DOCUMENT RESUME

ED 059 910 SE 013 452

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|---|---|
| TITLE | Mathematics Framework for California Public Schools, Kindergarten Through Grade Eight. The Second Strands Report. |
| INSTITUTION | California State Dept. of Education, Sacramento. |
| PUB DATE | 72 |
| NOTE | 128p. |
| EDRS PRICE | MF-$0.65 HC-$6.58 |
| DESCRIPTORS | Algebra; Arithmetic; *Curriculum; *Elementary School Mathematics; Evaluation Criteria; Geometry; Mathematics Education; Secondary School Mathematics; *State Curriculum Guides |
| IDENTIFIERS | California |

ABSTRACT

This report of a statewide Mathematics Advisory Committee outlines the kindergarten through grade eight mathematics curriculum in terms of nine "strands": Numbers and Operations, Geometry, Measurement, Applications of Mathematics, Statistics and Probability, Sets, Functions and Graphs, Logical Thinking, and Problem Solving. Broad goals for the entire program, and specific goals for each strand, are stated. Examples are given of activities and content leading to these goals. Material in the strands is not allocated to specific grade levels, but related to the students' overall development. An algebra course for grade eight is discussed, and the 1968 state criteria for the evaluation of textbooks is reprinted in an appendix. This document is intended to be of use to writers and publishers as well as teachers. (MM)

THE SECOND STRANDS REPORT

# Mathematics Framework

## FOR CALIFORNIA PUBLIC SCHOOLS

### Kindergarten Through Grade Eight

CALIFORNIA STATE DEPARTMENT OF EDUCATION
Wilson Riles, Superintendent of Public Instruction
Sacramento, 1972

THE SECOND STRANDS REPORT

# Mathematics Framework
## for California Public Schools
### Kindergarten Through Grade Eight

Prepared for the
California State Board of Education

and the
California State Curriculum Commission

By the
Statewide Mathematics Advisory Committee

The California State Board of Education authorized the compilation and publication of this report, which was funded under the provisions of Title III of the National Defense Education Act (NDEA). It was edited and prepared for photo-offset reproduction by the Bureau of Publications, California State Department of Education, and published by the Department, 721 Capitol Mall, Sacramento, California 95814.

Printed by the Office of State Printing
1972

## FOREWORD

When Julie Ann, who is seven years old, was asked to define mathematics, she said without hesitation, "Math is numbers." When her brother Danny, who is nine years old, was asked the same question, he said, "Math is fun." Then he added, "It's times and dividing and fractions, pluses and subtractions, expanding . . . and, oh, yeah, it's also prime numbers, equivalent fractions, clocks – that's telling what time it is – and word problems. I like word problems."

Julie Ann, who is enrolled in the second grade of one of our public schools, was also asked how she would teach her first lesson in mathematics if she were a teacher. Again, she did not hesitate. She said, "I'd write all the numbers on the board. Then I'd count all the numbers up to ten, because that's all the numbers I'd put up there. And after I'd counted up to ten, I'd point to a number and ask one of them (the children) to tell me what the number was, and the others would have to listen *well.*"

Then the little girl, who has sparkling brown eyes, was asked how she had learned her numbers. "When I first started learning math," she responded, "it was easy because Danny taught me, and I had fingers (to count on)."

I tell you of these two children because they identify for me the reason for our developing this mathematics framework. And I tell you of them because I am certain that neither Karl Friedrich Causs, the great German scholar, nor any modern mathematician could have given a clearer or more precise definition of mathematics than "math is numbers" or one that would have better expressed his feelings for mathematics than "math is fun." I am equally certain that I would have difficulty improving on Julie Ann's first mathematics lesson. She knows from her own experience – including her schooling – that the counting of things in groups (her fingers) is the root principle of mathematics – that one of the basic principles of learning is "listening *well.*"

Many fine people have worked many hours to develop this framework. I believe it has been developed with the best interests of the Julie Anns and Dannys in mind. I am hopeful that those responsible for curriculum planning in mathematics will find in these pages ideas that will lead to the development of even better instructional programs for our children.

Educational systems must constantly undergo change to meet the conditions of the times in which they exist. And the Statewide Mathematics Advisory Committee has prepared a framework designed to help in the development of mathematics programs that reflect the needs of the 1970s. I am deeply grateful for the work they have done.

As we develop our programs, let us remember that it is the direction in which we start the child that will determine his success. "Math is numbers." and we will not change that, but "math is fun" only because someone made it so.

Wilson Riles

*Superintendent of Public Instruction*

# PREFACE

In October, 1960, the California State Board of Education appointed an advisory committee on mathematics to the State Curriculum Commission. This committee served until 1962 and developed a comprehensive report and recommendations on mathematics curriculum and instruction, teacher preparation, and supplemental instructional materials. The report was submitted, in three parts, in March, 1962. One part comprised the framework for mathematics instruction and curriculum known as the first "Strands" Report. In conjunction with that framework, the committee then developed criteria to be used in the selection and adoption of mathematics textbooks for grades one through eight. The textbooks in this adoption arrived in California classrooms in September of 1964 for grades one, two, and seven, and in September of 1965 for grades three, four, five, six, and eight.

The advisory committee's complete report was published in December, 1963, by the California State Department of Education. It was entitled *Summary of the Report of the Advisory Committee on Mathematics to the State Curriculum Commission – The Strands of Mathematics; Mathematics Programs for Teachers; A Study of New Programs and Supplementary Materials.*

Six years after the appointment of the first advisory committee on mathematics, the State Board of Education appointed another committee, which included some members who had served on the 1960 committee. The Curriculum Commission, in anticipation of the next mathematics textbooks adoptions, believed that an evaluation and review of current mathematics textbooks and the 1962 Strands Report were called for, and it recognized the need for a new framework, which would take into consideration the requirements of California pupils in the 1970s.

The new committee met for the first time on January 27, 1967. It was asked by the Board to review the first Strands Report and assess it in view of current developments in the field of mathematics; to study and evaluate the mathematics program then in the schools as it related to that report; to review mathematical concepts and to indicate the most effective and feasible placement of those concepts in a mathematics framework; to help in developing criteria for evaluating mathematics textbooks and instructional materials for the next adoption and to study procedures that might be used in screening textbooks considered for use in California schools; to help develop ways of implementing mathematical concepts and applications in other fields of endeavor; to suggest means for evaluating the mathematics program in the schools as the new materials, textbooks, and framework were developed and used; to make recommendations and suggestions relative to inservice courses; and to make any suggestions believed to be helpful in implementing an ideal mathematics program in California.

The advisory committee completed its immediate task and produced a framework for kindergarten and grades one through eight – the second Strands Report. The framework was submitted to the State Board of Education in two parts, in preliminary form, in September, 1967, and approved by the Board. A criteria statement for textbook adoptions was prepared by the Advisory Committee and accepted by the Board in January, 1968. The Curriculum Commission based its forthcoming textbook selections and adoptions and its recommendations for supplementary mathematics teaching materials on the criteria. The mathematics textbooks were adopted for a five-year period beginning with the 1970-71 school year.

The State Board of Education, in approving the release of the preliminary 1967 Strands Report, authorized the California Mathematics Council to print it in its *Bulletin;* subsequently, it appeared in three parts in the October, 1967, and the January and May, 1968, issues. Reprints were then made available, for which there has been a steady demand from teachers, school administrators, textbook publishers, and teacher-training institutions in California and throughout the United States. Publication of the report in this final version will make possible its continued availability to those concerned with today's elementary mathematics curriculums and programs.

This report takes us into a second round of the school mathematics "revolution" – one that focuses on ways of teaching and learning mathematics. The "climate in the classroom" is the pervasive theme of education for the 1970s.

The Statewide Mathematics Advisory Committee, under the capable leadership of John L. Kelley, has made a tremendous contribution in preparing this document. The implications for mathematics educators and for authors and publishers of textbooks and other instructional materials are vast. We are indeed grateful for the dedicated efforts of the members of the Committee in completing this project.

EUGENE GONZALES
*Associate Superintendent of Public Instruction; and Chief, Division of Instruction*

MITCHELL L. VOYDAT
*Chief, Bureau of Elementary and Secondary Education*

# STATEWIDE MATHEMATICS ADVISORY COMMITTEE

The names of the members of the second Statewide Mathematics Advisory Committee and the positions they held at the time of the study follow:

John L. Kelley, *Chairman;* Professor of Mathematics, University of California, Berkeley

*G. Don Alkire, Professor of Mathematics, Fresno State College (deceased)

George F. Arbogast, Elementary Mathematics Supervisor, Los Angeles City Unified School District

Leslie S. Beatty, Director of Research and Curriculum Materials, Chula Vista City Elementary School District

William G. Chinn, Executive Director, Statewide Mathematics Advisory Committee Research Project, San Francisco City Unified School District

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*G. Don Alkire resigned in July, 1967, when the State Board of Education appointed his replacement, Robert A. Hansen.

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Chapter One

# INTRODUCTION

The report summarizing the productive work of the first Advisory Committee on Mathematics to the California State Curriculum Commission was completed in 1962 and subsequently published in 1963, and it became known as the "Strands Report."[1] That report gave major emphasis to a change in the *content* of the mathematics curriculum in kindergarten and grades one through eight. It recommended (1) that more emphasis be placed on the structure of mathematics; (2) that the elements of synthetic and coordinate geometry assume a central role in the curriculum; (3) that the language of sets be introduced; and (4) that mathematics and its applications should be related to the entire curriculum. That report continues to be a sound statement of the core of the elementary mathematics curriculum. These present recommendations refine and revise the cognitive aspects of the curriculum and extend the pedagogical requirement for improving the climate within the classroom.

The 1962 report was far ahead of its time in many ways. This farsightedness created many problems, two of which were crucial: there was no textbook series containing the entire mathematical content recommended in the report, and teachers had not had the opportunity to develop an adequate background for teaching a program in mathematics based on the recommendations. Today, California can expect to secure instructional materials that will meet the standards of the Strands Report. Many teachers in California have had preservice and inservice education requisite for the implementation of the programs while others are now ready and willing to undertake the additional inservice education that is necessary.

Based upon these premises, the State Department of Education should undertake a variety of experiments in the teaching of

[1] *Summary of the Report of the Advisory Committee on Mathematics to the California State Curriculum Commission: The Strands of Mathematics, Mathematics Programs for Teachers, A Study of New Programs and Supplementary Materials.* Sacramento: California State Department of Education, December, 1963.

mathematics and the inservice education of teachers. Research supported by the state's Mathematics Improvement Programs[2] will provide school districts with suggestions for classroom and school organizational patterns that will result in more effective utilization of the specialized training of teachers and with guidelines for inservice classes that will enable them to teach the mathematics program outlined in this report. It is further recommended that the preservice curriculum for elementary teachers be expanded to include a minimum of six semester or nine quarter units in mathematics.

Studies and experimentation since 1962 have identified mathematical content that can and should be taught to elementary pupils as well as effective ways in which pupils learn that content. This work, much of which has been done within California, constitutes an endorsement of those recommendations of the first advisory committee's report having to do with the mathematical content of the curriculum. Reports such as those from the Cambridge Conference on School Mathematics[3] have strengthened and confirmed a trend toward a broader mathematics curriculum in the kindergarten-grade eight program. The revisions and expansions of the content of the strands in this report are evidence of a commitment to this trend and constitute the recommendation of this advisory group for the content of the mathematics curriculum in kindergarten through grade eight for the next few years.

This document, in outlining the essential strands of mathematics, should be of use to writers, publishers, and teachers. It does not seek to be prescriptive to authors and publishers, nor is it intended as an outline for an inservice course for teachers. Within each strand, examples are given to convey the general scope and sequential development of the mathematical content. Writers, publishers, and teachers will find here a strong and clear statement of many subtle and crucial issues in curriculum planning together with recommendations on these issues. The obvious restrictions of space and time do not permit a full and definitive treatment of any strand.

The strands, as identified by the second advisory committee and which have been selected for special emphasis, split naturally into two categories. The first category includes the strands of Numbers and Operations, Geometry, Measurement, and Statistics and Probability. These strands are the basic cognitive subdivisions of the

[2]See Education Code sections 5799-5799.49.

[3]*Goals for School Mathematics.* The Report for the Cambridge Conference on School Mathematics. Published for Educational Services, Inc. Boston: Houghton Mifflin Co., 1963.

mathematics curriculum itself. The other category includes the strands of Applications, Sets, Functions and Graphs, Logical Thinking, and Problem Solving. These strands are catalysts, or processes which facilitate mathematical analysis to some degree in every mathematical enterprise. No single strand can stand by itself; together they constitute a strong, viable program. To slight one strand or to overemphasize any strand is to weaken this program significantly. A satisfactory curriculum will display and use these interdependent strengths in its development.

The next few years will see more and more students ready for a first course in algebra at the beginning of grade eight. Indeed, this is a conservative prediction if the program envisaged here has the merit claimed for it and if the program is adopted and implemented with a significant inservice training program. The day can be anticipated when every college-capable student will have had algebra before entering grade nine.

Algebra should not be identified as a strand in itself, any more than arithmetic should be considered a strand. The central ideas of a modern algebra course can easily be identified as an extension of the framework of the strands described here. This extension has not been indicated in each strand because the major emphasis of these recommendations concerns the mathematical training of students up to algebra. On the other hand, algebra is appropriate for grade eight, and it should be included for as many pupils as are prepared for it. For this reason, a brief outline of a suitable program for algebra in grade eight has been included in this report. It is important to recognize that a prerequisite for this algebra program is the content of the mathematics curriculum recommended here.

The most significant feature of the ongoing studies in mathematics education lies in pedagogy. How should mathematics be presented so that it will be best understood and most efficiently mastered? A growing body of evidence supports the belief that mathematics learning must involve each pupil in a participative activity employing manipulative materials as well as pure cognitive exercises.

Mathematics must be closely tied to its applications in every other discipline, from art to zoology. On the other hand, by its very nature mathematics is abstract. It is the abstract quality of mathematics that enables mathematics to be used in a wide scope of applications. Yet, a wide scope of applications does not "prove" the truth of mathematics. Mathematics has its own truth, independent of its power of application. However, each pupil should appreciate this abstraction, not just for abstraction's sake, but so that he can use these abstract ideas in a broad spectrum of situations. The capability to solve a variety of problems in all disciplines rests on the ability to recognize and use a general underlying principle. Often such a basic principle is either stated in pure mathematical language or is an abstract mathematical concept itself.

There are many valid ways to teach mathematics based upon the different modes by which pupils learn. The final word on mathematics education will never be written because society continually places new demands on mathematics. Moreover, changes in methods of instruction will also have a significant impact upon the total educational process. This report identifies some of the goals currently important and indicates how they may be achieved. Continued studies, experiments, and evaluations in this field are encouraged. The resulting evidence and judgments should form a basis for future development of more effective mathematics instructional programs in California.

Chapter Two

# BROAD GOALS

The elementary school mathematics program should be concerned with basic, pervasive, and fundamental mathematical concepts and skills. Priority rests with those concepts and skills that will be used, either directly or as prerequisites to other principles and techniques, in mathematics, in the physical and social sciences, and in the ordinary day-to-day decision making of every citizen.

Not all pupils are enthusiastic learners of mathematics. Often they will elect not to study mathematics, only to realize later that mathematics was an irreplaceable ingredient in their further learning. Therefore, the elementary school program must be designed so that pupils will be encouraged, enticed, and even cajoled, if necessary, into doing as much mathematics as they can. It is particularly important that the mathematics program be sufficiently flexible to accept at any time pupils who express new or deepened interest in mathematics.

The goals of the elementary school mathematics program are twofold: For those who will terminate their mathematical education at grade eight or grade nine, the program must provide the mathematics that an informed person must know; and for those who will continue to elect additional mathematics courses in high school, the program must provide a strong background to enable them to be successful in their advanced work. The latter group will include college-capable students who may be completing a full year of calculus in high school. In either case the program needs to be presented so that pupils will become enthusiastic and develop both an intellectual curiosity and a spirit of inquiry about mathematics. For this, pupils need learning experiences which give them an opportunity to explore, investigate, create, and recreate mathematics.

The following list of topics outlines broadly what these pupils should achieve and *appreciate* by the end of grade eight. Naturally, different pupils will have different levels of understanding and proficiency, and not all pupils can be expected to command knowledge in all of these areas:

*A sound background for algebra, including an introduction to:*

Numbers and operations with numbers
Conventional algorithms
Sets and functions
Mathematical sentences
Linear functions in a single variable
Solution of linear equations and inequalities
Quadratic equations, quadratic formulas, and the role of the discriminant

*A sound background for geometry, including experience with:*

Basic geometric configurations of plane and solid geometry
Simple straightedge and compass constructions
Congruence and similarity for plane figures
Translations and rotations
Measurement
Coordinate geometry in one, two, and three dimensions

*An appreciation of the development of mathematical systems, including:*

The real number system
Systems of logical thinking
Simple deductive systems and axiomatics

*The ability to use and apply mathematics, including familiarity with:*

Measurement
Statistics and probability
Extreme value problems; determination of maximum and minimum, both absolute and subject to constraints
The creation of a mathematical model as a description and means of analysis of a problem and the evaluation and interpretation of the result
Strategies and tactics for problem solving
Desk calculators

Chapter Three

# THE CLIMATE IN THE CLASSROOM

Perhaps the most significant feature of all scholarly endeavor — whether it is in mathematics, science, or the humanities — is its spirit of free and open investigation. There should be an infusion of this spirit into the classroom, for it has important implications for good pedagogy, especially for mathematics.

The learning of mathematics is a many-faceted enterprise. While the instructional program has definable goals, each pupil and his teacher must feel free to express and explore those facets which have particular meaning for him. The most striking feature of the best presentations of mathematics is the establishment of a classroom climate that, under the direction of the teacher, is pupil-oriented, self-directed, and nonauthoritarian in design. In this climate the teacher drops the role of an authoritative figure who passes judgment solely on what is right and wrong. He frames questions, plans work that excites curiosity, and encourages the pupils to exploit what they know and intuitively feel about the situation at hand. Within the definable instructional objectives, he assumes the role of a guide who conducts his pupils into regions uncharted to them.

There is, indeed, an analogy between the teacher in the classroom and a guide on a safari. The guide has been around the same wilds before, and he knows many of the pitfalls. He has the goal and major routes in mind. He also knows that his client wants to experience the thrill of the hunt and capture so he does not always direct his client down well-traveled paths. He does need to intervene when danger is imminent and to redirect his client when he is lost. He encourages him when weary and provides those extra insights that heighten the satisfaction and impact of the moment of truth. Above all, the guide acts as a pacer. Ultimately, he must see that his client manages to capture the big game. The teacher, too, is a guide and his pupils are the clients. Only the goal and the tools of pursuit are different.

The ideal classroom climate fosters the spirit of "discovery." It also provides a variety of ways for pupils to direct their own learning under the mature, patient guidance of an experienced, curiosity-encouraging teacher. Self-directed learning requires pupil involvement in creative learning experiences that are both pupil-motivated and teacher-motivated. These experiences are seldom accomplished by "Now, here's how it goes" lectures. Instead, the teacher encourages originality, recognizing that there is more than one way to solve a problem and accepting solutions in many different forms. In the ideal classroom, credit is given to learners for their own productive thinking even when it differs from the pattern anticipated by the teacher.

Thus, the climate in the classroom must provide an atmosphere of open communication between pupil and teacher. The teacher must encourage and accept problems from his pupils so that mathematics can be made as relevant as possible to pupil interests and needs, and he must provide leadership and mental discipline. Today, a rich variety of opportunities exists for learning mathematical concepts, for applying these experiences to situations, and for gaining technical proficiency in the allied schools.

Often, pupils ask questions the teacher is not ready to answer immediately. He should not hesitate to admit that he is not ready. He may use this opportunity to redefine, to rephrase, or to limit the problem, and in so doing he may illustrate to the class the technique he employes (as one individual) for attacking new problems. It would be an equally valid teaching technique for him to solicit information from his pupils and question them for leads.

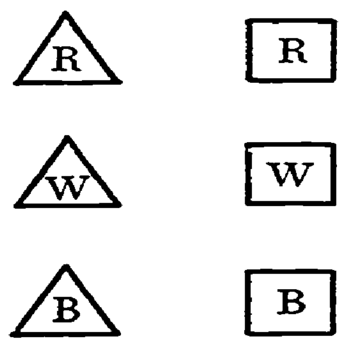
The members of one elementary class will spend approximately 1,000 hours together during one school year. Thus, learning is also a group experience. Group behavior affects the learning process, as pupils do learn from one another. Mathematics becomes a vibrant, vital subject when points of view are argued, and for this reason interaction among students should be encouraged. As pupils build mathematics together, they develop special pride in what they do, and their work gains momentum. A challenging problem will often serve as a special impetus for group projects as well as for individual research.

One of the exciting and effective ways of facilitating learning is the use of manipulative materials. The best of these materials are often simple things, which pupils may collect or make themselves. Manipulating sometimes means pushing a pencil or drawing a figure, but more often it means handling an object, comparing objects, or placing objects in various relations to each other.

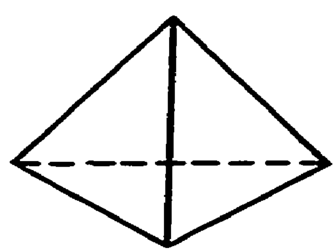
Learning is also facilitated through mathematically purposeful games, through the analysis of experiments and principles from the physical sciences, the social sciences, and the humanities, and through reinforcing previous mathematical experiences. The experiences should capitalize on each pupil's natural desire to see "why it works," to understand "how it works," and to find out "what comes next." The following are seven suggested class activities that facilitate learning:

1. *Sets, numbers, arithmetic operations, and geometry.* Each pupil constructs a set of congruent triangular cards, some of which he colors red, some white, and some blue. In addition, he constructs a set of congruent square cards, some colored red, some white, and some blue. These could be used in various ways; for example, by constructing a set of triangles and then a subset of red triangles. Sets of these figures are handy for illustrating addition, subtraction, multiplication, and division. An example is given in the following illustration of the concept of multiplication as an array. If there are red, white, and blue cards of each of two shapes, how many different colored shapes are there?

$3 \times 2 = 6$




2. *Geometry.* Join regular polygons (triangles, squares, or pentagons) to form regular polyhedra. For example, use red, white, or blue triangles, and join four of them to form a tetrahedron. If there are four triangles of each color, how many differently colored tetrahedrons can be formed?



3. *Mathematically purposeful games.* Play a game to sharpen skills. Here is a game to develop skills in estimation and number sense, a skill which is often overlooked in the desire for excessive accuracy. This game demands an estimate to the nearest hundred for

the sum of two numbers. A pair of numbers – for example, 275 and 517 – is given to each player. The winner is the one who first obtains the approximation of the answer; in this instance: $275 + 517 \sim 800 = 300 + 500$. Clearly, there are many variations of this type of game. Games should be introduced when there is a need for pupils to have a special skill.

4. *Applications of mathematics using functions and graphs.* Undertake a classroom experiment in growing radishes. Describe the growth by recording the data and making a graph. Express the growth by using the mathematical concepts of function and rate of change of growth, and determine whether the function is linear, quadratic, or exponential.

5. *Measurement.* To provide pupils with an intuitive grasp of conversion of units, each pupil should be given the opportunity to handle both pound and gram weights and to compare their heft. Physical handling of the weights can help him see the need for conversion of one system to another and help him carry out the conversion.

6. *Geometry and arithmetic.* Have the pupils explore the relationships between the number of sides, vertices, and diagonals of regular polygons, and between the number of edges, vertices, and faces of regular polyhedra.

7. *The number plane.* (a) Plot a sequence of points represented by pairs of numbers, and connect them in sequences by straight lines, forming interesting looking polygons. (b) Compute the area of a triangle as a function of the coordinates of the three vertices. (For simplicity, keep one edge of the triangle parallel to one of the coordinate axes). (c) Show how multiplication of a number by two relates to the graph of a line through the origin of slope 2.

Many of the requirements set forth here do not fit into the classical mold for textbooks. The pedagogical principles recommended change the role of a textbook from something each child should read and understand passively to a source book for experiences which each child undertakes actively.

The textbook materials submitted for future adoption should break the traditional textbook barrier to learning. Careful consideration should be given to the amount of verbalization and the reading skills required by the pupil to understand his text. (It is a paradox that mathematical concepts themselves are nonverbal, yet communication of these concepts requires sophisticated verbal skills and ability to use and understand symbolic representations.) A delicate balance must be maintained, and there should be a greater emphasis on the mathematical concepts themselves than on vocabulary or a pedantic insistence on certain symbolic conventions. On the other hand, the introduction of common mathematical terms or practices should not be overly delayed. For example, there is no reason why literal symbols as well as frames cannot be employed at an appropriate level.

Instructional materials adopted by the state to implement the mathematics program must be sufficiently flexible to be used with a variety of teaching methods and organizational plans. The ways in which the text can be personalized to the needs of the individual must be delineated. Whether or not ability grouping takes place, it is clear that in any classroom the rates of learning will vary, and the pacing of instruction must be planned accordingly. Perhaps of more significance, the pupils' modes of thinking will differ; some think best in concrete terms, others, in abstract formulations. The introduction of a new mathematical concept must be done in such a way as to appeal to each of these ways of thinking.

Chapter Four

# AN OVERVIEW OF THE STRANDS

In this section, a brief indication is given of the content of the nine strands of mathematics included in this report. These strands represent an extension and revision of the strands, eight in number, which were presented in the report made by the first Advisory Committee on Mathematics.[1] Fuller explanations of the technical terms used as well as the sequence and depth of the mathematical concepts, the appropriate vocabulary, and the symbolism are provided for each strand in chapters eight through sixteen.

### Strand 1. Numbers and Operations

The content of this strand is the heart of a traditional program of mathematics, and it remains central in the learning of pupils in kindergarten and grades one through eight.

The various number systems should be developed with their structure shown as an expanding sequence from the counting numbers through the rational numbers. Properties of the order relation should be studied on the number line, and this study must be related to strands of geometry and measurement. This relationship can be strengthened by use of the number plane.

The four fundamental binary operations of addition, subtraction, multiplication, and division must be presented. Subtraction is treated as the operational inverse of addition; and division, as the operational inverse of multiplication. Other interpretations of subtraction and division are also important for mathematical algorithms and applications. Attention should be given to the properties of commutativity, associativity, and distributivity and to the identity elements zero and one.

The system of rational numbers should be sequentially developed to show that:

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[1]*Summary of the Report of the Advisory Committee on Mathematics to the California State Curriculum Commission: The Strands of Mathematics; Mathematics Programs for Teachers; A Study of New Programs and Supplementary Materials.* Sacramento: California State Department of Education, December, 1963.

1. The rational numbers are constructed from the system of integers.
2. A rational number is denoted by a fraction, and many fractions can denote the same rational number.
3. Operations of addition and multiplication are defined so that the properties which hold for the integers remain true for rational numbers.
4. Subtraction of two rational numbers is related to solving an equation of the form $a + x = b$; similarly, division is related to solving an equation of the form $ax = b$.
5. Properties of order, betweenness, and density have geometric interpretations.
6. Not all points on the number line can be denoted by rational numbers.

Preparatory experiences for study of the full set of real numbers should include associating a real number with each point on a line and using decimal notation as a way of denoting a real number.

The study of numeration systems may be illustrated with a brief treatment of bases other than ten. This treatment should show the role of the place value numeration system and the invariance of the properties of the numbers under a change of notation.

### Strand 2. Geometry

The kindergarten and grades one through eight mathematics program should provide a strong intuitive grasp of basic geometric concepts: point, line, angle, plane, three-dimensional space, congruence and similarity, and coordinate geometry. The development of geometry must progress at each grade level. This program does not envisage a sequence of definitions, theorems, and proofs, but rather a wide assortment of informal geometric experiences. There must be a continual tie-in with the strands of numbers, measurement, applications, problem solving, and logical thinking. There will be times when short chains of deductive reasoning will be both time-saving and instructive for pupils.

