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ABSTRACT

The purpose of this study was to develop and evaluate procedures for validating a learning hierarchy from test data. An initial hierarchy for the computational skills of adding rational numbers with like denominators was constructed using Gagne's task analysis. A test designed to assess mastery at each of the 11 levels in this hierarchy was administered to a large sample of elementary school children. The pass-fail relationships from this test data were analyzed with seven learning hierarchy validation procedures, and seven hierarchical orderings of the 11 subtasks were determined. Fourth grade subjects, randomly assigned to seven treatment groups determined by the seven hierarchical orderings, worked 30 minutes a day on programmed materials until they were completed. Achievement tests measured acquisition of the terminal task. On the following day a transfer test on subtraction of rational numbers was administered. Two weeks later an alternate form of the achievement test was administered as a retention test. Results of the study indicate that sequence seems to have little effect upon immediate achievement and transfer. However, longer term retention seems quite susceptible to sequence manipulation. In general, the authors report that optimal instructional sequences can be devised using learning hierarchies validated from test data. (Author/JG)

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Analyzing Learning Hierarchies Relative to Transfer Relationships Within Arithmetic

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U. S. DEPARTMENT OF
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Summary

The purpose of this study was to develop and evaluate procedures for validating a learning hierarchy from test data. An initial hierarchy for the computational skills of adding rational numbers with like denominators was constructed using Gagne's task analysis. A test designed to assess mastery at each of the 11 levels in this hierarchy was administered to a large sample of elementary school children. The pass-fail relationships were analyzed using the following validation procedures adaptable for use with test data: (1) Pattern Analysis, (2) Guttman Scalogram Analysis, (3) the AAAS Approach, (4) Item Difficulty, (5) A correlational procedure developed using the phi coefficient, (6) Textbook Approach, (7) Randomization. Thus, 7 hierarchical orderings of the 11 subtasks were generated.

Programmed instructional materials were developed utilizing one lesson for each of the 11 subtasks. An evaluation of the efficacy of each validation procedure was conducted by actually sequencing the learning materials according to the hierarchies generated by each method and determining the effect of sequence upon time to complete the program, achievement, transfer, and retention. Fourth grade subjects were randomly assigned to the 7 treatment groups, defined by the 7 hierarchical orderings. Subjects worked independently through the programmed booklets devoting approximately 30 minutes per day to the materials until they were completed. Upon completion of the program each subject was given an achievement test measuring acquisition of the terminal task. On the day following completion of the booklet, a transfer test on subtraction of rational numbers was administered. Two weeks later an alternate form of the achievement test was administered as a retention test. An analysis of variance design was used to investigate the differential effects of sequence upon time to complete the program, achievement, transfer, and retention.

The results of this study indicate that the overall efficiency of the learning process can be affected by sequence manipulation. Sequence, even if random, seems to have little effect upon immediate achievement and transfer to a similar task. However, longer term retention seems quite susceptable to sequence manipulation. The mean retention score for the AAAS sequence group was significantly higher than those two of the other sequence groups. No significant differences were found across sequence groups on the mean number of minutes used to complete the program. In general, the results suggest that optimal instructional sequences can be derived using learning hierarchies validated from test data.

Introduction

One of the major problems encountered by both teachers and authors of instructional materials is the sequencing of instructional activities (Hartung, 1969; Hickey and Newton, 1964; Gagne, 1967 ; Briggs, 1968; Heimer, 1969). Gagne (1963) stated that "the design of an instructional situation is basically a matter of designing a sequence of topics." This statement suggests that there are optimal sequences of learning events for the acquisition of a given terminal task. Do such optimal learning sequences exist? How are they determined, and how are they verified?

There is substantial evidence to support the general theory of the hierarchical structure of knowledge. The results of several studies (Gagne and Brown, 1961; Gagne and Paradise, 1961; Gagne, Mayor, Garstens and Paradise, 1962) suggest that new skills and knowledge emerge from lower order knowledge, and that there is a significant amount of positive transfer from each successive subordinate level to the next higher level in a hierarchical ordering of such levels. In these studies Gagne and his associates did determine optimal learning sequences for different mathematical tasks.

The sequence of subordinate tasks in a learning hierarchy, after sufficient validation, should describe a teaching program that will effectively accomplish the instructional objectives. That is, an instructional sequence based on the levels in the hierarchy will represent an optimal route for acquisition of the terminal task by a sample of learners.

Empirical evidence supports an affirmative answer to the question, "Do optimal learning sequences exist?" Recent studies of sequencing (Niedermeyer, Brown & Sulzen, 1969; Brown, 1970) indicated that the logical sequence group (based on learning hierarchies) performed reliably better than the scrambled group relative to time to complete the instructional program, errors made on the program, and errors made on a criterion test of complex problem solving skills. Task analysis, though imprecisely defined, is a workable tool in the identification of the subordinate levels in a learning hierarchy. However, the question of validating the ordering of the levels in a hierarchy is a much more complex and illusive problem. Gagne (1968) stated that various methods have been tried but none seem entirely satisfactory as yet.

Gagne's (1962) approach to hierarchy validation assumed that all lower level tasks, on which a higher level task is dependent, must be mastered before the higher level task is mastered. Validating a hierarchy based on this assumption is a tedious undertaking. The primary method employed involves the use of programmed learning materials. Using such materials, studies were conducted both in the laboratory with individual subjects and in classrooms, where groups of subjects responded to the learning programs. Numerous variations in the basic program were investigated such as "high" or "low" guidance, "high" or "low" repetition and varying presentation orders of the frames. The proportion of positive transfer to each higher-level knowledge from the relevant lower-level knowledges was calculated by the formula:

$$P+ = \frac{A + B}{A + B + C}, \text{ where}$$

P+ = Proportion of positive transfer

A = Number of learners passing both lower level and adjacent higher level tasks

B = Number of learners failing both lower level and adjacent higher level tasks

C = Number of learners failing lower level tasks but passing adjacent higher level task

Perfect validity would be indicated by a ratio of 1.00. Gagne's work, after considerable revision, produced ratios of 1.00 or very near 1.00 and validated his hypothesized hierarchies.

The hierarchical analysis used by Gagne in association with the programmed learning sequences was adequate. However, Phillips and Kane (1970) found this procedure to be inadequate when applied to test data. Previous exposure of subjects to the subtasks of the hierarchy, confounded the issue of positive transfer. This study was directed toward investigating the use of pattern analysis, scaling techniques, and correlational approaches in validating learning hierarchies. In brief, the purpose of this research was to develop and evaluate indirect procedures for validating a learning hierarchy from test data. It seems imperative that efforts to develop efficient procedures for validating learning hierarchies be undertaken. Gagne's direct approach to learning hierarchy validation is valid but too tedious and costly to be undertaken in the cassroom or by publishers of commercial instructional materials.

An indirect validation procedure is defined as one based on the analyses of test data in contrast to a direct validation procedure based on the results of an instructional program. An outline of the steps undertaken to achieve these objectives is given below.

Using Gagne's task analysis a learning hierarchy for the computational skills of rational number addition involving like denominators was constructed. Based on this logical ordering of the subtasks, a test was constructed to assess mastery at each level in the hierarchy. The test was administered to 163 elementary school children in grades 4 through 6 to obtain a wide range of ability levels. The pass-fail relationships were analyzed using several indirect validation procedures including (1) Item difficulty (Nunnally, 1967), (2) The AAAS Approach (AAAS Commission on Science Education, 1968), (3) The Guttman Technique (Torgerson, 1958), (4) Pattern Analysis (Rimoldi and Grib, 1960) and (5) Correlational analysis (described in the procedures section).

Based on the logically ordered hierarchy a programmed instructional sequence was developed utilizing one lesson for each subtask in the hierarchy. To test the efficacy of each of the indirect validation procedures, the programmed lessons were sequenced according to the hierarchies generated by each. Fourth grade subjects (142) were randomly assigned to the following treatments.

(1) Logical Sequence - an ordering was based upon the sequence of subtasks generated by a task analysis of the instructional objectives. No empirical validation of this ordering was attempted. The index of agreement (Rimoldi & Grib, 1960) was calculated.

(2) Item Difficulty - an ordering was determined by arranging the subtasks from simplest to most complex based on item difficulty.

(3) Guttman Technique - an ordering was determined by the Guttman Scaling Technique. This procedure orders the subtasks such that the reproducibility coefficient is maximized.

(4) Correlational Analysis - an ordering was determined by arranging the subtasks so that the correlation between adjacent tasks was maximized.

(5) AAAS Approach - an ordering was determined by the AAAS hierarchy validation procedure.

(6) Textbook ordering - an ordering was determined by arranging the subtasks in the usual textbook sequence. This was accomplished by sequencing the subtasks in an ordering which gave closest fit to that given in several elementary mathematics texts.

(7) Random Sequence - an ordering was determined by randomly ordering the subtasks identified by task analysis.

Subjects worked through the programmed booklets independently devoting 30 minutes per day to the materials until they were completed. Upon completion of the program each subject was given the same achievement test measuring acquisition of the terminal task. On the day following completion of the booklet, a transfer test on subtraction of rational numbers was administered. Two weeks later a retention test (alternate form of the achievement test) was administered. An analysis of variance design was used to investigate the differential effects of sequencing on time to complete the program, achievement, transfer, and retention. Based on these results inferences were made concerning the adequacy of the indirect validation procedures for generating an optimal learning sequence.

Review of Related Research

This study was concerned with the development and evaluation of indirect procedures for validating learning hierarchies in mathematics. The research reviewed consists of studies (1) concerned with the sequencing of instructional materials in mathematics and (2) dealing with the psychometry of learning hierarchies.

Sequencing Instructional Materials in Mathematics

Instructional design in mathematics requires decisions about structuring the content and designing and ordering a set of instructional tasks. Gagne (1967) and Briggs (1968) have proposed the use of instructional sequences that require the learner to follow a specific route through a content structure. Basic to this theory is the assumption that instructional sequence is most fruitfully formulated in conjunction with content structure and that instructional sequence specifies the path the learner is to travel through the content structure. Thus in Gagne's learning hierarchy theory, task analysis is a major tool in designing instructional sequences.

Heimer (1969) indicated that there are no clear guidelines for the development of curricular materials. However, Gagne's theory and research pertaining to learning hierarchies and the sequencing of instructional materials have stimulated much research regarding sequencing variables. In the search for more definitive prescriptions for instructional design, seemingly contradictory results have been reported. Studies providing both evidence for and against using rigorous methods of content sequencing are reviewed.

Roe, Case, and Roe (1962) conducted a comparative study of sequencing utilizing a 71 item program on elementary probability. One group of students received a logically ordered form of the program, and one received a random version of it. A criterion test was administered to each subject immediately upon completion of the program. No significant differences were reported on time required for learning, errors during learning, criterion test score, or time required for criterion test. In a similar study using an extended version the same probability program, Roe (1962) found that the logically ordered sequence group performed significantly better on learning time, errors made during learning and on post learning measures. Roe (1962, p. 409) stated that "careful sequencing of items has a significant effect on student performance, at least for programs of some length and complexity."

Levin and Baker (1963) reported a study in which a 60 item geometry program for second graders was scrambled within 20 item blocks. The results showed no significant differences in measures of acquisition, retention, or transfer between those who worked through the logical program and those who completed the scrambled program.

Payne, Krathwohl, and Gordon (1967, p. 125) stated that "no one seems to doubt, that were one to scramble a whole course that learning would be retarded, so that in part, the size of the unit in which sequence is destroyed is a factor... there may be a continuum of dependence on sequence. At one extreme of the continuum, scrambling may have no effect on learning a set of spelling words which has no logical structure. At the other extreme, scrambling would be expected to result in considerable decrease in learning if the learning of one concept were prerequisite to learning the next in a logical hierarchy." Payne et al. (1967) designed a study to test the foregoing assumption. They examined the effects of scrambling upon the learning of three programs. The three programs were ranked by trained, independent observers from low to fairly high in logical interdependence. It was hypothesized that the effect of scrambling would be greatest for those programs dealing with tasks having the most logical development. The results of both immediate and delayed retention tests did not confirm this hypothesis.

Miller (1965) conducted a study in which a 98-frame program on topics in ratio and proportion was presented in logical and random sequences to seventh graders. The author reported substantial differences in error rates which supported the interdependency of the frames. The results, however, indicated that the scrambling of frames had little, if any, effect upon learning from the program.

Holland (1965) and Niedermeyer (1968) expressed concern over design and methodological weaknesses of the studies cited above. Holland (1965) pointed out that error rates for the two programs used in Roe et al. (1962) did not differ significantly suggesting that the items were not interdependent even in the logical sequence. Thus, no significant differences in criterion test scores should have been expected. Holland claimed that the items used in the Levin and Baker (1963) study were not hierarchical in structure. Further Levin and Baker suggested that the instructional materials used were relatively ineffective. Therefore, the results were not surprising.

Niedermeyer (1968) pointed out that error rates in the programs used in the Payne et al. (1967) study were not substantially different. Again raising the question of whether any learning difference should have been expected. The authors pointed out another flaw in their design. Many of the subjects already knew a considerable amount of the material presented in learning sequence. Thus, any meaningful assessment of sequence effect on learning was difficult to obtain.

Pyatte (1969) indicated a major problem with studies comparing logical and random ordered sequences. It is difficult to determine if the logical sequence is actually logical and the random sequence "random." That is, that the random sequence is completely unbiased in the sense that none of the subtasks remain in the hypothesized hierarchical ordering.

Several studies by Gagne and his associates, reviewed earlier, showed that when programmed materials in mathematics were sequenced according to learning hierarchies, acquisition of higher level tasks occurred only if all prerequisite subordinate tasks were mastered. King (1970) described some key studies utilizing programs based on hierarchies and well controlled learning situations avoiding methodological weaknesses pointed out in previous studies. These studies are reviewed below.

Using a program developed by Gagne and Brown (1961) for which there existed empirical data verifying a high degree of interdependence among its concepts, Niedermeyer, Brown and Sulzen (1969) conducted a study of sequencing. The program consisted of 70 introductory frames and 40 guided-discovery frames which required the learner to recall and use concepts learned in the introductory section. The terminal objective of the program was for the learners to discover a formula for the sum of n terms of a series. Three groups completed a logical, scrambled, and reverse form of the program. The authors found that only the logical sequence group performed significantly better than a control group on both a test of concepts and a transfer test. The scrambled sequence group performed significantly better than the control group on the concept test, but not on the transfer test. No significant differences among the three sequence groups were found on the posttests.

Wodtke, Brown, Sands, and Fredericks (1967) administered a 74-frame program on number bases in logical and scrambled versions. The hierarchical structure of the program was evidenced by the difference in error rates. On a pretest assessing foreknowledge of program material 90% of the subjects scored zero. The mean posttest achievement score was 18 on a 22 item test indicating the subjects actually learned from the program. However, there were no significant differences on achievement or aptitude-sequences interaction between those who studied the logical program and those who studied the scrambled program.

Miller (1969) conducted a study using eight program sequences on matrix arithmetic. The results showed that the scrambled sequences worked as well as the logical sequences for definitions and addition of matrices. However, in sequences where subjects were forced to learn matrix multiplication before learning definitions and matrix addition they performed significantly worse than those who learned needed definitions and matrix addition first. Miller concluded that mastery of individual tasks in a hierarchy can be accomplished in several ways including a scrambled programmed sequence. However, a logical sequence still appears to be best in terms of overall effectiveness and efficiency.

In a recent study Brown (1970) found that logical sequencing facilitated learning of programmed mathematical materials. Two sequence versions, logical and scrambled frame order, of the guided discovery program on number series used by Gagne and Brown in their 1961 study of concept formation, were compared. The logical sequence group performed reliably better than the scrambled group relative to time to complete the instructional program, errors made on the program, and errors made on a criterion test of complex, problem-solving skills. Brown concluded that when a sequence involves tasks that are complex problem solving behaviors that Gagne classifies as intellectual skills, ordering is an important factor in learning. Even for bright and relatively mature learners sequence can have an important effect upon learning.

In summary, it appears that mastery of individual subtasks in a hierarchy can be achieved in several ways, including learning from randomly programmed sequences. However, as Miller (1969) pointed out logical sequencing still appears to be best in terms of overall efficiency and effectiveness. Several of the studies reviewed here suggest that varying sequences of instructional stimuli which have high interdependency does not make much difference in effectiveness of instruction. However, many of these studies are plagued with design problems. Thus, before the results of research can be applied to the problems of sequencing instructional materials in mathematics for use in the ordinary classroom situation, substantial study of the effects of sequence upon time to achieve the terminal behavior, achievement, transfer, and retention should be undertaken.

The Psychometry of Learning Hierarchies

Validating a learning hierarchy is not a simple undertaking. Many researchers (Ausubel, 1963; Bruner, 1964; Gagne, 1965; Glaser, 1964; and Suppes, 1966) have long recognized that sequence is a critical variable in learning. The learner begins with simple tasks and progresses to increasingly complex tasks. However, both Gagne (1968) and Pyatte (1969) have pointed out that determination of this hierarchical ordering of subtasks from simplest to most complex is a major problem. Various procedures for validating the hypothesized ordering of the subordinate tasks in a learning hierarchy are reviewed below.

Gagne and Paradise (1961) were pioneers in learning hierarchy validation. Their approach was direct validation based on learners' responses to a programmed learning sequence and criterion tests administered immediately after the instructional program to establish pass-fail patterns for each component of the learning hierarchy. Consider the simple two-level hierarchy in Figure 1.

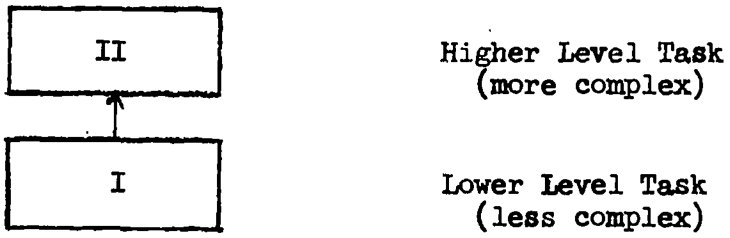


Figure 1. A Simple Two-level Hierarchy

Gagne's validation procedure was based on the assumption that task I must be mastered before task II can be mastered. Failure on task I would automatically produce failure on task II. Using + and - to represent pass and fail respectively, there are four possible pass-fail relationships which can be observed: (++), (+-), (--), (-+). For example, the first relationship signifies that the learner passed (performed to criterion) both task I and task II. Only the relationship (+-) is in direct contradiction of the theory and indicates a flaw in ordering. The relationship (-+) (passed lower level task but failed higher level task) indicates a weakness in the instructional program but provides no information concerning the validity of the hierarchy.

To validate a hierarchy Gagne analyzed the pattern of responses of each transfer in the hierarchy. That is, he constructed a contingency table of the observed responses to a higher level task and the task immediately prerequisite to it as illustrated in Figure 1. He calculated the following ratio to determine the degree of validity of the hierarchy.

$$\text{Proportion of Positive Transfer (P+)} = \frac{(++) + (--)}{(++) + (--) + (+-)}$$

Perfect validity would be indicated by a ratio of 1.00. If all learners contradicted the theory, having observed patterns (+-), then the ratio would be zero. Thus, P+ is bounded above and below by 1 and 0 respectively. The degree of validity of any hierarchy is measured by P+ with the lower limit of acceptability for P+ being .90.

Phillips and Kane (1970) investigated the efficacy of this ratio when applied to test data alone. Using Gagne's task analysis a learning hierarchy for the computational skills of whole number addition was constructed. Based on the hypothesized ordering of the subordinate levels, a test was constructed to assess mastery at each level. A second test utilizing a random ordering of the same items was constructed. Both tests were administered to a large sample of elementary school children in grades 3 through 6 in order to obtain a wide range of ability levels. The proportion of positive transfer between adjacent items on both tests was computed using Gagne's formula. The proportions between adjacent items on both tests were above .90, except in two instances. Thus both the hypothesized and the random hierarchies were validated by this procedure. The authors concluded that prior educational experiences confounded the issue of positive transfer when considering test data alone.

