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ABSTRACT

The plans for a field test of teaching tenth grade plane geometry through transformations is described. Text and homework materials have been written, and teachers have been trained. Eight intact classes are now being taught utilizing the materials. Data to be collected will include: (1) teacher descriptions of the course via questionnaire; (2) teacher assessment of course appeal to students and rate and level of comprehension on the part of the students, (3) student self-assessments as to comprehension and interest in the materials and (4) achievement test scores. The students will be classified by sex and I.Q. and data will be analyzed using a two-way analysis of variance with null hypothesis that the eight group means are not significantly different. A complete list of definitions, axioms, postulates, and theorems used in the materials is included. (JG)

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WSU-CORD

The Wisconsin State Universities Consortium of Research Development

Research Report

HIGH SCHOOL PLANE GEOMETRY THROUGH TRANSFORMATIONS: AN EXPLORATORY STUDY

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Cooperative Research

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INTRODUCTORY SECTION

SUMMARY

The primary purposes of this project were to:

1) Develop and field test a plane geometry course based strictly on transformations. Since the use of transformations is new, the field test was to be exploratory.

2) Draw conclusions as to the advantages, pedagogical and mathematical, that this approach to plane geometry offers.

At the time of this writing, the field test has not been completed so that final conclusions are still forthcoming.

The field test is being conducted with eight high school classes in tenth grade plane geometry. The field test will include a terminal achievement test which is criterion referenced rather than norm referenced, a teacher assessment by the teachers involved in the project and a student assessment.

INTRODUCTION

It may not be difficult to assume that geometry in some form will be assured a place in the high school curriculum of the future, but the problem seems to be what form it will take. That this is a problem of concern to mathematicians and educators we need only note that the Mathematical Association of America and the National Council of Teachers of Mathematics, at their joint meeting in January, 1967, devoted one whole portion of their convention to geometry in the secondary school. It was not their purpose to make specific recommendations but rather to explore the whole topic of geometry.

For years, instruction in plane geometry was based almost exclusively on Euclid's Elements. This was the case despite the known deficiencies in this mathematical system. In the 1950's, various national study groups began work on bringing geometry up to modern standards for a mathematical system. The efforts of the Commission on Mathematics of the College Entrance Examination Board are representative of this movement. Generally, the results were a better vocabulary, more complete axiom systems, and the inclusion of powerful algebraic techniques in the subsequent teaching materials. These changes were overdue and have been successful in improving the quality of instruction in geometry. At the same time, it should be noted that most of these new presentations of plane geometry were much in the spirit of Euclid's Elements for at least two reasons: (a) logic seems to take precedence in the arrangement and presentation of materials, and (b) emphasis on geometric facts and relationships of static figures.

Of late, there seems to be much less agreement that such modifications of Euclid's approach to plane geometry are the "best" for a high school student. It is possible to find individuals and groups who advocate supplementation or replacement by: coordinate approaches to geometry, vector approaches to geometry, transformation geometry, solid geometry, topics with emphasis on logical structure, spherical geometry, finite projective geometries, topology, and non-Euclidean geometries.

In order to make some valid decisions on what is to be included in the high school geometry course, we need some basic information. For any new teaching materials, we need information about whether we ought to teach it and whether it can be taught. Without this kind of information, valid curricular changes become a matter of whim or, as is generally the case, not made at all.

