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ABSTRACT

The arithmetic processes of aggregation and averaging are basic to quantitative investigations of employment, unemployment, and related concepts. In explaining these concepts, this report stresses need for accuracy and consistency in measurements, and describes tools for analyzing alternative measures. (BH)

AGGREGATION AND AVERAGING

By

IRVING H. SIEGEL

May 1968

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Foreword

This paper, *Aggregation and Averaging*, by Dr. Irving H. Siegel, of the Washington office of the Upjohn Institute, inaugurates a new series on Methods for Manpower Analysis. The series represents an expansion of the scope of the Institute's research and publication program.

The papers in the series are intended to reflect the state of art and to have tutorial value. They will deal with methods applicable to manpower analysis as well as to methods actually used. They will often take advantage of original research, as Dr. Siegel's paper does.

Harold C. Taylor
Director

Kalamazoo, Michigan
May 1968

Preface

The subject of this paper, the first in the new series on Methods for Manpower Analysis, has been of long interest to the author. It is fundamental to all quantitative investigations of employment, unemployment, and related concepts.

An effort has been made to appeal to the interests and needs of readers at different levels of sophistication. A quotation from *Alice's Adventures in Wonderland* that could have served as an epigraph to this paper guided the selection and presentation of material and references:

> And what an ignorant little girl she'll think me for asking! No, it'll never do to ask: perhaps I shall see it written up somewhere.

Comments from readers are invited so that the value of any subsequent version of this paper to makers and users of manpower measures may be enhanced.

Irving H. Siegel

Washington, D.C.
May 1968

Contents

The W. E. Upjohn Institute for Employment Research

THE INSTITUTE, a privately sponsored nonprofit research organization, was established on July 1, 1945. It is an activity of the W. E. Upjohn Unemployment Trustee Corporation, which was formed in 1932 to administer a fund set aside by the late Dr. W. E. Upjohn for the purpose of carrying on "research into the causes and effects of unemployment and measures for the alleviation of unemployment."

One copy of this bulletin may be obtained without charge from the Institute in Kalamazoo or from the Washington office, 1101 Seventeenth Street, N.W., Washington, D.C., 20036. Additional copies may be obtained from the Kalamazoo office at a cost of 50 cents per copy.

Aggregation and Averaging

I. Scope of Paper

Several reasons may be cited for linking the arithmetic processes of aggregation and averaging. First of all, these processes are basic to manpower measurement. They underlie other, more complex, sequences of numerical operations and, accordingly, are encountered daily and everywhere. They are close mathematical relatives, describe somewhat similar algebraic structures, and yield numbers that are readily convertible into each. They provide complementary, though partial, quantitative descriptions of an ensemble or population of discrete elements. The descriptions are partial in that they focus on but one common property or dimension of the elements at a time. They are complementary in that aggregation views the ensemble as a composite, a whole, while averaging characterizes the ensemble in terms of a representative element.

The goal of aggregation or of averaging is the provision of a single summary magnitude—an *aggregate*[1] in the first case, an *average* (also called a *mean* or *mean value*) in the second. Both processes combine measures for the elements. These measures refer to a common attribute, the one originally selected for an assignment of element numbers or a derived unit introduced by weighting. Measures derived for the elements by weighting are expressed in a unit or dimension that is presumably more appropriate for the purpose of comparison and combination. The final result is a *weighted aggregate* or a *weighted average*. Since aggregates and averages are single figures, they no longer tell anything about the dispersion of the numbers corresponding to the discrete elements.

When two or more properties of each element are of simultaneous interest, their measures have to be reduced to a common denominator

[1] The usage of *aggregate* in this paper differs from that sometimes encountered not only in mathematical literature but also in statistical writings (e.g., in books by A. C. Aitken and W. G. Cochran). In the early vocabulary of "set theory," which the standard treatises of Pierpont and Hobson on real-variable functions helped to propagate, the word was commonly used instead of *ensemble*. Enough other synonyms are available, however, for the latter—e.g., *class*, *group*, *population*, *set*, and *universe*. The prefix *sub* is also applicable to these words for a distinction between the ensemble and a part (larger than a discrete element).

While we are occupied with matters of terminology, we should also note that the ultimate (discrete) *elements* of an ensemble may be called *individuals*, *items*, or *members*. The common *property* in terms of which the elements are quantitatively evaluated in the first instance may also be designated an *attribute*, *characteristic*, *concept*, *quality*, *variable*, or *variate*. The resulting *numbers* are *figures*, *magnitudes*, and *measures*. In their subsequent transformations, the numbers remain expressed on a common *scale*—in a common *denominator*, *dimension*, or *unit*.

before aggregation or averaging can proceed. The required preliminary step may entail weighting or a more formal *scalarization of vectors*. Another possibility is the derivation of conversion coefficients for the several variables from a fitted multivariate regression (or response) function. This method lies outside the domain of the present paper.

This exposition is also circumscribed by two voluntary assumptions—that the elements are numerically describable without errors of observation and that sampling is not required. Other papers in this series will deal explicitly with issues of statistical inference—with the treatment or interpretation of data as samples drawn from real or hypothetical populations. Thus, in modern jargon, the viewpoint adopted here is *deterministic* rather than *stochastic* or *probabilistic*. Since connections with other provinces of analytical interest are too important to overlook, however, the reader of this paper will be reminded later that the statistical estimation of aggregates is a familiar and important topic of sampling theory and practice; and that the various kinds of averages may be viewed as "most probable" values for different "laws of error" or for different least-squares models.

Concentration on "100 percent samples" or on measurement without observational error hardly leaves a methodology paper devoid of significant issues. Matters of concept, definition, and dimensional propriety, for example, are always vital. They cannot be salutarily ignored since bad decisions or loose administration in data gathering may introduce biases for which subsequent compensation is not easy if at all possible. In this connection, it is pertinent to cite the position taken in the 1930's by a well-known statistician on a proposed enumeration of the unemployed; namely, that:

> . . . even a 100 percent sample could not give 5 percent accuracy because of differing ideas regarding definitions of unemployment and the interpretation of the questions. . . . Before it is profitable to talk of reducing sampling error to 5 percent, it would be necessary to reduce both the variability in response (by sharpening the definition) and the error of enumeration to magnitudes comparable with 5 percent accuracy.[2]

Long strides have since been taken in the design and use of official manpower statistics, but impressive nonsampling difficulties persist. Conspicuous gaps in industry coverage remain evident. Concepts and measures that are not strictly compatible have to be used frequently for want of better. Resort must often be had to indirect techniques for measuring concepts that are unclear in the first instance or that can be approximated only crudely at best. Many of these difficulties are re-

[2] Attributed to Frederick F. Stephan by W. E. Deming, *Some Theory of Sampling* (New York: Dover reprint, 1966), p. 39.

flected in hearings and prints of the Joint Economic Committee, in *Economic Reports of the President*, and in numerous other authoritative publications, such as the report issued in 1962 by a Presidential committee.[3]

Confusion may arise occasionally from the availability of more than one series for the "same" manpower concept, but this semblance of duplication is rare and is not necessarily deplorable. The Presidential committee dwelt at some length, for example, on problems engendered by the existence of two nonagricultural employment measures, one derived from a survey of firms and the other from a survey of households. A multiplicity of manpower measures, however, is atypical; and, besides, it can serve well the needs of discriminating analysis. If scarce statistical resources cannot realistically be redistributed in a clearly preferable manner, the remedy for apparent duplication is not the reduction of information but the better instruction of the public concerning the nuances of meaning.

A basic idea of this paper is that alternative choices of data, units, formula, and weights lead to measures for concepts that are cognate rather than the "same." These measures may seem equally eligible if no context or purpose is specified; but they actually are distinct members of a family, have variant meanings, and can differ significantly in magnitude.

Ideally, the use to be made of a manpower measure and a knowledge of the other variables to be measured jointly should govern our choices, but the "prefabricated" data or series that already exist are often the only ones that are practicably available. Limitations in the supply and quality of data and of series handicap analysis and should not be ignored in interpretation. In particular, the myth that existing statistics are "general-purpose" measures should not be taken too literally by the user.

Although this paper makes reference to various manpower concepts for which time series are available, the discussion does not focus on temporal change. Index numbers and manpower projections are principal subjects, instead, for other pamphlets of the Upjohn Institute. The treatment of aggregates and averages here is intended to lay a basis for the expositions of these and other more complex methods for manpower analysis.

II. A Prologue on Measurement

An observation made by John Locke in *An Essay Concerning Human Understanding* (1690) provides a fitting introduction to a paper on basic numerical processes. Concerning "unity or one," Locke remarked that

[3] See report of President's Committee to Appraise Employment and Unemployment Statistics, *Measuring Employment and Unemployment* (Washington: 1962), especially Chapter 4 and Appendix I; and Oskar Morgenstern, *On the Accuracy of Economic Statistics* (2nd ed.; Princeton: Princeton University Press, 1963), Chapter 13.

. . . it is the most intimate to our thoughts, as well as it is, in its agreement to all other things, the most universal idea we have. For number applies itself to men, angels, actions, thoughts,—everything that either doth exist or can be imagined.[4]

Of primary interest here is the applicability of number to "men"—specifically, to the summary description of such manpower concepts as labor force, employment, unemployment, payrolls, hourly earnings, weekly hours, man-hour productivity, and unit labor cost.

Varying degrees of commitment may be discerned in the application of number to manpower (and other) concepts.[5] The weakest degree is the use of number for mere identification and classification—for the differentiation of individuals from each other and for grouping them into more or less homogeneous categories. Thus, distinct serial numbers may be assigned to the employees on a company payroll, to the members of the military services, and to the registrants under the Social Security system. Coding digits, furthermore, may distinguish workers in one department from those in another, officers from enlisted personnel, men from women. A stronger use of number is to rank the members of an ensemble according to magnitude with respect to a common observed or derived property. Thus, with respect to a selected property, we may say that A is greater than ($>$), less than ($<$), or equal to ($=$) B; and that, if $A > B$ and $B > C$, then $A > C$. These relationships involve *symmetry* and *transitivity*, according to the terminology of logic.[6] A third variety of numerical application comes closer to true measurement: It permits the comparison of magnitudes, not as absolute totals, but with regard to differences. A unit that is fixed in meaning or nearly so has to be available for the determination of an excess or deficit.