The Staff of the American Association for the Advancement of Science (AAAS Commission on Science Education, 1968) have refined and extended Gagne's approach to hierarchy validation. Task analysis is used to generate hierarchies of subordinate subtasks. Learning sequences are designed to correspond to the hypothesized hierarchies. Pass-fail contingencies are used to test the dependency of each individual task on its immediate prerequisite subtasks. The Staff of AAAS pointed out that a high proportion of positive transfer - Gagne's statistics - is a necessary but not a sufficient condition for a valid hierarchy. Using the pass-fail relationships defined by Gagne, the AAAS defined the following three ratios:

$$(1)\quad \text{Consistency ratio} = \frac{(++)}{(++) + (+-)}$$

$$(2)\quad \text{Adequacy ratio} = \frac{(++)}{(++) + (-+)}$$

$$(3)\quad \text{Completeness ratio} = \frac{(++)}{(++) + (--)}$$

Ratio (1) is a measure of how consistent the data are with the hypothesized dependency. Ratio (2) is a measure of the adequacy of the identified subordinate tasks. Ratio (3) is a measure of the effectiveness of instructional materials designed to bring about learning. In the development of *Science -- A Process Approach*, the AAAS has considered high consistency, adequacy, and completeness ratios as the necessary and sufficient set of characteristics for a valid learning hierarchy. No significance test has been developed for either Gagne's ratio or those defined by the AAAS.

Cox and Graham (1966) used the Guttman (1944) Scalogram Analysis to develop a sequencially scaled achievement test. Essentially, the Guttman technique (Torgerson, 1958) orders items such that from knowledge of a learner's total score, his response pattern to the set of items can be predicted. The coefficient of reproducibility defined by:

$$\text{Rep} = 1 - \frac{\text{total number of errors}}{\text{total number of responses}}$$

indicates the degree to which a set of items forms a perfect scale. Error is defined as instances where a subject passes a higher level item after failing a lower level prerequisite item. Guttman suggested .90 as an acceptable lower limit for Rep. Cox and Graham reported a reproducibility coefficient of .97 for their final arrangement of items and thus concluded their hierarchical ordering as valid.

Several studies have been directed toward the validation of Bloom's *Taxonomy* (1956). Kropp, Stoker and Bashaw (1966) pointed out that the difficulty of the cognitive skills should increase as the hierarchy is ascended. Since mastery of lower level tasks is required before mastery of higher level more complex tasks, the difficulty of items should increase upwards along the hierarchy. Studies by Stoker and Kropp (1964) and Herron (1965) showed that the cognitive skills in the *Taxonomy* did form a hierarchy. Items assessing skills higher up in the *Taxonomy* were more difficult than those at lower levels in the *Taxonomy*. Stoker, Kropp and Bashaw concluded that their results based on item difficulty validated the *Taxonomy*.

There are several methods of hierarchical analysis reported in the literature which are used in the generation of hierarchies rather than the validation of deductively analyzed hierarchies. McQuitty (1960) developed a procedure for determining if a hierarchical structure underlies a set of items. He began with a large item pool with no *a priori* assumptions regarding the relationship between items in terms of complexity. The procedure consisted of combining pairs of items or variables which correlated highest with each other to form new items. This procedure is repeated until one pair of items remains. When items are plotted on a linear scale and successive pairs are connected by lines, the resulting diagram has a hierarchical structure. Smith (1968) employed McQuitty's method to investigate the hierarchical model underlying Bloom's Taxonomy. In general, Smith concluded that his analysis supported the Taxonomy rationale of a cumulative and hierarchical continuum of cognitive processes. McQuitty (1966) has further refined and improved his method of hierarchical syndrome analysis.

Multiple Scalogram Analysis (MSA) developed by Lingoes (1963) can be used to generate a hierarchical ordering from a large set of items for which there are no *a priori* assumptions made regarding order. The procedure essentially accomplishes the same goal as the Guttman Technique with built in controls against spuriously high reproducibilities as a function of extreme marginal values. The MSA method involves selecting an item from a set to be analyzed, finding that item among the remaining items which is most like it, determining the number of errors between the candidate item and all of its predecessors, and finally, applying a statistical test (X^2) of significance to adjacent item pairs. Whenever either the error or statistical criterion fails, the scale is terminated and another scale is started with a new item chosen from among those that remain, until the item set is exhausted. This procedure tends to produce several branches which have very little in common with one another.

Resnick and Wang (1969) have used MSA to generate various hierarchies. The methods developed by McQuitty (1960) and Lingoes (1963), as well as several other methods of hierarchical analyses outlined by Torgerson (1958), are not readily adaptable for use in validating hypothesized hierarchies. In these methods the data must speak for themselves with no *a priori* assumptions concerning order.

Carroll (Resnick and Wang, 1969) has developed a hierarchy validation procedure based on the correlation between items or subtasks. This method, like those of Gagne and the AAAS, begins with the construction of pass-fail contingency tables for all possible pairs of items in the hierarchy. The phi-phimax coefficients are then computed for each table. Phi is the correlation between two adjacent dichotomous items (Nunnally, 1967). Phimax is an estimate of the highest-possible phi coefficient given the marginals of the contingency table. The use of phimax in the denominator controls against artificial inflation due to extreme pass-fail rates, since this value becomes larger as the pass-fail rate becomes larger. This procedure is most useful in empirical searches for hierarchical relationships among large quantities of data. In discussing different approaches to hierarchy validation, Resnick and Wang (1969) cited no studies utilizing Carroll's method.

Grib and Rimoldi (1960) developed a procedure for comparing two patterns of responses from a number of subjects on a number of items. Responses to items may or may not be dichotomously scored. Listing each subject's responses to a set of items produces an observed matrix of responses with rows corresponding to subjects and columns corresponding to items. An expected matrix can be formed based on an operational definition of what response patterns are expected from a given set of items. The only restriction on the expected matrix is that the subjects total score on the expected pattern must equal his total score on the observed pattern. These two patterns can be compared and an index showing the amount of agreement between the two patterns can be computed using the following equations.

When the expected matrix of ones and zeros (1 = pass; 0 = fail) has been generated weights are calculated for each cell of the expected pattern. For the cells containing ones, the weights a_{ij} are given by

$$a_{ij} = \frac{R_i \; C_j}{\Sigma \; R_i} \quad \text{, where}$$

R_i = number of ones in the i^{th} row.

C_j = number of ones in the j^{th} column.

For the cells containing zeros, the weights $\overline{a_{ij}}$ are given by

$$\overline{a_{ij}} = \frac{\overline{R_i} \; \overline{C_j}}{\overline{\Sigma \; R_i}} \quad \text{, where}$$

$\overline{R_i}$ = number of zeros in the i^{th} row.

$\overline{C_j}$ = number of zeros in the j^{th} column.

The index of agreement I_a, indicating the amount of agreement or correlation between two patterns, is given by

$$I_a = \frac{A_t - {}_mA_t}{T - {}_mA_t}$$, where

T = sum of all a_{ij} and $\overline{a_{ij}}$

A_t = sum of a_{ij} and $\overline{a}_{ij}$ of the cells that are the same in the observed and expected patterns.

${}_mA_t$ = sum of a_{ij} and $\overline{a_{ij}}$ corresponding to minimum possible agreement.

The above formula is used when comparing patterns across subjects. A similar formula can be derived for comparing patterns across items. The index of agreement varies between zero (no agreement) and one (perfect agreement). Grib and Rimoldi compared values of I_a to the coefficients obtained using the pattern analysis developed by Green (1956) and found I_a gives conservative values. The authors report no significance test for the index of agreement.

In summary, Gagne's procedure for validating a learning hierarchy using programmed learning sequences is adequate and useful in research designed to gain empirical evidence to support the hierarchical structure of knowledge. However, for validating specific hierarchies to be used as guidelines for the sequencing of classroom instructional activities, less expensive indirect procedures using test data should be developed.

Several types of hierarchical analysis (McQuitty, 1960; Lingoes, 1963; Torgerson, 1958) are useful in generating hierarchies from a large item pool with no _a priori_ assumptions regarding the relationship between items in terms of complexity. In these methods the data must speak for themselves with no _a priori_ assumptions concerning order. Thus, these procedures are not readily adaptable for use in validating hypothesized hierarchies.

It seems that hierarchical analysis such as the Guttman Technique (Torgerson, 1958), Pattern Analysis (Rimoldi & Grib, 1960) and the AAAS Approach (AAAS Commission on Science Education, 1968) are quite adaptable for validating deductively analyzed hierarchies from test data. Other procedures based on the correlation between test items or levels of the hierarchy and the difficulty of items could be useful in validating a hierarchy from test data. The present study was designed to test the adequacy of each of these procedures for validating a learning hierarchy from test data by sequencing instructional materials according to the hierarchy generated by each and determining the effect of sequence upon achievement, transfer, retention and time to complete the instructional sequence.

Development of Materials and Procedures

The major theme of the present research was the differential effects of sequencing instructional programs according to a learning hierarchy validated by various procedures upon (1) time required to complete the program, (2) achievement, (3) transfer, and (4) retention. Will one sequence produce maximal achievement, facilitate the greatest amount of transfer and retention, and require less time to complete? This research was designed to answer these questions. Based upon the answers inferences were drawn concerning the adequacy of the indirect validation procedures used in generating each instructional sequence.

In order to accomplish effectively the objectives of this study, experimental decisions had to be made concerning the appropriate terminal task for the hierarchy and the sample of learners to participate in the experiment. These concerns are discussed below.

1. Terminal Task: The time of the year in which the study was conducted influenced the choice of the terminal task to be used in developing the learning hierarchy. Subjects were to work through the instructional materials in April. This meant a terminal task had to be selected which the subjects had not previously been exposed to yet at the same time ensuring that the learner had the necessary background for achieving the instructional objectives. Through the aid of textbooks and suggestions from elementary school teachers, the addition of rational numbers involving like denominators was chosen as the terminal task.

2. Samples: First, a sample of subjects for collecting data to be used in the hierarchy validation had to be selected. In order to obtain a wide range of ability levels, tests were administered to 163 elementary school children in grades 4 through 6. The fourth graders included in the sample were high achievers who had begun studying addition of rational numbers. Such a range was chosen so that both correct and incorrect responses were obtained at all levels in the hierarchy. Second, a sample of learners to work through the instructional material was selected on the basis

of two pretests. These pretests were designed to assess entering behaviors. Pretest I was designed to determine if the learners had acquired the necessary prerequisites for mastering the skills in the learning sequence. Pretest II was designed to weed out those learners who had already acquired the skills to be taught in the learning sequence. In this manner, 142 fourth graders were selected to participate in the experiment.

Throughout this report the reader will encounter some terms which may be unfamiliar. Operational definitions of these terms are given here.

1. Optimal Learning Sequence: An optimal learning sequence was defined as one which maximally facilitates achievement, transfer, and retention.

2. Valid Hierarchy: A valid hierarchy was defined as one which yields an optimal learning sequence.

3. Adequate Indirect Validation Procedure: An evaluation of the indirect validation procedures was made by actually sequencing instructional materials according to the hierarchies generated by each validation procedure. The hierarchy yielding the best approximation of an optimal sequence, as defined above, was considered valid. The adequacy of each indirect validation procedure was measured by the degree to which a hierarchy validated by the procedure yielded an optimal learning sequence.

Development and Analyses of the Hierarchy

Using Gagne's task analysis a learning hierarchy for the computational skills of rational number addition was constructed. The sequence of subtasks generated was reviewed by four authors of elementary mathematics texts. Based upon this formative evaluation of the adequacy and completeness of the hierarchy, the sequence of subtasks was revised until, to experts in mathematics education, there were no obvious flaws in the learning hierarchy. The hypothesized hierarchy is shown in Appendix B.

Based on the hypothesized ordering of the subordinate levels, a test was constructed to assess mastery at each level in the hierarchy. The test was designed to minimize chance or careless errors. A procedure of test construction similar to the "H-technique" (Stouffer, Borgatta, Hays, and Henry, 1952) was adopted. The test consisted of composite test items for each level in the hierarchy. Each composite item consisted of three items testing the same subordinate task. Pass at each level was defined as correct responses to at least two of the three items for the level. A sample test item for level VII is given in Figure 2; the entire test is given in Appendix D.

4 1/9	1 3/8	13 5/12
+5 2/9	+2 1/8	+ 6 5/12

Figure 2. Sample Test Item

The entire test consisted of 11 composite items making a total of 33 items. The internal consistency of the test was determined using the Kuder-Richardson Formula 20 (Nunnally, 1967).

The test was administered to 163 elementary school children in grades 4 through 6 to obtain a wide range of ability levels. It was administered by the classroom teachers and was completed by all subjects in one sitting. The test was not a timed test. Subjects were instructed to attempt all items and were given sufficient time to do so.

The pass-fail relationships were analyzed using various indirect validation procedures. These procedures are described below.

Pattern Analysis

The pattern analysis technique developed by Rimoldi and Grib (1960) was used to analyze the responses for the complete hierarchy on a subject by subject basis. As previously described, the index of agreement I_a indicates the amount of agreement or correlation between two patterns. In this case, I_a was calculated between the observed and expected patterns of pass-fail relationships.

The items on the test were sequenced according to the hypothesized ordering of the subordinate levels in the hierarchy. If this ordering was truly hierarchical, where each subtask was a necessary prerequisite to the next, once a learner failed a given level he would be expected to fail all subsequent levels. Thus the expected pattern was defined as one where no correct response followed an incorrect response. Using 1 and 0 to represent pass and fail respectively, a hypothetical observed matrix and the corresponding expected matrix are given in Figure 3. The responses of each subject were recorded on punch cards. A program was written which transformed each subject's observed pattern of responses into the expected pattern as indicated in Figure 3. Weights for each cell and the index of agreement were calculated using the formulas given earlier.

Items

Ss	1	2	3	4	5	6	7	8	9	10	11	Score
1	1	1	0	1	1	0	0	0	1	0	0	5
2	1	1	1	0	1	1	1	0	1	1	1	9
3	1	1	1	1	0	0	0	1	0	0	0	5

Observed Pattern

Items

Ss	1	2	3	4	5	6	7	8	9	10	11	Score
1	1	1	1	1	1	0	0	0	0	0	0	5
2	1	1	1	1	1	1	1	1	1	0	0	9
3	1	1	1	1	1	0	0	0	0	0	0	5

Expected Pattern

Figure 3. Sample observed and corresponding expected matrices.

Item Difficulty

The pattern of responses to the 11 items were analyzed to determine if the items were hierarchically ordered in terms of item difficulty. Required mastery of certain lower level necessary prerequisite subtasks before the next higher subtask can be mastered implied that the number of learners passing a lower level task must be greater than or equal to the number passing the next higher level task. In other words if the items were arranged hierarchically from simplest to most complex, the first item would be mastered by most learners tested. Each successive level or item would be more difficult in terms of the necessity of more recall from preceding prerequisite items. Therefore, the number of learners passing each higher level item would decrease. The difficulty of any dichotomus item (p-value) is the fraction of learners tested who passed that particular item or level (Nunnally, 1957). For instance, a p-value of .90 would mean that 90 percent of the learners tested passed the item. Thus, the observed p-values for the levels of the hierarchy should form a decreasing sequence of values. The maximum p-value should occur at the lowest level in the hierarchy and the minimum p-value should occur at the terminal task.

As previously stated, the test items were arranged according to the logically constructed hierarchy for the addition of rational numbers. A hierarchical chart of the 11 items or levels was drawn and the p-value for each was taken from an item analysis and entered in the hierarchy. This hierarchy of observed p-values was examined to determine if the expected pattern was exhibited. That is, the p-values should be higher at the lower levels of the hierarchy and decrease to a minimum at the terminal task. Serious deviations between the observed p-value rank and the expected p-value rank would indicate that the items were not hierarchically ordered.

It should be noted that validation based on p-values could only be made by comparison. There was no statistical test for significance of difference. One must appeal to his knowledge of the subject matter at hand rather than rigorous statistical procedures. For instance, suppose the following three items were hypothesized as being hierarchically ordered.

$$\text{I.}\quad \begin{array}{r} 4 \\ \underline{+5} \end{array} \qquad\qquad \text{II.}\quad \begin{array}{r} 14 \\ \underline{+23} \end{array} \qquad\qquad \text{III.}\quad \begin{array}{r} 27 \\ \underline{+19} \end{array}$$

Further, suppose the observed p-values for items I, II, and III were .94, .87 and .89 respectively. Since item III involves renaming and item II doesn't, item III would not be placed before item II based on this slight difference in p-values. However, if the difference was considerably greater, one would have to reorder the items and assume item II to be more difficult than item III. Also, the p-value at any level can be affected by simply constructing a more difficult item testing the same subtasks. However, this problem can be greatly controlled by careful test construction. Despite the crudeness of this procedure, it did provide a simple test for determining whether group performance, as a whole, validated the hierarchy in terms of predicted p-values.

<u>Phi Coefficient</u>

Gagne (1968, p. 3) stated, "A learning hierarchy represents the most probable expectation of greatest positive transfer for an entire sample of learners concerning whom we know nothing more than what specifically relevant skills they start with." In order to empirically validate a hierarchy of subordinate subtasks which possessed this characteristic, a procedure based upon the correlation between the levels was devised. If the 11 test items (levels in the hierarchy) were arranged hierarchically, there should be high correlation between adjacent items. Ordering the items based on the correlations between them would allow prediction of a learner's success on any given item based upon his performance on previous lower level items. For instance if a learner passed items 1 and 2 and item 3 correlated highly with item 2, the learner would be expected to pass item 3. On the other hand if the learner failed items 1 and 2, he would not be expected to pass item 3. This approach to hierarchy validation, based on the phi coefficient is similar to the Phi/Phimax procedure used by Carroll (Resnick & Wang, 1969). The correlational procedure used in this study is outlined below.

The product-moment correlation of two dichotomous distributions or test items is called "phi" (Nunnally, 1967). To illustrate the usefulness of the phi coefficient in assessing the degree of hierarchical relationship between two test items (or levels of a hierarchy), the definition of phi is given in terms of the pass-fail relationships shown in the contingency tables in Figure 4.

		Item 1	
		fail	pass
Item 2	pass	-+	++
	fail	--	+-

Table a

		Item 1	
		fail	pass
Item 2	pass	b	a
	fail	c	d

Table b

Figure 4. Pass-fail contingency tables.

Table a of Figure 4 indicates the four possible pass-fail relationships when a learner encounters two test items or subtasks of a learning hierarchy. That is, the learner may (1) fail item 1 and pass item 2; (2) pass both items 1 and 2; (3) fail both items 1 and 2 or; (4) pass item 1 and fail item 2. Symbolizing the four quadrants as shown in table b of Figure 4, phi is defined as follows:

$$\text{phi} = \frac{ac - bd}{(a + b)(c + d)(b + c)(a + b)}$$

By examination of the above formula, one sees that if the number of subjects falling in quadrants a (++) or c (--) is zero, then phi is negative. If an equal number of subjects fall in each of the four quadrants, then phi is zero. If all subjects tested fall in one of the four quadrants (leaving zeros in the remaining three), the numerator becomes zero yielding a zero phi-value. The larger the proportion of learners falling in either quadrants a or c becomes (short of N, the total number of learners tested), the larger phi becomes. Thus in order to obtain meaningful results in terms of hierarchical analysis, careful attention must be given to the data used in calculating phi. That is, one must ensure that a sufficient number of learners fall in each of the four quadrants of the contingency table b in Figure 4, and at the same time guard against too great a proportion of the learners tested having either the relationships (++) or (--). If the majority of the learners have achieved (or not achieved) both items, then phi is artificially inflated or decreased, shedding no light upon the hierarchical relationship of the two items. There are at least three ways in which this problem can be controlled to some extent.