One way of obtaining this basic information is through an exploratory study. In this project, such an exploratory study was conducted to determine how suitable a plane geometry course based on transformations would be for high school usage. A need for such a study is pointed out by the Cambridge Conference on School Mathematics in their report:

> In the further development of geometry, the motions of Euclidean space are to be treated, leading to the introduction of linear transformations and matrices and the eventual study of linear algebra. This led to the suggestion that the study of geometry could be based on transformations of the plane or space. It was felt that this suggestion warranted further study to see if an approach could be written up and whether it was appropriate at this level.[1]

In a recent article, Adler says:

> Goal 2 is an introduction to the role of transformations of space in the study of geometry. To achieve this goal, it is necessary to include in tenth-grade geometry the study of isometries of the plane. One way of doing this is to give the concept of an isometry a central role, as has been done in the schools of Denmark. It would certainly be worthwhile for some American schools to experiment with this approach, perhaps using as axioms those given in Guggenheimer's book, Plane Geometry and Its Groups.[2]

In a recent CUPM Report, Kelly concludes:

> Although I believe that convex figures should be included in the high school curriculum and hope that future recommendations will encourage this, I am even more concerned that transformations be included. The structure of the motion group of a geometric space is essential information about the space. Transformations are easy to teach and vital to ideas that are alive today.[3]

The transformational approach that was used for this study sought to retain many of the same theorems and results of traditional plane geometry. In that sense, it could be called old-fashioned. At the same time, it introduced a new viewpoint of plane geometry.

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[1]The Report of the Cambridge Conference on School Mathematics, Goals for School Mathematics, p. 47.

[2]Irving Adler, "What Shall We Teach in High School Geometry?" The Mathematics Teacher, March, 1968, p. 230.

[3]Committee on the Undergraduate Program in Mathematics, CUPM Geometry Conference-Part III: Geometric Transformation Groups and other Topics, p. 146.

METHODS

The first phase of this study was the writing of text materials. The materials were written so that traditional topics were included. These traditional topics were: angle relationships and perpendicular lines, parallel lines, and congruence. It was necessary to develop certain topics which were part of the transformational approach to plane geometry. Examples of these topics would include: mapping, invariance, one-to-one, etc. There were certain topics which are common to any plane geometry course. These topics include the general nature of plane geometry, deductive reasoning, inductive reasoning, elementary set theory, etc. An index of the text materials is included in the Appendix.

The writing of the text materials included the expository material, selected theorems and their proofs, and an item pool which was a source for homework assignments and for tests.

During the first semester of the 1969-70 school year, the material was taught to eight intact classes of high school geometry.

Research Design

After the writing of the materials, the training of the teachers, and the teaching of the course, the final evaluation is begun. This evaluation will be completed with data received from (a) teachers of the course and (b) students who received the instruction.

Data from the Teachers

The data from the teachers will be furnished in two modes: (1) A detailed questionnaire will be devised. This questionnaire will be given to the teacher at the beginning of his teaching of the course. In this way, the teacher will be expected to keep a running account of the course via the questionnaire. (2) The second mode of the data-gathering from the teachers will come at the end of the course. For this part, the teachers will be asked to write a general assessment. This assessment will not be long but will be designed to assess the overall effect of this approach to teaching plane geometry. Teachers will be asked to assess: the appeal of this approach both to the students and to the teachers, the rate of comprehension on the part of the students, the level of comprehension on the part of the students, and any other facets of the course which seem of importance to them.

Data from the Students

An inventory for the students will be devised. This inventory will consist of two parts: (1) students will make self-assessments as to comprehension and interest in the materials, and (2) a test of achievement.

In the self-assessment, the student will be asked to respond to a series of positive statements about the course materials. They will circle numbers from one to five depending on their strongly agreeing to strongly disagreeing with the given statement.

The students will be classified in two ways, by sex and by their membership in one of the quartile groups relative to IQ. A two-way analysis of variance will be used on the scores of this first self-assessment test. The null hypotheses being that the eight groups are drawn from the same population and that no significant differences occur in the means of the eight groups.

The terminal achievement test will be criterion referenced.

The construction of this achievement test involves the selection of test items, assigning weights to these items and setting the level of performance which will define success on the entire test.

In selecting test items, provision must be made for its validity or relevance to that which is supposed to be measured. Using the objectives of each lesson as a guide, the proposer will select at least four questions for each lesson. Then, the teachers in the experiment will be asked to rate the four or more questions from each lesson as to relevance. From the composition of the ratings, the proposer will make a final selection of two questions from each lesson.

