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ABSTRACT

The author presents material suitable for use by teachers of gifted students in the junior or senior year of high school. The mathematics presented includes mathematical induction, the binomial expansion, number theory and Pascal's triangle. The author weaves much of the history of mathematics into the materials. Included are student tests and bibliographies of related materials. (CT)

SOME MEANINGFUL MATHEMATICS IN TWO CHAPTERS

CHAPTER T1: The Binomial Expansion and Related Topics

CHAPTER T2: The Principle of Math Induction and Related Conjectures

SYOSSET PUBLIC SCHOOLS

USD 2

The Binomial Expansion and Related Topics

Chapter: TI

Level: 12X

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More Index, Chapter T1

NOTE I: All numbers in parantheses refer to the books in the Bibliography, Section 6.6, page 99 of Chapter T1.

NOTE II: Chapter T2 commences immediately after page 102.

Section 1: HYSTERICAL BACKGROUND

Outline of Section 1

The goals of section one lie primarily in the following areas:

1. An introduction and usage of such terms as algorithm, partial factoring, Math induction, recursiveness, tessaract, parallelopiped, propitious and false position.
2. An investigation of the ideas of Algebriac--Geometrical relatiomships, making assumptions based on a finite number of pieces of information, producing a "why" for each relationship given, dealing in the abstract using algebriac models and expansion of a binomial to answer a specific question.
3. Experience and maturation in dealing with non-definite, non-real mathematical concepts.

Note: Numbers in paranthеses refer to the numbered books in the bibliography (Section 6.6)

Section 1.1 Introduction

On June 19, 1623, in the small town of Clermont, in the province of Auvergne, in France, one Blaise Pascal was born. Monsieur Blaise Pascal was a sickly child and from the age of 17 until his death at 39 his wretched physique was subjected to attacks of acute dyspepsia (indigestion) and chronic insomnia. Yet this man was responsible for such diverse practical inventions as the wheelbarrow, the first mechanical adding machine and the barometer; he also wrote two of the greatesr works of early French Literature, the "Pensées" and the "Provincial Lettres";

and he did imaginative and creative work in three diverse areas of Mathematics. It is two of these areas in which we shall be working presently.

Pascal's mathematical efforts began at age 16, when he discovered the "Mystic Hexagram" along with 400 corollaries to the theorem which defined the hexagram; he thereby established the essential basis for a new non-metrical geometry. At age 30, he published his "Traite du Triangle Arithmetique", which embellished at great length the original triangle published in 1303 by the famous Chinese algebraist, Chu-Shi-Kie. And at age 31, along with another great mathematician, Pierre Fermat, he established the basis of combinatorial analysis and probability theory.

Throughout most of his life Pascal was in constant pain (a severe toothache caused him once to work 8 straight days on the theory of the cycloid curve, thereby recreating most of what the ancient Greeks had done); in fact, it has been reported that he had absolutely no sense of humor and never smiled. (Small Wonder!) However, in the ensuing subject matter, most of which was first produced by Pascal, you will find much material that will bring you great joy and happiness.

But first, and after you wipe that smirk from your face, we will look at some Greek mathematics. Hopefully, we will be able to relate this material to Pascal's Work later on.

Section 1.2 Square Root Algorithm

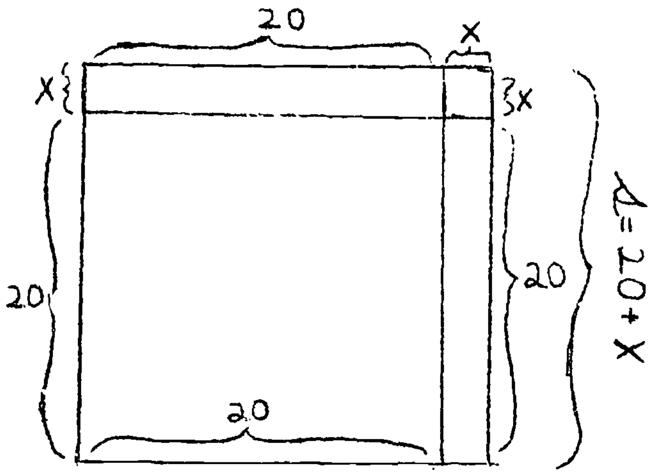
Most of you are familiar with the proof of the irrationality of $\sqrt{2}$, which is to say, the fact that no natural numbers (positive integers) exist whose ratio will be equal to the $\sqrt{2}$. When the ancient Greek Pythagorous (who lived most of his life in Italy) and his group was apprised of this fact, they were very much shook-up! It seems they had postulated that all observed natural phenomnenon could be described by using dimple algebraic (addition, multiplication) combinations of natural numbers. Well, clearly the diagonal of a square 1 unit on a side provides an immediate counter-example to their postulate!

As a result, some of the Pythagoreans are said to have committed suicide, but the more stable and pragmatic of them had another out. They merely geometricized all their mathematics. Numbers were not considered to have a pure existence (See page 10 for the rest of this paragraph.)

Now A.P(After Pythagorous), the Greeks would never talk about raising a number to the 4th or 5th power, as that would have no meaning. However, they would raise a number to the 2nd or 3rd power, ie., square the number, cube the number, since the 2nd power (the square) would represent the area of a (would you believe?) square and the 3rd power(the cube) would represent the volume of a (would you believe?) cube!!

Now, for the converse question; what does it mean to take the square root of a number? (Why don't we ask for the 2nd root of a number?) To most of us it means getting a slide rule or a set of tables or logs and obtaining some number, such that when the number is raised to the 2nd power(squared?) the original number results (or something close to it!). Well, to the Greeks, taking a square root meant to find the side of a square whose area was given; likewise, a cube root was considered the length of the side of a cube whose volume was given.

Their process for the former task(finding a <u>square</u> root) went something like this: (See page 10 for missing sentence) First, draw a picture of the given square, and within that given square find a square of largest area having for the length of its side a multiple of 10.



Since $10 \times 10 = 100 < 729$

$20 \times 20 = 400 < 729$

$30 \times 30 = 900 > 729$

The square in question is 20×20.

Place the square of known area in the lower left hand corner of the given square, and divide the remaining area up into two rectangles and one small square, (as shown in the diagram). Note that one of the dimensions of the rectangle is known (namely 20), whereas the small square is completely unknown.

Now, these two rectangles and the unknown square must account for the remaining area of the origianl square, namely, 729-400=329 square units. The area of each rectangle is 20 times x, and there are two of them, and the area of the unknown square is x^2. Therefore;

$$2\cdot(20\cdot x) + x^2 \text{ must equal } 329.$$

Now this equation, which is just a simple quadratic, lends itself to the following <u>partial</u> factoring:

$$(x)\left[2\cdot 20+x\right]= 329$$

The Greeks would now use the process of "false position" to solve this equation (educated guessing), or at least get the best approximation to a solution. Of course, since we have already taken the square whose side was the greatest multiple of 10 out of our original square, x must satisfy the inequality $0<x<10$. For instance, try $x=5$. Then:

$$(5)\ (2\cdot 20+5) = (5)(40+5)$$
$$= (5)(45) = 225 < 329$$

Therefore, try $x=6$. Then:

$$(6)[40+6] = (6)(46) = 276 < 329$$

Try $x=7$

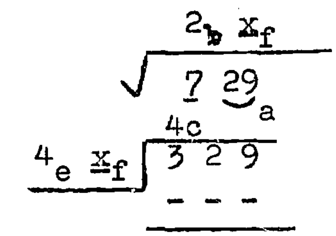
$$(7)(40+7) = (7)(47) = 329$$

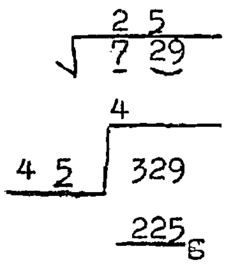
$\therefore$ $x=7$, and the square root of 729 is 27! (ie., 20 +7).

Have you ever noticed that 27= 20+7? Did you notice in the brackets above, that 40+7= 47? Will there always be a similar situation if this process is performed again? Will there always be a zero in the last place of the number being added to the number between 0 and 10? Of course, that's the way the process was set-up! The 40 was merely 2•20, where 20 was the multiple of 10! And since 20 was the greatest multiple of 10 which we could use, $0<x<10$ had to be so! In our decimal number system, when you add a multiple of 10 to a units digit, you merely juxtapose them (place them together, with the units digit covering the 0).

All these observations were made in the early 16th century by the various German <u>algorithmists</u> who were writing arithmetic books. Their task was to un-geometricize the Greek process. A process very similar to that outlined above was first published in 1513, and until a very few years ago, was a standard problem the Math 8 regents exam in New York State, (and therefore a part of the Math 8 curricubun)!

What algorithm (set of rules) did these German's come up with? Let's take a look. It went like this:





Step a. Mark off, left and right, two places from the decimal point.

Step b. Estimate the square root less than or equal to the first digit or pair of digits encountered.(ie., $2^2<7$)

Step c. Square your estimate and subtract from the first digit encountered.

Step d. Bring down the next pair of digits (ie., 29)

At this point you should realize that you have subtracted out the square whose area was 400 units, leaving 329 units to be accounted for.

Step e. Double your estimate made in step b, and place on line. (This represents 2 x 20)

Step f. Take any digit x, such that $0<x<10$ and place it in the two spots designated; note that when you juxtapose it to 4 you are actually adding x to 40; and when you juxtapose it to 2, you are actually adding it to 20.

Step g. Multiply x·4x; ie., (x)(40+x), so that the result is less than or equal to 329. Make certain you take the largest such x. In the present case , 7 works out very well, and there is no remainder, However, if there is some area left unaccounted for, the process can be continued (indefinitely) as long as you desire (many, many decimal places) or until all the area is accounted for.

Here is an example. Find the square root of 2.

```
        1. 4 1 4
      √ 2. 00 00 00
        1
 24   | 1 00
          96
 281  |   400
          281
 2824 |   11900
          11296
```

And now you might try an example; say; find the square root of 2237.29 (using the process, of course.) The answer is 47.3.

But you see, I'm not interested in the answer; that's why I gave it to you. What I'm interested in is whether or not you know the process? Do you know what the Greek geometrical process was all about? Do you see how the German algoithmists (rule makers) translated the geometrical process into an essentially algebraic one? This is what I want you to know!

And of course your rejoinder might be that you're not interested in what ancient Greeks and Germans did, and who needs this silly algorithm(or its explanation) when log tables are available, or even other means. Which is a very good rejoinder indeed. Why start off a chapter on the binomial expansion with an extinct algorithm for square roots?

Well, let me give you some partial answers. First of all, your parents have been feeling insecure ever since the advent of the new math, and here is a topic which they have been on the most intimate terms with in the past. I know my outline of the algorithm given above is a bit sketchy, so take the problem I just gave you home to your Mom and Dad and let them help you work out the process. This should help rebuild their shattered confidence to a certain extent, and also be a start towards building a bridge over your own personal generation gap between you and your parents. However, lest they get out of place, ask them why the process works! As they fumble about for an explanation you can once again assert your clear cut intellectual superiority over them, thereby arousing their undying enmity towards you.

But enough of social application! There are indeed some math lessons to be learned, and utilized, as well. We(you!) are subsequently going to develop a cube root algorithm much in the same fashion that the Germans developed their square root algorithm. But to do this we have to go back to the natural problem of breaking down the cube into different rectangular parallelopipeds(or as Mr. Wagner says, "Boxes"), just as the Greeks broke down the given square into a square of known area, two rectangles (1 dimension known, 1 unknown) and 1 small square of unknown dimensions. In other words, for our specific case above, the Greek approach consisted of the following(algebraic observation:

$729-400= 2\cdot(20\cdot x)+x^2$ is equivalent to

$729= 400+2\cdot(20\cdot x)+x^2$, or

$729= 20^2+ 2\cdot(20\cdot x)+ x^2$, or

$729=(20+x)^2$. For our case where $x=7$. $(20+7)^2=729$ was correct. That's the name of the game.

Section 1.2 More Algorithm

Now what about cube roots? How would we work on this? Let's take a cube of known volume, say V= 12,167(Obviously another propitious choice). Since $20^3=8000$ and $30^3= 27,000$, it is apparent that the largest cube with its side being a multiple of 10 whose volume is less than the given cube is a cube 20 units on a side with a volume of 8000 cubic units, leaving 12,167-8000=4167 cubic units unaccounted for. But again, there is some number x such that $0<x<10$ and such that 20+x will be the length of the side of the given cube! And then $12,167=(20+x)^3$ must be the case.

Which is to say, the given cube of volume 12,167 can be looked at (algebraically) as:

$$12,167=(20+x)^3 = 20^3 + 3\cdot(20^2\cdot x) + 3\cdot(20\cdot x^2) + x^3$$
$$12,167= 8000+3\cdot(20^2\cdot x) +3\cdot(20x^2) + x^3$$

The Greeks would not have obtained this expression algebraically as we have, but would have faked it by playing with blocks. More on this later.

At any rate, the Greeks would now use their process of "false position"(which is substitution using educated guesses) to obtain a solution; The German algorithmists would have used Partial factoring twice on the last three terms, and then have set up a spate of rules. Let's outline the evalution of the algorithm first, and then see how the breakdown of the given cube would make sense to the Greeks. Look at:

$$12,167 = 8000 +3(20^2\cdot x) +3(20\cdot x^2) +x^3$$
$$4,167= 3\cdot(20^2\cdot x) +3(20\cdot x^2) + x^3$$
$$=(x) \left[3\cdot 20^2 +3\cdot 20\cdot x+ x^2\right]$$
$$= x\left[3\cdot 20^2 + x \left[3\cdot 20 +x\right]\right]$$

Note: This idea of partial factoring is of extreme importance in giving a proof for the validity of the process known as Synthetic Division. Practically all texts foist off a demonstration that is complete hog-wash!

Now, if you still remember that $0<x<10$ and that a multiple of 10 added to x in our decimal system is merely equivalent to replacing the 0 with x, you should be able to outline an interesting algorithm for taking cube roots. Perhaps you might try to find the cube root of 12,167 which we have already started! (the answer is obviously 20+3=23; do you know why? Try cubing all the digits from 1 to 9 and see what happens!)

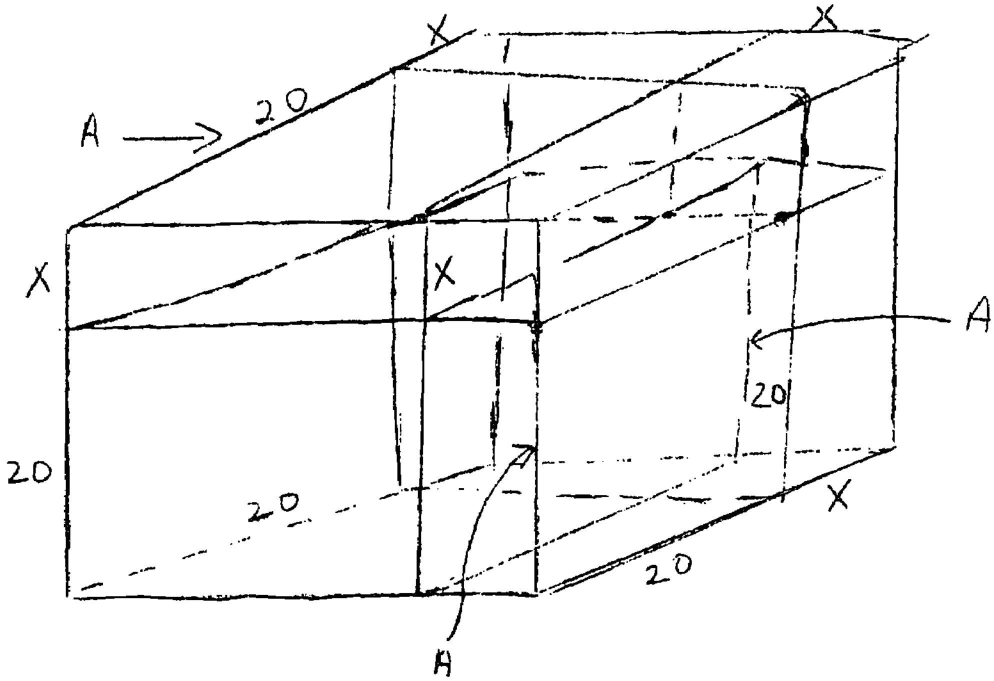
And now for the geometrical part! We have a given cube whose volume is 12,167 cubic units. We place a cube whose known volume is 8000 cubic units in the lower left hand corner; question: How do you break up the remaining space? Answer: The algebra tells you how! The remaining space(containing 4,167 cubic units) is to be broken down into 3 cereal boxes, two of whose dimensions are known and 1 unknown(ie., $3 \cdot 20^2 \cdot x$), 3 cigarette cartons, 1 of whose dimensions is known and 2 unknown (ie., $3 \cdot 20 \cdot x^2$) and 1 small cube of unknown dimensions(ie., x^3). You can readily see how the Greeks used various boxes (rectangular parallelopipeds) to fill up the left over space by inspecting the sketch below.

Sentence from Page 3:

... in themselves anymore, but were to be associated only with the measures of particular line segments. (Examples of numbers having a pure existence are given in Sec. 1.5.)

Sentence from page 4:

Given a square whose area is 729 units, find the length of a side of this square.



The cereal boxes are to be found one above the 20 by 20 cube, one to right and one behind this cube of known volume. The cigarette cartons are to be found in the <u>upper-front right</u> hand corner, the upper back left hand corner and standing on end in the lower back right hand corner. The cube of unknown dimensions is in the upper back right hand corner.

Perhaps if I had coordinized my cube (in three space, of course) I could have confused you more. But I imagine that would be impossible. Let's summarize these last results.

(see page 138 of (4) for another picture.)

The algebra we used to find a cube root has led us to a way of breaking down the space inside a cube so that we can use refined educated guesses for obtaining the length of one of its sides. The Greeks, clever as they were, undoubtedly faked this. In fact, how do we know which came first, the geometrical approach of the Greeks, or the algebraic approach of the German algorithmists? Obviously, historically, the Greeks came first .And that answers that question. But they would necessarily stop with finding cube roots, as the finding of a 4th root makes no sense; ie., it is not sense-ible,and therefore has no meaning! I don't know how far the German algorithmists went, but I doubt if they went past the square root algorithm. If any of you have successfully worked out a cube root algorithm, you'll know what I'm talking about.

I'm sure that at this point you're all still in a fog; but let's take a step backwards and inspect what we've done. The following chart might help.

Era	Problem	Process
Greek	Find the length of a side of a square of given area.	Break the square down into 1 known square, 2 partially known rectangles and 1 unknown square. Continue this process until "all" the area is accounted for.
German	Find the 2nd root (square root) of a given number.	Follow an algorithm blindly, the algorithm having been obtained by algebraicizing the Greek solution.
Greek	Find the length of an edge of a cube of	Break the cube down into 1 known cube, 3 partially known boxes,

Era	Problem	Process
Greek	given volume.	3 less partially known boxes and 1 unknown cube. Contihue until all space is accounted for.
German	Find the 3rd root (cube root) of a given number.	1. Algebraicize the Greek solution.
Greek	Find the length of an edge of a 4th dimensional cube of given content.	Non-sensical
German	Find the 4th root of a given number.	In renaissance Europe, if the Greeks couldn't do it, it couldn't be done.

The point to be understood here is that whereas historically the geometrical approach led to an algebraic approach, the algebraic approach is much more general and this partiuular algebraic approach was extensively significant in Isaac Newton's development of the calculus. However, we'll merely use it to develop a geometrical approach to finding the edge of a tessaract, the 4-dimensional perfect "cube". A tessaract is wha?

Section 1.3 The Tessaract

To the Greeks, a tessaract was non-sensible; to the Germans, they had nothing to work from. To us, we can easily "visualize" a tessaract and find the length of one of its edges because we can reason, and we have algebra that the Greeks didn't have. But first, what is a tessaract?

We will obtain the "picture" of a tessaract by developing it as an extension of the cube. To see how this is done, let's see how a line segment is the extension of a point, how a square is an extension of a line segment and how a cube is the extension of a square. Here is how the process is accomplished.

First start with a point P which is a zero-dimensional figure. If the point P is moved in a fixed direction to a new position P' a line segment ℓ is generated. A one-dimensional line segment is generated from a zero-denensional point. The segment consists of the end points P and P' and the measure of the path between them which is called length.

By moving or projecting the line segment ℓ in a direction perpendicular to ℓ and the same distance to a new position ℓ', it locus will be a two-dimensional square and its interior. Point P of the line segment ℓ moves to point P" generating line segment ℓ'' and P' moves to P"' generating line segment ℓ'''. The measure of the path between ℓ and ℓ' we call area.

By continuing in the same manner and projecting the square to a new position in a direction mutually perpendicular to the sides of the square and the same distance, one generates a three-dimensional figure or cube. The measure of the path between the square's original and new position is called volume. Note that each vertex of the square generated an edge of the cube and each side of the square a face of the cube.

To continue as before and project a cube into a new position leads to several unavoidable problems. How can one project a cube in a direction mutually perpendicular to the edges of the cube? In attempting to construct an intuitive model of a four-dimensional figure in our limited three-dimensional environment one must make use of perspective. One is very aware that a cube can be pictured in two dimensions. The picture is not a cube but does serve as a representation of it. One obtains this two-dimensional representation of a cube by projecting the square within the plane of the square in an arbitrary direction assumed to be perpendicular to the square's sides. It is assumed that the direction is perpendicular instead of actually projecting it perpendicular to itself for this is impossible in only two dimensions. Metric properties are destroyed by such means but at least we do have a picture of a model. Take the two-dimensional representation of the cube and again project it within the plane in a direction assumed to be mutually perpendicular to the edges. This establishes a two-dimensional picture of a four-dimensional tessaract. One can only rely on their imagination in having four lines mutually perpendicular at a point or vertex within a plane. One could and maybe in a more beneficial manner picture a tessaract by a model in space. If the cube is projected in a direction assumed to be mutually perpendicular to the edges of the cube to a new position in space and connecting the corresponding vertices to represent its path, then a model of a tessaract is formed in three dimensions. One could have projected the cube within the cube in the same manner as one could project a square within a square to represent a cube in two dimensions giving the effect of looking into a box.

In the analogue of projecting a cube into the fourth dimension to obtain a tessaract one should observe the relationships established thus far. In going from each figure to the next higher dimensional figure the vertices (called the point in the case of the zero-dimensional figure and the end points in the case of the line segment) became vertices in the higher dimensional figur generated. Hence, the number pf vertices of any figure is just twice the number of vertices of the figure with one less dimension. The point has one vertex, the line segment has two vertices,the square has four vertices, the cube has eight, and the tessaract should have sixteen.

