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ABSTRACT

The paper presents reasons for teaching topics from number theory to elementary school students: (1) it can help reveal why numbers "act" in a certain way when added, multiplied, etc., (2) it offers drill material in new areas of mathematics, (3) it can develop interest - as mathematical enrichment, (4) it offers opportunities for students to develop ideas of inductive and deductive reasoning, (5) it offers patterns which can be discovered, (6) it offers unsolved problems of mathematics, and (7) it lays a foundation for future work in algebra. Examples from number theory discussed include history, number patterns, prime numbers, unsolved problems, and divisibility rules. (JC)

NUMBER THEORY IN THE ELEMENTARY SCHOOL

a talk presented by

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There are a number of questions which might be asked of all teachers in general, and elementary teachers, specifically, in regard to the everyday teaching task. Some of these are: Do you have trouble with getting and keeping the interest of your students in mathematics? Do you find them losing interest in assignments which are no more than repetitious drill but which you know they need for practice? What do you do for the bright student in your class who always finishes his assignments early and then has nothing to keep him interested? What about those of all levels for whom arithmetic is drudgery? If you don't have these problems, then you should be congratulated. Either your class is exceptional or you are an exceptional teacher, or both. If you do have these problems then this paper suggests that the theory of numbers may offer a solution for you.

The plan of the paper is to give some values which can be accrued from the teaching of topics from number theory and then to present some examples which demonstrate these. Most topics presented are not those generally found in the elementary curriculum of today.

For example, factors, primes, and composites are not considered extensively.

What Is the Theory of Numbers?

First, you might be interested to know what the theory of numbers is. It has been characterized by various authors as: the descendent of Greek "arithmetica", number recreations and puzzles which interest students of higher mathematics, the purest branch of mathematics, the least applicable of all mathematics, one of the oldest branches of mathematics, the most difficult of all mathematical desciplines, and the science of numbers. It is any of these or all of these, depending on your viewpoint. It is an offspring from Greek arithmetica, yet today's number theory bears little resemblance to the number worship of the ancient Greeks. The theory of numbers is more than an idle pastime such as recreations and puzzles might suggest. Whether or not it is considered the most pure or least applicable of mathematics depends on whether you are a number theorist or not. Number theory is certainly one of the oldest branches of mathematics. It is also a science of numbers in that many discoveries have come about in it through experimentation and intuition such as the physicist or chemist uses.

A question which might have come to your mind is "If the theory of numbers is one of the most difficult branches of mathematics, then how can elementary school children and teachers be expected to study it?" The immediate answer is that there are many levels of difficulty and abstraction in number theory just as there are in algebra and geometry. We are suggesting a very intuitive, inductive level of study which is no more difficult than the many algebraic and geometric topics now studied in elementary school. The numbers dealt with are the familiar whole numbers and the concepts are at the level of the four fundamental operations.

What Values Are There for Elementary School Students?

Another question that can be asked is "What values do we expect to accrue from the study of number theory?" As we all know, the emphasis in contemporary mathematics programs is on "understanding" and the knowledge of the "why" of arithmetic as well as the "what" and "how." The theory of numbers can help reveal why numbers "act" in a certain way when added, multiplied, etc.

One problem facing all elementary teachers is how to get enough drill and review in fundamentals into their teaching. There is always the danger that excessive, repetitious drill will become meaningless and destroy initiative if students are assigned page after page of problems to give them practice in fundamentals. Number theory offers a nice solution to this dilemma. It is a good source of "incidental" drill material which focuses attention not on drill but on some interesting and new areas of mathematics. So, using number theory, the students can get the practice they need but in a painless manner. Also, the theory of numbers provides some concrete applications of whole numbers and students will be able to apply their skills to discovering some new properties of whole numbers. An example from set theory of practice offered by number theory is: the sets of even numbers, odd numbers, prime numbers and composite numbers offer some familiar examples of disjoint sets and union and intersection of sets.