Summoning all possible sagacity, the proposer will then assign weights to the items and designate the level of performance which will define success on the entire test.

After the test has been administered and scored, a reliability coefficient will be computed. The test will be split into halves using alternate items as a basis for the splitting. Then a reliability coefficient will be computed according to the formula $r = 1 - \frac{s_d^2}{s^2}$ where s^2 is the variance of scores on the test and s_d^2 is the variance of difference scores between the two halves. This reliability coefficient formula makes the assumptions that (1) the variation from true score in a series of measurements of a single individual is constant for all individuals in the group and (2) the errors of measurement in the two half-scores are uncorrelated. The first assumption is tenable when the group is fairly homogeneous. The second assumption is implicit in any split-half method of reliability coefficient computation.

Equivalence of treatment effects on the classes and teachers will be determined by using analysis of covariance on the results of a pretest and the final achievement test.

If the equivalence of treatment effects for classes and teachers can be demonstrated, then it will be possible to investigate differences between sexes and among IQ quartile groups and between parts of the test divided by item difficulty by using analysis of variance on the final achievement scores.

As stated previously the findings, conclusions and recommendations await the final collection of data. An addendum with final results will be submitted later.

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2 Definition 1: A set of points is collinear if there is a line which contains all of them.

2 Definition 2: A set of points is coplanar if there is a plane which contains all of them.

2 Definition 3: Two or more coplanar lines which are disjoint will be called parallel lines.

2 Postulate 1: Through any two distinct points A and B there is one and only one line.

2 Theorem 1: If two lines intersect they intersect in exactly one point.

3 Postulate 2: On every line there exist at least two distinct points. There exist at least three points which are not on the same line. There exist at least four points not on the same plane.

3 Postulate 3: Through any three points, not on the same line, there is one and only one plane.

3 Postulate 4: If two points lie in a plane then the line passing through them lies in that plane.

3 Postulate 5: If two planes intersect, then the intersection is a line.

3 Theorem 2: If a point lies outside a line, exactly one plane contains the line and the point.

3 Theorem 3: If two lines intersect, exactly one plane contains both lines.

6 Postulate 6: All points in a plane, but not on a given line, form two non-empty disjoint sets.

6 Definition 4: The two non-empty disjoint sets formed by a line in a plane will be called the half-planes of the line.

6 Definition 5: Point C is between A and B if A, B, and C are distinct, collinear points and there is a line d passing through C so that A and B are in different half-planes of d.

6 Definition 6: The set of all points between A and B together with points A and B make up the segment of A and B designated $\overline{AB}$. The line containing A and B will be designated as $\overline{AB}$.

6 Postulate 7: Two distinct points A and B are in different half-planes of a line ℓ if and only if $\ell \cap \overline{AB} \neq \emptyset$.

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7 Postulate 8: Of three distinct collinear points one is between the two others.

9 Definition 7: A relationship between elements in the domain of a mapping is said to be invariant under the mapping if this relationship also holds for the images in the range of the mapping.

9 Theorem 4: If the intersection of two sets is non-empty then the intersection of their images under a one-to-one mapping is also non-empty. If their intersection is empty then the intersection of their images is also empty.

12 Definition 8: A transformation of the plane will be defined as a one-to-one mapping of the set of points of a plane onto the same set of points.

16 Postulate 9: For any line 'a' in a plane there exists a reflection mapping designated M_a which maps the points of one half-plane onto the points of the other half-plane.

16 Postulate 10: M_a maps straight lines onto straight lines, and "betweenness" is invariant as a relation among points.

16 Definition 9: The product of M_a and M_b will be: $(M_bM_a)(A) = M_b(M_a(A)) = M_b(B) = C$ for points A, B, and C and lines 'a' and 'b' where $M_a(A) = B$ and $M_b(B) = C$.