In projecting each figure one should note that each line segment (called side in the case of a square and edge in the case of a cube) of a figure projects to a line segment and each vertex generates a different line segment in the new figure. Hence the number of line segments in any figure is twice the number of line segments in its corresponding figure of one less dimension plus the number of its vertiees. The number of sides of a square is twice the one generating line segment plus the two vertices or four. The number of edges of a cube is twice the four sides of a square plus the four vertices or twelve. The number of edges of a tessaract should be twice the twelve edges of a cube plus the eight vertices or thirty-two.

The measure of the path of a point we call length, the measure of the path of a line segment the area, the measure of the path of a plane region the golume, and mathematicians call the measure of the path of a solid the content. In projecting a square to obtain a cube each of the four sides of the square generates a face of the cube plus the two faces formed by the square reproducing itself; the cube has six faces

missing sentence below

in all. In obtaining a tessaract plus the six faces of the cube's original position and the six faces in its new position---or

Missing sentence; tessaract

a cube each of the twelve edges of the cube forms a face of the twenty-four faces in all.

In moving the cube in space to represent fouth dimension each of the cube's six faces will generate six solids in the tessaract plus the two solids formed by the cube reproducing itself. The tessaract will have eight solids in all and seems quite odd since we cannot comprehend a figure being bounded by eight solids, However, it was found by the foregoing discussion that the relationships of a figure and its corresponding figure of one more dimension are the same no matter what figure we start with. Each figure is bounded by the figures of one less dimension. Even without a true four-dimensional tessaract we know it is composed of sixteen vertices, thirty-two edges, twenty-four faces and eight solids. (The previous 8 paragraphs are from (4), pp. 133-135.)

And now that we know what a tessaract looks like, we can readily find the edge of a tessaract whose content C=279,841 tessa units. (obviously the content is obtained by multiplying the four dimensions of any hyper-prism, where the tessaract versus hyper-prism relationship is analogous to the cube versus rectangular parallelopiped or square versus rectangle relationship).Well since $20^4 = 160,000$ and $30^4 = 810,000$ the edge has a length between 20 and 30 linear units. Let's try to obtain a refined educated guess for the $0<x<10$ number by visualizing the breakup of the tessaract.

First, we'll put a tessaract with an edge of 20 in the lower left hand corner(wherever that might be) and a tessaract with an edge of x in the upper right hand corner(ditto). And of course there will be a number of partially known hyperprisms (3 known, unknown dimensions), less partially known hyperprisms(2 known, 2 unknown dimensions) and some least partially known hyperprisms (1 known, 3 unknown dimensions). In this last case, the content of each of these hyperprisms would be $20\cdot x\cdot x\cdot x$, or $20\cdot x^3$; the question, of course, is how many of them?

Now if you've followed the thinking of this dissertation thus far, you should at this point be making the suggestion that I take 20+x and raise it to the 4th power, which is what we'll do! If you perform the operation correctly, you should obtain the following results:

$$(20+x)^4 = 20^4 + 4\cdot 20^3\cdot x + 6\cdot 20^2\cdot x^2 + 4\cdot 20\cdot x^3 + x^4$$

Therefore, it is quite apparent that the interior of the tessaract should be broken up into 2 smaller tessaracts and 14 hyperprisms (where 4+6+4 = 14) of the various dimensions. And so, like the Greeks, we could be off and running with our "false position" process, substituting numbers in for x so that the sum of the contents of all the interior hyperprisms and tessaracts would be less than or equal to C=279,841 tessa units; and if we were German algorithmists we could immediately set up the following partial factoring of the expansion and go on from there.

Namely,

$$279{,}841 = 20^4 + 4\cdot 20^3\cdot x + 6\cdot 20^2\cdot x^2 + 4\cdot 20\cdot x + x^4$$

$$279{,}841 - 160{,}000 = 4\cdot 20^3\cdot x + 6\cdot 20^2\cdot x^2 + 4\cdot 20^3 + x^4$$

$$119{,}841 = x\{4\cdot 20^3 + 6\cdot 20^2\cdot x + 4\cdot 20\cdot x^2 + x^3\}$$

$$= x\left[4\cdot 20^3 + x\left[6\cdot 20^2 + x\left[4\cdot 20\cdot x + x^2\right]\right]\right]$$

$$= x\left[4\cdot 20^3 + x\left\{6\cdot 20^2 + x\left[4\cdot 20 + x\right]\right\}\right]$$

The algorithmists would then set up some rules and drive everyone insane. However, therei is a very neat way of setting up an algorithm for making "false position" guesses. See if you can follow this. Notice that in the partial factoring, above, (I never have defined partial factoring, but I'm assuming by now you have figured out what I mean!) if you substitute a number in for x, say $x=3$, and if you work your way out from the inside, the following listing occurs; first you add a number($4 \cdot 20$) to $x=3$; then you multiply that sum by $x=3$; next you add a number ($6 \cdot 20^2$) to your previous result, and then multiply by $x=3$ once more; you then add the number ($4 \cdot 20^3$) to your previous result and again multiply that result by $x=3$.

Now, did you notice the recursiveness of the operations add-multiply, add-multiply, add-multiply? Recursiveness is a very big mathematical word; it refers to the repetiveness of a process or set of processes, (in this case upon working from the inside out); ie., before step 2 can be performed, step 1 has to be performed first.(More, much more, on this later.)

Meanwhile, back at the algorithm. In evaluating the partially factored expression from the inside out, we saw the repeated use of the add-multiply process, where the multiplication was always done with $x=3$! Well, let's be clever and set up a two line algorithm for this process. First take the educated guess $x=3$ off on the right someplace since we'll always be multiplying by it. Next, put down the numbers 1, $4 \cdot 20$, $6 \cdot 20^2$, $4 \cdot 20^3$ in a row like so (1 is the coefficient of the x^4 term):

$$1 \quad 4 \cdot 20 \quad 6 \cdot 20^2 \quad 4 \cdot 20^3$$

$$x=3$$

Now, draw a line under this row, leaving room for another row of numbers; the plan is to perform all addition operations vertically and all multiplications by $x=3$ diagonally. Also, just re-write the 1 down below.(why?- Because we're being clever, that's why!)

1	$4\cdot 20$	$6\cdot 20^2$	$4\cdot 20^3$		
	$3\cdot 1$				$\lfloor x=3$
1	83				

Now multiply the 1 by x=3 and place the result diagonally above it, namely under the $4\cdot 20$; then add that number ($3\cdot 1=3$) to the $4\cdot 20$ and place the result vertically below the line. Repeat the process as outlined above; it should look like this:

1	$4\cdot 20$	$6\cdot 20^2$	$4\cdot 20^3$		
	3	249	7947	119,841	$\lfloor x=3$
1	83	2649	39,947		

Notice two things here: 1. The final number computed is the difference between the content of the given tessaract and the content of the tessaract of edge 20. How about that!(Obviously, another set-up.)

2. You should have noted that the algorithm I've outlined for evaluating educated guesses is none other than the infamous Synthetic Division; you might also notice that no division ever took place. <u>NONE WHAT-SO-EVER</u>! (Synthetic Division my left-eyebrow!)

Meanwhile, back at the problem. We have been attempting to find the length of the edge of a tessaract of content C=279,841 tessa units. We wanted to use the Greek technique of breaking down the interior of the given tessaract into some number of hyperprisms and smaller tessaracts so that we could get better and better refined educated guesses.(Because I've been using "nice" numbers for my cubes and tessaracts, it has not been apparent that the general processes outlined can be utilized again and again to get as good an approximation as desired for a cube root or 4th root of <u>any</u> number, including those which are not perfect cubes or 4th powers. To attempt a visual breakdown of the tessaract is absurd(in fact, I'm sure most of you don't believe that the Greeks broke down the cube without the algebra; but I assure you, they did, because they had no symbols in their algebra and consequently it took two paragraphs just to tell someone to add x to x.)

So what do we do? We rely on the expansion of $(20+x)^4$ to obtain both the hyper-prism breakdown and an algorithm to make the "false position" process. But is this valid? We can't see a tessaract; why should we believe that an algebraic expansion totally unrelated to the physical object should have any validity in describing how its interior should be broken down?

How do we know that there aren't 17 or 19 or 37 hyper-prisms of the various dimensions in the interior of the given tessaract? Perhaps you are going to tell me that the algebraic $(20+x)^2$ and geometrical (1 square, 2 rectangles, and 1 small square) breakdowns should also coincide for the tessaract. Since the expansion of $(20+x)^n$ did the job for n=1, 2 and 3 (1?) then certainly it should so the job for n=4,5,6 and 7. Is that what you're going to tell me?

NONSENSE. Sheer and utter nonsense. That's all assumption! Sheer, unadulterated assumption! In fact, it's presumption. You can't see tessaracts or 5-dimensional cubes, so how can you purport to tell me their internal breakdowns into hyper-prisms? You are reasoning inductively, inducing the results of extensions of known results, and foisting them off as truth!

Oh how we love to generalize! It beats thinkiig anyday, right?

On the other hand, wasn't it true that for the specific case of C= 279,841 the algebra led to an algorithm which gave us a correct value for the length of one of the edges of the tessaract? (By the way, did you ever dheck out the answer of 23?) And certainly if it works for one case, it must work for all possible cases, right?

Balogna. Let's take a look at Titterton's Theorem no. 1: in order to simplify a fraction of the form $\frac{ab}{bc}$, for instance $\frac{16}{64}$, it is only necessary to cancel out the b's, ie., $\frac{a\not{b}}{\not{b}c} = \frac{a}{c}$. Therefore, $\frac{1\not{6}}{\not{6}4} = \frac{1}{4}$ (The hard way: divide numerator and denominator by 16).

And $\frac{2\not{6}}{\not{6}5} = \frac{2}{5}$ (common divisor of 13)

And $\frac{1\not{9}}{\not{9}5} = \frac{1}{5}$ (common divisor of 19)

And even $\frac{4\not{9}}{\not{9}8} = \frac{4}{8} = \frac{1}{2}$ (common divisor of 49)

And so the theorem is proved since I have produced 4 cases which immediately verify the premise. What more could you ask?

Plenty more, that's what! If you accept Titterton's Theorem No.1 you're in bad shape. You know if it's Titterton's, it's got to be wrong! But the reasoning of the proof is certainly valid isn't it? Just about any topic in all Math we've ever learned has been put across to us with 4 or even a less number of examples. And what a lot of nonsense that's been. We truly and really need a criteria for ascertaining when a theorem or assertion can be validated. And this we will obtain.

What can we say, therfore, at this point, since we obtained the number 23 as the length of the edge of a tessaract of content 279,841? Basically, we can only say, "that's nice".

because we must first find a general validating principle and then use it to validate our algebraic process. Of course the interior breakdown of an n-dimensional cube can never visually be verified, but after all the evidence is in, we'll be in a much better position to accept our algebraic hypothesis for the internal breakdown.

But before we inspect a general principle for validating inductive assertions we need some tools. Without some handy tools we might as well forget it.

Section 1.4 Number Rules

Charlie Pythagorus, the ancient Greek mathematician, had a favorite saying: "Number rules the universe." As a catchy phrase it might not make it on Madison Avenue today, but Charlie and his crew used it as a reminder whenever they made any nature observations. For instance, it was common knowledge that the physical world consisted of only 4 elements; earth, fire, air and water. You might ask, "But how was this classification arrived at?" And Charlie would answer, "Number rules the universe." Which is to say: There are 4 dimensions of all form, namely point, line, surface and solid. And you might say "Which is which?" And the answer is a bit different than what you might expect. You see, there are 4 perfect solids; the tetrahedron, the octahedron, the icosahedron and the cube(hexahedron). And since the tetrahedron has very sharp vertices, it corresponded to fire. The icosahedron has very smooth vertices and therefore corresponds to water. Since the cube was very solid, it corresponds to earth and the octahedron(only one left) corresponds to air. Don't you see, if the postulate that "number rules the universe" is accepted, we get great insight into the structure of the 4 basic elements.

Of course, then somebody went and discovered a 5th perfect solid, the dodecahedron. But no sweat; it's immediately obvious that the dodecahedron merely corresponds to the structure of the universe as a whole.

And so it went, with all explanations aimed at verifying the sacred postulate. That is, until $\sqrt{2}$ came along. At which point, geometry took over and number theory took a back seat.

But just in case you don't think that number rules the universe, ponder these relationships. When Harry S. Dewey set up his decimal system for the classification of books in libraries, he arbitrarily assigned the number 512.81 to books of Mathematics written about number theory (relationships of integers; ie., the integers 3,4,5 are related in a very famous theorem). Little did he know that $2^9 = 512$ and $9^2 = 81$! Now, how about that?

Still not satisfied that number rules the universe? Try this. What is 11+2-1? Of course, the sum is 12. Well, watch this. ELEVEN + TWO-ONE = ELEVENTOW-ONE = LEVETW = TWELVE. And now you're convinced, right? (oh, no, we need at least 4 examples to prove a theorem. Sorry, I forgot.)

In the 11th verse of the 21st and last chapter of John's Gospel (New Testament), 153 fish are pulled into a boat. Well, since $153 = 1^3+5^3+3^3$, we have an immediate mathematical proof of the doctrine of the Trinity. (For those of you without a background in the doctrine of the Trinity forget it; just remember, number rules the universe!)

Of course, now that you have accepted the postulate, you might like to see some examples of the rule. Here's one such example: Perfect numbers are defined as any number which is the sum of its proper divisors. For instance, the proper divisors of 6 are 1,2 and 3. Since 1+2+3 =6, six is a perfect number. (Did you ever notice that you have 2 eyes, 2 ears, 1 nose and 1 mouth- a perfect number of sensors!) Through the ages, 28, 496 and nine more perfect numbers were found; Euclid had a formula for generating perfect numbers, namely, if 2^n-1 is prime, then $2^{n-1}(2^n-1)$ is a perfect number. (Obviously even). In 1952, the computer found 3 more perfect numbers for n=521,607 and 1279 in Euclids's formula ; ie., $(2^{520})(2^{521}-1)$ is a perfect number. Just check it out.

There are no known odd perfect numbers under 2 million. If you'd like to make a big splash in math circles, be the first on your block to find an odd perfect number greater than 2 million.

Say, did you ever notice that God created the world in just six days?

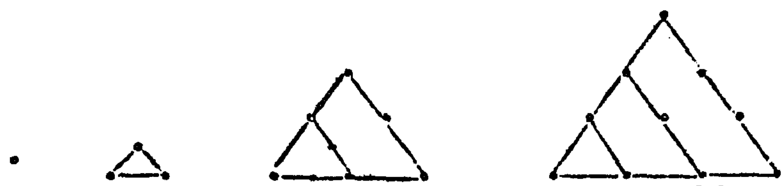
Now, the perfect numbers aren't too useful, unless you can somehow represent yourself as being the number 6 or 28 or such, and thereby claim perfection. However the amicable or friendly numbers are very useful.

Two numbers are said to be friendly if each is the sum of the proper divisors of the other. For instance, 284 and 220 are friendly numbers since 284 = 1+2+4+5+10+11+20+22+44+55+110, and each of the numbers $\{1,2,4,5,10,11,20,22,44,55,110\}$ divide 220; while 220 = 1+2+4+71+142 where each of the numbers $\{1,2,4,71,142\}$ divide 284. This pair of numbers was known to Pythagorus; they were "useful" in the sense that if you had a crush on someone, you merely showed her (him) that your name corresponded to the number 220 (making the letters of your name correspond to some set of natural numbers) and her(his) name corresponded to 284 (perhaps using a few fudge factors along the way), And therefore, instant love.

This pair of friendly numbers were the only pair known (in the western world) up until Pierre Fermat (a contemporary of Pascal) discovered another pair in 1636; namely 17,296 and 18,416. Of course Euler, 1747, made a systematic search and came up with 60 such pairs. There are now over 400 pairs known;

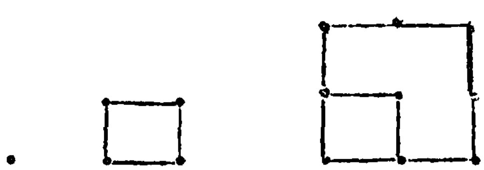
A very interesting story is that of the 16 year old Irish lad Nicolo Paganini who in 1886 found the friendly pair of numbers 1184 and 1210, which had somehow been overlooked by many of the world's greatest mathematicians. Perhaps you can find another pair overlooked by everyone!(But don't hold your breath).

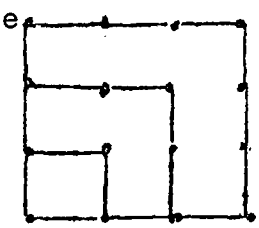
One last example(there are many more) of number ruling the universe are the figurate numbers which link natural numbers to geometry. The essense of these numbers is that they can always be written in a triangular, square, pentagonal, etc. array. For instance, 1,3,6,10,... are triangular numbers since



1 3 6 10

and 1,4,9,16,25,... are square numbers since





1 4 9 16

And so much for "Number rules the universe." Of course since the time of Pascal, Pythagorus' natural number approach has been replaced with a rather sophisticated probability-statistical analysis which essentially says that "Number rules the universe." If you've been smiling at my facetiousness in this last section, just keep on smiling; nothing's changed. Only now the absurdities are more sophisticated.

Section 1.5 Problems, Answers and Hints

1. Make a second attempt at writing an algorithm for finding cube roots, but this time include the "Synthetic Division"process.

2. Complete the following limerick:

A mathematician named Jay
Says extraction of cubes is childs play
You don't need equations
Or long calculations
_____ ___ ________ __ ___ _ ___ _______.

3. Build a model of a cube that breaks down into the 2 smaller cubes and 6 rectangular parallelopipeds described in Section 1.3.

4. Since it is possible to represent a cube (3 dimensions) on a 2-dimensional piece of paper, then it should be possible to represent a tessaract (4-dimensional) in 3 dimensional space. Build a (balsa wood) model of such a representation.

5. Make a chart showing the number of geometrical entities that are found in each of the n-dimensional cubes. The geometrical entities to be considered are points(vertices), line segments (edges), surfaces,solids, tessaracts, Wagners,Goudreaus, Van Horns, Tittertons, Eldies, Spadas and Cheneveys. (A wagner is a 5-dimensional cube, a Goudreau is a 6-dimensional cube, a Van-Horn

is a 7-dimensional cube(and you can believe that!), a Tittepton is an 8-dimensional cube(ie., $2^3=8$, or the square cubed-and you can believe that!),and Eldi is a 9-dimensional cube,and so on. Each of these n-dimensional cubes is generated from the (n-1)-dimensional cube as in the process outlined for the generation of the tessaract given in section 1.4.

6. An alternate sequence of generations can be formed from a point and line by considering the equilateral triangle as the third figure generated instead of the square. This sequence then gives rise to the generation of a tetrahedron, a pentatope and various elements named after members of Syosset's English Department. Make up a chart for this sequence of generations.

7. Check to see if Nicolo Paganini's amicable numbers (1184 and 1210) are indeed friendly, and then "develop" a friendly relationship between some two "objects" by an appropriate use of applied numerology.

8. Sometime during his lifetime(826-901), the ever popular Arabian Mathematician Tâbit ibn Qorrawitz discovered and published the following generating rule for amicable numbers. If $p= 3\cdot2^n-1$, $q= 3\cdot2^{n-1}-1$, and $r= 9\cdot2^{2n-1}-1$ are all primes for a particular value of n, then $2^n p\cdot q$ and $2^n r$ are a pair of amicable numbers. (This is the first known example of original Arabian mathematical work.) Verify Qorrawitz's formula for n=2 and n=4.

9. The figurates numbers $\{1,5,12,22,\ldots\}$ are considered as pentagonal numbers. Guess (if you can) three additional numbers in the sequence and verify that the entire sequence consists of pentagonal numbers by drawing appropriate figures as done for triangular and square numbers in section 1.5. How would you verify

your guesses otherwise?

Answers and or Hints for problems:

1. Read that part of Section 1.4 which clearly(?) explains the process for the 4th root problem.

2. My candidate:

"Just hot water to run on the tray." Gotcha, didn't I!

3. If your Dad has a table model or radial arm saw, this is not a difficult task. Please try to retain all your fingers if you attempt it.

4. See page 125 of (5) or page 139 of (4).

5. In the Wagner ,there would be 32 vertices ,80 edges, 80 surfaces, 40 solids, 10 tessaracts and of course, 1 Wagner. These results might be represented in the following ordered sex-tuple (32, 80, 80, 40, 10,1), The Goudreau would be described by the ordered sept-tuple (64, 192, 240, 160, 60,12, 1), where the 7th entry represents a Goudreau. If you do make the chart, note the many relations along the diagonal rows,

6. Refer to the symbolism of "Lord of the Flies" or "Catcher in the Rye". Also there is a discussion of this problem on page 137 of (4).

7. 1184= 1+2+5+10+11+22+55+110+121+242+605

1210= 1+2+4+8+16+32+37+74+148+296+592.

8. For n=2, $p=3\cdot 2^2-1 = 11$, $q= 3\cdot 2^{2-1}-1 = 5$ and $r= 9\cdot 2^{4-1}-1= 71$.
Therefore, $2^n p\cdot q = 2^2\cdot 11\cdot 5 = 220$ and $2^n r = 2^2\cdot 71 = 284$.
For n=4, $p= 3\cdot 2^4-1 = 47$ $q=3\cdot 2^{4-1} = 23$ and $r= 9\cdot 2^{8-1}-1 = 1151$
Therefore, $2^n p.q= 2^4\cdot 47\cdot 23= 17,296$ and $2^n r = 2^4\cdot 1151= 18416$
Note: for n=3, $p= 3\cdot 2^3-1= 23$, $q= 3\cdot 2^{3-1}-1= 11$ and $r=9\cdot 2^{6-1}-1= 287$.
But 287= 7·41 is not prime. Qorrawitz's formula does not purport

to generate <u>every</u> pair of amicable numbers; it only presumes to generate pairs of amicable numbers.

9. The three additional numbers in the sequence of pentagonal numbers would be 35,51 and 70. These numbers are readily found when it is recognized that the sequence 1,5,12,22,... is an arithmetic sequence of 2nd order (see (3), page 487, number 5) or by careful observation of the generation process visually demonstrated on page 57 of (2).