### Strand 3. Measurement

Measurement is a "doing" process and is best learned in this context. It provides a way to use the mathematical concepts and skills found in the other strands. The process can be used for teaching certain skills and concepts. Measurement is first presented as a way of comparing a common attribute (distance, area, number, or probability) of two things. Second, arbitrary units are selected for measuring. Finally, standard units are used as a means of communicating ideas so that there can be universal understanding.

All measurement processes should be presented from a point of view which recognizes that:

1. The measurement function assigns a number to an object to reflect a property of the object.
2. A unit is selected that has the same property as the object to be measured. A comparison is then made, and the number of these units present in the object is determined.
3. The choice of the unit is arbitrary.

The standard mensuration formulas should be developed from the three points mentioned above.

The English and the metric systems should be introduced in such a way that pupils will become mathematically bilingual in their use and will learn the advantages and disadvantages of each system. Pupils should develop a comparative sense about the two systems; for example, a centimeter is slightly less than but approximately equal to one-half inch.

### Strand 4. Applications of Mathematics

Concrete problems from the physical and social sciences should be presented hand in hand with pure mathematics. Constant exposure to concrete applications serves two purposes: It permits pupils to use the mathematics and the skills they have already developed to attack interesting problems, and it motivates them toward new and deeper mathematical insights. The application of mathematics should be presented as a process in which one does the following:

1. Constructs a mathematical model which reflects significant properties of the concrete problem
2. Formulates and analyzes a mathematical problem
3. Interprets the results of the mathematical analysis

Teachers should encourage pupils to propose areas for investigation that are of interest to them.

### Strand 5. Statistics and Probability

Today's society abounds with numerical data. Mathematics provides techniques to synthesize a conglomeration of numbers and to extract a few quantities that give an adequate picture of the whole. Often this process can be used effectively to predict behavior in similar and analogous situations. Awareness of the scope, power, and limitations of statistics and its theoretical sister, probability, constitutes vital knowledge for every citizen.

Some important topics to be introduced in the elementary grades are the following:

1. The rudiments of organizing data into standard graphs and charts
2. The meaning of the terms *average (mean), median,* and *mode*
3. The significance of variance and standard deviation as a crude measure of what portion of the total distribution is "close" to the average
4. The elementary notions of probability as the concept pertains to the "laws of chance" and to the physical situations from nature for which these are the correct mathematical models
5. The elementary notions of statistical inference

## Strand 6. Sets

An early introduction of the set concept is recommended to express the elementary notions of one-to-one correspondence and number. In this way a universal rationale can be given for the important properties of the integers. It is imperative to continue to use the language of sets in studying geometry, functions, solutions of equations and inequalities, number theory, and graphing. The set operations required are union, intersection, complementation, and Cartesian product. The properties of the set operations of union and intersection required are commutativity, associativity, and distributivity.

The language of sets should not be invoked for itself; rather, it should be used to make clear and precise the concepts and applications of mathematics. In particular, teachers should not require use of formal set terminology if a simple verbal phrase will suffice.

## Strand 7. Functions and Graphs

The function concept permeates all of mathematics as well as the real world. In intuitive terms, whenever one quantity is determined by others, a function is involved. For example, the area of a square is a function of the length of its side; the volume of a cone is a function of the radius of its base and its height; the speed of an automobile is a function of its acceleration, which in turn may be a function of the time the car is running. Functions are often pictured by graphs in one, two, or three dimensions. Functions and their graphs form powerful tools for studying mathematics and its applications.

The function concept and its generalization, the relation concept, should be developed, named, and used in the elementary school program. Graphs of functions and relations in one and two dimensions should be introduced in the early grades. In grades seven and eight, the teacher should employ standard functional notation, which denotes the value of the function, $f$, at the object, $x$, as $f(x)$; or prescribes the function, $f$, as $f: x \rightarrow f(x)$.

## Strand 8. Logical Thinking

A mathematical investigation has two sides: inductive and deductive. In kindergarten and grades one through eight, the program emphasizes and exploits induction by providing a variety of experiences with mathematical concepts. The strengths of mathematics, however, come from its deductive side. From certain assumptions, other properties and behaviors are inferred and deduced. This process of deductive reasoning, too, is a mathematical experience each pupil should share and appreciate. From a reasonable and pragmatic viewpoint, one may think of the use of logic and deductive reasoning as the application of "horse sense."

By the end of the eighth grade, school children should understand:

1. The logical connectives *and* and *or*
2. The meaning of sentences of the form "If $A$, then $B$" and the rule of inference which yields "$B$" if *both* "If $A$, then $B$" and "$A$" have been established
3. The role of negation
4. The role and scope of the quantifiers "For all . . . " and "There exists a . . . "
5. The notion of a "proof" as distinct from a "check"
6. The concept that equality is used in mathematics to denote two names for the same object or number

Strand eight indicates how logic can be introduced without using a heavy, formal system.

## Strand 9. Problem Solving

This strand recognizes that one of the major objectives of the mathematics program is the formulation and solution of problems. A sharp distinction should be made between the process of applying mathematics through the construction of mathematical models and the techniques of analyzing mathematical problems. A pupil learns much of his mathematics while he is engaged in the latter activity; he learns its significance while engaged in the former.

In problem solving, pupils should learn to use different strategies and tactics, such as:

- Constructing a diagram or using materials to illustrate the problem
- Guessing a reasonable answer
- Translating the conditions of the problem into mathematical sentences
- Performing the mathematical analysis and interpreting the answer

Any outline of strategies or sequential steps in problem solving should be viewed as optional. Under no circumstances should a pupil be forced to do a problem in a fixed way if he has discovered a method of his own which brings the problem to a correct solution.

Chapter Five

# THE KINDERGARTEN PROGRAM

The powerful, interrelated ideas of mathematics can be found in the experiences of young children. These early experiences provide intuitive background essential to the development of later mathematical content. Therefore, it is imperative that the instructional program in mathematics begin in kindergarten.

The envisioned kindergarten program is informal and exploratory in nature. For such a program to be effective, the teacher must have clearly in mind both its objectives and many possible ways of achieving them. Each kindergarten teacher should be provided with a guide that presents ways in which young children begin to develop mathematical ideas. This guide should assist teachers in planning suitable learning experiences. Because of different organizational plans in the schools and different needs in various regions, the guide for kindergarten should be made available also to districts that request it for teachers of the first grade.

The heart of the program should be activities involving children with physical objects that are usually found in the kindergarten environment; for example, blocks, objects in the playhouse or store, toys, pegboards, kindergarten beads, paints and brushes, clay, balls, and playground equipment. Other useful objects might include things brought from home for sharing or collected on neighborhood walks – rocks and shells, plants and animals, or story books. Activities may also be centered on materials especially designed to develop certain mathematical ideas. Children should have opportunities to compare objects; to classify and arrange them according to such attributes as shape, color, and size; to experiment with symmetry and balance; and to discover, continue, and create patterns. They should discover the relations of "more," "fewer," and "as many as" through the activity of matching *small* sets of objects. In these activities, the child develops understanding of such concepts as one-to-one correspondence and number. He should learn to name the cardinal number of a set, count at least through ten, and develop positional relationships such as *inside, outside, on; first, next, last; before, after, between; left, right;* and *above, below.*

Throughout their activities children should be encouraged to ask questions and talk about what they are doing, both with the teacher and among themselves. Children at this level are imitative and are interested in words. They are increasing their vocabulary rapidly. If the teacher introduces word and language patterns easily and naturally in situations that make their meanings clear, children will begin to assimilate the words and patterns into their own speech and thoughts. The key words here are *easily* and *naturally*. Children's own ways of conveying their ideas must be accepted at the time. But concurrently, they should have the opportunity to learn to express ideas with clarity and precision.

Chapter Six

# TECHNICAL PROFICIENCY

Facility and precision in arithmetic techniques are requisite for further mathematics and scientific work, as are algebraic and geometric techniques. Every pupil must learn to perform the four elementary arithmetic operations, to solve simple equations and inequalities, to make elementary geometric constructions, and to draw and interpret graphs.

The types of mathematical techniques required by every adult and the frequency with which he uses them are rapidly changing. Today's business and industrial procedures requiring only rudimentary arithmetical operations are frequently done or checked with a calculator or computer. It is less clear to what extent arithmetic operations and the skillful performance of the associated algorithms will permeate the daily life of tomorrow's citizens.

For these reasons those skills selected for precision and speed drills must tie into the whole mathematics curriculum. To implement the pedagogical philosophy which has been delineated, these drills must not be permitted to degenerate into stultifying and time-consuming routines, which stifle the receptive minds of elementary pupils. It is, of course, true that pupils must master facts and algorithms to a degree that will enable them to think through situations without being cluttered mentally by errors.

It is pedagogically important to accept correct algorithms invented by the pupil and to let him count on his fingers or use addition and multiplication tables, even though these procedures may be inefficient and time consuming. Certainly, it is hoped that pupils will eventually learn to replace these inefficient computational methods by more powerful ones. A pupil will develop better procedures if he first realizes that his initial methods are not wrong. Once he knows that what he is doing is acceptable, he may be directed or be self-motivated to learn faster and more efficient methods.

Of course, complete mastery of number facts and attainment of operational skills come only after continued *use* of numbers. Developing technical skills is an ongoing process. To acquire these skills, a child must learn more mathematics, have practice in applications of mathematics, or play mathematical games. It goes almost without saying that a game, in place of flash cards, makes any arithmetic problem more attractive. Any of the seven examples of ways to facilitate learning cited in Chapter Three will provide opportunities for practice in computational skills.

Attention to individual differences is never more important than in teaching addition facts. Some children learn best from tables, others by reasoning. Regrouping for combinations adding to ten is another method:

$$6 + 7 = 6 + (4 + 3) = (6 + 4) + 3 = 10 + 3 = 13$$

Some pupils will learn by regrouping for doubling:

$$6 + 7 = 6 + (6 + 1) = (6 + 6) + 1 = 12 + 1 = 13$$

In the decimal numeration system, only the addition and multiplication tables from 0 to 9 need ultimately be *learned*. As an aid in computation, commutativity can be exploited so that only half of the facts need be learned. Understanding of the principle of commutativity comes from analyzing pairs of addition facts such as $5 + 3 = 8$ and $3 + 5 = 8$. It should not be necessary to drill on both of these facts, however, since a general principle has been identified as an aid for computation.

Effective practice requires problem-by-problem reinforcement. Errors must be corrected as they occur. It is pointless to assign 20 exercises only to discover that one fundamental error has been repeated in the responses. Some exercises should explicitly call for practice with manipulative materials and simple geometric constructions. Cooperative work with a partner or in a small team provides another means of immediate feedback, and the techniques of programmed learning provide yet another. An acceptable textbook series will provide practice opportunities which can be adjusted to the needs of the learner. It must also provide immediate checks and reinforcements to help the pupil become responsible for his own learning.

## Chapter Seven

# EVALUATIVE PROCEDURES

Any educational program requires some evaluation procedure. We emphasize here that new programs require new evaluative instruments. This is particularly true in mathematics. Traditional training in mathematics emphasized technical skills, especially arithmetic skills. These skills are relatively easy to evaluate since an arithmetic skill is a very narrow concept; a column of figures is either added correctly or not. But the new programs in mathematics, including our present curriculum, have goals which are broader in scope and which seek understanding of more subtle mathematical concepts. Understanding of these concepts is not measured alone by how well a pupil performs a technique. For this reason, new tests and new test criteria must be developed to measure the success of a mathematics program designed to attain a wide spectrum of ends. The design of these instruments is not an easy problem, and the statistical evaluation of the reliability and validity of a particular instrument is complicated. We urge that as soon as possible, a special task force be created to deal with these problems.[1]

The goals of the mathematics program cannot be achieved overnight. The design of evaluative instruments is not an easy task. The statistical evaluations for the reliability and validity of a particular instrument are complicated. And it may well be that performance at the secondary level, after the pupils have been in the program for six to eight years, may be a better index of the elementary program than the results of a battery of tests.

[1]In 1968, under the Miller Mathematics Improvement Programs, the California State Department of Education contracted with Stanford University to undertake a Test Development Project for the construction of evaluative instruments. The project, completed in 1969, resulted in the development of a test instrument which is known as the "State of California Inventory of Mathematical Achievement" (SCIMA). The test, in three parts, comprises the model for state assessment of pupil mathematical achievement at grades three, six, and eight.

Chapter Eight

# STRAND 1. NUMBERS AND OPERATIONS

The major content of the study of numbers in the mathematics program for kindergarten through grade eight is a full development of the rational number system and an introduction to the real number system.

What is a number? In the discipline of mathematics, a number is an abstract object identified by its properties. However, in the instructional program in mathematics, numbers must be regarded as ideas derived from and applicable to situations and problems encountered by the learner. The program must provide activities that guide pupils from intuitive recognition of relationships through increasingly systematic ways of thinking. In developing ability to work with abstract symbols, most learners will need to "look through" symbols to the images of events in their experience which have given meaning to the symbols. These images may be of a physical activity in which a relation is recognized, a picture or a diagram that clarified a relation, or a pattern of symbols of a lower order of abstraction.

## Introduction to the Rational Number System

The system of rational numbers should be presented as a system of expanding ideas. In such a presentation the learner first encounters the set of whole numbers $\{0, 1, 2, \ldots\}$. These numbers are ordered and named by a positional system of notation, and the basic operations with their properties are introduced. Next, the system of positive rational numbers and zero (sometimes called the fractional numbers) can be studied. The system of whole numbers is then extended to include the integers by considering the set of negative whole numbers $\{-1, -2, -3, \ldots\}$.

To indicate the notation for positive and negative signed numbers, many textbooks employ the device of using a raised symbol ($^-5$, $^+3$, $^-1$) and the words "positive" or "negative," as differentiated from the lowered symbols and the words "minus" or "plus" to designate the operations on the set of integers ($^+5 - {}^-1 = {}^+6$). The use of the

raised symbol device is usually continued until discussion of the concept of opposites or additive inverses shows its use to be superfluous. At this point the authors explain that the device has served its purpose and drop its further use.

Finally, the study of such numbers as $-\frac{1}{4}$, $-\frac{7}{8}$, $-1\frac{1}{2}$ completes the system of rational numbers. The operations and properties, first developed for the whole numbers, are extended to each system as the study proceeds.

## Introduction to the Real Number System

Throughout this development, the number line should be employed to provide a geometric representation of number, leading to the idea that each rational number corresponds to a point on the number line. By grade eight pupils will know that, for example, $\sqrt{2}$ is not a rational number; yet, by a geometric construction there is a point on the number line which, in a natural way, should correspond to $\sqrt{2}$. The real number system may then be developed as a completion of the rational numbers on the number line, so that every point of the number line is in one-to-one correspondence with a number.

While the sequential development should in general be carried out as described above, there must be considerable overlapping. Preparatory activities for more formal study of each extension of the number system begin early and should continue through a long period of time. For example, a learner may have experiences with fractional parts of objects in kindergarten, several years before he works with rational numbers as such. At the upper level, preparation for study of the real numbers includes the decimal expansion of numbers. The decimal expansion of a rational number is periodic; e.g., 0.25000..., 0.3333..., 0.235235..., or 0.7143143..., and conversely, a periodic decimal names a rational number.

Comparatively few difficulties are met in developing the concepts and computational algorithms for the system of whole numbers. The development of the full system of rational numbers requires two stages. One stage is the introduction of the positive rationals (How can you divide 1 by 2? Why doesn't $\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$?). The other stage is the introduction of negative numbers (How can you subtract 3 from 1?). Introduction of negative numbers logically may either precede or follow the introduction of the positive rational numbers. Both stages are analogous, and yet each contains its own subtleties. A course is recommended that deals independently with these two stages, particularly in the early grades.

In many instances a few of the negative numbers may be introduced naturally $\{-10, -9, -8, \ldots, -1\}$ as a countdown procedure, or they may be used to label points on a temperature scale or on the other "half" of the number line. Children can handle easy arithmetic problems involving addition and subtraction of small positive and negative integers. (If the spacecraft was on automatic control from H − 2 hours until H + 1 hours, how long was it on automatic control?) At the same time, children will often need to use the familiar positive rational numbers $\{1/2, 1/3, 2/3, 1/4, 3/4\}$ to express parts of a whole. These should be introduced informally as the need arises, and experiences should be tied closely with physical examples. Here again it is possible to handle easy addition and subtraction problems without the complicated fanfare of the most general of rules. The more difficult arithmetic operations of multiplication and division are introduced later in the curriculum. The general rules for these operations as well as those for addition and subtraction in the full set of rational numbers are then presented. These topics require a fuller understanding of the number system.

Figure 1 is a diagram of the relationship of the sets of whole numbers, the integers, the rational numbers, and the real numbers.

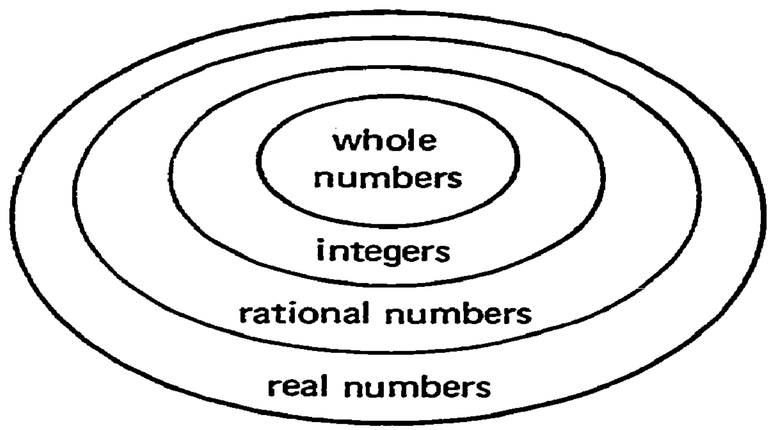



**Fig. 1. Relationships of sets of numbers**

## Unifying Ideas in Numbers and Operations

Activities that guide the learner to recognize and generalize the central unifying ideas in these number systems aid him to progress toward systematic thinking. Generalizations feed *back,* giving support to existing ideas, as well as *forward,* opening the way for new ideas.

**Early Stages of Development**

Early informal experiences are difficult to categorize. In kindergarten and the early primary grades, the child builds with blocks and other materials, handles objects of different shapes and notes characteristic features, sorts and classifies objects, fits objects inside others, arranges objects in order of size, experiments with a balance, recognizes positional relationships, recognizes symmetry, searches for patterns, and reasons out ideas. Such activities are highly important. Through them children develop ideas of relationships that contribute to Strand 1, Numbers and Operations, as well as to other strands of mathematics.

Activities in the classroom or on the playground — projects utilizing materials specifically designed to promote the development of certain ideas and the understanding of certain games and puzzles — provide experiences of the type described in the preceding paragraph. While engaged in these activities, children should be encouraged to talk about what they are doing, both with the teacher and with each other. These experiences provide a readiness for reading and writing mathematics.

The teacher should introduce new words or language patterns in close association with activities that make meanings apparent. Assimilation of vocabulary and language patterns into the children's own speech and thought is expected to occur gradually over a considerable period of time. Children first give evidence of understanding the teacher's usage; then they begin to use the words and patterns spontaneously in their speech. Often the first uses are imprecise. The teacher accepts and perhaps even repeats a child's speech. Then, sensitive to his intended meaning, the teacher guides him to precision through a skillful question, a statement of agreement that rephrases the idea, or a statement of amplification. Development of a child's ideas and language can be blocked by rejection of his first fumbling attempts to cope with what may be, to him, enormous complexities.

The principal unifying ideas in Strand 1 are:

- Order, counting, and betweenness
- Operations and their properties
- Identity elements
- Numeration systems
- Mathematical sentences

**Order, Counting, and Betweenness — First Unifying Idea**

For any pair of numbers, $a, b$, exactly one of three possible relations exists: $a < b$, $a = b$, or $a > b$.

Activities in which children compare the number of sets of objects without resorting to counting lead directly to the concept of the counting numbers. Learners can compare the numbers of two sets of objects by pairing the members one-to-one and discovering the possible relations. For example, if chairs and children are paired to see whether there is a chair for each child, one of the relations *fewer, as many as,* or *more* will be found to exist.

When a child can compare sets of objects and determine whether they are equivalent or nonequivalent, he is ready to learn relations between numbers. The learning of relations can be promoted by arranging sets of objects in sequential order. By this time many children have learned to associate the spoken names of numbers with the cardinal numbers of small sets of objects and zero with the number of the empty set. Experiences in arranging sets of objects in rank order help pupils to see the order of numbers and to learn the names of the natural numbers in counting sequence.

Counting requires matching the members of a set of objects with the members of an ordered set of names of the counting numbers. Some children require help in keeping the ordered set of number names in one-to-one correspondence with a set of objects they are counting, and they must move, touch, or point to the objects and say the words aloud if they are to count correctly. Children should have experience with many activities which involve counting in a variety of situations if they are to avoid confusions arising from partial learning.

The number line, which relates geometry and number, exhibits many important facts about numbers. At first, a number line can be marked or laid out on the floor with masking tape. A starting point (origin) is identified, and the children take steps along the line, counting as they step. Soon, numerals are associated with the points, with 0 (zero) as the origin. Numbers of steps taken along the line by different children are compared and relative positions discussed; e.g., *before, between,* and *after.* Later, movement along the line may be represented by marking a number line drawn on the chalkboard or paper.

When the number line is used, $a < b$ means that the point corresponding to $a$ is located to the left of (or below) the point corresponding to $b$. A number (point) $a$ is said to lie *between* two other numbers (points), $b$, $c$, if $b < a$ and $a < c$. For any three distinct numbers, $x$, $y$, $z$, exactly one of the three lies between the other two.

When pupils study the fractional representation for rational numbers, the number line helps them see that numbers whose fractions have the same denominator are ordered by the numerators of the fractions (Figure 2). They find that they can make a decision concerning the order of any two rational numbers by renaming them so that the fractions have the same denominators.

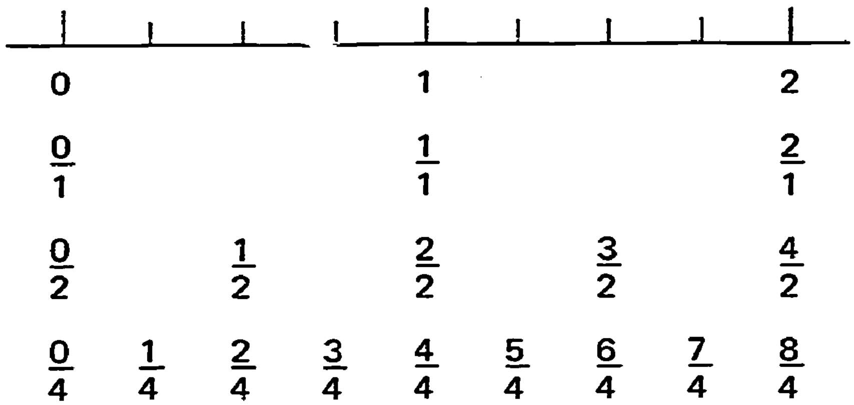



**Fig. 2. Representation of rational numbers with same denominator**

Children need to understand the concept of absolute value when both positive and negative numbers are used. The absolute value of the number $a$, denoted $|a|$, can be thought of as the distance from the point corresponding to the number $a$ to the origin. It may be defined by:

$$|a| = \begin{array}{l} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{array}$$

The number line should be used to point up the distinction between the set of integers and the set of rational numbers. The rational numbers are *dense;* between any two distinct rational numbers, there is always another rational number. Thus, the number $\frac{a+b}{2}$ is between $a$ and $b$.

The number line shows that any negative number is to the left of (or below) zero and is hence less than zero, and $-4$ is less than $-1$ (Figure 3).

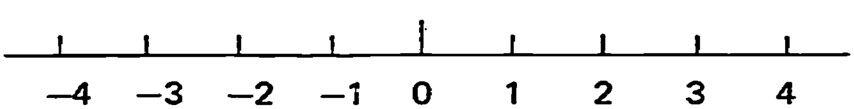



**Fig. 3. Number line showing relationships of numbers**

The concept of the number *plane,* which is necessary to the understanding of functions and graphing, should be introduced relatively early. Here, points in the plane are identified by ordered pairs of numbers. The number plane lends itself to interesting games and the study of patterns and should be related to map drawing. For further discussion of the coordinate plane, see Strand 2, Geometry; Strand 4, Applications of Mathematics; and Strand 7, Functions and Graphs.

**Operations and Their Properties — Second Unifying Idea**

Addition is an operation which assigns to an ordered pair of numbers, called *addends,* another number called their *sum.* For example, to the ordered pair of numbers (3, 4) is assigned 7, their sum. The fact that the ordered pair (4, 3) is *also* assigned the sum 7 is an example of the commutative property of addition. On the set of integers, subtraction is also an operation. Subtraction is the inverse operation of addition. If the sum and one addend are known, the other addend may be found by subtraction. Multiplication is an operation which assigns to an ordered pair of numbers, called *factors,* another number called their *product.* For example, to the ordered pair of numbers (3, 4) is assigned 12, their product. Division is the inverse operation of multiplication. If the product and one factor are known, the other factor may be found by division. On the set of nonzero rational numbers, division is also an operation.

Concepts of the operations and their properties have their roots in activities associated with the joining and separating of nonintersecting sets of objects. For example, suppose Terri and Charles are painting at easels and Sara wants to paint also. The teacher may ask, "How many children are at the easels now? How many will there be when Sara joins them? Let's count to see if we were correct." When the operation of addition can be associated with the numbers of the sets, the number sentence $2 + 1 = 3$ is written. Children now see once again that "2 + 1" and "3" are names for the same number.

In experiences leading to subtraction, the learner separates a set into two disjoint subsets and names the numbers. He finds that if he knows the numbers of the original set and one of the subsets, he can find the number of the other subset by subtraction. The learner is not expected either to follow or to give a wordy description of the relationship, which should be made clear through activities with such materials as sets of objects, scaled blocks or paper strips, and the number line. Pupils should experience various interpretations of subtraction including "take away," "comparison," "how many more," and the inverse relation of subtraction and addition.

Either addition or subtraction involves three distinct sets – two disjoint sets $A$ and $B$ and their set union, $C = A \cup B$. Any of the following describes the numbers in the sets:

$$n(A) + n(B) = n(C) \qquad n(C) - n(A) = n(B)$$

$$n(B) + n(A) = n(C) \qquad n(C) - n(B) = n(A)$$

Note also that the question, "What number must be added to 3 to equal 7?" has the symbolic form of an equation: $3 + \square = 7$. By definition the symbol $7 - 3$ names a number which, when added to 3, is 7.

In the learner's early experiences with multiplication, he counts the members of a set formed by joining disjoint sets that are equivalent. A rectangular array of objects is helpful. An appeal to the idea of repeated addition of equal addends may aid the learner in interpreting the operation. The number line is useful in illustrating this interpretation of multiplication.

The Cartesian product interpretation of multiplication extends the concept and makes it applicable to types of problem situations for which repeated addition is not an adequate model. For example, suppose one wants to find the number of different combinations one can make with three shirts and four pairs of slacks or the number of combinations that occur under the operation of addition with the numbers zero through nine.