Another ever present problem faced by elementary teachers is how to motivate students and to interest them in mathematics. The necessity of learning addition and multiplication facts and the algorithms of arithmetic offers a real challenge to the teachers to keep students' interest. Again the theory of numbers has some answers. The use of the history of mathematics, in which number theory plays a big part, is a very good interest-developer if handled correctly. Many topics from number theory besides the historical aspect offer interesting and challenging sidelights to the regular mathematics curriculum. It should not be inferred that such enrichment is applicable only to the case of the capable student who has finished his assignment early. Enrichment is possibly even more valuable for the slower student since he needs some relief from drill and drudgery. Number theory has much to offer both of these students as it can offer a challenge to the former and yield problems easily understood by the latter. Commonly, enrichment takes the form of allowing a capable student to move into materials studied in succeeding grades. If this is not desirable, the theory of numbers offers an alternative. Many topics which can be selected from this field are found nowhere in the elementary school program.

Modern mathematics programs place an emphasis on helping students to learn the structure of mathematics. Along with this, in the later elementary years students are exposed to some form of mathematical proof. This is a very difficult concept for teachers as well as students to grasp. Often "proofs" are given of statements which are obviously true and which students have accepted long before. For this reason, there is little understanding developed of the nature of proof and its place in mathematics. The theory of numbers is very fruitful in offering opportunities for students to develop ideas of inductive and deductive reasoning. In fact, they can formulate their own conjectures (guesses) about relationships between numbers and with practice make proofs of them.

Another important aspect of modern mathematics programs is the emphasis on discovery. The student should be allowed to experiment and search for patterns in seeking out mathematical structure. Many such patterns occur in number theory, even some demonstrating a relationship between arithmetic and geometry. By consideration of such patterns and by conjecturing from them students can get a feeling for the way that mathematicians work and an appreciation for what it means to really "do mathematics" and not just manipulate symbols.

The theory of numbers also offers assurances to students that mathematics is a vital growing subject and not just a stagnant body of rules from the past which are unchanging and unchangeable. Some of the most famous unsolved problems of mathematics are in the field of number theory. Their most valuable facet for elementary school mathematics is that they are easily understood by elementary school pupils and can demonstrate to them that there are still "fields to conquer" in mathematics.

Looking to the future of the student, another value to be obtained from number theory in the elementary schools is that it helps to lay a foundation for future work in algebra. It offers some good applications of variables and generalizations of the type vital to algebra.

Some Examples

Now we consider some examples from number theory to support the claims made for it. First we consider the historical point of view as enrichment.

The History

If you have studied any history of mathematics, you know that sooner or later the ancient Greeks must be mentioned. We shall begin with them. For the theory of numbers, the most important group of ancient Greeks was a society known as the Pythagoreans. This group was first formed as a school for liberal arts training by the Greek mathematician Pythagoras (c. 550 B.C.) You have probably heard his name mentioned in relation to a famous theorem in geometry. An interesting sidelight is that the famous philosopher Plato was a student at the school. This school developed into a secret brotherhood with secret rites and observances. The members were sworn to secrecy on the subject of any of their discoveries. Out of admiration for Pythagoras, they attributed many of their discoveries to him.

Their main beliefs rested on the assumption that whole number is the cause of the various qualities of matter. For this reason, they attributed mystical properties and human characteristics to numbers. For example, even numbers were thought to be soluble, feminine and pertaining to the earthly, and odd numbers were regarded as indissoluble, masculine, and of heavenly nature. The number one stood for reason, 2 for opinion, the number 4 for justice since it was the first product of equals (2 x 2 = 4), and five suggested marriage, the union of the first odd, masculine number, 3, and the first even, feminine number, 2. They did not consider one as odd but as the source of all numbers since 1 + 1 = 2, 1 + 1 + 1 = 3, etc. Whatever the superstitious beliefs of this group, they contributed much to mathematics and it is generally conceded that they took the first steps in the development of number theory.

Number Patterns

Number patterns play a role in the theory of numbers. Two interesting multiplication patterns are as follows:

1 x 1 = 1	1 x 1089 = 1089
11 x 11 = 121	2 x 1089 = 2178
111 x 111 = 12321	3 x 1089 = 3267
1111 x 1111= 1234321	4 x 1089 = 4356

Students could be presented these and then asked: Can you see the pattern? Does it continue on from here? If it continues, how far? They would be motivated to see how far, and get drill in multiplication while investigating it.

Some of the number relationships that the Pythagoreans either discovered or with which they worked were given colorful and suggestive names: friendly or amicable numbers, excessive numbers, defective numbers, and perfect numbers. These characterizations were related to the divisors or factors of the numbers.