Section 1.6 <u>Student Test</u>

I. In this question, merely fill in the blanks:

The Greeks of long ago were extremely clever people when it came to synthesizing numerical solutions to geometrical problems. For instance, in order to find the edge of cube of a given volume, they developed a a) ________ techinque which depended upon the breaking up of the interior of a cube into b.)________ rectangular parallelopipeds and one small cube after a cube of c.)________ dimensions had been removed from the original. Some of these rectangular parallelopipeds had 2 known dimensions and d.)________ unknown dimensions, while the others had 1 known dimension and e.)________ unknown dimensions.Using their process of f.)________, they would find a value of the unknown dimension such that the g.)________ of the volumes of all the h.)________ and the unknown cube would be i.)________ or equal to the unaccounted for volume of the given cube. Of course if there was still some volume left unaccounted for, they would now try to once again fill in the remaining space with a more refined set of parallelopipeds and small cube. This process would be carried on j.)________ until the entire volume of t e cube was accounted for, or until the desired accuracy was achieved.

II. Define, in your own words, the following terms:

1. algorithm
2. recursiveness
3. Mathematical induction
4. partial factoring
5. tessaract

III. Choose the correct answer in each of the following:

1. The number of vertides of any n-dimensional cube has exactly a.) the same number b.) twice as many c.) three times as many d,) twice as many plus the number of edges ..... as the (n-1)-dimensional cube.

2. The number of edges of any n-dimensional cube has exactly a.) the same number b.) twice as many c.) three times as many d.) twice as many plus the number of vertices .... as the (n-1)-dimensioanl cube.

3. The number of surfaces of any n-dimensional cube has exactly a.) the same number b.) twice as many c.) three times as many d.) twice as many plus the number of edges ...of the (n-1)-dimensional cube.

4. The number of pth structural members (vertices, edges, surfaces, solids, etc), exclusive of vertices, of any n-dimensional cube has exactly a.) the same number b.) twice as many c.) three times as many d.) twice as many plus the number of (p-1)-structural members... as the (n-1)-dimensional cube.

5. Blaise Pascal was born in

a.) Italy b.) Ireland c.) France d.)Israel

6. The reason why there is no geometrical word equivalent to 4th is because

a.) Pythagorus repressed the concept

b.) There is no fourth dimension

c.) You can't visualize a 4th dimension

d.) Who needs two words to say the same thing.

7. The German algorithmists were a.) an early European folk-rock group b.) fisherman sailing the foggy Baltic Sea c.) men who made rules so that thinking was unnecessary d.) necessary to the development of mathematics.

8. The tessaract a.) is impossible to visualize b.) can be represented in 3-dimensional space c.) is a figment of the imagination d.) has a precise mathematical description.

9. The digit which when cubed yields a units digit of 3 is a.) 3 b.) 5 c.) 7 d.) 9.

10. The fact that the existence of a tessaract in nature is impossible does not phase the mathematician; he merely describes the figure in terms of previously defined figures. This description process depends upon the concept of a a.) recursiveness b.)partial factoring c.) math induction d.) algorithm

11. After the tessaract has been constructed (conceptually, of course), the interior breakdown is established a.) visually by experimentation b.) by an algorithm c.) by the expansion of a binomial d.) by teacher edict.

12. The objectives of this first section have been to a.) confuse the student b.) teach world history better than in the Social Studies Department c.) give a working knowledge of some new, unusual and abstract concepts d.) indicate to the student how

abject and complete his mathematical ignorance is.

13. The results of this first section have been to a.) confuse the student b.) teach world history better than the Social Studies Department c.) give a working knowledge of some new, unusual and abstract concepts d.) indicate to the student how abject and complete his mathematical ignorance is. d.) none of these.

Answers to Student Test

I. a. recursive b. six c. known d. one e. two f. false position g. sum h. rectangular parallelopipeds i. less than j. indefinitely

II. Since the definitions are to be "in your own words", I can't very well put an answer down for this.

III. 1. b 2. d 3. d 4. d 5. c 6. c 7. c 8. a,b,c,d 9. c 10. a 11. c 12 a,b,c,d 13. ?

Section 2 ONE-TWO, BUCKLE YOUR SHOE

Outline of Section 2.

The goals of Section 2 fall into the following categories:

1. Establish a raison d'être for the existence of Pascal's Triangle.

2. Establish a correspondence between counting subsets and making selections.

3. Develop counting techniques.

4. Attack once again the concept of sloppy math inuction.

Section 2.1 Counting Subsets

Do you remember the definition of a set? A set is a collection of well-defined objects. How do we know when an object is well-defined? When we can tell whether it ought to or ought not to be in a given set. Sets, of course, are described by either (or both) of two methods: a listing of the elements (objects) are given, or the objects (elements) belonging to the set are carefully defined. For instance, you might talk about the set consisting of the first four letters of the alphabet (English alphabet, that is) or you might just say $\{a,b,c,d\}$.

Well, I assume you've noticed the change of pace. No tessaracts or partial factoring here, just good ole sets. I wonder if they're all connected somehow?

At any rate, let's get back to sets. Hurray for modern Math! (It's all so easy.) One area the chapter in the Math 11 book just touched lightly on (and Glicksman and Ruderman, too) was a discussion of subsets. The set A is a subset of the set B if every element of A is also an element of B. This is symbolized as $A \subseteq B$, and is read, A is contained in B. Enough then of review.

Let's investigate the number, I say, the <u>number</u> of subsets and the character of the subsets of a given set. For instance, look at B={a,b,c,d}. You should remember that { } and B are both subsets of B. Also, {a}, {b}, {c}, and {d} are single element subsets of B. What are the 2-element subsets of B? Why {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d} of course. There are six of them if I didn't miss any. Did you notice my technique for obtaining all possible 2-element subsets of B? First I fixed a and exhausted all second possibilities, then I fixed b and exhausted all second possibilities, etc. This is <u>not</u> the only technique, but it's a good one (if I do say so myself).

Now you use a similar technique to list the 3-element subsets (you should find 4 of them-- <u>obviously</u>).

And so, for a four element set, there is one no-element subset, 4 1-element subsets, 6 2-element subsets, 4 3-element subsets and 1 4-element subset. You might notice that the total number of subsets is 16.

Before you go on to the next section, stop! Find the breakdown of the 1,2, and 3 element sets into appropriate subsets. List your results, and make some guesses as to the general situation if you can. Remember, the key phrase for the rest of this chapter is "Observe, Explore, Discover!"

Section 2.2 <u>Symmetry Counting</u>

Let's look at the set S={a,b,c,d,e}. We should immediately note that there is 1 no-element subset and 5 1-element subsets. However, the number of 2-element subsets is a matter of careful counting. Let's use the exhaustion technique: with "a" fixed we get {a,b}, {a,c}, {a,d}, {a,e}; with b fixed we get {b,c}, {b,d}, {b,e}; with c fixed we get {c,d} and {c,e}; and with d fixed we get {d,e}. Therefore, there are 4+3+2+1 = 10 2-element subsets of set S.

Observe, explore, discover. Did you notice that using this technique we get a sequence of numbers 4,3,2,1 which when added together form an arithmetic progression. (Does anyone remember the formula for the sum of an arithmetic progression?) It appears that a generalization might be made here (if we only knew that silly formula! oh, shucks!)

And now back to the counting of the 3-element subsets of set S. Remember that wise-guy "obviously" I threw in when I told you that the number of 3-element subsets of a 4-element set was 4? Well, it applies to the question of the number of 3-element subsets of a 5-element set also; obviously the number is 10! Wha?

Didn't we just spend a great deal of time counting the 2-element subsets of a 5-element set? Well, check this. Let the 2-element subset {a,b} correspond to the 3-element subset {c,d,e}; ie., let subset A={a,b} correspond to its complement, A'={c,d,e}. Let B={a,c} correspond to the 3-element subset ^ [missing sentence: see page 46] there are exactly 10 complements of the 2-element subsets; and therefore, there are exactly 10 3-element subsets of set S.

Pretty neat, eh? If you understood that, then you should be able to tell how many 4-element subsets of the 5-element set S there are, <u>without</u> counting. Since there are 5 1-element subsets, there are 5 complements to each of these 1-element subsets, and therefore 5 4-element subsets of the set S. List them if you don't believe me!

I call this technique symmetry counting. Of course it is of no value unless previous information is given (it's recursive!). For instance, you can't tell me the number of 7-element subsets of a 9-element set unless you first know the number of 2-element subsets of the 9-element set. Or in general, the number of r-element subsets of an n-element set is the same as the number of (n-r)-element subsets of the n-element set. Does this generalization agree with your observations, explorations and discoveries?

Section 2.3 More Techniques

We now have (supposedly) a chart or listing of the number of subsets of 1,2,3,4 and 5-element sets. We have seen that we only need half of these numbers to get the other half (for any given set). Now let's obtain the subset breakdown of a six element set $T=\{a,b,c,d,e,f\}$. We shall make use of a purely recursive technique. We first notice that T and $S=\{a,b,c,d,e\}$ (of the previous section) differ only in the letter f belonging to T. Therefore all the (10) 2-element subsets of S are certainly 2-element subsets of T, plus all those 2-element subsets formed by taking all the elements of S and adjoining to them the element f of T; namely $\{a,f\}$, $\{b,f\}$ $\{c,f\}$, $\{d,f\}$, and $\{e,f\}$.

And we have again exhausted all the possible 2-element subsets of T; any 2-element subset of T without an f was counted in the 10 we took from S; any 2-element subset of T with an f was counted in the 5 new subsets formed; therefore, the total number of 2-element subsets of set T is 10+5 = 15.

Let's try it again. How many 3-element subsets does set T have? Set S has 10, all of which will be included in the count for

T; also, set S has 10 2-element subsets; to each of these adjoin the element f of T, thereby obtaining 10 3-element subsets of T. The total is 10+10 = 20.

If you didn't follow all of the above, try one or all of these three things: 1. Re-read the section very carefully, being careful to distinguish between the words set and subset.

2. Try to find a 3-element subset of T which has not been included in the listing given above, (ie., either in the set of 3-element subsets of S or the 2-element subsets of S with an f adjoined).

3. Read the next section.

Section 2.4 Pascal at Last

In the previous section I have outlined a recursive technique for counting subsets. Let's see what we've got and use the technique to enumerate the number of r-element subsets of a 7 element set, where we'll let r vary from 0 to 7. Inspect the following table of results:

Number of Elements in the Set	Number of r-element subsets r: 0	1	2	3	4	5	6	7
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	_	_	_	_	7	1

On the last row, I have already filled in the obvious; namely, 1 no-element subset, 7 1-element subsets, 1 7-element subset and (by the symmetry concept) 7 6-element subsets. What about the rest?

From the previous section, we saw that to obtain the number of 2-element subsets of an n-element set when the number of 1-element and 2-element subsets of the (n-1)-element set is known, we merely add the number of 1 element and 2-element subsets of the (n-1)-element set together. Believe it or not, that's exactly what we established by our observe, explore, discover methods in Section 2.3. Take another look if you don't believe me! And read carefully.

So. The number of 2-element subsets of a 7-element set is therefore equal to 6+15, or 21. Likewise, the number of 3-element subsets of a 7-element set is merely 15+20, or 35. As far as the table is concerned, to find the number of r-element subsets of an n-element set, you merely go to the line above (the (n-1)th line) and add the r and the (r-1)th number together. Simple, eh?

For those of you who have been around, you should have by now recognized both Pascal's Triangle and the generation process thereof! Alas, Pascal has finally arrived (or should I say Chu-Shï-Kié?)! The table does not give the triangle in its most popular form, but it is in the form which Pascal used.

Entire paragraph missing: See page 46.

Stick around. We're going to put it all together yet.

Section 2.5 <u>Problems and Answers</u>

1. We noted in section 2.1 that the total number of subsets for the 4-element set was 16. What is the total number of subsets of each of the n-element sets in the table of section 2.4? From this

information, can you generalize a rule for finding the number of subsets of an n-element set?

2. How many 2-element subsets does an 8-element set have? That question's too easy; try finding the number of 2-element subsets of a 37 element set. Of a 49 element set? a 123 element set? A 150 element set?

3. If you can handle #2, try this: How many 3-element subsets does a 37 element set have? If you're crazy enough to use the recursive scheme to obtain that answer, try finding the number of 3-element subsets of a 49, 123 and 150 element set!

4. Anyone for finding the number of 4-element subsets of a 37 element set?

5. In section 1,we investigated the tessaract. To get a (mathematical) idea of a possible interior structure of the tessaract we took $(20+x)^4$ and expanded it, obtaining 1 known tessaract, 4 hyper-prisms of one variety, 6 of another, 4 of a third variety and 1 unknown tessaract.

By now these numbers should have some significance to you; but what does subsets have to do with the possible structures of tessaracts?

Answers to Problems

1. I will give the answers as ordered pairs (a,b), where a will equal the number of elements in the set and b the total number of subsets of the set with n elements. The set of answers therefore would be: $\{(1,2), (2,4), (3,8), (4,16), (5,32), (6,64), (7,128)\}$

It might appear to you that there is indeed a general rule, namely $(n,2^n)$, but that's all hogwash. Just look at the following example:

Given a circle O with a point A on the circumference, we say that the interior of the circle is "divided" into one region (trivial case). Given circle O with two points A and B on the circumference, the line segment AB divides the interior of the circle into 2 regions. Given circle O with three distinct points A,B and C on the circumference, the line segments AB,AC,BC divide the interior of the circle into 4 regions.(Draw the picture and see for yourself). Given circle O with four distinct points A,B, c and D on the circumference, the line segments AB,AC,AD,BC,BD and CD divide the interior of the circle into 8 regions. Continue this process. The set of relations (number of points on circumference, number of regions in the circle) denoted by (p,r) consists of $\{(1,1),(2,2),(3,4),(4,8),(5,16), \ldots\}$ where the dots can be filled in as needed. Obviously the rule is $(p, 2^{p-1})$.

Obviously my left eye-brow! The actual rule is $(p, 1+(1/24)(p)\cdot(p-1)(p^2-5p+18))$. Try it and see! For p=6, the number of regions is $1+(1/24)(6)(5)(24)$ or 31. Which is the first case for whidh the purported case breaks down! Draw a circle, put 6 distinct points on the circumference, draw all possible connecting line segments (how many would that be?) and count the regions carefully. You should get 31.

In other words, after 5 specific cases, the apparent rule breaks down to be replaced by a monstrosity (where did it come from?) In the case for the total number of subsets we can guess the rule as $(n, 2^n)$, but how do we know it doesn't break down for the very next case? Our table only has the first seven values; can we be sure that the sum of all the subsets of an 8-element set is 2^8? Not at all, based on the previous example.

There is, of course, a constructive proof as to why $(n,2^n)$ is indeed the correct relationship; but unless you can give it, you've got no right to assume the relationship is true.

2. Let's look at the 37 element set. How many 2-element subsets? Suppose we use the exhaustion technique; take some element, say a, from the set; match it with each of the other 36 elements; then take the b element and match it with each of the other 35 elements (it's already been matched with the a). Continuing in this fashion you will obtain a sequence of numbers 36,35,34,...3,2,1 whose sum is the total number of 2-element subsets of a 37 element set.

Since you still haven't remembered the formula for the sum of an arithmetic progression, let's derive it. Form the sum of 36+35+34+...3+2+1; note that 36+1=37; 35+2=37; 34+3=37; etc. There are obviously 36/2 pairs of these sums; Therefore (36/2)(37)=666 is the number of 2-element subsets of a 37 element set.

The derivation of the formula for the sum of the arithmetic progression is obvioulsy "first plus last" (a_1+a_n) times the number of terms divided by 2(n/2-the number of pairs of sums). Therefore, the sum equals $(n/2)(a_1+a_n)$, where a_1 is the first term of the sequence and a_n is the n^{th} or last.

sentence missing: page 46

For our case (the counting of 2-element subsets consisting of a adjoined to all the other elements) and the value of a_n is 1, while there are (n-1) terms. Therefore, the sum is $((n-1)/2)(n-1+1)=(n)(n-1)/2$.

For n=49, the number of 2-element subsets is 1176; n=123, 2-element subsets number 7503; n=150, 2-element subsets number 11,175.

There are other clever techniques which also give rise to the same general expression.

3. The general rule here is $\frac{(n)(n-1)(n-2)}{3 \cdot 2}$. Where'd I get if from? Read on.

4. The general rule for finding 4-element subsets of an n-element set is $\frac{(n)(n-1)(n-2)(n-3)}{4 \cdot 3 \cdot 2}$. Check it out for n=4,5,6 and 7. If you believe this constitutes a proof, you've missed the whole message in Problem 1 above.

5. Nothing at all. Counting subsets apparently has something to do with the expansion of a binomial, but subsets and tessaracts have nothing in common. It's the expansion that we want to investigate, so read on.

Section 2.6 Pre-Test

I. If a particular sequence of numbers 3,7,9,13,9,7,3 give rise to a second sequence of numbers 3,10,16,22,22,16,10,3 where the rule of generation is that of a)________ triangle, the sum of the second sequence is exactly b)________ the sum of the first sequence. This is quite c)________ so since each member of the original sequence is used twice in the d)_______ of the new sequence. For instance, the first 9 is added to the 7 to get 16 and is added to the e)_______ to get 22, and so the first 9 appears twice in the new sequence. Even the first 3 is used twice; once by itself and once with the 7 to yield f)_________.

Although the sequence 3,7,9,13,9,7,3 has nothing to do with Pascal's Triangle (except perhaps that it is g)________), the above discussion does provide the essence of the h)_________ proof alluded to in the solution of problem 2.2. Of course the more i)_________student will attempt to develop this proof to fit the case in question(namely, that the j)_______ of the subsets of an n-element set is 2^n.)

II. Define in your own words:

a) English alphabet

b) Observe

c) Explore

d) Discover

e) Arithmetic Progression

f) Symmetry

g) Pascal's Triangle

III. 1. In the counterexample given in the answer to question 2.1, when 4 points were put on the circumference, they determined a) 4 line segments b) 6 line segments c) 8 line segments d) 5 line segments.

2. As in problem 1 immediately above, 6 points on the circumference would determine a) 15 line segments b) 10 line segments c) 20 line segments d) 6 line segments

3. As in problem 1 and 2 immediately above, 12 points on the circumference would determine a) 12 line segments b) 66 line segments c) 132 line segments d) 42 line segments

4. The best technique for finding the answer to question 3 above is to a) use the exhaustive procedure for counting 2 element subsets b) extend Pascal's triangle to the 12th row and read off the answer c) guess d) apply the formula $\frac{(n)(n-1)}{2}$ as derived from the exhaustive technique.

5. The derivation of the formula found in potential answer 4 d above depended upon a) teacher edict b) student complaisance c) knowledge of the formula for the sum of an arithmetic progres-n d) having a good text as resource material.

6. The problem of finding how many triangles are determined by 5 points, no 3 of which are collinear, is equivalent to a) getting up at 6 o'clock on Monday morning b) finding all 3 element subsets of a 5 element set c) trisecting any angle with compass and straight edge d) the football team beating Hicksville.

7. Eight points, no 3 of which are collinear, will determine a) 56 triangles b) 28 triangles c)15 triangles d) 35 triangles

8. The best method for answering question 7 above is a) to extend Pascal's triangle to the 8th row and read off the answer b) to use the formula given in the answer to question 2.3 c) to guess d) to wait until a sophisticated notation and formulation is introduced in Section 4.5.

9. Assuming 1 5 6 3 6 5 1 is the n^{th} row in Pascal's triangle the $(n+1)^{th}$ row would be a) 1 6 11 9 9 11 6 1 b)1 6 9 11 11 9 6 1 c) can't be found d) of no value; so why find it?

10. Assuming 1 5 6 3 6 5 1 is the n^{th} row in Pascal's Triangle the $(n-1)^{th}$ row would be a) 1 4 2 1 2 4 1 b) can't be found because the generation process isn't commutative c) can't be found because the middle term (3) is less than the value on either side.
d) ½ 9/2 3/2 3/2 9/2 ½

Answers to Pre-Test

1. a) Pascal's b) twice c) obviously d)generation e) 13 f)10 g) symmetric h) constructive i) clever?; inquistive?; mathematically talented? brownnosing?;enterprising?; masochistic? j) sum

11. a) Definition would consist of a 2500 word paper discoursing on the historical development of the alphabet as we know it.
b) Ouverez les yeux! c)Get your hands dirty! d) Say, " ah, HAAA"

e) Common difference between terms f) Mirror image? g) an array of cleverly generated numbers?

III. 1. b 2. a 3. b 4. d,a,b, in that order 5. c 6. b 7. a 8. a or d, but not b. You have no right to use any formula unless it has been validated for you. Said the blind man to his friend, "So I picked up my hammer and saw. " 9. a 10. c b is incorrect because the generation process is commutative.

From page 36:

b,d,e . Well, do you get the picture? There are exactly 10 2-element subsets;

From page 39:

And so if you're interested in knowing all about subsets, their numerousness and character, we have found an array of numbers which would indeed be very useful. Of course, the question might still remain, "Who needs it?"

From page 42:

of an n-element set), the value of a_1 is (n-1) (for n = 37, there were 36 subsets

Section 3 EXPANSIONS

Outline of Section 3.

The contents of this section include:

1. The expansion of all binomials in a quick, efficient fashion.
2. More lessons in Observe, Explore, Discover techniques.
3. The built-in review of all algebraic rules.
4. Specifically learning the expansion of $(a+b)^2$ so that no student will ever get it wrong again. (Keep dreaming!)

Section 3.1 Busy Work

Expand $(a+b)^k$, for k= 1,2,34,...,8,9. That's exactly what you must do. The letters "a" and "b" merely represent dummy variables (a very descriptive phrase, right?). Actually, after about four or five expansions, your observations and explorations should lead you to a discovery. AH--HAAA!

Section 3.2 Discoveries and Assumptions

So,Pascal's Triangle strikes again! Look at those nice coefficients. I wonder if it would be a rash assumption to assume that the coefficients of $(a+b)^n$ will be found on the nth row of Pascal's Triangle? You bet it would! You've no right to assume that the coefficients of $(a+b)^{12}$ will be found on the 12th row of Pascal's Triangle. Only if you can present a constructive proof or establish a general criteria for proving out such questions can you assume that the use of Pascal's Triangle for finding coefficients of binomial expansions is valid.

You might notice another difficulty also. Suppose you're asked to find the 5th coefficient in the expansion of $(a+b)^{12}$.

If you have validated the usage of Pascal's Triangle for this purpose, you'll still have to extend the triangle to the 12th row; a somewhat tedious job just too find one number. Fortunately, there are two alternate procedures for obtaining a particular coefficient; one is recursive and the other is direct. We'll discuss the recursive procedure below and the direct approach in Section 5.