In the Cartesian product interpretation of multiplication, we pair the elements of one set with the elements of another and then enumerate the number of ordered pairs that results. Figures 4-a and 4-b illustrate the possible pairings of a set of three objects with a set of four in a rectangular array. Each of the pairings can then be represented more simply. We can determine the number of elements in the array by a variety of methods – counting, repeated addition, or multiplication.

| | * | ● | ◆ | # |
|---|---|---|---|---|
| □ | □ * | □ ● | □ ◆ | □ # |
| ○ | ○ * | ○ ● | ○ ◆ | ○ # |
| △ | △ * | △ ● | △ ◆ | △ # |

**Fig. 4-a. Pairings of sets of objects in rectangular array**

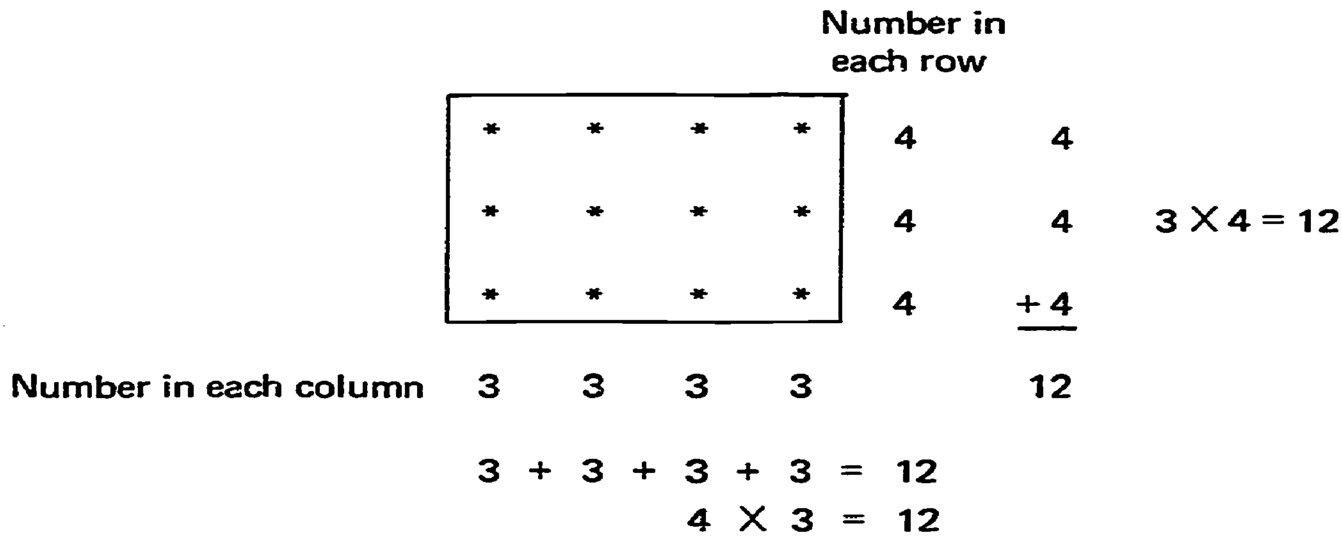



**Fig. 4-b. Pairings of sets of objects illustrated mathematically**

Division is the operation for finding one factor if the product and the other factor are known. In the early stages of learning, the child begins to comprehend the relation when he separates a set into equivalent subsets and determines the number of them. Appeal to the idea of successive subtraction may aid children in interpreting division. The number line is useful in illustrating this interpretation of division.

Note that division is the operation which solves, for example, the equation $3 \times \square = 12$. The symbol $12 \div 3$ names, by definition, a number which when multiplied by 3 is 12.

Another important aspect of division is "division with a remainder." Within a set of numbers (for example, the set of positive integers), it is not always possible to divide one number by another. (Indeed, it is this deficiency of the integers that motivates the construction of the rational numbers.) Thus, for example, 14 cannot be divided by 3. There is no integer whose product with 3 is 14; the equation $3 \times \square = 14$ has no solution in the set of integers. We can, however, write $14 = 3 \times 4 + 2$. An array such as the one in Figure 5 illustrates this relationship.

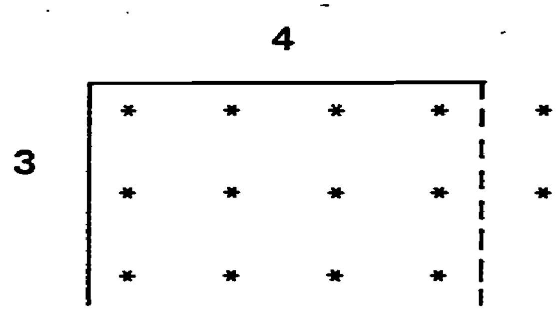
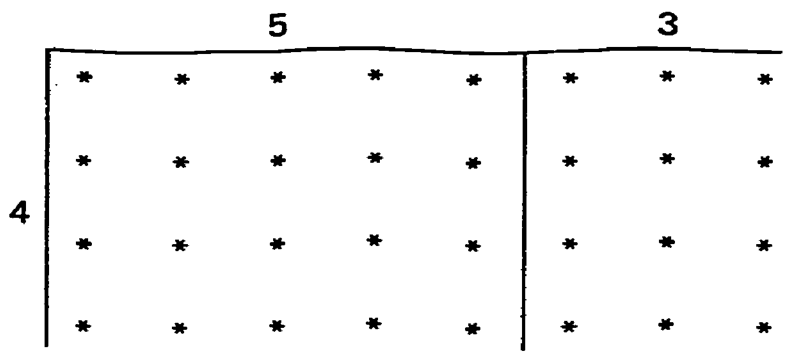



**Fig. 5. Illustration of equation $14 = 3 \times 4 + 2$**

Within the set of rational numbers, division, with the exception of division by zero, is always possible. The solution of $3 \times \square = 14$ is $^{14}/_3$ or $4\frac{2}{3}$.

Both addition and multiplication are commutative — the order in which two numbers are added or multiplied does not affect the result. Both addition and multiplication are also associative — the way in which the numbers are grouped for addition or for multiplication does not affect the sum or the product. Subtraction and division are neither commutative nor associative. Multiplication is distributive over addition. Figure 6 illustrates that the product of a number and the sum of two numbers is the sum of the products of the first number and each of the addends.




$4 \times (5 + 3) = 4 \times 8 = 32$

$(4 \times 5) + (4 \times 3) = 20 + 12 = 32$

**Fig. 6. Distributivity of multiplication illustrated**

Distributivity of multiplication over addition and subtraction is an extremely useful and important property, since it is basic to all multiplication algorithms. It should be emphasized in the elementary school program.

Division distributes over addition and subtraction in the form, for example,

$$(8 + 12) \div 4 = (8 \div 4) + (12 \div 4)$$

since

$$20 \div 4 = 2 + 3$$
$$5 = 5.$$

On the other hand,

$$20 \div (2 + 2) \neq (20 \div 2) + (20 \div 2)$$

since

$$5 \neq 10 + 10.$$

The teacher should know and understand mathematical terms; e.g., *commutative, associative, distributive, identity element, opposite, inverse,* and *absolute value.* When the learner gives evidence of familiarity with an idea, the teacher can begin to use a new term easily and informally and give him opportunity to build it into his own speech. Later, attention should be given to it in the reading vocabulary. Mathematics books can provide an excellent source of material for teaching the reading of mathematical content, and might, on occasion, be included in the reading program. Mathematical experiences are also a rich source of ideas for children's writing.

By the end of the elementary school mathematics program, most pupils should be able to understand and read the standard terminology and language patterns for the mathematical concepts they have learned. Learning to understand and read specialized language takes place through many opportunities for use rather than through memorization and parrot-like repetition.

It is doubtful that either sound knowledge of underlying structure and principles or facility in computation can stand alone. One enhances the other, not only in its usefulness but also in its attainment. Skills gained in the absence of understanding are soon forgotten and not readily transferable to different situations, and concepts attained without the support of skills are frequently not operational. The learner should have experiences that will enable him to develop a degree of facility in computation that gives him confidence in his ability to deal with numbers and their applications. Since much of the arithmetic of daily life involves estimating and computing without the use of pencil and paper, these are important aspects of computation. (For more discussion of estimation, see Strand 3, Measurement.)

Sets of exercises should be arranged so that patterns may be discovered to assist in the development of generalizations. When children have stated a general idea, such questions as "Does this idea always work?" "Does the idea work with larger numbers?" and "Can you find an example in which the idea does not work?" stimulate practice that reinforces knowledge of basic facts and develops confidence in working with greater numbers.

**Identity Elements —**
**Third Unifying Idea**

The numbers zero and one have special properties with respect to addition and multiplication. The addition of zero to a number results in the same (the identical) number. Thus, $a + 0 = 0 + a = a$ for all numbers $a$. Hence, zero is called the identity element with respect to addition.

Similarly, one (1) is the identity element with respect to multiplication. This concept plays an important role in computation with fractions. Through experiences with objects, the number line, and geometric figures separated into congruent regions, learners may be guided to see that 1 may be expressed as $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$, . . . . Then they can discover why $\frac{2}{8}$ is another name for $\frac{1}{4}$, $\frac{6}{9}$ another name for $\frac{2}{3}$, and so on. For example:

$$\frac{1 \times 1}{1 \times 4} = \frac{2 \times 1}{2 \times 4} \quad \text{and} \quad \frac{1 \times 2}{1 \times 3} = \frac{3 \times 2}{3 \times 3}$$

The reason is, of course, that multiplying both the numerator and the denominator by the same counting number amounts to multiplying the fractional number by 1. Thus:

$$\frac{a}{b} = \frac{k \times a}{k \times b} = \frac{k}{k} \times \frac{a}{b} = 1 \times \frac{a}{b} = \frac{a}{b}$$

Figure 7 may be useful for comparing the properties of the rational numbers under addition and multiplication.

| Addition (+) | Properties | Multiplication (×) |
|---|---|---|
| $a + b = b + a$ | Commutativity | $a \times b = b \times a$ |
| $a + (b + c) = (a + b) + c$ | Associativity | $a \times (b \times c) = (a \times b) \times c$ |
| 0: $0 + a = a + 0 = a$ | Identity element | 1: $1 \times a = a \times 1 = a$ |
| $a + (-a) = 0$ | Inverses | If $a \neq 0$, then $a \times \left(\frac{1}{a}\right) = 1$ |
| If $a + b = a + c$, then $b = c$ | Cancellation | If $a \neq 0$, and $a \times b = a \times c$, then $b = c$ |

**Fig. 7. Comparison of properties of rational numbers**

Under the distributive law:

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$(b + c) \times a = (b \times a) + (c \times a).$$

Another property of the real numbers is that of order. For all $a$, $b$, exactly one of the alternatives, $a = b$, $a < b$, or $b < a$, is true.

If $a < b$, then $(a + c) < (b + c)$.

If $a < b$ and $0 < c$, then $(a \times c) < (b \times c)$.

**Numeration Systems – Fourth Unifying Idea**

Symbols are necessary both for purposes of communication and for efficiency and economy of thought. A numeration system is a set of symbols for naming and recording numbers. A *number* is a mathematical concept. A *numeral* is a symbol for a number, and it names the number. For example, the number eight, the concept which is associated with any set having the property of containing eight members, has infinitely many names. Among them are eight, 8, VIII, $5 + 3$, $32 \div 4$, $^{24}/_{3}$, $2^3$, and $10_{\text{eight}}$.

Certain rules or principles govern the use of the symbols in any system of numeration. The two major principles of a place-value system of numeration are base and place. The numeration system which we commonly use is the *decimal place-value* system. In this system, with its ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9), place value is the fundamental principle for organizing the naming of numbers beyond nine. The ten digits can be placed in many positions according to ordered powers of base ten to express any number in the system of whole numbers. The notational scheme can be extended to express every real number.

A learner's concept of the decimal place-value numeration system has its beginning in preschool or kindergarten experiences when the pupil first names numbers. The concept is refined and extended throughout the mathematics curriculum. Among the ideas developed as the pupil gains understanding of decimal place-value notation are:

- The positional relationships, left and right
- Concepts of the whole numbers zero through nine and their order
- Symbolizing of value by the position of a digit in a numeral
- Naming of numbers in different ways; for example, naming 532 as

| | |
|---|---|
| 5 hundreds, 3 tens, 2 ones | 5 hundreds, 2 tens, 12 ones |
| 53 tens, 2 ones | 4 hundreds, 13 tens, 2 ones |
| 532 ones | 4 hundreds, 12 tens, 12 ones |

- Operations of addition and multiplication; for example:

$$532 = (5 \times 100) + (3 \times 10) + (2 \times 1),$$

or

$$0.532 = (5 \times \tfrac{1}{10}) + (3 \times \tfrac{1}{100}) + (2 \times \tfrac{1}{1000})$$

- Multiplying and dividing by ten and powers of ten, using exponential notation; for example:

$$532 = (5 \times 10^2) + (3 \times 10^1) + (2 \times 10^0)$$

$$0.532 = (5 \times 10^{-1}) + (3 \times 10^{-2}) + (2 \times 10^{-3})$$

Consideration of systems of numeration that do not utilize the principle of place value can lead to appreciation of the advantages of a place-value system and of the historical development of numeration systems. Study of place-value systems with bases other than ten can contribute to an understanding of the principles of base and place in the familiar decimal system. Recognition that properties of numbers are not dependent on a specific numeration system should be a goal of such studies, but numeration systems other than the decimal system should not become a major part of the mathematics program.

**Mathematical Sentences — Fifth Unifying Idea**

When a learner has had many experiences with joining and separating sets of objects and can recognize and name the numbers of the sets, he can begin to use number sentences to record and communicate his ideas. He may, for example, find that the union of two disjoint sets, one of four blocks and the other of two blocks, is a set of six blocks. The sentence associated with the numbers of the sets and the operation of addition is "Four plus two equals six." Expressed with numerals and signs, the sentence is "4 + 2 = 6"; this form is shorter and easier to write. Many experiences with objects, pictures, and the number line lead to learning the basic pattern ___ + ___ = ___. Initially pupils count or simply recognize the numbers of small sets to find the numbers they will name in a sentence. Through such experiences pupils develop understanding of addition and begin to learn number facts, which enable them to perform the operation more efficiently. Basic sentence patterns for the other operations are established in similar fashion; that is, from experiences in "real" situations, the child proceeds to spoken sentences concerning the numbers and then to written symbolism.

A number sentence is simply a way of making a statement about numbers, and when it contains only numbers and relational symbols, it is either true or false. In the preceding example, the statement asserted the equality (logical identity) of the number named by $4 + 2$ and 6. The statement is true. The statement $4 + 2 = 5$ is false, since $4 + 2$ and 5 do not name the same number. The statements of inequality $4 + 2 \neq 5$ and $5 < 4 + 2$ are true statements.

In number sentences various symbols such as ___, □, △, ?, *x*, *y*, *a*, *c*, and *k* are used to represent numbers from specified sets. Equations or inequalities that contain such symbols are called open sentences. Their truth or falsity depends upon the numbers used to replace the symbols.

Experiences with sentences having variables in different positions extend the child's understanding of operations and properties and provide an intuitive background for algebra. Geometrically shaped frames are useful in early experiences, since children can write numerals within them and are not hampered by the necessity of rewriting. However, as soon as children can easily distinguish between numerals and letters, they should have experiences with the conventional use of letters for variables. Solution of number sentences can form the basis for interesting puzzles and games which generate both fun and learning.

Number sentences are particularly useful in guiding learners to generalize relationships and properties of numbers. They provide a form for recording ideas in which patterns may be discovered and for making general statements concerning them. For example, pupils may be challenged to find the "secret" for solving such sentences as $5 + \triangle = 5$ to gain insight into the role of zero as the identity element in addition. Such statements as $(12 + \square) - \square = 12$ give insight into "doing" and "undoing" — that is, into inverse relations. When the secret has been discovered, a generalized statement may be formulated: $(\triangle + \square) - \square = \triangle$, or $(a + b) - b = a$ for all numbers *a* and *b*.

Sentences should increase in complexity as children develop ability and as they are faced with the need to analyze more complicated problems. The use of parentheses requires special attention. Probably parentheses will first appear in the associative law: $2 + (3 + 5) = (2 + 3) + 5$. Parentheses must be introduced in expressions involving several operations; e.g., $(8 - 4) + 2 = 4 + 2 = 6$, while $8 - (4 + 2) = 8 - 6 = 2$. Parentheses serve as a sign to "do this first."

Certain conventions are followed to avoid a cumbersome use of parentheses. For example, additions and subtractions are performed in the order in which they occur, as are multiplications and divisions. Multiplications and divisions are performed before additions and subtractions unless otherwise specified by symbols of enclosure. For example, $8 - 4 + 2$ always means $(8 - 4) + 2$. To express $8 - (4 + 2)$, the indicated parentheses must be used.

On the other hand, $8 + 5 - 2$ means $(8 + 5) - 2 = 13 - 2 = 11$. Here the alternative $8 + (5 - 2)$ is also 11.

Another example is $6 + 4 \times 3$; thus, $6 + (4 \times 3) = 6 + 12 = 18$. To express $(6 + 4) \times 3$, the indicated parentheses must be used, as in $(6 + 4) \times 3 = 10 \times 3 = 30$.

Consistent use of parentheses to indicate order of the operations establishes background for later formulation of rules. Usually, brackets and braces to indicate that the terms enclosed are to be treated as single terms are not used until the later grades.

Relationships between mathematical functions appear in sentence form. The mathematical sentences used to characterize the function that expresses the distance between two points are a particularly important example. The following type of mathematical sentence in geometry is suitable for advanced pupils in grades seven and eight:

> Let two points be $a$ and $b$. Let the distance between the two points be denoted by $d(a, b)$. For example, if $a$ and $b$ are opposite vertices of a square of side 1, then $d(a, b) = \sqrt{2}$. Three properties characterize the function $d(a, b)$ as expressed by the following three mathematical sentences:
>
> (1) $d(a, b) = 0$ if and only if $a = b$.
>
> (2) For every two points $a$ and $b$, $d(a, b) = d(b, a)$.
>
> (3) For every three points $a$, $b$, and $c$, $d(a, b) \leq d(a, c) + d(c, b)$.
>
> Property (3) states that for any three points $a$, $b$, and $c$ (see Illustration 1-1), the distance between $a$ and $b$ is less than or equal to the sum of the distances between $a$ and $c$ and $b$ and $c$.
>
> When $a$, $b$, and $c$ are points on a number line (Illustration 1-2),
>
> $d(a, b) = |a - b|$.
>
> Then (3) becomes the important relation
>
> $|a - b| \leq |a - c| + |c - b|$.

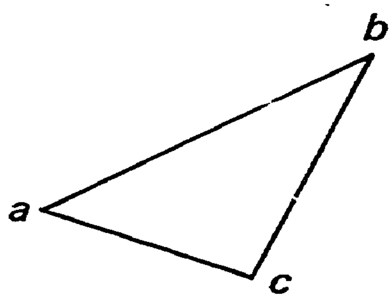



Illustration 1-1

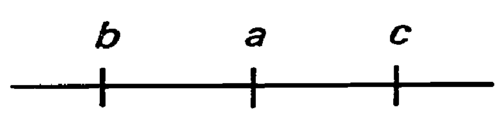



Illustration 1-2

In the upper grades, it will be important to analyze and solve more complicated mathematical sentences in a single variable. This may be done from the point of view of solution sets, or as a series of equivalent mathematical sentences. Here is an example:

For what numbers $x$ is $2x + 3 < (2 + 5x) - 8$?

(1) $2x + 3 < (2 + 5x) - 8.$

Now, consider the right hand side of (1).
For all $x$, $(2 + 5x) - 8 = 5x + 2 - 8 = 5x - 6$. Hence for every $x$, (1) is equivalent to

(2) $2x + 3 < 5x - 6.$

That is, for every $x$, (1) is true if and only if (2) is true. We can also say that (1) and (2) have the same solution set.

(3) $2x + 3 + 6 < 5x - 6 + 6.$

Mathematical sentences (2) and (3) are equivalent, and (3) is equivalent to

(4) $2x + 9 < 5x.$

Next, (4) is equivalent to

(5) $-2x + (2x + 9) < -2x + 5x.$

Now, for all $x$, $-2x + (2x + 9) = (-2x + 2x) + 9 = 9$. For all $x$, $-2x + 5x = (-2 + 5)x = 3x$. Thus, (5) is equivalent to

(6) $9 < 3x.$

Finally, (6) is equivalent to

(7) $3 < x.$

Thus, (1) is equivalent to (7), and the answer to the original question is: "All numbers greater than 3" or $\{x: x > 3\}$; equivalently, the solution set of (1) is $\{x: x > 3\}$. In practice many of these steps can be skipped.

The concept of the mathematical sentence provides an important frame of reference from which to view the development of the integers from the whole numbers and the development of the rational numbers from the integers. Here is a sketch of this development, which should be of interest to teachers and many pupils in grades seven and eight. Basically, problems (equations) that involve division and subtraction as well as multiplication and addition need to be solved. Often these problems are of such a complexity that neither the system of whole numbers nor the integers is adequate.

There are, of course, solutions for equations like $y + 2 = 6$ and $2x = 6$ within the system of whole numbers. There is a need, however, to solve problems such as $y + 2 = 1$ and $2x = 1$.

The system of integers provides a solution for all equations of the form

$$y + c = d,$$

where $c$ and $d$ are *whole numbers.* The integer which is the solution is denoted by $d - c$. The defining property of this number is that

$$(d - c) + c = d.$$

The system of rational numbers provides a solution for all equations of the form $ax = b$ $(a \neq 0)$, where $a$ and $b$ are integers. The rational number which is the solution is denoted by $\frac{b}{a}$. The defining property of this number is that $\frac{b}{a} \times a = b$. The symbol $\frac{b}{a}$ is called a fraction; it is a numeral, a name for a rational number.

## Problems with Rational Numbers

The problems that now arise in the development of the rational numbers are these:

- When does $\frac{b}{a} = \frac{u}{v}$?
- How are rational numbers added and subtracted?
- How are rational numbers multiplied and divided?
- How can a greater-than, less-than relation be introduced within the set of rational numbers?

Answers to these questions can all be motivated from the point of view that a rational number is a solution to an equation. For example, consider the question, How should rational numbers be added?

Given $\frac{a}{b}$ and $\frac{u}{v}$, what is $\frac{a}{b} + \frac{u}{v}$?

We know that $a/b$ is a number $x$ for which $bx = a$.

We know that $u/v$ is a number $y$ for which $vy = u$.

We are asking for $x + y$. We need to find an equation for which $x + y$ is a solution.

Now, since $v \neq 0$, $bx = a$ is equivalent to $v(bx) = va$.

Since $b \neq 0$, $vy = u$ is equivalent to $b(vy) = bu$.
Hence, adding these two equations yields

$$vbx + bvy = va + bu,$$

or,

$$bv(x + y) = va + bu.$$

Thus, the equation for which $x + y$ is a solution is:

$$x + y = \frac{va + bu}{bv}, \text{ or } \frac{a}{b} + \frac{u}{v} = \frac{(va + bu)}{bv}.$$

It is an important theorem that having all rational numbers, both positive and negative, we can now solve *all* equations of the form $rx = s (r \neq 0)$ and $u + y = v$, where $r$, $s$, $u$, and $v$ are any *rational* numbers.

The development of the rational numbers must demonstrate these structural properties. Equally important, rational numbers and the four fundamental operations must be viewed in a physical context. Thus, to answer the question, "When we measure $2\frac{1}{2}$ yards in feet, how many feet do we get?" we divide the number of yards given by the number of yards in a foot. That is:

$$\begin{aligned} 2\tfrac{1}{2} \div \tfrac{1}{3} &= (2 + \tfrac{1}{2}) \div \tfrac{1}{3} \\ &= (2 \div \tfrac{1}{3}) + (\tfrac{1}{2} \div \tfrac{1}{3}) \\ &= 6 + \tfrac{3}{2} \\ &= 6 + 1 + \tfrac{1}{2} \\ &= 7\tfrac{1}{2} \end{aligned}$$

or

$$\begin{aligned} 2\tfrac{1}{2} \div \tfrac{1}{3} &= \tfrac{5}{2} \div \tfrac{1}{3} \\ &= \tfrac{15}{2}. \end{aligned}$$

The search for solutions to equations will be continued throughout the entire mathematical program. Equations like $x^2 = 2$ will bring irrational numbers. Equations like $x^2 = -1$ will bring complex numbers. Functional equations like $f(xy) = f(x) + f(y)$ will bring logarithms. Indeed, placing a spotlight on equations provides a guiding light to all of mathematics.

Chapter Nine

# STRAND 2. GEOMETRY

The world around us is highly dependent on geometric shape and form. The interrelations of these geometric forms affect and structure our thoughts and actions. The universality of this geometric presence means that geometry must be made an important part of the mathematics program for kindergarten through grade eight. In his early experiences, the child should develop an intuitive grasp of geometric concepts which will contribute to the success of his later studies in geometry. A singularly important role is played by coordinate geometry as it fuses arithmetic, algebra, and geometry into a primary tool of science and mathematics.

Because geometry is universal, every child can find many physical models of geometric concepts. Because these models are readily available and because they have a demonstrable utility, they will help make geometry easy and interesting for the young learner. A study of geometry will help each child to appreciate the role of patterns, shapes, and forms in everything around him, and the language of geometry enables him to communicate what he perceives to others. His ability to perceive a given geometric situation will develop his powers of analysis and classification and will enable him to apply known information in new situations.

The geometry program in the elementary school is a program of informal geometry. Here the word "informal" refers not to a casual manner of presentation or emphasis but to the absence of a formally developed subject using an axiomatic approach and stressing formal proofs. The goal of the geometry program from kindergarten to grade eight should be to provide the foundation for later formal study. When appropriate, the teacher may present short deductive arguments.

The use of physical materials which have been carefully selected will introduce experiences with specific geometric concepts. A perspective drawing of a rectangular prism in a geometry textbook will have more meaning to the learner if he has first held a block or a box in his hand, described its various parts, analyzed it to determine what parts he would have to make to construct a similar box, and then actually made the model. Reconstruction activities lend themselves to the utilization of problem solving skills.

The following are examples of activities that require this sort of analysis:

1. Have the pupils draw a polygon with exactly seven sides. Then, on another part of the paper, have them designate seven points, $A$, $B$, $C$, $D$, $E$, $F$, and $G$, and connect "adjacent" points $A$ and $B$, $B$ and $C$, . . . , $F$ and $G$, $G$ and $A$, with straight line segments. Note that this problem leads into a discussion of convex and nonconvex polygons.
2. Ask the pupils to determine in how many different ways six congruent squares may be connected in the plane so that they can be folded up into a cube. For example, can the six squares of Illustration 2-1 be folded into a cube?

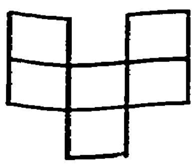
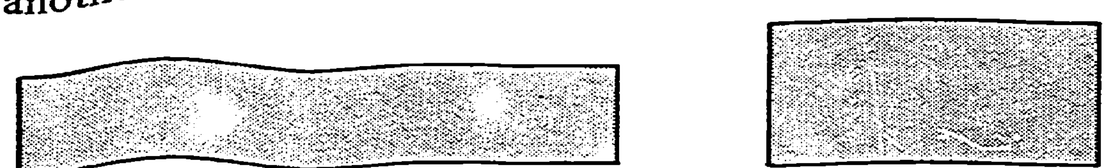


**Illustration 2-1**

The *language* of geometry is an important part of classroom instruction. In the early school years, the introduction of geometric terms should be quite informal. For example, as the pupil meets geometry for the first time teachers should accept his use of the term "triangle" for "triangular region." If teachers themselves use the correct terms, then pupils will eventually learn to be precise in using descriptive geometric terms. Pupils should also learn to describe a geometric shape or form in relative or approximate terms. For example, in describing the two rectangular regions in Illustration 2-2, the pupil should be able first to recognize that both figures are bounded by quadrilaterals which are rectangles. Second, the pupil should be able to describe (a) as having one side that is very much longer than the other, and (b) as having one side just a little longer than another.



(*a*) (*b*)

**Illustration 2-2**

Following this initial intuitive approach, the teacher will introduce the use of formal descriptive terms utilizing line segments that will prepare pupils for a more sophisticated abstract analysis.

