_1$ and $\underline{P}_2$ differ somewhat, reflecting the fact that factorial complexity greater than one--for variables 3,5, and 8--and more broadly defined factors are present at occasion 2, but not occasion 1. Again, two 50 x 9 matrices of random error--mean zero, variance .25 by columns--$\underline{E}_1$ and $\underline{E}_2$ of (28), were constructed, and were added to $\underline{XP}_1'$ and $\underline{XP}_2'$, respectively. The resulting matrices were then column rescaled to have unit variances, yielding, respectively, $\underline{Z}_1$ and $\underline{Z}_2$.

Clearly, although the underlying three components (the value of three for $\underline{r}$ was again confirmed from analyses of $\underline{R}_1$ and $\underline{R}_2$) are stable over the two occasions, the variables themselves have changed more than randomly between occasions, as can be seen from the $\underline{R}_{12}$ matrix. The initial pattern matrices were obtained using (35), after the initial matrix of component scores had been computed using (39) in connection with the Hotelling procedure discussed earlier. The initial patterns were combined using (40) and a normal varimax rotation was performed on the resulting $\underline{F}$* matrix. The transformed patterns, as well as transformed component scores, were obtained by (41).

As is seen in Table 3, the three obtained vectors of component scores were very close to the "true" scores, and the estimated rotated pattern matrices for both occasion 1 and occasion 2 were extremely close to the corresponding "true" patterns. Once again, however, the great similarity between "true" and estimated elements in this example is likely, to some extent, a function of the artificial nature of the data. If, however, the experimenter wishes to hypothesize--for theoretical or conceptual reasons--unchanging underlying components, although the variables themselves can be observed to change somewhat from occasion 1 to occasion 2, the Case III solution may be

TABLE 3

Correlations within and between Occasions, Correlations between True and Estimated Component Scores, and True and Estimated Rotated Pattern Matrices for Each Occasion for a Case III Solution Using Artificial Data (Decimal Points Omitted)

Correlations of Variables

Occasion 1 ($\underline{R}_1$)

	1	2	3	4	5	6	7	8	9
1	100								
2	72	100							
3	72	68	100						
4	-06	09	07	100					
5	-08	06	02	66	100				
6	01	07	09	70	62	100			
7	15	04	14	01	04	04	100		
8	08	02	10	06	06	-08	70	100	
9	06	05	-02	09	09	01	61	68	100

Occasion 2 ($\underline{R}_2$)

	1	2	3	4	5	6	7	8	9
1	100								
2	66	100							
3	50	58	100						
4	-10	-08	42	100					
5	06	13	38	47	100				
6	-23	-09	31	69	44	100			
7	14	04	17	07	46	-09	100		
8	54	52	41	10	49	-08	48	100	
9	04	-06	04	-01	42	-18	68	47	100

Correlations between Variables on Occasion 1 (Rows) and Occasion 2 (Columns) ($\underline{R}_{12}$)

	1	2	3	4	5	6	7	8	9
1	73	75	43	-14	06	-22	15	56	03
2	68	70	58	-06	10	-05	12	53	00
3	65	75	48	-05	03	-16	07	46	-02
4	-09	05	52	66	57	63	05	08	-11
5	-03	-01	46	73	54	61	10	09	09
6	-06	10	47	77	50	60	02	04	-02
7	20	11	12	01	50	-23	72	54	69
8	09	09	-01	-12	51	-15	67	41	61
9	03	02	09	-08	48	-04	53	45	67

Correlations between True and Estimated Rotated Component Scores

True Component	Estimated Component I	II	III
I	968	-042	069
II	042	967	-053
III	-060	064	960

Rotated Pattern Matrices

Occasion 1

	True ($\underline{P}_1$) I	II	III	Estimated ($\hat{\underline{P}}_1$) I	II	III
1	80	00	00	85	-14	12
2	75	00	00	83	00	05
3	70	00	00	69	-04	04
4	00	80	00	08	81	00
5	00	70	00	03	69	06
6	00	75	00	10	78	-04
7	00	00	75	06	00	76
8	00	00	65	-01	-02	74
9	00	00	70	-03	04	61

Occasion 2

	True ($\underline{P}_2$) I	II	III	Estimated ($\hat{\underline{P}}_2$) I	II	III
1	80	00	00	84	-14	13
2	75	00	00	89	-02	05
3	50	50	00	67	53	07
4	00	80	00	-02	88	-05
5	00	50	50	09	65	59
6	00	75	00	-11	82	-19
7	00	00	75	05	07	83
8	50	00	50	59	07	60
9	00	00	70	-08	-02	86

used to implement this hypothesis.