True measurement is stricter than any of the foregoing applications of number, for it assumes a scale having both an origin or zero and a rigid unit. It assumes that the fundamental operations performable on pure numbers have clearly interpretable or manifestly plausible counterparts in the treatment of our manpower concepts. The operations are addition, subtraction, multiplication, and division (except by zero). In the language of mathematics (especially of set and group theory) and of logic, an *isomorphism*, or structural equivalence, is assumed between the domain of possible magnitudes of a manpower concept and the co-domain of

[4] Book II, Chapter 16, Article 1, of Locke's classic.

[5] This paragraph and the next two take some account of ideas presented by S. S. Stevens, "On the Theory of Scales of Measurement," *in* Arthur Danto and Sidney Morgenbesser, eds., *Philosophy of Science* (New York: Meridian Books, 1960), pp. 141–199; N. R. Campbell, *Foundations of Science: The Philosophy of Theory and Experiment* (New York: Dover reprint, 1957); P. W. Bridgman, *The Way Things Are* (Cambridge: Harvard University Press, 1959), pp. 135–137; and M. R. Cohen and Ernest Nagel, *An Introduction to Logic and Scientific Method* (New York: Harcourt, Brace, 1934), Chapter 15.

[6] Cohen and Nagel (footnote 5), pp. 297–298.

"real" (rational and irrational) numbers. A "one-to-one mapping" or "injection" associates every manpower magnitude with the same number in the co-domain.[7]

For manpower measurement, it is preferable to adopt this strict scale in the first instance and subsequently to temper or to downgrade the implications of numerical operations (if necessary) by appeal to additional information and to common sense. The *ratio scale*, as it is sometimes called,[8] permits one to say not only that *A* is larger or smaller than *B* by a given amount but also how many units each one contains and what the relative magnitudes are. When crude data or techniques of estimation have been used, however, or when computations are subject to severe rounding, it is desirable not to "squeeze the numbers too hard." Judgment is always in order even when it may be out of fashion. The comment cited earlier on the manpower census of the 1930's is also pertinent here.

More than one ratio scale may be of interest in aggregation and averaging. It was noted at the outset that numbers expressed in a conventional or natural unit are assigned to the members of a manpower ensemble; that the original numbers may be converted, by the introduction of weights, to a common denominator deemed more relevant or more stable for the problem under consideration. Employment figures, for example, may be stated originally as numbers of people, but the problem may require translation of such figures into man-hour units. Within the definition of workers or man-hours that is adopted, the aggregation or averaging process strictly implies that any worker or man-hour is equivalent to, and quantitatively exchangeable with, any other. A proper discount may have to be made, however, in interpretation.

Corresponding to the measurement of totals (from zero) and of differences between totals are the notions of *stock* and *flow*. These two terms, occasionally encountered in writings on economic time series,[9] are adaptable to the discussion of manpower aggregates and averages. A stock refers to a status or inventory—to a total quantity that is fixed at a point in time or selected as typical of a period. An example is the number of workers reported by an establishment on Form 790 of the U.S. Bureau of Labor Statistics for the pay period including the 12th

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[7] *Ibid.*, pp. 137–141, on *isomorphism*. The term is also mentioned by Stevens and Bridgman (see footnote 5). Important in advanced mathematics, the concept is treated in standard works on higher algebra (e.g., by Birkhoff and MacLane), on sets and groups, and on matrices. Among the writings consulted in the preparation of this paper were J. A. Green, *Sets and Groups* (London: Routledge & Kegan Paul, 1965) and F. E. Hohn, *Elementary Matrix Algebra* (New York: Macmillan, 1958), especially the appendix on "The General Concept of Isomorphism," pp. 288–290.

[8] Stevens (footnote 5), pp. 147–148.

[9] Among the few modern works using the terms *stock* and *flow* are R. G. D. Allen, *acro-Economic Theory: A Mathematical Treatment* (London: Macmillan, 1967), pp. 2–3.

day of a given calendar month. We may average such stock figures for 12 consecutive months to obtain one stock estimate for characterization of a whole year. A flow represents a (gross or net) change that is recorded during an interval in a real or imaginable stock. An example is the number of man-hours worked during a month—a gross addition to a conceivable initial (zero or positive) stock. Such flow figures may, unlike stock figures, reasonably be combined into an aggregate for an interval of time; thus, an annual total of man-hours worked is a meaningful measure. A change in the number of workers on consecutive payrolls is also a flow—a net difference between two stock figures. Not only are flow figures cumulable but they may also plausibly be averaged —for example, a "typical" monthly man-hour total may be derived for a particular year from 12 monthly figures.

Stock and flow figures of the same genus are connected by a formula. For example:

Number of workers reported for a given month = Number of workers reported for preceding month + Gains − Losses.

The two reported worker totals are status figures or stocks. The gains represent the gross inflow of workers from one pay period to the next; the losses represent the gross worker outflow. The difference between gains and losses is the net flow (plus or minus).

It is a familiar plaint that measurement in the human disciplines lacks the definitiveness apparently achievable in the physical sciences and in the world of objects in general. Such employment units as the worker or man-hour, indeed, lack the stability or homogeneity of the meter (a unit of length), the second (time), and the degree Kelvin (absolute temperature). Another way of stating the situation is that manpower measurement, however precisely accomplished and however refined the unit we choose, still fails (a) to reflect cogently and comprehensively the essence of a multidimensional social phenomenon or (b) to reflect what measurement in terms of some other important property might be expected to show.

In manpower and other aggregation and averaging, it is desirable to distinguish between "literal" and "verbal" algebra.[10] The details of composition and structure of a summary measure are not strictly divorce-

[10] The distinction between literal and verbal algebra has been made by I. H. Siegel in various places—e.g., in "Systems of Algebraically Consistent Index Numbers," *1965 Business and Economic Statistics Section Proceedings of the American Statistical Association*, pp. 369–372; "On the Design of Consistent Output and Input Indexes for Productivity Measurement," in *Output, Input, and Productivity Measurement* (Princeton: Princeton University Press, 1961), pp. 23–41; and *Concepts and Measurement of Production and Productivity* (Washington: U.S. Bureau of Labor Statistics, 1952).

able from the circumstances in which such a measure is to be used. Different formulas and weighting schemes yield results that may be significantly dissimilar. Two sets of numerical assignments in the "same" unit differently defined are not necessarily proportional to each other. Multiple plausible measures may be devised for a family of related concepts, but they are not casually interchangeable. These recognized and unrecognized dangers give importance to literal algebra, which is concerned with the design of a measure to accord with the purpose or context of use. It is also concerned with correct interpretation. It pays due regard, therefore, to the ingredients and manner of construction of whatever measure happens to be used for want of better. Verbal algebra, on the other hand, is content with names and labels. "Any old" aggregate or average that permits a proper "cancellation of words" is uncritically accepted. Among the possible unsatisfactory consequences of verbal algebra are dimensional eccentricity, to which some attention will later be given, and the mistaking of noise for message.

Since the assignment of numbers to concepts is usually far from ideal, attention will also be devoted in this paper to algebraic tools for analysis of the relationship between alternative summary measures. As already noted, concepts, units, formulas, and weights should preferably be related to the purpose or context of measurement, which should also dominate the choice of adjustment procedures for overcoming limitations of available data or series. Furthermore, common sense may dictate a preference for one aggregate or average over another on mere dimensional grounds. Since differences in content, form, and method may significantly affect the numerical results, the maker or user of summary measures should consider the sources, magnitude, and direction of possible divergence.

III. Aggregation

Definition

In brief, aggregation may be described as the *derivation and subsequent combination into a sum of commensurable numbers corresponding to the elements of an ensemble*. The summed numbers are defined on a ratio scale. They are commensurable in that they have a common denominator; they do not literally have to be integers, exact multiples of a prescribed unit.

More explicitly, aggregation entails (a) the assignment of numbers to a common property of the elements of an ensemble; (b) the adjustment of these numbers, if necessary, to overcome limitations of the underlying data or to reflect a refinement of concept; (c) the weighting of the original or adjusted numbers, if necessary, to reduce them to a common denominator that is deemed more homogeneous, more stable, or more relevant to the aim of an exercise; and (d) the summation of the original, adjusted, or weighted figures. The sum is the aggregate, the final result of the process. It measures the *size* of an ensemble with reference to a pertinent observable or derived property.

Aggregates differ according to the original choice of a common attribute of ensemble-elements and the subsequent modes of refinement and weighting. When no weights are introduced, the numbers originally assigned to the elements are, in effect, equally weighted. Sometimes, it is analytically useful to rewrite a weighted or unweighted aggregate in an equivalent expanded form—that is, with "telescoping" weights.

Formulas

Counting is the simplest and most familiar variety of aggregation.[11] Every element of any group has at least the attribute of discreteness, of "oneness." Thus, if a group has n elements, the most obvious aggregate is $1 + \cdots + 1 = \sum^{n} 1 = n \times 1 = n$. (On summation symbols, see Appendix.)

Although n—or any other aggregate measure—usually stands by itself, without a designated unit, it is not really a pure or abstract number. Even in the simplest case, it has an implied unit—e.g., "elements" or "things." It may also represent a sum of people, employees, man-hours, unemployed persons, or payroll dollars—i.e., a sum referring to a manpower characteristic.

When the elements have been grouped into subclasses, a weighted sum of subclass measures may be substituted for a completely fresh count. Each subclass is treatable as a complex element; it has a "oneness," but its content of ultimate elements provides the weighting factor, n_i, needed for a much better determination of the size of the ensemble. The symbol for the sum of elements in the s subclasses of unequal size is $\sum_{i=1}^{s} n_i = n$. Again, a common dimension is implied—elements, things, or some more explicit characteristic, such as number of employees.

When employment is expressed initially in terms of workers and the preferred common unit is man-hours, a transformation of the original numbers is achievable with weights representing hours per worker. Thus, if n_i employees in the i^{th} industry work, say, h_i hours per week and if there are s industries altogether, the corresponding weighted aggregate is $\sum_{i=1}^{s} n_i h_i$. The subscripts may be dropped if no confusion would result: Σhn.