The *properties* of geometric figures can be presented through experiences in classification. Classifying geometric shapes according to size and shape makes pupils aware of similarities and differences between these shapes. Further classification based on other properties will permit pupils to analyze a geometric shape according to the properties of a triangle, square, rectangle, parallelogram, pentagon, or hexagon. These properties are often used in measurement. For example, to find the perimeter of a regular polygon such as a square, the pupil needs to measure only one side if he is aware that all sides are the same length.

Other properties find interesting practical applications; for example, in construction work: Why does a carpenter, checking a rectangular frame, measure the length of the two diagonals?

Pupils should have experience in interpreting pictures of geometric shapes and forms in both two- and three-dimensional space. Practice in the construction of geometric shapes with a straightedge and a compass should be included in the instructional program. The making of a pictorial model should be encouraged even though there may be an adequate model in the textbook. Such construction often provides clues to the solution of problems. As a part of the study of polyhedra, we suggest an informal treatment of Euler's formula. This formula relates the number of faces, $F$; the number of edges, $E$; and the number of vertices, $V$, of a polyhedron: $F + V = E + 2$.

As soon as pupils have had adequate experience with the geometry of the physical world in which they have been able to touch and see actual models of the concepts being developed, they are ready to enter into a study of geometry that cannot be directly visualized. In some respects, one might call it "make believe" geometry, for the pupils have to perceive such concepts as a point, line, and plane as ideals that exist but cannot actually be represented in totality by a physical model. Pupils in the primary grades usually have difficulty bringing true meaning to the concept of a point, since it is something they cannot see or touch. They are often unable to perceive the set of all points in space or on a number line since their concepts of quantity may not extend to such infinite dimensions. This, in itself, may justify delay in introducing these abstract concepts until grades three to eight. In a spirally developed curriculum, however, learners in grades one through three should have initial experiences that intuitively develop the idea of a point as a fixed location and a line as a set of points which is infinite. The study of maps in the early grades provides many opportunities for this intuitive development, as does the use of the number line.

The study of such abstractions as point, line, ray, line segment, plane, and angle shows they are related to models in the physical world. Every attempt should be made to clarify these concepts through physical and pictorial models. Although it is difficult to make blanket statements for all pupils as to what they should know at a given level, it is assumed that, for example, pupils by the end of grade six will be familiar with and be able to describe a given line segment as the subset of a line ($\overleftrightarrow{AB}$) which is formed by two endpoints ($A$, $B$) and all of the points between them.

A basic concept in geometry is that of *congruence*. The concept of congruence grows out of classification activites and of certain experiences in the measurement of angles, line segments, the area of geometric shapes, and the volume of geometric solids. Two geometric forms or figures are said to be congruent if one can be superimposed on the other by a rigid transformation. (A rigid transformation is one which preserves the distance between two points. Such a transformation can be accomplished by a sequence of rotations about a point, reflections about a line, and translations. In particular, no stretching or squeezing is permitted.) Pupils should understand the differences between congruence ($\cong$) and equality ($=$) as the terms apply to geometry, since this distinction is often required in the axiomatic approach in later geometry.

Pupils should have experiences in comparing similar geometric figures. They should see the relationship of the radii in each of several increasingly larger circles. They should be involved in problems that require them to see the relationship between the length of the radius and the circumference or area of the circle. In studying the volume of cylindrical containers, pupils should see the effect of changing the height as compared with changing the radius of the base. They will begin to see that if the height doubles, so will the volume; whereas if the radius is doubled, the volume is quadrupled. Such experiences introduce the concept of ratio and proportion. These are enhanced by construction of similar figures. (Have the pupils draw a triangle that has the same size angles as a given triangle, but with sides twice as long.) Drawing maps to scale also illustrates the idea of similarity. More generally, scale drawings lead to an understanding of similarity and yield interesting applications of mathematics to physical situations.

A particularly good exercise for the class is to estimate area from a map. Choose, for example, a map of California. The map will have a linear scale showing number of miles to the inch. One method the children may use for estimating the area in square miles is to block off the state into rectangularly shaped parts, measure each part in inches, convert to miles, multiply the dimensions of each rectangle, and then add together the areas of all the parts. This method works very well for states whose boundaries are mostly straight lines. For other states, like Alaska, the children may try a second method: Trace the map on graph paper (16 squares to the square inch is good enough). Count the number of squares inside the boundary, estimating those that are part in and part out of the state. Determine the correct scale factor to multiply the number of squares counted. (A variant of either method entails cutting up the traced map and fitting pieces together to form larger rectangular regions.) Finally, compare the estimates from the various methods with the figure given in any standard resource book.

Many elementary classroom experiences in science and social sciences provide uses for the coordinate plane. Pupils may work with ordered pairs and graph these ordered pairs on a coordinate plane. Mathematical activities with the coordinate plane provide pupils with opportunities to solve problems, to translate mathematical information from a table to a graph (or vice versa), and to practice basic facts of arithmetic in game-like situations.

*Coordinate* geometry should be introduced at the primary level. Pupils in the primary grades can plot points in the first quadrant and graph data recorded in science experiments. Successive experiences will involve all four quadrants, and by grade eight pupils should know how to graph linear and quadratic equations as well as write the mathematical sentence that describes the graph. These simple concepts of Cartesian geometry have many applications in real life and provide the best possible basis for pupil choice in the study of more sophisticated mathematics at a later time.

*Metric* geometry brings the strands of geometry, operations, and measurement together. Practice in determining length, perimeter, area, volume, and angular measurement provides learners with a basis for the application of concepts and skills presented in the other strands. Not only should pupils have an opportunity to measure geometric shapes and figures with standard instruments such as the ruler and compass, but they should also have opportunities to make general measurements. In the latter, they should express the measurements in terms of greater-than or less-than relationships without a quantitative designation. General measurements are made by visual analysis, such as placing the model of one angle on top of another to determine whether it is larger or smaller.

Chapter Ten

# STRAND 3. MEASUREMENT

Measurement is so much a part of everyday life that we are often unaware of its extensive use and involvement in our thoughts, observations, and decisions. Measuring is a key process in the applications of mathematics since it is a connecting link between mathematics and our physical and social environment. For these reasons it is vitally important that the elementary school program include a detailed study of the measuring process. As a pupil learns about measurements, he should be actively engaged in activities that utilize both standard and arbitrary units of measure. Since the measuring process involves numbers, especially an extensive use of rational numbers, substantial attention should be given to its study in the elementary school mathematics curriculum. Attention should also be given to measurement in other subjects, however, such as geography and science.

Measurement is a process whereby numbers are assigned to an object or a set to represent certain quantitative attributes of the object or set. For example, answers to the following questions require measurement: "How wide is this table?" "How big is a page of this book?" "How heavy is this rock?" "How many words are on this page?" "How many seats are in this classroom if there are five rows with six seats in each row?"

### Beginning the Study of Measurement

In the introductory stage of the study of measurement, the pupils become familiar first with the sets to be measured, such as line segments, solids, weight, and time periods; and then with common units of measurement, such as the inch, foot, yard, meter, mile, pint, quart, gallon, liter, ounce, gram, pound, minute, hour, day, week, month, and year. Initially, comparisons are at a "greater than," "less than," or "equal to" level. The refinement and complexity of the various units of measure determine the grade level in which each one is introduced in the school program.

The analytical stage, which follows the recognition and identification stage, can begin with the development of some fundamental

concepts. It is important that the pupils recognize that certain types of measurement may lend themselves more readily than others for a particular purpose, and that the decision as to which type of measurement is most applicable be based upon the pertinent physical attribute of the object. For example, length is an essential property associated with a line segment. Thus, measurement associates a number with a line segment; this number is called the length of the segment. The measure of a segment, a number, tells how many times a "unit" segment can be fitted into the segment being measured.

In the initial stages, the "unit" chosen may be some handy object; for example, a pencil, an eraser, a piece of string, the span of a hand, or a bucket. Soon, however, pupils should be led to see that while the choice of a "unit" is arbitrary and may be a locally accepted way of measuring, it is necessary, for purposes of accurate communication, to choose one or more standard units.

**Making Approximations in Measurements**

An understanding of the approximate nature of measurement is essential. Exact measurements exist as ideals and are obtainable only in discrete situations in which the measuring process consists of counting the number of elements in the set to be measured. When a segment is measured, a scale based on the unit appropriate to the purpose of measurement is selected. Every measurement is made to the nearest unit. By using smaller units, more precise measurement is obtained. Inherent in a measuring process are two types of errors. One is caused by the instrument employed in the measurement. (For example, have a pupil measure the length of a table top once with a foot ruler and once with a yardstick. Expect different results.) Another error is introduced by the person making the measurement. As an instructive exercise, ask each member of the class to measure the width of the classroom to the nearest quarter inch using a foot ruler. It is most likely that many different answers will result, and making a decision as to the best approximation to the length will require the use of statistical procedures.

Approximations in measurements should be related to estimations of numbers in arithmetic, so both of these underlying notions reinforce each other. Insight into the operation of estimating should involve much more than simply obtaining an answer. Pupils should have experiences in which they have to make an estimate of a product or sum based upon their knowledge of the nature and order of numbers. For example, a pupil should be able to estimate that the product of 5 × 193 will be less than 1,000. He is able to sustantiate this by thinking that 193 is almost 200 and 5 × 200 = 1,000.

A sense for number should be nurtured so that the learner does not make blind estimations. For example, to estimate $^{69}/_{23}$, the conventional rule for rounding off replaces 69 by 70 and 23 by 20, so that $^{69}/_{23}$ is estimated as $^{70}/_{20} = {}^{7}/_{2}$. Estimating $^{69}/_{23}$ as $^{60}/_{20}$ yields $^{6}/_{2}$, a better estimate and by happy accident the correct answer. In the first estimation, increasing the dividend has the effect of increasing the quotient, and decreasing the divisor further increases the quotient. On the other hand, in the second estimation the effects of the estimation are compensatory.

Associated with these principles in numerical calculations are the problems of escalating errors in measurement through the use of mensuration formulas. For example, suppose that the sides of a rectangle have been measured and found to be 124 centimeters and 32 centimeters, measured to the nearest centimeter. What is the area? What errors are introduced in applying the formula $A = L \times W$? We know only that $123.5 \leq L \leq 124.5$ and that $31.5 \leq W \leq 32.5$ (see Illustration 3-1).

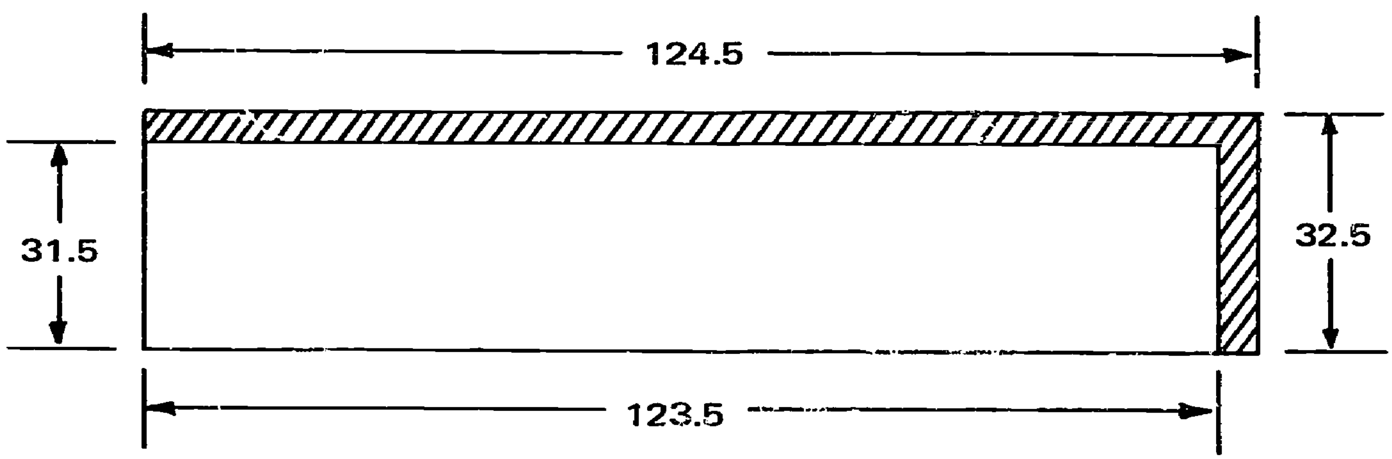



Illustration 3-1

Thus, $123.5 \times 31.5 \leq A \leq 124.5 \times 32.5$. Here the difference in the extremes is 156, so that use of this formula may give a measurement which may be in error by as much as 156 square centimeters. The product of $124 \times 32$ might be estimated as $100 \times 30 = 3{,}000$. A better estimate is achieved by compensatory approximations: $100 \times 40 = 4{,}000$.

## Learning the Metric System

In addition to the common units of measure the pupil encounters in his immediate environment, he should become familiar with the metric system, which is commonly used in the natural sciences and is being generally accepted throughout the world. Indeed, the United States Government is considering an official date for converting to use of the metric system of weights and measures as standard. It is easy to convert from one metric unit to another metric unit, and the decimal system of numeration identifies the metric system as a "natural" system to use. For example, a length measuring 3 meters, 8 decimeters, 4 centimeters is expressed in centimeters by the place value numeral 384. After they are introduced to the metric system, the pupils should use both the metric and the English systems interchangeably and frequently so that the metric system will become as familiar to them as the English system is. In this respect, the children can become mathematically bilingual. The exercise of converting from the metric system to the English system should be avoided. Rather, approximations of frequently used units might be established; for example, a meter as a little longer than a yard, or an inch as a little more than two and one-half times as long as a centimeter. It is especially important that every pupil have the opportunity to use actual meter sticks, gram weights, and liter containers in taking measurements, as well as yardsticks, ounce weights, and pint containers.

For convenience in computation and ease in comparing the two measurements, and to express the degree of precision claimed for a measurement, scientific notation is important. It is to be expected that scientific notation, including negative exponents, will be studied before the end of the eighth grade.

## Making Other Measurements

The same conceptual sequence used in the measurement of line segments can be followed in the measurement of angles, areas, and volumes. For simple angles, start with developing an intuitive device for comparing the size of a pair of angles. Thus, an angle is "larger than," "smaller than," or "the same size as" another angle. Models of the angles may be made of wire so that they can be superimposed, or one angle may be copied onto another. In comparing two angles, pupils should know that if two angles have a vertex and side in common, the angle whose interior contains the side of the other angle is the larger. Angles may be measured with arbitrary "unit angles" by "filling in" their interiors with these unit angles. Again the measure is approximated to the nearest unit. Markings on a clock suggest that circles can be divided into subsets of equal measure and thus used as units to measure angles. This leads to the development of angles of standard unit measure in degrees and then to the construction and use of a protractor.

For the measurement of area and volume, a start is made by identifying the interior regions of plane figures and solids and developing intuitive means of comparing the sizes of these interiors. Pupils can make models of the interiors of plane figures to be compared, and they can see if one can be fitted inside the other, cutting one of the models into pieces if necessary. They may follow similar procedures for comparing solids. Solids may be thought of as containers, and pupils may follow the same intuitive approach for comparing their volumes by ascertaining which model holds more water or sand. They may measure areas or volumes by completely filling the plane figure or solid with arbitrary units and determining how many such units are needed. Graph paper may be used to advantage in finding the areas of plane figures. Pupils should gain an understanding of why standard units in the form of squares or cubes are chosen for measuring area or volume, respectively. The development and use of formulas for the calculation of areas and volumes is particularly important at this time.

Measurement of other phenomena such as weight, time, and heat can be studied in a similar fashion. Computation with measures and conversion of units should evolve from actual situations (as distinct from verbal descriptions of actual situations) in order that it be done meaningfully and the results interpreted reasonably in terms of significant digits. Ample opportunity should be provided for estimation so that the teacher can determine if the process has meaning for the pupil.

## Understanding Concepts of Measurement

The development in kindergarten and grades one through eight of Strand 3, Measurement, should lead to general concepts of measurement such as the following:

- Measurement is a comparison of the object being measured with a "unit" and yields a number to be attached to the object as the measure of the object. (Measurement may thus be conceived as resulting in a pairing of objects with numbers. This pairing is of a type that leads to the notion of a function. Therefore, measurement may be treated as a special case of a function.)
- The choice of a measurement "unit" is arbitrary, but standard units are agreed upon for accurate communication and simplified computation.

- Measurement is approximate, and the precision of the measurement depends upon the measurement unit employed.
- Any process of measurement has the following basic properties:

  If object $A$ is part of object $B$, the measure of $A$ is less than or equal to the measure of $B$.

  If objects $A$ and $B$ are congruent, then their measures are equal.

  If objects $A$ and $B$ do not overlap, then the measure of the object consisting of the union of $A$ and $B$ is the sum of the measures of $A$ and $B$.

It is important to identify these basic properties as they are used in developing the mensuration formulas. For example, the development of the formula

$$A = \frac{1}{2} b \cdot h$$

for the area of a triangle (as expressed in standard units) can be indicated by a cutting and matching exercise together with the properties mentioned above.

First, consider a right triangle as shown in Illustration 3-2. Construct a rectangle of width $h$ units and length $b$ units. Cut it along a diagonal. By placing these two triangles together on $T$, show that all three triangles are congruent. Thus, each has the same area. Since, moreover, two of them exactly cover a rectangle of area $b \cdot h$, it follows that each triangle has an area which is equal to one-half its base times its altitude.

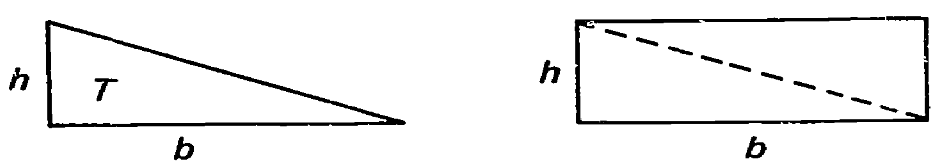



**Illustration 3-2**

Now, in a standard fashion as indicated in Illustration 3-3, obtain the same formula for the area of any triangle by computing the sum (or difference) of two right triangles.

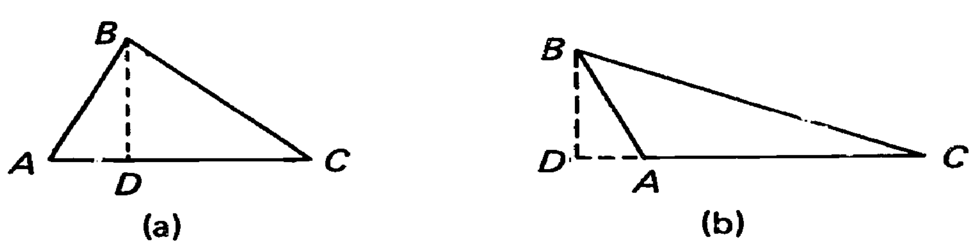



**Illustration 3-3**

Let $\triangle ABC$ denote the triangular region determined by the triangle $ABC$. Then, corresponding to Illustration 3-3a:

$$\triangle ABC = \triangle ABD \cup \triangle BCD$$

and corresponding to Illustration 3-3b:

$$\triangle ABC = \triangle BDC - \triangle ABD.$$

Another interesting exercise is the construction of a ruler from a blank straightedge made of some heavy material like wood or cardboard. The only "tool" needed other than the blank straightedge is a sheet of parallel ruled paper from a tablet or binder. The problem is to subdivide the blank into a number of equal segments. The blanks distributed to the pupils should be of varying lengths, and no blank should be exactly the same length as the ruled paper.

The pupils should be given free rein in their approach to the problem. In particular, each pupil should select the number of equal segments for his ruler.

As a sample problem, suppose it has been decided to divide a blank into ten equal parts. As shown in Illustration 3-4, ten spaces are counted on the ruled paper, and the blank is laid diagonally on the ruled lines so that the ends match ten spaces. The blank is then marked where the lines cross.

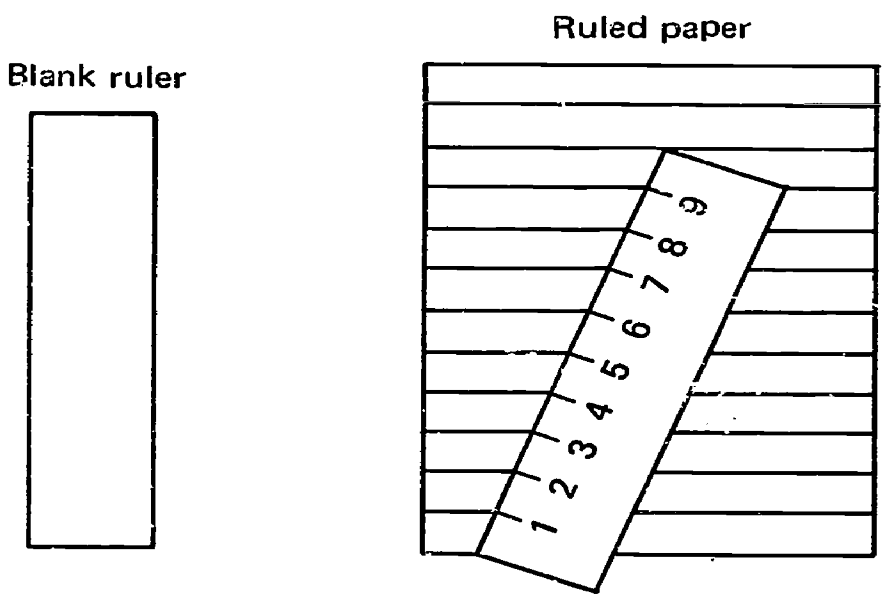



**Illustration 3-4**

Preliminary discussion will have brought out the idea that units are arbitrary. Blanks of different lengths, of course, yield units of different lengths. Each child will enjoy naming his own unit. Further use of these rulers will yield different measures of, for example, the length of a table top. This exercise demonstrates well why some customary units are selected and used universally to facilitate communication. In addition, this exercise provides a good springboard for the properties of parallel lines which have been assumed and implicitly used here.

Chapter Eleven

# STRAND 4. APPLICATIONS OF MATHEMATICS

Children have a natural curiosity about phenomena found in the physical world. As they work with physical materials, they unconsciously pose problems that need solution or ask questions such as "Why does this work?" This approach to the learning of mathematical concepts is not only more interesting to pupils, but it also provides them with practice in using the basic skills of arithmetic and geometry.

Mathematics does not literally deal directly with the raw physical condition, but only with a refined *model* of the situation. It does not divide ten apples by five children. Mathematics divides the number ten by the number five; the answer, two, is interpreted as meaning that if ten apples are evenly distributed among five children, then each child will have two apples. It is technically incorrect to divide ten apples by five children and obtain two apples per child, but this procedure is one that follows standard scientific practice in the determination of units such as "miles per hour." This distinction should be made a part of the teacher's manual so that the teacher will appreciate the process. In more complex situations, especially when the model does not fit the situation so exactly, the distinction between the model and its origin will be crucial.

This strand, Applications of Mathematics, is concerned with problems which arise in the context of some natural event. Often this will be a phenomenon arising in the natural sciences, in the social sciences, or in the humanities. The problems selected for study should be so meaningful to each pupil that he may retain an honest expectation of experiencing the problem himself. To the child who looks at a tree and says, "That's a tall tree," the question, "How tall is that tree?" will have meaning.

### Rationale of Applied Mathematics

The rationale of applied mathematics runs like this:

1. Given a concrete situation for analysis, a study of the situation identifies the features that are significant to the central problem.

2. Those features that are amenable to mathematical analysis are abstracted, and relations between these mathematical quantities are expressed in a way which characterizes those in the original concrete situation. Any other features are ignored. These abstracted features form the mathematical model of the problem.

3. The formulation of the problem is then subjected to a mathematical analysis. This analysis may be informal or transparent; on the other hand, it may be quite complex. If an analysis cannot be given, then additional simplifying assumptions and modifications of the model must be made. (Strategies for mathematical analysis will be discussed in Strand 9, Problem Solving.)

4. Finally, the mathematical solution is reinterpreted in the light of the original problem. Predictions for future behavior of the situation are made. The validity and usefulness of the model (step 2) is judged by how well it can predict future behavior in actual practice.

The steps outlined in the preceding paragraph should be discussed in the teacher's manual of the textbook series. However, they should never be presented as a problem-solving plan, which must be discussed and detailed in each problem. In particular, each pupil and teacher should feel free and should be required often to rephrase the initial problem, reselect the properties to be modeled, reconstruct the model, and redo the analysis. Formalistic procedures should never block intuition or flashes of genius. The teacher should strengthen intuition at every opportunity by encouraging informed guessing and estimation. On the other hand, pupils may be confused and inhibited if the teacher fails to make explicit some of the steps which have been outlined here. Also, some pupils who can easily do the mathematical analysis may nevertheless be totally unprepared to construct the correct mathematical model.

## Reverse Process of Model Building

Up to this point, stress has been placed on the strategy of model building as leading from physical problems to more abstract mathematical models and their analyses. The reverse process — given a mathematical problem, construct a physical realization for it — is pedagogically important. Such a physical model is a significant aid to understanding mathematics.

The nature of the application of mathematics will vary greatly from kindergarten through grade eight. At each level applications should include not only those in which the model is arithmetical but also those in which the model is geometrical. Some applications should make use of mathematical principles already developed, and others should motivate new principles. As often as possible, base the applications on physical situations that the learner may experience himself rather than on hypothetical situations.

As the pupils advance in age and maturity, the sophistication of the problems they can handle will be limited only by the opportunities they have to participate in real situations that present problems for solution. The approach to a specific problem should evolve from a self-determined pattern, as derived by the pupils from their previous mathematical experiences.

## Examples of the Modeling Process

Here are four examples of the modeling process; each is a child-sized version of a practical problem.

EXAMPLE 1: A school is four blocks from Paul's home – one block north and three blocks east. How many different routes may Paul choose to walk to school?

We first make a street map of the local area (Illustration 4-1). We will ignore the fact that a sidewalk exists on both sides of each street, with its complication of two routes which use the same street. We introduce the simplifying assumption that Paul will not "go out of his way" nor will he take a shortcut. Mathematically, we can interpret this to mean that the length of his route will always be four blocks long and always follow streets. Now the analysis may be done from the map. Draw arrows to show the alternatives at each corner, and complete the analysis by counting the possibilities. This counting operation, first systematizing and then counting all possibilities, is an extremely useful type of application. For the elementary school pupil, it will provide important readiness for later work in probability.

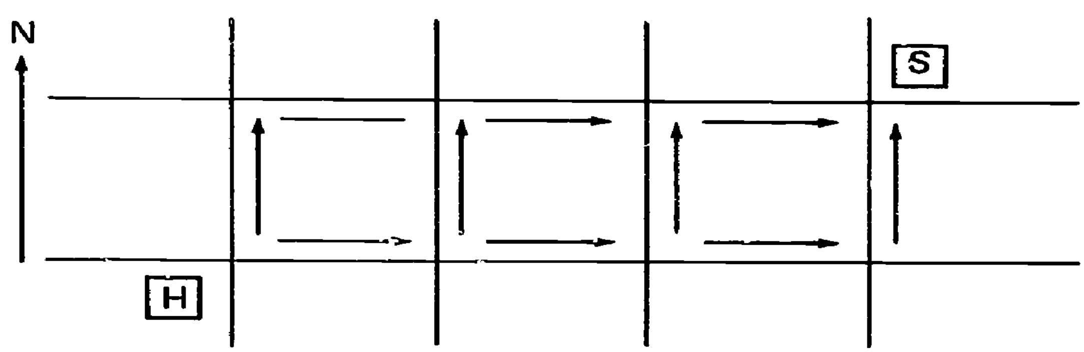



**Illustration 4-1**

EXAMPLE 2: Carpeting is to be laid to cover the classroom floor. How much carpeting should be ordered?