Section 3.3 Cooke's Law

Let's use our observation, exploration and discovery technique to establish a unique recursive relationship between the coefficients of an expansion. For instance, $(a+b)^6 = a^6 + 6a^5b + 15a^1b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Let's summarize our observations:

1. Every term of the expansion consists of three factors; a numerical coefficient, some power of a and a power of b (in case you're wondering, a^6 can be written as $1 \cdot a^6 \cdot b^0$, since 1 is the multiplicative identity and $b^0=1$)
2. The exponents on the a's run "downhill" from 6 to 0; the exponents on the b's run "uphill" from 0 to 6.
3. The sum of the exponents of each and every term is six.
4. There are 7 terms in the expansion.

Enough of observations. Now, let's explore. I've already clued you in that we're going to develop a recursive relationship between coefficients, so keep that in mind. My second clue is that the exponent on the a factor is going to be involved.

Take a look. The second term has a coefficient of 6 and the exponent on the a is 5; the 3rd term has a coefficient of 15. So what? Well, $6 \cdot 5 = 2 \cdot 15$, right? Quick, let's try the next case.

As we just noted, the coefficient of the 3rd term is 15 while the exponent on the a is 4; the coefficient of the 4th term is 20. Well, does 4·15 = 2·20? No, not quite. 4·15 =3·20. Hmmmm. First a factor of 2, then a factor of 3. Let's try another: 20-coefficient of 4th term; 3-exponent on a ; 15-coefficient of 5th term; 20·3=4·15. And now a factor of 4. Oh, now I see!

If you don't see it yet, try my third clue: The number of the term is involved.

And so we have discovered Cooke's Law (named after Paul Martin Cooke, former teacher at Syosset High School, who was the first one to discover this rule------- in a text book, that is!) What is the law? See if you can write it down in general; I'll bring it up later on.

Section 3.4 Summary

The immediately preceding paragraphs are again very foolish. Once more we have used just one example to discover a relationship. We have absolutely no way of knowing whether this recursive scheme is valid for cases other than $(a+b)^6$. We must find some way so that all our observations, explorations and discoveries will be of general use.

Likewise, the big question in this section is, "Can we truly and really use Pascal's Triangle to find coefficients of a binomial expansion?" What is the nexus between the two? What do they have in common? Is there some constructive proof that will validate a relationship? In section 2.4, I have given an example of what I consider a constructive proof relating subset counting to Pascal's Triangle. A similar approach can be used to relate the coefficients of the binomial expansion to Pascal's Triangle.

However, we still need a general approach. We are getting there, but we still need to introduce one more key tool: the ability to locate any number in Pascal's Triangle without writing down any lines of the triangle at all. After we've obtained that generalization, we'll be able to tie everything together.

Section 3.5 Problems and Answers

1. Take $(a+b)^n$ for n= 2,3,4,5,6. Leave a alone and let b=1 both before and after the expansion. Observe, explore, discover.

2. Take $(a+b)^n$ for n= 2,3,4,5,6. Let a=1 and b=1 both before and after the expansion. Observe, explore, discover.

3. Take $(a\mp b)^n$ for n= 2,3,4,5,6, Let a=1 and b=-1 both before and after the expansion. Observe, explore, discover. (Watch out for corollaries to this one.)

4. Take $(a+b)^n$ for n=1,2,3,4,5,6. Let a=10 and b=1 both before and after the expansion. Observe, explore, discover.

5. Obtain a copy of Courant and Robbins' What is Mathematics? (see (3) of the bibliography). Turn to page 16: the constructive proof (with diagram) for validating the use of numbers from Pascal's Triangle as coefficients in the binomial expansion is right there before you. Drink it all in. Live a little. (By the way, from this point on, I will assume we have made a valid connection between the numerical coefficients of the binomial expansion and the numbers of Pascal's Triangle.)

6. Write down a general rule for Cooke's Law.

Answers and or Hints to problems

1. The overall effect of substituting b=1 into the expansion of

$(a+b)^n$ is to get an expression in a with numerical coefficients only; i.e., the b factor in each of the terms of the expansion "disappears". It's really still there but I to any power is 1 and the multiplicative identity very nicely "disappears" under the operation of multiplication.

2. Now, both the a and b factors "disappear" from the expansion leaving only the numerical coefficients with plus signs between them. Since $(a+b)^n$ becomes $(1+1)^n = 2^n$, it is evident that the sum of the numbers of the nth row of Pascal's Triangle (the numerical coefficients of $(a+b)^n$) is 2^n. This is a valid proof of this relationship only after you have done problem 3.5.5 above.

3. The result of letting a=1 and b=-1 in the expansion is to get the numerical coefficients connected by alternating signs; when substituted into $(a+b)^n$, $(1-1)^n = 0$. The real discovery to make here however is that the sum of every other number from the nth row of Pascal's Triangle is equal to the sum of those skipped over. This is obvious for n= odd number, but n= even number is a bit different. For instance, for n=6, the numerical coefficients of $(a+b)^6$ (the numbers from the nth row of Pascal's Triangle) are

1 6 15 6 1 ; and

1+15+15+1 = 32, while 6+20+6 = 32. How about that?

4. Since 10+1 = 11, $(10+1)^n$ should give powers of 11. Therefore, when the 4th row Pascal's Triangle is regarded not as a sequence of numbers 1,4,6,4,1, but as a number in decimal form, 14,641, it should equal 11^4. Which it does since 14,641 equals $1 \cdot 10^4 + 4 \cdot 10^3 + 6 \cdot 10^2 + 4 \cdot 10 + 1$ or $(10 + 1)^4$.

Of course a bit of difficulty arises when you take $(10+1)^5$, since our decimal system has only 10 digits in it. However, if you know what you're talking about, 1 5 10 10 5 1 can readily be put into the decimal form 16,051. Try finding 11^6 the "easy" way (Answer: 1,771,561)

5. There are four copies of Courant and Robbins in t e library and the Math Department has 15 copies. Get busy and get a copy.

6. If $n \cdot a^p \cdot b$ is the kth term of the expansion $(a+b)^n$, then the coefficient of the (k+1)th term is $(n \cdot p)/k$.

For instance, $21 \cdot a^5 b^2$ is the 3rd term of $(a+b)^7$; therefore n=21, p=5 and k=3. The (k+1)th coefficient is therefore $(21 \cdot 5)/3$, or 35.

Section 3.6 Pre-Test

I. Pascal's Triangle is a rather unique array of numbers which is apparently very useful in a a)_______ of applications The b)______ we have seen so far consist of finding r-element c)_______ of n-element sets, finding the numerical coefficients of binomial d)________ and prophesizing possible structural breakdowns of n-dimensional e)_________.

Or course there is one major drawback in using Pascal's "Triangle Arithmetique". If your problem is somewhere near the top of the triangle, no f)________. But if n=16 or so, it is some mess (because of the g)________ nature of the generation process) to arrive at a solution.

And therefore there is still a task before us. It would be h)________ to have a general non-recursive method for obtaining any entry in the i)_______ of numbers called j)_________ Triangle.

II. Define: a) coefficient

b) term

c) factor

d) Hmmm.

e) Ah-haaa.

III. 1. In the expansion of $(a+b)^9$, the sum of the coefficients is a) 2^9 b) 512 c) the first part of the number in the Harry S. Dewey library classification-system for number theory d) $2 \cdot 16^2$

2. The numerical coefficient of the 4th term from the left in the expansion $(a+b)^9$ is a) 84 b) 126 c) 36 d) 512

3. In making up a 7th root algorithm, I want to make use of an idealized 7-dimensional "cube". The breakdown of the interior of this 7-dimensional "cube" would include two smaller 7-dimensional "cubes" and n hyper-prisms. The value of n would be a) 128 b)62 c) 32 d) 126.

4. In problem 3 above, how many of the 7-dimensional hyper-prisms will have 4 known dimensions and 3 unknown dimensions? a) 7 b)21 c) 35 d) none

5. The sum of the numerical coefficients of the expansion $(a-b)^7$ is a)-14 b) 0 c) 14 d) 32 e) 128

6. If $495a^8b^x$ is the 5th term of a binomial expansion, then the numerical coefficient of the next term is a) 792 b) 792 c)792 d)792 e) none of these.

7. If $495a^8b^x$ is the 5th term of a binomial expansion, then the numerical coefficient of the previous term is a) 792 b) 495 c) 220 d) 676 e) none of these

8. If k is the coefficient of the pth term of an expansion where the exponent on the 1st factor is n, then the next coefficient is

a) $(n \cdot p)/k$ b) $(k \cdot n)/p$

c) $(k \cdot p)/n$ d) none of these

9. Eleven cubed equals

a) 121 b) 1331 c) 14,641 d) 156,051

10. Every term of a binomial expansion has exactly a) two factors b) three factors c) four factors d) five factors

Answers to Pre-Test

I. a) variety b) three c) subsets d) expansions e) cubes f) sweat g) recursive h) nice?, swell?, lovely? i) array j) Pascal's

II. a) Some people go hunting, some people coefficient.

b) Separated from other terms by either plus or minus signs.

c) Every product is made up of at least two factors.

d) Sound made while exploring.

e) Sound made when discovering.

III. 1. a,b,c,d 2. a 3. d 4.c 5. b 6. d 7. c 8. b 9. b 10. b

Section 4 NEW NOTATION

Outline of Section 4.

The contents of this section include:

1. The introduction of direct notation for finding the numbers of Pascal's Triangle, and all the necessary concepts leading to the discovery.

2. Using the notation to solve previously encounted problems.

Section 4.1 From Here to Wetson's to Home

You're about to leave here and head for home one afternoon, but decide to stop off for some of the finer gourmet delicacies at one of the most famous eating spots on Long Island found right here in Syosset. However, you realize you can't afford it and head for Wetson's instead.

Now you have four possible choices of the means of transportation for getting from the High School to Wetson's: you could take one of those big Yellow Dragons,go by Shanks Mare, use your Roller Skates or Hop on your Skate Board(since it's downhill all the way). After you've partaken of your earthly reward at Wetson's, you have the possibility of getting a ride home in someone's car (perhaps an ambulance), or perhaps you might Fly, but more than likely you'd end up Crawling home.

At any rate, you've got the possibility of a dozen totally different means of arriving home. Is that right? Let's count'em up.

Suppose you take a Yellow Dragon to Wetson's; then you might ride, fly or crawl on home. That's three different possible approaches for getting home. And if you'd have used Shanks Mare to get from the High School to Wetson's, you would still have three choices for the rest of the way home. In other words, for each means you use to travel the first part of the trip, there are three ways of taking the second part of the trip. And since there are four initial choices, there would be four times three totally different ways of getting home.

What I have done above is merely a romanticization(?) of a concept called the multiplication principle; viz., if there are m ways of getting form A to B and n ways of getting from B to C, then there are m•n ways of getting from Ato C. This principle applies to more than just taking trips as we will see immediately below.

Section 4.2 Counting Stripes on a Wall

I have a little office down the hall painted a rather drab green and I thought I might like to spice it up a bit with a bright paint job. I eventually located six different-color paints around the department; namely, Van Horn Vanilla, Wagner White, Chenevey Chartreuse, Ralph Red, Bernie Blue and Elegant Eldi. (The latter is a pastel shade of sex-appeal.)

Well, I finally decided to use all six colors and paint vertical stripes of equal width on one wall. In this way I wouldn't offend anyone and every color would obtain an equal coverage.

But now I had another decision to make; what order should I choose? Which color should I use first? After all, with such a sensitive group of colors I had best be careful how I ordered t[illegible] colors on the wall. Should I use a political ordering (left to right, of course)? An intellectual ordering? An order based on good looks? (No good-not enough room on the ugly part of the wall!) A size ordering? A height ordering? A disposition ordering? How should I order the colors? No matter what order I chose, someone would make an interpretation for it --- and I'd be in hot water. Maybe I could just paint it Terrific Titterton and forget the whole project! (Terrific Titterton is a bright shade of outburst.)

Well, let's not panic. Let's investigate the problem. The first stripe could be painted any one of six colors, right? And then I would still have five colors to choose from for the second stripe; by the multiplication principle that's 30 choices right there! For the third stripe there would still be four choices to choose from, for the 4th stripe there would be three colors to choose from, for the fifth stripe there would be two colors to choose from and for the sixth stripe there would be one possible color left; no choice.

But by the recursive use of the multiplication principle, that's 6·5·4·3·2·1 or 720 different possible orderings of those six colored stripes on the wall. Certainly I could find one of those 720 that could not be interpreted in a derogatory fashion. And so I ended up with Wagner White(a deep, dark shade of black) followed by Van Horn Vanilla, Elegant Eldi, Ralph Red, Chenevey Chartreuse and Bernie Blue. A lovely display to say the least.

In mathematical circles, an ordered listing of any set of objects is called a permutation or arrangement of these objects. If just any two OBJECTS are interchanged we have a new ordered listing, which is to say, a new permutation. Of course, the number of permutations of six objects(where all six are to be utilized in the ordered list is

6·5·4·3·2·1 or 720 as we have established in the above discussion.

By the way, rather than write out 6·5·4·3·2·1 each time, let's help ourselves (and the typist) by merely using the symbol 6! (read "six factorial") to mean the same thing. In general, $n! = (n)(n-1)(n-2)(n-3)\cdots\cdots 3\cdot 2\cdot 1$, and $0!=1$, while $1!=1$ also. These last two definitions are very convenient (and non-contradictory) to our later theory. Please note that $(n+3)! = (n+3)(n+2)(n+1)!$ and that $(n+r+1)! = (n+r+1)(n+r)(n+r-1)!$ are merely applications of the original definition to special cases. Think them through and understand them thoroughly. (Hint: try numbers for n and r and verify the breakdown).

Meanwhile back in my little office. The one wall was such a hit that I decided to go on to the others. However , the two side walls could only accomadate 4 of the same size stripes as the back wall. I wasn't going to worry about which paints got left out, but that old order question was still bugging me. How many different orders could there be this time?

Certainly, there could still be six choices of colors for the first stripe, five others for the second stripe, 4 for the third stripe and 3 for the 4th stripe. Therefore, there would be 6·5·4·3 or 360 possible permutations (ordered listings) to choose from. No sweat.

Hey, I thought we were going to help out the typist and use that factorial notation! But can we use it now? Six factorial has two factors too many, namely 2 and the 1. If we wrote 6!, we'd have to divide out the 2·1. We could write our answer of 360 as 6!/2!. But what connection does this have to our original problem counting the number of permutations of 6 objects when taken 4 at a time?

Ah. Just take a look. Certainly the 2 in the denominator is equal to (6-4), since there were 4 stripes to fill in, and therefore (6-4) left alone. Let's use the following notation for the problem: ${}_6P_4$ will represent the number pf permutations of six objects taken four at a time, and ${}_6P_4=(6!)/(6-4)!$ is the manner used to compute the number in question.

In general, therefore, ${}_nP_r= (n!)/(n-r)!$ is the number of permutaions of n objects taken r at a time.

Section 4.3 Special Case

Another office was also to be painted, with the same width stripes on a same sized wall, but a problem arose: the Bernie Blue and Elegant Eldi were all used up. And so there were now six stripes to be painted with only four colors. Again the question appears: how many didtinct ordered listings of stripes on this wall could there be?

Well, let's look at this choice of paints. Since there is three times as much Van Horn Vanilla as there is Ralph Red, Chenevey Charteurse or Wagner White, we'll use the Vanilla paint for 3 stripes and the others for just one each. (Note: each pair of stripes is separated by a thin black line so that two Vanilla stripes next to each other are distinguishable.) Look at this choice: V.H. Vanilla, Ralph Red, Chenevey Chartreuse, V.H. Vanilla, Wagner White and V.H. Vanilla. If all six colors were different, there would be 6! or 720 different ordered listings of the stripes. But for each of the permutations as listed above, there are 3! permutations of the V.H. Vanillas which don't change the overall order of the arrangement at all.

In other words, the total number of possible permutations (if all colors were different) must equal the number of distinct permutations (all colors not different) times the number of permutations per set of same colors. Which is to say, the number of distinct permutations is equal to the number of all possible permutations divided by the number of un-noticeable permutations. In the case above, there would be 6!/3! distinct permutations, or 120.

Let's try another example to help(possibly) clarify the idea above, but let's use words instead of paints. For instance, take Bill.(Please!) There would be 4! or 24 arrangements of the letters B,I,L and L; but half of these would be indistinguishable from the other half, because when you interchange the two L's, nothing visible happens; no apparent change takes place. Therefore the number of distinct permutations is 4!/2! or 12.

Section 4.4 More Stripes?

Back to stripes. I'm not afraid any more, I don't care about the order of my stripes. Nobody even noticed the order of the stripes on my wall! But I still have a problem; there is still one wall left in my office but with only room for three stripes. We've been re-supplied with all six paints and now I have to make a (strictly aesthetic) choice of some three paints from the six. In other words, I must select three paints from the six; and my problem is that there are 20 ways of doing this, and it's going to be a tough decision because I have no aesthetic sense at all----- my taste is in my mouth. Such problems,eh?

How did I figure that there were twenty possible decisions to make? Look at this reasoning! For each of the selections I make (you're not supposed to know how many there are altogether), there are 3! pr 6 permutations. For instance, if I were to choose Wagner White, Bernie Blue and Ralph Red, there would be six ways of rearranging these colors on the wall; that is 3! permutations per selection. Well now, the number of selections times the number of permutations per selection would equal the number of permutations of 6 objects taken 3 at a time. But I know the number of permutations of 6 objects taken 3 at a time($_6P_3$ =(6!)/(3!) = 120). Hurray; now I can find the number of selections; merely divide the number of permutations(known) by the number of permutations per selection (known) and bingo!You've got the number of selections, 120/6=20, just like I said.

In case you hadn't noticed, the reasoning here is the same as in section 4.3. However, the type of problem proferred in section 4.3 has no definite descriptive formula for computation of a solution. But the problem of counting the possible number of selections does. Let $_nC_r$ represent the number of choices (selections--we've already used S to represent sets) of n things taken r at a time. For each choice of r objects, there are r! permutations per choice, and there are $_nP_r$ permutations of the n objects taken r at a time. (Remember, in a permutaion the order of the listing is important; in making a selection, the order of the elements has no significance.) Therefore, by our reasoning above, $_nC_r \cdot r! = {}_nP_r$. Since $_nP_r = n!/(n-r)!$, then $_nC_r \cdot r! = n!/(n+r)!$; and finally, $_nC_r = n!/(r!)(n-r)!$. We therefore now have formulae for both permutation counting and selection counting.

Section 4.5 Let There be Light

Perhaps you've noticed (and perhaps you haven't) that the question "How many selections of three paints from the six are there?" might very readily be translated to "How many 3-element subsets of the 6-element set of paints are there?" Indeed, counting the number of selections is nothing more than counting the number of subsets of a set. There is absolutely no difference whatsoever! And what then have we accomplished?

We now have a formula not only useful for counting particular selections but also for counting subsets; but that means our formula should yield the numbers to be found in Pascal's Triangle; but that means we have a way of finding the coefficients of any binomial expansion with<u>out</u> referring back to Pascal's Triangle or Cooke's law! We have really made a huge jump foreward! Let's investigate our discovery a little bit!

As we saw above ${}_6C_3 = 6!/3!3! = 20$ can be considered as the number of selections we obtain when we choose 3 elements from 6. Or it can now be considered as the number of 3-element subsets of a 6-element set. Or it can be considered as the 4th number in the 6th row of the Pascal's Triangle as illustrated in section 2.4 (where n=6 and r=3, how about that!) Or it can be considered as the numerical coefficient of the 4th term in the expansion of $(a+b)^6$.

In fact, the expression n!/r!(n-r)! is much too important a number to be tied down to the expression ${}_nC_r$. Therefore, we'll define a new notation; from here on out, $\binom{n}{r} = n!/r!(n-r)!$ will be a number representing the following values:

1. $\binom{n}{r}$ equals the number of selections of r objects taken

from a set of n objects.

2. $\binom{n}{r}$ equals the number of r-element subsets of a n-element set.

3. $\binom{n}{r}$ equals the (r+1)th number in the nth row of Pascal's Triangle (as defined in Section 2.4).

4. $\binom{n}{r}$ equals the (r+1)th numerical coefficient of the expansion $(a+b)^n$. And thus we have found a neat way of finding and or locating numbers in each of the 4 cases above, a direct, immediate method which in essence ties together several areas of apparent diversity. We will show how extensively the $\binom{n}{r} = n!/(r!)(n-r)!$ notation and computaion can be used in the next section.

The roof of my office? I painted it Stupefying Student, a drab color of constant complaint.

Section 4.6 Problems and Such

1. In painting my office, I ended up with a Wagner White, Van Horn Vanilla, Elegant Eldi, Ralph Red, Chenevey Chartreuse and Bernie Blue ordering of the stripes. Make an interpretation for this arrangement.

2. One of the orderings I thought of for the paints was an intellectual ordering; of course, this immediately fixes Bernie Blue in the first stripe and Van Horn Vanilla in the last stripe. How many possible permutations are there under these conditions?

3. A second ordering that I pondered was an excess-fat ordering. Of course, this immediately fixes Van Horn Vanilla and Wagner White in the first two positions and Bernie Blue in the last position. How many different arrangements are there under these conditions?

4. Another possible ordering occurred to me on the basis of toughness. Right away the first three stripes were fixed with Chenevey Chartreuse, Ralph Red, and Bernie Blue respectively. How many arrangements are there under these conditions?

5. Evaluate each of the following:

a. $2!$ b. $3!$ c. $4!$ d. $5!$ e. $6!$ f. $7!$ g. $8!$ h. $8!\cdot 21!+7!\cdot 22!$

i. $6!\cdot 23!-7!\cdot 22!$ j. $0!$

6. How many distinct arrangements of the letters of the word "Titterton" are there? of the letters of the word "Chenevey"? Of the word "Mississippi"?

7. How many choices are there for selecting 5 people from a group of 10? How many choices are there for selecting 4 people from 2300? In the later case, with so many possible choices, how come we end up with such a collection of bombs for student Government? (Maybe this year will be different? Wanna give me odds?)

8. More computational practice: You must become computationally capable! Compute $\binom{n}{r}$ for n= 4,5,6 and r= 0,1,2,3,...,n. (Note: r can't be greater than n, right? So that's a pretty shrewd way of telling you to perform 18 different problems. 18?)