17 Postulate 11: For line 'a', $M_a^2(A) = (M_aM_a)(A) = M_a(M_a(A)) = A$ for all points A in the plane.

17 Postulate 12: For all points A in line 'a', $M_a(A) = A$.

19 Theorem 5: The reflection mapping is one-to-one and onto.

19 Definition 10: Ray AB will consist of point A, the points between A and B, point B, and all points C so that B is between A and C. It will be designated $\overrightarrow{AB}$.

19 Definition 11: An angle is a union of two distinct rays with a common vertex. $\overrightarrow{AB} \cup \overrightarrow{AC} = \angle BAC = \angle CAB$.

20 Postulate 13: For any angle there exists a unique line called the angle bisector such that a reflection in this line will map one leg of the angle onto the other leg.

20 Definition 12: If $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$ then the interior of $\angle ABC$ is equal to the intersection of the half-plane of $\overleftrightarrow{BA}$ containing C and the half-plane of $\overleftrightarrow{BC}$ containing A. The set of exterior points include those points which are neither part of the angle nor are interior points of the angle.

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21 Definition 13: If the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are collinear and $\overline{BA} \neq \overline{BC}$ then the angle formed is a straight angle. The interior of $\angle ABC$ then is either half-plane of the line formed by $\overrightarrow{BA} \cup \overrightarrow{BC}$.

23 Definition 14: A line 'a' is perpendicular to line 'b' ($a \perp b$) if and only if $M_a(b) = b$. It will be assumed that $a \perp b$ implies $b \perp a$. It must also be assumed that $a \neq b$.

23 Postulate 14: For any pair of points A and B there exists a unique line b which is the perpendicular bisector of $\overline{AB}$. This line b is defined so that $M_b(A) = B$.

23 Definition 15: The mid-point $\overline{AB}$ will be $b \cap \overline{AB}$ where 'b' is the perpendicular bisector of $\overline{AB}$.

23 Theorem 6: The mid-point of $\overline{AB}$ exists and is unique.

26 Definition 16: A finite product of reflection is a motion. We will designate an unspecified motion as F.

26 Postulate 15: If a motion leaves a line pointwise invariant then it is either the identity transformation or the reflection in the line. A line will be assumed pointwise invariant under a transformation if two of its points are invariant.

26 Postulate 16: If line 'a' is parallel to line 'b' and $d \perp b$ then $d \perp a$.

26 Definition 17: Let F represent a particular motion. If the motion S is such that $SF = I$ then we will say that S is the inverse motion of F and designate S as F^{-1}.

27 Theorem 7: For an unspecified motion F and line 'a', $FM_aF^{-1} = M_{a'}$, where $a' = F(a)$.

27 Theorem 8: 'a' is perpendicular to 'b' if and only if $M_aM_b = M_bM_a$.

28 Corollary: If $M_a = M_b$ then $a = b$.

28 Theorem 9: Perpendicularity as a relation between two lines is invariant under any motion.

29 Theorem 10: There is one and only one line perpendicular to a given line through a given point.

33 Definition 18: Two plane sets are congruent ($\cong$) if and only if one is the image of the other under some motion. Imagine R and R' to be some congruent geometric sets, then there exists some motion F such that $F(R) = R'$ but then F has an inverse F^{-1} and consequently $R = F^{-1}(R')$. Thus if there is a motion for which R' is the image of R then there is also a motion for which R is the image of R'.

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35 Theorem 11: If $M_s(A) = B$ for some line 's' then for any $P \in s$ it follows that $\overline{PA} \cong \overline{PB}$.

35 Theorem 12: If $\overline{AB} = \overline{AC}$ and $\overline{AB} \cong \overline{AC}$ then $B = C$.

36 Definition 19: $\overline{AB} < \overline{CD}$ if and only if $\overline{AB} \cong \overline{CB'}$ and B' is between C and D. Furthermore $\overline{AB} < \overline{CD}$ will imply that $\overline{CD} > \overline{AB}$.