A Case IV Solution

In contrast to the examples used to illustrate Cases I and III, our data for the application of Case IV procedures were those gathered by Meyer and Bendig [1961], and subsequently reanalyzed by Harris [1963] and Corballis and Traub [1970]. As with the latter study, the number of components (factors, in the Corballis and Traub study) was determined to be two. Separate component analyses of the $\underline{R}_1$ and $\underline{R}_2$ matrices showed one latent root greater than one for $\underline{R}_1$ and two, for $\underline{R}_2$. The canonical correlation analysis subsequently conducted yielded the following canonical correlations: .89, .74, .58, .45, .37. Thus, by finding maximally congruent linear composites, we found two components in each set that correlated highly enough to suggest that we were likely dealing with the same two constructs on each occasion. The $\underline{R}_1$, $\underline{R}_2$, and $\underline{R}_{12}$ matrices, as well as the other matrices of interest, are displayed in Table 4.

The unrotated pattern matrices in Table 4 were obtained by the procedures given in (44) through (47). A normal varimax transformation was applied to these patterns, with virtually no improvement in simple structure. Inspection of the plane spanned by the I-II vectors for each occasion revealed that no true simple structure resolution is possible with these data. It will be noted that although the two component cross-occasion correlations were altered in the transformation, their sum was unaltered. Inspection of the two unrotated (or rotated) pattern matrices reveals that a high degree of congruence was obtained, with the interpretation of the components likely identical on the two occasions. Finally, some of the difference between the present solution for these data and that given by Corballis and Traub [1970] is due to the fact that the latter authors derived components in terms of disattenuated correlations.

TABLE 4

Correlations within and between Occasions, Correlations between Occasion 1 and Occasion 2 Components, and Occasion 1 and Occasion 2 Pattern Matrices for a Case IV Solution Using the Meyer and Bendig Data (Decimal Points Omitted)

Correlations of Variables

	Occasion 1 (R_1)					Occasion 2 (R_2)					Between Occasion 1 (Rows) and Occasion 2 (Columns) (R_{12})				
1	100					100					81	35	42	41	24
2	37	100				34	100				35	65	32	14	15
3	42	33	100			46	18	100			49	20	75	40	17
4	53	14	38	100		56	06	54	100		58	-04	46	73	15
5	38	10	20	24	100	24	15	20	16	100	32	11	26	19	43

Correlations between Occasion 1 and Occasion 2 Components

	Unrotated Components		Rotated Components	
	Occasion 2 Component		Occasion 2 Component	
Occasion 1 Component	I	II	I	II
I	886	000	859	056
II	000	743	056	770

Pattern Matrices

	Occasion 1				Occasion 2			
	Unrotated		Rotated		Unrotated		Rotated	
	I	II	I	II	I	II	I	II
1	87	11	73	48	91	-02	83	38
2	48	64	15	78	38	77	00	85
3	77	04	68	37	77	-05	72	29
4	69	-62	89	-26	64	-57	82	-23
5	41	04	35	21	31	09	24	22

4. Conclusions

Procedures have been developed and in some cases illustrated for the factor analytic treatment of data matrices, involving the same subjects, obtained on two occasions. The specific techniques have differed in terms of whether or not the component scores and/or pattern matrices are hypothesized to be stable or unchanging over time. Good reasons can be advanced for the hypothesis of stable scores _and_ patterns (our Case I) in some situations where only constructs that are constant relative not only to scores of the subjects measured, but also to the composition of these constructs in terms of linear combinations of the observed variables are of any theoretical interest. On the other hand, it is conceivable that situations may arise in which only one of these matrices may reasonably be expected to be constant. That matrix may be the component pattern (Case II), with primary interest in changing scores of the subjects (relative to the respective group mean; changes in group elevation are not of concern) over time on these constructs whose composition over time is hypothesized not to change, or it may be the matrix of unchanging scores (Case III), in which the interest lies in the changing composition (relative to the observed variables) of these unchanging constructs (relative to the subjects involved). If the experimental situation suggests that both scores of the subjects and composition relative to the variables can be expected to change over time, our Cases IV and V may be appropriate, Case IV if we wish to maximize the congruence of the changing scores, and Case V if we wish as similar composition of the (changed) factors as possible. In each case, provisions have been made for a simple structure resolution. In all cases except Case IV, least-squares estimates have been provided for all hypothesized matrices.

Although the procedures developed for Cases I, II, III, and V are least-squares, we have not attempted to provide techniques for _evaluation_ of the hypotheses embodied in the various cases. It appears conceivable that likelihood ratio procedures could be developed to estimate the goodness-of-fit of the reproduced data arrays--in terms of the estimated matrices--to the observed $\underline{Z}_1$ and $\underline{Z}_2$ matrices.

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