9. For the binomial expansion, we now have three ways of obtaining the numerical coefficients: Pascal's Triangle, Cooke's Law and the direct use of the notation $\binom{n}{r}$. When is it most propitious to use a. Pascal's Triangle?

b. Cooke's Law?

c. The $\binom{n}{r}$ formula?

<u>Answers of Such to the Problems</u>

1. Obviously it's as simple as two Dutchman being separated from o Frenchman by Alsace and Lorraine(Would you believe Laurel and Hardy?

Dick and Jane?), two "gumbahs" from the ould sod.

2. There would be $4!=24$ possible arrangements; there are only 4 slots open.

3. Just $3!=6$; there are only 3 slots open, and where they are doesn't matter at all.

4. Likewise, $3!=6$. Three open slots anywhere on the wall yields an answer of $3!$

5. a. 2 b. 6 c. 24 d. 120 e. 720 f. 5040 g. 40,320
h. $7!21!(8+22)$ or $30\cdot 7!\cdot 21!$ i. $6!\cdot 22!(23-7)$ or $16\cdot 6!\cdot 22!$ j. 1

6. Titterton: $(9!)/(4!) = 15{,}120$. Chenevey: Just one; there is only one Chenevey. Mississippi: $(11!)/4!4!2! = 34{,}650$

7. $\binom{10}{5} = 252$; $\binom{2300}{4} = (575)(2299)(383)(2297)$; Law of Natural Selection.

8. See the 4th, 5th and 6th row of Pascal's Triangle (as defined in Section 2.4)

9. When the expansion is raised to the 5th power or less, you'd probably use Pascal's Triangle: by now those numbers should be very familiar to you. For higher powers you'd probably use the direct computation approach, although Cooke's Law is always the way if only 3,4 or 5 of the first terms to appear are the terms desired.

Section 4.7 <u>Enrichment?</u>

Additional counting problems can be obtained from any good textbook. These counting prøblems are amoung the most difficult problems to do because of the diverse interpretations we can give to the questions. The English language gets involved; and that's trouble!

However, a specific list of problems follow:

1. For warm ups and relatively straight forward problems, try (10) page 197,2-8, 11-13; page 203,,4-11,15,17,19; page 200, 1-3,6,11,14.
2. For additional warm ups, try (11); page 338, 1-19; page 340, 1-16.
3. In the minor leagues we find these problems from (12); page 299, 1-10; page 304, 1.3-8; page 308, 1-4, 6-10.
4. In the mojor leagues we have the problems from(13); page 430, 1-30; page 433, 1-30. (They're all mad!).
5. More major leagues; from (14); page 252, 2-4; pgge 254, 2-15; page 259, 1-15.

This last mentioned set of problems includes extensions of the theory initiated in section 4.3. This theory yields numbers which are found to be coefficients in a trinomial expansion.

In addition, for those of you who enjoy difficult challenges, Sections 1 and 4, Chapter 3 of (9), give a different, sophisticated and powerful approach to the derivation of the relationships we found in the material above. It is not easy to interpret (9)'s reasoning, but it is really and truly fantastic.

Section 4.8 Pre-Test

I. In the answer part of section 2.5, for problem 2.5.2, we used some clever techniques (summation formula for an arithmetic progression) to find that the number of 2-element subsets of an n-element set was (n)(n-1)/2. In the answers to problems 2.5.3 and 2.5.4, it was asserted that the number of 3-element and 4-element subsets of an n-element set could be found by using the formulae (n)(n-1)(n-2)/3·2 and (n)(n-1)(n-2)(n-3)/4·3·2 respectively.

Verify that all 3 formulae are correct.(And how you know why I'm so smart.)

II. Define: a. Shanks Mare

b. Yellow Dragon

c. Disposition

d. Derogatory

e. Computationally capable

III. 1. The number of distinct arrangements of the word "distestincts" is a. 6,652,800 b. 2,371,280 c. 126 d. none of these.

2. Seventeen points in 3-space, no three of which are collinear, determine x lines. The value of x is a. 126 b. 136 c. 146 d. 156.

3. Seventeen points in 3-space, no three of which are collinear, determine y triangles. The value of y is a. 720 b. 91 c. 680 d. 961

4. If I choose 5 colors from 9, I have the possibility of making any one of a. 136 decisions b. 70 decisions c. 84 decisions d. 126 decisions.

5. If I were to count all possible selections of none or more of 9 paints then I would have any one of x decisions to make. Of course, x equals a. 512 b. 255 c. 128 d. 1024

6. A committee of 13 people is to be split into 4 subcommittees of 5,3,3, and 2 members. The number of ways these subcommittee assignments can be made is a. 2,367,200 b. 1441,440 c. 729 d. 367,212

7. A teacher is going to give A's to 6 of his 21 students and F's to the others (Nice guy, eh!). The number of ways he can do this is a. 1 b. 1327 c. 54,264 d. 64,268

8. When the expression 6!•23!-7!•22! is re-written as a single term, it should look like a. 13! b. -1•22! c. -1 d.16•6!•22!

9. Another teacher is going to give A's to 6 of his 21 students, B's to 8 others and C's to the rest(all gifts!); he can do this in x ways. x= a. 21!/6!8!7! b. 21!/6!7!8! c.21!/6!8!7! d. if you compute this number your're mad!

10. I have a nickel, dime, quarter, fifty-cent piece and a dollar bill in my pocket(I wish). The number of different (actual) sums I can make up with these is a. 16 b. 32 c. 23 d. 31

<u>Answers to the Pre-Test</u>

I. $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{(n)(n-1)(n-2)!}{2!(n-2)!} = \frac{(n)(n-1)}{2}$

$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{(n)(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{(n)(n-1)(n-2)}{3\cdot 2}$

$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{(n)(n-1)(n-2)(n-3)(n-4)!}{4!(n-4)!} = \frac{(n)(n-1)(n-3)}{4\cdot 3\cdot 2}$

II. a. Shanks Mare means à pied.

b. Yellow Dragon K+S variety.

c. Ed Kranepool would say "Sometimes I play disposition, sometimes I play datposition, but mostly I sit on da bench."

d. Derogatory: See Webster

e. Computationally capable: being able to add 1+1

III. 1. a 2. b 3.c 4. d 5. a 6. b 7. c 8. d 9. a 10. d

Section 5 NOTABLE NOTATION

Outline of Section 5

The goals of this section include:

1. The introduction of sigma notation.

2. The generalized algebraic "proof" of utilizing the $\binom{n}{r}$ notation to describe the numbers of Pascal's Triangle and related counting problems.

3. Finding the solutions of sophisticated-looking problems.

Section 5.1 Sophistication

Since $\binom{n}{r}$ describes the (r+1)th numerical coefficient of the expansion $(a+b)^n$, we can use our new notation to describe the expansion of $(a+b)^n$ in the following fashion:

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots$$
$$\binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0b^n$$

For n=6, the expansion of $(a+b)^n = (a+b)^6$ becomes

$$\binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 +$$

$\binom{6}{6}a^0b^6$ which, except for the uncomputed numerical coefficients, is exactly the same as the expansion appearing in section 3.3. As we did in that section, let's once again summarize our observations relative to the expansion: (I find that these relationships cannot be emphasized enough; please pay attention to them this time!)

1. Every term in the expansion consists of three factors; a numerical coefficient denoted by $\binom{n}{r}$, a raised to some power and b raised to some power. Again, each term is the product of three factors. Learn it!

2. The exponents on the a run "downhill" from n to 0; the exponents on the b run "uphill" from 0 to n.

3. The sum of the exponents of each and every term is n.

4. There are n+1 terms in the expansion, each separated from the other by a plus sign.

Now let's take a look at some two consecutive terms in the expansion of $(a+b)^n$; inspect $\binom{n}{2}a^{n-2}b^2$ and $\binom{n}{3}a^{n-3}b^3$. How do they differ? Obviously, the first has the number 2 written in three positions, while the second has the number three written in those <u>same</u> three positions. If instead of particular number I were to write an i in each of those three positions, $\binom{n}{i}a^{n-i}b^i$, and let i= 2,3, then I would have two terms of my expansion all in one swell foop! (Or is it one fell swoop?) At any rate, notice that the sum of the exponents of the a and b <u>factors</u> is n-i+i or n, which it must be.

You see, if I now say let i vary from 0 to n, ie., let i=0,1, 2,...,n, I would have all the terms of the expansion. But there's still a problem; each of these terms is seperated from another by a plus sign. What I need therefore, if I'm going to write out this entire expansion in a neat notation (which is my goal), is a plus sign generator. And that's what I'm going to call the Σ sign (read "sigma" sign) which I am now introducing. With one additional convention, I can now write the entire expansion of $(a+b)^n$ as $\sum_{i=0}^{n} \binom{n}{i}a^{n-i}b^i$. The additional convention is that instead of writing "let i vary from o to n" or "let i=0,1,2,...,n ", I now have the same understanding by writing i=0 below the sigma sign and n above it. The convention means that you start with i=0 and keep going until you reach i=n; substitute each of these values for i into the three spots of the general $\binom{n}{i}a^{n-i}b^{i}$ term, making certain that after each substitution of a value of i into all three spots that the plus sign generator (the sigma-sign) produces a plus sign to separate the terms. Now, that's quite a mouthful; let's see some action.

We'll expand $(a+b)^6$. By our above definitions,
$(a+b)^6 = \sum_{i=0}^{6} \binom{6}{i}a^{6-i}b^{i}$ (In other words, substitute 6 in for n). Now, let's see if the expression on the right does indeed yield the familiar expansion $(a+b)^6$.

The first thing you're supposed to do is let i= 0; the first term therefore becomes $\binom{6}{0}a^6b^0$; now the sigma-sign produces a plus sign so that we have $\binom{6}{0}a^6b^0$ + ; next we let i=1, and the second term becomes $\binom{6}{1}a^5b^1$; then the sigma-sign produces a plus sign, and away we go! What a neat device, eh?

Did I hear you say you don't care for it? Well, you asked for it! For your defiance and impertinence you will expand $(a+b)^{150}$ tonight for homework! And even if you don't compute the numerical coefficients, that ought to keep you busy for quite a while.

But really I'm all heart. So if you'd like, you can use the sigma-notation. That should take about 20 seconds of your time; watch: $(a+b)^{150} = \sum_{i=0}^{150} = \binom{150}{i}a^{150-i}b^{i}$.

I don't know if you realize it or not, but I just did your homework for you! All 151 terms are neatly stacked one on top of the other in that expression on the right. That's right! As far as I'm concerned I've got 150 plus signs and 151 terms all written out on the right hand side. Of course if you're still stubbornly fighting the concept of that sigma-sign, be my guest, and write out all 151 terms explicitly.

As we shall see in the problem section below, the sigma notation is very useful in solving many problems. And in Chapter T2 we will use the sigma notation for relationships other than the binomial expansion.

Section 5.2 Cooke and Pascal in Generalization

Throughout the previous sections I've continually harped on the necessity of a general algebraic proof or a constructive proof to validate our many discoveries. For instance, we noted certain relationships among the coefficients, exponents and number of the term in the binomial expansion and called it Cookes's Law. Of course all our observations were made relative to the expansion of $(a+b)^6$, just one example. But now we have the ability to express the relationship for $(a+b)^n$. Let's set it up.

Since $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$, let's investigate two successive terms, namely the (r+1)th and the (r+2)th. These terms are equal to $\binom{n}{r} a^{n-r} b^r$ and $\binom{n}{r+1} a^{n-(r+1)} b^{r+1}$ respectively. Remember that the (r+1)th term has the numerical coefficient $\binom{n}{r}$ because we start counting with r=0. Now Cooke's Law says that the exponent of the a factor multiplied by the coefficient of the (r+1)th term, divided by (r+1), the number of the term, yields the coefficient of the (r+2)th term. In our new notation this would be

$\frac{(n-r)}{(r+1)} \binom{n}{r} = \binom{n}{r+1}$, where (n-r) is the exponent on the a, (r+1) is the number of the term, $\binom{n}{r}$ is the numerical coefficient of the (r+1)th term and $\binom{n}{r+1}$ is the numerical coefficient of the (r+2)th term (By the way, see how this description of Cooke's Law compares with your answer to Problem 3.5.6. Which is more concise, yours or the one above?)

We'll work on the left hand side of the expression only and see if we can't manipulate it to look like the right hand side. Certainly $\frac{(n-r)}{(r+1)} \cdot \binom{n}{r} = \frac{(n-r)}{(r+1)} \frac{n!}{r!(n-r)!}$ by the definition of $\binom{n}{r}$. But $(r+1)r! = (r+1)!$ (See that part of Section 4.2 where the factorial notation was introduced if you don't follow the above statement; or try it with numbers. If r=6, r+1 = 7.) Therefore, $\frac{(n-r)}{(r+1)} \binom{n}{r} = \frac{(n-r)\ n!}{(r+1)!(n-r)!}$. But now we note that $(n-r)! = (n-r)(n-r-1)!$ or $(n-r)! = (n-r)(n-(r+1))!$ Now the expression $\frac{(n-r)\ n!}{(r+1)!\ (n-r)!}$ becomes $\frac{n!}{(r+1)!\ (n-(r+1))!}$ since the factor $(n-r)$ is divided out. But the final expression is exactly what we are looking for since $\binom{n}{r+1} = \frac{n!}{(r+1)!\ (n-(r+1))!}$ And Cooke's Law does indeed work for all values of n.

If you didn't follow that bit of algebra you'd best go back over carefully, because there's more to come.

In section 2.3 I demonstrated what I called a constructive proof to verify that the numbers of Pascal's Triangle (and the generating process thereof) do indeed apply to the counting of subsets. I went through it twice back there but it wasn't that thorough a job. We did two cases and obviously the procedure could be generalized to show the correspondence between the counting of all subsets and the generation of all Pascal's Triangle.

But the procedure wasn't specifically generalized; no general terms were given, no general algebraic rules were made up. Constructive proofs contain just a little too much arm waving, even when they are valid, as is the proof given in section 2.4 relating subset counting to Pascal's Triangle and the proof given in Courant and Robbins (Problem 3.5.5) relating the binomial coefficients to Pascal's Triangle. However, now we will provide a clean, algebraic, concise, powerful and general argument that will validate in one fell swoop(or is it one swell foop?) the use of numbers from Pascal's Triangle (and the generation process thereof) to count subsets, determine binomial coefficients and to walk the dog.

From definition 3 of section 4.5, we know that the (r+1)th number on the nth row of Pascal's Triangle is denoted by $\binom{n}{r-1}$. Similarly, the (r+1)th number of the (n+1)th row is denoted by $\binom{n+1}{r}$. (If this is unclear, see the Pascal's Triangle found in section 2.4). The generation rule for Pascal's Triangle would be expressed as: $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. It is this statement which we are to verify if the connection between Pascal's Triangle and making selections is to be generally validated.

Again I will work on the left side and hope to end up with an expression equivalent to the number $\binom{n+1}{r}$ found on the right side. Actually, this problem is just as easy as adding two fractions: you merely find a common denominator and add. Simple, eh? Certainly $\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!\,(n-(r-1))!} + \frac{n!}{r!\,(n-r)!}$ by definition of the $\binom{n}{r}$ notation. Now, what is the common denominator? Well, since $r! = (r)(r-1)!$, we merely need a factor of r introduced into the denominator of the left hand fraction to make those two factors equivalent; and since $(n-(r-1))! = (n-r+1)! = ((n-r)+1)! = ((n-r)+1)(n-r)!$, we need only introduce the factor (n-r+1) into the denominator of the right hand fraction to make those two factors equivalent. That is,

$$\frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!\,(n-r)!} = (\tfrac{r}{r})\frac{n!}{(r-1)!(n-} \quad \overline{\quad)!} +$$

$$\frac{n!}{r!(n-r)!}\frac{(n-r+1)}{(n-r+1)} =$$

$$\frac{r \cdot n!}{r!\,(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\,(n-r+1)!}$$

Since the denominators are now the same, we can add! But first, note the common factor of n! in both numerators. If we factor it out before we add, the sum will look like this:

$$\frac{n!\,[r+n-r+1]}{r!\,(n-r+1)!}, \text{ or } \frac{n!\,(n+1)}{r!\,(n-r+1)!}; \text{ which is to say,}$$

$$\frac{(n+1)!}{r!\,(n+1-r)!}.$$ We have therefore shown that --

$$\binom{n}{r-1} + \binom{n}{r} = \frac{(n+1)!}{r!\,(n+1-r)!} \quad ; \text{ but } \quad \binom{n+1}{r} = \frac{(n+1)!}{r!\,(n+1-r)!}$$

Therefore, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. Q.E.D.

And in case you've never wondered, Q.E.D. means Quod Erat Demonstradum; which means "Which was to be demonstrated."

Of course, some people think it means Quite Enough Done. I'll concur at this point, Let's look at some problems.

Section 5.3 Problems

1. Express as an ordered triple the 1st, 3rd and 5th terms of the expansion $(\frac{1}{3}x^2 + 5y)^6$.

2. What is the 19th term in the expansion of $(2x-y)^{20}$?

3. I am very musical; I play 7 different instruments (piano, trumpet, harmonica, bongo drums, indian drum, conch shell and bass fiddle). How many different combo's can I advertise if I can manage to play all 7 one or more at a time?

4. As I mentioned previously, the sigma-sign has additional uses other than the binomial expansion. Referring to the definition of the sigma-sign and related notation compute:

a. $\sum_{i=1}^{4} i^2$ b. $\sum_{i=2}^{4} (2i-1)^2$ c. $\sum_{i=4}^{6} i^3$ d. $\sum_{i=2}^{5} (3i+5)^3$

5. By now, most of us are aware that $(a+b)^6$ can be written as $\sum_{i=0}^{6} \binom{6}{i} a^{6-i} b^i$, but very few if any of us are aware that $\sum_{i=0}^{6} \binom{6}{i} a^{6-i} b^i = (a+b)^6$. We are so used to looking left to right in Mathematics that we find if nigh impossible to look right to left (even those of us who have had Hebrewlessons!). That is why you'll have so much trouble doing these problems:

Solve for x:

a.) $\sum_{i=0}^{9} \binom{9}{i} 3^{9-i} x^i = 0$ b.) $\sum_{i=0}^{7} \binom{7}{i} 4^{7-i} x^i = 0$

c.) $\sum_{i=0}^{19} \binom{19}{i} x^{19-i} 4^i = 0$ d.) $\sum_{i=0}^{8} (2)^{8-i} \cdot \binom{8}{i} \cdot x^i = 0$

e.) Simplify: $\sum_{i=0}^{5} \binom{5}{i} 3^{5-i} 2^i$

f.) Find the integer equivalent to : $\sum_{i=1}^{6} \binom{6}{i} (2)^i$

g.) Reduce to lowest terms: $\dfrac{\sum_{i=0}^{7} \binom{7}{i} 4^{7-i} 3^i}{\sum_{i=0}^{11} \binom{11}{i} 2^{11-i} 5^i}$

h.) Reduce to lowest terms: $\dfrac{\sum_{i=0}^{6} \binom{6}{i} 3^{6-i} \cdot 3^i}{\sum_{i=0}^{8} \binom{8}{i} 2^i 4^{8-i}}$

i.) Reduce to lowest terms: $\dfrac{\sum_{i=0}^{5} \binom{5}{i}\cdot 4^{i}}{\sum_{i=0}^{3} \binom{3}{i}\, 2^{3-i}\cdot 3^{i}}$

6. Problems #1 and #2 above are easy enough(?), but now try these :

a.) Find the 5th term of $(x+1)^{13}$

b.) Find the 3rd term of $(x^{2/3} - \frac{1}{x})^{5}$

c.) Find the 4th term of $(2x^{3/2} + x^{-2/3})^{7}$

d.) Find the x^2 term of $(x^2 - \frac{3}{x})^{7}$

e.) Find the x^{-3} term of $(1 - \frac{1}{x})^{5}$

f.) Find the middle term of $(\frac{1}{x} - \frac{x^2}{4})^{6}$

7. I have a million problems to see if you understood the algebraic masterpiece of section 5.2. For if $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, then certainly

a.) $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ and

b.) $\binom{n+1}{r-2} + \binom{n+1}{r-1} = \binom{n+2}{r-1}$

Verify these two relationships <u>in general</u>.

8. Even if you can't do the algebra of #7 above, certainly you comprehend the relationships, n'est- ce pas? In which case you can solve the following for x:

a.) $\binom{29}{5} + \binom{29}{6} = \binom{x}{6}$ b.) $\binom{31}{17} + \binom{31}{16} = \binom{32}{x}$

c.) $\binom{47}{19} + \binom{47}{x} = \binom{48}{20}$ d.) $\binom{x}{14} + \binom{x}{15} = \binom{39}{15}$

Answers to Problems

1. $(x^{12}/729,\ \frac{125}{27}x^{8}y^{2},\ \frac{3125}{3}x^{4}y^{4})$ Hint: Take i=0,2,4. The rest is algebra and arithmetic.

2. Since we start counting with $i=0$, the 19th term has $i= 18$. The fact is just a little bit important. Just a little bit. The 19th term therefore is $\binom{20}{18}(2x)^{20-18}(-y)^{18}$, or $760x^2y^{18}$.

3. $\sum_{i=1}^{7}\binom{7}{i} = \sum_{i=0}^{7}\binom{7}{i} - \binom{7}{0} = 2^7 - 1 = 127$

4. a.) $1^2 + 2^2 + 3^2 + 4^2 = 30$

   b.) $3^2 + 5^2 + 7^2 = 83$

   c.) $4^3 + 5^3 + 6^3 = 405$

   d.) $11^3 + 14^3 + 17^3 + 20^3$

5. a.) $(3 + x)^9 = 0$; $x= -3$

   b.) $(4 + x)^7 = 0$; $x= -4$

   c.) $(x + 4)^{19} = 0$; $x= -4$

   d.) $(x + 2)^8 = 0$; $x= -2$

   e.) $(3 + 2)^5 = 3125$

   f.) $(1 + 2)^6 = 729$

   (Hint: Remember problem 3.5.1)

   g.) $\dfrac{(4 + 3)^7}{(2 + 5)^{11}} = 7^{-4}$

   h.) $\dfrac{(3 + 3)^6}{(2 + 4)^8} = 6^{-2}$

   i.) $\dfrac{(1 + 4)^5}{(2 + 3)^3} = 25$

6. a.) Take $i=4$, let $a=x$, $b=1$. Answer: $715x^9$

   b.) Take $i=2$, let $a = x^{2/3}$, $b= -\frac{1}{x}$. Answer: 10

   c.) $560x^4$

   d.) Be clever: $2835x^2$

   e.) Be clever: $-10x^{-3}$

(If you expanded the entire binomial to obtain the answers for d, e and f, consider yourself a clod!)