1. When using the phi coefficient to indicate the hierarchical relationship between items, test a large sample of learners representing a wide range of ability and achievement levels. This will ensure a greater balance of the number of learners falling in each of the four pass-fail categories. Thereby reducing the artifact.

2. Censor the data. That is, define operational rules pertaining to the data used. Control of the number of learners falling in each of the four quadrants can be exercised by eliminating data from specific learners. For instance, if some learners have already mastered too many or too few of the items do not include these data in the analyses.

3. Use some procedure such as Carroll's phi-phimax method. Phimax is an estimate of the highest-possible phi coefficient given the marginals of the contingency table. Since phimax would become larger as the pass or fail rate of either items became more extreme, the use of phimax in the denominator controls against artificial inflation of the index due to extreme pass or fail rates.

The first control method was used in this study for three reasons: (1) It is difficult to develop an adequate algorithm for discarding data and even more difficult to defend. Such a formula would almost have to be defined in terms of a given sample, (2) a search of the literature provided no evidence in support of the Carroll method. No empirical "tryout" of the efficacy of this procedure was reported, (3) due to the wide range of ability levels tested, it was felt that extreme pass or fail rates would have minimal inflationary effects upon the phi coefficient.

In order to further exemplify how phi was used to validate the hierarchy used in this study, an example is given. Suppose that five items (levels in a hierarchy) are hypothesized as being hierarchically ordered 1, 2, 3, 4, 5. Item one is defined as the easiest item on the test or lowest level in the hierarchy. Phi is calculated between item 1 and each of the other four items.

ϕ (1.2)
ϕ $\underline{(1.3)}$
ϕ (1.4)
ϕ (1.5)

ϕ $\underline{(3.2)}$
ϕ (3.4)
ϕ (3.5)

ϕ (2.4)
ϕ $\underline{(2.5)}$

Hypothesized ordering	Ordering based on phi
1	1
2	3
3	2
4	5
5	4

Figure 5. Hypothetical Ordering Via Phi.

Suppose $\phi_{(1.3)}$ (phi between items 1 and 3) is the largest. Now phi is calculated between items 2, 4, 5 and the largest phi underlined. The process is repeat k-2 times, where k is the number of items. The resulting hierarchy generated by phi is given in Figure 5.

In a hierarchy with k levels there are k! possible orderings. Ordering the levels via phi seeks the ordering which maximizes phi between adjacent levels as described above. If there are no *a priori* assumptions made concerning the hierarchical ordering the levels, then phi must be calculated between item 1 and the remaining k-1 levels choosing the highest phi and repeating the process k-2 times as shown in Figure 5. However, in validating a deductively analyzed hierarchy, there are *a priori* assumptions made concerning transfer from one item (level) to another. In Figure 6, the clusters of items, in which high positive transfer between adjacent items was expected, are shown.

Hypothesized ordering	Cluster I	Cluster II	Cluster III
1. 2/9 + 3/9			
2. 1/10 + 3/10 + 5/10	1	5	9
3. 2 + 1/2	2	6	10
4. 3 1/7 + 2 3/7	3	7	11
5. 6/9 =	4	8	
6. 3/16 + 5/16			
7. 4 1/9 + 5 2/9			
8. 3 1/6 + 4 1/6 + 2 2/6			
9. 7/4 =			
10. 3 5/8 + 2 7/8			
11. 7 2/5 + 4 1/5 + 2 4/5			

Figure 6. Hypothesized clusters of items having high correlations between adjacent items.

This procedure for ordering items by the phi coefficient was applied within these clusters of items. Of the k! orders within each cluster, the one which maximized the phi coefficient between adjacent items was chosen as the hierarchical ordering for that cluster. This procedure lead to the hierarchical ordering of all 11 items.

The formula given here for phi can be derived as a special case of the usual product-moment formula. When correlating two dichotomous distributions or items, exactly the same results is obtained from phi that would be obtained from the product-moment formula. Thus, in order to utilize existing computer programs, phi values were calculated from the product-moment formula. No statistical test of differences between the phi's was used. Decisions concerning ordering of items were based strictly upon the numerical values of phi.

The Guttman Scalogram Analysis

The Guttman Scalogram Analysis (Torgerson, 1958) was used to determine the extent to which the 11 items could be arranged in an order such that passage of a certain item reliably predicted passage of all items lower in the hierarchy. A hypothical set of perfectly scaled items is shown in Figure 7.

Items

Ss	1	2	3	4	5	6	7	8	Total Score
1	1	1	1	1	1	1	1	1	8
2	1	1	1	1	1	1	1	0	7
3	1	1	1	1	1	1	0	0	6
4	1	1	1	1	1	0	0	0	5
5	1	1	1	1	0	0	0	0	4
6	1	1	1	0	0	0	0	0	3
7	1	1	0	0	0	0	0	0	2
8	1	0	0	0	0	0	0	0	1
9	0	0	0	0	0	0	0	0	0

(1 = correct; 0 = incorrect)

Figure 7. A perfect Guttman Scale

Subjects are listed down the side; items are listed along the top. Note that once a subject fails an item all subsequent items are failed. Similarly, if a learner passes a given item, he has passed all prerequisite items. Thus from knowledge of a learner's total score, his response pattern to the set of items can be predicted or reproduced. The proportion of responses to the items that can be correctly reproduced is a measure of how well a set of items can be ordered such that the response patterns form a triangular array as in Figure 7. The proportion of responses to the items that can be correctly reproduced is defined as the coefficient of reproducibility (Rep).

$$\text{Rep} = 1 - \frac{\text{Total number of errors}}{\text{Total number of responses}}$$

Error is defined as a case where a subject passes a higher level item after failing a lower level item. For instance, if one is scaling 4 items, patterns like (-+++), (--++), (+-+-) represent 1, 2 and 1 errors respectively. The value of .90 was suggested by Guttman (1944) as an acceptable lower limit for Rep.

Since hierarchically ordered items should exhibit the triangular pattern of a perfect Guttman scale, the Guttman scaling technique should be of some value in hierarchy validation. From a high Rep ($\geq .90$), the hierarchical relationship of the entire set of 11 items could be inferred. Thus, the object was to arrange the 11 items in such an order so as to maximize Rep. A computer program for the Cornell technique of scalogram analysis (Guttman, 1947) was employed. The program analyzed the pass-fail patterns of subjects to the 11 items and formed permutations of the ordering of these until Rep was maximized. The print-out gave the optimal ordering of the items and the Rep for the optimal ordering.

<u>AAAS Procedure</u>

The AAAS Commission on Science Education (1968) procedures for validating a learning hierarchy were modified for use with test data to investigate the dependency of each individual subtask on its immediate prerequisite. The AAAS approach has been used successfully when using test data in conjunction with instructional materials. This approach is outlined below.

Consider a simple two-level hierarchy with a terminal task and two subordinate subtasks (Figure 8). If subjects

TERMINAL TASK

Subordinate Subtask   Subordinate Subtask

Figure 8. Simple Two-Level Hierarchy

are asked to respond to test items assessing mastery of each of the three cells of this hierarchy, there are 8 possible patterns of performance outcomes which can be observed. Using the code 0 and 1 for fail and pass respectively, the 8 possible configurations are:

1 1 1 1 0 0 0 0
1 1, 0 1, 1 0, 0 0, 1 1, 0 1, 1 0, 0 0

The assumption being that for those learners passing the terminal task, it is highly probable that they are also able to pass the two subordinate subtasks.

In Table 1, the patterns of zeros and ones are translated into descriptions of mastery of the terminal task and mastery of the subordinate subtasks.

Table 1. Possible pass-fail patterns in a two-level hierarchy.

(++)	(+-)	(-+)	(--)
1 1 1	1 1 1 0 1, 1 0, 0 0	0 1 1	0 0 0 0 0, 0 1, 1 0

The relationships, (++) passed both the terminal task and the subordinate subtasks, (+-) passed the terminal task and failed one or both subordinate subtasks, (-+) failed the terminal task and passed the subordinate subtasks, (--) failed the terminal task and failed one or both subordinate subtasks, are represented by the four columns.

The following three ratios are defined in terms of these relationships.

$$(1)\quad \text{Consistency ratio} = \frac{(++)}{(++) + (+-)}$$

The consistency ratio is defined as the quotient of the number of patterns consistent with the hypothesis divided by the total number of subjects who acquired the terminal task. The value of this ratio is a measure of how consistent the data are with the hypothesized dependency.

$$(2)\quad \text{Adequacy ratio} = \frac{(++)}{(++) + (-+)}$$

The adequacy ratio is defined as the quotient of the number of patterns consistent with the hypothesis divided by the sum of the number of subjects acquiring both the terminal and subordinate tasks and the number of subjects acquiring the subordinate subtasks but not the terminal task. The adequacy ratio is a measure of the adequacy of the identified subordinate tasks.

$$(3)\quad \text{Completeness ratio} = \frac{(++)}{(++) + (--)}$$

The completeness ratio is defined as the quotient of the total number of patterns consistent with the hypothesis divided by the sum of the number consistent with the theory and the number of subjects failing both the terminal and subordinate tasks. The completeness ratio is a measure of the effectiveness of instruction.

In Science - A Process Approach high (.90 or above) consistency ratio, adequacy ratio and completeness ratio for every dependency relationship are considered the necessary and sufficient set of characteristics for a valid hierarchy. However, when attempting to validate a deductively analyzed hierarchy from test data alone, the above procedure must be modified. In both the AAAS procedure and Gagne's work, the relationships (-+) (lower level passed, higher level failed) suggests that there may be something inadequate in the instructional material related to acquiring the terminal task and that these relationships should not be included in a measure of support for the hypothesis. However, when analyzing test data this relationship does support the hierarchical nature of the subtasks. With test data, there is no basis for expecting a learner to pass a superordinate task when all subordinate tasks are passed as in an instructional program. The learner may simply have reached his achievement level. One would expect, however, that once a learner failed an item he would fail all subsequent higher level items. Thus with test data alone only (+-) of the four possible relationships between adjacent items contradicts the hierarchical theory.

The consistency ratio does indicate how consistent the test data are with the hypothesized ordering. A low consistency ratio between two adjacent levels would indicate a problem in ordering since this ratio is 1.00 (perfect) if no one contradicts the theory and decreases as the number of subjects contradicting the theory increases. A low adequacy ratio may simply indicate that a large proportion of learners tested reached their achievement level at this point in the hierarchy. If this ratio is too low it would indicate that a wider range of ability and achievement levels should be included in the sample tested. The relationship (--) is in accord with the theory when using test data. That is, no higher level task is passed after failure on the lower level tasks. Thus, the completeness ratio is of very little value in determining the hierarchical ordering of task from test data. Of course, this ratio is of great importance when using data from an instructional program.

In summary, when validating a hierarchy from test data the consistency ratio is of major importance. An acceptable level for this ratio in the present study was set at .85. In the AAAS work the level of acceptability was .90. However, when analyzing the dependency of adjacent items from test data alone it seemed reasonable to accept a lower value as evidence of a valid hierarchical ordering. The adequacy ratio indicates, more than anything else, a flaw in sampling. Since the relationship (-+) is in accord with the hierarchical theory when using test data, it was not expected or necessary for this ratio to be as high (.90) as that recommended in the AAAS report. An acceptable level for this ratio in the present study was set at .70. Again, the completeness ratio when using test data indicated a flaw in sampling and not a flaw in ordering. Since the relationship (--) is in accord with the hierarchical theory and many subjects at the upper levels of the hierarchy would be expected to fall in this category, an acceptable level for this ratio was set at .50.

In using the modified AAAS procedure to validate the hierarchy used in this study, the items were arranged so as to maximize the consistency ratio and to keep the adequacy ratio at an acceptable level. No attempt was made at controlling or maximizing the completeness ratio. In addition to computing the three ratios of the AAAS procedure, the proportion of positive transfer used by Gagne was calculated. Another ratio defined by the authors as the order ratio was computed. The order ratio indicated the proportion of

$$\text{Order ratio} = \frac{(++) + (--) + (-+)}{(++) + (--) + (-+) + (+-)}$$

learners' response patterns which were consistent with the theory. The level of acceptability for this new ratio, was set at .90 as a lower limit.

Textbook Sequence

The 11 instructional lessons were integrated and sequenced according to the "usual textbook ordering." The "usual textbook ordering" was determined by examining different elementary mathematics texts. The following three fourth grade texts were used in determining this ordering:

(1) Deans, E., Kane, R. B., McMeen, G. H., & Oesterle, R. A. *Understanding Mathematics*. New York: American Book Company, 1968.

(2) Duncan, E. R., Capps, L. R., Dolciani, M. P., Quast, W. G., & Zweng, M. *Modern School Mathematics: Structure and Use*. New York: Houghton Mifflin Company, 1967.

(3) Keedy, M. L., Dwight, L. A., Nelson, C. W., Schluep, J., & Anderson, P. A. *Exploring Elementary Mathematics*. New York: Holt, Rinehart & Winston, Inc., 1970.

The order of presentation of the 11 subtasks in these 3 texts was very similar. The textbook sequence was determined by arranging the subtasks in an ordering which gave closest fit to that given in these texts.

The subtasks were, of course, not presented in as isolated and pure form as in the instructional materials developed. Often, 2 or 3 of the subtasks were presented in one section or lesson. Also, addition and subtraction of rationals were presented almost simultaneously. Thus in order to keep the program length the same for comparative purposes and have the lessons sequenced in the "usual textbook ordering", the logically ordered lessons were integrated to form new lessons. That is, one or more of the lessons were combined to form a new lesson. However, the same type and number of responses were required of the student. This sequence was used in order to gain some useful information in answering the question, "Do authors of instructional materials need to be concerned with more rigorous and logical organization of subtasks within chapters or sections?"

Random

A random ordering of the 11 items (subtasks) was included for two reasons. First, an instructional sequence based on a random ordering of the lessons could serve as a control group in the analysis of variance model. Second, comparison of the random group with each of the other groups could provide useful information in determining if an optimal ordering of the 11 items was achieved. Research evidence on the sequencing of instructional activities is somewhat contradictory. Several studies yield results which support the theory of careful sequencing while others yield results that suggest, that careful sequencing has little effect upon learning.

Development of Instructional Materials

A programmed text format was chosen for two reasons: first, it could provide uniform instruction for students in different groups. Thus variance due to sources not of primary interest in this research could be reduced. Second, this format allowed complete random assignment of subjects to treatment groups. In order to conduct this study in the public schools, intact classes had to be utilized. The use of programmed instructional sequences permitted subjects, rather than classes, to be randomly assigned to treatment groups. Thus, strengthening the generalizability of the results.

The instructional materials consisted of an eleven lesson programmed booklet on the addition of rational numbers with like denominators (Appendix C). The program utilized one lesson for each of the 11 levels in the learning hierarchy. The procedure used in writing these lessons employed techniques similar to those of Crowder and Skinner. The method employed was similar to what May (1965) called hybrid programming. Conventional exposition and problem sets were combined with the Skinner mode. Each of the 11 lessons was designed to develop the specific skill represented by the corresponding hierarchy level. The lessons were from 2 to 3 pages in length making a total program length of 29 pages. The program called for frequent responses from the learner which he wrote in blanks provided. A page from the booklet is given in Figure 9 to illustrate the format used.

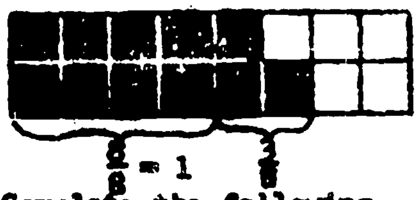
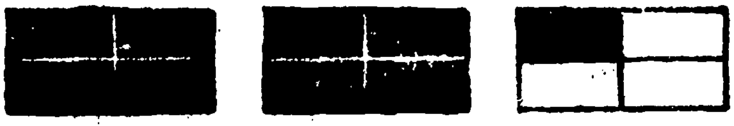
	ANSWERS
4. Use the shaded portions to help you write a mixed numeral that names the same number as $\frac{11}{8}$.	
	
$\frac{11}{8} = \frac{\square}{8} + \frac{\bigcirc}{8} =$ ____	8, 3, $1\frac{3}{8}$
5. Complete the following.	
(a) $\frac{9}{7} = \frac{7}{7} + \frac{\square}{7} = \bigcirc + \frac{\square}{7} = \bigcirc\frac{\square}{7}$	(a) 2, 1, 2, $1\frac{2}{7}$
(b) $\frac{12}{9} = \frac{9}{9} + \frac{\square}{9} = \bigcirc\frac{\square}{9}$ Is $\frac{3}{9}$ in lowest	(b) 3, $1\frac{3}{9}$
terms? ____ $\frac{3}{9} = \frac{\square}{3}$	no, 1
Thus $\frac{12}{9} = 1\frac{\square}{3}$	1
(c) $\frac{10}{8} = \frac{8}{8} + \frac{\square}{8} = \bigcirc\frac{\square}{8} = \bigcirc\frac{\square}{4}$	(c) 2, 1, 2, $1\frac{1}{4}$
(d) $\frac{9}{7} =$ ____ + ____ = ____	(d) $\frac{7}{7}$, $\frac{2}{7}$, $1\frac{2}{7}$
6. Use the shaded portions to help you write a mixed numeral that names the same number as $\frac{9}{4}$.	
	
$\frac{9}{4} = \frac{\square}{4} + \frac{\square}{4} + \frac{\square}{4}$	4, 4, 1
$\frac{9}{4} =$ ____ + ____ + $\frac{\square}{4}$ = ____ + $\frac{\square}{4} = \bigcirc\frac{\square}{4}$	1,1, 1, 2, 1, $2\frac{1}{4}$

Figure 9. Sample instructional lesson.

The first draft of the instructional sequence was written with the lessons (frames) sequenced according to the logically constructed hierarchy. The lessons were written as concisely and briefly as possible while still achieving the instructional objectives of the lesson. This was done to keep the program at a realistic length and to minimize the reading load.

The first draft was reviewed by three authors of elementary mathematics texts. Judgments were made concerning the appropriateness and correctness of language, adequacy and effectiveness of art work, layout and format, and in general the overall expected effectiveness of the learning sequence in teaching the skills of rational number addition. Based upon the suggestions of these authors, the materials were revised.

Since the subjects were accustomed to using commercially produced materials, steps were taken to make the learning sequence as attractive in appearance as possible. The use of ditto masters was selected for reproducing the materials. This method allowed the use of colors, easy correction of typographical errors, and a relatively simple procedure for producing the needed art work. All drawings were in red and green and the textual print was in blue. Lessons were reproduced on 8 1/2 x 11 inch paper with each page having a 2 inch answer column at the right. Each lesson (frame) began on a separate sheet indicating major segments of the program and allowing for easy reordering of the lessons for each treatment group.

In order to assess the effectiveness of the learning program, a tryout was conducted utilizing five fourth grade students. Pre- and post-test achievement was measured by an 18 item test. The pretest mean was 1.80; the posttest mean was 13.60. The difference between the means was significant at the .01 level. Note was taken as to where these students had difficulty in understanding terminology, interpreting drawings and examples, and following the exposition. Based on these observations, the materials were revised again. Based on the results of this tryout and the inspection by experienced authors, the materials were judged as adequate for achieving the instructional objectives. The lessons were ordered according to the hierarchies generated by the seven indirect validation procedures. The lessons and pages of each sequence were then appropriately numbered. From the exterior all booklets appeared the same. They seemed adequate in terms of attractiveness, durability, and usability for fourth grade students.

Experimental Procedures

Fourth grade subjects were selected to participate in the study on the basis of two pretests. Pretest I (Appendix D) was designed to determine if the learners had mastered the necessary prerequisites for successfully achieving the skills presented in the programmed text. The test concentrated on the concept of fraction, recognizing parts of a whole, reading and writing fractional numerals, whole number addition, and simple whole number division. Pretest I was administered to 175 fourth grade subjects one week prior to initiation of the learning sequence. The students were at two different schools in the same Indiana county. Subjects were grouped according to ability. Each school had one class of high, medium, and low ability students. Pretest II was administered only to those students judged, on the basis of pretest I, ready to undertake the programmed materials.

