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ABSTRACT

The author discusses major issues related to the teaching of elementary school mathematics and draws upon research results for his analysis. The elementary school mathematics curriculum is discussed in terms of selection of content, educational objectives and content arrangement. Within the realm of mathematics learning, discovery learning, the use of manipulative materials, student attitudes, the use of practical applications, and the need for real understanding in mathematics are examined. The author discusses the organization of classes and pupils for instruction. In this area he examines grouping practices for gifted and remedial students both within classes and school-wide as well as the usage of mathematics specialists in the classroom. Finally, specific questions regarding teaching practices are considered including computer-assisted instruction (CAI), programmed learning, and the effects of the "new mathematics" programs. (Author/CT)

WHAT RESEARCH SAYS TO THE TEACHER

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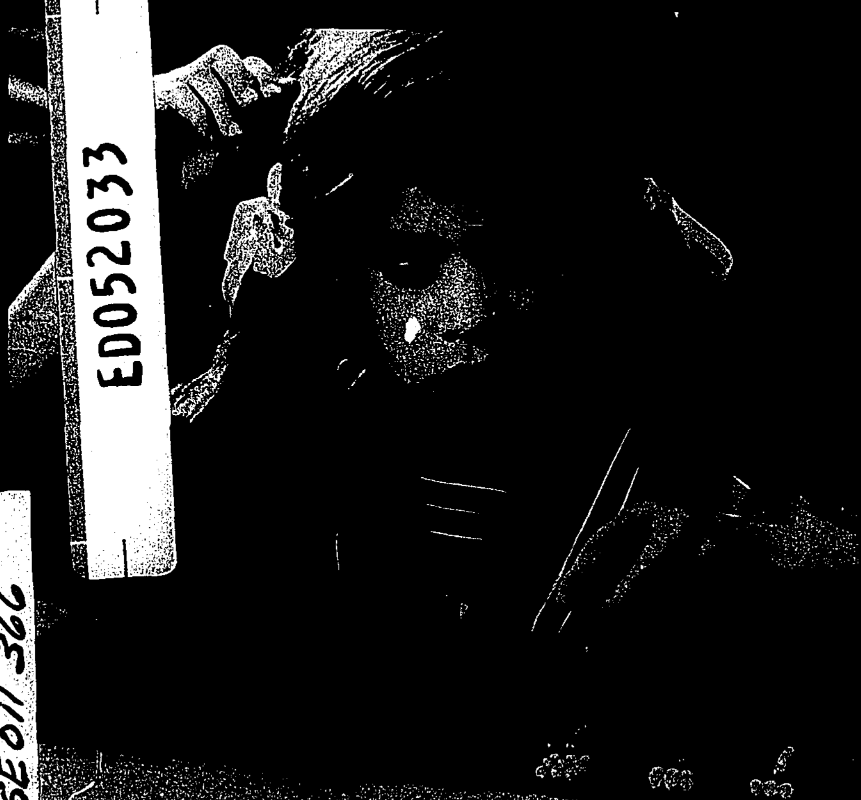
Teaching Elementary School Mathematics

Herbert F. Spitzer



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EXPLANATION AND ACKNOWLEDGMENTS

Most sections of this 1970 edition of the booklet are retained in substance from *Teaching Arithmetic,* which was written by Herbert F. Spitzer of the State University of Iowa in 1962. The content of these sections is as applicable to concepts of elementary school mathematics in 1970 as it was to arithmetic instruction in 1962.

The following sections, also written by Professor Spitzer, who is now at the Southwest Educational Development Laboratory, are completely new in this edition: TV Instruction, Individualization of Instruction, Programmed Learning, Computer-Assisted Instruction, Evaluation of Achievement, and the Selected References.

The manuscript for this edition was reviewed by James D. Gates, executive secretary, National Council of Teachers of Mathematics, and C. Alan Riedesel of Pennsylvania State University.

Editorial production was handled by Mrs. Kirsten H. Gardner of the NEA Publications Division.

THE ELEMENTARY SCHOOL MATHEMATICS CURRICULUM

THE ELEMENTARY SCHOOL MATHEMATICS CURRICULUM is far from fixed, even though arithmetic is one of the oldest school subjects. While some content is almost universally agreed upon and has been a part of the mathematics curriculum for many years, changes have been made in content from time to time. In recent years many rather radical changes in both content and method have been markedly affecting the curriculum.

How Is the Content To Be Selected?

How the content of elementary mathematics is to be selected is a problem as important today as it was 25, 75, or 100 years ago. The solutions proposed for this problem, like the solutions proposed in the past, are greatly affected by another question, namely, "What are the objectives of elementary mathematics instruction?" The two preceding questions in combination with two others, "What shall be the arrangement of the content for instruction?" and "How shall the content be graded?" comprise the four major problems of curriculum construction in elementary school mathematics.

What Are the Objectives of Elementary Mathematics Instruction?

Since the objectives of instruction influence selection of curriculum content, the second of the four major problems will be considered first. For many years elementary mathematics was considered primarily a tool subject, and, as a result, the content was slanted toward the most frequently used aspects. Within the past 30 years the objectives of elementary mathematics instruction have gradually stressed aspects other than the strictly utilitarian.

Some authorities have relegated social utility to a minor role, approaching elementary school mathematics as a foundation for further study in the whole field of mathematics or in light of other objectives. However, since the goal of using elementary school mathematics either as a foundation for further study of mathematics or as an essential tool in the study of science or other fields is also utilitarian, the social utility theory has apparently not been abandoned, but its base or foundation has been radically altered.

It is clear from recent research that the objectives of mathematics—especially as seen by leaders in mathematics, in education, in industry, and in government—not only encompass the computational aspects of the subject and the provision of an essential element in the education of a broadly educated person but also include the provision of a base for the study of all mathematics and science as well as for advanced study in a host of other fields.