7. Trivial problem. Just ask your teacher to do them. They should

only take a few moments of his (her) time! (Ho-ho-ho!).

8. a.) 30 b.) 17 c.) 20 d.) 38

Section 5.4 <u>Pre-Test</u>

1. $\sum_{i=2}^{4} i^2 =$ a.) 16 b.) 30 c.) 55 d.) 29

2. The 5th term in the expansion of $(2x-3)^4$ is a.) 81 b.) $-128x^3$ c.) $216x^2$ d.) $162x$

3. The 4th term of the expansion $(\frac{x^3}{2} - \frac{1}{x})^6$ is a.) $-\frac{5}{2}x^6$ b.) $20 + \frac{x^9}{8} - \frac{1}{x^3}$ c.) $-\frac{x^6}{8}$ d.) $-\frac{5}{2}x^3$

4. $\sum_{i=2}^{5} (2i+3)$ is equal to the integer a.) 20 b.) 40 c.) 48 d.) 218

5. The x-term of the expansion $(x^2 - \frac{3}{x})^5$ is equal to a.) $-27x$ b.) $90x$ c.) $-270x$ d.) $-90x$

6. When all 132 terms of the expanded $(a+b)^{131}$ are implicitly written down, you've got a.) some headache b.) a piece of paper full of terms c.) 3 general factors preceded by an anotated signa-sign d.) a lot of numbers, letters and exponents.

7. The constant term of $(x^2 + \frac{1}{x^2})^2$ is equal to a.) 1 b.) 2 c.) 3 d.) 4

8. If the equation $\sum_{i=0}^{15} \binom{15}{i} 3^i \cdot x^{3-i} = 0$ is solved for x, the answer will be a.) -3 b.) 3 c.) 15 d.) 12

9. When simplified, the trivial fraction $\dfrac{\sum_{i=0}^{6} \binom{6}{i} 3^{6-i} 5^i}{\sum_{i=0}^{4} \binom{4}{i} 7^{4-i}}$

a.) 16 b.) 8/7 c.) 64 d.) $(8/7)^2$

10. If $\binom{29}{13} + \binom{29}{14} = \binom{30}{x}$, then x equals a.) 13 b.) 14 c.) 30 d.) 29

Answers to Pre-Test

1. d 2. a 3. a 4. b

5. c 6. c 7. b 8. a

9. c 10. b

Section 6 MATH IS FUN

Section 6.1 Summary(?) of Chapter Tl

Not this kid? If you want a summary, just look over the outlines of each section. Or do all the problems again.(Or try doing them for the first time!) But chapter summaries are for the birds; what I've done instead is put together this section 6. It's got a fractured history, Pascal's own work, songs, (which are indeed summaries) another pre-test and finally a bibliography from which I stole all my information (except for a few aberrations and mental spasms).

If this whole section isn't summary enough, then you'll just have to summarily write your own.

Section 6.2 A Tragic Tale

On the cold bright morning of November 23, 1654, Blaise Pascal was to be found in a four in hand(carriage pulled by two horses) traveling along a road running parallel to the beautiful Seine River just outside of Paris. He was approaching the town of Neuilly and the brisk air felt good on his face, helping him to forget the almost sleepless night he had just passed and mitigating the dull ache that he constantly had in his stomach. The thirty-one year old bachelor was on his way to visit a friend in Neuilly where they were going to look over the work that Blaise had done the year before utilizing his arithmetic triangle.

The better part of the previous year had been spent using the arithmetic triangle to solve a problem posed to Pascal by a gambler acquaintance of his, a fellow by the name of Antoine

Gombaud, who was better known as the Chevalier de Mere.

It seems that Gombaud had had a bit of an altercation with a fellow gambler when, in the middle of a particular game of some sort, they were forsed to disperse. The argument centered around the qestion of who should get the better share of the pot------ and how much of the pot. Antoine took the problem to Pascal, and Blaise set upon not only the problem given to him, but as is typical with most good mathematicians, took on the task of generalization as well. In this way he essentially laid all the foundations of Probability Theory.

Pascal wrote to Pierre Fermat and transmitted Gombaud's problem to him, and they worked the original problem out together. Whereas Pierre (the founder of number theory as such) used some of his sophisticated techniques to solve the problem, Pascal used his arithmetic triangle and his version of the $\binom{n}{r}$ notation we introduced in section 5. (I say "his version" since the factorial notation was not introduced until another Frenchman named Christian Kramp used it in 1808 in order to help the printer out. Previous to 1808, the notation for n! was $1^{n/1}$. How about that?) . They wrote to each other throughout the year 1654 and both their solutions agreed in essentials, although Pascal had made a few arithmetic errors. Which goes to prove something or other!

At any rate, the sequel to the story is that when Pascal informed Gombaud of his solution, the ever gracious Chevalier was anything but thankful. It seems the mathematical analysis of the problem went against his intuitive notion of the solution (and against his wallet too, I imagine), and he ended up writing and publishing a paper in which he discussed at great length the worthlessness of science in general and arithmetic in particular.

At any rate, as Pascal was approaching Neuilly, something startled the horses and away they went-just as you've seen a thousand times in a thousand western movies. But this time there was no hero to jump onto the buckboard and rein in the horses. In fact, in Pascal's case not only did the horses not stop but as they approached the bridge across the Seine leading into Neuilly, they failed to negotaite the last turn and vaulted over and through the railing of the bridge, dropping into the cold, cold Seine below. Pascal, however, remained above, precariously perched in the tottering carriage, staring into the cold, treacherous Seine far below. A person of his poor physique and physical health would have had little chance of surviving the Seine---- assuming that he would have survived the fall in the first place. Although a very carefull mathematician, Pascal had almost "jumped to a conclusion."

For the rest of his life Pascal was haunted by hallucinations of a precipice before his feet; he carried a bible with him constantly; he declared that he would retire from public life and spend his time "contemplating the weakness and misery of man"; and from then on "he regarded the pursuit of all sdience as a vanity to be eschewed for its derogatory effects on the soul." You see, he could only conclude that his near demisal from this earth had been a message from God; to wit, that he stop playing with arithmetical triangles and such things and that he should shape up---otherwise he would be shipped out.

Now, most people of that age did not think in such terms, but Pascal had had some pretty tough problems to cope with during his lifetime. He was such a frail child that his father Etienne (who was an amateur mathematician himself; the limaecon of Pascal (which you should encounter when you study polar coordinates) is named after the father Etienne and not after Blaise) decided that he should not be subjected to the difficult study of mathematics; it would be too hard on the poor lad. However at age 12, little Blaise was chafing at the bit; he demanded to know what geometry was all about (just like all the students (?) at Syosset High School). His father gave him a somewhat succinct but complete explanation of the subject matter, and Pascal sat down to play with all these matters. He soon had re-created much of Euclid's geometry, including the theorem on the sum of the angles of a triangle---without any previous knowledge of the relationship's existence. At age 14, he was admitted to the weekly meetings of an elite group of French mathematicians which eventually evolved into the French Academy of Sciences. And at age 16 he made the discovery of his "mystic hexagram".

This "most beautiful theorem in the whole range of geometry" goes something like this: If a hexagon(convex or concave) is inscribed in a conic (circle,ellipse,parabola, hyperbola) then the points of intersection of the three pairs of opposite sides are collinear and conversely. In other words, suppose we number the six points on an ellipse 1,2,3,4,5,6. Then Pascal's theorem of the "mystic hexagram" says that the intersections of the pairs of lines 12,45;23,56;34,61 are collinear. Give it a try and see if it works out.

Note: if you don't choose your points propitiously, you'll need a very large piece of paper to verify the theorem. Experiment and you'll see what I mean.

In section 1.1 I mentioned that Pascal had worked out some 400 corollaries to his theorem. Here are a couple you might try to verify.

1. A pentagon 12345 is inscribed in a conic; the pairs of lines 12,45;23,51;34 and the tangent at 1 intersect in three collinear points. (If you still haven't figured how to construct this theorem, stop and think a moment; each pair of lines intersect in one and only one point; there are three such intersections and therefore three points; Pascal's theorem states that these three points lie on one line!)

2. The pairs of opposite sides of a quadrangle inscribed in a conic, together with the pairs of tangents at opposite vertices, intersect in four collinear points.

3. If a triangle is inscribed in conic, then the tangents at the vertices intersect the opposite sides in three collinear points.

4. Given three points on a conic and the tangents at two of them, the third tangent can be constructed.

Now, the best way to see what these corollaries say is to draw a circle, read carefully, and try to draw in the given information. Just remember that they were first discovered by a 16 year old boy in 17th century France, and you're a big deal 17 year old in 20th century U.S.A.

Aside from the set of 4 examples I've given above, there have been other numerous and attractive consequences discovered through an almost unbelievable amount of research. For instance, there are 60 possible ways of forming a hexagon from 6 points on a conic (see if you can verify that!) and, by Pascal's theorem, to each hexagon corresponds a Pascal line. Furthermore, these 60 Pascal lines pass three by three through 20 points, called Steiner points which in turn lie four by four on 15 lines, called Plucher lines. The Pascal lines also concur three by three in another set pf points, called Kirkman points, of which there are 60. Corresponding to each Steiner point there are three Kirkman points such that all four lie upon a line, called Cayley line. There are 20 Cayley lines, and they pass four by four through 15 points, called Salmon points. There are also many other extensions and properties of the configuration, but at this point I'm sure we'd all be too dizzy to even read them! I can't even prove Pascal's Theorem, although I have read that the number of such proofs is legion.

Of course, any projective geometry book had a proof ((7), p66), but my excuse is that the proof is non-metrical; there are no numbers involved and no algebraic manipulations to be made. (a lie; (7), page 143). That, as a matter of a fact, is the real beauty of Pascal's Theorem; it deals only in points, lines and conics. No algebra need be utilized.

And now I'm certain you know why at the age of 17 Blaise Pascal developed acute dyspepsia (say, now I know where they got the name Pepsia Cola-----especially the diet variety). In fact, his digestive tract gave him so much trouble that when he was working in his father's office a year later (his father was essentially a tax collector; Blaise used to help him keep the books straight) he found it difficult to keep his mind on the long additions that had to be done. To get the job done he therefore invented the first adding machine.

The instrument was able to handle numbers not exceeding six digits. It contained a sequence of engaging dials, each marked from 0 to 9, so designed that when one dial of the sequence turned from 9 to 0 the preceding dial of the sequence automatically turned one unit. Thus the "carrying" process of addition was mechanically accomplished. Pascal eventually had over 5o of these machines manufactured, and a couple of them can still be found in a Paris museum more than 300 years later. Apparently "Built to Last" was Pascal's trademark.

But Pascal wasn't built to last; at age 23 his digestive track was in such bad shape that he suffered temporary paralysis. At this same time a brand of religious fervor was sweeping France (called Jansenism) which required the rejection of the corrupt material world and a "conversion" to the spiritual. Pascal figured that the temporary paralysis was a sure sign that he had been dabbling in the devil's own backyard; he therefore converted to thinking of his soul instead of the mystic hexagram and such. During this period he wrote his famous "Pensees" which were supposed to be introspective excursions into the depths of his soul. Whatever they were, they were foundations of modern French literature.

Of course six or seven years later Pascal slipped back into "sin" and did his work on the arithmetic triangle and probability theory. But the near initiation into the Polar Bear's club changed all that! He only fell from grace once more; that was when a toothache drove him to work eight straight days on the cycloid. (The cycloid is the curve traced out by the motion of a fixed point on the circumference of a wheel rolling along a straight line on a flat pavement). This was the last work that Pascal did; he fell grievously sick and died four years later.

In the work on the cycloid he determined the areas and volumes of sections and rotations of sections about various axes which depended on summation formulas (which he derived by using his arithmetic triangle). He published many of the results of his cycloid work in the form of challenge problems for other mathematicians. However, he didn't sign them as Blaise Pascal as he had supposedly eschewed the pursuit of such nonsense; he therefore signed them as Amos Dettonville or its anagram (letters re-arranged) Louis de Montalte. Clever fox, eh?

The summation formulas alluded to above were very useful and necessary in the discovery of the Calculus by Isaac Newton and Leibnizt. We'll see some of these derivations in the next chapter, after we investigate in depth the principle of Math Induction.

Needless to say, the principle of Math Induction was first presented in an incidental way in (would you believe?) Pascal's paper on the arithmetic triangle!

I've included this little treatise on the life of Blaise Pascal for a couple of reasons. First, his work and discoveries run throughout the material in these two chapters, all analysis and all mathematics; and we should have some feeling for the humanity of the man responsible for all of this. Secondly, I hope all you Math geniuses will learn a lesson from this tragic figure and live your lives with more direction and meaning.

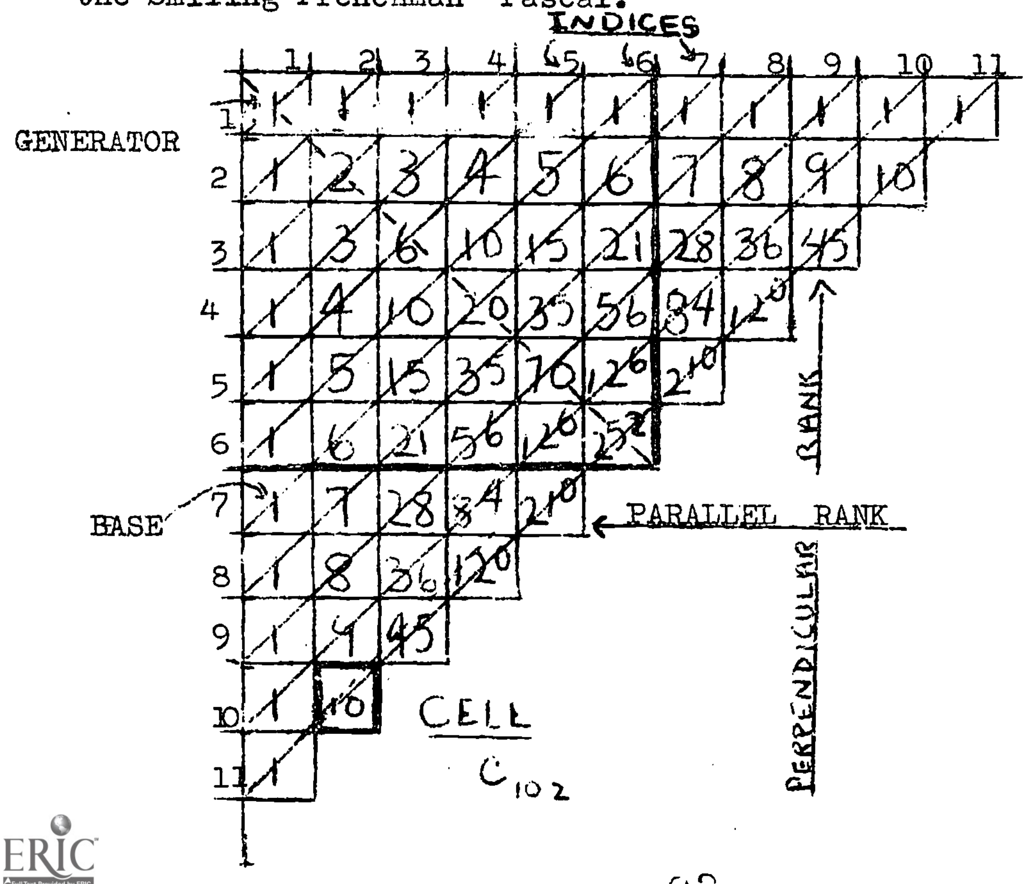
In particular, lay off the pepsi and potato chips, or you'll [end] up with paralysis of the brain too!

Section 6.3 The Arithmetic Triangle.

At this point, we've talked about Pascal's Triangle so much that it's incumbent upon us to see just exactly what he did. Below you will find my translation of the French found on page 67 of (8), which in turn was a translation of the Latin in which all important Mathematics and other disciplines were written up to the 19th century (so that all mathematicians could read a given work without some weak translation getting in the way.)

Speaking of weak translations, what I've tried to do is merely gather all the pertinent relationships; they are presented here for your perusal. Some of them require a great deal of attention before they yield any meaning.

But now, the "Traité du Triangle Arithmetique", by Blaise "the Smiling Frenchman" Pascal.



Definitions:

1. Each cell has a name, c_{ij}, where i is the appropriate index from the parallel rank and j is the appropriate index from the perpendicular rank, (i.e., c_{43} is the cell in the 4th parallel rank (row) and 3rd perpendicular rank (column)).

2. Cells of the same base: (Base 4) c_{41}, c_{32}, c_{14}.

3. Cells of the dividend: c_{11}, c_{22}, c_{33}, etc. (the main diagonal).

4. "The cells of the same base equally distant from its ends are called reciprocals, as c_{42}, c_{24} and c_{32}, c_{23}, because the index of the parallel rank of the one is the same as the index of the perpendicular rank of the other, as is apparent in the example just given. It is quite easy to show that those cells which have their indices reciprocally equal are in the same base and equally distant from its extremities.

It is also quite easy to show that the index of the perpendicular rank of any cell whatsoever, added to the index of its parallel rank, exceeds by unity the index of its base.

For example, cell c_{43} is in the third perpendicular rank, and in the fourth parallel rank, and in the sixth base; and the two indices of the ranks 3+4 exceed by unity the index of the base 6, which arises from the fact that the two sides of the triangle are divided into an equal number of parts; but this is rather understood than demonstrated."

Rule

"Now the numbers which are placed in each cell are found by this method:

The number of the first cell, which is in the right angle, is arbitrary; but when that has been decided upon, all the others necessarily follow; and for this reason, it is called the generator of the triangle. Each of the others is determined by this rule:

The number of each cell is equal to that of the cell which precedes it in its perpendicular rank, added to that of the cell which precedes it in its parallel rank.

From these facts there arise several consequences. Below are the principal ones, in which I consider those triangles whose generator is unity; but what is said of them will apply to all others."

Corollary 2. In every arithmetic triangle, each cell is equal to the sum of all those of the preceding parallel rank, comprising the cells from its perpendicular rank to the first, inclusively.

Consider any cell c_{34} : I assert that it is equal to $c_{21}+ c_{22}+ c_{23}+ c_{24}$, which are cells of the parallel rank above, from the perpendicular rank of c_{34} to the first perpendicular rank.

This is evident by defining the cells, merely, in terms of the cells from which they are formed.

For $c_{34} = c_{33} + c_{24}$

$$c_{33} = c_{32} + c_{23}$$

$$c_{32} = c_{31} + c_{22}$$

$$c_{31} = c_{21}$$

Therefore,

$$c_{34} = c_{21} + c_{22} + c_{23} + c_{24}$$ (Obviously!)

Corollary 3. In every arithmetic triangle, each cell is equal to the sum of all those of the preceding perpendicular rank, comprising the cells from its parallel rank to the first, inclusively.

Corollary 4. In every arithmetic triangle, each cell diminished by unity is equal to the sum of all those which are included between its perpendicular rank and its parallel rank, exclusively.

Corollary 5.- In every arithmetic triangle, each cell is equal to its reciprocal.

Corollary 6.- In every arithmetic triangle, a parallel rank and a perpendicular one which have the same index are composed of cells which are respectively equal to each other.

Corollary 7.- In every arithmetic triangle, the sum of the cells of each base is twice those of the preceding base.

Corollary 8. -In every arithmetic triangle, the sum of the cells of each base is a number of the geometric progression which begins with unity, and whose order is the same as the index of the base.

Corollary 9.- In every arithmetic triangle (the sum of), each base diminished by unity is equal to the sum of all preceding bases.

Corollary 10.-In every arithmetic triangle, the sum of as many continuous cells as desired of a base, beginning at one end, is equal to (the sum of) as many cells of the preceding base, (plus) taking as many again less one.

Corollary 11.-Every cell of the dividend is twice that which precedes it in its parallel or perpendicular rank.

AND NOW FOR THE REALLY BIG SHOW:

Corollary 12.-In every arithmetic triangle, if two cells are contiguous in the same base, the upper is to the lower as the number of cells from the upper to the top of the base is to the number of those from the lower to the bottom, inclusive.

Lemma 1: which is self-evident, that this proportion is met with in the second base; for it is apparent that c_{21} is to c_{12}

as 1 is to 1.

Lemma 2: that if this proportion is found in any base, it will necessarily be found in the following base.

(THIS IS THE FIRST FORMAL PRESENTATION OF THE PRINCIPAL OF MATH INDUCTION IN THE HISTORY OF MATHEMATICS!!!)

Corollary 13. - In every arithmetic triangle, if two cells are continuous in the same perpendicular rank, the lower is to the upper as the index of the base of the upper is to the index of its parallel rank.

Corollary 14. - In every arithmetic triangle, if two cells are continuous in the same parallel rank, the greater is to the preceding one as the index of the base of the preceding is to the index of its perpendicular rank.

Corollary 15. - In every arithmetic triangle, the sum of the cells of any parallel rank is to the last cell of the rank as the index of the triangle (of the base of the triangle) is to the index of the rank.

Corollary 16. - In every arithmetic triangle, (the sum of) any parallel rank is to the rank below as the index of the rank below is to the number of its cells.

Corollary 17. - In every arithmetic triangle, any cell whatever added to all those of its perpendicular rank is to the same cell added to all those of its parallel rank as the number of cells taken in each rank.

Corollary 18. - In every arithmetic triangle, two parallel ranks equally distant from the ends are to each other as the number of their cells.

Corollary Final. In every arithmetic triangle, if two cells in the dividend are continuous, the lower is to the upper taken four times as the index of the base of the upper is to a number greater (than the base) by unity.

"Thence many other proportions may be drawn that I have passed over, because they may be easily deduced, and those who would like to apply themselves to it will perhaps find some, more elegant than these which I could present".

Other Discoveries (To be Made):

1. Add the thirty-six numbers displayed in the square (heavy dark lines). Try to locate their sum in the Pascal triangle, and then formulate a general theorem.

2. Try to recognize and locate in the Pascal triangle the numbers involved in the following relation:

$$1 \cdot 1 + 5 \cdot 5 + 10 \cdot 6 + 10 \cdot 4 + 5 \cdot 1 = 126$$

3. Try to recognize and locate in the Pascal triangle the numbers involved in the following relation:

$$6 \cdot 1 + 5 \cdot 3 + 4 \cdot 6 + 3 \cdot 10 + 2 \cdot 15 + 1 \cdot 21 = 126$$

Observe (or remember) analogous cases, generalize.