More complex situations are often encountered in manpower ag-

[11] Stevens (footnote 5), p. 147, observes:

Foremost among the ratio scales is the scale of number itself—cardinal number—the scale we use when we count . . . . This scale of the numerosity of aggregates is so basic and so common that it is ordinarily not even mentioned in discussions of measurement.

gregation, and they are easy to handle. Thus, a need may arise to distinguish the different companies within an industry and the different occupations or departments within a company—especially because of dissimilarities with respect to number of workers, hours of work, or some other relevant characteristic, such as hourly remuneration. More summation signs—or more subscripts, at least—then have to be introduced.

Let us consider a specific case involving a fixed number (s) of industries, a variable number (f) of companies in each industry, a variable number (g) of departments in each company, and variable numbers of workers (n) and hours per worker (h) in each department of a company. The expression for total man-hours may then be written very explicitly as:

$$\sum_{i=1}^{s} \sum_{j=1}^{f_i} \sum_{k=1}^{g_{ij}} n_{ijk} h_{ijk}.$$

The symbols direct that we first sum man-hours by department (subscript k) for each company (subscript j), then sum the company results within each industry (subscript i), and finally sum the industry figures into a grand total. The procedure may be visualized simply in a "tree" diagram (see Appendix).

When there is no ambiguity, it is sufficient to write $\Sigma n_{ijk} h_{ijk}$. This stripped version of the expression presented above focuses attention on workers and hours in department "cells." The cells may be identified exhaustively, unequivocally, and without duplication by means of permutations of the industry-company-department subscript numbers. Accordingly, aggregation may be accomplished directly and completely at the department level if we do not need also to have company and industry subtotals.

This is a good place to underscore two points made earlier about alternative aggregate measures for the same ensemble. Obviously, the size of a company as represented by man-hours worked exceeds the size in terms of the number of workers. Second, a percentage distribution of man-hours by company department differs from a distribution based on numbers of workers if hours of work are not uniform.

The next observation provides a bridge to the discussion of averages; but a determined crossing will not be made just yet. If a company's hours of work are uniform from department to department, the measure $\sum_{k=1}^{g} n_k h_k$ simplifies to $\bar{h} \Sigma n_k$, where $\bar{h}$ is the constant companywide figure for hours and Σn_k, of course, is the company's number of workers. If subscripts are dropped and the two expressions are written as an identity, the result is

$$\bar{h} = \frac{\Sigma n \bar{h}}{\Sigma n},$$

which has the form of a weighted average. The expression on the right, of course, is mathematically otiose.

An examination of alternative forms of a *given* aggregate (fixed in dimension and magnitude) also yields valuable insights. It assists in the proper design of averages and of algebraically consistent aggregates. It protects against dimensional impropriety in the construction of summary measures. Consider the fact that at least six different multiplicative verbal identities may be written for a payroll aggregate:

Payroll ≡ Workers × Hours per Worker × Hourly Earnings
≡ Man-hours × Hourly Earnings
≡ Output × Unit Labor Cost
≡ Workers × Hours per Worker × Hourly Productivity × Unit Labor Cost
≡ Man-hours × Hourly Productivity × Unit Labor Cost
≡ Output × Unit Labor Requirements × Hourly Earnings

Accordingly, if sufficiently detailed information is available for the commodity output, labor input, and worker remuneration of a company, industry, or larger sector of the economy, it becomes possible to express the corresponding payroll aggregate as a sum in at least six different ways (subscripts omitted):

$$\Sigma nhe$$
$$\Sigma me$$
$$\Sigma qc$$
$$\Sigma nhpc$$
$$\Sigma mpc$$
$$\Sigma qre$$

The meanings of the italic letters become clear upon comparison of verbal and algebraic equivalents.

Can the payroll aggregate also be expressed in different ways as a product of aggregates and averages for the very same characteristics that enter into the verbal identities? The answer is yes, provided that averages and aggregates are weighted with care; they cannot be of "any old" variety. The problem demands meticulous literal algebra; casual verbal algebra will not do. Dimensional sense imposes an additional constraint. This paper later shows how a payroll identity may be written in terms of appropriate aggregates and averages.

Although sums of unweighted and weighted logarithms have not been discussed here, they are encountered, as well as aggregates involving ordinary numbers, in manpower analysis. When logarithms are summed, the aggregate represents the logarithm of a product; and a weighted sum of logarithms corresponds to the logarithm of the product of numbers that have been raised to powers (the powers are the weights). Logarithmic aggregates, as shown below, are pertinent to the discussion of geometric averages.

Estimation From Samples

Since population aggregates often have to be estimated from sample information, some illustrations are offered. The topic also belongs to the province of survey techniques and accordingly is eligible for treatment in other contributions to this series on Methods for Manpower Analysis.

The usual objective in estimating an aggregate is to obtain a figure that is unbiased and has a tolerably small, if not the least possible, sampling variance. The procedure is unbiased if its "expected" result, in a statistical sense, is the same as the population total. The sampling plan and the estimation procedure have to be coordinated closely if bias is to be avoided, restricted, or compensated, and if the variance is to be kept within acceptable bounds.[12]

Suppose that F companies comprise an industry and that f of them are to be sampled with a view to estimation of total employee man-hours. If all companies have equal probabilities of inclusion in the sample, and if man-hour data are obtained and used for all the workers in a selected company (m_i), then $(F/f)\sum_{i=1}^{f} m_i$ provides an unbiased estimate of total industry man-hours. The "blowup factor," $\frac{F}{f}$, is also called a "sampling ratio," a "weighting factor," or an "expansion ratio."

Other sampling schemes may also yield unbiased estimates. Suppose that in each of f companies selected with equal probability an employee is picked at random. If his hours are h_i', and if his company has n_i employees altogether, an unbiased estimate of total industry man-hours is given by $(F/f)\sum_{i=1}^{f} n_i h_i'$.

If the F companies in the industry have unequal probabilities of selection (p_i), and if f companies are drawn at random, we may design an unbiased estimate of aggregate man-hours for this case also. The estimate is $(1/f)\sum_{i=1}^{f} \frac{m_i}{p_i}$, where m_i represents the man-hours of the i^{th} selected company. It can be shown that, if only one company were to be sampled (i.e., $f = 1$), the variance of the estimated aggregate is reduced to zero when the p_i are proportional to the m_i. Attempts accordingly are made in practice to approximate such p_i values—e.g., on the basis of man-hour figures derived from earlier surveys.

We conclude this section with an illustration from "multistage sampling," a technique largely developed for population surveys but easily adapted to manpower studies in general. A monograph issued in 1947 by the U.S. Bureau of the Census shows the following formula (original

[12] Concerning this paragraph and the next three, see, for example, Deming (footnote 2), pp. 87–99.

symbols) for an unbiased estimate, x', of "the population contained within the [city] area covered by block-sampling":

$$x' = \sum_{i=1}^{R} \frac{M_i}{m_i} \sum_{j=1}^{m_i} \frac{N_{ij}}{n_{ij}} \sum_{k=1}^{n_{ij}} x_{ijk}.$$

Here, the subscript i refers to one of the R strata of the city, j to one of the M city blocks, and k to one of the N dwelling places in the city; the m_i and n_{ij} refer to the sampled blocks and sampled dwelling places, respectively; and the sampling ratios, $\frac{M_i}{m_i}$ and $\frac{N_{ij}}{n_{ij}}$, are assumed to be known. The formula may obviously be adapted to the estimation, for example, of employment in an ensemble of R industries by means of a sample of workers in n_{ij} companies chosen within each of m_i particular industries.[13]

IV. Averaging

Definition

The process of averaging yields a number that is of the same order of magnitude and is expressed in the same unit as the numbers for a common property of the ensemble elements. The element numbers are the ones originally assigned or obtained by a subsequent adjustment, but they do not yet reflect weighting. The effect of any weighting that is introduced to make the element numbers most meaningfully additive for aggregation has to be reversed, in a sense, in the course of averaging.

An average (of positive numbers) is smaller than an aggregate; yet it, too, characterizes an ensemble. Thus, its derivation takes account of every item in a group, although the original element numbers are not retrievable. Furthermore, it is sensitive to the choice of weights and to the structure of the combining formula. Most important is the representativeness of an average from a mathematical standpoint: its substitutability for the number corresponding to each element.

The last point is usually reflected in definitions of averaging and averages. Cognizance is taken of it when averaging is defined as *the process of deriving, from an aggregate measure or from the measures assigned to the elements according to a selected common attribute, a single number that is representative of the elements and is mathematically substitutable for each.* More simply, averaging is *the derivation of a representative number that leaves an aggregate unchanged when it is used in lieu of the measures assigned to all the elements with respect to a selected common property.* Other criteria have also been employed in the definition of averages, but the notion of substitutability serves our need adequately.

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[13] On this paragraph, see *A Chapter in Population Sampling*, by the Sampling Staff of the U.S. Bureau of the Census (Washington: 1947), pp. 16-20.

Formulas

The criterion of mathematical substitution may be stated formally: X is an average of $x_1, \cdots, x_n$ relative to a function, ϕ, of these n measures if $\phi(x_1, \cdots, x_n) = \phi(X, \cdots, X)$.[14] Thus, $X = A$, the *arithmetic mean*, when $x_1 + \cdots + x_n = A + \cdots + A$; for then, $\Sigma x = nA$, so that $A = \Sigma x/n$. The *geometric mean* corresponds to $X = G$ when $x_1 \cdots x_n = G \cdots G$; for then, $\Pi x_i = G^n$, so that $G = \sqrt[n]{\Pi x}$. For the *harmonic mean*, we have

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} = \frac{1}{H} + \cdots + \frac{1}{H}, \quad \text{or} \quad \Sigma(1/x) = n/H, \text{ so that}$$
$$H = n/\Sigma(1/x) = n/\Sigma x^{-1}.$$

(The exponent -1 signifies a reciprocal.)