How Is the Content of the Curriculum To Be Identified?

Instead of searching the common everyday uses of mathematics for sources of elementary school mathematics content, present-day research workers are analyzing the fields of mathematics (especially number theory) and the basic principles governing number operations, algebra, and geometry. This search is for content that may be adapted, modified, or used as it is in elementary school instruction.

In the actual selection of content, three factors seem to be dominant. *First*, it is now reasonably clear that with the changes in objectives that have occurred in the past three decades the field of mathematics is the major factor in the selection of content for the elementary school curriculum: that is, the criterion for the selection of material is whether the material will either contribute to the understanding of mathematics or help to provide a foundation for further study of mathematics. *Second*, although usefulness in life outside the school is no longer considered as important in the selection of content as it formerly was, many experts in instruction conclude, after critical study, that demonstrable usefulness is a powerful motivating factor. Consequently, usefulness in life outside the school is a factor in the selection of content. The *third* factor is the difficulty of the concept. While it has been demonstrated that it is possible to teach some aspects of even very difficult concepts at any grade level, including the primary grades, it is often more practical to omit or postpone some topics because they are too difficult.

What Is the Best Arrangement of the Content?

In attempting to determine the best arrangement of content for instructional presentation, many proposals have been and are being tried. One such proposal has been appropriately called the spiral form of presentation. By this scheme, the material on a topic, such as addition of whole numbers, was studied a number of times during a single year and for several successive years. Since pupils frequently returned in their study to the same topic, but at a little higher level, the content was said to be spirally (not circularly) arranged. While all elementary mathematics curriculums follow to some extent a spiral arrangement of content, there has been a decided trend toward a concentrated treatment of topics, especially within a grade.

Another proposed arrangement is that of presenting mathematics primarily in connection with projects, activities, and the like. While once vigorously promoted and still frequently mentioned, this procedure does not receive much serious consideration today, since research has shown that pupil achievement under this arrangement is inferior. Even though the total presentation of mathematics content through activities has not produced desirable achievement, one aspect of this kind of arrangement of content has proved itself: Motivation of pupil study often appears easier when projects or activity settings are used than when direct study of the number operation involved is undertaken. Furthermore, project settings often assist greatly in clarifying the operational procedures being taught. The use of projects or activities to illustrate mathematics may, then, if kept in proper perspective, have an important role in mathematics instruction.

Another plan for arranging content for instruction is known as the problem method. While considered a superior plan for arranging content in the fields of science and social studies, this plan has not until recently received much consideration in mathematics. In many experimental programs, especially where exploration is encouraged, such genuine problem topics as, "What is the best way to check multiplication with two-digit multipliers?" have been introduced.

Some of the new mathematics programs present an arrangement of content designed primarily to develop recognition and understanding of the major principles governing the operations of arithmetic. For example, in addition the intent of the arrangement of content might be to focus attention on the computative and associative aspects of the operation, on the identity element, and the like. Since the study of such principles and characteristics of addition is not dependent upon any physical setting (life problem situations), such programs tend to provide little experience with illustrations of the everyday uses of arithmetic.

The enthusiastic endorsement given such programs by mathematicians indicates that there is probable merit in arranging arithmetical content to emphasize its mathematical aspects. Although none of the programs emphasizing mathematical content has been in operation long enough to produce much evaluative data, the brief reports available make it seem likely that this form of arrangement of content will become more popular.

In view of the information presented on the various means of arranging content for instruction, there is apparently no one best arrangement of content. The better programs tend toward a concentrated (rather than a spiral arrangement) treatment of topics within a grade, with emphasis on the mathematical rather than the utilitarian aspects of topics. The arrangement of content in these better programs is also influenced by the use of life situations to illustrate the concepts and operations. Such illustrations often provide the needed familiar setting for study and discussion and are also a powerful motivating factor. The arrangement of content in these better programs may also at times make marked use of the problem method of study.

How Shall the Content Be Graded?

The introduction of new content through new projects and as a result of an intensive analysis of the fields of mathematics and other recent innovations has brought to curriculum makers new problems and practices with regard to grade placement of topics. Since most of the "new mathematics" has resulted from moving down or adapting mathematics content from high school and college courses, a climate especially favorable to including topics formerly thought too difficult at lower grade levels has been established. The placement of mathematics topics at lower grade levels has also been fostered by the findings of research concerning practices in European schools. There is, then, a definite trend in 1970 to place mathematics topics at a lower grade level than was the case in 1950.

The dramatic changes resulting from the substitution of the analysis of mathematics for the analysis of everyday uses of arithmetic (most often of a business nature) as the major factor in the selection of content has created more than normal uncertainty concerning not only the content but also the grade placement of that content in the elementary mathematics curriculum.

On the basis of current and emerging practices, it appears that such factors as the likelihood of pupil interest and the inherent difficulty of topics will continue to be major factors in grading. However, identification through analysis of basic mathematical ideas will lead to introducing some mathematics at lower grade levels than is now common practice. Thus, the factor of difficulty may become less important in the grading of content.

FACTORS CONSIDERED PARAMOUNT IN LEARNING MATHEMATICS

How Important Is Understanding?

The findings of early twentieth century research studies tended to focus attention on eliminating those aspects of mathematics that were little used in ordinary life situations. Consequently, instructional procedures of that period tended to emphasize those phases of the subject, especially the difficult, which research had shown to be useful in life situations. These instructional procedures used drill extensively. Reaction to this type of mathematics instructions led to formulation of the so-called meaning theory of mathematics instruction. In the meaning theory, the words *meaning* and *understanding* were the guides.