We will need to know something about the shape and area of the room. In a real situation, one would also need to know how the carpeting was to be purchased – by the square foot or square yard or by the linear foot from a roll of a specified width. Also, one would consider to what extent the carpet pattern might affect the way in which the material could be cut. In this problem, however, these considerations will be disregarded in order to simplify the task of determining the amount of carpet to cover the area of the room. We will further assume the floor is rectangular in shape, ignoring any special features of the room such as an entryway or a closet. A geometrical model may be constructed, therefore, simply by representing the floor as a rectangular region and assigning units of measure to the sides of the rectangle. The mathematical analysis and computation will yield the area. This number is then converted into the amount of carpeting to be ordered.

EXAMPLE 3: A toy electric train runs around an oval track. What is its speed?

We might assume, first, that only "average speed" is wanted. Such an assumption requires that speed be measured when the train has been running long enough for a "steady state" to be reached. Changes in speed as the train rounds the curved ends may be ignored. Another simplifying assumption might be that one circuit of the train around the track can be accurately timed.


**Illustration 4-2**

To help visualize the problem, a geometric model of the track can be constructed as in Illustration 4-2, showing that segments AB and CD are parallel, and arcs AC and BD are semicircles. Make the appropriate measurements on the actual track. First, determine the circumference of the oval and interpret this number as the length of the track. Now, with stopwatch in hand, time one complete run of the actual train around the actual track. As a check on accuracy, let the train run for several laps, time the total elapsed time of the run, and then compute an average time for one lap. Discuss with the class ways in which the experimenter can reduce the measurement errors inherent in determining the time of one circuit of the track. This analysis uses the fact that distance ($d$) traveled at a constant rate of speed ($r$) for a period of time ($t$) is the product of $r \cdot t$ (the units representing $r$ multiplied by the units representing $t$). The distance traveled in a period of time can be expressed as a mathematical function: $d(t) = r \cdot t$. It is important to point out that a graph of this function yields a straight line.

EXAMPLE 4: How fast do bacteria grow?

The science curriculum provides a situation for this problem. Obtain some petri dishes and source material and some bacteria culture, and attempt to determine the amount of bacteria growth at 24-hour intervals. Ignore conditions such as the effect of imperfectly controlled heat and light and the amount of bacteria already present in the host material. The growth will have to be estimated from an approximation of the area of the region of the petri dish covered by the growth. Record the growth data for the 24-hour intervals on a graph (see Illustration 4-3). (This is in itself a modeling exercise.)

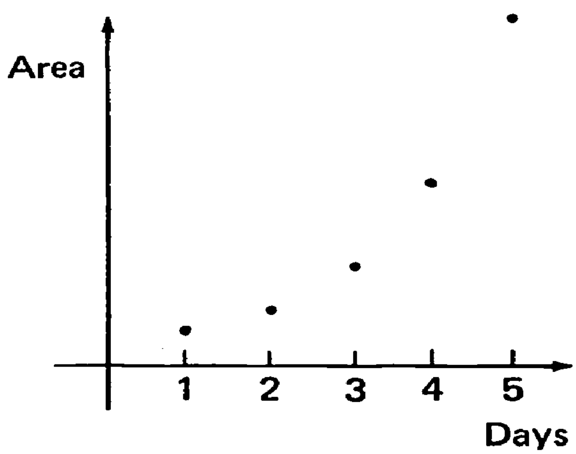



**Illustration 4-3**

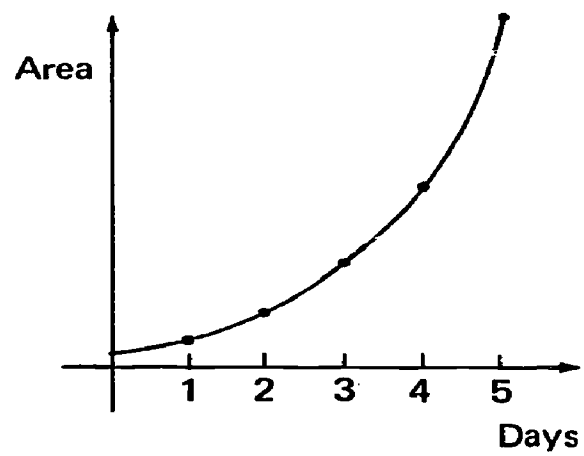



**Illustration 4-4**

How can the growth of these bacteria be described? Is there a mathematical relationship that can be used to predict future growth? If it is also assumed that the growth is continuous, then a smooth curve can be drawn between the plotted points (see Illustration 4-4). For the mathematical analysis of this graph, we find a function that describes this growth. The graph may show that the amount of bacteria doubled each day. The functional relation may then be expressed by the equation $AN = k \cdot 2^N$, where $k$ is the amount initially present, or half the amount present after one day. Further experimentation might be done to confirm the general correctness of this relation. On the other hand, the graph might show that the notation $AN = k \cdot 3^N$ better describes the function.

The mathematical model for growth of bacteria is thus a functional relationship which describes the amount of bacteria after a fixed time. If the model has validity, then upon determining the constant ($k$) by observing the growth during one period, one can use it to predict what growth should be expected at any time during the growth cycle of the bacteria. It is important that the pupil learn from this experiment and analysis that *growth* is an *exponential* function of time, here denoted by the letter $t$. A functional relationship of the type $A(t) = k \cdot b^t$ (where $b$ and $k$ must be determined) represents almost all growth processes. It is, of course, only a mathematical model since many of the factors controlling growth have been neglected. Basically, it lies behind the population explosion. The "explosion" part exists since the function *is* of the type $A(t) = k \cdot b^t$, and *is not* of the type $A(t) = b \cdot t + k$ or $A(t) = b \cdot t^2 + k$.

Chapter Twelve

# STRAND 5. STATISTICS AND PROBABILITY

We live in a world of statistics. Advertising agencies bombard us with information indicating the superiority of their products and quote their research findings as proof. Governmental departments report findings to substantiate the need for expenditures. Medical research is made available to us in reports of data that have been gathered in the furtherance of a better cure for a disease. Statistical data point to smoking as a possible cause of lung cancer because there exists a high correlation between heavy cigarette smoking and the incidence of lung cancer. Newspapers and magazines contain graphs, tables, and charts, and their stories reflect an interpretation of the data. The citizen of today and tomorrow needs to understand and interpret the data with which he is confronted; he will have to make decisions based on analysis of such data. The permeating influence of statistics and its theoretical sister, probability, compels students to understand enough about these aspects of mathematics to assess information intelligently.

The ability to interpret data can come only after many experiences in collecting data, learning how to organize data so that conclusions can be made, and in interpreting data. Therefore, experiences in collecting, organizing, and interpreting data should be included in an elementary school mathematics program. Experiences should begin in kindergarten and should be part of the instructional program at each succeeding level through the eighth grade.

## Collecting and Organizing Data

Collecting data should be the outgrowth of experiences involving observations. The elementary classroom abounds with sources of data. The children themselves provide data regarding their names, their heights, their preferences, the books they read, their birthdates, or their summer vacation travels. Science experiences provide opportunities for gathering two types of data – quantitative and observational. Quantitative data are those in which something can be measured or assigned a numerical quantity to denote change or growth. Observational data are of a descriptive nature and denote more of a general physical change; for example, the stages of development of a butterfly.

Data collecting should involve the pupil. It is of greater value to the pupil to collect the daily temperature readings for a period of time at school or at home than it is for him to turn to a report in a book where similar data are available. The information he will be using is the result of something in which he himself has been involved. Once the pupil has had sufficient direct experiences with collecting data, then data from textbooks can take on more meaning.

When a purpose has been established, data collecting begins with the recognition of the sources. As a pupil gathers data, his interest is aroused. He learns to be observant and discriminating and begins to develop analytical ability. He recognizes and examines discrepancies in the data he collects. His "mathematical eye" is often opened for the first time. He is involved in the problem-solving process.

Both simple data collection and organization of data weave into the strand on functions: the first, by one-to-one correspondence between an element in a sequence and its statistic; the second, by the display of a table or graph.

Organizing data is an art that the pupil must learn. The information in a table, graph, or chart must be presented in such a way that it complements the purpose for which the data were originally gathered — it must tell a specific story. A part of the time spent in the elementary school mathematics program should be devoted to learning the fundamental procedures for making the many different types of tables, graphs, and charts. Pupils should become familiar with the type of graphic presentation that can best be used for each of the different kinds of data, and they should be able to read the graph, table, or chart and decide what story it really tells. This is central to what might be considered simple statistical inference.

Many activities may be initiated that will interest children in grades four, five, and six. For example, they may make a table of the heights and weights of the class members, or of the number of feet each can kick a ball. When the data are broken into boy-girl classifications, some interesting conclusions appear. Compare the class data for height and weight with similar data for the whole school. As a variation of this problem, plot points (height, weight) for each pupil in the class. What conclusions can be drawn?

Another excellent project is to plot the average monthly temperatures in your city. Graph the average monthly temperatures with the months identified along the horizontal axis and the temperatures along the vertical axis. For San Francisco, a graph would have its lowest point around January, rise to a peak around August or September, then decline as it approached December. Obviously, discussions can bring out the characteristic behavior for the winter and the summer months. Next, present the same kinds of information with more than one city graphed on the same set of axes. A comparison of different locations such as San Francisco and San Diego may reveal that, while the general tendency (low in winter, high in summer, intermediate in spring and fall) may be observed to be similar for both, one curve is more peaked and the other more shallow. The lesson may be extended by reviewing other data and observing that the range between maximum and minimum points is smaller for stations closer to the equator (local irregularities minimized); statistics for some other stations in the equatorial regions may be produced to verify such a conjecture. To develop further the winter-summer theme, temperature graphs for stations in the southern hemisphere (such as Sidney, Australia) may be studied.

## Interpreting the Data

Interpreting data involves a very critical and analytical study of the facts. Pupils should be acquainted with the scale used in constructing a graph, should understand the purpose of data collecting, and should be able to comprehend the purpose of such indications of central tendency as *mean, median, mode, variance,* and *standard deviation*. Learning experiences should provide the pupils with opportunities for making predictions and for drawing conclusions from the data given. Pupils should learn to challenge the source of the data and the manner in which they are pictured.

These concepts are presented in Table 1 with four sets of data, which presumably represent noontime temperatures on ten successive days in each of four California cities. For convenience, the temperature readings are arranged in decreasing order. This ranking is not important, but it helps serve in lieu of a graph and will enable us to appreciate more easily what is happening.

The median and the mode serve principally as quick approximations to the mean since little numerical calculation is required. In comparison with the mean, they give us a little indication as to whether more of the observations are below average than above. In these examples we find that the average tells us little about the overall distribution. Other statistical measures — the variance and its square root, the standard deviation — give us some indication of the amount of spread in the data around the average. As we see in the accompanying tables, a great deal of spread will produce a high variance (as shown for Los Angeles in Table 2); while only a small spread (as for San Francisco in Table 3) will show a small variance.

To compute the variance, subtract the mean from each temperature observation, square this number, and add the squares for all observations. Finally, divide the total obtained by the number of observations. Compute the standard deviation by taking the square root of the variance.

The variance and the standard deviation are keys to how much confidence we can place on the average. For example, for a trip to Los Angeles or San Francisco, should you take heavy or light clothing? The average temperature for both cities is 70° F. At San Francisco the variance is 2.2; we can be confident that the temperature upon our arrival will be close to 70 degrees. But at Los Angeles the variance is 180, and it cannot be certain that the temperature will be close to 70 degrees.

## Experimentation to Produce Statistics

Experimentation is a natural way to produce sets of statistics. For example, pupils can plot the length of a suspended spring against the weight applied to the spring; the graph will show a straight line up to a certain limit, which is reached when the spring has been permanently deformed. As another example, the growth of beans can be plotted to give height as a function of the number of days from germination. Experiments with simple pendulums of various lengths may be undertaken to illustrate the inverse of the squaring function, obtaining a constant multiple of the square root with the formula $T = \sqrt{\frac{l}{g}}$ where $T$ is the period, $l$ the length, and $g$ a constant due to gravity.

Clearly, empirical data are not likely to yield a smooth curve precisely. Thus, deviations or anomalies from uniformity suggest the need for curve-fitting procedures. The intuitive level of curve fitting considered appropriate at this time need not go beyond drawing a smooth curve that fits most of the plotted points.

The central role played by normal distributions in statistical problems should be explained at appropriate times. Many examples which have distributions of this type should be studied. These examples should display the important role played by the standard deviation in normally distributed populations since 95 percent of the population falls within the interval bounded by two standard deviations from either side of the mean.

TABLE 1
**Noontime Temperatures in Four California Cities**

| | Los Angeles | San Diego | San Francisco | San Bernardino |
|---|---|---|---|---|
| | °*F.* | °*F.* | °*F.* | °*F.* |
| | 100 | 80 | 72 | 75 |
| | 80 | 80 | 72 | 74 |
| | 80 | 80 | 71 | 73 |
| | 80 | 80 | 71 | 72 |
| | 60 | 80 | 71 | 71 |
| | 60 | 60 | 69 | 69 |
| | 60 | 60 | 69 | 68 |
| | 60 | 60 | 69 | 67 |
| | 60 | 60 | 68 | 66 |
| | 60 | 60 | 68 | 65 |
| Totals . . . . . . . | 700 | 700 | 700 | 700 |
| Average or mean . | 70 | 70 | 70 | 70 |
| Mode . . . . . . . . | 60 | 60 or 80 | 69 or 71 | Any number |
| Median . . . . . . | 60 | 70 | 70 | 70 |

TABLE 2
**Computing Variance in Temperatures for Los Angeles**

| | Los Angeles observations | Observations minus means | Squares of observations minus means |
|---|---|---|---|
| | °*F.* | | |
| | 100 | 30 | 900 |
| | 80 | 10 | 100 |
| | 80 | 10 | 100 |
| | 80 | 10 | 100 |
| | 60 | −10 | 100 |
| | 60 | −10 | 100 |
| | 60 | −10 | 100 |
| | 60 | −10 | 100 |
| | 60 | −10 | 100 |
| | 60 | −10 | 100 |
| Average . . . . . . | 70 | | |
| Total . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . | | | 1,800 |
| Variance . . . . . . . . . . . . . . . . . . . . . . . . . . . . | | | 180 |
| Standard deviation ($\sqrt{180}$) . . . . . . . . . . . . . . . . . . . . . | | | 13.4 |

TABLE 3
Computing Variance in Temperatures for San Francisco

| | San Francisco observations | Observations minus means | Squares of observations minus means |
|---|---|---|---|
| | °F. | | |
| | 72 | 2 | 4 |
| | 72 | 2 | 4 |
| | 71 | 1 | 1 |
| | 71 | 1 | 1 |
| | 71 | 1 | 1 |
| | 69 | −1 | 1 |
| | 69 | −1 | 1 |
| | 69 | −1 | 1 |
| | 68 | −2 | 4 |
| | 68 | −2 | 4 |
| Average | 70 | | |
| Total | | | 22 |
| Variance | | | 2.2 |
| Standard deviation ($\sqrt{2.2}$) | | | 1.48 |

**Statistical Sampling**

In statistical sampling, a part is taken at random from a whole, and the composition of the sample is studied in order to arrive at an estimation of the composition of the original population from which the sample was drawn. For example, a sample of 59 cars is taken from an assembly line producing 10,000 cars. Three of these 59 cars are found to have defective brakes. How many cars of the 10,000 produced in this assembly line are likely to have defective brakes? Clearly, a model for this kind of problem is the following: 25 marbles are drawn from a large jar of 1,500 red and white marbles. Of the 25 marbles drawn, 20 are red and 5 are white. What is likely to be the original color composition of marbles in the jar?

**The Probability Theory**

The probability theory reverses the sampling process; the probable composition of the sample is deduced from the known composition of the original population. For example, if it is known that 1 percent of all people in the United States are left-handed, how many people in a United States city with a population of 800,000 are left-handed? Or, it may be known that 80 percent of the crop in a field produces fertile seeds. From a packet of 200 seeds collected from this field, how many can be expected to germinate? Corresponding to these are models such as the following: In a container are 100 marbles, 80 of which are red and 20 of which are green. If a sample of five marbles is drawn at random from this container, how many are expected to be red? If the original composition is 80 red and 20 green, then the probability that one marble drawn at random is red would be

$$\frac{80}{80 + 20} \text{ or } \frac{4}{5}.$$

The probability theory should not be developed extensively in the elementary grades, but some experience with the laws of chance is invaluable and necessary. One reason is that the mathematical models of many scientific and economic problems lie within probability theory. Experiences with probability appropriate in the elementary school include drawing marbles randomly out of containers, spinning spinners, flipping coins, and throwing dies. Many children's games come supplied with spinning pointers. These games can be adapted for classroom experiments in probability by varying the areas of the sectors. Illustration 5-1 pictures a spinning pointer, which may point to red, yellow, or blue sectors that are, respectively, one-third, one-fourth, and five-twelfths of the region. What is the probability of the spinner's pointing to yellow on a spin? In tossing dies, are we likely to find that a die with four faces painted red, one face yellow, and one face blue will produce different results in tallying the color of faces than a die with one red, two yellow, and three blue faces?

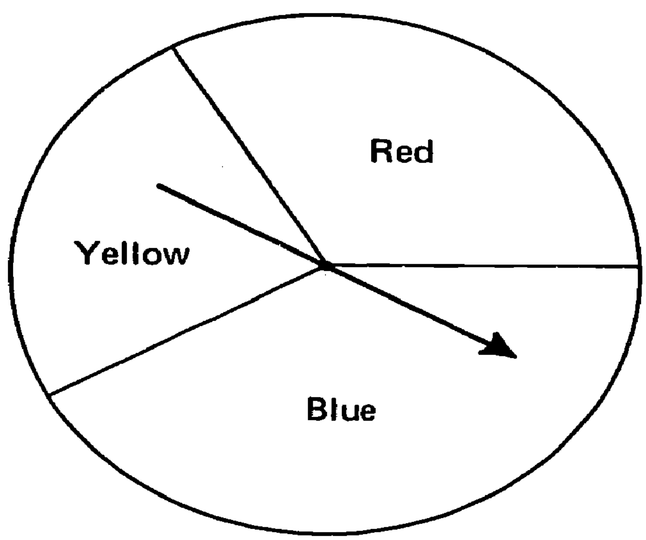



**Illustration 5-1**

In the seventh and eighth grades, experiments can be more complex. The following are some suggestions for suitable experiments:

1. Conduct a series of repeated trials of the same experiment.

2. Combine outcomes of different experiments; for example, consider the result of a spinner experiment together with the result of tossing a coin; or combine the results of two spinner experiments.

3. Experiment with drawing marbles from a container, considering the effect of not replacing the drawn marble as against replacing the marble before the next draw.

4. Tally the results of repeated tosses of a coin. In a pair of tosses of a coin, the results may be:

hh: heads on the first toss followed by heads on second toss
ht: heads on the first toss followed by tails on second toss
th: tails on the first toss followed by heads on second toss
tt: tails on the first toss followed by tails on second toss

Each pair of tosses may then be tallied under one of four possible outcomes: hh, ht, th, or tt. When the Hardy-Weinberg principles and the Mendelian law are discussed in the science class, results of the coin-tossing can be compared with the theoretical results in genetics.

Repeated tosses of a coin may be further extended by taking each triplet of tosses as a single outcome. The possible outcomes may then be given by hhh, hht, hth, htt, thh, tht, tth, ttt, where, for example, thh would mean tails in the first toss followed by heads in the second toss and heads in the third toss.

All of the ramifications in statistical inferences need not be formally treated until high school, but an intuitive feeling for statistical inference may be acquired earlier, in the elementary grades, through a wealth of experiences in probability. These experiences lead, for example, to insight into the understanding of actuarial events.

With a carefully designed curriculum, it is possible and desirable to introduce aspects of probability theory that include dependent and independent trials and aspects of statistics that include simple statistical inferences. Concepts of probability and statistical sampling form an important foundation to the natural and physical sciences and to the social sciences and business as well.

Chapter Thirteen

# STRAND 6. SETS

The concept of a *set* is fundamental in communicating ideas in mathematics, just as it is in our everyday life. We commonly speak of teams, committees, classes, groups, congregations, armies, flocks, herds, and so on. The concept of a set should be introduced early and used throughout the kindergarten through grade eight mathematics curriculum. Once the concept is introduced, it is particularly important that it be used effectively in subsequent mathematical development. Indeed, if no such use can be made, then the idea should not be presented at all. As the playwright Chekhov said, a cannon should not be brought on stage unless it is to be fired.

From kindergarten on the language of sets should be used as needed to gain clarity, precision, and conciseness in mathematical communication. The symbols of set operations may be introduced to avoid using longer, more complicated verbal equivalents. At no time is it suggested, however, that set theory be developed as an end in itself. Thus, it is not recommended that the curriculum include a treatment of Boolean algebra or axiomatic abstract set theory.

The development of this strand benefits, as do other strands, from a spiralling escalation of material wherein old concepts are rediscovered along with newer ones. For kindergarten and the first grade, it is probably sufficient to introduce the notion of a set as a collection of things. Children should be encouraged to identify natural examples of sets and to form new sets from old by joining (set union) and intersecting. Sets of objects can be indicated by many devices such as placing cutouts on a flannel board or drawing pictures of the members of the set. If several sets are to be pictured together, they can be separated by surrounding each set with a circle-like curve. At this introductory level, the teacher will use simple terms correctly and encourage their use by the pupils, but will not require a mastery of this vocabulary from the students.

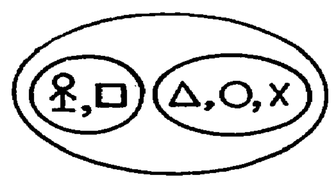
Later, in the intermediate grades, the teacher will need to use a more sophisticated device than a picture. To indicate a set and its members, according to the recommended convention, the elements of a set are enclosed between braces. Thus, the set of the first four positive odd integers is denoted: $\{1, 3, 5, 7\}$. The order of the listing or the number of times an element appears in the list does not affect the set: thus, $\{1, 3, 5, 7\} = \{7, 1, 5, 3\} = \{3, 5, 7, 5, 1, 3\}$. At the same time, examples of larger sets should be presented in which it is inconvenient or impossible to list all the elements. Two such examples are the set of all positive odd integers less than $\sqrt{1049}$ and the set of all positive odd integers.

Teachers of arithmetic, measurement, and geometry at all grade levels should make extensive use of the basic concept of one-to-one correspondence, commonly called matching. This concept relies implicitly on a preconceived notion of set to describe what is being matched. The abstract nature of the number four can be learned by considering many different sets and by noting that the elements of some sets can be matched in a one-to-one correspondence with the elements of a particular set, $\{\bigcirc, \triangle, \varnothing, \square\}$, to which we assign the number four.

In presenting arithmetic and geometry to intermediate or junior high school pupils, the teacher should use set operations and correct symbolism: *set union* ($\cup$), *set intersection* ($\cap$), and *set difference* ($-$). Here are some examples of their use. In the first two, an alternative example suitable for elementary pupils is suggested.

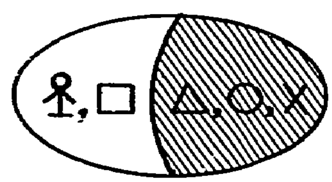
- In addition (set union):

$$\underset{\{x, y\}}{2} \;\underset{\cup}{+}\; \underset{\{a, b, c\}}{3} \;\underset{=}{=}\; \underset{\{x, y, a, b, c\}}{5}, \text{ or}$$



- In subtraction (set difference):

$$\underset{\{a, b, c, x, y\}}{5} \;\underset{-}{-}\; \underset{\{a, x, y\}}{3} \;\underset{=}{=}\; \underset{\{b, c\}}{2}, \text{ or}$$



- In solving equations (solution set):
  1. Solve for $\square$: $\square + 5 = 7$. The solution set is $\{2\}$.
  2. Solve for $x$: $2x + 1 = 21$. The solution set is $\{10\}$.

- In solving inequalities (solution set):
  1. What whole numbers satisfy $4 < \square \leq 8$? The solution set is $\{5, 6, 7, 8\}$.
  2. What real numbers satisfy $2x^2 + 5 < 7$? The solution set is $\{x: |x| < 1\}$.

- In addition and subtraction of fractions:

$$\frac{1}{18} + \frac{1}{30} = ?$$

1. Solution using least common multiple —

   The set of multiples of 18 = $A$ = {18, 36, 54, 72, 90, 108, 126, 144, 162, 180, . . .}.
   The set of multiples of 30 = $B$ = {30, 60, 90, 120, 150, 180, . . .}.
   The set of common multiples of 18 and 30 = $A \cap B$ = {90, 180, . . .}.
   The least common multiple of 18 and 30 is 90. Hence,

$$\frac{1}{18} + \frac{1}{30} = \frac{5}{90} + \frac{3}{90} = \frac{8}{90}.$$

2. Solution using greatest common divisor —

   The set of divisors of 18 = $C$ = {1, 2, 3, 6, 9, 18}.
   The set of divisors of 30 = $D$ = {1, 2, 3, 5, 6, 10, 15, 30}.
   The set of common divisors of 18 and 30 = $C \cap D$ = {1, 2, 3, 6}.
   The greatest common divisor of 18 and 30 is 6. Hence,

$$\frac{1}{18} + \frac{1}{30} = \frac{1}{6}\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{1}{6} \times \frac{8}{15} = \frac{8}{90}$$

- In geometry (set intersection):

  Two different straight lines intersect in at most one point (see Illustration 6-1).

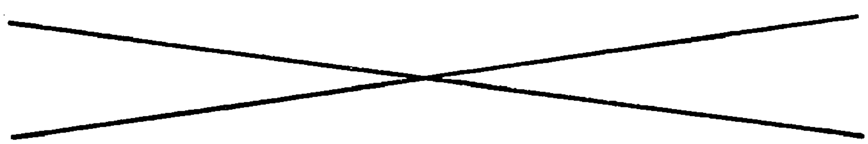


**Illustration 6-1**

- In geometry (one-to-one correspondence):

  *Theorem:*

  Two line segments have the same number of points (see Illustration 6-2).

  *Proof:*

  Let line segments $\overline{AD}$ and $\overline{BC}$ intersect at O. The point P on $\overline{AB}$ corresponds to the point Q on $\overline{CD}$, which is the intersection of the line determined by the two points P and O and $\overline{CD}$.

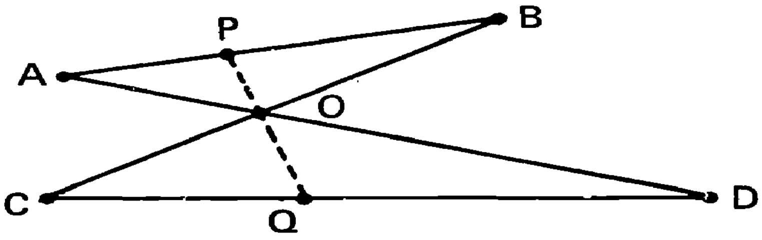



**Illustration 6-2**

The concept of a *subset* of a set follows naturally. The set $A$ is said to be a subset of the set $B$ if and only if every element of $A$ is an element of $B$. For example, the set of even integers is a subset of the set of all integers; a line segment is a subset of a line; and the solution set of $-1 \leq x - 1 \leq 0$ is a subset of the solution set of $0 \leq x^2 \leq 1$. The notation $A \leq B$ may be used to denote that $A$ is a subset of $B$.

For multiplication and the number plane, the notion of an ordered pair of elements is important. For example, how is multiplication involved in answering this problem: If I have three shirts and four pairs of slacks, how many different combinations can I wear? To answer this, count the number of sets (shirts, slacks). This set is really ordered by the nature of the garment: shirt first, slacks second. (The choice of which is first and which is second is not important.)