Pretest II (Appendix D) was designed to determine if the students had already mastered the skills to be taught in the instructional sequence. The test consisted of one item for each of the 11 levels in the hierarchy. Pretest II was an alternate form of the achievement test administered at the completion of the instructional program. Only those subjects judged, on the basis of pretest II, to have mastered an insignificant number of the skills in the instructional program were included in the study. Of the 175 students tested, 142 met the criterion on both tests. Thus, the sample of learners participating in the study knew the necessary prerequisite skills and knew, essentially, none of the skills presented in the learning sequence. In order to control any changes in behavior due to time elapse, pretest II was administered on Friday of one week and the programmed sequence began on the following Monday.

Since the students were already stratified on ability, subjects were assigned randomly to treatment groups within the three strata high, medium, and low ability. This procedure strengthened the unbiasedness of the samples since essentially the same number of high, medium, and low ability students were in all treatment groups. By the principle of randomization all treatment groups were considered equal.

All 175 fourth grade students at Rossville Elementary School, Rossville, Indiana and Kyger Elementary School, Frankfort, Indiana, completed the programmed instructional booklet. Of this number, only 142 students were actually included in the study. Thirty did not satisfy the criteria on pretest I and II and three were lost due to experimental mortality.

The students worked through the programmed booklets independently devoting approximately 30 minutes per day to the materials until they were completed. The experimenters explained how students were to use the materials and assisted the students in any problems they encountered for the first two days of the study. Following this, the teacher supervised the students' work until completion of the study.

The students entered their responses to questions directly in the booklets using a cardboard cover-up for the answer column. Subjects were instructed to keep the answer column covered until they entered their response then pull the cardboard down to reveal the correct response. If their response was correct, they were instructed to continue to the next question and repeat the process. If their response was incorrect, they were instructed to draw a line through the incorrect response and enter the correct response above it. Before moving to the next frame, students were advised to attempt to determine why their response was incorrect and the authors' response was correct. Much of the time, their mistakes were due to carelessness and they could see them. However, if they could not, they were instructed to ask the teacher for help at this point. Teachers were instructed to give help only in the context of each child's material. For instance, if one child's sequence had a frame which involved writing the simplest name for fractional numerals but no preceding frame dealt with the definition of simplest name or the mechanics of renaming, the teacher did not show the student the manipulations involved in renaming. The students were guided in using only the information and art work provided in the given frame. This procedure led to considerable frustration on the part of some students in treatment groups where necessary prerequisites were not provided. However, teachers had to adhere to this procedure if the effects of sequencing upon achievement, retention, and transfer were to be measured.

The booklets were distributed by the teacher at the beginning of each work session. However, due to other school activities the 30 minute work session length varied somewhat from day to day. The teachers kept a log book of the exact number of minutes spent on the booklets each day. As each child finished his booklet, the date and time of completion was entered in his booklet. At the completion of each session, the booklets were collected by the teacher. No student was allowed to take his booklet home or study it other than during the supervised work session. In order to keep students from rushing through the materials, they were reminded at the beginning of each session that they were to study the materials and try to remember what they did, not just copy in the correct responses. They were also told that they would be tested upon completion of the booklet. For the duration of the study, the students had no mathematics instruction other than these work sessions. Teachers provided other mathematics activities for each student who finished his booklet until the whole group had finished.

An achievement test measuring acquisition of the terminal behavior was administered to each student on the day following completion of the programmed sequence. The test consisted of two items per hierarchy level except for two levels which pertained to renaming. An alternate form of this test was administered as a retention test two weeks after the administration date of the achievement test. During this two week period, students studied mathematical topics other than operations with fractions. A transfer test on subtraction of rational numbers with like denominators was administered on the day following administration of the achievement test. This test consisted of 10 items analogous to those covering the rational number addition. No renaming in the subtraction process such as required in 3 1/8 - 5/8 was included on the test. However, as on the addition test, students were required to reduce answers to lowest terms. The achievement, transfer and retention tests are included in Appendix D.

Analysis of variance for multiple groups, unequal n's model, (Winer, 1962) was used to investigate the differential effects of sequencing on each of the four dependent variables achievement, transfer, retention, and time to determine if the mean scores of the 7 sequence groups differed significantly. Other statistical procedures used are discussed in the next section.

Results

This research was directed toward the development and evaluation of procedures for validating a learning hierarchy from test data. An evaluation of the efficacy of each procedure was conducted by actually sequencing learning materials according to the hierarchies generated by each method and determining the effect of sequence upon achievement, transfer, retention, and time to complete the program. In order to accomplish the objectives of this study, various methods of hierarchical analyses were used. Before reporting the results of sequence effects, the results of these hierarchical analyses are presented.

Hierarchical Analyses

A test designed to assess mastery at each level in the hierarchy for addition of rational numbers was administered to 163 elementary school children in grades 4 through 6 (Appendix B). The internal consistency (Kuder-Richardson Formula 20) of the test was .81. The pass-fail relationships were analyzed by several scaling techniques and methods of hierarchical analyses, adaptable for use with test data, to generate hierarchical orderings of the 11 levels. The results of these hierarchical analyses are presented below.

Pattern Analysis

To provide evidence that the logical ordering was indeed logical, the pattern analysis technique developed by Rimoldi and Grib (1960) was used. The index of agreement was .87. No reordering of the 11 items was attempted.

Item Difficulty

Validating a learning hierarchy using item difficulties is based on the assumption that items at the lower level of the hierarchy are simple and easy and that they get increasingly more complex and difficult moving up the hierarchy. The observed p-values for the original ordering of the 11 items in Table 2. Tables 2 through 31, in which statistical results are displayed may be found in Appendix A. Inspection of these values indicated that items 3 and 7 were out of order. That is, there were items above them in the hierarchy which had greater p-values indicating they were easier items.

The hierarchical ordering generated from the observed p-values is shown in Table 3. The items were rearranged so that the p-values formed a decreasing sequence of values.

Phi Coefficient

The results of the validation procedure using the phi-coefficient described previously are shown in Tables 4 through 6. Three clusters of items were hypothesized as having high dependency upon one another. The correlation matrix for the items within each cluster was calculated. By inspection of these matrices, the items within clusters were arranged so that the correlation between adjacent items was maximized. The correlations between all adjacent items in the validated ordering were significant at the .01 level. The resulting hierarchical ordering of the 11 items is shown in Table 7.

Guttman Technique

Student responses to the 11 items were analyzed using Guttman Scalogram Analysis. This procedure was used to determine the extent to which the 11 items could be arranged in an ordering such that passage of a certain item reliably predicted passage of all items lower in the hierarchy. The proportion of responses to the item that can be correctly reproduced is a measure of how well a set of items can be ordered such that the response patterns form a triangular array of a perfect Guttman Scale. This proportion is defined as the coefficient of reproducibility (Rep).

The pass-fail patterns of subjects to the 11 items were analyzed and permutations of the ordering of these were formed until Rep was maximized. The revised ordering yielded a Reproducibility Coefficient of .94. The hierarchical ordering validated by the Guttman Technique is given in Table 8.

AAAS Procedure

The modified AAAS procedure described earlier was used to measure the dependency of each item on its immediate prerequisite. The items were arranged so that the consistency ratio was maximized. The consistency ratios between adjacent items of the revised ordering are listed in Table 9. All of these values, except one, were greater than or equal to .85.

The adequacy ratios for the revised ordering of the 11 items are listed in Table 10. This ratio indicated, more than anything else, a flaw in sampling. Since the relationship (-+) is in accord with the hierarchical theory when using test data, it was not expected or necessary for this ratio to be as high (.90) as that recommended in the AAAS report. All of the values in Table 10 were greater than or equal to .70.

The completeness ratios for the revised ordering are listed in Table 11. No attempt was made at controlling or maximizing this ratio. All values in Table 11, except one, were greater than or equal to .50.

The proportions of positive transfer and the order coefficients as defined by Gagne and the authors respectively are listed in Tables 12 and 13. Gagne's notion of proportion of positive transfer was modified for use with test data to derive the formula for the order ratio. This ratio indicated the proportion of the subjects' response patterns which were consistent with theory. The level of acceptability for this ratio was set at .9(, as a lower limit. All values listed in Table 13, except one, were greater than or equal to .90.

Textbook Ordering

Three elementary school mathematics texts were examined to determine this ordering. The order of presentation of the 11 subtasks in these 3 texts was very similar. In examining these texts, it was found that all subtasks pertaining strictly to renaming skills were presented in the chapter preceding operations with rational numbers. None of the 11 subtasks were treated in as isolated form as they were represented in the hierarchy. For instance, adding with two fractions having like denominators and adding with three such fractions was presented as one unit. Adding with mixed numerals (either 2 or 3 addends) which did or did not require reducing the fractional parts to lowest terms were all treated together.

All 3 texts presented addition with mixed numerals where the sum of the fractional parts is greater than 1 as the most complex skill in addition of rational numbers with like denominators. However, such additions involving 2 and 3 addends were all treated together.

There were slight differences in how the three texts presented the 11 subtasks in the hierarchy. The textbook ordering used in the present study was determined by choosing an ordering which gave the closest fit to that presented in the 3 texts. The ordering resulting from this procedure is shown in Table 14. In more than one instance, two or more of the subtasks were integrated to form new subtasks. Thus, the textbook ordering contained only 7 subtasks. The instructional material for this sequence, however, was the same length in terms of student responses as for the other sequences.

Random Ordering

The 11 subtasks of the hypothesized hierarchy were randomized to form this ordering. The resulting ordering is shown in Table 15 along with the orderings generated by the other procedures.

The index of agreement was calculated for each of the 7 orderings. These values are given in Table 16. The index of agreement for the random ordering was .61.

Pretests

Subjects to participate in the experimental study of the effect of sequence on achievement, transfer, retention, and time to complete the program were selected on the basis of their performance on two pretests. These tests were designed to assess entering behavior. Pretest I was designed to answer the question "Does the learner have the necessary prerequisite skills needed to master successfully the skills presented in the learning program?" Pretest II was designed to answer the question "How many of the skills presented in the learning program has the learner already mastered?"

Some of the items on Pretest I were judged more crucial than others. However, in no instance was a learner accepted who responded incorrectly to more than 7 items. Thus, an acceptable score was defined as one ranging between 24 and 17 on a 24 point test. The proportion of subjects obtaining each acceptable score on Pretest I are listed in Table 17. Note that 87% of the subjects included in the study obtained scores of 20 or higher with 23% having perfect scores. Only 13% of the subjects gave incorrect responses to 5 or more items on Pretest I. The mean score on Pretest I was 21.84.

Pretest II consisted of 11 items assessing mastery at each of the 11 levels in the hierarchy upon which the learning program was based. The proportion of subjects obtaining each acceptable score is given in Table 18. Note that 71% of the subjects tested were unable to respond correctly to any of the 11 items. Ninety-two percent of the subjects gave correct responses to 2 or less of the 11 items. The mean score on Pretest II was 0.563.

One hundred forty-two subjects met the pretest criteria. That is, their scores were between 17 and 24 on Pretest I and between 0 and 4 on Pretest II.

The Effects of Sequence

The major purpose of the present research was an investigation of the differential effects of sequencing instructional materials according to a learning hierarchy validated by various procedures upon time to complete the program, achievement, transfer, and retention. This research was designed to answer the question, "Will one sequence require less time to complete and maximally facilitate achievement, transfer and retention?" The results of the effect of sequence upon these four variables are presented below. The internal consistency coefficients for all tests used appear in Table 19.

Achievement

An achievement test displayed in Appendix C, was administered one day following completion of the instructional program. Of major interest was the comparison of the logical sequence derived using Gagne's task analysis, and the random sequence group with all sequence groups. Planned comparisons (Hays, 1963) were made between the mean of the logical sequence group on the achievement test and the means of all other sequence groups. No significant differences, at the .05 level, were found between the mean achievement score of the logical sequence group and the other sequence groups. Similarly, no significant differences were found between the mean achievement score of the random sequence group and the other groups.

The differential effects of sequence upon achievement were investigated using an analysis of variance design. Identification of the groups is given in Table 20. The one-way analysis of variance on achievement is shown in Table 21. No overall significant differences were found.

Transfer

A transfer test (Appendix D) on the subtraction of rational numbers was administered on the day following completion of the achievement test. Planned comparisons between the mean transfer score of neither the logical nor random sequence groups with all other groups showed any significant differences at the .05 level. The one-way analysis of variance on transfer is shown in Table 22. No overall significant differences were found.

Retention

An alternate form of the achievement test administered at the completion of the program was administered two weeks after completion of the learning program as a retention test. Using planned comparisons the mean retention score of the logical sequence group was compared with all other means. The difference between the logical sequence group mean (7.50) and the textbook sequence group mean (4.95) was significant at the .05 level. All other comparisons were nonsignificant. Similarly, comparisons between the random sequence group mean and all other means were not found to be significant.

The one-way analysis of variance on retention is shown in Table 23. The F-ratio of 2.12 was very near the critical value 2.15 for significance at the .05 level. Thus, post-hoc comparisons of all means using the Duncan Multiple Range Test (Winer, 1962) were made. The difference between the means of the AAAS sequence group (8.52) and the textbook sequence group (4.95) was statistically significant at the .05 level. Also the difference between the means of the AAAS sequence group and the item difficulty sequence group (5.37) was statistically significant at the .05 level. All other comparisons were nonsignificant.

Time

Planned comparisons between the mean number of minutes to complete the programmed instructional booklet of the logical and random sequence groups and all other groups were made. At the .05 level, only one of these comparisons was statistically significant. Namely, the difference between the mean time required to complete the program by the logical sequence group (103.86) and the correlational sequence group (135.33). The range of the number of minutes required by each group to complete the instructional program is listed in Table 24.

The one-way analysis of variance on time is shown in Table 25. No overall significant differences were found at the .05 level.

Further Investigations of Sequence Effects

Responses were first marked as incorrect if they were not written in simplest form. Responses such as 9/12 or 3 11/8 were considered incorrect since they were not written in simplest form. Examination of the tests revealed that many students scoring very low had actually mastered the skills involved in rational number addition. However, due to not following directions or having not mastered the renaming skills subjects failed to write the answers in simplest form. In order to assess more accurately the differential effects of sequencing, two other scoring procedures were used. Too many low scores due solely to not renaming could obscure the true effects of sequencing. A second scoring algorithm gave one-half credit for responses which were correct but not reduced to lowest terms. A third scoring algorithm disregarded renaming.

The one-way analyses of variance on achievement allowing partial credit and disregarding reducing to lowest terms in scoring are shown in Tables 26 and 27 respectively. No overall significant differences were found in either case. Planned comparisons between the mean achievement scores of the logical and random sequence groups and all other means revealed no significant differences in the means at the .05 level.

The one-way analysis of variance on transfer allowing partial credit in scoring is shown in Table 28. The difference between treatment means was not significant at the .05 level. Comparisons of the logical and random sequence group means with all other group means showed significant differences between only two pairs of means. The difference in the means of the random sequence group (6.89) and the phi coefficient sequence group (4.76) was statistically significant at the .05 level. A significant difference was found between the means of the random sequence group (6.89) and the textbook sequence group (4.52).

The one-way analysis of variance on transfer disregarding reducing to lowest terms in scoring is shown in Table 29. The F-ratio was significant at the .05 level. The Duncan Multiple Range Test indicated significant difference between two pairs of means. The differences between the means of the random sequence group (8.26) and both the phi coefficient sequence group (5.10) and the textbook sequence group (5.19) were significant at the .05 level.

The one-way analyses of variance on retention allowing partial credit and disregarding reducing to lowest terms in scoring are shown in Tables 30 and 31 respectively. The F-ratio in the first case was near the critical value for significance at the .05 level. The Duncan Multiple Range Test indicated significant differences between two pairs of means. The AAAS sequence group mean was significantly greater than that of both the item difficulty and the textbook sequence groups.

When disregarding reducing to lowest terms, the F-ratio was significant at the .05 level. Significant differences between two pairs of means were found using the Duncan test. The AAAS sequence group mean was significantly greater than those of both the item difficulty and the textbook sequence groups.

Discussion

The reader's attention was directed toward two troublesome problems with many studies of the effects of sequence reported in the literature. First, in comparing the effects of a logical and a random sequence upon learning, it was not demonstrated that indeed a logical sequence and an unbiased random sequence were being used. Second, in many of the studies reported, it was suspected that too many of the subjects already knew much of the material presented in the learning program. This study was designed to minimize the possibility of these pitfalls.

The index of agreement was used to determine if the hypothesized ordering developed through the use of task analysis was indeed logical. That is, that it was hierarchical in structure. The index of agreement was .87 which indicated that the observed response patterns of the subjects correlated highly with the expected patterns indicating that the logical ordering was logical. The index of agreement for the random order was .61. Thus, the logical ordering appeared to have markedly more of the characteristic of hierarchical structure than did the random ordering. With the exception of the textbook ordering all other sequences were validated empirically using various procedures. The indices of agreement for all validated orderings were above .85 indicating high correlations between observed and expected response patterns. The index of agreement for the textbook ordering was only .62, however.

Subjects included in the study had to meet stringent criteria on two pretests. Namely, they had to have the necessary prerequisites for undertaking study of the skills presented in the instructional program, and they could not have already mastered the skills to be taught. Thus, the probability was very low that outcomes attributed to sequence were affected by the aforementioned problems.

On the other hand due consideration must be given to two sources of artifact over which we had less control than would have been desirable.

1. Teachers were instructed on the type and amount of student help to provide. However, they reported that learners in the random sequence group, who were asked to perform certain tasks when they had not mastered necessary prerequisites, were very frustrated. In these instances, the teachers may have provided too

much instruction making assessment of sequential effects difficult. This could have accounted for the absence of significant differences among the mean achievement scores of the seven sequence groups.

2. Examination of subjects' responses revealed that many students did not write the answers in lowest terms. Again, the teachers were instructed to stress directions and be sure all learners understood what was expected of them. Thus, it might be concluded that the lessons pertaining to reducing to lowest terms were not adequate in terms of allowing for enough practice and repetition. However, when allowing partial credit or disregarding reducing to lowest terms in scoring, still no significant differences were found on immediate achievement.

Neither planned nor post hoc comparisons showed any significant differences between the logical sequence group and the other sequence groups on achievement, transfer, or retention. The logical sequence group did require significantly less time to complete the program than did the correlational sequence group. This suggests that careful task analysis of instructional objectives can be a powerful tool in devising optimal instructional sequences. In fact it may mean, in terms of overall cost, that careful analysis of the instructional objectives to reveal the prerequisite subtasks is an adequate procedure for developing a valid hierarchy.

Conclusions

Within the limitations of this study the results seem to justify the following conclusions:

1. The overall efficiency of the learning process, using programmed instructional materials, can be affected by changing the sequential ordering of the subtasks.

2. Sequence, even if random, has little effect upon immediate achievement.

3. Retention appears to be the variable, of the four under study, most susceptable to sequence manipulation.

4. No sequence maximally facilitated achievement, retention, and transfer, and required less time to complete. However, based on the group means, the AAAS procedure yielded the best sequence overall.

5. Textbook authors may need to give more careful consideration to the sequencing of subtasks within major topics or subdivisions of a chapter.

6. Optimal instructional sequences can be derived using learning hierarchies validated from test data.

Recommendations

The results of the present study suggest that sequence effects the overall efficiency of the learning process. The fact that no differences in immediate achievement were found between the random sequence group and the other hierarchically ordered groups may have resulted from sample size and the complexity of the skills involved. The effects of sequence should be investigated by replications with more complex skills involving longer learning sequences and larger samples.