In the more than 40 years since new emphasis began to be put on mathematics meanings, a number of experimental studies have shown that a meaningful approach to learning is superior to approaches that either ignore meaning completely or at least do not emphasize meaningful aspects. The observational reports of competent observers have been far more valuable than the findings of experimental studies in showing the importance of meaningful learning. There is a definite belief on the part of those who have observed pupil study that, where meanings and understanding are emphasized, such learning is superior to that found where no such emphasis exists. Also, mathematicians and other specialists who have made a critical analysis of mathematics instruction in the post-Sputnik era have concluded unanimously that understanding is essential to good instruction.

What Are the Roles of Exploration and Discovery in Learning?

Early in the period when meaningful teaching of mathematics began to receive attention, there was advocacy of an exploratory type of instruction stressing pupils' seeking of answers. This method (sometimes called "teaching with emphasis on discovery") emphasized pupil experimentation and accented verification of statements and solutions. It gradually became more popular and recently has become the chief characteristic of some of the so-called modern mathematics programs developed in the period following Sputnik I. Controlled research studies, comparing pupil achievement in programs emphasizing an exploratory type of procedure with programs of a nonexploratory nature, have given a slight edge to the exploratory programs. When comparisons of outcomes are made through other data-gathering means, such as observations of pupil resourcefulness, confidence, and general interest in mathematics, the results have been even more definitely in favor of programs emphasizing exploration and discovery. Such evidence indicates that use of exploration and discovery creates a type of learning situation that appeals to some students of mathematics teaching.

Should Several Solutions or Just One Solution Be Emphasized?

For those who hold that the best way to ensure meaning and understanding is through use of minute step-by-step explanation (demonstrated or shown) of the operation that the pupil is to learn and to use later in solving the exercises and word problems of mathematics, of course only one solution will be emphasized. Representatives both of the "new" and of the "conventional" practices adhere to this theory of mathematics teaching.

On the other hand, in instructional programs emphasizing exploration and discovery, it is inevitable that there will be more than one possible method of solution or suggested solution; in fact, this is a major characteristic of such programs. As previously pointed out, exploration and discovery are major characteristics of some of the "modern" programs developed after Sputnik.

How Important Are Concrete Representations to Learning Elementary Mathematics?

To facilitate pupil understanding of elementary mathematics concepts and procedures, there has in recent years been increased use of special devices, such as sets of blocks or rods with proportions corresponding to the relations that exist among the first 10 counting numbers. One set of such blocks combines color with proportions. While experimental studies comparing the achievement of pupils using concrete representations of number with the achievement of other students are lacking, teachers using the blocks give enthusiastic reports of their value.

On the basis of the reports of competent observers and of analyses of instructional programs emphasizing the use of colored sticks or rods and other concrete representations of number, it may be concluded that these programs have some value in helping pupils to develop cardinal ideas of number and number relationships. However, a time quickly comes in pupil learning when concrete number representations (including color) become handicaps to efficient use of numbers. They are, then, at best only one of many instructional aids that efficient teachers may use, but they should not be the exclusive instructional device.

Does Direct Study Make for Better Interest and Achievement?

While research has shown that mathematics projects and activities do not result in superior achievement, they are valuable for generating interest and providing familiar settings for study. In recent years, however, some students of mathematics instruction have again challenged this view, claiming that greater interest and achievement result from direct study of mathematical operations without reference to applications or uses of operations.

The role that applications or uses of mathematics has played in teaching the subject has varied considerably. There have always been, as one reviewer of 60 years ago put it, some extremists who believe that the subject can best be taught solely through consideration of applications. There also always seem to be other extremists who believe that elementary mathematics can best be taught by direct and systematic study of number and the operations of number without any or, at best, with only incidental reference to uses of these operations. No results of comparison studies of these two extreme points of view are available. An examination of textbooks of the past hundred years reveals that books have been published which consisted almost wholly of problems and, conversely, other books which contained, primarily, computational exercises. The fact that neither type of book ever became very popular is some evidence that neither position is superior and that a combination of the two points of view is probably the best procedure.

In the past 40 years some books have emphasized social situations (an example of uses) and have therefore seemed to give prominence to teaching elementary mathematics through applications. These books, however, contained many computational exercises and therefore cannot be cited as good illustrations of an attempt to teach mathematics through applications alone.

The emphasis on "social" mathematics has at times made the mathematics program appear to consist more of economics, geography, and other social sciences than of mathematics. Reaction against this social mathematics, as well as other factors, has resulted in a recent return to a much greater emphasis on direct study of number and number operations. Some programs use problem settings to introduce study of new procedures, but major study is directed toward consideration of the operation being taught. In other programs, no word problem settings are used; consideration is directed immediately to the number operation.

Observational data from the latter programs indicate that pupil interest in study of this type is high and that the resulting learnings are valuable. Since it has frequently been assumed that study of computational procedures without reference to applications is considered lacking in pupil appeal, reports of high pupil interest may be surprising. However, many studies, one as early as 1909, report that pupils enjoy working with numbers (pure computation) more than they do solving mathematical word problems.

Part of the high interest of pupils in the direct study of number operations may be due to the novelty of the teaching situation. It should also be noted that demonstrated usefulness has been found to be a powerful motivating factor. It would seem, then, that continued use of applications to motivate and to provide settings for the introduction of systematic and thorough study of operations is the best procedure to adopt.

Does Pupil Attitude Toward Mathematics Affect Learning?

Many teachers have believed that a pupil's attitude toward elementary mathematics definitely affects achievement—that is, those pupils who like the subject generally do better. Recent research tends to confirm this belief. The measures used in research show that the relationship between favorable attitudes and achievement, although low, is clearly positive. The creation and maintenance of a favorable attitude toward elementary mathematics should be, therefore, a major concern of the classroom teacher.