In mathematics we frequently pay attention to the order. An ordered pair is a sequence consisting of two terms listed in order. Thus, the ordered pair (1, 2) is different from the ordered pair (2, 1). Indeed, these two ordered pairs are plotted as distinct points (see Illustration 6-3) on the number plane. An ordered pair has a first element, $a$, and a second element, $b$; thus, the ordered pair $(a, b)$ is distinct from the ordered pair $(b, a)$ unless $a = b$. Ordered pairs are the means by which a function is carefully defined.

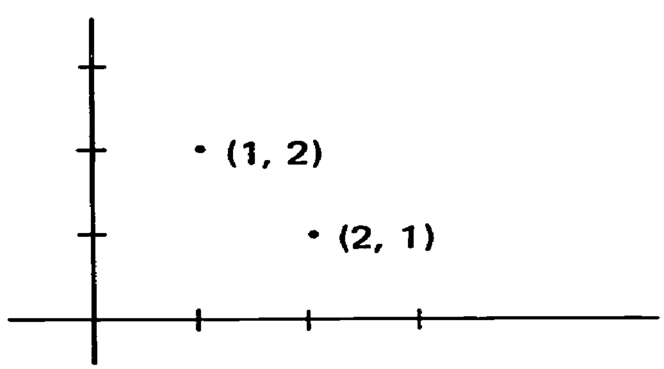



**Illustration 6-3**

By grades seven and eight, pupils may be expected to use correctly the elementary terms for set concepts. At this level they will probably need a better way of expressing sets. A way which has proved to be extremely useful is the *set builder* notation. For example, the solution set of $2x + 3 < (2 + 5x) - 8$ was shown to be the set of all numbers greater than 3. In the set builder notation, this is written as $\{x:\ 2x + 3 < (2 + 5x) - 8\} = \{x: x > 3\}$, and the latter is read, as has been indicated, "$\{x: x > 3\}$ is the set of all numbers $x$ such that $x$ is greater than 3."

Using this scheme, one can write the set of all positive odd integers less than $\sqrt{1049}$ in various ways, such as:

1. $\{x:\ 0 < x < \sqrt{1049}$ and $x$ is an odd integer$\}$.
2. $\{x:\ 0 < x,\ x^2 < 1049$, and $x$ is an odd integer$\}$.
3. $\{x:\ x = 2y + 1$ for some integer $y$ and $0 < x < \sqrt{1049}\}$.
4. $\{2y + 1:\ 0 < y,\ (2y + 1) < \sqrt{1049}$, and $y$ is an integer$\}$.

In general, a set may be described by notation, enclosed in braces, consisting of a symbol for a typical element of the set (a "John Doe"), followed by a colon (which stands for "such that"), and followed in turn by a list of conditions which the element must satisfy. In this vein, an expression for the set of possible candidates for President of the United States in 1972 is $\{x$: $x$ is a native-born citizen of the United States and $x$ is at least 35 years old$\}$.

In example 4 above, instead of $x$ there appears the expression $(2y + 1)$ as an element in the set, followed by a description of the condition on the symbol $y$ appearing in the expression.

As a final example, we define the set of Pythagorean triples:

$$\{(a, b, c):\ 0 < a < b < c,\ a^2 + b^2 = c^2,\ a, b, c, \text{ integers}\}.$$

In the final developmental stage of the language of sets in kindergarten and grades one through eight, the pupil is expected to use these set concepts, terms, and notations correctly and with reasonable precision as he studies inequalities, solution sets, informal geometry, functions, probability and statistics, and the applications of mathematics.

Chapter Fourteen

# STRAND 7. FUNCTIONS AND GRAPHS

Most of mathematics is concerned with relations. The young child learns early to relate certain objects or sounds with other objects. For example, he relates a child with the child's parent. Intuitively, we recognize without formal articulation certain ordered pairs of objects. Thus, the child learns pairs of the form (name, object) in discovering language. Sets of related pairs of objects are studied throughout the elementary mathematics program. They are often described by graphs and tables, and the coordinate plane of geometry is described in these terms. These notions lead directly to the concept of function, which permeates all of mathematics and science. This concept should be developed, named, and used in the elementary school program.

The process of forming certain pairs should be introduced early in the mathematics program. In beginning arithmetic the pupil learns to associate each set of objects with a number. He learns to count by pointing to the objects in sequence and pairing them with the set of ordered number words. He finds that counting is a way of determining what number is to be associated with a certain collection of objects; thus, counting determines certain related pairs (set, number). While different sets of objects may be paired with the same number, a finite set of objects is paired with one and only one number.

Another example is seen in the relationship *greater than.* A number may be related to the numbers which are greater, and ordered pairs may be thought of in the form (number, greater number). This is an example of a relation in which a single number is related to many other numbers.

Use of a pictorial description of relations between certain pairs of objects should begin early in the mathematics program. Plotting and graphing are ways to make pictures of relations. Plotting may be initiated with games such as tic-tac-toe. A class may graph the height of a plant on successive days of a month, or it may record temperatures for each day. The experience of such graphing reinforces the concept of the number line, both vertical and horizontal, presents a picture for linear relationships, and provides an excellent way of grasping relationships intuitively. Graphing is also invaluable in the applications of mathematics. A class may record the length of a spring (rubber band) as weights of successive sizes are suspended, or, at a more advanced stage, the period of a simple pendulum as the length of the string is varied. In the study of measurement, a class may record the number of ounces (or any standard unit) of water required to fill cylindrical jars of various diameters to a fixed depth. In the study of geometry, a class may plot circumference as related to diameter for different circles.

While a relation between numbers is described by plotting related pairs on the plane, the plane is also described in terms of all ordered pairs of numbers. This connection (point in the plane with ordered pair of numbers) is basic to the mathematics which connects geometry and algebra. It is also basic to the understanding of maps and more generally to scale drawings. The elementary programs should include the study of the coordinate plane.

One sort of relation is of particular importance to science and mathematics: for each member of a class of objects, there is just one object to which it is so related. For example, for each child there is just one natural mother. Such a relationship is called a functional relationship, or simply a function. Thus, the related pairs that form a functional relationship have the property that there is, at most, one related pair with a given first member. It should be emphasized that order is most important. As indicated above, the relationship in which each child corresponds to his mother is a function; thus, the set of ordered pairs of the form (child, mother) is a function. The relationship in which each mother corresponds to each of her children (mother, child) is not a function, since a single mother may have several children.

The pupils should become familiar with another conceptual description of function. Since a function relates one object to a unique second object, we may think of it as a sort of machine, as shown in Figure 8 .

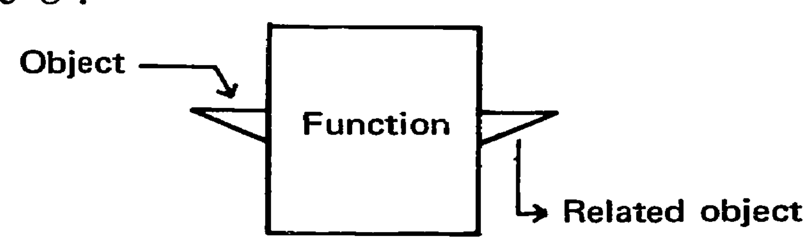



**Fig. 8. Concept of a function as a machine**

Figure 9 demonstrates the squaring function in which every number corresponds to its square, and Figure 10 illustrates the counting function.

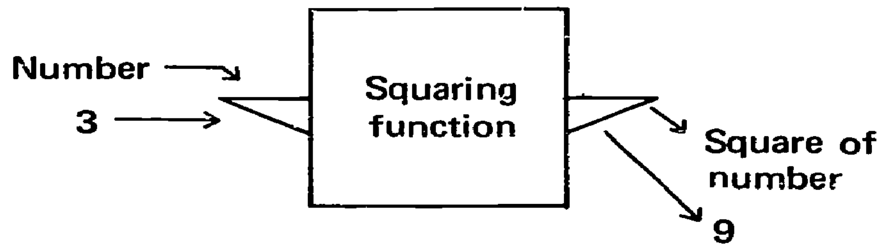



**Fig. 9. Illustration of the squaring function**

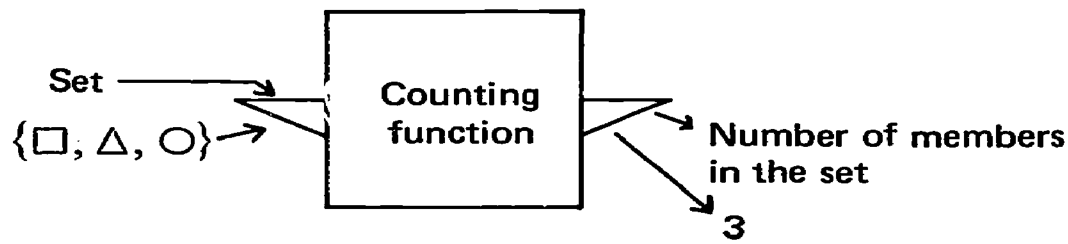



**Fig. 10. Illustration of the counting function**

Functional notation should be used systematically by the end of the eighth grade. Several notational schemes may be suggested, and different notational schemes should be used upon occasion since different notations are suggestive of different aspects of the function concepts. The following are some illustrations of the notations possible for the squaring function:

$$S(2) = 4,\ S(3) = 9,\ S(4) = 16$$

or,

$$S: \begin{array}{l} 2 \rightarrow 4 \\ 3 \rightarrow 9 \\ 4 \rightarrow 16 \end{array}$$

or, as a table,

| $\Delta$ | $S(\Delta)$ |
|---|---|
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |

The description, or formula, $S(n) = n^2$ for each number $n$ is also useful and appropriate.

The function concept includes mathematical operations. Elementary pupils encounter binary operations in learning the number facts although the terminology of binary operation is probably new to them. The basic multiplication facts, for example, describe this binary operation on the set of whole numbers. The operation is binary because only two whole numbers are involved. Thus, a binary operation on numbers, such as addition or multiplication, is a special way of relating two numbers with a third number. The function may be presented in a table, such as the familiar addition or multiplication tables. In multiplication, the number pair (3, 4) is paired with the number 12; the number pairs (4, 3), (2, 6), (6, 2), (12, 1), and (1, 12) are also paired with the number 12 in the multiplication operation. This indicates that many different number pairs may be paired with the same single number in a binary operation, but a given number pair is associated with only one number.

Intuitive experiences will pave the way toward conceiving a function as a set of ordered pairs in which no two pairs have the same first element. The pupil should realize that functions can be presented or described by statements, formulas or equations, tabulated data, and graphs. He is then on his way toward understanding many mathematical ideas that have far-reaching applications. The mathematics program should familiarize the pupil with the functional notations given here and should enable him to plot linear and quadratic functions as well as functions with jumps, such as the greatest integer functions.

Chapter Fifteen

# STRAND 8. LOGICAL THINKING

The solution of many problems often depends on the logical organization of known information or on the careful selection of alternatives. A typical problem will begin with the question, "What will happen if . . . ?" Sometimes it is important to know the consequences of a change in the rules by which a system operates.

Logical thinking or reasoning ability has its roots in the elementary school program, especially in the elementary mathematics program where the emphasis is on helping pupils learn to organize ideas, to understand what they learn, and to think for themselves. Logical thinking at the elementary school level does not imply development of formal proofs or a study of logic per se. Logical thinking and deductive reasoning at this level are a matter of well-organized common sense. Common sense can be sharpened by the use of standard logical techniques such as Venn diagrams and truth tables. Children should be able to decide whether a particular mathematical construct fits a definition and to recognize a specific instance of a general principle. The following examples will illustrate the types of logical thinking that can be started in the elementary school:

1. Draw a Venn diagram as shown in Illustration 8-1. Discuss with the class examples that fit the diagram, showing that all positive odd numbers are whole numbers and that one or more whole numbers exist that are not odd. (An important aspect of this example is the study of the quantifiers "all" and "there exist.")

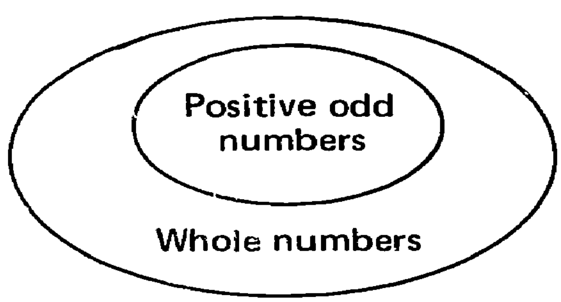



**Illustration 8-1**

2. Cut out pieces of paper as geometric models for numbers as shown in Illustration 8-2. Use these in combination to make generalizations and inferences about odd and even numbers. The even numbers are represented in the illustration by rectangles two squares in width, while the odd numbers greater than 1 are represented by rectangles with a square attached. By fitting the pieces together, pupils may discover (1) that the sum of two odd numbers is an even number; (2) that the sum of an odd number and an even number is an odd number; and (3) that the sum of two even numbers is an even number. Later on, pupils see that $2n$ and $2n + 1$ are ways of designating odd and even numbers. In this example, the model is used to justify the statements (conclusions or generalizations) that are made. This technique can be applied in studying the behavior of odd and even numbers under multiplication.

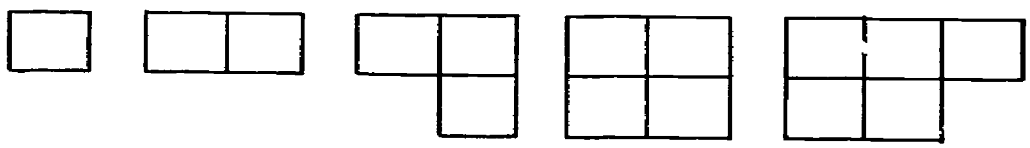


**Illustration 8-2**

3. Establish $A$ and $B$ as two points on a number line, $B$ representing a number greater than $A$. Locate point $H$ between points $A$ and $B$, closer to point $B$ than to point $A$. What statements can be made about the numbers represented by points $A$, $B$, and $H$?

4. Suppose that the following facts are given about three numbers: (a) Each number is a positive integer; (b) the sum of the three numbers is an odd number less than 12; (c) one of the numbers is three times another; and (d) the product of any two of them is an odd number. Which of the following statements are consistent with all of the facts (a), (b), (c), and (d)? Of the consistent statements, which, together with the given facts, determine the numbers uniquely?

(1) The numbers are all equal.

(2) Two of the numbers are odd. The other is even.

(3) The sum of two of the numbers is one less than the third number.

(4) The numbers are all different.

5. Suppose there are 33 members in a class, and some of them ride bicycles to school, others bring their lunches, and some do

both. If 21 ride bicycles and 15 bring their lunches, how many do both? (The Venn diagram of Illustration 8-3 helps in the analysis.)

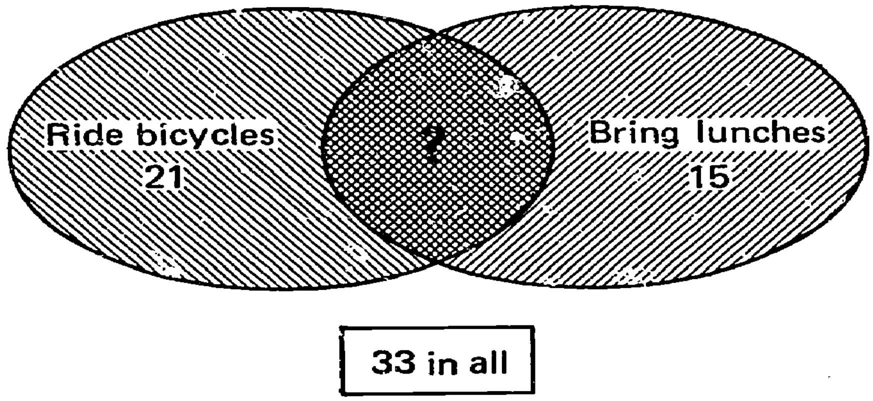



**Illustration 8-3**

At the elementary school level, the study of logical thinking follows the meaningful buildup of principles and generalizations. In most instances, these generalizations are the outgrowth of critical analysis of many related problems in which a common element or happening occurs. The program for kindergarten and early grades should provide many opportunities for children to identify likenesses and differences, classify and categorize objects according to characteristic features, find patterns, and state generalizations.

At the lower elementary level, the use of many definitions of mathematical terms should be nontechnical in nature. For instance, in introducing the term *less than,* the teacher should demonstrate that two is less than five by attempting to establish a one-to-one correspondence between the objects in a set of two objects and a set of five objects. Next, introduce the symbol meaning "less than" (<), and describe it in operational terms. For example, $2 < 5$ means that two is named before five in the counting sequence (or appears at the left of five on the number line). A start toward a mathematical definition is made when a pupil understands that $2 < 5$ means there is a counting number that can be added to two to get five. The formal definition can wait until more precise algebraic terminology becomes available. For some pupils this point may be reached in grade eight.

The use of variables is another example of the nontechnical use of symbols in mathematics in the elementary school mathematics program. The symbols □, ◇, *n, x,* ?, ○, as used in a mathematical sentence, indicate that something is unspecified in that sentence. There is something that we want to find out from the information that has been given us. When we do find it out, we may write it in place of the variable or symbol. The missing element may be a numeral or an operational sign; in most instances it stands for an undetermined number. Later on, the variable will take on the meaning of representing any of several numbers that can be used in the given sentence. In elementary mathematics, a variable is just a symbol used in a mathematical sentence to stand for a mathematical object (element, number, point, line, or function, depending on the context) as yet unspecified or to be determined. Such a symbol is just a generic name, like "John Q. Public," for a typical member of the set under consideration. The mathematical sentence of which a variable is a part places conditions or gives properties which may, of course, determine the element uniquely or show how all members of the set behave under these conditions. (No definition for the term *variable* should be attempted, and indeed the term itself may be avoided altogether.)

Logical thinking may also be based upon the definitions or agreements that are given to some of the words used in mathematics. In the early grades pupils become acquainted with the use of the words *all* and *some*, which are known as quantifiers in logic. They are able to distinguish between the meanings of the two statements, "All balls are red," and "Some balls are red." In mathematics, the word *some* means "there is at least one."

Terms such as *and, or, if . . . , then,* and *not* are important logical terms and should be informally introduced in the elementary grades. To see why in mathematics it is important to agree on the meanings of these terms, examine the multiple meanings of those words as they are used in the English language. Pupils need to distinguish between numbers that are both odd *and* prime and numbers that are either odd *or* prime and to recognize that *or* does not exclude *and.* The meaning of the "if . . . , then" sentence can be analyzed in simple situations. For example, in the sentence, "If Helen goes to the movies, then Jim goes also," we note that the statement is false only if Helen went to the movies and Jim did not go. In the upper elementary grades, the "if . . . , then" logic can be extended further to geometric applications such as in testing the accuracy of the statement, "If a plane figure has four congruent sides, then it is a square."

Everyone should come to realize that mathematics is concerned with establishing the truth or falsity of sentences like "If *A*, then *B*." It is only when such a sentence is used that the pupil is concerned with the truth of *A*. The main deductive scheme of logic states that the truth of *B* follows from the knowledge that *both* "If *A*, then *B*" and "*A*" are true. Unless these two separate truths are established, the truth of *B* cannot be deduced from this argument.

It is very important for each pupil to learn how to express the negation of a statement. For example, the negation of the statement, "All isosceles triangles are equilateral," is "There exists a triangle which is isosceles but is not equilateral." The negation of the statement, "There is a largest whole number," is "For every whole number, *N*, there is a whole number which is greater than *N*."

Elementary schoolchildren seem to enjoy simple syllogistic arguments. These arguments may be given in humanistic situations, such as:

> A girl who is elected "Queen of the May" is a pretty girl.
> Lana is this year's "Queen of the May."
> Therefore, Lana is a pretty girl.

Other examples should be offered in mathematical context:

> Prime numbers larger than 3 are odd numbers.
> The number 41 is a prime number.
> Therefore, 41 is an odd number.

In addition, these syllogisms should be accompanied by the appropriate Venn diagrams. For example, in Illustration 8-4 the position of the dot labeled "41" shows that 41 is an odd prime greater than 3, while that of the dot labeled "24" shows that 24 is not an odd prime.

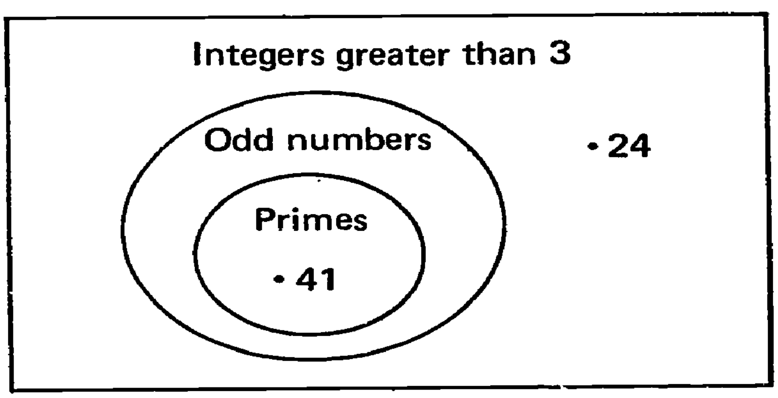



**Illustration 8-4**

Logic is also concerned with developing other types of patterns for drawing valid inferences. Simple inference schemes should be informally introduced and kept at a level that makes sense to pupils. For example, what conclusion or conclusions could be drawn from each of the following situations?

1. John is the tallest boy in his class. John and Pete are in the same class.
2. Three pieces of chalk are put into two empty containers while you are not looking.
3. If $\frac{2}{3}$ of a number is 6, then $\frac{1}{3}$ of the number will be $\frac{1}{2}$ of 6, or 3. What is the number? (Many other deductive reasoning situations similar to this one occur in the arithmetic program.)

The development of logical thinking is an integral part of the instruction in all phases of the elementary mathematics program. A separate and distinct treatment of logic or logical thinking is not intended at this level. The teachers' editions of the textbooks should point out specific situations in which the teacher can extend the lesson beyond the mere acquisition of facts and skills into the level of logical thinking.

Chapter Sixteen

# STRAND 9. PROBLEM SOLVING

Mathematics exists to pose and to solve problems – all kinds. Problems come from the "real" world and from the "ideal" world of mathematics. Some problems have profound implications, others are merely riddles; some problems have "solutions," others only lead to more problems. Here the long view is taken that a "problem" is the articulation of something that requires analysis before understanding. Of course, this criterion will vary from person to person. Because analysis is required and because analysis is personal, understanding does not come without some burst of invention within the mind. How can this creative mental process be facilitated for elementary school children?

## The Concern in Strand 9

In this strand some methods for facilitating understanding are offered. It is important to differentiate this strand from Strand 4, Applications of Mathematics. In that strand we were concerned with problems arising outside the domain of mathematics and with the construction of mathematical models. Pupils need many experiences in applying mathematics to situations that arise in other fields of study; hence, the inclusion of Strand 4. However, pupils need the technical ability to solve the mathematical models that have been constructed. They also need to solve the mathematical problems generated within mathematics itself. Finally, they need experiences with problems that are contrived to build proficiencies in mathematics itself and to develop ability in problem solving. These aspects are emphasized in this strand, Problem Solving.

The ability to formulate meaningful problems has more value in the marketplace than just the ability to solve problems. It cannot be emphasized too often that each attempt on the part of a pupil to formulate a problem should be encouraged. A high point of any teacher's career comes when his pupil asks a question the teacher has not previously considered. Of course, it is a subtle task to be sure that a question has been reasonably formulated. There will be no attempt to define a "reasonably formulated" question; certainly pupils should be given considerable experience in formulating questions. Do not insist that all irrelevant data be trimmed away; indeed, until a solution is obtained, exactly what is relevant is unknown. Even data judged irrelevant can suddenly lead to a new insight or method in solving the problem itself.

Most elementary textbooks do not contain real problems. They do contain "verbal" problems, which often are thinly disguised computational exercises ("John has four marbles. Dick has six marbles. How many do they have together?"). Exercises of this type are boring for the learner who can read, and the difficulty that a slow reader has may have little to do with mathematical concepts. More problems should be included which require pupils to explore, analyze, and investigate – problems that are open-ended in the sense that they invite conjectures and generalizations. Several examples of this type of problem are included at the end of this strand.

## Problem Solving Strategy and Tactics

The basic objectives of the mathematics program should be to provide pupils with opportunities for problem solving and to assist them in devising means for attacking problems with an expectation of success. In this strand a distinction is made between problem-solving *strategy* and problem-solving *tactics*. Strategy means a general plan of attack. Tactic means a single technique which will help with a part of the problem. This distinction is, of course, a relative matter; a strategy in one grade may well be a tactic in another. For example, regrouping for easy addition might be a strategy in the early grades, but a tactic in the upper grades when it is used to simplify mathematical sentences involved in a more complex mathematical analysis.

The strategic principles which are presented here should be expanded in the teacher's manual and introduced to each pupil. It is very important that these *not* be made a format to be followed or become a "step-by-step" procedure to which all solutions must conform. It cannot be emphasized enough that in problem solving, a creative method of solution is more valuable than a routine answer. At the same time, if a pupil has flashed to a correct answer, his thought process should not be hindered by having to conform to unnecessary formalism.

### Understanding the Problem

A first step in any strategy is to make sure the problem is understood. Regardless of the origin of the problem, the solver must understand it so well that he can restate it in his own words. He should be able to identify given data or the hypotheses and to pinpoint the object of the problem. At this stage it is sometimes useful to guess an answer and try it out.

This first step is one which has proved to be the toughest for the beginner. It is upon the *first* reading of the problem that despair sets in. Here the teacher must begin to build confidence. Anything the pupil says can be helpful, even repeating the statement verbatim. Every pupil should be assured that any comment, even the seemingly prosaic or trivial, is a step in the right direction. Indeed, good teachers have found that no comment is really trivial or prosaic; each provides feedback to the teacher as to how the pupil has heard the problem. With such information a teacher can better help his pupil. The teacher can stimulate discussion by suggesting information, techniques, experiences, skills, or procedures that are already known that might help the solver in his particular problem.

Several tactics are available to the solver at this point. He may construct a diagram or picture of what is required. (Children in the early grades are fond of inserting many superfluous details in such pictures. These should be tolerated, for they help to point out what a bare bones skeleton the mathematical model really is.) He may construct a physical model or use materials to simulate the problem; the use of aids should be encouraged. As practice and skill increase, the pupil should be led to the creation of models which concisely present the essence of the problem. The pupil may also find the construction of a graph a handy medium to describe various features of the problem.

*Example:* Janet has seven cents. She goes to the store to buy one candy cane that costs one cent and as many jumbo packages of bubble gum as she can. If each package of bubble gum costs two cents, how many can she buy?

The pupil might draw a picture such as the one given in Illustration 9-1. This problem might be further analyzed in a series of pictures (see Illustration 9-2).

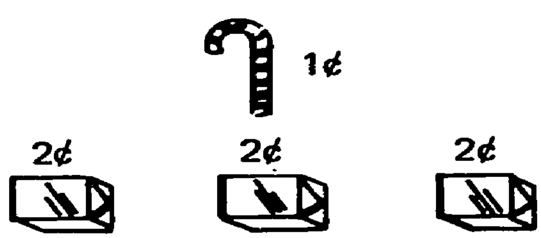



**Illustration 9-1**

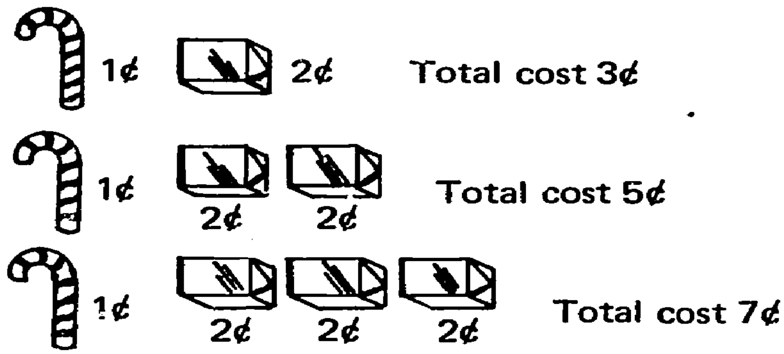



**Illustration 9-2**

The tactic of making a picture of any mathematical situation might well become the habitually accepted thing to do. This habit can be established in pupils by imitation of the teacher who will always do this or often suggest to the class that they do it together. The tactic should be started with the first mathematical experiences, when pupils are learning concepts and skills, and be carried on to more complex problems requiring intricate diagrams and graphs.