36 Theorem 13: If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are rays in the same half-plane of $\overleftrightarrow{AD}$ and $\angle BAD \cong \angle CAD$ then $\overrightarrow{AB} = \overrightarrow{AC}$.

37 Definition 20: $\angle ABC < \angle DEG$ if and only if there is a motion F such that $F(\angle ABC) = \angle HEG$ and H is in the interior of $\angle DEG$.

39 Definition 21: A right angle is defined by a pair of perpendicular rays. An angle less than (<) a right angle is acute. An angle less than a straight angle but greater than (>) a right angle is obtuse.

39 Theorem 14: All right angles are congruent.

39 Theorem 15: All straight angles are congruent.

40 Definition 22: Two angles which have a common vertex and a common leg but no interior points in common will be called adjacent angles. The rays not common to the two angles will be called the exterior sides or exterior legs of the two angles.

40 Definition 23: (Angle Addition) The "sum" of two adjacent angles will be defined as the angle formed by their exterior sides.

43 Definition 24: Two angles are supplementary if they are congruent respectively to two adjacent angles whose exterior sides form a straight line. Two angles are complementary if they are congruent respectively to two adjacent angles whose exterior sides are perpendicular.

43 Theorem 16: If two angles are complementary to the same angle or to congruent angles, then they are congruent.

43 Theorem 17: If two angles are supplementary to the same angle or to congruent angles, then they are congruent.

44 Definition 25: Rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are opposite rays if A, C, and B are collinear and C is between A and B.

44 Definition 26: Two angles are vertical angles if the sides of one are opposite rays of the sides of the other.

44 Theorem 18: Vertical angles are congruent.

46 Definition 27: Triangle ABC ($\triangle ABC$) is isosceles if and only if two of the sides are congruent. The angle between the congruent sides is the vertex angle. The other two angles are the base angles of the isosceles triangle.

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46 Theorem 19: The angles opposite the congruent sides in an isosceles triangle are congruent.

46 Corollary: An equilateral triangle is equiangular.

46 Theorem 20: The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

47 Theorem 21: If $M_a(A) = B$ and $M_a(C) = D$ where $A \notin a$ and $C \notin \overleftrightarrow{AB}$, then C and D are in the same half-plane of $\overleftrightarrow{AB}$.

47 Theorem 22: If two angles of a triangle are congruent then the triangle is isosceles.

47 Corollary: An equiangular triangle is equilateral.

50 Theorem 23: (asa) If a side and its adjacent angles or one triangle are congruent respectively to a side and its adjacent angles of another triangle then the two triangles are congruent.

51 Theorem 24: (sas) If two sides and the included angle of one triangle are congruent respectively to the two sides and included angle of another triangle then the two triangles are congruent.

51 Theorem 25: (sss) If the three sides of one triangle are congruent respectively to the three sides of another then the two triangles are congruent.

55 Definition 28: A right triangle is a triangle with a right angle. The side opposite the right angle is the hypotenuse. The other two sides are the legs.

55 Theorem 26: (LL) If the legs of one right triangle are congruent to the legs of another then the right triangles are congruent.

55 Theorem 27: (HL) If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, the triangles are congruent.

57 Definition 29: An altitude of a triangle is a line passing through any vertex of a triangle perpendicular to the line of which the opposite side is a segment.

57 Theorem 28: The perpendicular bisectors of the sides of a triangle intersect at a unique point called the circumcenter.

58 Definition 30: If a line passes through B and is perpendicular to line 'd' then the intersection of the perpendicular and the line 'd' will be called the projection of point B onto line 'd'. It will be designated B^d.

58 Theorem 29: Segments determined by any point on the bisector of an angle and its projections onto the legs of the angle are congruent.