THE NEW MATHEMATICS

What Characterizes the New Mathematics?

The term *new mathematics* refers primarily to experimental programs produced since 1957. These programs are characterized by the inclusion of more mathematical content than appeared in previously existing programs and by an emphasis on the study of the mathematical structure of operations and concepts. This mathematical emphasis in the elementary school has (a) possible advantages in that pupils acquire mathematical insights, skills, and procedures that should enable them to proceed more rapidly; and (b) possible disadvantages in that pupils generally have difficulty in assimilating abstract ideas presented through verbal communication in the classic mode of mathematical presentation. Perhaps the major distinguishing characteristic of the new programs is the attempt to develop mathematical concepts, operations, and principles from a mathematical point of view with little or no reference to their use, even during the introductory consideration of a topic.

Analysis reveals that the most important content of the new mathematics programs consists primarily of content adapted from what was formerly considered secondary school (algebra and geometry) and college (number theory) mathematics.

Despite differences in practice, the methodology of the various new programs in general purports to promote thinking through use of a problem-solving approach which emphasizes exploration, experimentation, and discovery. The stress on study of mathematical structure, concepts, and principles leads naturally to use of methods that resemble the classic mathematical modes of presentation with their precise language, well-defined terms, deduction, and other characteristics.

The variety of content and procedures revealed by analysis of the different new programs points to the fact that there is not any *one* new mathematics program but that instead there are a number of programs which have, as has been indicated, some of the general characteristics noted.

Evidence from observation indicates that the new programs have interest appeal for both pupils and teachers. The interest of teachers seems especially high when they are given help in acquiring background knowledge. Whether this interest appeal is intrinsic or due to novelty and the extra effort that normally accompanies experimental programs is not yet clear. Evidence primarily from observation also shows that the learning of the new aspects of the subject (such as sets, equations, ratios, numeration and notation in other bases, and geometric constructions) is good, but not enough times has elapsed to evaluate this early study of former secondary and college mathematics.

What Contributions Do the New Programs Make?

Analysis of recent instructional practices reveals that incorporation of new procedures and content into conventional programs has proceeded rather rapidly. This is evidence of one of the major contributions made by the new mathematics movement—the creation of an educational climate that is receptive to change and to new ideas.

The number line may be cited as an illustration of the increased acceptance of a relatively new instructional aid through the new mathematics. Although long used in algebra in connection with positive and negative numbers and suggested as a useful teaching device in arithmetic in two professional books prior to 1950, this device was used very little until the new mathematics programs gave it publicity. Other illustrations of the speed-up of the introduction of useful mathematical symbols, terms, and the like into current arithmetic curriculums, partly as a result of the impetus given by the new mathematics, are the inclusion of the following in some current instructional materials: (a) the sign of inequality ($\neq$), (b) the use of arrays ($\begin{smallmatrix}\circ&\circ&\circ\\ \circ&\circ&\circ\end{smallmatrix}$), (c) the term *set,* (d) the use of more than one name for a number, and (e) equations.

The climate created in part by the new mathematics has fostered greater willingness to engage in experimentation. This important source of new ideas and new practices is therefore much more extensively engaged in than was formerly the case.

In the small number of research studies that have been made, achievement in the new mathematics programs as measured by conventional tests shows little or no superiority over older programs of instruction. While this type of evidence is meager, it suggests that the new programs are not panaceas for the problems of elementary mathematics instruction.

Critical analysis of the new programs has shown the usual exaggerated claims, the lack of refinement, the problems, the errors, and other shortcomings that are the almost inevitable accompaniment of hastily assembled plans. As testing and further study of these new programs continue, these shortcomings should be eliminated.

Critical analysis of the new programs has also revealed rather widespread misinterpretation of their general function. The erroneous belief is held that these new programs represent something distinctly new and superior which, when substituted for but not incorporated into the old programs, results in a panacea for all the problems of elementary mathematics instruction. Actually, the original purpose of all the new programs has been to experiment with untried procedures and new content for the purpose of improving current programs.

ORGANIZATION OF CLASSES AND PUPILS FOR INSTRUCTION

What Modifications Are Necessary for the Gifted?

The great interest in the gifted or superior pupil, which developed in the late 1950's and continues at the present time, has resulted in analysis of the regular or common programs of instruction in elementary mathematics to determine how well these programs meet the needs of gifted children. This analysis has revealed that the gifted or superior child is especially handicapped in mathematics when only the common program is available. This is true because in mathematics, unlike reading, social studies, science, and other major elementary school curricular areas, little supplementary material is available for study.

Attempts to meet the obvious special needs of the mathematically superior pupil have affected the organization of the elementary mathematics class. One rather popular procedure involves alteration of the instructional program. Under this procedure one of two general types of program is followed. In one type, the pupil is given material for study that would not, in the course of the regular program, be considered until a year or several years later. In the other, the superior pupil's efforts are directed to more extensive study of the topic under consideration. Extensive study might involve much more difficult material than that found in the regular program, but theoretically the material would not be of a type that the pupil would encounter in the regular program of study in succeeding years. The first of these programs became known as vertical extension or enrichment and the second, as horizontal extension or enrichment. When schools first began to attempt to meet the needs of the superior elementary school pupil in mathematics, the use of vertical extension of subject matter was almost universally adopted. It was probably the more feasible plan, since materials from upper grade and secondary school mathematics were available, whereas materials for horizontal extension were either nonexistent or in very short supply.