As an illustration, let us consider 284 and 220. The set of divisors of 220 (excluding 220) is

$\{1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110\}$

and of 284 (excluding 284)

$\{1, 2, 4, 71, 142\}$

If we add up the divisors of each, we get an interesting result.

$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$

$1 + 2 + 4 + 71 + 142 = 220$

The divisors of each add up to the other!! Such a pair of numbers were called friendly or amicable numbers by the Pythagoreans. They attached a mystical aura to this pair and believed that perfect friendship would be sealed between any two people wearing these as an ornament. You might ask if there are more such pairs. Curiously enough, that question seems to have not been answered until 1636, over 2000 years after Pythagoras. Some other pairs are 1184 and 1210, 2620 and 2924, and 5020 and 5564. An interesting fact about the pair 1184 and 1210 is that after it had gone undiscovered despite the efforts of mathematicians for hundreds of years, a

16-year old boy discovered it in 1866.

Numbers were also named by the Pythagoreans as to how they themselves compared with the sum of their divisors (less than themselves). If we consider 6, 8, and 12, we see the following:

n	Sets of divisors (smaller than n)	Sum of divisors
6	1, 2, 3	1 + 2 + 3 = 6
8	1, 2, 4	1 + 2 + 4 = 7
12	1, 2, 3, 4, 6	1 + 2 + 3 + 4 + 6 = 16

We see that 6 is equal to the sum of its divisors

8 is greater than the sum of its divisors, and

12 is less than the sum of its divisors.

For the reason that 6 is equal to the sum of its divisors it was called a perfect number. This was also the reason, it was claimed, that God created the world in 6 days, because 6 is a perfect number. For the reason that 8 is greater than the sum of its divisors, it was called excessive. Also since 12 is less than the sum of its divisors it was called defective. All numbers were likewise characterized as either perfect, excessive, or defective.

It becomes obvious if one tries a few examples that most numbers are either defective or excessive. For example, between 1 and 100 there are only 2 perfect numbers, 6 and 28. We can easily show that 28 is perfect. The set of its divisors less than itself is

$\{1, 2, 4, 7, 14\}$

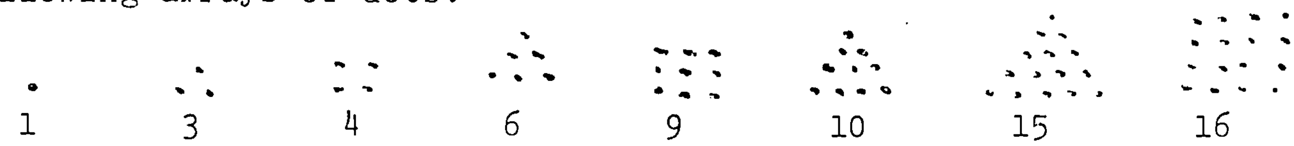
The sum of these is $1 + 2 + 4 + 7 + 14 = 28$

All other numbers between 1 and 100 are either excessive or defective. In fact, only 22 perfect number were known up to May, 1963. There are formulas known which help in this search and of course, high speed electronic computers can be used for the long calculations necessary, so this number is probably outdated. Before the invention of computers, men spent many years working out calculations by hand in the search for perfect numbers. It is amazing that even 22 are know today considering the size of some of them, e.g., one of these numbers discovered in 1883 has 37 digits. One surprising fact is that all of the 22 known perfect numbers are even numbers. It is not known whether odd perfect numbers exist or not.

There are also interesting patterns obtained from addition of positive integers. Consider the following:

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$
$$1 + 2 + 3 + 4 + 5 = 15$$

So far this is not too interesting. Let us return to the ancient Greeks. In early Greek days, notably by the Pythagoreans, numbers were recorded by dots. These dots were arranged in arrays which suggested names for the numbers and also allowed properties of the numbers to be derived from the geometric configurations. These numbers have been given the name figurate numbers. Consider the following arrays of dots.



From the configurations such numbers as 1, 3, 6, 10, and 15 were called triangular numbers and such numbers as 1, 4, 9, and 16 were called square numbers. The list can be continued on indefinitely for each type of number. Note that the number 1 is both types. There were also pentagonal numbers, hexagonal numbers, etc. You might wish to experiment and see what numbers would have these names.