(This is taken from (8), page 67. The problems are from (6), page 87.)

Section 6.4 T's Sing Along

Say, how about some entertainment? Didn't I tell you back in section 1.1 that you'd be all full of joy and happiness after going through this chapter? Well, here it is! Instant joy and happiness! We have songs to sing!

The first is sung to the tune of Bye-Bye-Blackbird (here again is where your parents will come in handy; they know the song very well even if you have never heard of it!) The song is certainly just as good as anything Mad magazine has put out---but then again, maybe that 's not saying too much.

But here it is: the entire first chapter summarized in "Blaises's Blues in B-flat."

Counting stripes on a wall, and subsets, one and all
You can use the Binomial Expansion.
Finding roots, both cube and square, Pascal's numbers everywhere
And in the Binomial Expansion.
Everywhere you look you're bound to find'em
One-- Two---One and all the rest behind'em.
When in doubt and you must guess
Use these numbers and their recursiveness
And in the Binomial Expansion.

Another set of lyrics which I have penned depends upon the tune of "Spoonful of Sugar" from the ever-lovin' Mary Poppins story. As a Math teacher here at Syosset it has been my woeful task to witness again and again the complete ineptness of most students in their quest for obtaining the rth term of an $(a+b)^n$ expansion. Despite the list of careful observations given in both sections 3.3 and 5.1, most students persist in putting plus signs between factors, in having the sum of the exponents unequal to n and in hard core cases, some have even forgotten to include the numerical coefficient. In an attempt to correct these absurdities and disgraces (they are really sins), I herewith present my song entitled, "Spoonful of Helpfulness".

Verse 1

In every test you'll ever take, It is quite probable you'll make
Perhaps, perchance or maybe a mistake
But if you li-'.n to this song, You'll likely not go wrong
So just relax perk up and sing along.

Chorus

Just a tenshun to the details and you'll get the problem right
You'll get the problem right, My ---what a delight
Just a tenshun to the details and you'll get the problem right
And have less homework every night.

Verse 2

There is a problem that we know, I haven't any doubts
We're all familiar with its in's and out's
But getting it right we cannot do, Or at most a very few
The results - by and large - are strictly Pee U.

Verse 3

And yet the problem of which we speak. That has us up the creek
Is simple as any problem we might seek
And if we wish to be more jovial, Make T. less patrimonial
Then we must - we must - learn the ex- pansion binomial

Verse 4

In every term you've got to see. Three factors - one, two, three
four or more or less you must agree

The first's a factor numerical, forget it, I'll get hysterical
You'll flunk - you'll flunk - your mark will be quite sperical.

Verse 5

If you would keep your teacher sane. And save yourself great pain
Then learn the rules for powers, signs and such
Know how to use the sigma sign
Think TWICE--- there'll be sunshine
And I - quite surely - will love you very much.

Section 6.5 More Entertainment, or History Pre-Test

The following is a sample exam which you can use to ascertain whether you have absorbed the concepts and facts of section 6.2.

1. Pascal's version of Roger's and Hammersteins's "Surrey with the Fringe on the Top" (from the musical "Oklahoma") a. is in ¾ horse time b. never made the top ten c. has a precipitous ending d. gets carried away by excessive enthusiasm of some of the principles involved.

2. Pascal was called "The Smiling Frenchman" because a. he was a pepsi cola salesman b. people in pain always smile; it only hurts when they laugh. c. he was a "blaise" of fire. d. Was a picture of robust health and earthy humor.

3. Descarte, a contemporary of Pascal, was the founder of coordinate geometry, more correctly called Des Cartesian coordinate system. There is a story told, however, about how Descarte brought much grief to a circus performer whose horse was computationally gifted. This horse could add, multiply, subtract, divide and extract roots (Sassafras was extremely tasty that year). But when someone gave the horse a coordinate geometry problem to do, he balked and had a

nervous breakdown. The reason for this was obviously because a. he chafed at the bit b. the amount of work to be done was in-ordinate c. the horse was cross-eyed and couldn't handle an ordered pair d. you don't put Descarte before the horse.

4. Pascal did not join his horses in the river because of a. Gene Autry b. Roy Rogers c. B.F. Goodrich non-skid tires d. an aversion to cold water.

5. When Pascal solved the Chevalier de Mere's problem, the Chevalier was a. not too sweet b. ecstatic c. gave up gambling d. wrote letters of praise to his friends, Amos and Dettonville.

6. When Pascal said that "The vanity of earthly pursuits were to be eschewed," he meant that a. Pierre Fermat could do his own Math homework from then on b. he would have to masticate (or Fletcherize) his food more thoroughly c. he was going to antique his sister's vanity with a new color called "Eschew" d. earthly pursuits were okay, but e-width shoes were necessary.

7. Pascal's father, Etienne, was also a famous mathematicain, and his name was given to a. a special type of lemon, the limaçon b. a curve described by revolving tops called limaçons c. the French equivalent of leprechaun, lemaçon d. curve co-discovered by a Chinese mathematician's son, Li-ma.

8. The mystic hexagram that Pascal discovered concerned itself with a. conics, corollaries, collinearity and cosmic confusion b. with an 8-letter word containing an x useable in a game of Scrabble if your opposition lets you cheat c. a six word telegram discussing heavenly happenings d. a six-sided ship moored in the Connecticutt River.

The corollaries to Pascal's Theorem are all very a. obvious

b. trivial c. enervating d, hard to understand.

10. The story of Pascal clearly indicates that if you drink enough pepsi, you will suffer a. temporary paralysis b. permanent paralysis c. stomach paralysis d. brain paralysis

Answers to History Pre-Test

1. obviously c, although d comes galloping close.
2. You're on the ball if you chose c.
3. d; and don't change horses in the middle of a stream, either.
4. a. Monsieur Eugene Autry had left his umbrella at the part of the bridge where the horses went through the railing; it stuck in one of the wheels of the carriage and prevented it from going any further.
5. a) i.e., he worked up a sweat over Pascal's solution.
6. b) Dr. Fletcher wrote a great many articles and books during the period from 1910 to 1920. It is believed that the quote from Pascal served as the basis of his theory for complete digestion of food-stuffs.
7. c) the Irish didn't invent everything!
8. a) obviously
9. c) because you don't know how to interpret that word either!
10. b) try it and see.

Section 6.6 Bibliography

1. Men of Mathematics, by Eric T. Bell, Simon and Schuster 1937 or Simon and Schuster 1961 (paperback). This book contains witty, alive and readable biographical sketches of the world's greatest mathematicians. The works of these men are also treated in an intuitive fashion and great insight can be obtained into what they

have done.

2. An Introduction to the History of Mathematics, by Howard Eves (Revised Edition). Holt, Rinehart and Winston 1965.

A fantastic book which contains hundreds of the problems that have involved mathematicians over the centuries. The treatment is extensive and readable.

3. What is Mathematics? , by Courant and Robbins. Oxford University Press, 1963.

This book contains more non-analysis mathematical information than practically any other, You don't read this book page by page; you read it line by line.

4. Geometrical Models and Demonstrations, by Donald L. Bruyr. J. Weston Walch, box 1075, Portland, Maine.

If you' like to build geometrical models, this is the book for you. But far more important, in the discussion and outline of the building techniques, thorough explanations of the whys are given.

5. Mathematics and the Imagination, by Kasner and Newman. Simon and Schuster 1940.

This book is one of the wittiest and most enjoyable books you can mathematically imagine. The extent of the problems and puzzles proferred and solved is beyond comparison. The book is a wealth of insights, descriptions and explanations- and all at an elementary level.

6. Mathematical Discovery, Volume 1. by George Polya, Wiley and Sons, 1962.

The problem sections in this book are as long as the disserta-ns. Each section is designed along a discovery approach; with an extensive solution section to help those who get bogged down. This is another author who brings a vast amount of background knowledge into play when he discusses a particular subject.

7. Fundamental Concepts of Geometry, by Bruce E. Meserve, Addison-Wesley, 1959.

This book has been used to teach projective geometry to students in the Intro class here at Syosset. It's lucid and readable, but it covers subject matter which you've never encountered before.

8. A Source Book in Mathematics, by David Eugene Smith, Dover (paperback), 1959. Two Volumes.

These two volumes contain the translations of the original manuscripts of the great mathematicians. A cursory reading is an eye opener as to the wonderful development of our mathematical notation.

9. Introduction to Finite Mathematics, by Kemeny, Snell and Thompson. Prentice Hall, 1966. Second Edition.

This is a college level text containing simply beautiful problems. The authors are wise guys, and serve up many, many curve balls.

10. Advanced Algebra, by Myron White. Allyn and Bacon, 1961.

This is our old Advanced Algebra text. It is a revision of the book that I used in high school and that's how old it is!

11. Modern Algebra and Trigonometry, by J. Vincent Robison. McGraw Hill, 1966.

This is our 11th year Math text. It contains more than book (10).

12. Integrated Algebra and Trigonometry, by Fisher and Zeibur, Prentice Hall, 1958.

This is a college level text with many unigue developments.

Cooke's Law can be found in it.

13. Principles of Mathematics, by Allendoerfer and Oakley. McGraw-Hill, 1963.

What a book! Our normal 12x bbok, it is loaded. Most teachers and students are too simple to handle it though.

14. Fundamentals for Advanced Mathematics, by Glicksman and Ruderman. Holt, Rinehart, Winston. 1964.

This book makes Allendoerfer and Oakley (known as Carl and Cletus to the initiated) look like Dick and Jane. Our 12x Junior book, it utilizes sophisticated notation far above the call of duty(or common sense, for that matter).

15. Elementary Mathematical Analysis, by Herbig-Bristol. D.C. Heath, 1967.

The Math 12x book of the future. If any of the previous ones confuse you, get ahold of Herbig-Bristol. They believe in writing so so that you can read it (Not like me.)

THE PRINCIPLE OF MATH INDUCTION AND RELATED CONJECTURES

Chapter: T2

Level: 12X

by

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Index

Chapter T2

Note: all numbers in parentheses refer to the bibliography, section 6.6, of Chapter T1.

Section 1: Pascal Revisited

Outline of Section 1

In this section the goals consist of:

1. Familiarization and comprehension(?) of the Principle of Math Induction
2. Verification, utilizing the Principle of Math Induction, of a multitude of formulae and divisibility relationships.

Section 1.1 The Spoiler

A prime number is a positive integer which is divisible only by itself and 1. The opposite of "prime" is "composite". The number 1 is neither, but stands alone.

I mention these definitions because one of the many discoveries yet to be made in mathematics is that of a prime number generator. By generator I mean a function whose range will consist only of prime numbers (although not necessarily each and every prime number) for some specific domain.

For instance, for $n \in$ Integers, 2n is an even integer; likewise, 2n-1 is an odd integer. These are trivial examples of even integer and odd integer generators. We have run into a rather sophisticated generator in problem 1.6.8 of Chapter T1; remember Tabit Ibn Qorrawitz's amicable number generators? We also found that Qorrawitz's formulae did not yield each and every pair of amicable numbers; but it did yield only amicable numbers.

Now I'm discussing all this here because I've been doing a little work looking for a prime number generator, and I do believe I've found one! Unlike Qorrawitz's complicated mess, I found a very simple expression: For n BELONGING TO positive integers, $n^2 - n + 41$ will always be prime!

Pretty neat, eh? Let's check it out. For n = 1, $p(n) = n^2 - n + 41$ equals $p(1) = 1 - 1 + 41 = 41$, a prime number. Likewise, $p(2) = 4 - 2 + 41 = 43$, a prime number; $p(3) = 47$ (jumped right over 45---great, right?); $p(4) = 53$ (skipped the 49 and 51); $p(5) = 61$ (sO I skipped 59; didn't say I'd get each and every prime number; just said all my outputs are prime); $p(6) = 71$, $p(7) = 83$, $p(8) = 97$, $p(9) = 123$, $p(10) = 131$, etc.

I haven't had too much free time lately, but I did check the generator function using values for n up to 23. $p(23) = 547$ is prime, but it takes 8 divisions to verify that fact. And who needs all that grief?

I learned the lesson of Chapter T1 though, and I'm not about to announce my discovery to the world until I can come up with some general approach to verify my formula. In doing my research for that short history of Pascal I wrote in section 6.2 of Chapter T1 I came across the following curious quote which gave me a couple of ideas:

"Although this propostion contains infinitely many cases, I shall give for it a very short proof, supposing two lemmas.

The first lemma asserts that the proposition holds for the first case of n = 1, which has been checked.

The second lemma asserts this: if the propostion happens to be valid for any case, say n, it is necessarily valid for the next case, namely n + 1.

We see hence that the proposition holds necessarily for all cases, for all values of n. For it is valid for n = 1 by virtue of the first lemma; therefore, for n = 2 by virtue of the 2nd lemma; therefore, for n = 3 by virtue of the second lemma, for n= 4 likewise, and so on ad infinitum.

And so nothing remains but to proove the second lemma."

Now my proposition fits this case very well; for an unlimited number of choices for n, my generator will always produce prime numbers. In case you're faked out by the word "lemma", just think of it as a small trivial theorem.

And here's why I found this quote to be apropos my problem; ...nt through the task of checking out 23 cases, Now I'm pretty sure that the next case, n = 24, is going to produce a prime number too---but that 's because I <u>believe</u> in my generator function; and I've had 23 sucessful trials, which is certainly more than enough for certainty! But Pascal says that if we can prove the n = 24 case not by going through all the necessary divisions but by merely referring back to the previous case, the n = 23 case, then we've got it made. Becasue if every case can be proven by merely working (in a general way) from the previous case, then we know the rule will be valid for all n $\in$ positive integers.

Let's be more specific; I have a function $p(n) = n^2 - n + 41$; p(23) was prime; let 23 = K; then n = 24 will be represented by K + 1, right? I want to use the K's for two reasons: first, numbers tend to get in the way and obscure what's happening; second, if I use the general term K, it need not represent the number 23, but might just as well represent 1 or 5 or 11 or 19 or 37. (In which case k + 1 will represent 2 or 6 or 12 or 20 or 38.). Since $p(23) = (23)^2 - 23 + 41 = 547$ was prime, we will now consider $p(K) = K^2 - K + 41$ a prime number i.e., $K^2 - K + 41$ = rib 1, where rib 1 is prime. Now, if we can show that $p(K + 1) = (K+1)^2 - (K+1) + 41$ is another prime, and if our verification depends only on the previous or Kth case, then we will have established a general procedure for verifying the prime-ness of all numbers of the form

$(K + 1)^n - (K + 1) + 41$.

Let's see if we can establish this procedure: remember, we are assuming that $p(K) = K^2 - K + 41$ = rib 1, a prime number. We inspect the $n = K + 1$ case; namely, $p(K+1) = (K+1)^2 - (K+1) + 41$. By algebraic manipulation, $p(K + 1) = K^2 + 2k + 1 - K - 1 + 41 = (K^2 - K + 41) + 2K$. Notice that I have not combined the $-K$ and the $2K$; this is because I know something about the expression $K^2 - K + 41$, namely that it is equal to rib 1, a prime number. Therefore replace the expression $(K^2 - K + 41)$ by rib 1; we need only show then that (rib 1) + 2K is always another prime number, say rib 2, and by Pascal's observations we will have proved that the expression $n^2 - n + 41$ is indeed a prime number gererator! We'll have made history!!

It is true, isn't it, that any even number (2K is always even) added to a prime number gives a prime? Ut oh, I think maybe not. I know of a case; $7 + 2 = 9$, and 9 ain't quite prime. Oh well, back to the drawing board.

Do you realize what we've done? We've shown that somewhere along the line my generator is bound to break down; it becomes obvious that when we give our attention to the previous case and use it to prove the next case, that the production of only prime numbers is not assured.

And now that I look at my generator, $p(n) = n^2 - n + 41$, it becomes immediately obvious that I don't have a prime number generator. How silly of me!

On the other hand, what about the expression $n^2 + n + 17$? Or maybe $n^2 - 79n + 1601$? They look pretty good! I'll have to check them out later. Maybe I've really got something this time!

Section 1.2 Math Induction

Pascal's observation as quoted above is commonly referred to as the "Principle of Math Induction" (abbreviated PMI). It is not magic, it doesn't produce anything and it's never saved anyone's life; it merely purports to verify conjectures about an infinite number of cases. The PMI is divided into two steps: 1. Verify that the n = 1 case holds, then

2. Assume that the n = K case is true, and using this information, show that the n = K + 1 case holds.

This procedure does indeed verify the given conjecture because K is general; and for K = 1, the K + 1 case refers to n = 2; but then K can be considered to be equal to 2 and the K + 1 case refers to n = 3. And this continues, as Pascal says, ad infinitum. Therefore verifying the conjecture for all cases.

Here's another conjecture: a little man told me that

$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{(n)(n+1)}{2}$ (See the answer to problem 2.5.2 of Chapter T1.)

We'll use the PMI to verify it's truth or non-truth. Lemma 1 says check out the n = 1 case. Does $\sum_{i=1}^{1} i = \frac{(1)(1+1)}{2}$?

i.e., substitute n = 1 into both sides of the conjecture. Well, $\sum_{i=1}^{1} i = 1$, and $\frac{(1)(1+1)}{2} = 1$, so the conjecture is true for the n=1 case.

Note: to get a feeling for the problem you might try checking out a few more cases, say n = 2 or 3. Also, I'll use LHS and RHS to mean Left Hand Side and Right Hand Side respectively. For n = 2, the LHS becomes $\sum_{i=1}^{2} i = 1 + 2 = 3$ and the RHS becomes $\frac{(2)(2+1)}{2} = 3$. For n = 3, the LHS becomes $\sum_{i=1}^{3} i = 1 + 2 + 3 = 6$ and the RHS becomes $\frac{(3)(3+1)}{2} = 6$.

Now we investigate lemma 2: Assume the n = K case is true; i.e., $\sum_{i=1}^{K} i = 1+ 2+\ldots+ K = \frac{(K)(K+1)}{2}$. Utilizing only this information we must show that the n = K + 1 case is true; i.e.,

$$\sum_{i=1}^{K+1} i = 1 + 2 +\ldots + K +(K+1) = \frac{(K+1)((K+1)+1)}{2}$$

There are several approaches to this problem; here are two procedures. The first works more or less backwards, using a major substitution. The second merely requires that you have the gift of prophecy. Here's the first.

Lemma 2: We have assumed that

1.) $\sum_{i=1}^{k} i = \frac{(K)K+1)}{2}$

We must <u>show</u> that

2.) $\sum_{i=1}^{K+1} i = \frac{(K+1)(K+1+1)}{2}$

We will work on the LHS of 2). Now, $\sum_{i=1}^{K+1} = (\sum_{i=1}^{K} i) + (K+1)$; But $\sum_{i=1}^{K} i = \frac{(K)(K+1)}{2}$ by our assumption 1). Therefore, substitute $\frac{(K)(K+1)}{2}$ into the LHS of 2) getting $\frac{(K)(K+1)}{2} + (K+1)$.

These two terms have a common factor of (K+1); after we factor it out and add the remaining two terms, we have $(K+1)(\frac{K+2}{2})$. But that is exactly the RHS of 2).

The second procedure is as follows: We have assumed 1); namely, that $\sum_{i=1}^{K} i = \frac{(K)(K+1)}{2}$. From out of thin (or even fat, for that matter) air we pick the expression K+1 and add it to both sides of 1). This gives us $\left(\sum_{i=1}^{K} i\right) + (K+1) = \frac{(K)(K+1)}{2} + (K+1)$ which is true of course by one of the most basic axioms of geometry. Now, $\sum_{i=1}^{K} i + (K+1) = \sum_{i=1}^{K+1} i$ while $\frac{(K)(K+1)}{2} + (K+1) = \frac{(K+1)(K+2)}{2}$ and we have shown that $\sum_{i=1}^{K+1} i = \frac{(K+1)(K+2)}{2}$. Let's look at another conjecture, and verify it without any of the verbal interference as in the above cases.

Conjecture: $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$

Verification Number 1.

I. Let n = 1.

$$\sum_{i=1}^{1} i^2 = 1 \ ; \ \frac{(1)(1+1)(2(1)+1)}{6} = 1$$

Therefore, the n = 1 case is true.

II. Le6 n = K case be true:
i.e., 1.) $\sum_{i=1}^{K} i^2 = \frac{(K)(K+1)(2K+1)}{6}$

2.) Show: $\sum_{i=1}^{K+1} i^2 = \frac{(K+1)(K+1+1)(2(K+1)+1)}{6}$

Working on LHS of 2):

$$\sum_{i=1}^{K+1} i^2 = \sum_{i=1}^{K} i^2 + (K+1)^2$$

Substitute $\frac{(K)(K+1)(2K+1)}{6}$ for $\sum_{i=1}^{K} i^2$ using assumption 1.).

Then LHS of 2) becomes:

$\sum_{i=1}^{K+1} i^2 = \frac{(K)(K+1)(2K+1)}{6} + (K+1)^2$ Factoring out (K+1), we get

$(K+1)\left(\frac{2K^2+K}{6} + \frac{6}{6}(K+1)\right)$ or $\frac{(K+1)}{6}(2K^2 + 7K + 6)$

or $\frac{(K+1)}{6}(2K+3)(K+2)$

But $\frac{(K+1)(K+2)(2(K+1)+1)}{6}$ is the RHS of 2), and the verification is complete.

Veaification Number 2.

I. Let n= 1. Then

$$\sum_{i=1}^{1} i^2 = 1 \text{ and } \frac{(1)(1+1)(2(1)+1)}{6} = 1$$

Therefore lemma 1 is satisfied.

II. Assume the n = K case is true;

i.e., $\sum_{i=1}^{K} i^2 = \frac{(K)(K+1)(2(K)+1)}{6}$ Now from thin air, $(K+1)^2$.

Add: $\sum_{i=1}^{K} i^2 + (K+1)^2 = \frac{(K)(K+1)(2K+1)}{6} + (K+1)^2$

Using algebra and notation: $\sum_{i=1}^{K+1} i^2 = \frac{K+1}{6}(2K^2+K+6K+6)$

$$\sum_{i=1}^{K+1} i^2 = \frac{(K+1)(K+2)(2K+3)}{6}$$

which was to be proven. Q.E.D.

If you still don't follow this procedure, do the following:

1. Re-read the entire section.
2. See the Flow-Chart which makes up Section 1.4
3. Re-read the entire section. Very carefully, this time!