Both types of program have, while solving some problems, created others. One of the most serious problems created by vertical enrichment programs is the duplication of content that the student later encounters in his study of the regular advanced courses in mathematics. The student who has followed a vertical enrichment program in elementary school will have had experience with the most interesting parts of higher arithmetic, algebra, geometry, and number theory and will not then find the offering in regular advanced courses very interesting. If interest is to be maintained, different material must be provided, which means either presenting still more advanced mathematics or providing horizontal extension.

The most serious problem created when horizontal enrichment is attempted is the lack of suitable instructional materials of this type at the level needed. The production of such materials is hampered because it is difficult to distinguish between what might be used as horizontal enrichment of a topic and what might be vertical extension of the same topic. To illustrate this difficulty, consider whether the "casting out of nines" as a check for multiplication might be horizontal enrichment for a fifth grade pupil or whether this check might in an integral part of the regular program in sixth, seventh, or eighth grade.

In spite of the difficulties encountered in the identification and production of horizontal enrichment material, extensive research by teachers, curriculum production staffs, and others is rapidly producing a large body of such materials. As a result of the increasing body of material and because it seems inherently superior to vertical enrichment, horizontal enrichment appears to be the more acceptable proposal to elementary and secondary school teachers, supervisors, and curriculum directors. Also, publishers have adopted essentially the horizontal rather than the vertical extension of material in attempting to supply instructional programs for superior pupils. On the other hand, many specialists from other fields, such as secondary and college mathematics teachers and psychologists, who have recently become interested in mathematics in the elementary school seem to favor vertical extension rather than horizontal as a means of meeting the needs of the superior or gifted pupil. Data from research studies are badly needed to determine the ways that either type of enrichment program is definitely superior to the other.

While the initial impetus for this recent reorganization of instructional materials came in response to pressure to meet the needs of the gifted, the provision of varying and extensive materials is beneficial to all levels of ability.

Does Grouping of Pupils Facilitate Learning?

In addition to altering instructional materials in attempts to provide for the superior, various ways of organizing classes or of grouping pupils within classes to provide a mathematics program suited to differences in ability are being used.

Placement of the gifted or superior in separate classes, with the less able in other classes, is being used in some schools where there are enough pupils to permit such an organization. This plan, however, has not been attractive to many administrators because of many extraneous factors, most of which are of an administrative nature. Assessment of the merits of this plan is difficult because of its limited use and various extraneous factors.

Another plan designed to adapt instruction to the varying ability levels of pupils is within-class grouping. This plan provides for organization of groups (usually 2 to 4 in a class) ranging from highest to lowest in ability. The high ability group receives more material for study, much of which in the best programs is also more difficult than the regular materials, while the pupils of lowest ability receive directions for work (assignments) that are less extensive. Theoretically, this type of organization for instruction has much merit; in practice, it has been difficult to do because of the great effort required of teachers to direct the work of several groups. This difficulty stems primarily from a shortage of instructional materials for use with the high and low ability groups and from lack of classroom methodology in directing this kind of study. From the reports of competent observers, it is obvious that this kind of grouping can hardly reach its potential when only one textbook is the basis of the instructional materials for all groups.

In schools where the within-class grouping of pupils is most successfully used, extra materials for the high ability group are more often provided through horizontal extension. This horizontal extension is especially likely to be found where the program has been in operation for several years. Within-class grouping is most successful where the grouping is flexible and where the pupils have an opportunity to choose their group (even the superior group); however, the difficulty and extent of the material studied determine the composition of groups. In the most outstanding classrooms where such grouping is used, pupils in all groups begin the study of each new phase of arithmetic in the curriculum as one group and generally study simultaneously in the same areas but at varying levels of difficulty and extension. The organization and direction of such group study within a class requires superior teaching skill and a wealth of instructional materials. On the basis of the reports of competent observers, one can predict that the flexible plan of within-class grouping will be much more widely used in the future.

Do Special Teachers Make for Better Learning?

The use of special teachers (departmental type organization) of elementary mathematics is another organization plan for improving mathematics instruction that has received increasing attention in recent years. This plan, once popular in elementary schools in the 1920's and 1930's, is advocated primarily by mathematician-educators who have recently become interested in elementary mathematics instruction. They argue that, by using special teachers, the instruction will be given by persons with much more training and with greater likelihood of interest in mathematics than is possible in the self-contained or modified self-contained type of organization. Research studies of 20 to 25 years ago comparing the achievement of pupils in schools of departmental with schools of self-contained types of organization failed to show superiority of the departmental type of instruction. Also, a more recent study based on a special-teacher plan specifically set up to bring out superior achievement in elementary mathematics failed to show greater achievement on the part of the pupils instructed by special teachers. In these studies, however, the mathematics teachers had received far less training in mathematics than mathematics educators consider to be desirable.

Since a departmental type of organization for instruction creates administrative and curricular problems, it is not looked on favorably by many supervisors and administrators.

What Suggestions for an Effective Remedial Program?

The creation of special classes or groups within a class to receive remedial instruction is another of the many organizational plans that have been used to provide instruction that is suited to the individual needs and abilities of pupils. Efforts to identify pupils in need of such remedial instruction have led to the development of tests that are of assistance in diagnosing elementary mathematical deficiencies. The creation of classes designed to provide remedial instruction has also led to the development of special materials for use in such classes.

The use of remedial instruction, especially in upper elementary school grades, has received increased attention in recent years. Among the marked improvements that have resulted from this increased attention is a change in the plan of identifying pupils in need of remedial instruction. The new plan emphasizes providing each pupil with evidence that reveals his deficiencies and encourages him to volunteer to use materials designed for remedial instruction. Such involvement of the pupil in making decisions regarding participation in remedial instruction is also being carried into the selection of the actual instructional materials. Pupils are permitted to choose from the variety of available remedial materials.