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58 Theorem 30: The three angle bisector of the angles if a triangle intersect in a unique point called the incenter.

59 Theorem 31: (HA) Two right triangles are congruent if the hypotenuse and an acute angle of one are congruent to the hypotenuse and acute angle of another.

62 Theorem 32: No product of two reflections is equal to one reflection.

62 Theorem 33: If three lines intersect in a unique point or are parallel then the product of the three reflections defined by these lines is equal to one reflection.

66 Theorem 34: If a rotation has more than one fixed point then it is the identity.

67 Theorem 35: Two pairs of lines, all of which intersect at a common point, define the same rotation about that point if there exists a product of reflections in two lines for which one pair of lines is the image of the other pair of lines.

71 Definition 31: Two oriented angles are congruent if the pair of lines defining one oriented angle is the image of the pair of lines defining the other oriented angle in a product of two reflections.

72 Definition 32: If α, β, and γ represent oriented angles then:

(a) $\gamma = \alpha + \beta$ means $R_{(0,\gamma)} = R_{(0,\beta)}R_{(0,\alpha)}$.

(b) $\gamma = -\alpha$ means $R_{(0,\gamma)} = R_{(0,\alpha)}^{-1}$.

(c) $\gamma = 0°$ means $R_{(0,\gamma)} = I$.

(d) $\gamma = 180°$ means $R_{(0,\gamma)} = M_bM_a$ where $a \cap b = 0$ and $a \perp b$.

75 Definition 33: The rotation $R_{(0, 180°)}$ will be called a reflection in point 0 and will be designated M_0.

75 Theorem 36: $M_0^2 = M_0M_0 = I$ for any point 0 in the plane.

75 Theorem 37: Every line passing through 0 is mapped onto itself by M_0.

75 Theorem 38: For all points A in the plane, 0 is the mid-point of the segment formed by A and $M_0(A)$.

75 Theorem 39: If 'a' is a line which contains 0 and B is a point disjoint from 'a' then $M_0(B)$ is in the opposite half-plane of 'a' from B.

75 Theorem 40: M_0 maps lines not passing through 0 onto parallel lines.

78 Theorem 41: If a given point and line in a plane are disjoint then there is one and only one line through the given point parallel to the given line.

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78 Theorem 42: If 'a', 'b', and 'c' are distinct, coplanar lines such that a||c and b||c then a||b.

78 Theorem 43: If a line intersects one of two parallel lines then it intersects the other also, provided all three lines are coplanar.

80 Definition 34: A transversal is a line which intersects each line of a set of lines but does not belong to the set itself.

80 Definition 35: Interior angles on opposite sides of the transversal with different vertices will be called alternate interior angles.

80 Definition 36: Alternate exterior angles are exterior angles on opposite sides of the transversal with different vertices.

80 Definition 37: Corresponding angles are angles on the same side of a transversal, one interior and one exterior, that have different vertices.

80 Theorem 44: When two parallel lines are intersected by a transversal then the alternate interior angles are congruent.

81 Theorem 45: When two parallel lines are intersected by a transversal then the corresponding angles are congruent.

81 Theorem 46: If two parallel lines are intersected by a transversal then the interior angles on the same side of the transversal are supplementary.

84 Theorem 47: If two lines are intersected by a transversal so that the alternate interior angles are congruent then the two lines are parallel.

84 Theorem 48: If two lines are intersected by a transversal so that the corresponding angles are congruent then the lines are parallel.

84 Theorem 49: If two lines are intersected by a transversal so that interior angles on the same side of the transversal are supplementary then the lines are parallel.

87 Definition 38: A translation is the transformation formed by the product of reflections in two parallel lines.

87 Theorem 50: If lines 'a', 'b', 'c', 'd', 'x', and 'y' are all parallel and if $M_y M_x(a) = c$ and $M_y M_x(b) = d$ then the translation identified by 'a' and 'b' is the same as that of 'c' and 'd'.