Further research should attempt to determine the effects of sequence upon the total learning process with students at different achievement levels in mathematics. That is, consideration should be given to the effects of sequence upon the attitudes and anxieties experienced by learners in different sequence groups, the interaction effects between sequence and ability, and the effects of sequence upon immediate achievement, transfer, and long term retention. The effects of sequence upon learning mathematics should be investigated at both the secondary and elementary school levels. The effects of carefully sequenced instructional materials according to validated learning hierarchies on the performance of the slow learner and the remediational value of such instructional sequences should be investigated thoroughly.

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APPENDIX A
STATISTICAL TABLES

Table 2. Observed P-values.

Item	Number Ss passing Item	P-value
1	158	.97
2	159	.98
3	129	.79
4	145	.89
5	113	.69
6	94	.58
7	85	.52
8	90	.55
9	69	.42
10	66	.40
11	66	.40

Table 3. Hierarchical ordering of P-values.

Item	Number Ss passing Item	P-value
1	158	.97
2	159	.98
4	145	.89
3	129	.79
5	113	.69
6	94	.58
8	90	.55
7	85	.52
9	69	.42
10	66	.40
11	66	.40

Table 4. Correlation matrix for culster I and the hierarchical ordering.

	1	2	3	4
1	1.00	.89	.17	.50
2		1.00	.11	.45
3			1.00	.44
4				1.00

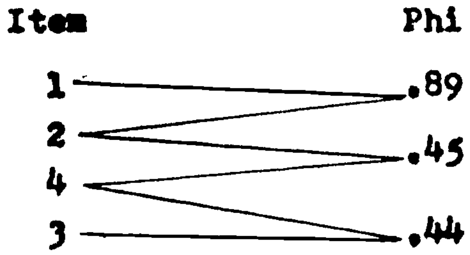

ordering

Table 5. Correlation matrix for cluster II and the hierarchical ordering.

	5	6	7	8
5	1.00	.59	.69	.72
6		1.00	.64	.67
7			1.00	.89
8				1.00

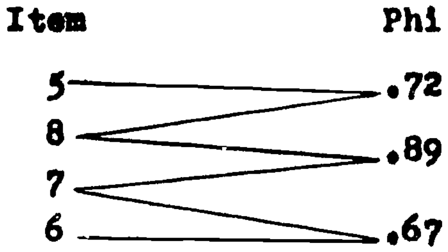

ordering

Table 6. Correlation matrix for cluster III and the hierarchical ordering.

	9	10	11
9	1.00	.72	.74
10		1.00	.94
11			1.00

ordering

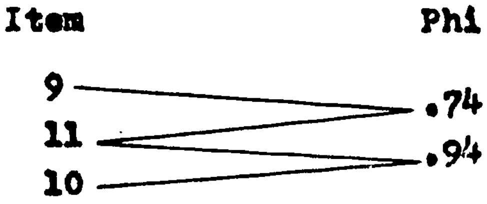


Table 7. Hierarchical ordering generated by the phi coefficient.

Hypothesized ordering	Validated ordering using Phi
1	1
2	2
3	4
4	3
5	5
6	8
7	7
8	6
9	9
10	11
11	10

Table 8. Hierarchical ordering generated by the Guttman Technique.

Hypothesised ordering	Validated ordering using Guttman Technique
1	2
2	1
3	4
4	3
5	5
6	6
7	8
8	7
9	9
10	10
11	11

Table 9. Consistency ratios.

Dependency Relationship between items	Frequency of pass-fail patterns; higher-lower				Consistency ratio $\frac{++}{(++) + (+-)}$
	++	--	-+	+-	
1-2	158	4	0	1	0.99
2-4	145	4	14	0	1.00
4-3	125	11	22	5	0.96
3-5	93	14	39	17	0.85
5-6	89	43	17	14	0.86
6-7	79	54	24	6	0.93
7-9	67	67	23	6	0.92
9-10	57	84	12	10	0.85
10-8	68	66	4	25	0.73
8-11	65	66	26	6	0.92

Table 10. Adequacy ratios.

Dependency Relationship between items	Frequency of pass-fail patterns; higher-lower				Adequacy Ratio $\frac{(++)}{(++) + (-+)}$
	++	--	-+	+-	
1-2	158	4	0	1	1.00
2-4	145	4	14	0	0.91
4-3	125	11	22	5	0.92
3-5	93	14	39	17	0.71
5-6	89	43	17	14	0.84
6-7	79	54	24	6	0.77
7-9	67	67	23	6	0.74
9-10	57	84	12	10	0.83
10-8	68	66	4	25	0.94
8-11	65	66	26	6	0.71

Table 11. Completeness ratios.

Dependency Relationship between items	Frequency of pass-fail patterns; higher-lower				Completeness ratio $\frac{(++)}{(++) + (--)}$
	++	−−	−+	+−	
1-2	158	4	0	1	0.98
2-4	145	4	14	0	0.98
4-3	125	11	22	5	0.92
3-5	93	14	39	10	0.87
5-6	89	43	17	14	0.67
6-7	79	54	24	6	0.59
7-9	67	67	23	6	0.50
9-10	57	84	12	10	0.40
10-8	68	66	4	25	0.51
8-11	65	66	26	6	0.50

Table 12. Proportions of positive transfer.

Transfer relationship between items	Frequency of pass-fail patterns: higher-lower				Proportion of positive transfer
	++	--	-+	+-	$\frac{(++) + (--)}{(++) + (--) + (+-)}$
1-2	158	4	0	1	0.99
2-4	145	4	14	0	1.00
4-3	125	11	22	5	0.96
3-5	93	14	39	17	0.86
5-6	89	43	17	14	0.90
6-7	79	54	24	6	0.96
7-9	67	67	23	6	0.96
9-10	57	84	12	10	0.93
10-8	68	66	4	25	0.84
8-11	65	66	26	6	0.96

Table 13. Order coefficients.

Dependency relationship between items	Frequency of pass-fail patterns: higher-lower				Order coefficient $\frac{(++) + (--) + (-+)}{(++) + (--) + (-+) + (+-)}$
	++	--	-+	+-	
1-2	158	4	0	1	0.99
2-4	145	4	14	0	1.00
4-3	125	11	22	5	0.97
3-5	93	14	39	17	0.90
5-6	89	43	17	14	0.91
6-7	79	54	24	6	0.96
7-9	67	67	23	6	0.96
9-10	57	84	12	10	0.94
10-8	68	66	4	25	0.85
8-11	65	66	26	6	0.96

Table 14. Textbook ordering.

Hypothesized ordering	Textbook ordering
1	5
2	9
3	3
4	1,2
5	6
6	4,7,8
7	10,11
8	
9	
10	
11	

Table 15. Hierarchical orderings generated by each indirect validation procedures.

		Procedures				
Logical (Task analysis)	Item Difficulty	Correlation (Phi coefficient)	Guttman	Textbook	AAAS	Random
1	1	1	2	5	1	3
2	2	2	1	9	2	6
3	4	4	4	3	4	2
4	3	3	3	1,2	3	7
5	5	5	5	6	5	1
6	6	8	6	4,7,8	6	10
7	8	7	8	10,11	7	8
8	7	6	7		9	5
9	9	9	9		10	4
10	10	11	10		8	11
11	11	10	11		11	9

Table 16. Indices of agreement.

Validation technique	Index of agreement
Logical (task analysis)	.87
Item difficulty	.88
Correlation (phi coefficient)	.87
Guttman technique	.88
Textbook ordering	.62
AAAS approach	.87
Random ordering	.61

Table 17. Proportion of subjects obtaining acceptable scores on Pretest I (N = 142, 24 point test).

Acceptable score	Proportion of Ss obtaining score	Mean score on Pretest I
24	.23	21.84
23	.18	
22	.24	
21	.11	
20	.11	
19	.05	
18	.04	
17	.04	

Table 18. Proportion of subjects obtaining acceptable scores on pretest II (N = 142; 11 point test).

Acceptable score	Proportion of Ss obtaining score	Mean score on Pretest II
0	.71	0.563
1	.12	
2	.09	
3	.04	
4	.04	

Table 19. Reliability of measurements.

Test	KR-20
Hierarchical analyses	.81
Achievement	.94
Transfer	.91
Retention	.93

Table 20. Identification of groups.

Number of group	Description
1	Logical sequence
2	Guttman Technique
3	Random
4	Item difficulty
5	Phi coefficient
6	Textbook
7	AAAS approach

Table 21. One-way analysis of variance on achievement.

Source of Variation	MS	df	F	P
Treatments	18.83	6	1.45	.1992
Experimental error	12.98	135		

Group Means

1	2	3	4	5	6	7
6.23	4.84	6.47	5.21	6.05	5.14	7.62

Table 22. One-way analysis of variance on transfer.

Source of Variation	MS	df	F	P
Treatments	11.30	6	1.43	.2051
Experimental error	7.88	135		

Group Means

1	2	3	4	5	6	7
5.05	4.63	5.26	3.95	3.67	3.57	5.24

Table 23. One-way analysis of variance on retention.

Source of Variation	MS	df	F	P
Treatments	33.61	6	2.12	.0542
Experimental error	15.84	135		

Group means

1	2	3	4	5	6	7
7.50	7.32	6.89	5.37	7.62	4.95	8.52

Table 24. Time required to complete instructional program.

Group	Time in minutes		
	minimum	maximum	range
1	70	205	135
2	70	120	50
3	67	191	124
4	66	230	164
5	75	230	155
6	87	230	143
7	65	230	165

Table 25. One-way analysis of variance on time.

Source of Variation	MS	df	F	P
Treatments	2931.20	6	1.96	.0757
Experimental error	1498.99	135		

Group means

1	2	3	4	5	6	7
103.86	101.95	114.16	126.11	135.33	117.24	122.86

Table 26. One-way analysis of variance on achievement (partial credit).

Source of variation	MS	df	F	P
Treatments	43.21	6	1.688	.1278
Experimental error	25.60	135		

Group means

1	2	3	4	5	6	7
9.64	7.42	9.89	7.05	8.67	7.81	11.05

Table 27. One-way analysis of variance on achievement (reducing disregarded).

Source of variation	MS	df	F	P
Treatments	71.43	6	1.70	.1259
Experimental error	42.12	135		

Group means

1	2	3	4	5	6	7
11.91	9.68	13.05	8.47	11.00	11.10	14.05

Table 28. One-way analysis of variance on transfer (partial credit)

Source of Variation	MS	df	F	P
Treatments	19.97	6	1.89	.0870
Experimental error	10.59	135		

Group means

1	2	3	4	5	6	7
6.27	5.89	6.89	5.00	4.76	4.52	6.86

Table 29. One-way analysis of variance on transfer (reducing disregarded).

Source of Variation	MS	df	F	P
Treatments	31.43	6	2.17	.0465
Experimental error	14.31	135		

Group means

1	2	3	4	5	6	7
7.05	6.84	8.26	5.95	5.10	5.19	7.86

Table 30. One-way analysis of variance on retention (partial credit).

Source of Variation	MS	df	F	P
Treatments	56.32	6	2.14	.0525
Experimental error	26.35	135		

Group means

1	2	3	4	5	6	7
10.91	10.26	9.70	7.84	10.81	7.57	12.10

Table 31. One-way analysis of variance on retention (reducing disregarded).

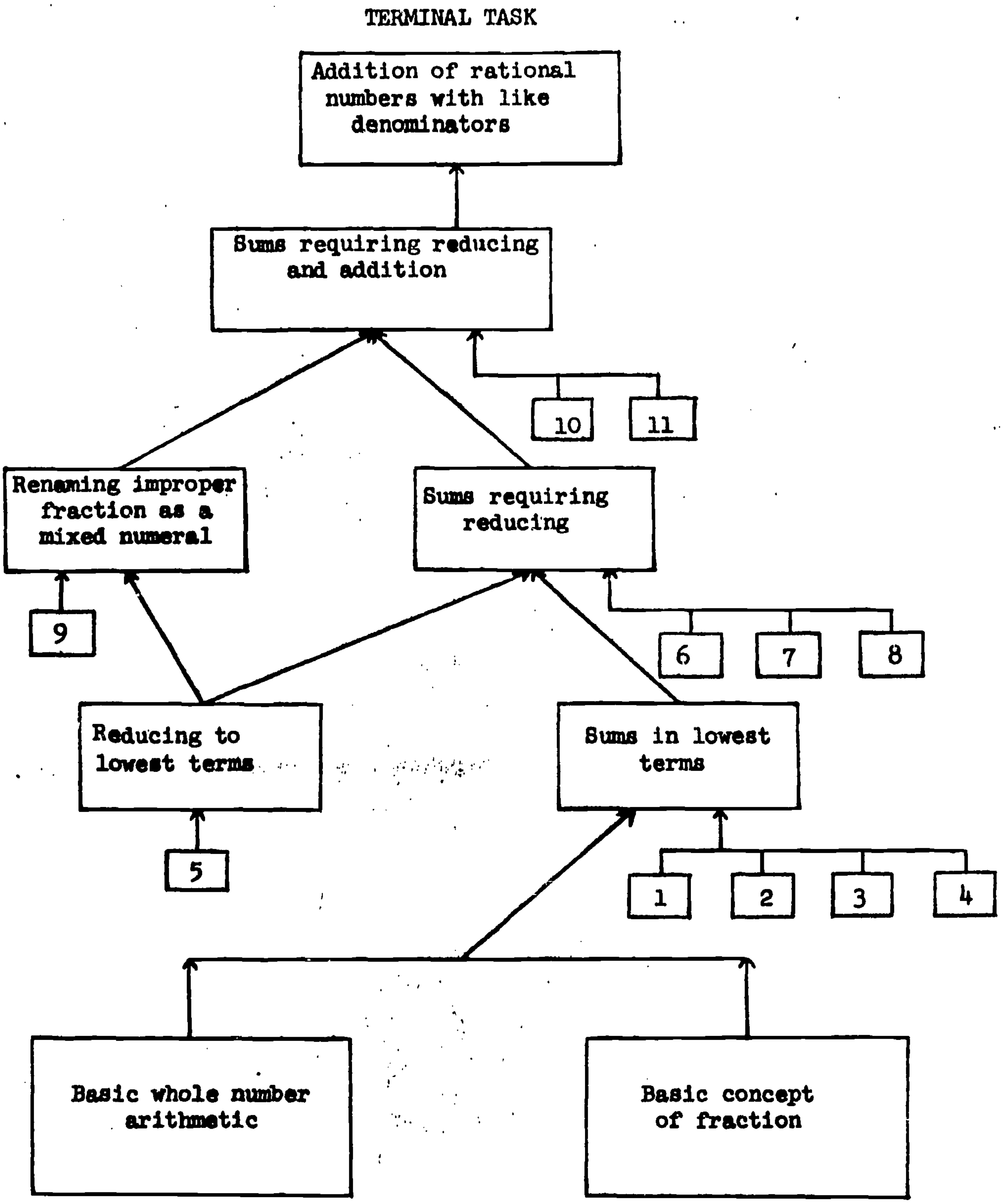
Source of variation	MS	df	F	P
Treatments	95.08	6	2.25	.0416
Experimental error	42.24	135		

Group means

1	2	3	4	5	6	7
14.23	12.84	12.53	9.68	13.90	10.00	15.48

APPENDIX B

HIERARCHY FOR THE COMPUTATIONAL SKILLS OF RATIONAL NUMBER ADDITION



IDENTIFICATION OF HIERARCHY CELLS

1. Adding with two fractions having like denominators where the sum requires no reducing.

$$\begin{array}{r} \frac{2}{9} \\ \underline{\frac{3}{9}} \end{array}$$

2. Adding with three fractions having like denominators where the sum requires no reducing.

$$\begin{array}{r} \frac{1}{10} \\ \frac{3}{10} \\ \underline{\frac{5}{10}} \end{array}$$

3. Adding a rational number named by a fraction and a whole number.

$$\begin{array}{r} 2 \\ \underline{\frac{1}{2}} \end{array}$$

4. Adding with two mixed numerals having like denominators.

$$\begin{array}{r} 3\frac{1}{7} \\ \underline{2\frac{3}{7}} \end{array}$$

5. Finding equivalent fractions by dividing both numerator and denominator by the same number.

$$\frac{6}{9} = \frac{\square}{3}$$

6. Adding with two fractions with like denominators where the sum requires reducing to lowest terms.

$$\begin{array}{r} \frac{3}{16} \\ \underline{\frac{5}{16}} \end{array}$$

7. Adding with two mixed numerals with like denominators where the sum of the fractional parts requires reducing to lowest terms.

$$\begin{array}{r} 4\frac{1}{9} \\ \underline{5\frac{2}{9}} \end{array}$$

8. Adding with three mixed numerals where all denominators are alike and the sum of the fractional parts requires reducing to lowest terms.

$$\begin{array}{r} 3\frac{1}{6} \\ 4\frac{1}{6} \\ \underline{2\frac{2}{6}} \end{array}$$

9. Changing names from an improper fraction to a mixed numeral.

$$\frac{12}{8} = \frac{8}{8} + \frac{\square}{8} =$$

10. Adding with two mixed numerals where the denominators involved are alike and the sum of the fractional parts is greater than 1.

$$\begin{array}{r} 3\frac{5}{8} \\ \underline{2\frac{7}{8}} \end{array}$$

11. Adding with three mixed numerals where the denominators involved are alike and the sum of the fractional parts is greater than 1.

$$\begin{array}{r} 7\frac{2}{5} \\ 4\frac{1}{5} \\ \underline{1\frac{4}{5}} \end{array}$$

APPENDIX C

PROGRAMMED INSTRUCTIONAL SEQUENCE

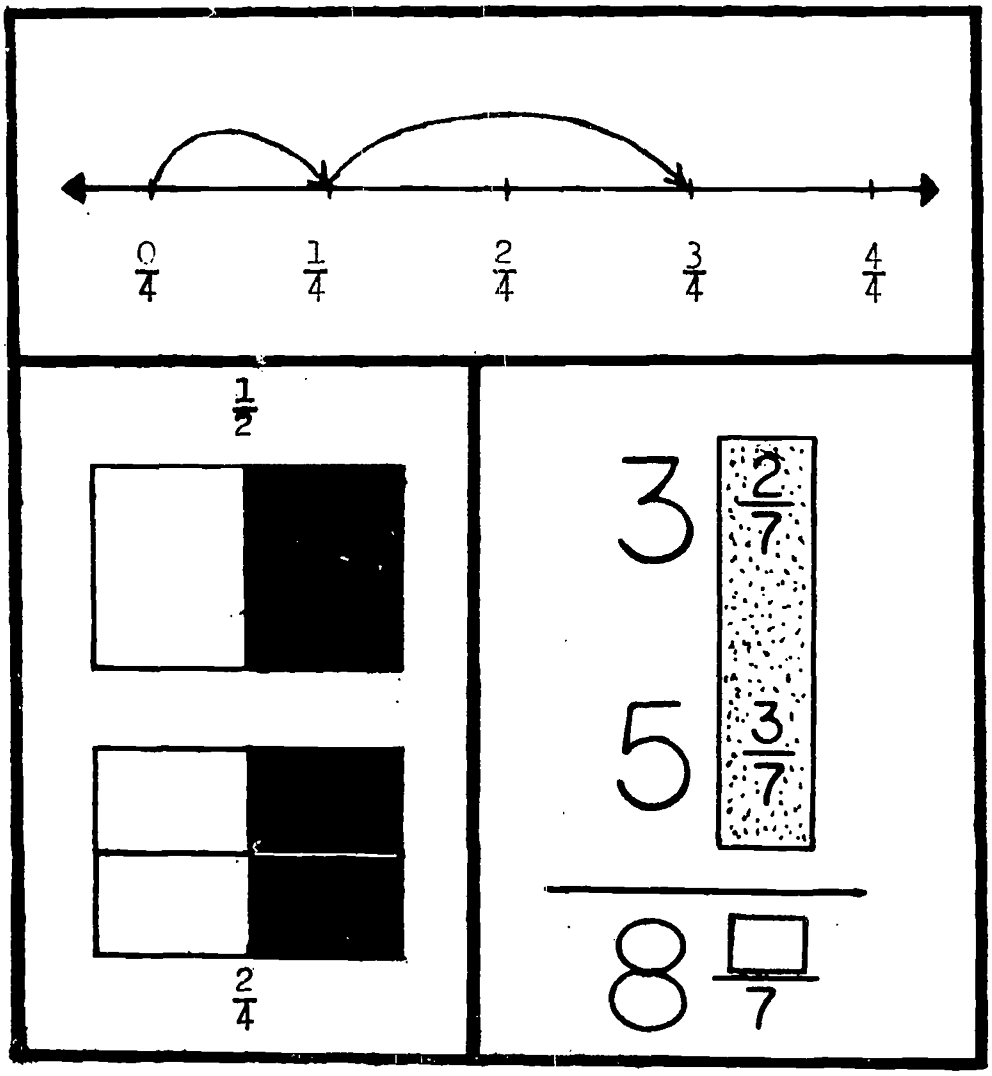
ADDITION OF FRACTIONS

PROGRAMMED

SKILL

BUILDING

SEQUENCE

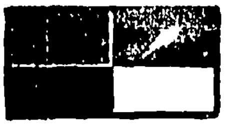


NAME ______________________________

SCHOOL ______________________________

Lesson 1

1. Study this picture.



(a) How many parts of the same size are there? ______

(b) How many parts are shaded? ______

(c) What part of the whole shape is shaded? ______

(d) A name for three fourths is $\frac{3}{4}$.