ANSWERS TO SPECIFIC QUESTIONS REGARDING MATHEMATICS TEACHING

Is Use of Mental Arithmetic a Good Instructional Practice?

The term *mental arithmetic* as used here refers to the solution of arithmetical questions without use of pencil and paper. Since most such solutions involve computation, this kind of arithmetic is sometimes called "mental computation." Another name is "oral arithmetic." The use of mental arithmetic, as shown by the examination of professional writings and by an examination of the teachers' editions of pupil texts, has increased markedly in the last decade. This increase may be traced to several factors. *First*, it is being recognized that experience with mental computation emphasizes the salient features of our numeration system, the relationship between numbers, and the relationship between processes; and it puts a premium on thinking of the type that life often requires. *Second*, instruction using mental arithmetic is becoming more popular because of its psychological advantages. In the usual classroom non-pencil-and-paper situation, pupils are seldom confronted with records of mistakes. For each new situation all pupils have a clean slate, and, if they are to be proficient, pupils are forced to look for and to take advantage of regroupings (e.g., $21 \times 19 = 20 \times 19 + 1 \times 19$) and to use other procedures that mark the numerically sophisticated. Other factors have made mental arithmetic more popular: (a) it makes for easy inclusion of unrelated topics of mathematics, (b) it is an efficient way to give practice since no time is lost in copying exercises or writing answers, (c) it requires little in the way of pupil materials or teacher presentation, and (d) it is readily adaptable to use of the tape recorder and other machine-type presentations.

Authorities who recommend more emphasis on mental arithmetic do not look upon this as an encroachment on pencil-and-paper arithmetic, but rather as a complement to or as an integral part of that program

Analysis of textbooks has shown that their materials are insufficient for an adequate program of instruction in oral arithmetic. Supplementary materials are therefore needed. Teachers' editions of pupil texts are beginning to supply such material.

How Can Verbal Problem-Solving Ability Be Improved?

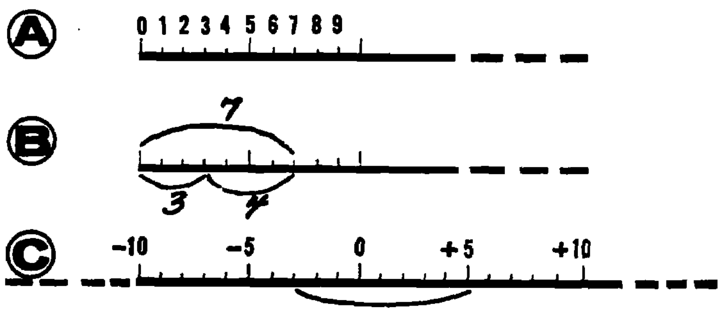
Methods of improving pupil facility in the solution of verbal or word problems have probably been the subject of more investigations than has any other arithmetic topic. The sheer number of these research studies points to both the difficulty and the importance of this area of instruction.

The results of these many investigations are not easy to assess, and there is no indication that the final answers to the many aspects of the teaching of verbal problems can yet be formulated. There does, however, seem to be fair agreement that the use of formal analysis (What is given? What is to be found? etc.) is of doubtful value as a problem-solving improvement program. It also seems clear that children use no one pattern in the solution of verbal problems and that the textbook programs for improving word problem solving are too meager to expect satisfactory growth in this area of elementary mathematics. On the positive side, there is fair evidence that use of oral presentation and non-pencil-and-paper solution tends to increase facility in solving verbal problems, that pupil understanding of mathematics and the ability to solve problems show a high relationship, and that intensive use of a few specific problem-solving improvement procedures (e.g., finding the number question of problems, the formulation of word problems, etc.) over a relatively long period of time (one semester) will markedly improve pupil achievement.

Some of the new mathematics introduced during the period 1957-61 tends to ignore, and even to discredit, the use of word problems. At this time the evidence is not clear that playing down of word problems is sound pedagogically.

How Important Is the Number Line in Teaching?

The number line (*see* A in illustration opposite), a device that has recently become rather well known in arithmetic, has, as stated earlier, been used for a long time to teach negative numbers in algebra. While it is now being used in arithmetic to some extent for teaching negative numbers, this aspect of its application in arithmetic is negligible. Its important uses in present-day arithmetic are in counting, in determining the sum or difference of two amounts, in reading numerals, in giving a notion of the nature of the number series, and in giving background experience for later work in addition and subtraction of numbers written with numerals. The number line is a valuable teaching aid for graphic representation of the relations between numbers as in $3 + 4 = 7$ (*see* B in illustration) or as in $5 - 8 = -3$ (*see* C in illustration), showing that the number series may extend indefinitely in either direction (positive or negative) from a given point.



In connection with number lines the question is frequently asked, "Should the symbol zero be used to label that point on the line?" Whether the zero point on a number line should be identified depends upon whether the symbol for zero might play any part in exercises for which the line is to be used. For counting exercises, the zero would be detrimental; for showing addition, subtraction, multiplication, and division as these operations are used with whole numbers in elementary school, use of zero is of little consequence; for showing negative numbers and the extension of the number series in both directions, the zero is essential.

Which Method of Subtraction Is More Effective?

Current practice, as shown by analysis of textbooks and publications on elementary mathematics teaching, is overwhelmingly in favor of the subtractive or take-away method of subtraction. This popularity is not a reflection of the superiority of the subtractive over the additive as shown by the achievement of pupils taught by the respective methods. In fact, such studies, while perhaps giving a slight edge to the subtractive, do not indicate that the additive method is ineffective. The swing to the subtractive method during the past 30 years has been largely due to the emphasis given to meaning and understanding in teaching during this period. Because the take-away method is more logical for the pupil who is not very sophisticated mathematically and because it can be more easily shown with objects and drawings in teaching, it has seemed to be superior to those teachers and writers who were interested in promoting meaning and understanding.