88 Theorem 51: A translation has no fixed point unless it is the identity.

88 Theorem 52: A translation can be defined by the product of two point reflections.

88 Theorem 53: The product of any two point reflections defines a translation.

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91 Theorem 54: Any translation can be given by a pair of points one of which can be chosen arbitrarily.

91 Theorem 55: The product of two translations is another translation.

92 Theorem 56: Under any translation a line is mapped onto itself or onto a parallel line.

92 Theorem 57: If a translation T maps A into A′ and B into B′ then $AA' \| BB'$.

94 Theorem 58: For any three points A, B, and C there exists a point D so that $M_C M_B M_A = M_D$.

94 Corollary: If A, B, and C are any three points then $M_C M_B M_A = M_A M_B M_C$.

94 Theorem 59: The product of two translations is commutative.

95 Theorem 60: If 'a' and 'b' are any two parallel lines and 'c' and 'd' are perpendiculars to these two parallel lines then the segment produced by the intersection of 'c' with 'a' and 'b' is congruent to the segment produced by the intersection of 'd' with 'a' and 'b'.

96 Theorem 61: If A′ is the image of A and B′ is the image of B under a translation then $\overline{AA'} \cong \overline{BB'}$.

100 Definition 39: A vector, designated $\|AB\|$ is the translation which maps A to B.

100 Definition 40:

1. If $T_1 = \|AB\|$ and $T_2 = \|CD\|$ then $\|AB\| + \|CD\| = T_2 T_1$.
2. If $T_1 = \|AB\|$ then $\|AB\| \neq \|BA\| = T_1^{-1}$.
3. $\|AB\| = I$ if and only if A = B.
4. If $T_1 = \|AB\|$ then $n\,\|AB\| = T_1^{\,n}$.

102 Definition 41: A parallelogram is a four-sided figure with vertices A, B, C, and D so that $\overline{AB} \| \overline{CD}$ and $\overline{BC} \| \overline{AD}$. It will be denoted ▱ABCD.

102 Theorem 62: If $T = M_B M_A$ then $T = 2\,\|AB\|$.

102 Theorem 63: For vectors $\|AB\|$ and $\|BC\|$, $2\,\|AB\| + 2\,\|BC\| = 2(\,\|AB\| + \|BC\|\,)$.

103 Theorem 64: The points A, B, C, and D are the vertices of a parallelogram if and only if $M_D M_C M_B M_A = I$.

104 Theorem 65: The figure formed by joining the mid-points of any four-sided figure (quadrilateral) is a parallelogram.

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106 Theorem 66: The sum of the oriented angles of a triangle is 180°.

107 Definition 42: In $\triangle ABC$, if $\overrightarrow{BA}$ and $\overrightarrow{BX}$ are opposite rays then $\angle CBX$ is an exterior angle of $\triangle ABC$.

107 Definition 43: In $\triangle ABC$ if $\angle CBX$ is an exterior angle then the angles $\angle A$ and $\angle C$ are the remote interior angles relative to $\angle CBX$.

107 Theorem 67: As oriented angles, an exterior angle of a triangle is equal to the sum of the two remote interior angles.

107 Definition 44: If in $\triangle ABC$ the points D, E, and G are the mid-points of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively then $\overline{DC}$, $\overline{EA}$, and $\overline{GB}$ are the medians of the triangle $\triangle ABC$.

107 Theorem 68: The diagonals of a parallelogram bisect each other.

107 Theorem 69: A line through the mid-point of a side of a triangle parallel to a second side intersects the third side at its mid-point.

108 Corollary: A line joining the mid-points of two sides of a triangle is parallel to the third side.

108 Theorem 70: The medians of a triangle intersect at a unique point called the centroid of a triangle.

111 Theorem 71: The product of reflections in lines which intersect in pairs in three points, or such that two of them are parallel and the third is a transversal, is a glide reflection.