$\frac{3}{4}$ The 3 tells how many parts we are using.
The 4 tells the number of parts in the whole.

The numeral $\frac{3}{4}$ is called a fraction.

3 ⟷ numerator
4 ⟷ denominator

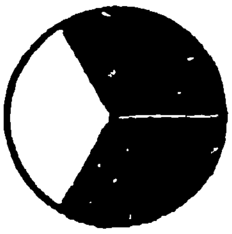
Each fraction is made up of two numerals which name whole numbers. The number named above the bar is called the ________. The number named below the bar is called the ________.

2. What part of each shape is shaded?

(a)

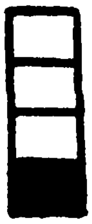
______

(b)



______

(c)



______

(d)

______

3. Name the numerators.

(a) $\frac{1}{2}$ ______ (b) $\frac{3}{4}$ ______ (c) $\frac{5}{8}$ ______

Name the denominators.

(a) $\frac{2}{9}$ ______ (b) $\frac{1}{5}$ ______ (c) $\frac{4}{7}$ ______

ANSWERS

(a) 4

(b) 3

(c) $\frac{3}{4}$

numerator
denominator

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{1}{4}$ (d) $\frac{5}{6}$

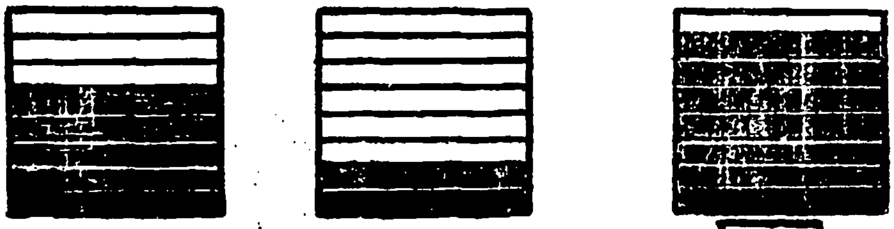
(a) 1 (b) 3 (c) 5

(a) 9 (b) 5 (c) 7

4. Complete the table.

	Fraction	We read
(a)	$\frac{2}{3}$	two thirds
(b)	$\frac{5}{8}$	______
(c)	$\frac{4}{9}$	______
(d)	$\frac{1}{2}$	one half
(e)	______	three fourths
(f)	______	five sevenths

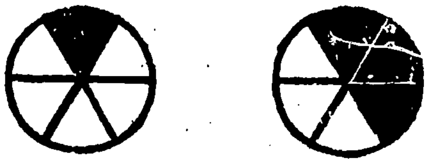
5. How can we add? Use the pictures to help you find the sum of $\frac{5}{8}$ and $\frac{2}{8}$.



$$\frac{5}{8} + \frac{2}{8} = \frac{\square}{8}$$

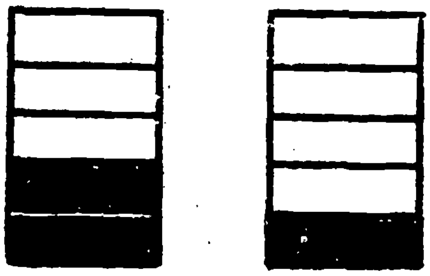
6. Use the shaded portions to help you find these sums.

(a)



$\frac{1}{6} + \frac{3}{6} = \frac{\square}{6}$

(b)



$\frac{2}{5} + \frac{1}{5} =$ ______

ANSWERS

(b) five eighths

(c) four ninths

(e) $\frac{3}{4}$

(f) $\frac{5}{7}$

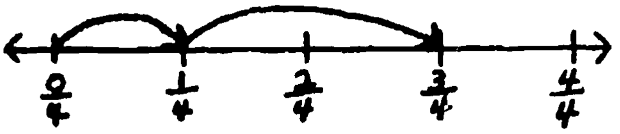
7

(a) 4

(b) $\frac{3}{5}$

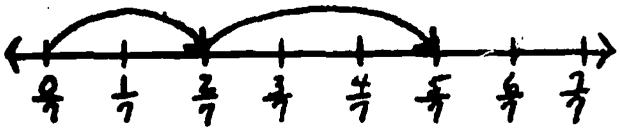
7. Use the number line to help you find these sums.

(a)



$\frac{1}{4}+\frac{2}{4}=\frac{\square}{\bigcirc}$

(b)



$\frac{2}{7}+\frac{3}{7}=\frac{\square}{\bigcirc}$

8. What have we discovered? When the denominators are the same, we add the numerators. We keep the same denominator.

Example: $\frac{4}{9}+\frac{3}{9}=\frac{4+3}{9}=\frac{\square}{9}$

$$\begin{array}{r} \frac{4}{9} \\ +\ \frac{3}{9} \\ \hline \frac{\square}{9} \end{array}$$

9. Find the sums. Remember! When the denominators are the same, we add the numerators and keep the same denominator.

(a) $\begin{array}{c} \frac{2}{9} \\ \frac{3}{9} \\ \hline \end{array}$ (b) $\begin{array}{c} \frac{1}{7} \\ \frac{2}{7} \\ \hline \end{array}$ (c) $\begin{array}{c} \frac{3}{8} \\ \frac{4}{8} \\ \hline \end{array}$ (d) $\begin{array}{c} \frac{2}{6} \\ \frac{5}{6} \\ \hline \end{array}$ (e) $\begin{array}{c} \frac{4}{11} \\ \frac{5}{11} \\ \hline \end{array}$

ANSWERS

(a) $\frac{3}{4}$

(b) $\frac{5}{7}$

7

7

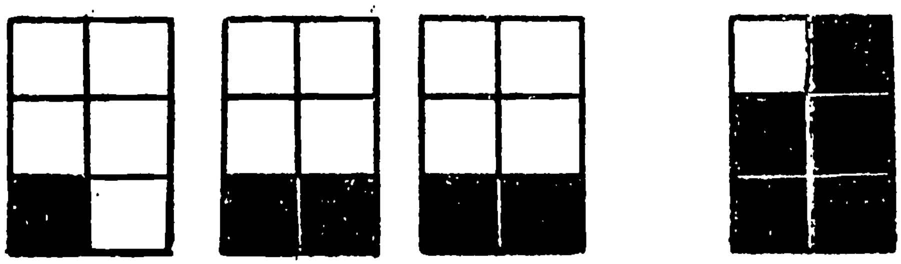
(a) $\frac{5}{9}$ (b) $\frac{3}{7}$

(c) $\frac{7}{8}$ (d) $\frac{7}{6}$

(e) $\frac{9}{11}$

Lesson 2

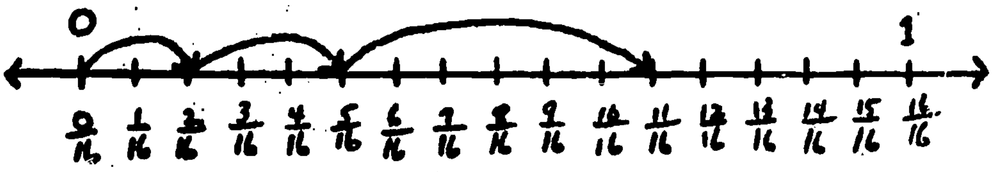
1. Use the shaded portions to help you find the sum.



$$\frac{1}{6} + \frac{2}{6} + \frac{2}{6} = \frac{\square}{6}$$

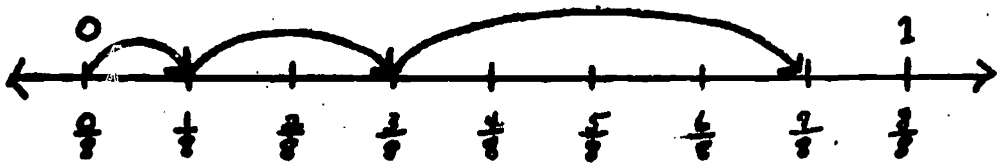
2. Use the number line to help you find the sums.

(a)



$$\frac{2}{16} + \frac{3}{16} + \frac{6}{16} = \frac{\square}{\bigcirc}$$

(b)



$$\frac{1}{8} + \frac{2}{8} + \frac{4}{8} = \frac{\square}{\bigcirc}$$

ANSWERS

5

(a) $\frac{11}{16}$

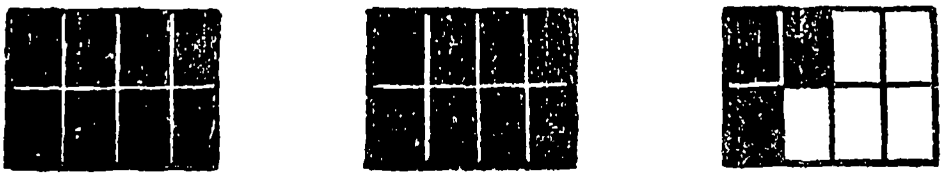
(b) $\frac{7}{8}$

	ANSWERS
3. How do you add with two fractions having like denominators? Add the ________________ and keep the same ________________.	numerators denominator
4. Do you add the same way with three fractions having like denominators? ______	yes
Example: $\frac{2}{9} + \frac{4}{9} + \frac{1}{9} = \frac{(2 + 4) + 1}{9}$ $= \frac{\square + 1}{9} = \frac{\square}{9}$	6, 7
$\begin{array}{r} \frac{2}{9} \\ \frac{4}{9} \\ \underline{\frac{1}{9}} \\ \frac{\square}{9} \end{array}$	7
5. Find the sums. (a) $\begin{array}{r} \frac{1}{10} \\ \frac{2}{10} \\ \underline{\frac{4}{10}} \\ \frac{\square}{10} \end{array}$ (b) $\begin{array}{r} \frac{2}{9} \\ \frac{4}{9} \\ \underline{\frac{1}{9}} \end{array}$ (c) $\begin{array}{r} \frac{3}{13} \\ \frac{2}{13} \\ \underline{\frac{6}{13}} \end{array}$ (d) $\begin{array}{r} \frac{2}{7} \\ \frac{1}{7} \\ \underline{\frac{2}{7}} \end{array}$ (e) $\begin{array}{r} \frac{1}{8} \\ \frac{2}{8} \\ \underline{\frac{4}{8}} \end{array}$	(a) 7 (b) $\frac{7}{9}$ (c) $\frac{11}{13}$ (d) $\frac{5}{7}$ (e) $\frac{7}{8}$

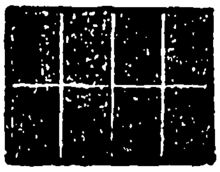
Lesson 3

ANSWERS

1. How do you add 2 and $\frac{3}{8}$? Use the pictures to help.



Note that



represents one whole.

Does $\frac{8}{8} = 1$? ______

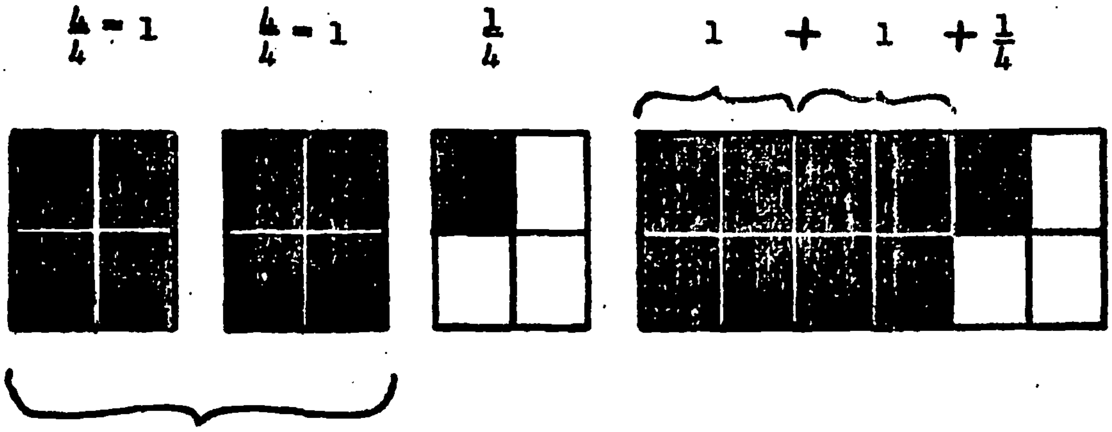
yes

In the picture above we have two wholes and $\frac{3}{8}$ of another.

Does $2 + \frac{3}{8} = 2\frac{3}{8}$? ______.

yes

2. Use the pictures to help you find the sum.



$$2 + \frac{1}{4} = 2\frac{\boxed{}}{4}$$

1

Does $2\frac{1}{4}$ mean $2 + \frac{1}{4}$? ______

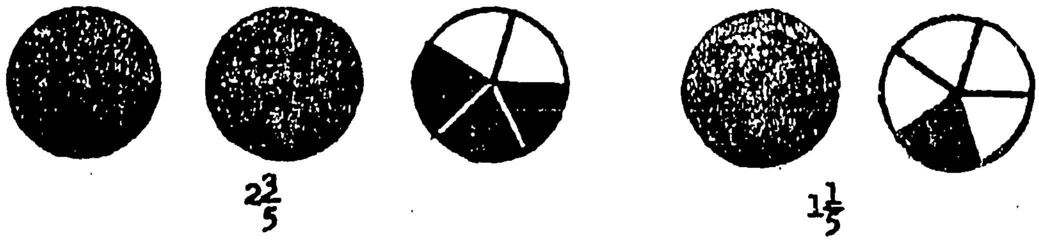
yes

	ANSWERS
A numeral like $2\frac{1}{4}$ is called a <u>mixed numeral</u>. Can you guess why? $2\frac{1}{4}$ is read "two and one fourth."	
3. What have we discovered? The number represented by $2 + \frac{1}{4}$ is the same as the number represented by the mixed numeral $2\frac{1}{4}$.	
(a) What mixed numeral names the same number as $4 + \frac{5}{6}$? ____	$4\frac{5}{6}$ (we say "four and five sixths")
(b) Does $4\frac{5}{6} = 4 + \frac{5}{6}$? ____	yes
4. Complete the following.	
(a) $3\frac{2}{5} = 3 + \frac{\boxed{}}{5}$	2
(b) $2\frac{7}{9} = 2 +$ ____	$\frac{7}{9}$
(c) $5\frac{4}{7} =$ ____	$5 + \frac{4}{7}$
(d) $4 + \frac{5}{8} = 4\frac{\boxed{}}{8}$	5
(e) $1 + \frac{3}{5} =$ ____	$1\frac{3}{5}$
(f) $7\frac{3}{4} =$ ____	$7 + \frac{3}{4}$
5. Add:	
(a) $\begin{array}{r}2\\ \frac{3}{7}\\ \hline\end{array}$ (b) $\begin{array}{r}4\\ \frac{1}{3}\\ \hline\end{array}$ (c) $\begin{array}{r}8\\ \frac{2}{9}\\ \hline\end{array}$ (d) $\begin{array}{r}11\\ \frac{3}{10}\\ \hline\end{array}$	(a) $2\frac{3}{7}$ (b) $4\frac{1}{3}$ (c) $8\frac{2}{9}$ (d) $11\frac{3}{10}$
(e) $\begin{array}{r}\frac{1}{2}\\ 4\\ \hline\end{array}$ (f) $\begin{array}{r}\frac{2}{3}\\ 5\\ \hline\end{array}$ (g) $\begin{array}{r}\frac{5}{13}\\ 12\\ \hline\end{array}$ (h) $\begin{array}{r}\frac{3}{16}\\ 32\\ \hline\end{array}$	(e) $4\frac{1}{2}$ (f) $5\frac{2}{3}$ (g) $12\frac{5}{13}$ (h) $32\frac{3}{16}$

ANSWERS

Lesson 4

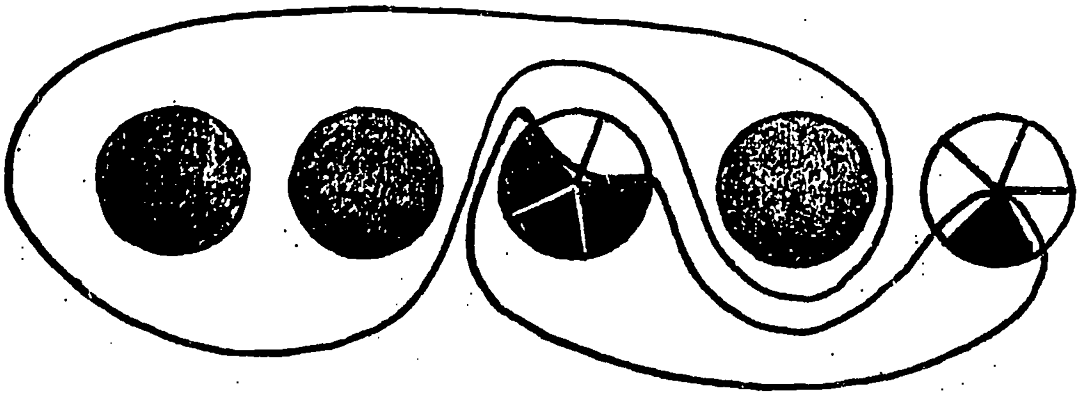
1. Mrs. Jones has $2\frac{3}{5}$ apple pies and $1\frac{1}{5}$ cherry pies.