The second definition of averaging, the simpler statement that emphasizes the invariance of an aggregate to substitution, obviously yields equivalent results. Thus, the replacement of every x_i by $A = \Sigma x/n$ in the aggregate Σx produces no change. If we start with the aggregate $\Sigma \log x$ and substitute $\log G = (1/n) \Sigma \log x$, for every $\log x_i$, we return to $\Sigma \log x$. Similarly, Σx^{-1} is invariant to the substitution of H^{-1} for every x_i^{-1}.

A general formula is available that includes these three classical means (and others too, such as the root mean square) and that meets the substitution criterion. This formula is:

$$X_t = \left(\frac{\Sigma x^t}{n}\right)^{1/t},$$

where t may take any value. When $t = 1$, this expression reduces to A; when t approaches zero, the limiting value of X_t is G; when $t = -1$, the result is H. (The root mean square corresponds to $t = 2$.) If every x_i is replaced by $A = \Sigma x/n$, the generalized expression yields $X_t = A$. The expression yields $X_t = G$ and $X_t = H$ when G and H, respectively, are substituted for every x_i.[15]

A famous inequality for aggregates, due to Hölder, may be modified slightly to refer explicitly to generalized unweighted means. For two sets of variates, x_i and y_i, the relation,

[14] See E. L. Dodd, *Lectures on Probability and Statistics* (Austin: University of Texas Press, 1945), pp. 20, 29, and 40.

[15] The generalized formula is cited by E. V. Huntington, "Mathematical Memoranda," *in* H. L. Rietz, ed., *Handbook of Mathematical Statistics* (Boston: Houghton Mifflin, 1924), p. 6. It is also shown by Milton Abramowitz and Irene F. Stegun, eds., *Handbook of Mathematical Functions*, Applied Mathematical Series, No. 55 (Washington: National Bureau of Standards, 1964), p. 10; and G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge: Cambridge University Press, 1934), pp. 12–13.

$$\frac{\Sigma xy}{n} \leq \left(\frac{\Sigma x^t}{n}\right)^{1/t} \cdot \left(\frac{\Sigma y^{t'}}{n}\right)^{1/t'}, \text{ holds}$$

when $t > 1$, $t' > 1$, and these exponents satisfy the conjugacy condition, $1/t + 1/t' = 1$. The Cauchy-Schwarz inequality is obtained when $t = t' = 2$. When $t = t' = 1$, the expression becomes

$$\frac{\Sigma xy}{n} \gtrless \frac{\Sigma x}{n} \cdot \frac{\Sigma y}{n};$$

and the left member exceeds, is less than, or equals the right member according as the correlation between x_i and y_i is positive, negative, or zero. In particular, when the x_i and the y_i are unequal and similarly ordered (i.e., correspond exactly in rank), the left member must exceed the product of the two unweighted arithmetic means on the right (Chebyshev's inequality). Reference will be made again to Hölder's inequality.[16]

Several remarks on the familiar unweighted averages are in order:

1. All are *internal* means; that is, each average lies between the least and the greatest of the x_i. Externality is a familiar hazard in work with index numbers, as a later paper will show. Furthermore, as will be noted below, the possibility of externality is sometimes built into mathematical "production functions," which usually attempt to relate output to the contributions of manpower and capital. Finally, when aggregates and averages are improperly matched in multiplicative identities, an attempt to adjust the averages (to assure identity) could also lead to externality.

2. It still does not seem to be generally known that an unweighted arithmetic mean may be written as a specially weighted harmonic mean; and, conversely, that a harmonic mean may be viewed as a weighted arithmetic mean. This fact is not only of pedagogic interest but also has analytical value.

3. Since ancient times, it has been known that, when the x_i are unequal (and positive), the arithmetic mean exceeds the geometric mean, which in turn exceeds the harmonic mean. Instances in which this proposition is applicable, however, are not always recognized.

4. When the x_i are equal, all the unweighted means are equal to x_i. This is an extreme case of the substitution criterion.

Before turning to weighted averages, we note two other measures of "central tendency" or "location" that are treated in the first chapters of an elementary statistics text: the *mode* and the *median*. The mode of a frequency distribution or "histogram" is the value of the variate cor-

[16] On Hölder's and Chebyshev's inequalities, see, for example, Hardy, Littlewood, and Polya (preceding footnote), pp. 24–26, 43–44. The generalizability of Hölder's expression to three or more variables is noted by C. R. Rao, *Linear Statistical Inference and Its Applications* (New York: Wiley, 1965), p. 44.

responding to maximum frequency. This measure is most meaningful when only one maximum clearly exists—a case often encountered in manpower statistics. It is not meaningful when distributions exhibit no bunching at all or when two or more major concentrations of frequency are evident. Some authors would distinguish the "absolute mode" from "relative modes" when a multimodal distribution literally has one peak.[17] The median divides the total frequency of a distribution into two equal parts. When the number of values is even rather than odd, the determination of the median involves some arbitrariness. A mathematical generalization is available that embraces the median, mode, and unweighted arithmetic mean.[18]

The notion of substitutability applies to weighted, as well as unweighted averages; and the inequalities that hold for unweighted means also hold for positively weighted ones. The invariance of the weighted aggregate Σwx to a replacement of every x_i by $\Sigma wx/\Sigma w$, the weighted arithmetic mean, is immediately evident. Similarly, the weighted geometric mean, $(\Pi x^w)^{1/\Sigma w}$, may be introduced in lieu of the x_i into Πx^w without any mathematical effect. This replacement is equivalent to the use of $\Sigma w \log x/\Sigma w$ for every log x_i in the aggregate $\Sigma w \log x$. The weighted harmonic average, $\Sigma w/\Sigma(w/x)$, likewise satisfies the substitution criterion.

Even as one formula may be written to embrace the familiar unweighted averages, a generalization exists that subsumes the common weighted varieties and many others:

$$\bar{X}_{w,t} = \left(\frac{\Sigma wx^t}{\Sigma w}\right)^{1/t}$$

Again, the specialization of t is the key to the different means.[19] This formula is of interest in the study of production functions.

A short accompanying table illustrates the variation in mean values achievable for a simple array with different formulas and weights. The numbers being combined are 1, 2, 4, and 8. In one case, no weights (i.e., equal weights) are used. In the second case, the lowest number in the array receives double weight. In the final case, the highest number is doubly weighted. The order of the several means remains unchanged from case to case. Although the table may not reflect the variability to which manpower calculations are actually subject, it does support our view that users, as well as makers, of statistics should give due attention

[17] Bernard Ostle, *Statistics in Research* (2nd ed.; Ames: Iowa State University Press, 1963), pp. 58–59.

[18] Huntington (footnote 15), pp. 6–7, cites Dunham Jackson's elegant expression for the median and a generalization due to Jackson and R. M. Foster.

[19] Hardy, Littlewood, and Polya, *op. cit.*, pp. 13–18. This work cites other averages, such as Muirhead's (pp. 44–45), and refers to the important notion of *convexity*, which illuminates the study of inequalities among averages.

to the choice of formulas and weights and to the analysis of inter-mean differences.

Alternative Means[a]

Mean	*Unweighted*	*Weighted: I*[b]	*Weighted: II*[c]
Harmonic, $t = -1$	2.13	1.74	2.50
Geometric, $t \rightarrow 0$	2.83	2.30	3.48
Arithmetic, $t = 1$	3.75	3.20	4.60
Generalized, $t = 2$	4.61	4.15	5.46
$t = 3$	5.27	4.89	6.03

[a] The numbers being averaged are 1, 2, 4, and 8.

[b] The weights assigned to the four numbers are 2, 1, 1, and 1, respectively.

[c] The weights are 1, 1, 1, and 2.

The progression exhibited by the means for increasing values of t in each of the columns is not an accident of data selection. Higher values of t do, indeed, correspond to higher weighted and unweighted means (for positive x_i and w_i). Statisticians who recognize this theorem may state it in terms of expected values or *moments*.[20]

Dimensional Propriety

In manpower analysis, weighted arithmetic averages are encountered all the time, and weighted harmonic means are often used without being identified as such. If firms or industries have different hours of work, a "logical" average of these hours is of the arithmetic variety and incorporates employment weights. The resulting expression, $\Sigma nh/\Sigma n$, is dimensionally very acceptable; the numerator is expressed in man-hours, a conventionally additive unit, and the denominator is expressed in employment, another conventionally additive unit. Furthermore, verbal algebra, which features the cancellation of words, makes it clear that the formula provides a measure of hours of work. Note that $\Sigma nh/\Sigma n$ may also be written as $\Sigma nh/\Sigma(nh/h)$; it is a "telescoped" version of a harmonic mean of hours of work with man-hour weights. Hence, the harmonic mean is also "logical" for combining hourly figures for different firms or industries, but it has to incorporate suitable weights.

Is it always easy to tell if an average is "logical" for combining the measures of elements with respect to a certain attribute? Yes, two tests are applicable, even though we cannot always implement our preferences. First, unless a context prescribes otherwise, both the numerator and denominator ought to be expressed in additive units. The joint measure-

[20] *Ibid.*, pp. 26–27; and Michel Loève, *Probability Theory* (3rd ed.; Princeton: D. Van Nostrand, 1963), p. 156.

ment of several variables within the context of a verbal identity may sometimes oblige acceptance of some curious aggregates, but eccentrically weighted means should not be sought for their own sake. Second, the ratio must, on performance of the indicated verbal algebra, disclose the property selected for averaging. Suppose, for example, that payroll weights are used in an arithmetic mean of hours of work. The numerator becomes a dimensional mess (Σnh^2e) that no reasonable verbal identity would require; it is not expressed in a meaningful unit although the denominator is. This awkwardness should be enough to rule out the average even though, according to the verbal algebra, it is a composite indicator of hours of work: and even though the operations of arithmetic are also correctly performed.

Do these remarks suggest that payrolls have no place in the measurement of average hours of work? Not at all, but the approach has to be cautious. Let us revert to the first of the payroll identities shown in the discussion of aggregation formulas, namely:

Payroll ≡ Workers × Hours per Worker × Hourly Earnings.

Within this framework, all three characteristics may be measured compatibly for the same ensemble. Multiple measures may be devised for each characteristic, and at least one set makes good dimensional sense for all three.