**Expressing the Problem in Mathematical Language**

Once the problem is understood, a next step might be to translate the statements of the problem into appropriate mathematical language and symbolism. This step is a crucial one, for in it the conditions of the problem must be made explicit and correct. This step in problem solving is essentially that of constructing a mathematical model of the problem. The process of model building was discussed in Strand 4, Applications of Mathematics.

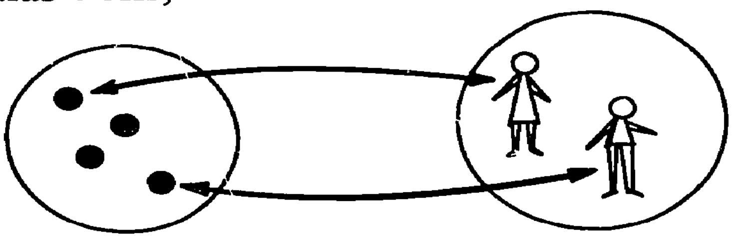
Each of the preceding eight strands provides tools for this part of problem solving. Each strand has something to say about a language for expressing and relating mathematical concepts.

A statement expressing relations between mathematical concepts is called a mathematical sentence. Such a sentence often consists principally of symbols.

*Examples:*

1. Solve: $2\square + 1 = 7$.
2. Solve: $2\square + 1 = \triangle$.
3. Angle $ABC$ is a right angle, since $AB \perp CD$.
4. The area (of a rectangle) $= L \cdot W$.
5. If $x^2 \leq 4$, then $-2 \leq x \leq 2$.

One of the tactics that teachers can help children master is the translation of English sentences into mathematical sentences and, conversely, the interpretation of mathematical sentences as English sentences. For example, the English sentence, "If Billy has four marbles and Jane has two dolls, then Billy has more marbles than Jane has dolls," could be translated variously as:

1. 
2. $4 > 2$
3. $4 = 2 + n$

The mathematical sentence, $4 > 2$, might be translated into many English sentences, such as "A car with four wheels has more wheels than a bicycle with only two wheels."

Complex mathematical sentences should be dissected so that the student comes to learn what each part of the sentence means. Such discussions will point out the different roles played by symbols that indicate operations or relations:

$+, -, \times, \div, <, \leq, \perp, \parallel, \sim$

and symbols that denote numbers, points, or lines:

$x, a, \square, \triangle, \diamond, P.$

These discussions will also bring out the need for the use of parentheses and the role of the associative law. For example, compare $6 - (3 - 2)$ with $(6 - 3) - 2$. The notion of equivalent mathematical sentences will arise naturally when free rein is given to pupils to construct mathematical statements corresponding to English ones. At this point the teacher may introduce the concept of a *solution set.*

Use of mathematical sentences may assist a pupil to make generalizations in the early stages of his mathematical education. If he observes that $0 + 1 = 1$, $0 + 2 = 2$, $0 + 3 = 3$, and so on, he should be able to generalize that for every whole number $n$, $0 + n = n$.

A single mathematical sentence will often unify different problem situations. For example, the so-called three cases of percent can be united in the single sentence:

$$\frac{a}{100} = \frac{c}{b},$$

which relates the percent, $a$, the base, $b$, and the percentage, $c$.

Thinking through a problem situation may result in the formulation of a mathematical sentence. When a pupil learns to describe a problem in his own words and then to state the relationships by use of a mathematical model in sentence form, he has a sound approach to problem solving. The process of building a mathematical model should not be mechanical but should lead to a deep understanding of the mathematical process. Pupils who understand and use this approach to problem solving seldom ask, "Do I multiply or divide?" The pupil should establish his own procedural steps in problem solving through his ability to verbalize what he did in any given situation. He should not be restricted to following a prescribed set of steps. Allow him to think for himself, permit him to get into situations that result in failure, ask him questions that lead him to redirect his thinking, and eventually he will attain a solution. Then ask the solver to describe the situation to others. The teacher is always there to listen.

It is important to notice that different situations may lead to the same mathematical sentence. These translation projects provide good opportunities for the class and the teacher to think out loud together. The problem about Janet and her seven cents leads to the mathematical sentence, $2x + 1 = 7$. The following problems also lead to the same sentence:

1. Hale High School has played seven football games. One game ended in a tie. If Hale won half of the other games, how many did they win?
2. Jim lives seven blocks from school. He met Bill one block from school. When Jim met Bill, Jim had walked twice as far as Bill. How far does Bill live from school?
3. There are seven children in the Hardy family. Tom is the only boy. Each of the girls has a twin sister. How many pairs of twins are there in the Hardy family?

There are, of course, many equivalent ways to formulate a mathematical sentence associated with a specific problem. Some alternate and equally natural ways to formulate a mathematical sentence associated with the above problems are:

For problem 1: $7 = 1 + x + x$

For problem 2: $7 - 1 = 2x$

For problem 3: $x = \dfrac{7 - 1}{2}$

The following system of sentences, which correspond to the trial and error method of solving the candy cane — bubble gum version of the problem, is exactly the sequence a pupil in the fourth grade followed to work the problem verbally:

$$7 - 1 = 6 > 2$$

$$7 - 1 = 6 > 2 + 2$$

$$7 - 1 = 6 = 2 + 2 + 2$$

**Analyzing the Problem**

A later step in a strategy may concern the *analysis* of the mathematical model or of the mathematical sentences which express the hypotheses of the problems. The need for this analysis may motivate the discussion of new mathematical principles. Some helpful tactics include reformulating the conditions to be analyzed, studying associated functions, considering extreme cases for any of the parameters involved, or creating new aids such as graphs, models, or pictures.

*Example:* Suppose the mathematical condition required a solution for $2x + 1 = 7$. We could:

1. Try various numbers for $x$.

2. Solve a similar, but easier, equation: Replace $2x$ by $y$, solve $y + 1 = 7$, and finally, note that $2x = y$, or $x = \frac{1}{2}y$.

3. Write an equivalent sentence: $2x = 6$.

4. Make a graph (Illustration 9-3) of $2x + 1 = y$ from the information obtained in (1).

5. Change the sentence to $0 \leq 2x + 1 \leq 7$. Do this from the graph in (4).

6. Change the sentence to $0 \leq 2x + 1 \leq 7$. Do graphically on the number line.

7. Change the sentence to $2x = 7$ or $2x - 1 = 7$. What are the effects of these changes on the graph shown in (4)?

The information we can read from the graph is, "If $1 \leq 2x + 1 \leq 7$, then $0 \leq x \leq 3$."

**Interpreting the Answer**

A final part of a strategy is the interpretation of the answer. An answer must seem reasonable, and a necessary part of this interpretation is "checking your answer." It is important to differentiate between verifying the *accuracy* of the problem analysis and the computational work, and checking the solution because some *logical* step in either the model building itself or in the analysis (e.g., the introduction of extraneous roots) requires it. An accuracy check is carried out because of a personal desire on the pa_t of the solver (or his teacher), but a logical check is a part of the problem itself.

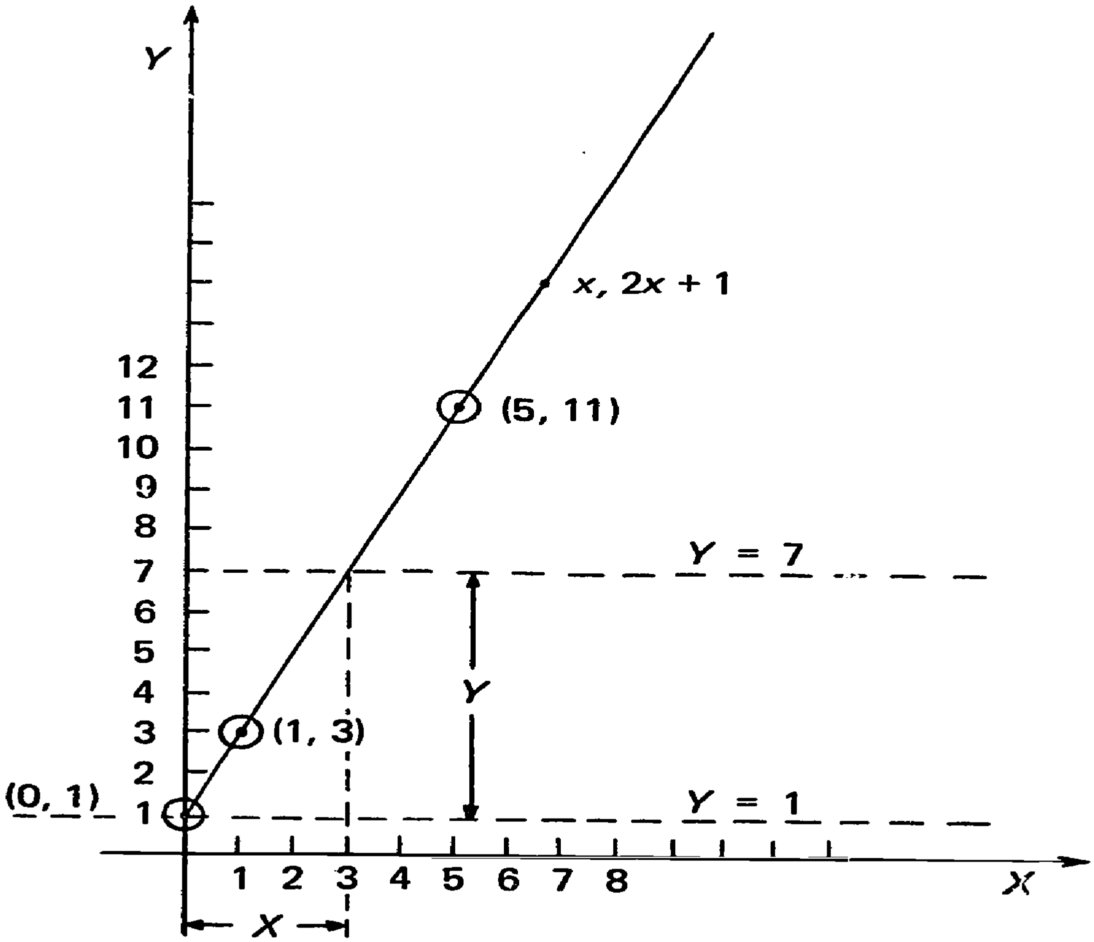



**Illustration 9-3**

## Need for Basic Skills in Problem Solving

In summary, the pupils must be helped to acquire the basic skills necessary to make the kinds of analyses which will lead to a profitable attack on a problem. The teacher should help the pupils to recognize that a problem exists; the pupils will find solutions with a minimum of interference in their experimentation. Necessary arithmetic techniques can be introduced as they are needed, but gathering information for analysis is the pupil's task. The pupils may devise graphs, flow charts, and other visual aids to help them in visualizing problems and sometimes their solutions as well. Conditional statements should be used to a much greater degree than they have been in the past. The logical developments to be expected by self-questioning should be investigated and used. When number manipulation is required, it should be done carefully and the result investigated. In most cases numerical results should be anticipated by estimating in advance. This technique should be introduced early in the mathematical experience of all pupils and should become second nature to them. Possibly situations will arise, and they definitely can be contrived, in which an estimate will reveal an error in analysis or translation before any numerical manipulation is started. Results should never be accepted without testing. Many times the post-discussion of a problem is as valuable as the preliminary approach, and opportunities for teaching from such discussion should not be overlooked.

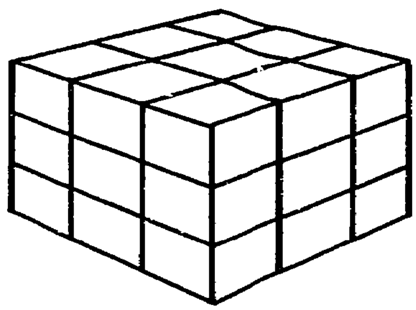
In the course of the discussions, both at the time of problem analysis and during solution analysis, the teacher must be receptive to all pupil suggestions and comments. Rejection of suggestions at any time may defeat the purpose of the discussion, at least for the individual pupil, and could lead to abandonment of the particular problem and possibly a regression in the whole problem-solving technique. A pupil's self-image must be maintained at a level acceptable both to him and to the group if discussion is to be kept profitable.

## Chapter Seventeen

# EXAMPLES OF OPEN-ENDED PROBLEMS

**Problem 1**

Assemble a number of 1-inch cubes into a 3-inch cube and paint the surface area. Now find out how many of the 1-inch cubes are painted on one side, how many on two sides, and so on.



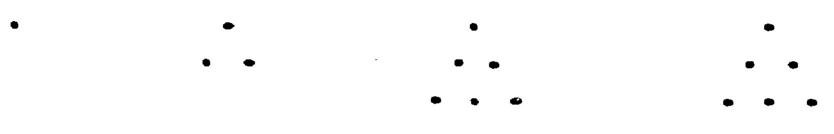
Complete the following chart:

| Number of 1-inch cubes painted on | | | | | | Volume |
|---|---|---|---|---|---|---|
| No sides | One side | Two sides | Three sides | Four sides | More than four sides | Number of 1-inch cubes |
| | | | | | | |

Solution: 1 6 12 8 0 0 27

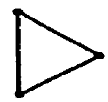
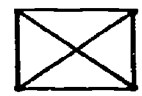
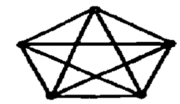
**Problem 2**

Triangle numbers are those that can be shown by dots, for example, arranged in triangular form. The first four triangle numbers are one, three, six, and ten. Find the first ten triangle numbers.

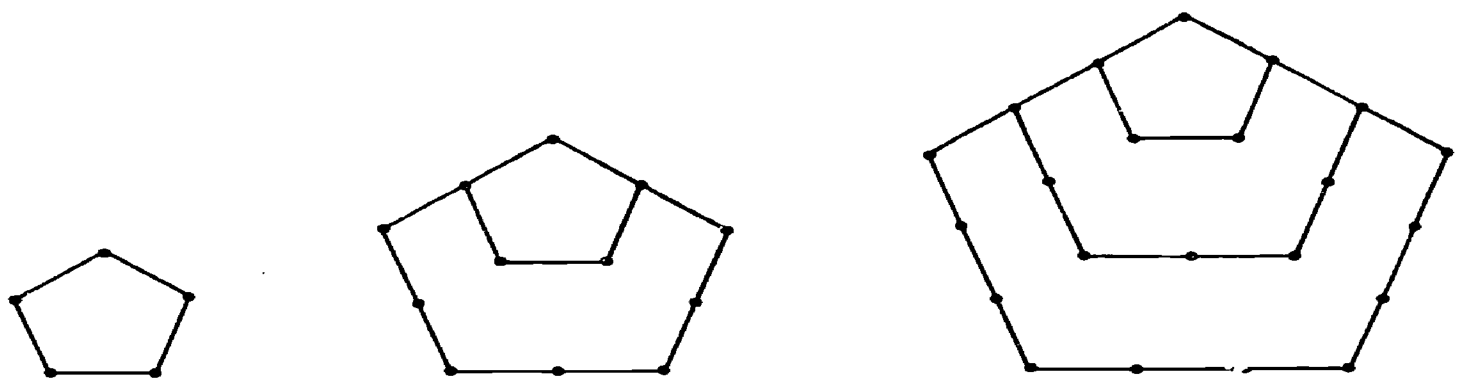


1 3 6 10

The triangle numbers are found in many patterns; for example, the number of line segments connecting two, three, four, and five noncolinear points, and so on.

| Figure |  |  |  |  |
|---|---|---|---|---|
| Number of points | 2 | 3 | 4 | 5 |
| Number of connections | 1 | 3 | 6 | 10 |

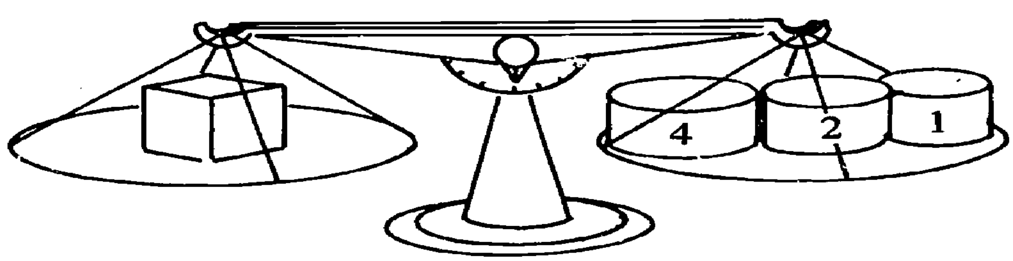
Similarly, there are many patterns for square and pentagonal numbers. Here are the first three pentagonal numbers. What are the first six pentagonal numbers?



5 12 22

**Problem 3**

Richard has a balance and a set of 1-, 2-, 4-, and 8-pound weights. He puts three of the weights on one side of the balance and a box on the other. The diagram shows that the two sides balance exactly. If the weights on one side total 7 pounds, this shows that the box weighs 7 pounds also. The equation for this is $7 = 4 + 2 + 1$.



Write the equations that show how boxes of 1, 2, 3, 4, . . ., 15 pounds can be weighed using the balance and different combinations of weights. As an alternate problem, use weights of 1, 3, 9, 27, . . ., pounds. These problems use base 2 and base 3 respectively.

**Problem 4**

Jane has 17 coins that total $1.00. What might they be? One solution is two quarters, three dimes, two nickels, and ten pennies. What is another solution?

**Problem 5**

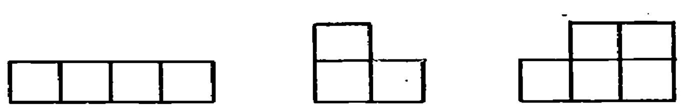
Bill has 16 cents. What coins may he have? Complete the chart showing all possible solutions.

*Solution:*

| 10 cents | 5 cents | 1 cent |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 6 |
| 0 | 3 | 1 |
| 0 | 2 | 6 |
| 0 | 1 | 11 |
| 0 | 0 | 16 |

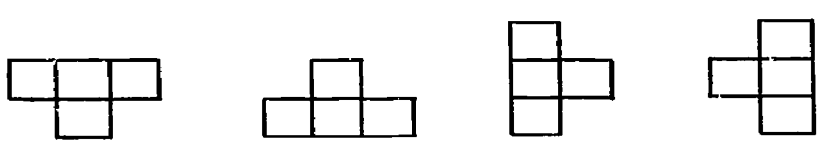
**Problem 6**

Connected squares are squares having at least one side congruent to another square. The following squares are connected sets:



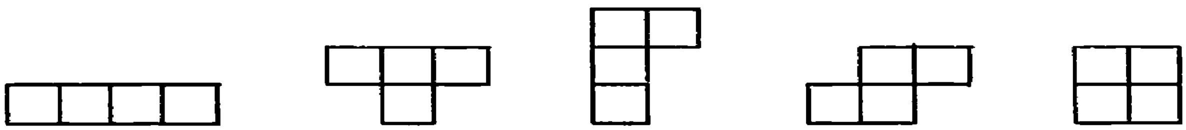
The following squares are *not* connected sets:

Note that the following connected squares are all the same set, placed in different positions:



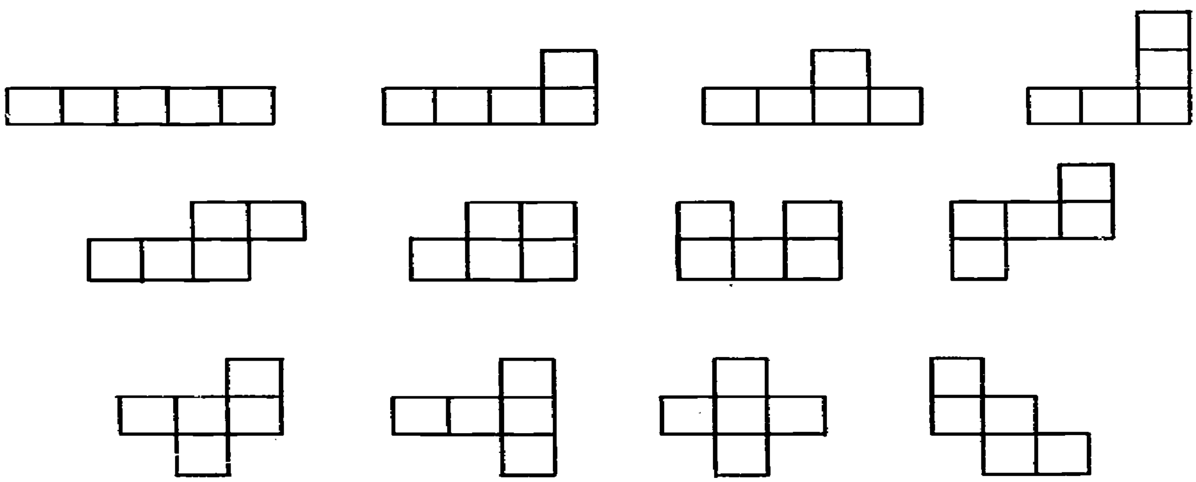
a. On graph paper ruled in 1-inch squares, draw all possible connected sets of four squares. (It may be helpful to cut out 1-inch squares with scissors and experiment by arranging the sets of four squares in different positions.)

*Solution:*



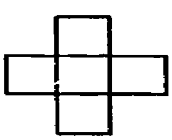
b. Similarly, draw all possible connected sets of five squares.

*Solution:*

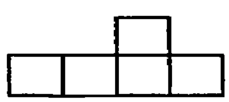


c. Cut out the following sets of five squares from graph paper. Which can be folded into an open box?

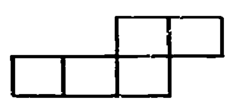
A



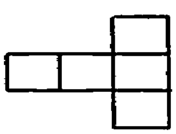
B



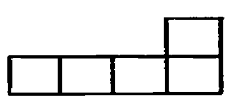
C



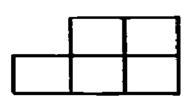
D



E

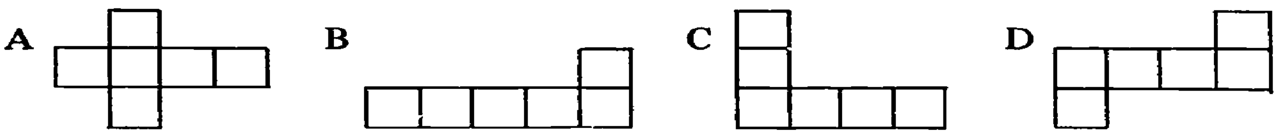


F



*Solution:* A, B, C, D, and E

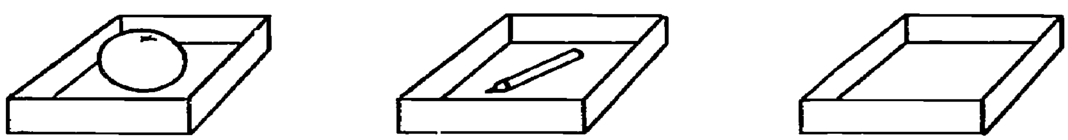
d. Cut out the following sets of six squares. Which sets can be folded into cubes?




*Solution:* A and D

**Problem 7**

Sam had an orange and a pencil and three empty boxes. He put the orange and the pencil in two of the boxes as shown in the illustration. Jim took the pencil and put it in the same box as the orange. Make a chart showing the ways Jim and Sam put the articles in the boxes, and work out the equation for each. Complete the chart, showing all possible ways to put two articles in three boxes. (Ignore the fact that the articles are different and consider only the numbers 1, 2, or 0.)



Box A Box B Box C

*Solution:*

| | Box A | Box B | Box C | Equation |
|---|---|---|---|---|
| Sam | 1 | 1 | 0 | 1 + 1 + 0 = 2 |
| Jim | 2 | 0 | 0 | 2 + 0 + 0 = 2 |
| | 0 | 1 | 1 | 0 + 1 + 1 = 2 |
| | 1 | 0 | 1 | 1 + 0 + 1 = 2 |
| | 0 | 2 | 0 | 0 + 2 + 0 = 2 |
| | 0 | 0 | 2 | 0 + 0 + 2 = 2 |

Work out the number of arrangements possible, using more boxes and more objects to put in them. (This type of problem is important in probability and number theory.)

## Chapter Eighteen

# ALGEBRA FOR GRADE EIGHT

An algebra course for grade eight should appear as a natural outgrowth of the concepts presented in strands 1 through 9. For this reason we need primarily to list topics and prescribe proficiencies which should be achieved. This section of the publication identifies ways of looking at traditional topics in algebra in which the primary concerns are the basic mathematical concepts.

One of the immediate goals for a first course in algebra is a study of the important properties of the real number system which pertain to and make possible the solution of equations and inequalities. While this is essentially an extension of Strand 1, Numbers and Operations, solutions must be approached through the avenues of functions and be related to geometry by graphing on the real number line and in the real number plane. Meaningful applications must be presented to motivate the use of the class of equations considered.

It is also appropriate in a first course in algebra to review the important field properties of the rational numbers. (See page 34 for a list of the properties.) A useful discussion can be centered on the set of numbers: $\{a + b\sqrt{2}: a \text{ and } b \text{ rational}\}$. This set of numbers is closed under addition and multiplication and satisfies all of the properties.

### Increased Emphasis on Functions

A first course in algebra should contain an increased emphasis on functions. What follows appears to be a promising approach. For eighth grade students, a concrete approach should be made through the use of graphs; by this means, pupils develop an intuitive grasp of the algebraic structure of functions. A mathematics framework for grades nine through twelve is a projected task of the Statewide Mathematics Advisory Committee. If that task is assigned, the Committee would obviously give additional attention to the first course in algebra.

The set of functions studied in this first course in algebra is almost exclusively real valued functions. It is appropriate at this point to give a careful definition of a real valued function from the point of view of ordered pairs of real numbers, to identify the domain and range of a function, and to stress the mapping concept associated with a function. Processes of constructing new functions from old ones should be identified. These processes should be linked with techniques of graphing. The important operations are:

*Addition:* If $f$ and $g$ are functions, the function $(f + g)$ is defined by $(f + g)(x) = f(x) + g(x)$.

*Multiplication:* If $f$ and $g$ are functions, the function $(f \cdot g)$ is defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.

*Composition:* If $f$ and $g$ are functions, the function $(f \circ g)$ is defined by $(f \circ g)(x) = f[g(x)]$.

These definitions do, of course, require some restrictions on the domain and range of the functions. For addition and multiplication, the functions $f$ and $g$ must have a common domain. The domain of $(f + g)$ and $(f \cdot g)$ is the intersection of the domains of $f$ and $g$. For the composition $(f \circ g)$ to have meaning, the range of $g$ must be contained in the domain of $f$. For this first introduction, however, mathematical correctness can be preserved with little loss in utility if only those functions are considered whose domain is the set of all real numbers, or, at least, the set of positive real numbers. Sufficient examples should be given to show that composition is not commutative.

**Example**

If $f: x \to x^2$ and $g: x \to x + 2$,

then

$f \circ g: x \to (x + 2)^2$,

while

$g \circ f: x \to x^2 + 2$.

Thus,

$f \circ g \neq g \circ f$.