(a) How many whole pies does she have? ____ (a) 3

(b) Note that each small piece is $\frac{1}{5}$ of a pie. How many pieces of this size are there? ____ (b) 4

2. How can we find how many pies Mrs. Jones has in all? Study the pictures.



(a) First, add the whole pies. 2 + 1 = ____ (a) 3

(b) Next, add the pieces of pie. $\frac{3}{5} + \frac{1}{5} =$ ____ (b) $\frac{4}{5}$

(c) Mrs. Jones has ____ whole pies and ____ of another pie. In all she has $3 + \frac{4}{5} =$ ____ pies. (c) 3 $\frac{4}{5}$ $3\frac{4}{5}$

	ANSWERS
3. Remember! $2\frac{3}{5} = 2 + \frac{3}{5}$ and $1\frac{1}{5} = 1 + \frac{1}{5}$.	
Thus, $2\frac{3}{5} = 2 + \frac{\boxed{}}{5}$	3
$+1\frac{1}{5} = 1 + \frac{\boxed{}}{5}$	1
$3 + \frac{\boxed{}}{5} = 3\frac{\boxed{}}{5}$	4, 4
Mrs. Jones has _____ pies in all.	$3\frac{4}{5}$
(a) Name the sum represented in each shaded box.	
$\begin{array}{rr} 3 & \frac{1}{9} \\ +\ 2 & \frac{4}{9} \\ \hline & \frac{\boxed{}}{9} \end{array}$ $\qquad$ $\begin{array}{rr} 3 & \frac{1}{9} \\ +\ 2 & \frac{4}{9} \\ \hline \bigcirc & \end{array}$	5, 5
Does $3\frac{1}{9} + 2\frac{4}{9} = 5\frac{5}{9}$? _____	yes
(b) What have we discovered? To add with two mixed numerals, first add the ______. Second add the ______ numbers.	fractions whole
Example: $\begin{array}{r} 4\frac{2}{7} \\ +5\frac{3}{7} \\ \hline \end{array}$	
Step 1. $\begin{array}{r} \frac{2}{7} \\ +\ \frac{3}{7} \\ \hline \frac{\boxed{}}{7} \end{array}$ $\quad$ Step 2. $\begin{array}{r} 4 \\ +5 \\ \hline \end{array}$ $\quad$ Step 3. $\begin{array}{r} 9 \\ +\ \frac{5}{7} \\ \hline \bigcirc\frac{\boxed{}}{7} \end{array}$	5 9 $9\frac{5}{7}$ (1) 5 (2) 9 (3) $9\frac{5}{7}$

4. **Find the sum.**

(a) $\begin{array}{r} 2\frac{4}{8} \\ 1\frac{1}{8} \\ \hline \end{array}$ (b) $\begin{array}{r} 9\frac{3}{10} \\ 2\frac{4}{10} \\ \hline \end{array}$ (c) $\begin{array}{r} 6\frac{5}{9} \\ 4\frac{3}{9} \\ \hline \end{array}$

(d) $\begin{array}{r} 11\frac{2}{9} \\ 7\frac{3}{9} \\ \hline \end{array}$ (e) $\begin{array}{r} 14\frac{3}{11} \\ 15\frac{5}{11} \\ \hline \end{array}$ (f) $\begin{array}{r} 10\frac{3}{7} \\ 5\frac{3}{7} \\ \hline \end{array}$

ANSWERS

(a) $3\frac{5}{8}$

(b) $11\frac{7}{10}$

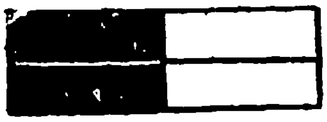
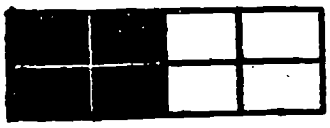
(c) $10\frac{8}{9}$

(d) $18\frac{5}{9}$

(e) $29\frac{8}{11}$

(f) $15\frac{6}{7}$

Lesson 5

	ANSWERS
1. Study these pictures.	
(a) What part is shaded? ______ 	(a) $\frac{2}{4}$
(b) What part is shaded? ______ 	(b) $\frac{1}{2}$
(c) Does $\frac{2}{4}$ and $\frac{1}{2}$ represent the same part of the region? ______	yes
Does $\frac{2}{4} = \frac{1}{2}$? ______	yes
(d) Does $\frac{2 \div 2}{4 \div 2} = \frac{1}{2}$? ______	yes
$\frac{2}{4}$ and $\frac{1}{2}$ are called <u>equivalent fractions</u>. They name the the same number.	
2. (a) How many eighths are shaded? ______ 	(a) 4
(b) How many sixteenths are shaded? ______ 	(b) 8
(c) A fraction equivalent to $\frac{8}{16}$ is ______.	(c) $\frac{4}{8}$
(d) Another fraction equivalent to $\frac{8}{16}$ is ______.	(d) $\frac{1}{2}$
(e) Does $\frac{8 \div 8}{16 \div 8} = \frac{1}{2}$? ______	(e) yes
What have we discovered?	
The number the fraction represents does not change when you divide both ______________ and ____________________ by the same number.	numerater denominator

	ANSWERS
3. A fraction is in lowest terms if there is no whole number other than 1 that will divide both the numerator and denominator.	
Examples: (a) $\frac{1}{4}$ is in lowest terms since no whole number other than 1 will divide both the numerator 1 and the denominator 4.	
(b) $\frac{6}{8}$ is not in lowest terms since 2 will divide both the ______ 6 and the ______ 8, $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$.	(b) numerator denominater
Are $\frac{6}{8}$ and $\frac{3}{4}$ equivalent fractions? ______	yes
Is $\frac{3}{4}$ in lowest terms? ______	yes
Is $\frac{3}{9}$ in lowest terms? ______	no (Since 3 will divide both 3 an 9
4. Find the missing numeral for each $\square$ to show how to rename these numbers in lowest terms.	
(a) $\frac{4 \div \square}{6 \div \square} = \frac{2}{3}$ (b) $\frac{5 \div \square}{10 \div \square} = \frac{1}{2}$	(a) 2 (b) 5
(c) $\frac{6 \div \square}{15 \div \square} = \frac{2}{5}$	(c) 3
5. Write these fractions in lowest terms.	
(a) $\frac{4}{8} = \frac{\square}{2}$ (b) $\frac{3}{18} = \frac{\square}{6}$	(a) 1 (b) 1
(c) $\frac{6}{8} = \frac{3}{\bigcirc}$ (d) $\frac{3}{12} = \frac{\square}{\bigcirc}$	(c) 4 (d) $\frac{1}{4}$

(e) $\frac{5}{10} =$ (f) $\frac{3}{9} =$

(g) $\frac{4}{16} =$ (h) $\frac{8}{12} =$

(i) $\frac{5}{20} =$ (j) $\frac{24}{30} =$

(k) $\frac{36}{48} =$ (l) $\frac{18}{32} =$

ANSWERS

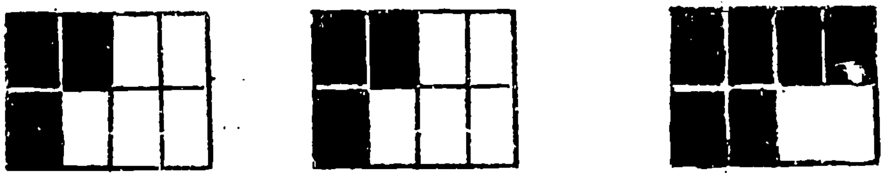
(e) $\frac{1}{2}$ (f) $\frac{1}{3}$

(g) $\frac{1}{4}$ (h) $\frac{2}{3}$

(i) $\frac{1}{4}$ (j) $\frac{4}{5}$

(k) $\frac{3}{4}$ (l) $\frac{9}{16}$

Lesson 6

	ANSWERS
1. How do we add $\frac{3}{8}$ and $\frac{4}{8}$? When the ______________	denominators
are the same, we add the __________ ________.	numerators
We keep the same ______________.	denominator
So, $\frac{3}{8} + \frac{4}{8} = \frac{\square + \bigcirc}{8} =$ ______	3,4, $\frac{7}{8}$
Is $\frac{7}{8}$ in lowest terms? ______	yes
2. Let's add $\frac{3}{8}$ and $\frac{3}{8}$. Use the pictures to help you find the sum.	
	
$\frac{3}{8} + \frac{3}{8} = \frac{\square}{8}$	6
Is $\frac{6}{8}$ in lowest terms? ______	No, since 2 will divide both 6 and 8.
What fraction is equivalent to $\frac{6}{8}$ (Use the picture to see the answer)? ______	$\frac{3}{4}$
$\frac{3}{4}$ is the simplest name for $\frac{6}{8}$.	
If your best friends were named Laura Melinda Williamson and Daniel LeRoy Applegate, you would not call them by their full names. Instead you would simplify their names to Laura and Danny. When we add with fractions, we will name the sum by its simplest name. We write the answer in ______________ ______________.	lowest terms

Thus,

$$\frac{3}{8}+\frac{3}{8}=\frac{6}{8}=\frac{6\div\square}{8\div\square}=\_\_\_\_\_$$

3. To add with fractions having like denominators: (1) add the numerators (2) keep the same denominator, and (3) write answer in __________ __________.

Example: $\frac{3}{16}+\frac{5}{16}=\frac{8}{16}=\frac{8\div 8}{16\div 8}=\frac{1}{2}$

$\frac{8}{16}$ in lowest terms is $\frac{1}{2}$. They are ______________ fractions.

Add. Remember! Write answers in lowest terms.

(a) $\begin{array}{r}\frac{1}{9}\\ \frac{5}{9}\\ \hline \frac{6}{9}\end{array}=\frac{6\div\square}{9\div\square}=\frac{2}{\bigcirc}$

(b) $\begin{array}{r}\frac{3}{8}\\ \frac{1}{8}\\ \hline \end{array}$

(c) $\begin{array}{r}\frac{3}{10}\\ \frac{5}{10}\\ \hline \end{array}$

(d) $\begin{array}{r}\frac{4}{15}\\ \frac{8}{15}\\ \hline \end{array}$

(e) $\begin{array}{r}\frac{3}{16}\\ \frac{1}{16}\\ \hline \end{array}$

(f) $\begin{array}{r}\frac{13}{24}\\ \frac{8}{24}\\ \hline \end{array}$

ANSWERS

2, $\frac{3}{4}$

lowest terms

equivalent

(a) 3,3

(b) $\frac{1}{2}$

(c) $\frac{4}{5}$

(d) $\frac{4}{5}$

(e) $\frac{1}{4}$

(f) $\frac{7}{8}$

Lesson 7

1. Let's add $4\frac{3}{8}$ and $5\frac{1}{8}$.

(a) Does $4\frac{3}{8} = 4 + \frac{3}{8}$? ______

(b) Does $5\frac{1}{8} = 5 + \frac{1}{8}$? ______

(c) $4\frac{3}{8} = 4 + \frac{\square}{8}$

$5\frac{1}{8} = 5 + \frac{\square}{8}$

$9 + \frac{\square}{8} =$ ______

(d) Is there a simpler name for $\frac{4}{8}$? ______

(e) Write $\frac{4}{8}$ in lowest terms. $\frac{4}{8} =$ ______

So, $4\frac{3}{8}$

$+\ 5\frac{1}{8}$

$9\frac{4}{8} =$ ______

2. Add. Write answers in lowest terms.

(a) $9\frac{1}{10}$

$10\frac{3}{10}$

$19\frac{4}{10} = 19 + \frac{4}{10} = 19 + \frac{4 \div \square}{10 \div \square} = 19 + \frac{⬡}{\bigcirc} =$ ______

(b) $17\frac{3}{6}$

$5\frac{1}{6}$

$22\frac{4}{6} = 22\frac{\square}{3}$

(c) $8\frac{1}{4}$

$25\frac{1}{4}$

ANSWERS

(a) yes

(b) yes

(c) 3, 1, 4 $9\frac{4}{8}$

(d) yes

(e) $\frac{1}{2}$

$9\frac{1}{2}$

(a) 2, $\frac{2}{5}$

$19\frac{2}{5}$

(b) 2

(c) $33\frac{1}{2}$

(d) $3\frac{1}{6}$

$$\begin{array}{r} 1\frac{1}{6} \\ \hline \end{array}$$

(e) $3\frac{4}{15}$

$$\begin{array}{r} 5\frac{2}{15} \\ \hline \end{array}$$

(f) $15\frac{3}{14}$

$$\begin{array}{r} 27\frac{4}{14} \\ \hline \end{array}$$

(g) $8\frac{7}{16}$

$$\begin{array}{r} 3\frac{5}{16} \\ \hline \end{array}$$

ANSWERS

(d) $4\frac{1}{3}$

(e) $8\frac{2}{5}$

(f) $42\frac{1}{2}$

(g) $11\frac{3}{4}$

Lesson 8

1. Name the sum.

$$\begin{array}{r} 3\frac{1}{6} \\ 4\frac{1}{6} \\ 2\frac{2}{6} \\ \hline \end{array}$$

adding the fractional parts

Step 1: $\frac{1}{6} + \frac{1}{6} + \frac{2}{6} = \frac{(1 + 1) + 2}{6} = \frac{\square}{6}$

Reducing to lowest terms

Step 2: $\frac{4}{6} = \frac{4 \div \square}{6 \div \square} = \frac{\square}{3}$

adding the whole number parts

Step 3: $3 + 4 + 2 =$ ______

adding whole number sum to fractional part sum

Step 4: $9 + \frac{2}{3} =$ ______

ANSWERS

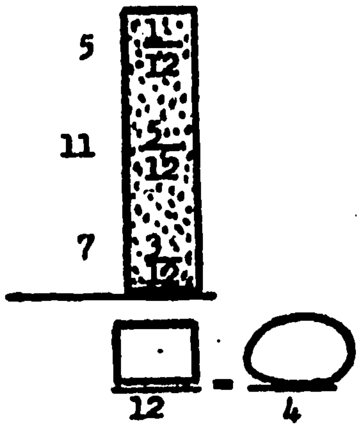
4

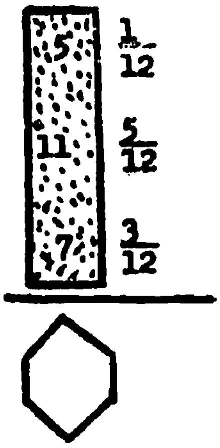
2, 2

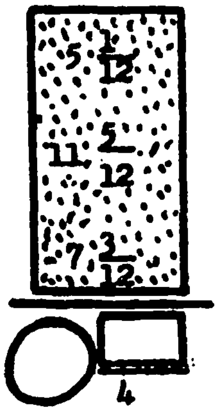
9

$9\frac{2}{3}$

(b) Name the sum represented in each shaded box.

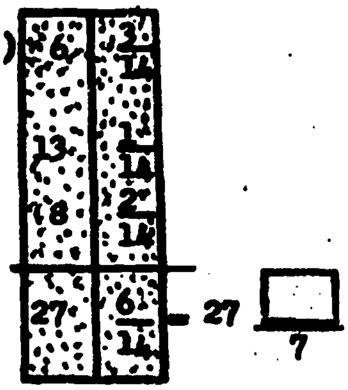






2. Add. Write answers in lowest terms.

(a)



(b)

$$\begin{array}{r} 16\,\frac{2}{21} \\ 4\,\frac{4}{21} \\ 3\,\frac{3}{21} \\ \hline \end{array}$$

ANSWERS

9, 3, 23

$23\frac{3}{4}$

(a) 3

(b) $23\frac{3}{7}$

(c) $$\begin{array}{r} 23\frac{2}{9} \\ 9\frac{1}{9} \\ \underline{8\frac{3}{9}} \end{array}$$

(d) $$\begin{array}{r} 4\frac{3}{16} \\ 7\frac{5}{16} \\ \underline{10\frac{4}{16}} \end{array}$$

(e) $$\begin{array}{r} 14\frac{3}{8} \\ 1\frac{1}{8} \\ \underline{9\frac{2}{8}} \end{array}$$

(f) $$\begin{array}{r} 43\frac{5}{18} \\ 4\frac{7}{18} \\ \underline{13\frac{3}{18}} \end{array}$$

(g) $$\begin{array}{r} 6\frac{2}{15} \\ 2\frac{1}{15} \\ \underline{5\frac{7}{15}} \end{array}$$

(h) $$\begin{array}{r} 9\frac{1}{10} \\ 3\frac{3}{10} \\ \underline{12\frac{1}{10}} \end{array}$$

(i) $$\begin{array}{r} 3\frac{5}{36} \\ 8\frac{7}{36} \\ \underline{6\frac{9}{36}} \end{array}$$

ANSWERS

(c) $40\frac{2}{3}$

(d) $21\frac{3}{4}$

(e) $24\frac{3}{4}$

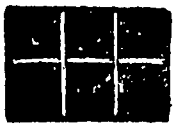
(f) $60\frac{5}{6}$

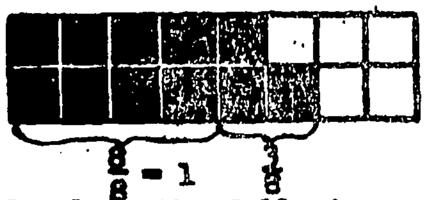
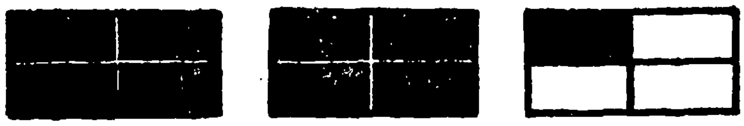
(g) $13\frac{2}{3}$

(h) $24\frac{1}{2}$

(i) $17\frac{7}{12}$

Lesson 9

	ANSWERS
1. Use the pictures to help you answer these questions.	
(a)  How many thirds are shaded? _____	3
(b) How many $\frac{1}{3}$ boxes in 1 box? _____	3
(c) $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{\square}{3}$.	3
(d) Does $\frac{3}{3}=1$? _____	yes
(e)  How many sixths are shaded? _____	6
(f) $\frac{\square}{6}=1$	6
2. Complete the following.	
(a) $\frac{4}{4}=\square$ (b) $\frac{7}{7}=\square$	(a) 1 (b) 1
(c) $\frac{2}{\square}=1$ (d) $\frac{5}{\square}=1$	(c) 2 (d) 5
(e) $\frac{\square}{8}=1$ (f) $\frac{\square}{15}=1$	(e) 8 (f) 15
What have we discovered? When both the numerator and the denominator of a fraction are the same, the fraction names the whole number _______.	1
3. Jim has $\frac{7}{4}$ candy bars. Use the shaded portions to help you find a simpler name for $\frac{7}{4}$. 	
(a) How many fourths are shaded red? _____	4
(b) Four $\frac{1}{4}$ candy bars make _____ _____ candy bar.	one whole
(c) How many fourths are shaded green? _____	3
(d) How many fourths are shaded in all? _____	7

	ANSWERS
(e) $\frac{7}{4} = \frac{4}{4} + \frac{3}{4} = \square + \frac{3}{4} = 1\frac{\square}{4}$	1, 3
(f) Jim has one and ______ ______ candy bars.	three fourths
4. Use the shaded portions to help you write a mixed numeral that names the same number as $\frac{11}{8}$.  $\frac{8}{8} = 1$ $\frac{3}{8}$ $\frac{11}{8} = \frac{\square}{8} + \frac{\bigcirc}{8} =$ ______	8, 3, $1\frac{3}{8}$
5. Complete the following.	
(a) $\frac{9}{7} = \frac{7}{7} + \frac{\square}{7} = \bigcirc + \frac{\square}{7} = \bigcirc\frac{\square}{7}$	(a) 2, 1, 2, $1\frac{2}{7}$
(b) $\frac{12}{9} = \frac{9}{9} + \frac{\square}{9} = \bigcirc\frac{\square}{9}$ Is $\frac{3}{9}$ in lowest	(b) 3, $1\frac{3}{9}$
terms? ______. $\frac{3}{9} = \frac{\square}{3}$	no, 1
Thus $\frac{12}{9} = 1\frac{\square}{3}$	1
(c) $\frac{10}{8} = \frac{8}{8} + \frac{\square}{8} = \bigcirc\frac{\square}{8} = \bigcirc\frac{\square}{4}$	(c) 2, 1, 2, $1\frac{1}{4}$
(d) $\frac{9}{7} =$ ______ + ______ = ______	(d) $\frac{7}{7}$, $\frac{2}{7}$, $1\frac{2}{7}$
6. Use the shaded portions to help you write a mixed numeral that names the same number as $\frac{9}{4}$.  $\frac{9}{4} = \frac{\square}{4} + \frac{\square}{4} + \frac{\square}{4}$	4, 4, 1
$\frac{9}{4} =$ ______ + ______ + $\frac{\square}{4}$ = ______ + $\frac{\square}{4} = \bigcirc\frac{\square}{4}$	1, 1, 2, 1, $2\frac{1}{4}$

	ANSWERS
7. Write mixed numerals for each. Remember! Write fractional part to lowest terms.	
(a) $\frac{12}{8} = \frac{8}{8} + \frac{4}{8} = 1 + \frac{\boxed{}}{2} = 1\frac{1}{2}$	1
(b) $\frac{9}{5} = \frac{5}{5} + \frac{\boxed{}}{5} =$ ____	4, $1\frac{4}{5}$
(c) $\frac{13}{6} = \frac{6}{6} + \frac{6}{6} + \frac{\boxed{}}{6} = 1 + 1 + \frac{\boxed{}}{6} = \bigcirc\frac{\boxed{}}{6}$	1, 1, $2\frac{1}{6}$
(d) $\frac{15}{7} =$ ____	$2\frac{1}{7}$
(e) $\frac{9}{8} =$ ____	$1\frac{1}{8}$
(f) $\frac{10}{4} =$ ____	$2\frac{1}{2}$
(g) $\frac{15}{12} =$ ____	$1\frac{1}{4}$
(h) $\frac{7}{5} =$ ____	$1\frac{2}{5}$

Lesson 10

1. Name the sum.

(a)
$$\begin{array}{r} 3\frac{5}{7} \\ 5\frac{6}{7} \\ \hline 8\frac{\boxed{}}{7} \end{array}$$

(b) Note that the numerator of the fraction $\frac{11}{7}$ is

__________ than the __________. Fractions of this

type can be renamed as __________ __________.