Is there any manpower characteristic for which payrolls constitute a most "natural" weight? Of course, for hourly earnings, but a harmonic formula has to be used. Letting y represent the payroll of the i^{th} company or industry of an ensemble, we write $\Sigma y/\Sigma(y/e)$. This harmonic expression is transformable into a weighted arithmetic mean of hourly earnings with man-hour weights—which association is hinted in the second of the six identities presented earlier for payrolls. The equivalence is clear if we first rewrite the harmonic average as $\Sigma me/\Sigma(me/e)$ and then simplify to obtain $\Sigma me/\Sigma m$. Clearly, the numerator has the dimension of payrolls, and the denominator refers to man-hours. Each of these units is conventionally additive. Verbal algebra verifies that the quotient represents average hourly earnings.

Do weights and the numbers being averaged have to be perfect dimensional "mates"? Preferably, yes; and, when approximations to the logically desirable weights have to be used, the choice still ought to make tolerable dimensional sense. For example, employment weights might plausibly substitute for man-hour weights in an arithmetic mean of hourly earnings; but it would be foolish to weight by, say, man-hour productivity instead, or by its reciprocal, unit labor requirements. Awareness that a relevant accounting identity should be satisfied provides a guide to (a) good literal algebra (which is concerned with the content and structure of measures) and to (b) good dimensional sense while it also assures (c) satisfaction of the less stringent demands of verbal algebra.

Production Functions

Since "labor" is one of the two major inputs explicitly entering into the production functions of econometrics, the role of averaging (and aggregation) in these formulas merits a few remarks. In the construction of such functions, fine points of algebraic compatibility—both "within" and "between" the measures of the macrovariables—are typically overlooked. But we wish to address another matter, not to evaluate the customary treatment of output (Q), labor (L), and capital (K) as simple homogeneous properties to which "any old" single numbers are supposedly assignable without a qualm regarding content and structure.

In the famous Cobb-Douglas production function with constant returns to scale, the labor-capital core is a geometric mean; and the scale coefficient (d) too is an internal average, of labor productivity and capital productivity. Inspection of

$$Q = dL^b K^{1-b}$$

shows at once that labor and capital are being combined into a geometric mean; the sum of the weights, the exponents, is one. Furthermore,

$$d = \frac{Q}{L^b K^{1-b}} = \left[\frac{Q}{L}\right]^b \left[\frac{Q}{K}\right]^{1-b};$$

that is, the scale coefficient is a geometric mean of labor productivity and capital productivity and hence is a measure of composite factor productivity.[21] By measuring L and K in proper "efficiency units,"[22] we may recalibrate the production function and force d to become unity:

$$d = 1 = \left[\frac{Q}{dL}\right]\left[\frac{Q}{dK}\right]^{1-b}.$$

When the notion of constant returns is abandoned (so that the sum of the exponents may exceed or fall short of one), d is no longer just a geometric mean of labor and capital productivity factors. It is then a geometric mean multiplied by another term; and the extra term could be sufficiently large or small to cause externality, to force d outside the range of the two productivity factors. "Increasing returns" or "decreasing returns" of such magnitude seem most unreasonable—especially for a production function that is *static*, that incorporates no time trend.

From Hölder's inequality, we may derive a mathematical statement[23] that is useful in the comparison of a Cobb-Douglas function for an in-

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[21] I. H. Siegel, "Partitioning a Gross Change into Additive 'Explanatory' Components," in *1966 Business and Economic Statistics Section Proceedings of the American Statistical Association*, p. 407.

[22] On "efficiency units" (a concept invoked earlier by Marshall, Pigou, Joan Robinson, and Keynes), see Allen (footnote 9), especially Chapter 13.

[23] Hardy, Littlewood, and Polya, *op. cit.*, p. 27.

dustry as a whole with the aggregate of similar functions for individual firms. According to this statement,

$$\Sigma L^b K^{1-b} < (\Sigma L)^b (\Sigma K)^{1-b};$$

that is, the core of the industry production function exceeds the sum of the cores of the corresponding establishment functions.

The CES (constant elasticity-of-substitution) production function, which has recently come into prominence,[24] also assigns an important place to an average. This average may be harder to recognize than the Cobb-Douglas geometric mean, but it, too, is comprehended in the generalized weighted formula shown earlier. Thus, the CES function has the form

$$Q = d[bL^{-t} + (1 - b)K^{-t}]^{-1/t},$$

and the expression in square brackets is a weighted arithmetic mean of L^{-t} and K^{-t} that is itself raised to the power $-1/t$. The expression accordingly is an internal average of L and K.

If the exponent of the CES term in square brackets (but not of L or K) is changed to $-u/t$, the function is permitted to reflect nonconstant returns to scale. This change (i.e., the non-homogeneous introduction of u) could again lead to externality of the average of L and K—and of d, the composite productivity ratio.

Probability Theory of Averages[25]

Gauss, Edgeworth, Keynes, and many other famous contributors to the literature of probability have associated various kinds of averages with laws of error. These averages, conceived as parameters of particular distributions, may be derived by the minimization of certain functions of the deviations (or "errors") in both directions. Nowadays, the different means are more often regarded as linear estimates of population parameters that minimize the sum of squared deviations (i.e., the "variance") without reference to any specified error distribution.

When the simplest linear model, with only one constant, is fitted by the method of least squares, the familiar averages are quickly derivable as "best" mean values. Suppose that we have n observations on X of the form $x_i = X + v_i$, where the v_i are residuals. If we minimize the sum

[24] Allen (footnote 9), pp. 52–55; Marc Nerlove, "Recent Empirical Studies of the CES and Related Production Functions," *in* Murray Brown, ed., *The Theory and Empirical Analysis of Production* (New York: National Bureau of Economic Research, 1967), pp. 55–122; and K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow, "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, August 1961, especially pp. 228–231.

[25] See Edmund Whittaker and G. Robinson, *The Calculus of Observations* (New York: Dover reprint, 1967), Chapter 9; and any more recent intermediate or advanced text in statistics (e.g., by Kendall and Stuart, Draper and Smith, or Plackett) or in econometrics (e.g., by Goldberger, Christ, or Malinvaud).

of the squared deviations or residuals, we obtain $\frac{d\Sigma v^2}{dX} = \Sigma(x - X) = 0$, or $X = \Sigma x/n$, the arithmetic mean. If the observations have the weights w_i, the result of minimization of Σwv^2 is $\Sigma w(x - X) = 0$, or $X = \Sigma wx/\Sigma w$, the weighted arithmetic mean. Setting $w_i = 1/x_i$, we obtain $\Sigma(1/x)(x - X) = 0$, or $X = n/\Sigma(1/x)$, the harmonic mean. When $w_i = w_i'/x_i$, we obtain a weighted harmonic mean; here, the weights are w_i'. Additional means, including the unweighted and weighted geometric varieties, are obtainable from $\Sigma(x - X) = 0$ and $\Sigma w(x - X) = 0$ by the substitution of logarithmic and other expressions for x and X.

V. Consistent Aggregates and Averages

Identities Versus Economic Criteria

When aggregates and averages are designed or selected for use in conjunction with each other, an effort should be made to satisfy the same identities that constrain the corresponding measures for elements. These identities, which are definitional, were illustrated for payrolls and related variables in the discussion of aggregation formulas. Among the sets of measures that are compatible in concept, composition, and structure, we should ordinarily prefer those that best avoid dimensional eccentricity.

The position adopted here is much less stringent than the one stated or implicit in occasional studies of the "aggregation problem" by economic theorists or econometricians. The ambitious objectives sought by them—and not significantly achieved—do not necessarily subsume our limited goal of algebraic or accounting consistency. A recent essay has stated well and succinctly a challenge that greatly exceeds our modest intentions:

> The aggregation problem in economics is concerned with the relationship between four kinds of things. These are micro-variables (the variables which impinge directly on the individual decision maker), micro-relationships (the results of individuals' actions with respect to micro-variables), macro-variables (variables which have lost some of the labeling, e.g., with respect to individual decision makers, individual commodities, or individual time periods that characterize micro-variables), and macro-relationships (relationships holding between variables, at least one of which is a macro-variable). The aggregation problem arises because the macro-variables are functionally related to the micro-variables, the micro-variables to each other, and the macro-variables to each other. There are more relationships than can be chosen independently and the problem is that of consistency between them.

The same author observes that differing circumstances will dictate different approaches; but, "if any conclusion is to be drawn at all, it is that problems of aggregation in economics are usually swept under the rug."[26] Our view is that accounting consistency should not be neglected and ought even to be pursued in its own right at both the microlevel and the macrolevel.

Illustrations of Compatible Measures

Let us return to the second of the identities shown for payrolls. This aggregate may be written as the product of a term for man-hours and another for hourly earnings in two distinct ways:

$$\Sigma me \equiv \frac{\Sigma me}{\Sigma e} \cdot \Sigma e$$

$$\equiv \Sigma m \cdot \frac{\Sigma me}{\Sigma m} .$$

In the first of these two variants, man-hours are averaged and hourly earnings are aggregated; in the other, man-hours are aggregated and hourly earnings are averaged. The latter variant makes more dimensional sense than the first and, other things being equal, should be preferred; the addition of man-hours of different workers is much more realistic than the addition of their earnings per hour.

Any aggregate may be written as the continued product of as many telescoping formulas as we please, but common sense and the nature of a problem should guide our diligence and our choice. Thus, even though we start with Σme for payrolls, we are at liberty to recognize more than two factors—for example, by breaking man-hours (m) into employment (n) and hours per worker (h), as in the first of the identities shown earlier in this paper. But the literal algebra for three factors points to six possible alternatives, not all of which are equally attractive:

$$\Sigma me \equiv \Sigma nhe \equiv \Sigma n \cdot \frac{\Sigma nh}{\Sigma n} \cdot \frac{\Sigma nhe}{\Sigma nh}$$

$$\equiv \Sigma n \cdot \frac{\Sigma nhe}{\Sigma ne} \cdot \frac{\Sigma ne}{\Sigma n}$$

$$\equiv \frac{\Sigma nh}{\Sigma h} \cdot \Sigma h \cdot \frac{\Sigma nhe}{\Sigma nh}$$

$$\equiv \frac{\Sigma nhe}{\Sigma he} \cdot \Sigma h \cdot \frac{\Sigma he}{\Sigma h}$$

$$\equiv \frac{\Sigma ne}{\Sigma e} \cdot \frac{\Sigma nhe}{\Sigma ne} \cdot \Sigma e$$

$$\equiv \frac{\Sigma nhe}{\Sigma he} \cdot \frac{\Sigma he}{\Sigma e} \cdot \Sigma e$$

[26] Kelvin Lancaster, "Economic Aggregation and Additivity," *in* Sherman R. Krupp, ed., *The Structure of Economic Science* (Englewood Cliffs: Prentice-Hall, 1966), pp. 202, 214. See also H. Theil, *Linear Aggregation of Economic Relations* (Amsterdam: North-

Since the first of these six variants makes more dimensional sense than the rest, it should be preferred in situations allowing choice and as a model for approximation. The second set of products is second best.