As soon as composition as an operation on functions is mentioned, the concept of an inverse for a function should be introduced: If $f$ is a function, a function $g$ is called the *inverse* of $f$ if $g \circ f$ is the identity function; that is, $g \circ f\ (x) = x$ for all $x$ in the domain of $f$. The inverse function must, of course, be clearly differentiated from the reciprocal function. The simple relation between the graph of a function and the graph of its inverse should be emphasized.

**Example**

If $f$ is the function $f(x) = x + 2$, then the inverse of $f$ is the function $g$ defined by $g(x) = x - 2$, and the reciprocal function is the function $h$ defined by

$$h(x) = \frac{1}{x + 2}.$$

Thus, if $g$ is the inverse function for $f$, then $g \circ f$ is the identity function. If $h$ is the reciprocal function for $f$, then $f \cdot h$ is the constant function 1.

## Study of Linear Functions

A first study can be restricted to the class of linear functions. These are functions of the form $f: x \rightarrow ax + b$ or $f(x) = ax + b$. Many applications of mathematics lead to linear functions. Problems involving rate, time, and distance are in this category.

Many functions can be reexpressed as (linear) functions in this compact form. The traditional problems of collecting "likes with likes" are included here.

**Example**

The following functions are all linear:

$$f(x) = 2x + 5 - x$$

$$g(x) = 6 - \frac{7}{2} + \frac{3}{4}x$$

$$h(x) = (x - 1)(x + 1) - x^2 + 2x$$

Justification for the steps in the simplification used here should be discussed, and some exercises designed to achieve proficiency in these techniques should be included.

Except for the constant functions (when $a = 0$), these functions have inverses that are also linear. Linear functions are closed under the operations of addition and composition, but not under multiplication. Thus, the mathematical system of linear functions has many properties of the integers under addition.

It is important to relate these functions to their graphs in the number plane. An informal demonstration should be given to show that, based on the geometric concepts of similarity already introduced in the elementary school program, the graph of a linear function is a straight line, and, with the obvious exceptions, the converse is also true. The usual trouble with lines which are parallel to the $y$-axis arises, but from such a discussion it is possible to introduce relations and linear systems of equations and inequalities in two indeterminates.

Linear inequalities should be solved graphically as well as algebraically to provide a display of the results both on the number line and in the number plane.

**Example**

For what values of $x$ is $-2 < 2x + 1 < 2$?

*Graphic solution:*

Graph the function $f\colon x \to 2x + 1$ as in the accompanying figure.

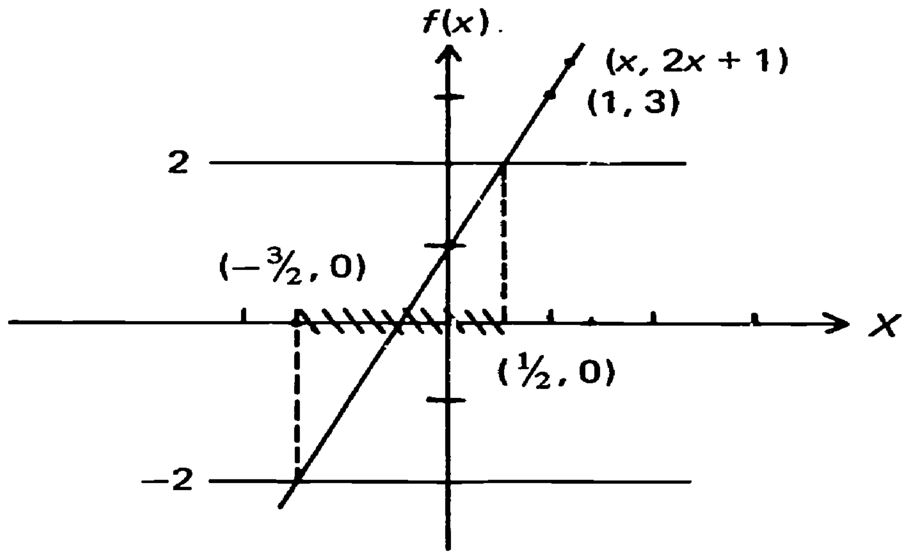



**Fig. 11. Graph of function $x \to 2x + 1$**

Identify the interval $(-2, 2)$ in the range of the function. Find the corresponding interval in the domain. Conclude (see region shown with hash marks) that if $-\frac{3}{2} < x < \frac{1}{2}$, then $-2 < 2x + 1 < 2$.

*Algebraic solution:*

1. Add $(-1)$ to each term of the inequality $-2 < 2x + 1 < 2$. (This is possible because of the rule, if $a < b$, then $a + c < b + c$ for real numbers.) Obtain: $-3 < 2x < 1$.

2. Multiply each term of the inequality $-3 < 2x < 1$ by $\frac{1}{2}$. (This is possible because of the rule that if $a < b$ and $c < 0$, then $ac < bc$ for real numbers.) Obtain: $-\frac{3}{2} < x < \frac{1}{2}$.

Another challenging and useful area of applications of linear equalities and inequalities is that of linear programming.

**Example**

Suppose that a dog's diet must contain at least 4.8 pounds of carbohydrates and six pounds of protein each week. Brand X dog food provides 40 percent carbohydrate and 60 percent protein and costs 25 cents per pound. Brand Y provides 60 percent carbohydrate and 40 percent protein and costs 20 cents per pound. How much of each brand should be purchased each week so that the combined diet will meet the minimum requirements and yet keep the total cost at a minimum? (Assume that there is no upper limit on the total number of pounds that may be consumed, although obviously an amount close to 4.8 + 6 = 10.8 is desirable.)

## Study of Quadratic Functions

A study of quadratic functions should follow the discussion of linear functions. Quadratic functions are those of the form:

$$f: x \rightarrow ax^2 + bx + c.$$

By permitting $a = 0$, we could include the linear functions in this class. We recommend, however, the convention of using the term "quadratic functions" to mean that $a \neq 0$. To recognize which functions are quadratic or to have facility in operating with them, pupils need considerable practice in the manipulation of these expressions in $x$. It is important that pupils study these functions as affected by the parameters $a$ and $c$. Thus, compare:

$$g(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$$

and

$$f(x) = ax^2 + bx + c = a \cdot g(x).$$

Compare also:

$$h(x) = x^2$$

and

$$k(x) = x^2 + c = h(x) + c.$$

We shall, of course, need the discriminant and the quadratic formula to determine the real zeros of quadratic functions. A heavy emphasis should be placed on the process of "completing the square" because that process is more useful than the quadratic formula itself.

For example, given both the functions

$$f(x) = x^2 + 2x + 3$$

and

$$g(x) = (x + 1)^2 + 2,$$

we can easily prove that $f(x) = g(x)$ for all $x$. But it is more important to be able to transform the form of the function

$$x^2 + 2x + 3$$

into the form

$$(x + 1)^2 + 2,$$

and to know that such a procedure will be helpful. Through these techniques every pupil should learn that any quadratic function is essentially like the squaring function $x \to x^2$.

For the class of quadratic and linear functions, it is important to study the operations of addition, multiplication, and composition and the existence of inverses of functions. The comparison of the behavior of all quadratic functions with the function $x \to x^2$ should be related to the effect of a translation of coordinates.

A host of interesting and important applications of quadratic functions can be approached by considering various isoperimetric problems involving rectangles.

**Example**

A boy has some money to buy fencing to build a pen for his pet rabbit. He has decided to build a rectangular pen since this simplifies construction problems. He may build it either within the open area of the yard or along a two-foot stretch of the neighbor's fence. If he builds in the open area of the yard, he may use arbitrary dimensions (except that of course he can't exceed the length of fencing he has money to pay for). If he builds along the neighbor's fence, he can save that two-foot stretch of fencing, but in this case he must build a pen that is only two feet wide. Where and in what shape should he build the pen to make most effective use of his fencing; that is, how can he get the biggest area for his money?

Discussion of this problem will include the extreme value problems of a little and a lot of fencing. What shape fence should the boy build if he builds in the yard? We should conclude that, of all rectangles with a fixed perimeter, the square has the largest area. The discussion will bring out the corollary that the geometric mean is smaller than the arithmetic mean:

$$\sqrt{xy} < \frac{x+y}{2}$$

by comparing rectangles of size $x$ by $y$ and the square of side

$$\frac{x+y}{2}.$$

In connection with quadratic functions, it would be natural to include a brief introduction to complex numbers. If we denote a solution of $x^2 = -1$ by $i$, we may study the set of numbers

$\{a + bi:\ a,b \text{ real}\}$

under the usual definitions for addition and multiplication. It is easy to show that these numbers, like the set

$\{a + b\sqrt{2}:\ a,b \text{ rational}\}$

satisfy the field properties listed on page 34, except for the ordering properties. If the complex numbers are introduced, it could be shown that every equation of the form $ax^2 + bx + c$ where now $a$, $b$, and $c$ may be complex numbers has a solution in the field of complex numbers. The geometric representation of complex numbers as points in the complex plane might be included as well.

## Introduction of Other Functions

In this first course in algebra, the full scope and power of the class of all polynomial functions need not be stressed. In particular, it is not recommended that polynomials be treated as expressions or forms in an indeterminate. All discussions should be kept as concrete as possible.

A class of functions that should be introduced are those involving exponents and radicals. In developing the traditional sequence of functions of the type $f\colon x \to x^n$ where $n$ is a positive integer, the essential difference of the even and odd cases should be examined. It is important to study the behavior of these functions as they depend on the parameter $n$. Of course the standard rules for operating with exponents must be covered.

Next we study functions of the form:

$$f\colon x \to x^{1/n}.$$

A variety of motivations can and should be used to investigate these two classes of functions. One that should be included is that of seeking an inverse for all or part of the functions $f(x) = x^n$ where $n$ is a positive integer. In particular, we follow the convention that $x^{1/2}$ denotes the positive number, if any, whose square is $x$. Thus,

$$(y^2)^{1/2} = |y|.$$

In studying exponential functions, it is important that some time be spent in computing with rational values that "work out nicely." In this way the construction and use of a table of the integral powers of 2 is a useful learning experience.

It is but a short step to fractional exponents; that is, functions of the type:

$$h:\ x \rightarrow x^{p/q},$$

where $p/q$ is a rational number. The law (valid for positive numbers $x$) which states that

$$(x^p)^{1/q} = (x^{1/q})^p$$

may also be viewed as stating a result about the composition of the functions:

$$f: x \rightarrow x^p$$

and

$$g: x \rightarrow x^{1/q}\ (x \geq 0).$$

This result is that $h = f \circ g = g \circ f$.

Finally, negative exponents should be introduced.

There are many subtleties in the treatment of exponents. One concerns the standard laws:

$$a^r a^s = a^{r+s}$$

$$(a^r)^s = a^{rs}$$

$$a^r b^r = (ab)^r.$$

These hold for certain combinations of $a$, $b$, $r$, and $s$, but not for other combinations. They all do hold when $a$ and $b$ are positive. On the other hand, we contrast the standard paradox:

$$-1 = (-1)^1 = (-1)^{2/2} = [(-1)^2]^{1/2} = 1^{1/2} = 1$$

with:

$$-1 = (-1)^1 = (-1)^{3/3} = [(-1)^3]^{1/3} = (-1)^{1/3} = -1.$$

This subtlety must be explored.

Another subtlety concerns inequalities and order. Note that

$\sqrt{x} < x$ if $x > 1$,

but

$x < \sqrt{x}$ if $x < 1$.

For what combinations of $x$ and rational $r$ is $x^r > x$? On the other hand, the exponential functions are monotonic; that is, if $x < y$, then $x^r < y^r$ for positive exponents $r$.

It is desirable that the pupil have an intuitive grasp of the continuity of the functions:

$$f\colon x \to x^n$$

and

$$g\colon x \to x^{1/n}$$

for integral $n$.

Sufficient examples should be provided so that each pupil will get a feel for the fact that, for example, if $a$ and $b$ are close together, then $a^n$ and $b^n$ are close together, and *conversely*. Each pupil should know that each positive real number has an $n^{th}$ root, and he should know some techniques for finding an approximation. Such familiarity is important before tackling such problems as the meaning of $2^{\sqrt{2}}$. Irrational exponents should be discussed but in no great detail. We do recommend that a basis for later work with logarithms be laid. An excellent way to do this is to construct a table of the powers of 10 using estimates like $10^3 \sim 2^{10}$ and so $2 \sim 10^{.3}$. A complete treatment of this approach is given in *Goals for School Mathematics.*[1]

The study of rational functions of the form:

$$r\colon x \to \frac{2x^2 + 3x + 5}{x - 3}$$

or more generally of the form:

[1]*Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics.* Published for the Educational Services, Inc. Boston: Houghton Mifflin Co., 1963 (paperback), pp. 73-76.

$$s: x \to \frac{f(x)}{g(x)}$$

where $f$ and $g$ are polynomial functions, should be undertaken from the point of view of what properties the function inherits from those possessed by $f$ and $g$. Many of the traditional algebraic skills are acquired by drill in simplifying functions of this type. The similarity with rational numbers should be observed in carrying out these manipulations.

The crucial subtlety in dealing with rational functions is the determination of the domain of the function. Teachers should be correct but not pedantic. Students should learn to take care in making combinations, but pedantic artificiality should not pass for mathematical sophistication. For example, suppose there are given two functions, $f$ and $g$:

$$f: x \to \frac{x^2}{x-1} \text{ and } g: x \to \frac{1}{x-1}.$$

For each of these functions, we exclude (1) from the domain. But what of the function $f - g$?

$$f - g: x \; \frac{x^2}{x-1} - \frac{1}{x-1}.$$

We see that $\dfrac{x^2}{x-1} - \dfrac{1}{x-1} = \dfrac{x^2-1}{x-1}$ for all $x \neq 1$.

The last expression is, of course, not reduced to "simplest terms." Do this to obtain $x + 1$. It is true that $f - g: x \to x + 1$ for all $x \neq 1$, but we have no mathematical right to claim that $(f - g)(1) = 2$. The important question is, for what $x$ is:

$$\frac{x^2-1}{x-1} = x + 1\,?$$

This is probably too subtle a question to be considered in a first course in algebra. Certainly, it should not be made an issue, but it should be pointed out that "identities" like the one above hold only for numbers in the domain of the two functions. To do otherwise leads to nonsense like $\frac{0}{0} = 2$.

Some topics from elementary number theory should be included in this algebra syllabus. This subject can provide an appropriate setting for many principles of logical thinking. A nontrivial exercise in logical thinking and number theoretic problems is to show that, to determine whether an integer, for example 1,107, is a prime, it is necessary to check as possible factors only those primes less than or equal to its square root; in this case, the prime integers up to and including 31.

Chapter Nineteen

# GOALS

The following statement of goals for the student is in general terms; refer to the main body of this report for indications of depth and extent of coverage. The objectives can be stated and tested in behavioral terms.

1. *Numbers and Operations*

To use effectively the fundamental operations of arithmetic, computing with fractions and with decimals; to understand and utilize the properties of the operations (commutative property and so forth) and the properties of order and absolute value; and to understand the structure of the several number systems and the special properties of each

To read and understand mathematical sentences involving operations, exponents, and letters, and to formulate and use such sentences in the analysis of mathematical problems

2. *Geometry*

To recognize and use common geometric concepts and configurations; to utilize compass and straightedge for simple constructions; and to understand and to construct simple deductive proofs

To know and use the elementary quantitative geometric techniques, such as measure of angles, area, and volume; to employ the concepts of similarity and congruence in applications such as plans and maps; and to utilize the coordinate plane

3. *Measurement*

To make measurements; to understand the notion of unit of measurement, and to use and interpret various units; to understand the degree of accuracy of an approximate measurement; to estimate measurements and the results of simple calculations involving measurements; and to conceive and use forms of measurement as functions

4. *Applications*

To analyze concrete problems by using an appropriate mathematical model; to employ graphs, scale drawings, sentences, formulas, computations, and reasoning in studying the mathematics of such a model; to interpret mathematical consequences in concrete terms; and to examine the concrete results of such an analysis in terms of reasonableness and accuracy

5. *Statistics and Probability*

To construct and read ordinary graphs

To collect and organize data by means of graphs and tables; to interpret data using concepts describing central tendency, such as mean, median, and mode; and to understand statistical variance as a measure of central tendency

To understand, at a simple level, the idea of sampling, and to interpret and predict from data samples

To understand rudimentary notions of probability theory and of chance events

6. *Sets*

To understand and use routinely the basic set concepts, notations, and operations

7. *Functions and Graphs*

To use the coordinate plane to display relations and to organize data; to recognize and utilize the concept of function, or functional relation; and to use functions and the usual functional notation in analysis and problem solving

8. *Logical Thinking*

To understand, to appreciate, and to use precise statements; to understand and use correctly the simple logical connectives such as *and* and *if . . . , then;* to distinguish, conceptually and in operations, between the "for some" and "for all" quantifiers; and to follow and to construct simple deductive arguments

9. *Problem Solving*

To devise and apply strategies for analysis and solution of problems, and to use estimation and approximation to verify the reasonableness of the outcome

# Appendix

## CRITERIA FOR EVALUATING BASIC AND SUPPLEMENTARY MATERIALS IN MATHEMATICS, KINDERGARTEN AND GRADES ONE THROUGH EIGHT

(Adopted by the California State Curriculum Commission on November 17, 1967, and approved by the State Board of Education on January 12, 1968)

I. Manner of Presentation

A. Pedagogical approaches

1. Textbooks shall facilitate active involvement of pupils and encourage investigation and discovery of mathematical ideas.
2. Task-oriented problems, commensurate with the pupils' maturity, shall be included at all levels.
3. Steps in a process shall be presented in such a way that all pupils can recognize and either explain or demonstrate reasons for them.
4. Some answers shall be provided to permit student checking and prevent reinforcement of errors.
5. Manipulative materials and pictures shall be provided at kindergarten through grade three levels; diagrams, graphs, and tables shall be utilized to develop and clarify ideas at upper levels.
6. Correct mathematical vocabulary, appropriate to the grade level, shall be used, but its use shall not be unduly stressed.
7. Exposition and vocabulary shall be such that difficulty in language and reading does not interfere with pupils's learning of mathematical ideas.
8. Adequate exercise and word problems shall be included for introduction, reinforcement, diagnosis, and reteaching in each area.

B. Organization

1. The strands of mathematics, identified in the *Report of the Statewide Mathematics Advisory Committee, 1967–68,* shall be presented in a spiral organization.

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[1]These criteria formed the basis for the mathematics textbook adoptions of 1969. The 1969 adoptions appeared in California classrooms in the 1970-71 school year. This adoption was for a five-year period.

2. Suggestions for introductory activities related to all the strands shall be included in the teachers guide for kindergarten. The textbooks for succeeding grades shall continue the development of all the strands.
3. The textbooks shall utilize opportunities for the strands to reinforce each other.
4. Ideas and skills previously introduced shall be maintained through appropriately spaced practice.
5. Concepts and skills shall be extended in such a way that pupils relate what they are now learning to prior learnings.
6. Adequate drill materials shall be provided at each step, and problem materials shall be provided to include questions and problems for the students who are above grade level in mathematical competence and experience. Many of these types of material should be expendable at all levels.
7. Mathematical sentences shall be introduced early and shall be employed within all the strands, where appropriate.
8. The vocabulary and symbolism shall be consistent within the texts and in agreement with current usage.
9. Material shall be organized and presented in a way that provides for individual differences.
10. Historical references to the development and uses of mathematics shall be included where appropriate and in a manner consistent with the grade level.
11. Textbooks for grade one through grade three shall preferably be expendable.
12. Textbooks shall include tables of contents, indices, glossaries, and appropriate mathematical tables.

II. Scope of Content—Grade One Through Grade Eight

A. Numbers and operations—The content shall include:

1. One-to-one correspondence between sets as the basis for developing concepts of number, counting, and order
2. The number line and number plane to be used as an aid in development of numbers and operations
3. The operational properties of commutativity, associativity, distributivity, closure, identity element, and inverse element as an integral part of the development of each number system
4. The development of the four binary operations of addition, subtraction, multiplication, and division, and the inverse relationships, and, where appropriate, other interpretations of the operations
5. Stress on the multiplicative structure of numbers, to include factoring and prime numbers

6. Rates, ratio, and percent as special applications of rational numbers
7. The expanding number systems, from the natural numbers through the rational numbers to the real numbers, to meet mathematical needs
8. A thorough development of place value and exponents in the decimal numeration system
9. A brief consideration of other systems of numeration for further understanding of the roles of base and place, the properties of numbers independent of the numeration system, and computational logarithms

B. Geometry—The content shall include:

1. An intuitive, informal development of basic geometric concepts of point, line, angle, plane, space, and various subsets of three-dimensional space.
2. The introduction of geometry in the early grades through the use of objects and figures
3. Classification of plane geometric configurations based upon their properties
4. The introduction of congruence and similarity, including the utilization of scale drawings and maps
5. Experiences in geometric construction with the compass and straightedge.
6. The use of physical models in two- and three-dimensional space, including the construction of such models through drawing, paper folding, and cut-and-paste activities
7. The development of geometric arguments in the form of short chains of deductive reasoning
8. Coordinate geometry, beginning with the plotting of points and graphing of simple linear equations
9. The introduction of geometric terms concurrently with geometric concepts.
10. Equality and inequality relationships in metric geometry involving length, perimeter, volume, area, angular measure
11. An introduction to the Pythagorean theorem and its use in determining simple trigonometric ratios
12. The concept of closed and open curves and the concepts of inside and outside of regions bounded by simple closed curves

C. Measurement—The content shall include:

1. The development of measurement as a series of activities which involve the pupil in the measuring process and which include the use of instruments for measuring
2. An emphasis on data gathering and recording rather than computation and conversion skills

3. Measurement as a function, assigning a number to an object to reflect a property (attribute) of the object
4. Flexibility in the choice of a unit for measuring, with introduction to arbitrary units preceding introduction to standard units
5. Presentation of standard units as a uniform way of communicating measurements
6. Experiences with both English and metric units and their relative measures
7. An understanding of the approximate nature of measurement
8. Development of skills and practice in estimation in all grades
9. The development and use of formulas for determining perimeter, area, and volume
10. Activities involving drawing to scale and map reading
11. Study of precision and accuracy commensurate with increasing depth of understanding by the pupils

D. Applications of mathematics—The content shall include:
1. Applications from the natural and social sciences in the form of problems posed and experiments to be performed
2. Open-ended applications to encourage conjecture and generalization

E. Statistics and probability—The content shall include:
1. Provision for experiences in organizing data into tabulations, charts, and graphs
2. Development of concepts of different measures of central tendency; i.e., average (mean), median, and mode, as aids to interpretation of data
3. The introduction to variance and standard deviation and their usefulness
4. The introduction of predictions from data
5. Elementary concepts of probability as they pertain to "laws of chance" and to related situations

F. Sets—The content shall include:
1. The set concept, to be introduced informally, in the use of one-to-one correspondence as a basis for understanding of numbers, counting, and order
2. The language of sets used when appropriate in the study of geometery, functions, solutions of equations and inequalities, number theory, and graphing
3. Set operations, including union, intersection, difference, and Cartesian product

G. Functions and graphs—The content shall include:
1. The function concept and its generalization, and the relation concept
2. The concept of a function in connection with measurements (area, volume, distance) and experimental inference, and other examples from applied mathematics
3. Construction and reading of graphs representing data from activities of pupils to clarify the concept of function

H. Logical thinking—The content shall include:
1. The "roots" of *informal* logic, to begin in the kindergarten and expand in sophistication throughout the grades
2. The meaning of such words or phrases as "and," "or," "for some," "for every," "all," "some," "if . . . then," "not," "both" as they are used in mathematics, to be developed commensurate with the vocabulary of the pupils.
3. Inductive reasoning, to be employed from the early grades through the use of models, recognition of patterns and relationships, and arriving at generalizations
4. Deductive reasoning, to be introduced and developed with a "light touch" within the vocabulary of the pupils
5. Mathematical sentences, to be introduced early and used consistently throughout the program
6. The concept that equality means that different names are used for the same thing

I. Problem solving—The content shall include:
1. Practice exercises for reinforcement of skills and work problems involving analysis, differentiated from each other in the textbooks
2. Suggestions for various problem-solving strategies and tactics, such as construction of diagrams, elimination of irrelevant material, guessing at reasonable solutions, translation of conditions into mathematical sentences, and so forth
3. Problems which suggest several alternative strategies for solution
4. Open-ended and challenging problems to encourage conjecture, data recording, analysis and discerning of patterns, and making of generalizations

III. Accelerated Program—Algebra (Grade Eight)

A. Organization
1. The system of real numbers shall be studied from the point of view of equations, inequalities, and functions.

2. Definitions, field, order, equality postulates, and proof of equivalence theorems are essential for the development of the mathematical system.
3. The system of complex numbers may be introduced as an extension of the system of real numbers in discussing the solution of the quadratic equation.

B. Scope of content
1. Equations and inequalities—The content shall include:
   a. The study of systems of equations and inequalities by means of concept of solution sets
   b. Graphical and algebraic solutions of equations, including quadratic equations and the role of the discriminant
2. Functions—The content shall include:
   a. The function concept developed as a correspondence (mapping, operations) and as a set of ordered pairs
   b. The systematic use of current terminology, including the concepts of domain and range
   c. The use of the coordinate plane to visualize functions and to discuss zeros of polynomial functions
3. Exponents and radicals—The content shall include:
   a. The introduction of laws of exponents to study integral exponents, including negative integral exponents
   b. Introduction to rational exponents
4. Polynomial functions—The content shall include operations on polynomial functions to parallel the operations of integers
5. Rational functions—The content shall include:
   a. The fundamental algebraic operations on the class of rational functions
   b. Investigation, in solving equations, of the zeros of rational function
6. Sets, mathematical sentences, and logic—The content shall include:
   a. The consistent use of the current terminology and the further development of concepts associated with sets
   b. Mathematical sentences to be used systematically in solving problems of equations and inequalities
   c. The truth function of number sentences and of mathematical sentences, with variables as an aid in understanding
   d. The logical concepts of definition, quantification, and implication (The concepts of converse, inverse, and contrapositive of a statement may be included.)
7. Problem solving and applications—The content shall include:

a. Problems presented in relation to processes, problems designed to develop and maintain skills, and problems to illustrate and develop desired understandings
b. Problems to be arranged in gradually increasing order of difficulty and problems accurate in content and realistic for the age group for which the book is intended
c. Problems which lead to "extraneous roots"

IV. Teachers Edition

A. In the plan for each lesson, the teachers edition shall reflect the pedagogical philosophy of the *Report of the Statewide Mathematics Advisory Committee, 1967–68,* Part I, so that the teacher may serve as guide and facilitator for the pupils.

B. The teachers edition shall provide suggestions for:

1. The use and construction or sources of manipulative materials related to the instructional program in the text
2. The flexible use of textbooks to meet individual needs and ungraded classes, particularly with regard to practice materials
3. Avoidance of rigidity in the use of formal mathematical terminology and accepted forms of answers
4. Development of the understanding that mathematical symbols or notations may have more than one interpretation, definition, or use
5. Ways in which teachers may utilize situations and data from classroom lessons in the natural and social sciences as a source for teaching problem-solving
6. A variety of ways in which a given concept or skill may be presented
7. Development of a kindergarten program in mathematics
8. Extensive use of the number line and the number plane as instructional aids in developing abstract mathematical concepts
9. Evaluating pupil progress at terminal points of various areas of instruction
10. Supplementary activities to develop proficiency, awareness, creativity, and self-confidence in all mathematical areas
11. Differentiation between material to be read independently by pupils and that to be read and discussed with the teacher's guidance

C. The teachers edition shall contain:

1. Adequate information concerning the mathematical background underlying each lesson

2. A listing of the skills or concepts presented at each grade level and the pages where those lessons are found
3. A summary of the scope and sequence of the total program from kindergarten through grade eight in each teachers edition
4. Indices, glossaries, and answers in a form most convenient for teacher use

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