(c) $\frac{11}{7} = \frac{7}{7} + \frac{\boxed{}}{7} = 1\frac{\boxed{}}{7}$

(d) $8\frac{11}{7} = 8 + 1\frac{\boxed{}}{7} = \bigcirc\frac{\boxed{}}{7}$

$9\frac{4}{7}$ is a simpler name for $8\frac{11}{7}$.

2. Complete the following.

Rename as a mixed numeral

Reduce to lowest terms

$$\begin{array}{r} 10\frac{5}{8} \\ 7\frac{7}{8} \\ \hline \bigcirc\frac{12}{8} \end{array} = \bigcirc + 1\frac{4}{8} = \bigcirc + 1\frac{\boxed{}}{2} = \bigcirc\frac{\boxed{}}{2}$$

ANSWERS

(a) 11

(b) greater
denominator
mixed
numerals

(c) 4, 4

(d) 4, $9\frac{4}{7}$

17,
17, 17, 1

$18\frac{1}{2}$

	ANSWERS
3. $\begin{array}{r} 12\ \frac{9}{14} \\ 27\ \frac{11}{14} \\ \hline \end{array}$	
Step 1. $\frac{9}{14} + \frac{11}{14} = \frac{\boxed{}}{14}$	20
Step 2. $\frac{20}{14} = \frac{14}{14} + \frac{\boxed{}}{14} = 1 + \frac{\boxed{}}{7} = 1\frac{\boxed{}}{7}$	6, 3, 3
Step 3. $12 + 27 =$ ______	39
Step 4. $39 + 1\frac{3}{7} =$ ______	$40\frac{3}{7}$
4. Add. Write answers in simpliest form.	
(a) $\begin{array}{r} 15\ \frac{7}{15} \\ 4\ \frac{11}{15} \\ \hline \end{array}$	(a) 18, 3
$19\frac{\boxed{}}{15} = 19 + 1\frac{\boxed{}}{15} = 19 + 1\frac{\boxed{}}{5} =$ ______	1, 20 $\frac{1}{5}$
(b) $\begin{array}{r} 3\ \frac{3}{4} \\ 9\ \frac{1}{4} \\ \hline \end{array}$	
$12\frac{\boxed{}}{4} = 12 +$ ____ $=$ ____	(b) 4, 1, 13
(c) $\begin{array}{r} 8\ \frac{7}{9} \\ 16\ \frac{4}{9} \\ \hline \end{array}$	
$\bigcirc\frac{\boxed{}}{9} = 25\frac{\boxed{}}{9}$	(c) 24, 11, 2

(d) $\begin{array}{r} 25\frac{4}{7} \\ \underline{18\frac{5}{7}} \end{array}$

(e) $\begin{array}{r} 11\frac{3}{10} \\ \underline{4\frac{9}{10}} \end{array}$

(f) $\begin{array}{r} 2\frac{7}{8} \\ \underline{16\frac{3}{8}} \end{array}$

(g) $\begin{array}{r} 43\frac{13}{16} \\ \underline{35\frac{5}{16}} \end{array}$

(h) $\begin{array}{r} 7\frac{9}{11} \\ \underline{8\frac{4}{11}} \end{array}$

(i) $\begin{array}{r} 8\frac{17}{18} \\ \underline{37\frac{7}{18}} \end{array}$

ANSWERS

(d) $44\frac{2}{7}$

(e) $16\frac{1}{5}$

(f) $19\frac{1}{4}$

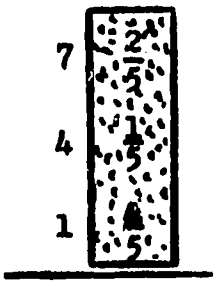
(g) $79\frac{1}{8}$

(h) $16\frac{2}{11}$

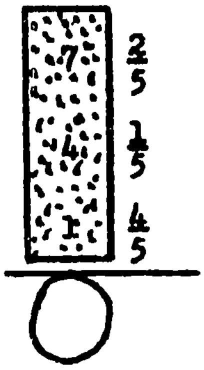
(i) $46\frac{1}{3}$

Lesson 11

1. Name the sum represented in each shaded box.



$\frac{\boxed{}}{5} = 1\,\frac{\boxed{}}{5}$



$1\frac{2}{5} + 12 =$ ______

Thus

$$\begin{array}{rl} 7 & \frac{2}{5} \\ 4 & \frac{1}{5} \\ 1 & \frac{4}{5} \\ \hline 13 & \frac{\boxed{}}{5} \end{array}$$

2. Add. Write answers in lowest terms.

(a)

$$\begin{array}{rl} 13 & \frac{5}{12} \\ 5 & \frac{7}{12} \\ 7 & \frac{11}{12} \\ \hline \bigcirc & \frac{\boxed{}}{12} \end{array} = \bigcirc + 1\,\frac{\boxed{}}{12} = \underline{}$$

ANSWERS

7, 2,
12

$13\frac{2}{5}$

2

(a) 25 , 23
25 , 11,
$26\frac{11}{12}$

	ANSWERS
(b) $21\frac{7}{8}$ $9\frac{3}{8}$ $12\frac{5}{8}$ ______	(b) $43\frac{7}{8}$
(c) $7\frac{3}{4}$ $4\frac{1}{4}$ $2\frac{3}{4}$ ______	(c) $14\frac{3}{4}$
(d) $41\frac{5}{6}$ $14\frac{4}{6}$ $8\frac{2}{6}$ $\bigcirc\frac{\square}{6} = \bigcirc + 2\frac{\square}{6} = \bigcirc + 2\frac{\square}{3} =$ ______	(d) 63, 14, 63, 2 63, 1 $65\frac{1}{3}$

	ANSWERS
(e) $\begin{array}{r} 8\frac{4}{5} \\ 5\frac{3}{5} \\ 16\frac{4}{5} \\ \hline \end{array}$	$31\frac{1}{5}$
(f) $\begin{array}{r} 13\frac{5}{7} \\ 15\frac{3}{7} \\ 24\frac{2}{7} \\ \hline \end{array}$	$53\frac{3}{7}$

APPENDIX D

TESTING INSTRUMENTS

RATIONAL NUMBER ADDITION

Name ______________________ Grade ______________________

Teacher ______________________ School ______________________

Add:

1.	$\begin{array}{r}\frac{2}{9}\\ \frac{3}{9}\\ \hline\end{array}$	$\begin{array}{r}\frac{1}{5}\\ \frac{2}{5}\\ \hline\end{array}$	$\begin{array}{r}\frac{2}{7}\\ \frac{4}{7}\\ \hline\end{array}$
2.	$\begin{array}{r}\frac{1}{10}\\ \frac{3}{10}\\ \frac{5}{10}\\ \hline\end{array}$	$\begin{array}{r}\frac{2}{8}\\ \frac{3}{8}\\ \frac{2}{8}\\ \hline\end{array}$	$\begin{array}{r}\frac{4}{13}\\ \frac{6}{13}\\ \frac{1}{13}\\ \hline\end{array}$
3.	$\begin{array}{r}2\\ \frac{1}{2}\\ \hline\end{array}$	$\begin{array}{r}3\\ \frac{3}{5}\\ \hline\end{array}$	$\begin{array}{r}7\\ \frac{2}{9}\\ \hline\end{array}$
4.	$\begin{array}{r}3\frac{1}{7}\\ 2\frac{3}{7}\\ \hline\end{array}$	$\begin{array}{r}4\frac{2}{5}\\ 9\frac{2}{5}\\ \hline\end{array}$	$\begin{array}{r}10\frac{4}{11}\\ 3\frac{5}{11}\\ \hline\end{array}$

5. Write these fractions in lowest terms:

$\frac{6}{9}$ = ______ $\frac{6}{15}$ = ______ $\frac{5}{20}$ = ______

Add. Write answers in lowest terms.

6.

$$\begin{array}{ccc} \frac{3}{16} & \frac{7}{15} & \frac{2}{10} \\ \frac{5}{16} & \frac{2}{15} & \frac{3}{10} \\ \hline \end{array}$$

7.

$$\begin{array}{ccc} 4\frac{1}{9} & 8\frac{7}{16} & 13\frac{3}{4} \\ 5\frac{2}{9} & 3\frac{5}{16} & 7\frac{1}{4} \\ \hline \end{array}$$

8.

$$\begin{array}{ccc} 3\frac{1}{6} & 7\frac{5}{12} & 22\frac{3}{15} \\ 4\frac{1}{6} & 11\frac{3}{12} & 5\frac{7}{15} \\ 2\frac{2}{6} & 6\frac{1}{12} & 17\frac{2}{15} \\ \hline \end{array}$$

9. Write mixed numerals that stand for the same number as each of these fractions.

$\frac{7}{4}$ = $\frac{24}{10}$ = $\frac{16}{6}$ =

Add. Write answers in lowest terms.

10.	$3\frac{5}{8}$	$15\frac{4}{9}$	$4\frac{3}{7}$
	$2\frac{7}{8}$	$6\frac{7}{9}$	$8\frac{5}{7}$
11.	$7\frac{2}{5}$	$13\frac{1}{4}$	$9\frac{5}{12}$
	$4\frac{1}{5}$	$5\frac{3}{4}$	$10\frac{7}{12}$
	$2\frac{4}{5}$	$7\frac{2}{4}$	$11\frac{11}{12}$

RATIONAL NUMBER ADDITION

Pretest I

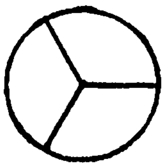
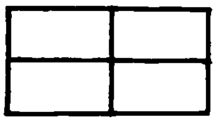
Name ______________________ Teacher ______________________

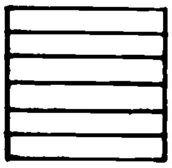
Grade ______________________ School ______________________

1. Circle the numerals that are fractions.

1 $\frac{1}{2}$ $\frac{3}{5}$ 4 $\frac{2}{3}$

2. Circle the pictures which show thirds.



3. Match each numeral with the shaded part of a shape as shown.

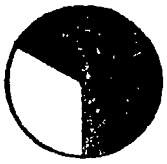
$\frac{1}{2}$

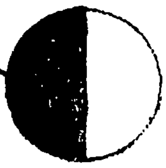
$\frac{2}{3}$

$\frac{1}{4}$

1

$\frac{3}{5}$





4. Write the numeral that names three eighths. ________.

5. Write the words that name $\frac{4}{5}$ ____________ ____________.

6. Fill in the $\boxed{}$ to make these sentences true.

$1 = \frac{\boxed{}}{6}$ $\frac{4}{4} = \boxed{}$

7. Add.

$$\begin{array}{r} 4 \\ 5 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 21 \\ +\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 14 \\ +\ 27 \\ \hline \end{array} \qquad \begin{array}{r} 39 \\ +\ 86 \\ \hline \end{array} \qquad \begin{array}{r} 605 \\ 3 \\ +\ 74 \\ \hline \end{array}$$

8. Divide.

$4 \div 2 =$ $9 \div 3 =$ $12 \div 4 =$

$18 \div 3 =$ $42 \div 3 =$ $54 \div 2 =$

RATIONAL NUMBER ADDITION

Pretest II

Name ______________________ Teacher ______________________

Grade ______________________ School ______________________

Add.

1. $\begin{array}{r} \frac{1}{5} \\ + \frac{3}{5} \\ \hline \end{array}$

2. $\begin{array}{r} \frac{1}{9} \\ \frac{2}{9} \\ + \frac{4}{9} \\ \hline \end{array}$

3. $\begin{array}{r} 4 \\ + \quad \frac{3}{4} \\ \hline \end{array}$

4. $\begin{array}{r} 2\frac{2}{7} \\ + 4\frac{3}{7} \\ \hline \end{array}$

5. Reduce to lowest terms: $\frac{6}{8} =$

Add. Write answers in lowest terms.

6. $\begin{array}{r} \frac{1}{9} \\ + \frac{5}{9} \\ \hline \end{array}$

7. $\begin{array}{r} 9\frac{3}{10} \\ + 4\frac{2}{10} \\ \hline \end{array}$

8. $\begin{array}{r} 5\frac{1}{8} \\ 7\frac{2}{8} \\ + 8\frac{3}{8} \\ \hline \end{array}$

9. Write as a mixed numeral: $\frac{15}{4} =$

Add. Write answers in lowest terms.

10. $\begin{array}{r} 11\frac{7}{9} \\ + 15\frac{5}{9} \\ \hline \end{array}$

11. $\begin{array}{r} 12\frac{4}{7} \\ 5\frac{6}{7} \\ + 3\frac{5}{7} \\ \hline \end{array}$

RATIONAL NUMBER ADDITION

Achievement Test Form A

Name ____________________ Teacher ____________________

Grade ____________________ School ____________________

Add: Write answers in lowest terms.

1. $\begin{array}{r} 8\frac{2}{7} \\ \underline{9\frac{6}{7}} \end{array}$

2. $\begin{array}{r} 6\frac{1}{5} \\ \underline{3\frac{2}{5}} \end{array}$

3. $\begin{array}{r} 8\frac{7}{12} \\ 3\frac{11}{12} \\ \underline{9\frac{8}{12}} \end{array}$

4. $\begin{array}{r} \frac{3}{8} \\ \underline{\frac{2}{8}} \end{array}$

5. $\begin{array}{r} 8\frac{5}{13} \\ \underline{10\frac{6}{13}} \end{array}$

6. $\begin{array}{r} 2 \\ \underline{\frac{1}{8}} \end{array}$

7. $\begin{array}{r} \frac{5}{9} \\ \underline{\frac{2}{9}} \end{array}$

8. $\begin{array}{r} \frac{3}{10} \\ \frac{4}{10} \\ \underline{\frac{2}{10}} \end{array}$

9. $\begin{array}{r} 11\frac{5}{8} \\ \underline{4\frac{7}{8}} \end{array}$

10.
$$\begin{array}{r} \frac{1}{11} \\ \frac{4}{11} \\ \underline{\frac{5}{11}} \end{array}$$

11.
$$\begin{array}{r} 12\ \frac{4}{5} \\ 4\ \frac{3}{5} \\ \underline{16\ \frac{1}{5}} \end{array}$$

12.
$$\begin{array}{r} \frac{2}{6} \\ \underline{\frac{2}{6}} \end{array}$$

13.
$$\begin{array}{r} 4\ \frac{3}{8} \\ 3\ \frac{1}{8} \\ \underline{5\ \frac{2}{8}} \end{array}$$

14.
$$\begin{array}{r} 15\ \frac{4}{18} \\ 3\ \frac{5}{18} \\ \underline{5\ \frac{3}{18}} \end{array}$$

15.
$$\begin{array}{r} 4\ \frac{7}{15} \\ \underline{6\ \frac{2}{15}} \end{array}$$

16.
$$\begin{array}{r} 15\ \frac{2}{10} \\ \underline{7\ \frac{3}{10}} \end{array}$$

17.
$$\begin{array}{r} \frac{5}{14} \\ \underline{\frac{3}{14}} \end{array}$$

18.
$$\begin{array}{r} 7 \\ \underline{\frac{1}{2}} \end{array}$$

RATIONAL NUMBER SUBTRACTION

Transfer Test

Name ______________________ Teacher ______________________

Grade ______________________ School ______________________

Subtract. Write answers in lowest terms.

1. $\begin{array}{r} \frac{5}{7} \\ -\ \frac{2}{7} \\ \hline \end{array}$

2. $\begin{array}{r} \frac{7}{9} \\ -\ \frac{4}{9} \\ \hline \end{array}$

3. $\begin{array}{r} 5\,\frac{4}{5} \\ -\ 3\,\frac{1}{5} \\ \hline \end{array}$

4. $\begin{array}{r} \frac{11}{16} \\ -\ \frac{4}{16} \\ \hline \end{array}$

5. $\begin{array}{r} 9\,\frac{7}{8} \\ -\ 2\,\frac{5}{8} \\ \hline \end{array}$

6. $\begin{array}{r} 7\,\frac{5}{7} \\ -\ \frac{3}{7} \\ \hline \end{array}$

7. $\begin{array}{r} \frac{9}{10} \\ -\ \frac{1}{10} \\ \hline \end{array}$

8. $\begin{array}{r} 5\,\frac{1}{2} \\ -\ 2 \\ \hline \end{array}$

9. $\begin{array}{r} 21\,\frac{13}{15} \\ -\ 3\,\frac{4}{15} \\ \hline \end{array}$

10. $\begin{array}{r} 15\,\frac{12}{13} \\ -\ 6\,\frac{4}{13} \\ \hline \end{array}$

RATIONAL NUMBER ADDITION
Achievement Test Form B
(Retention Test)

Name ______________________ Teacher ______________________

Grade ______________________ School ______________________

Add. Write answers in lowest terms.

1. $\frac{1}{8}$
$\underline{\frac{3}{8}}$

2. $7\frac{7}{9}$
$\underline{6\frac{4}{9}}$

3. $12\frac{4}{6}$
$\underline{5\frac{5}{6}}$

4. $10\frac{5}{12}$
$9\frac{3}{12}$
$\underline{7\frac{1}{12}}$

5. $\frac{1}{7}$
$\underline{\frac{4}{7}}$

6. $6\frac{1}{14}$
$2\frac{3}{14}$
$\underline{8\frac{3}{14}}$

7. 5
$\underline{\frac{4}{7}}$

8. $7\frac{15}{16}$
$4\frac{11}{16}$
$\underline{2\frac{10}{16}}$

9. $7\frac{3}{9}$
$\underline{2\frac{1}{9}}$

10. $$\begin{array}{r} 5\frac{2}{7} \\ 13\frac{1}{7} \\ \underline{17\frac{6}{7}} \end{array}$$

11. $$\begin{array}{r} \frac{2}{8} \\ \frac{4}{8} \\ \underline{\frac{1}{8}} \end{array}$$

12. $$\begin{array}{r} \frac{7}{16} \\ \underline{\frac{5}{16}} \end{array}$$

13. $$\begin{array}{r} 11\frac{3}{10} \\ \underline{4\frac{4}{10}} \end{array}$$

14. $$\begin{array}{r} \frac{2}{9} \\ \frac{3}{9} \\ \underline{\frac{2}{9}} \end{array}$$

15. $$\begin{array}{r} 8\frac{1}{12} \\ \underline{3\frac{5}{12}} \end{array}$$

16. $$\begin{array}{r} 4 \\ \underline{\frac{3}{4}} \end{array}$$

17. $$\begin{array}{r} 9\frac{4}{9} \\ \underline{14\frac{2}{9}} \end{array}$$

18. $$\begin{array}{r} \frac{3}{10} \\ \underline{\frac{4}{10}} \end{array}$$