The algebraic possibilities are obviously numerous, for each one of the six verbal identities shown earlier for payrolls (Section III) allows variants. It should be concluded that the algebraic or accounting criterion is flexible and accommodates a wide variety of analytical situations. Specifically, this criterion allows a reflection of the context of measurement in the content and structure of measures. The very characteristics that have to be appraised jointly provide an important part of the context. To the extent feasible, furthermore, dimensional awkwardness should be avoided; and, if rigidly followed, this principle would drastically narrow the range of acceptable formulations.

Speaking of flexibility invites another reminder that weighted arithmetic means are transformable into differently weighted harmonic means. Only arithmetic versions of averages have been presented here in the two-factor and three-factor displays, but translation into harmonic equivalents occasionally helps us to take fuller advantage of the data available for approximations.

An adjustment might be considered when available measures for different characteristics are algebraically incompatible. Suppose that, for the first of the six variant formulations shown above for Σnhe, an adequate measure of Σn exists but the companion measures of average hours per worker and average hourly earnings do not have appropriate weights. The two available averages may be modified in various ways to eliminate the difference between Σnhe and the product of all three factors. Thus, a constant could be added to the two averages; or, instead, a multiplier or exponent might be applied to each. One must be alert, however, to the danger of externality; a substantial adjustment can force an intended average outside the range of the numbers assigned to the elements.

Another kind of adjustment would aim at harmonization of the multiple measures yielded by variants of the identities. Thus, if adequate measures are available for the first two of the six sets of expressions shown for Σnhe, we may wish to compute geometric means of corresponding pairs. It is also tempting to contemplate, as in the theory of index-number bias,[27] the harmonization of all six variant expressions by the computation of the sixth roots of the products of corresponding formulas. Although the derivation of such geometric means of all the vari-

Holland Publishing Company, 1965); H. A. J. Green, *Aggregation in Economic Analysis* (Princeton: Princeton University Press, 1964); and A. A. Walters, "Production and Cost Functions: An Econometric Survey," *Econometrica*, January-April 1963, especially pp. 5-11.

[27] I. H. Siegel, "The Generalized 'Ideal' Index-Number Formula," *Journal of the American Statistical Association*, December 1945, pp. 520-523.

ants would eliminate structural bias, the cost is high. Thus, aggregates and averages would be confounded for each characteristic, and all measures would be treated as equally acceptable despite departures from dimensional sense. (There is also the question, of course, of data availability.)

VI. Tools for Analyzing Alternative Measures

Some algebraic devices useful in exploration of differences between alternative aggregates and alternative averages are now briefly examined. These tools help reveal the consequences of choice. Awareness of them should encourage discriminating approximation as well as careful selection among the alternative measures that are available or constructible.

Tools for Aggregates

A weighted aggregate is equivalent to the sum of two terms, one of which incorporates a Pearsonian correlation coefficient:

$$\Sigma wx = \frac{\Sigma w \Sigma x}{n} + n\sigma_w \sigma_x r_{w \cdot x}.$$

Only the correlation coefficient, $r_{w \cdot x}$, may assume zero or negative values. When its value is zero, the weighted aggregate reduces to the product of (a) the sum of weights and (b) the unweighted average of the numbers assigned to the elements of the ensemble. When its value is negative (positive), the weighted aggregate is smaller (greater) than this product.

The same algebraic statement is readily adaptable to analysis of the difference between two dissimilarly weighted aggregates. If the difference is first written as $\Sigma w'x - \Sigma wx = \Sigma(w' - w)x = \Sigma \omega x$, all that is required is the substitution of ω_i for w_i in the preceding paragraph. Some of the ω_i—and $\Sigma\omega$ too—may be negative.

When the weighted aggregates involve three variables, as in some of the payroll identities shown earlier, a weighted correlation coefficient may advantageously be used. Writing $r_{w:x \cdot y}$ for this coefficient,[28] we have:

$$\Sigma wxy = \frac{\Sigma wx \Sigma wy}{\Sigma w} + \Sigma w \sigma_{w:x} \sigma_{w:y} r_{w:x \cdot y}.$$

The difference between weighted aggregates in which *both* the weights and element numbers are dissimilar may also be analyzed readily. Suppose, for example, that we wish to study the difference between

[28] The weighted correlation coefficient is often associated with the name of L. von Bortkiewicz. See two papers by I. H. Siegel in the *Journal of the American Statistical Association:* "Note on a Common Statistical Inequality," June 1943, pp. 218–219; and "The Difference Between the Paasche and Laspeyres Index-Number Formulas," September 1941, pp. 344–346.

totals for man-hours worked by men and by women in the same group of occupations. The hours of work presumably differ, and so do the two occupational distributions. Writing Σnh for men and $\Sigma n'h'$ for women, we have:

$$\Sigma n'h' - \Sigma nh = \Sigma n(h' - h) + \Sigma h(n' - n) + \Sigma(h' - h)(n' - n).$$

The third term on the right may be distributed equally between the other two, to yield a familiar symmetrical decomposition formula allegedly showing the "contributions" of the hours difference and employment difference to the total disparity. A symmetrical decomposition formula, however, has to be used with circumspection. Thus, it can yield bizarre results when the $h_i' - h_i$ and the $n_i' - n_i$ tend to have opposite signs.[29] Besides, it has no organic significance and does not really show the independent effects of these two sets of differences.

The same tool is adaptable to discussion of the difference between aggregates containing three or more factors. When payrolls are expressed as products of three variables, the right-hand side has seven components. When four variables are used (e.g., workers, average hours, productivity, and unit labor cost), fifteen components emerge. Again, the second-order and higher-order difference terms may be distributed symmetrically, but the "contributions" of the various factors to the total disparity are not really separable and identifiable in this simple manner. The results, moreover, are sometimes mischievous; the caveat expressed above for the simpler case of two factors still applies.[30]

Tools for Averages

The analytic devices presented for aggregates may readily be adapted to the study of differences between averages. Additional tools will also be noted below.

The utility of the weighted correlation coefficient may be illustrated for the case in which two averages of x_i have the weights $w_i y_i$ and w_i, respectively. The difference between these two means is reflected in the following expression:

$$\frac{\Sigma wxy}{\Sigma wy} = \frac{\Sigma wx}{\Sigma w} + \frac{\Sigma w}{\Sigma wy}\,\sigma_{w:x}\sigma_{w:y}r_{w:x\cdot y}.$$

The mean on the left exceeds the one on the right when the weighted correlation coefficient for x and y is positive; and it is smaller when the coefficient is negative. When the coefficient vanishes, $\Sigma wxy/\Sigma wy = \Sigma wx/\Sigma w$.

The applicability of partition formulas may be illustrated for the most

[29] See paper by I. H. Siegel cited in footnote 21, p. 405.

[30] *Ibid.*, pp. 405–406.

general case, in which two means differ not only in the weights but also in the characteristics being compared. Let us suppose that two distinct properties for an ensemble are measured by x_i and z_i and that the appropriate corresponding weights are w_i and y_i. (We may, for example, wish to investigate the relationship between an average of weekly hours with employment weights and an average of hourly earnings with man-hour weights for the same ensemble of elements.) A compromise partition formula may be written that formally breaks the difference between the compared means into two identifiable parts:

$$\frac{\Sigma wx}{\Sigma w} - \frac{\Sigma yz}{\Sigma y} = \Sigma\left[\frac{y}{\Sigma y}(x - z)\right] + \Sigma\left[x\left(\frac{w}{\Sigma w} - \frac{y}{\Sigma y}\right)\right].$$

The first term on the right is associated with dissimilarities between the two sets of property measures; the second reflects dissimilarities of the two weighting structures. A reminder is needed, however, that these components of a partition formula should not be interpreted in causal terms. An alternative compromise formula for the very same case may be written as:

$$\frac{\Sigma wx}{\Sigma w} - \frac{\Sigma yz}{\Sigma y} = \Sigma\left[\frac{w}{\Sigma w}(x - z)\right] + \Sigma\left[z\left(\frac{w}{\Sigma w} - \frac{y}{\Sigma y}\right)\right].$$

The multiplicity of compromise formulas should itself induce caution in interpretation.

Related to expressions involving a correlation coefficient are many others that reach into matrix and vector algebra. Let us consider the sign of the difference between two averages of hourly earnings (e_i), one of which is weighted by man-hours ($n_i h_i$) and the other, intended as an approximation, weighted by employees (n_i). The difference between these means, Δ, is expressible as

$$\Delta = \frac{\Sigma nhe}{\Sigma nh} - \frac{\Sigma ne}{\Sigma n} = \frac{\begin{vmatrix}\Sigma nhe & \Sigma nh \\ \Sigma ne & \Sigma n\end{vmatrix}}{\Sigma nh \cdot \Sigma n},$$

and the sign of the difference is given by the sign of the determinant in the numerator. The matrix of this determinant is the product of two rectangular arrays:

$$\begin{bmatrix} n_1 h_1, & \cdots, & n_s h_s \\ n_1, & \cdots, & n_s \end{bmatrix} \text{ and } \begin{bmatrix} e_1, & \cdots, & e_s \\ 1, & \cdots, & 1 \end{bmatrix}.$$

The rule for matrix multiplication (or the generalized Lagrange identity of vector algebra) permits us to rewrite the determinant as a double sum (see Appendix):

$$\sum_{i=1}^{s-1} \sum_{j=i+1}^{s} \begin{vmatrix} n_i h_i & n_j h_j \\ n_i & n_j \end{vmatrix} \cdot \begin{vmatrix} e_i & e_j \\ 1 & 1 \end{vmatrix} = \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} n_i n_j (h_i - h_j)(e_i - e_j).$$

At a minimum, this double sum tells us that Δ must be (a) positive when the Spearman coefficient of rank correlation between the h_i and the e_i assumes the value $+1$, (b) negative when the coefficient has the value -1, and (c) zero when the coefficient vanishes. This statement accords with Chebyshev's inequality, cited earlier. The critical double sum may also be introduced into expressions involving weighted correlation coefficients.[31]

The tools mentioned above have been borrowed largely from index-number literature. This fact should not be surprising, for the index formulas normally used are weighted arithmetic, harmonic, or geometric means of relatives.