Recently a number of writers have contended that, because the additive method is more logical for some word problem situations, both methods should be taught. A few mathematicians and mathematically minded educators who are responsible for the new mathematics seem to advocate that the subtractive method be abandoned and that only the additive method be taught. In fact, some have even gone so far as to recommend that subtraction per se not be taught, but that it be considered only as the inverse of addition. This recommendation is too new to have been the subject of much research.

How Estimate the First Quotient Digit with Two-Digit Divisors?

The question of how to estimate the first quotient number has been the subject of many rather extensive investigations. The earliest studies attempted primarily to determine by what method the smallest number of errors would occur. The two methods usually compared in these investigations have been the apparent and the increase-by-one methods. In the former, the tens number of the two-digit divisor is used in estimating the quotient. In the latter, with divisors ending in 1, 2, 3, 4, and sometimes 5, the apparent rule is used. For divisors ending in 6, 7, 8, and 9 or 5, 6, 7, 8, and 9, a second rule, namely, rounding to the next ten and using that tens number in making the quotient estimate, is used. Thus, the second plan is a two-rule plan, while the first is a one-rule plan. In general, the practice has been to begin instruction with the apparent method and then eventually to shift to the second method with the applicable divisors.

Since use of these two procedures has not been very satisfactory, a search for modifications in methods of presentation goes on continually. Recently a proposal to use another one-rule method, that of rounding up only to get the number used in making the quotient estimate, has been proposed. In using this rule, the quotient estimate when not the true quotient is always too small and therefore need not be discarded; instead a second division with remainder is undertaken, and, when completed, the two quotient digits are added. The possible merits (a one-rule procedure, no erasing, illustration-of-division idea) of this plan indicate that it will be investigated thoroughly in the future.

TV INSTRUCTION

TV instruction was widely heralded 10 to 15 years ago as having great possibilities for the teaching of elementary school mathematics. Like its predecessor, the film, this plan of presenting mathematics instruction encountered some inherent difficulties that seem to be unsolvable.

A recent study indicated that Patterns of Instruction, a program that utilized TV instruction, was liked by pupils and teachers and that "results generally favorable to the televised course were obtained." Such restrained language suggests that the TV instruction was not too effective. Why is a procedure that was thought to have such potential for mathematics instruction a few years ago used so little today?

Many factors could be cited in answering this question. Perhaps the most significant of these factors is that TV or film presentation of mathematics instructional material makes for poor pupil involvement. The pupil too often becomes a passive viewer. In addition, there isn't an opportunity for interplay between teacher and pupil and between members of the class.

Instruction in mathematics via TV does have value. However, the full potential for mathematics instruction of TV, like other new instructional means, is not likely to be realized with the first or second attempts. Refinements on the basis of experience and further study are essential to the production of the best instructional materials. It appears that the teaching of mathematics via TV will more likely be successful if development is directed toward producing materials to be used as part of ongoing programs instead of as complete replacement programs.

INDIVIDUALIZATION OF INSTRUCTION

To fit instructional materials and procedures to the abilities of children has long been a major concern of many educators. Individualization of instruction, like the grouping of pupils according to ability, is only one of many means for dealing with the individual differences in ability that exist in every classroom. Programmed instructional material and computer-assisted instruction are two other procedures now being advocated as means for dealing with differences.

The current interest in individualization of instruction in elementary school mathematics instruction stems primarily from the Individually Prescribed Instruction (IPI) mathematics materials developed at the Learning Research and Development Center in Pittsburgh. Probably because these materials are relatively new and rather expensive, comparative and descriptive research studies by others than those directly involved in either the development or the promotion of the materials are not yet available in the educational literature.

The IPI materials, like the materials of programmed learning, are based on the idea of a learning hierarchy. The pupil is expected to demonstrate mastery of one learning task before he is allowed to proceed to the next task. In addition, the IPI materials are similar to the programmed instruction approach in their use of behavioral objectives based on mathematics activity analysis.

The major features of the IPI program, however, differ markedly from programmed instruction. Learning is directed by guides and materials that are similar to those used in conventional instruction. A key feature of the program is the use of tests to determine the level where the pupil is to begin his study of the materials on a topic. For example, a fourth-grade pupil's first study of addition is determined by the score this pupil makes on an addition placement test. Once study of a topic is begun, progress is determined by the scores made on performance tests. This combined testing and learning procedure appears to make for truly individually prescribed instruction. Furthermore, as is true in the use of programmed instructional materials, a pupil proceeds at his own rate of learning.

Another superior feature of the IPI materials is the fact that they eliminate grade-level designations without involving any reassignment of pupils. A fourth-grade pupil whose placement tests suggest that he begin study of subtraction with instructional material typical of grade 2 begins such study without a change of room or teacher. Of course, some of his classmates may be using fourth-grade subtraction material.

The effective use of IPI mathematics materials calls for a great deal of bookkeeping and guidance by a teacher. In fact, a teacher plus an assistant per class is considered essential by some users of the materials.

The mathematical content and procedures for study of IPI materials leave much to be desired. Modern mathematics receives far less emphasis than would normally be expected in a relatively new program. There is a noticeable lack of attention to precision in the presentation of materials. For example, missing addend exercises of the type $3 + n = 7$ are labeled as addition exercises. Very frequently the learning exercises differ little from exercises that were labeled as merely drill 20 years ago. A much more serious limitation of the individually prescribed instruction idea is the fact that this procedure practically eliminates the interplay of ideas between pupils, and even between teacher and pupil. Some previous attempts at individualizing instruction in a manner similar to that attempted in the IPI approach (Winnetka, Dalton) floundered, and one of the stumbling blocks was this loss of interplay in the classroom.