Notice the triangular numbers, 1, 3, 6, 10, and 15. These are the ones we found by successively summing positive integers! There is a pattern here. If we were presenting this to youngsters, there are many fruitful questions we could ask. For example: "Do you see the pattern developing? What would be the next triangular number after 15? After that? Look at the rows of the arrays. Do you see a similar pattern? The first triangular number is 1, the second 3, the third 6, etc. What would be the 7th? the 8th? Could you generalize to the nth triangular number?"

There is an anecdote told about the famous mathematician, Karl Frederick Gauss (1777 - 1855). When he was a small boy in school, his teacher, as teachers are sometimes wont to do, gave the class the the task of adding up the first one hundred positive integers. That is, they were to find the sum

$$1 + 2 + 3 + 4 + \ldots + 96 + 97 + 98 + 99 + 100.$$

Karl Frederick dismayed the teacher by obtaining the answer in record time! (She probably had hoped for at least a ten minute respite.) He had discovered a shortcut to the answer, and in fact a shortcut to the answer to any such problem, as follows. Instead of the sum in the form

$$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100,$$

he had seen it in the form

$$\begin{array}{l} 1 + 2 + 3 + 4 + \ldots + \\ \underline{100 + 99 + 98 + 97 + \ldots} \\ 101 + 101 + 101 + 101 + \ldots \end{array}$$

There are 50 such sums of 101 so that result is (50)x(101) = 5050. This result can be easily generalized for the sum of the first n positive integers. The result of the Gauss problem can be viewed as $(50)\times(101) = \frac{100}{2} \times 101 = \frac{(100)\times(101)}{2}$.

More generally for the sum of the first n positive integers we have

$$\frac{n(n + 1)}{2}$$

Gauss supposedly discovered this. Whether he did or not, it would be an interesting problem and result to have your students to consider. Note also, that it answers the question "What is the nth triangular number?"

Now consider the square numbers.

$$1 = 1 \cdot 1$$
$$4 = 2 \cdot 2$$
$$9 = 3 \cdot 3$$
$$16 = 4 \cdot 4$$

Do you see the pattern? What is the next square number after 16? What is the 1st square number? the second? the fourth? the fifth? the tenth? the nth?

If you recall, in algebra you use an exponent to indicate the number of times a number is taken as a factor. For example,

$2 \times 2 = 2^2$, $3 \times 3 = 3^2$, When we read it, we say 2 "squared" and 3 "squared". What do you suppose the origin of this is? It surely could have come from figurate numbers.

You may ask whether the square numbers can be obtained as a pattern of sums, as the triangular numbers were. Consider the following sums:

$1 = 1$

$1 + 3 = 4$

$1 + 3 + 5 = 9$

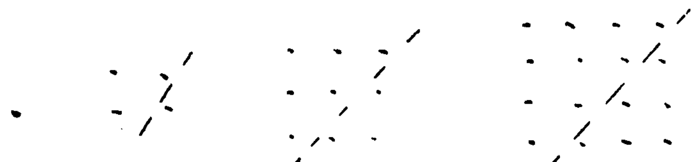
$1 + 3 + 5 + 7 = 16$

What are the numbers that we added? The odd numbers are added. What are the numbers that are obtained? The square numbers!

Another interesting result is shown in the following sums of consecutive triangular numbers:

	1	3	6	10	15	21
+		1	3	6	10	15
	1	4	9	16	25	36

Each square number is the sum of two consecutive triangular numbers! This relationship between square and triangular numbers can also be seen in their figurate patterns. Consider the lines drawn in the patterns for square numbers below.



Each square number is shown to be divided into two triangular numbers. By observing these patterns we could ask questions leading to a generalization answering the question, "Of which two triangular numbers is the nth square number the sum?" This illustrates the point that some topics from the theory of numbers offer many opportunities for discovering patterns and induction leading to proof.