VII. Toward Better Design and Use of Manpower Measures

Our discussion of aggregation and averaging has disclosed or confirmed various needs and opportunities for improving the quantity, quality, variety, relevancy, and understanding of manpower measures. Challenges are discernible for statistics producers and consumers—both public and private—at the levels of compilation, processing, and interpretation. An effort is now made to summarize potentials for improvement under three heads: (a) the scope, composition, and structure of manpower measures; (b) the consistency of these measures with each other and with nonmanpower measures jointly required for the same purpose or context; and (c) the difference between alternative measures, especially the difference between preferred and practical ones. Although no attempt is made to take account of matters more pertinent to papers on, say, sampling, index numbers, or time series in general, the needs and opportunities that are noted below have fairly universal import since aggregation and averaging are fundamental quantitative processes.

Importance of Measurement Details

With respect to the first head, it seems impossible to overstate the case for understanding what a manpower measure "really means." Attention should accordingly be paid to the intended and actual scope of a measure, the definition of the attribute being measured, the weights, the unit of aggregation, the combining formula, the adjustments, the substitutions and approximations, and so forth.

In one way or another, at some time or some place, it does matter whether or not "hours of work" refer to normal schedules, to actual time, to paid time, or to available time. Similarly, it matters whether or not "employment" (a) includes this or that category or industry, (b) has been adjusted to show full-time equivalents or to reflect remunera-

[31] On this paragraph, see papers by I. H. Siegel cited in footnote 28 and his monograph on *Concepts and Measurement of Production and Productivity* (footnote 10).

tion differences, (c) refers to payroll records or to a given day, (d) has been derived from establishments or households, (e) includes self-employed persons and family workers. We should care whether or not "unemployment" includes (a) underemployment and other "disguised unemployment", (b) discouraged workers no longer seeking work, (c) social dropouts who should be won back to the active labor force. We ought to know whether or not "earnings" include (a) fringe benefits, (b) payments in kind, (c) unpaid accruals, (d) withheld taxes.

For such reasons table titles, headnotes, footnotes, other descriptive small print, and technical manuals require scrutiny. They are no more dispensable than the sales contracts and warranties relating to tangible products. Thus, greater use should be made of technical notes supplied by the Bureau of Labor Statistics in *Handbook of Methods for Surveys and Studies* (Bulletin No. 1458) and in issues of *Employment and Earnings* and *Monthly Report on the Labor Force*; by the Joint Economic Committee in supplements to *Economic Indicators*; by the Office of Business Economics in supplements to the *Survey of Current Business*; and so forth.

Matching of Measures

The second point concerns compatibility beyond the requirements of verbal algebra. "Any old" aggregate or average having an agreeable short title and permitting a proper cancellation of words is not really equivalent to any other measure that carries the same short designation. Measures that are used in conjunction should preferably be matched for conceptual and structural consistency; and the ideal algebraic requirements should be kept in view when attempts have to be made to substitute or approximate. Explicit accounting identities provide useful guides for methodical exploration of the multiple alternatives and for making orderly retreats from the preferred to the practical. Wherever possible, dimensional eccentricity should be avoided in the choice of weights even though other standards of literal algebra are satisfied.

Comparison of Measures

Finally, with the aid of algebraic tools and in cases favorable for testing, it is worth while to investigate the direction, magnitude, and sources of difference between alternative measures. When multiple measures are available for the "same" concept, variability in their reports is to be expected. Sensitivity to the difference in their meanings is demanded, not frustration over the need for informed choice. Besides, approximations and substitutes should not routinely be assumed to serve as well as "the real things." In short, better design and use of manpower aggregates and averages are vital to any agenda for general improvement of manpower statistics and of methods for manpower analysis.

Appendix

Summation Symbols and Rules

This supplement deals with aspects of aggregation often accorded insufficient attention in the statistical and mathematical literature. Authors frequently do not pause to discuss summation symbols and operations at all; and those who do tend to confine themselves to the simplest of cases. The reader is apparently assumed to be familiar with whatever elliptical or compact notation is employed; or to be able to proceed with sureness after minimal preliminary instruction.[1]

The following remarks deal mostly with summation, but they also have implications for averaging. Although not employed in this paper, the bar symbol is commonly used for an average—e.g., $\Sigma x/n = \bar{X}$. Reference will be made below to the combination of bar and dot notation for averages, as in the statistical technique called "analysis of variance."

Notation for Summation

Of the alternative symbols used for summation, Σ is the most familiar. Since Euler's time, it has been identified with the process of aggregation, but it has also had challengers.

A rival of Σ is S, which Laplace apparently preferred and which is still used in some statistical works for typographical convenience. Occasionally, S is used in addition to Σ—say, to distinguish the aggregation of sample data from aggregation for a population.

Another alternative to Σ is the square-bracket notation employed by Gauss in his presentation of least-squares "normal equations." His practice became standard among later writers on curve-fitting. Like Σ, however, the Gaussian symbol is being displaced to some degree by still more compact matrix notation in contemporary intermediate and advanced statistical literature.

In matrix algebra, tensor calculus, and mathematical physics, partiality is sometimes shown for the *summation convention*, which Einstein introduced in a paper on relativity in 1916. Thus, we may write $n_i h_i$ instead of $\sum_{i=1}^{s} n_i h_i$ as the sum of weekly man-hours for all s categories of n_i workers occupied h_i hours per week. When the context does not make the limits of summation clear, explicit information on the range (i.e., $i = 1, \cdots, s$) may be added. The advantage of the convention is revealed more fully in instances requiring multiple subscripts and superscripts and multiple summation. The convention actually directs that

[1] For a patient treatment of summation, see the appendix on "The Notations Σ and Π" in F. E. Hohn, *Elementary Matrix Algebra* (New York: Macmillan Company, 1958), pp. 271–281. A more exhaustive review is presented by Jerome Cornfield and W. Duane Evans in their *Theory of Sampling Surveys: Part I, Fundamental Tools*, processed, 1951, Section 3.

summation be performed only with respect to the repeated subscripts or superscripts in the prototype expression shown for an element. A repeated symbol is called a *dummy* or *umbral* index or suffix; any remaining unrepeated, or *free*, index may assume any value within its own range (but one value at a time and without implying further aggregation).[2]

In the *dot notation* encountered in statistical and matrix literature, a dot is substituted for the index with respect to which summation or averaging is intended. Other subscripts, if necessary, are retained. The system is applicable when the letters have more than one subscript. The range need not be stated when no ambiguity results; and the notation may be combined with the summation convention to dispense with Σ. Illustrative of the dot notation for sums are the following: $\sum_{i=1}^{s} a_{ij} = a_{\cdot j}$; $\sum_{j=1}^{r} a_{ij} = a_{i\cdot}$; $\sum_{i=1}^{s}\sum_{j=1}^{r} a_{ij} = \sum_{j=1}^{r} a_{\cdot j} = \sum_{i=1}^{s} a_{i\cdot} = a_{\cdot\cdot}$; and $\sum_{j=1}^{r}\sum_{k=1}^{r} a_{ij}a_{hk} = a_{i\cdot}a_{h\cdot}$, for $i \neq h$. In the analysis of variance, the symbol $\overline{X}_{\cdot j}$ is often used for the mean of entries in the j^{th} column of a table of measurements (i.e., the summation takes place over the rows); $\overline{X}_{i\cdot}$ for the mean of the entries in the i^{th} row (summation occurs across the columns); and $\overline{X}_{\cdot\cdot}$ for the grand mean of all the table entries. The notation is extensible to cases involving more than two subscripts.[3]

Scalar Multiplication as Aggregation or Averaging

The *scalar product* (also called *inner product* and *dot product*) of two vectors is a sum, and this fact fruitfully associates aggregation and averaging with vector algebra—and with matrix algebra too. Since scalar multiplication is a cornerstone of Grassman's calculus of "extension" and has more than a century of history, it is best regarded as an independent procedure adaptable to aggregation and averaging.

The notation for a scalar product is reminiscent of Gaussian brackets. Two letters, usually printed in boldface, are juxtaposed; and each stands for a vector. Punctuation—a dot or a comma—sometimes separates these

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[2] On the summation convention, see, for example, *International Dictionary of Applied Mathematics* (Princeton: D. Van Nostrand, 1960), p. 913; T. L. Wade, *The Algebra of Vectors and Matrices* (Reading, Mass.: Addison-Wesley, 1951), pp. 67–68; W. L. Ferrar, *Algebra* (2nd ed.; London: Oxford University Press, 1957), pp. 41–42; F. E. Satterthwaite, "Concise Analysis of Certain Algebraic Forms," *Annals of Mathematical Statistics*, March 1941, pp. 77–83; and D. F. Lawden, *An Introduction to Tensor Calculus and Relativity* (2nd ed.; London: Science Paperbacks and Methuen, 1967), pp. 25–26.

[3] On the dot notation for sums, see S. R. Searle, *Matrix Algebra for the Biological Sciences* (New York: Wiley, 1966), pp. 10–11, 16–17. On the dot notation for averages in the analysis of variance, see Churchill Eisenhart, "The Assumptions Underlying the Analysis of Variance," *Biometrics*, March 1947, pp. 1–21; P. R. Rider, *An Introduction to Modern Statistical Methods* (New York: Wiley, 1939), pp. 142–144; or any recent statistics manual.

vectors. When a comma is used, parentheses may also enclose the multiplicands. When no punctuation is used to separate the vectors, the first of the two is occasionally written with the superscript "T" or with a stress mark; these symbols denote "transposition," or the conversion of a column vector into a row vector, a step strictly required for scalar multiplication.