PROGRAMMED LEARNING

The evidence from research studies is inconclusive on whether programmed learning materials or conventional materials make for greater learning. One writer reported that of 13 studies reviewed, 3 indicated greater achievement for pupils using programmed instructional materials and 3 showed greater achievement on the part of pupils using traditional materials. The remaining 7 found the differences so slight that they were considered negligible. Another reviewer of the evidence of research concluded that "programmed instruction can be used to present many topics effectively." The fact that a great many of the investigations of the effectiveness of programmed materials produced conflicting evidence prompted a critical reexamination of some of the basic assumptions of programmed learning. From such examinations some researchers have concluded that a well-defined algorithm for producing learning hierarchies (one of the tenets of programmed learning) does not exist. Some researchers also produced evidence to show that it is not necessary for a pupil to master one level before proceeding to the next.

An interesting observation of researchers on programmed learning is that effective use of programmed material in the classroom is very dependent on teacher direction or guidance. Thus, instead of replacing the teacher, as its early advocates claimed it could, programmed learning in elementary school mathematics *requires* a skilled and interested teacher. Accordingly, some studies have attempted to find the best combination of teacher-directed learning and programmed instruction. From the reports of other studies it is clear that not all pupils profit equally well from use of programmed material. Furthermore, as one researcher observed, a major contribution of programmed instruction to learning for some children is not the greater achievement that results. Rather, it is freedom from the disturbing constraint of trying to keep up; pupils are able to work at a convenient speed.

It appears, then, that while programmed instructional material may not produce greater achievement than will conventional instructional material, it nonetheless has some merit and may well become an integral part of good instructional programs.

COMPUTER-ASSISTED INSTRUCTION

Computer-assisted instruction (CAI), one of the newest of the technical schemes for teaching, has not yet been well tested for elementary school mathematics. The absence of research reports is due in part to the newness of this development and also to the enormous effort and expense required to produce and utilize CAI in elementary school mathematics. A report on the major program in CAI mathematics instruction, that at Stanford, points out that significant gains in student achievement were observed.

Attention is called to the fact that computer costs decrease with volume. Therefore, classroom computer use may someday be practical. Moreover, some textbook publishers are already concerned with the development of CAI materials in elementary school mathematics.

Since CAI materials are essentially an extension of the programmed learning plan of instruction, the same factors that limit programmed learning instruction must be overcome before the CAI plan will become very effective.

EVALUATION OF ACHIEVEMENT

There has always been some concern on the part of many teachers and others about the adequacy of the instruments used in measuring mathematical achievement. Attention is called to the use of the term *achievement* instead of the broader term *growth*. Achievement is used because most measuring instruments seem to be based on the assumption that growth in mathematical achievement is the only growth sought in mathematics teaching. But the changes in content in recent years have been accompanied by a shift in objectives. The new objectives stress that study of mathematics should produce, in addition to mathematical achievement, interest in mathematics, a desire to learn, creativeness, and a foundation for further study of mathematics.

With the shift in emphasis there has been a great increase in concern about the adequacy of measuring instruments. One critical review of recent research stated, "In most mathematics studies inappropriate or inadequate measuring devices were used." The inappropriateness and inadequacy was due primarily to the changes in content and objectives that have occurred since the measuring instruments were designed.

There is another significant factor (perhaps the *most* significant factor) that contributes to the inadequacy of measuring instruments in mathematics education. This is the belief on the part of educators that all the outcomes of mathematics teaching can be measured by tests, and those primarily of the objective pencil-and-paper type. That there is evidence other than that provided by objective tests (e.g., reports of observed activities, records of the production of pupils, especially the creative and the unique, etc.) which can be used as measures of pupil growth has long been recognized. Such evidence, however, is not easily amenable to the commonly used statistical treatments; therefore, it receives little attention by the professionals in the field of educational measurement. As a result, available test-type measuring instruments and the new ones being devised will continue to ignore many worthwhile avenues to knowledge.

Not only are test-type measuring instruments inadequate for obtaining a complete measure of mathematical growth, they do not even measure achievement well. This is due not just to the inability of the test designer to produce tests that measure well what they purport to measure, but also to the inability of the curriculum developers to translate the goals of the mathematics curriculum into clear objectives. It is one of the cardinal rules of educational measurement that good measuring instruments can be devised only for clearly stated objectives. To secure such objectives for use as guides in constructing tests to measure achievement has long been a major concern of test makers. What is needed, say the professional test makers, is a description of the behavior that pupils should exhibit as evidence of learning. The statement of objectives in terms of the learner's behavior has been used extensively in animal learning experiments, and there have been attempts to formulate behavioral objectives for elementary school mathematics. The first reaction of most teachers and students of mathematics education to behavioral objectives has not been nearly as enthusiastic as has been that of men in the field of measurement. According to the critics, these objectives place too much emphasis on the acquisition of knowledge and skill. As one critic points out, the behavioral objectives approach to curriculum study is similar to the plan of curriculum building based on activity analysis and its accompanying specific objectives that was championed by some educational reformers of 40 to 50 years ago. A curriculum built on activity analysis or the related behavioral objectives approach (what some people think a pupil should learn) may be very well for a training program, but it hardly provides a good base for an educational program. The latter involves interplay between teacher and pupil and between pupil and the subject. The educational program should foster creativity and thinking. The making of judgments, not the recall of specific subject matter, is among the most significant characteristics of the educated. As this writer sees it, the use of behavioral objectives does little to foster the development of the major characteristics of the mathematically educated; therefore, the attempt to improve measurement in mathematics by adopting behavioral objectives in curriculum construction is not likely to be successful.

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