Prime Numbers

We will not consider too much of prime numbers, but will mention in passing that the Pythagoreans studied these numbers extensively. Another Greek mathematician, the famous geometer Euclid, (c. 300 B. C.) proved that there are an infinite number of these and Eratosthenes (c. 230 B.C.), also Greek, devised an alogrithmic device for finding all primes less than any given number. This device is called the sieve of Eratosthenes and is very combersome to work with for very large numbers. Its operation is carried out as follows. Begin by writing down the positive integers in order (a square array works nicely).

1	2	3	4	5	6	7	8	9	10
11	12								100

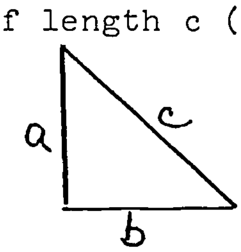
We note the first prime, 2, circle it and cross out every second number thereafter. (The number 1 is usually not considered to be prime.) These numbers of course have a factor of 2 and are not prime. We note the first number not crossed out, 3, circle it and cross out every third number thereafter. (These all have factors of 3.) Five is the next number not crossed out, we circle it and cross out every fifth number thereafter. This process is repeated until every number has either been circled or crossed out. The circled numbers are the primes. We leave it to the reader and his or her pupils to discover when we can stop the procedure and be assured of having found all primes in the list.

Mathematics As An "Alive" Subject: Unsolved Problems

One conclusion about mathematics to which most students seem to come sometime in their education is that mathematics is a fixed and unchanging body of knowledge in which all problems are solved and no questions still unanswered. They feel that mathematics is stagnant and unrewarding to study. All too often we perpetuate this misconception by over emphasis on rules which must be accepted without question and drills which must be carried out. We must try to do just the opposite. It is necessary that students leave us with the impression that mathematics is a vital, growing subject, that there are many unsolved problems and new areas to explore. We have some simple examples in number theory.

One of these comes to us from a famous French mathematician of the seventeenth century, Pierre de Fermat. He could have been credited with many important discoveries in mathematics, but he was not interested in publishing his results. His famous unsolved problem probably resulted from this disinterest.

The story of this problem really goes back to one of the ancient Greeks, Diophantus of Alexandria. He wrote a work called the Arithmetica which brought together the algebraic knowledge of the Greeks. In his work, there was a discussion of a theorem well-known to anyone who has studied plane geometry in school. This is the so-called Pythagorean theorem. In a right triangle with legs of length a and b and hypotenuse of length c (see figure) $a^2 + b^2 = c^2$.



In the Arithmetica, there was a discussion of triples of integers a, b, and c which satisfy the above relation. Two examples of such triples are:

$a = 3, b = 4, c = 5$ and $a = 5, b = 12, c = 13$.

since $3^2 + 4^2 = 9 + 16 = 25 = 5^2$

and $5^2 + 12^2 = 25 + 144 = 169 = 13^2$

Obviously enough, such triples are called Pythagorean triples. Fermat had obtained a translation of Diophantus' work and was very intrigued by it. He studied the Pythagorean triples and tried to make generalizations. Out of this came his famous unsolved problem. In the margin of his copy of Diophantus, he wrote

> It is impossible to have 3 integers a, b, and c such that
> a^3 and $b^3 = c^3$
> or a^4 and $b^4 = c^4$
> Or in general, for any n greater than 2, it is impossible to have three integers a, b, and c such that $a^n + b^n = c^n$. I have discovered a truly womderful proof for this but the margin is too small to contain it.

He never published his proof. Mathematicians, both brilliant and and not so brilliant, have been trying ever since then to prove or disprove his conjecture. It has been proved for some particular values of n up to 250,000,000 but not in general. Doubt persists whether Fermat himself knew a proof, but he certainly was brilliant enough to have discovered one. Also any statement which he said he had proved has never been disproved since his time. Perhaps some day an amateur may stumble on the proof which has evaded mathematicians for centuries.

Another famous unsolved problem, even more simple to state than Fermat's is called Goldbach's conjecture. Goldbach was a Prussian mathematician of the 18th century. His conjecture was that any even number can be written as the sum of two primes. E.g.,

2 = 1 + 1 (if 1 is considered prime)
4 = 1 + 3 "
6 = 3 + 3
8 = 3 + 5
10 = 3 + 7
12 = 5 + 7
.
.
.

Unlike Fermat, Goldbach made no claim of having proved his conjecture. Mathematicians have been trying to prove it for over 200 years and have succeeded in proving it true for all even numbers up to 100,000 and many larger numbers, but have not shown it true in general.