Let $\mathbf{n}$ be a vector specifying s different numbers of employees; and let $\mathbf{h}$ be the corresponding vector for average hours worked by the s personnel categories. The scalar product, $\mathbf{nh} = \mathbf{n} \cdot \mathbf{h} = (\mathbf{n}, \mathbf{h}) = \mathbf{n}^T\mathbf{h} = \mathbf{n}'\mathbf{h}$, is equivalent to $\sum_{i=1}^{s} n_i h_i$, the man-hours total. If $\mathbf{n}$ refers to $n_i/\Sigma n$ instead of $n_i (i = 1, \cdots, s)$, it becomes a weight vector; and the scalar product of this new $\mathbf{n}$ and the original $\mathbf{h}$ represents the weighted arithmetic average of hours worked by employees in all s categories.[4]

Double Summation

The text contains a double sum in which the limits are linked. Such sums are more difficult to handle than those shown above to illustrate the dot notation.

The brief discussion that follows should make three lessons clear. First, the order of summation is procedurally important though mathematically immaterial. Second, the designation of the limits of summation requires care, for algebraic statements that seem similar may yield dissimilar results. Finally, compensatory adjustments may be made in the instructions for summation and in the choice of limits so that different expressions do yield identical results.

The double sum shown in the text referred to the difference between two means of s figures for hourly earnings (e_i), one weighted by employees (n_i), and the other by man-hours ($n_i h_i$). The sum was expressed in the form $\sum_{i=1}^{s-1} \sum_{j=i+1}^{s}$; the lower limits of i and j are interdependent. The sum has important connections, as the text indicates, with weighted correlation and with the algebra of vectors and rectangular matrices.

Our double sum has ${}^sC_2$ terms. The preferred order of summation is to fix i first; then, for each i taken in turn, we run through the range of j.

Two other expressions involving interdependent limits may be substituted for our double sum. They have the forms $\sum_{j=1}^{s-1} \sum_{i=j+1}^{s}$ and $\sum_{j=2}^{s} \sum_{i=1}^{j-1}$. The first equivalent indicates that i and j may be interchanged without

[4] See two papers by I. H. Siegel in the *Journal of the American Statistical Association*: "The Difference Between the Paasche and Laspeyres Index-Number Formulas," September 1941, pp. 343–350; and "Note on a Common Statistical Inequality," June 1943, pp. 217–222. See also R. A. Barnett and J. N. Fujii, *Vectors* (New York: Wiley, 1963), Chapter 2; and F. B. Hildebrand, *Methods of Applied Mathematics* (2nd ed.; Englewood Cliffs: Prentice-Hall, 1965), p. 23.

mathematical effect; thus, we may fix j first and then, for each j taken in turn, run through the range of i from $j + 1$ to s. The second alternative indicates that the upper limits for i and j, rather than the lower ones, may be made interdependent; and, in this version, it is convenient to fix j and then, for each j taken in turn, let i run through its appropriate set of values.

The double sum may be rewritten with inequalities for the lower limits of i and j. Thus, the sum may be cast into the form $\sum_{i<}^{s-1}\sum_{j}^{s}$. It may also be converted into $\sum_{i>}^{s}\sum_{j}^{s-1}$ because of the symmetry of the prototype product term (i.e., invariance of the term to an interchange of i and j). Both of these alternatives may be combined into $\frac{1}{2}\sum_{i\neq}^{s}\sum_{j}^{s}$, a form that permits easy reversal of the order of summation and that takes cognizance of the duplication of terms when summation proceeds independently with respect to i and j.

The last expression is also equivalent to $\frac{1}{2}\sum_{i}^{s}\sum_{j}^{s}$. Again, the coefficient $\frac{1}{2}$ reflects the fact that, when summation is undertaken with respect to i and j independently, the appropriate ${}^{s}C_2$ terms are duplicated. We may dispense with the inequality symbol for i and j since the terms corresponding to $i = j$ vanish.

Expressions for double sums may sometimes be rendered more elegant by the introduction of the *Kronecker delta* or its complement as a coefficient.[5] The Kronecker symbol, δ_{ij}, has the value 1 when $i = j$ and 0 when $i \neq j$. Accordingly, its complement, $1 - \delta_{ij}$, has the value 0 when $i = j$ and 1 when $i \neq j$. This complement could be introduced into various expressions for our double sum, but no gain in compactness would be achieved.

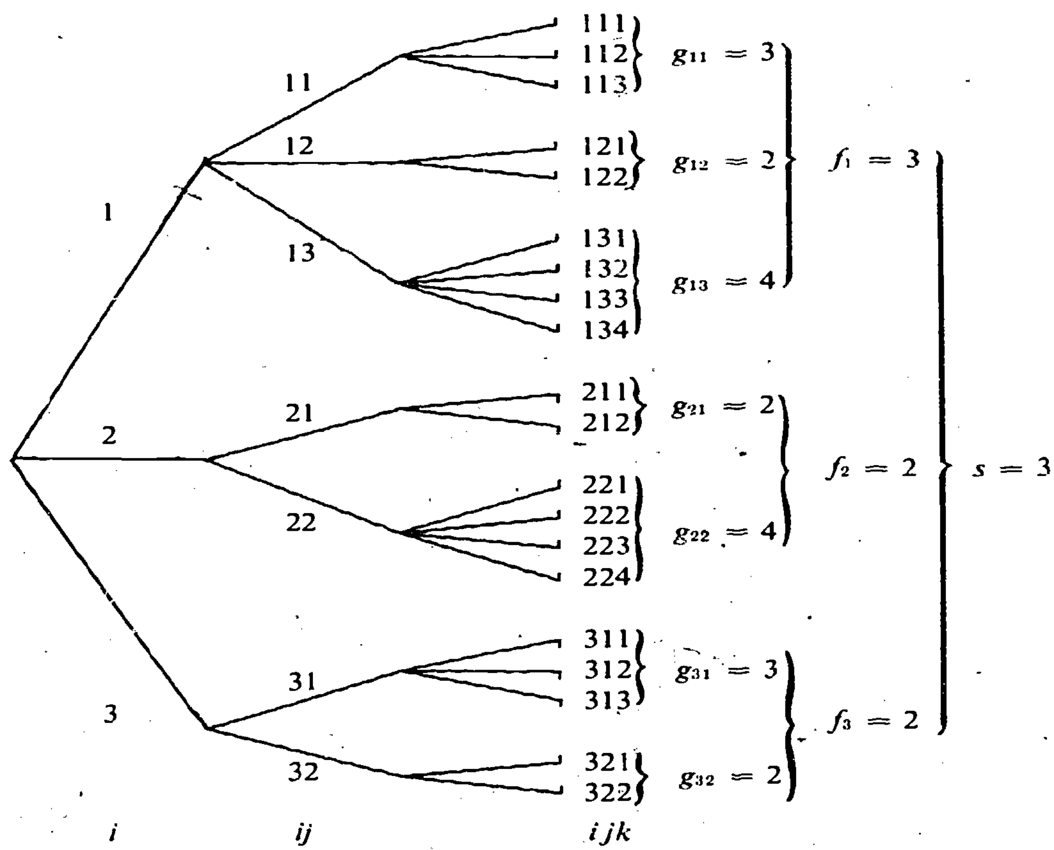
Finally, it should be acknowledged that a single symbol is often used in instances strictly demanding a double Σ. Carelessness, convenience, or the assumption that a reader is already familiar with the operations involved and that communication is assured may encourage the use of, say, $\sum_{i>j}^{s}$ to indicate a sum of ${}^{s}C_2$ terms.

A Tree for Three

Multiple summation (for samples as well as populations) may be visualized readily with the aid of *tree diagrams*, which are tools commonly used in the exploration of logical possibilities, decision sequences, branching probabilities, game outcomes, and so forth in "finite mathematics."

[5] Among the many books in which the Kronecker notation is mentioned and illustrated is Hildebrand (footnote 4), pp. 15–16.

Illustrative Tree Diagram for a Triple Sum (3 Industries, 7 Companies, 20 Departments)



(The 20 ordered triplets corresponding to the permissible values of i, j, and k represent man-hours in the 20 departments; the 7 ordered pairs corresponding to the values of j and k represent man-hours in the 7 companies.)

All of these applications involve the nesting of subsets in sets, and of sets in still larger sets; and the levels of aggregation may be as numerous as we please.[6]

Usually, tree diagrams are thought to "start" at a trunk at the left then to "branch" outward to the right, but the opposite direction is the natural

[6] On tree diagrams, see J. G. Kemeny, Arthur Schleifer, Jr., J. L. Snell, and G. L. Thompson, *Finite Mathematics with Business Applications* (Englewood Cliffs: Prentice-Hall, 1962), pp. 19–22 and *passim*.

one for depicting the aggregation process. It is not the filiation or proliferation of paths that is of interest in summation but their combinability, their convergence.

Let us return to the triple sum shown for man-hours in the text. The j^{th} company in industry i has k departments. The total number of companies is not the same for every industry (i.e., the upper limit of j is f_i, a variable with respect to industry): and the total number of departments is not the same for every company (i.e., the upper limit of k is g_{ij}, a variable with respect to company and industry).

The hierarchical structure of this triple sum is exhibited in the accompanying tree diagram. The total number of industries, s, is taken as 3 for simplicity. Nested within the first industry are 3 companies (f_1), while the second industry contains 2 companies (f_2), and so does the third (f_3). Another branching shows, for the first industry, 3 departments in the first company (g_{11}), 2 departments in the second (g_{12}), and 4 departments in the third (g_{13}). The varying numbers of departments in the companies of the second industry and of the third industry are indicated by the remaining third-order branches of the tree.

Since the 20 departments in the diagram are distinctly identified by their triple subscripts, or *ordered triplets*, aggregation may be performed directly as well as by successive combinations. Thus, we may combine the man-hours in the 20 departments at once; or we may alternatively obtain the g_{ij} company sums first, add these company figures to obtain the f_i industry sums, and finally add the industry sums to obtain the grand total for the s industries.