These two examples demonstrate an important point about mathematical proof which we should make certain is appreciated by children. The point is: to prove something in the mathematical sense, no number of specific cases will suffice. If we prove something true in

1,000,000 specific cases, we cannot be mathematically sure that it will be correct in the one million and first case.

Divisibility Rules

Moving away from history, we can find many other topics from the theory of numbers which offer sources of enrichment. The rules for divisibility are always of interest to youngsters. Most of you probably know how to determine whether we can exactly divide a given number by 2, 5, and 10. But do you also know that there are ways of telling whether numbers are divisible by other numbers, for example 3, 4, 9, 11? For example:

for 3 and 9, just add the digits. If the sum is divisible by 3 or 9, then the original number is also.

Example: 144

1 + 4 + 4 = 9.

144 divides by 9 and 3 exactly.

Example: 852

8 + 5 + 2 = 15

3 divides 15 so 3 divides 852

9 does not divide 15 so 9 does not divide 852

The old rule of checking computation answers by casting out 9's is related to this.

For 4, if the number made up of the last 2 digits is exactly divisible by 4 then the number itself is. E.g., for 1084 and 9063

$84 \div 4 = 21$ so 4 divides 1084

$63 \div 4 = 15$ r 3 so 4 does not divide 9063

The rule for 11 is of a similar nature to that for 3 and 9. These rules are very closely related to the base and place value properties of our decimal numeration system. This is an important point which could be easily demonstrated to middle and late elementary children and would help them understand our numeration system better. It would be even more valuable if they were first impressed and intrigued by the seeming mystery that these rules work. They could then appreciate more the power of mathematics to explain to them why the rules work.

Some Interesting Numbers

Some final specific examples of enrichment topics come from what are sometimes called "interesting numbers." 58 and 1089 are two of these. A skillful teacher could make quite a bit of magic out of these. If we take any three digit number, except for ones with all three digits the same or with the first and last digits the same and perform 4 operations, we will always end up with 1089. For example: Consider 826

1. We reverse the digits of the number getting 628

2. Subtract the smaller from the larger,

$$\begin{array}{r} 826 \\ 628 \\ \hline 198 \end{array}$$

3. Reverse the digits of this answer and get 891.

4. We add this final number to the one just before it.

$$\begin{array}{r} 198 \\ 891 \\ \hline 1089 \end{array}$$

This occurs with any three-digit number with the restrictions mentioned. One thing for which you should be on guard is a number which when the digits are reversed and the subtraction performed, 99 is obtained. If this occurs, consider this as 099 and reverse the digits to get 990 on the next step.

Example: 574

Reverse 475

Subtract 574-475 = 099

Reverse 990

Add 990 + 099 = 1089

The number 58 is called interesting for another reason. Choose any number, say 243.

Sum the squares of its digits

$$2^2 + 4^2 + 3^2 = 4 + 16 + 9 = 29$$

Sum the squares of the digits of the answer:

$$2^2 + 9^2 = 4 + 81 = 85$$

Etc.

$$8^2 + 5^2 = 64 + 25 = 89$$

$$8^2 + 9^2 = 64 + 81 = 145$$

$$1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42$$

$$4^2 + 2^2 = 16 + 4 = 20$$

$$2^2 = 4$$

$$4^2 = 16$$

$$1^2 + 6^2 = 1 + 36 = 37$$

$3^2 + 7^2 = 9 + 49 = 58$, our interesting number.

This always happens. No matter what number we begin with, with one exception, we end up with 58. (The exception is when we get 1 any time in the above iterated process. In this case you could tell your students to multiply the 1 by 2, or in fact any number, and continue. This may even add more magic to it.) Some numbers take longer to get to 58, but we always end up there.

There are many other interesting numbers and number patterns which would intrigue youngsters and lead them to try and discover on their own. These can be found in the sources in the bibliography.

Conclusion

In closing, we reiterate our conviction that the theory of numbers is a field of mathematics that can make important contributions to the teaching and learning of elementary school mathematics. Its importance and value as a motivational device and as a representation of mathematical beauty and truth should not be overlooked. It is hoped that this discussion has convinced some of you and you will try it in your teaching.

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