And then you can practice on the following:

Conjecture Group I (CGI):

1. Problem 1, page 92. of (13).
2. Problem 2, page 92 of (13).
3. $\sum_{i=1}^{n}(4i-3) = 1+5+\ldots+(4n-3) = 2n^2-n$
4. $\sum_{i=1}^{n}(5i-4) = 1+6+\ldots+(5n-4) = \frac{1}{2}(5n^2-3n)$
5. $\sum_{i=1}^{n}(6i-5) = 1+7+\ldots+(6n-5) = 3n^2-2n$
6. $\sum_{i=1}^{n}(8i-7) = 4n^2-3n$
7. $\sum_{i=1}^{n}(ai-(a-1)) = \frac{an^2-(a-2)n}{2}$

   where a is any integer greater than 1.
8. $\sum_{i=1}^{n}(3i+2) = \frac{1}{2}(3n^2+7n)$
9. $\sum_{i=1}^{n}(5i-2) = \frac{1}{2}(5n^2+n)$
10. $\sum_{i=1}^{n}(7i-3) = \frac{1}{2}(7n^2+n)$
11. $\sum_{i=1}^{n}(4i+1) = 2n^2+5n$
12. $\sum_{i=1}^{n}(12i-11) = 6n^2-5n$

13. $\sum_{i=1}^{n} (5i-3) = \frac{1}{2}(5n^2-n)$

14. $\sum_{i=1}^{n} (7i-5) = \frac{1}{2}(7n^2-3n)$

CG II:

1. $\sum_{i=1}^{n} 5\cdot 3^{i-1} = 5 + 15 + \ldots 5\cdot 3^{n-1} = \frac{1}{2}(5\cdot 3^n-5)$

2. $\sum_{i=1}^{n} 2^{i-1} = 1 + 2 + \ldots + 2^{n-1} = 2^n-1$

3. $\sum_{i=1}^{n} (3^{i-1} + 5^{i-1}) = 2 + 8 + 34 + \ldots + (3^{n-1} + 5^{n-1}) = \frac{1}{4}(2\cdot 3^n + 5^n - 3)$

4. $\sum_{i=1}^{n} (2^{i-1} + 5^{i-1}) = \frac{1}{4}(2^{n+2} + 5^n - 5)$

5. $\sum_{i=1}^{n} (a r^{i-1}) = \frac{a(1-r^n)}{1-r}$, where a, r are constants, $r \neq 1$

6. Problem 12, page 93 of (13).

C G III:

1. Problem 7, on page 92 of (13).

2. Problem 8, page 92 of (13).

3. $\sum_{i=1}^{n} (i^4) = (1/30)(6n^5 + 15n^4 + 10n^3 - n)$

4. $\sum_{i=1}^{n} (i^5) = (1/12)(2n^6 + 6n^5 + 5n^4 - n^2)$

5. $\sum_{i=1}^{n} (2i-1)^2 = (1/13)(4n^3-n)$

6. $\sum_{i=1}^{n} (2i-1)^3 = 2n^4 - n^2$

7. $\sum_{i=1}^{n} (3i-1)^2 = \frac{1}{2}(6n^3+3n^2-n)$

C G IV:

1. $\sum_{j=2}^{n} j\cdot\left(\sum_{i=1}^{j-1} i\right) = 2\cdot 1 + 3\cdot(1+2) + 4\cdot(1+2+3)+\ldots+n(1+2+\ldots+(n-1)) =$

$(1/24)((n-1)(n)(n+1)(3n+2))$

2. $\sum_{j=1}^{n} (1+6\sum_{i=1}^{j}(i-1)) = \frac{(n)(n+1)}{2}^{2}$

3. $\sum_{i=1}^{n} \frac{1}{(i)(i+1)} = \frac{n}{n+1}$

4. $\sum_{i=1}^{n} (i)(i+1)(i+2) = \frac{(n)(n+1)(n+2)(n+3)}{4}$

5. $\sum_{i=1}^{n} (i)(i+1) = (1/3)((n)(n+1)(n+2))$

6. $\sum_{i=1}^{n} (i\cdot i!) = (n+1)! - 1$

C G V:

1. $\sum_{i=1}^{n} (2i) = n^2 + n + 2$

Section 1.3 Divisibility

The PMI can also be used to verify the divisibilty of certain expressions by given integers. This section will dwell on the procedures which you might use to verify some of these conjectures.

Once again, I'll not tell you where these conjectures have come from: at this point we'll merely verify them. Also, to facilitate matters, I will introduce the following notation: the phrase "a divides a^2" will symbollically be written $a \mid a^2$. The vertical line is not a division line but is merely translated as the word "divides". Also, if $a \mid p(n)$, I will assume that there exists some integer q such that $a \cdot q = p(n)$. Now for a few verifications.

Example 1.

Show that $2 \mid (n)(3n-1)$.

I. Let $n=1$; then $(1)(3(1)-1) = 2$. Certainly for $q=1$, $2 \cdot q = 2$.

II. Assume that $2 \mid (n)(3n-1)$ for $n=K$; i.e., $2 \mid (K)(3K-1)$. Therefore, by my convention stated above there exists some integer q_1 such that 1.) $(K)(3K-1) = 2 \cdot q_1$ We must show that $2 \mid (n)(3n-1)$ for $n = K+1$; i.e., we must find some integer q_2 such that 2.) $(K+1)(3(K+1)-1) = 2q_2$, utilizing only assumption 1.); namely, $(K)(3K-1) = 2q_1$.

Working on the LHS of 2) we obtain $(K+1)(3K+2) = 3K^2+5K+2$ When re-grouped this expression becomes $(3K^2-K) + (6K+2)$ (i.e., add the propitious zero K-K). But $3K^2-K = 2q_1$ by 1), and therefore by substitution the LHS of 2) becomes

$2q_1 + 6K + 2 = 2(q_1 + 3K + 1)$. And $q_2 = q_1+3K+1$ is certainly an integer. Q.E.D.

The alternate procedure is to take 1), $(K)(3K-1) = 2 \cdot q_1$, and add the "out of thin air" expression 6K+2 to both sides.

Then $(K)(3K-1) + 6K+2 = 2q_1 + 6K+2$ or

$(K+1)(3K+2) = 2(q_1+3K+1)$ or

$(K+1)(3(K+1)-1) = 2q_2$ Q.E.D.

Example 2.

Show that $5 \mid 7^n - 2^n$.

i.e., Find an integer q such that $7^n - 2^n = 5q$ for all $n \in I^+$.

Now in the previous example the real crux of the matter was to [illegible]duce a propitious zero (we used K-K) that enabled us to use our previous information. This of course was also true in the "out of thin air" procedure, because although no propitious zeros appeared in the exposition, yet we certainly utilized one in the experimental stage of the verification (that part of the problem which we did on the side to come up with the lovely expression 6K +2).

And now we will show that $5 \mid 7^n - 2^n$, or $7^n - 2^n = 5q$.

I. Let n=1 $7-2 = 5$. Certainly for q = 1, $5 = 5 \cdot 1$

II. Assume that $7^n - 2^n = 5q_1$ for n = K. Therefore,

1.) $7^K - 2^K = 5q_1$ We must show that

2.) $7^{K+1} - 2^{K+1} = 5q_2$, where q_2 is some integer.

Well, $7^{K+1} = 7 \cdot 7^K$ and $2^{K+1} = 2 \cdot 2^K$. The key to this entire exposition, however, is to introduce a propitious zero consisting of either of two "mixes":

A.) $7 \cdot 2^K - 7 \cdot 2^K$

or B.) $2 \cdot 7^K - 2 \cdot 7^K$.

Using A.), the LHS of 2) becomes $7^{K+1} + 7 \cdot 2^K - 7 \cdot 2^K - 2^{K+1}$.

If we now give our undivided attention to the first <u>and</u> third terms and then to the second and fourth terms we can make the following partial factorings:

$7^{K+1} - 7 \cdot 2^K = 7(7^K - 2^K)$ and $7 \cdot 2^K - 2^{K+1} = 2^K(7 - 2)$.

The LHS of 2) becomes therefore $7(7^K - 2^K) + 2^K(7 - 2)$; upon the substitution of $5q_1$ for $7^K - 2^K$ (using 1.)) and simplification of the expression 7 - 2, the LHS of 2) becomes

$7 \cdot 5q_1 + 5 \cdot 2^K$ or $5(7q_1 + 2^K) = 5q_2$, since $7q_1 + 2^K$ is certainly an integer. Q.E.D.

If instead of A.) we had used B.) , the LHS of 2) would become

$7^{K+1} - 2 \cdot 7^K + 2 \cdot 7^K - 2^{K+1}$; after partial factoring and simplification this becomes $7^K(7 - 2) + 2(7^K - 2^K)$

After the substitution of 1) we obtain $5(7^K) + 2(5q_1)$ or $5(7^K + 2q_1) = 5q_2$. Q.E.D.

Of course if we had done all this preliminary work on the side somewhere, the "out of thin air" procedure would look like this: Given 1), $7^K - 2^K = 5q_1$, first multiply both sides by 2, obtaining $2 \cdot 7^K - 2^{K+1} = 2 \cdot 5q_1$. Then add the term $5 \cdot 7^K$ to both sides, obtaining $2 \cdot 7^K - 2^{K+1} + 5 \cdot 7^K = 2 \cdot 5q_1 + 5 \cdot 7^K$. Then rearrange and simplify thusly:

$$2 \cdot 7^K - 2^{K+1} + (7-2)7^K = 5(2q_1 + 7^K)$$

$$2 \cdot 7^K - 2 \cdot 7^K + 7 \cdot 7^K - 2^{K+1} = 5q_2$$

2) $7^{K+1} - 2^{K+1} = 5q_2$ Q.E.D.

And how for some more problems to keep you out of mischief.

Conjecture Group I (C G I):

1. Problem 3, page 92 of (13).
2. Problem 4, page 92 of (13).
3. Problem 9, page 92 of (13).
4. Problem 17, page 93 of (13).
5. Show that $6 \mid (n)(n+1)(2n+1)$
6. Show that $2 \mid 6n^3 + 3n^2 - n$
7. Show that $2 \mid 5n^2 - 3n$
8. Show that $4 \mid n^4 + 6n^3 + 11n^2 + 6n$
9. Show that $3 \mid 4n^3 - n$
10. Show that $3 \mid n^3 + 6n^2 + 2n$
11. Show that $3 \mid n^3 + 3n^2 + 5n$
12. Show that $3 \mid n^3 - 3n^2 + 8n$
13. Show that $4 \mid n^4 + 2n^3 = 15n^2 - 10n$
14. Show that $5 \mid n^5 - 5n^3 + 4n$

15. Show that $3 \mid n^3 - 6n^2 + 17n$

16. Show that $5 \mid n^5 + 9n$

17. Show that $5 \mid n^5 + 15n^3 - 6n$

18. Show that $6 \mid (-n)(-n+1)(-n+2)$

19. Show that $6 \mid (n)(n+1)(n+2)$

20. Show that $24 \mid (n)(n-1)(n^2-5n+18)$

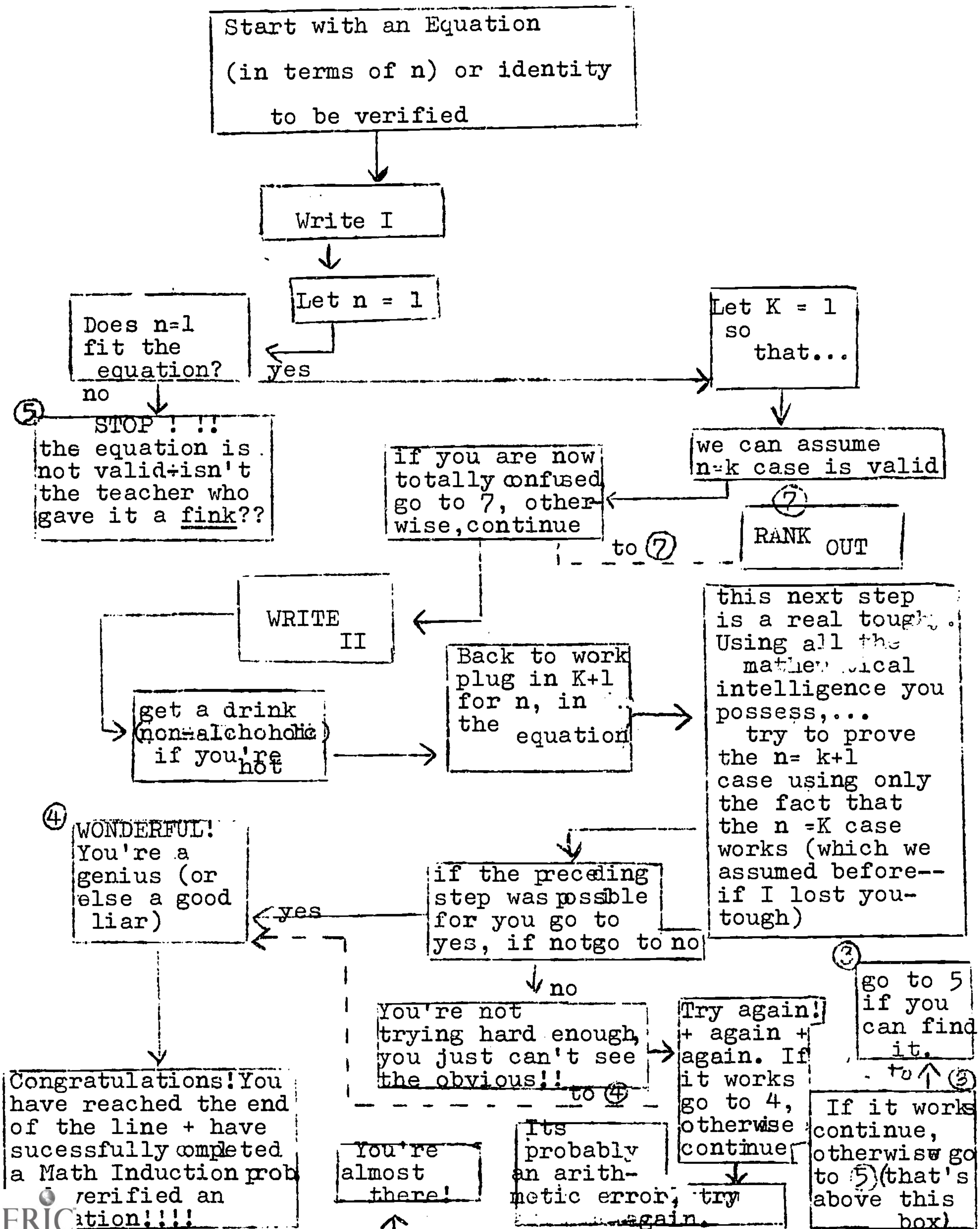
21. Show that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$

<u>C G III</u>:

1. Problem 10, page 92 of (13).
2. Show: $4 \mid (5^n - 1)$
3. Show: $7 \mid (8^n - 1)$
4. Show: $3 \mid (37^n - 34^n)$
5. Show $7 \mid (6^{2n} - 1)$
6. Show: $4 \mid (3^{2n} - 1)$
7. Show: $6 \mid (5^{2n} - 1)$
8. Show: $13 \mid (8^{2n} - 5^{2n})$
9. Show: $7 \mid (5^{2n} - 2^{2n})$
10. Show: $15 \mid (11^{2n} - 4^{2n})$
11. Show: $4 \mid (2^{n+2} + 5^n - 5)$
12. Show: $8 \mid (3^{2n+1} + 5^{2n+1})$
13. Show: $7 \mid (2^{2n+1} + 5^{2n+1})$
14. Show: $133 \mid (11^{n+2} + 12^{2n+1})$
15. Show: $57 \mid (7^{n+2} + 8^{2n+1})$
16. Show: $31 \mid (5^{n+2} + 6^{2n+1})$
17. Show: $13 \mid (3^{n+2} + 4^{2n+1})$
18. Show: $91 \mid (9^{n+2} + 10^{2n+1})$
19. Show: $(x + y) \mid (x^{2n+1} + y^{2n+1})$
20. Show: $(x + y) \mid (x^{2n} - y^{2n})$
21. Show: $(x - y) \mid (x^n - y^n)$

Note: For 20 and 21, $x>y$. For all three (19, 20 and 21), $x, y \in I^+$.

Section 1.4 The Iris Schoenberg Flow Chart



Section 2: Conquering the Conjectures

Outline of Section 2

The contents of this section include:

1. Indications, both specific and general, of where all the formulae, conjectures and other relationships found in Section 1 came from.
2. Factorization galore.
3. Other insights.

Section 2.1 Where Did Those Blankety-Blank Summation Formulae Come From ?

The immediate answer to the above question for many of the examples given in Section 1.2 is "Out of a text book, dopey!" However, it is not the intent of this section to merely list a set of textbooks (that's been done already in Section 6.6 of Chapter T1; one bibliography should be enough). The actual question is, "Where did the authors of these text books get their problems?"; and if you answer that they stole them from somebody else's textbook, you're probably right. Of course this process can't go on ad infinitum; it had to start somewhere. If you guess that the originals were found on the back of the tablets on which the Ten Commandments were written, you're probably right!

Actually, Chapter 3 of (6) is loaded with schemes which mathematicians have used over the centuries to "Observe, Explore and Discover" some of these relations. In fact Polya (author of (6)) does an extensive treatment on how Pascal obtained the formulae of CG III, Section 1.2. These formulae were used by Pascal to compute the areas of sections of his cycloid curve (remember the toothache?), and later on by Newton to discover the Calculus. However, I will give you a more general approach than Pascal's (as found in Polya's book).

But first let's look at the conjectures of CGI, Section 1.2. All 14 of these formulae are derived from the same basis: they are all arithmetic progressions. If you expand the LHS of any of these expressions you will have an arithmetic progression; the RHS is merely the formula for the sum of an arithmetic progression for the specific case involved. (Remember we derived that formula in the answer to problem 2.5.2 of Chapter T1; first plus last times number of terms divided by two).

For instance, problem 12 of CGI, section 1.2, says that $\sum_{i=1}^{n} (12i - 11) = 6n^2 - 5n$. The first term of the summation is 12 - 11 or 1 while the last term is 12n - 11. Since there are n terms, the summation formula gives $(1 + 12n - 11)(n/2)$, or $(12n - 10)(n/2) = 6n^2 - 5n$. Simple, eh? Check out a few more.

The problems found in CGII of section 1.2 have a very similar background. Conjectures 1, 2 and 5 are merely the sum of a geometric progression; conjectures 3,4 and 6 are merely doubles. That is to say, 3,4 and 6 were manufactured by taking two geometric progressions and summing them together. If you've forgotten the formula for the summation of a geometric progression, just look at problem 5 of CGII, section 1.2. Of course I know you've forgotten (who ever learned it?) the derivation of the formula, but I've got a sophisticated derivation coming up in section 2.2 of this chapter. Not only will you forget the formula now but you'll have a derivation you won't understand either.

And now on to CGIII; these are beauties! All seven conjectures in this group can be found from the same general process (as can all the conjectures of CG I, section 1,2). Let's look at a specific case and work it out. But remember, the procedure is general.

The first thing to do is to form a sequence of partial sums; a wha?

Well, let's look at conjecture 6 which says:

$\sum_{i=1}^{n} (2i - 1)^3 = 2n^4 - n^2$. Now a sequence is just a listing of numbers; partial sums refers to the fact that I am going to let n on the sigma-sign get progressively larger, and I am going to take a sum each time I change the n.

Watch. For n = 1, $\sum_{i=1}^{1} (2i-1)^3 = 1$; for n = 2, $\sum_{i=1}^{2} (2i-1)^3 = 1^3 + 3^3 = 28$; for n = 3, $\sum_{i=1}^{3} (2i-1)^3 = 1^3 + 3^3 + 5^3 = 153$ (my favorite number); likewise, for n = 4, the partial sum is 496; n = 5 yields the number 1225; n = 6 yields 2556 (you need 11^3 toobtain that number; anybody know how to get 11^3 the easy way?); n = 7 yields 4753 and n = 8 yields 8128. The sequence of partial sums that I tried to define above is the set of numbers $\{1, 28, 153, 496, 1225, 2556, 4753, 8128, \ldots\}$. This sequence can extend indefinitely since n can increase without bound.

Okay. Thus far we have completed Step 1: we have formed the first few terms of a sequence of partial sums utilizing the LHS of our conjecture. Now, take <u>differences</u> of these numbers until you get a sequence of constants --- that is, a sequence made up of all the same numbers. Watch: the given sequence is 1 28 153 496 1225 2556 4753 8128 ...

The first "difference" sequence is

27 125 343 729 1331 2197 3375 ...

That is, 28 - 1 = 27, 153 - 28 = 125, 496 - 153 = 343, etc.
The fact that the "difference Sequence" consists of perfect cubes should not be surprising simce that's how we formed the sequence of partial sums in the first place!

Now, the second "difference" sequence is

98 218 386 602 866 1178 ...

The third "difference" sequence is

120 168 216 264 312 ...

The fourth "difference" sequence is

48 48 48 48 ...

Hurray, the constant sequence finally arrived! And since it took us four tries to obtain the constant difference sequence, we know that the sequence of partial sums is an arithmetic sequence of order 4. The definition of an arithmetic sequence of order 4 is given in what we did above; we had to form 4 difference sequences before we came up with the constant sequence; we therefore call the original sequence of partial sums an arithmetic sequence of order 4. (A slight variation of this definition, where a recursive scheme is used, is given on page 487, number 5 of (3), as was mentioned in the answer to problem 1.5.9, Chapter T1.) By the way, please notice that the scheme of taking differences can be reversed so that additional terms of the original sequence can be obtained without cubing a number. Try it and see.

Meanwhile, back at the conjecture. In step 1 we formed a sequence of partial sums; in step 2 we formed "difference" sequences in order to find out that our sequence of partial sums is an arithmetic sequence of order 4. WE can now assume (and be certain of its existence) that there is a polynomial of degree 4 which will produce the numbers of the sequence of partial sums when it is evaluated at the various n's. In other words, we now know that

$$\sum_{i=1}^{n} (2i-1)^3 = an^4 + bn^3 + cn^2 + dn + e$$

for some particular values of a,b,c,d and e.

At this point you've got to be saying, "Where'd that polynomial in n come from?" Well, my answer is that I was all along going to assume the existence of some polynomial function that would give me a means for computing elements in the sequence of partial sums, but I just didn't know what degree to choose! After all, why not?

Now the proof that an arithmetic sequence of partial sums of order 4 can be "described" by a polynomial of degree 4 depends upon a theorem of the Calculus which says that "If the derivatives of any two functions are equal, then the functions differ by at most a constant". Actually what we've used in the above example is an extension of this theorem: if the 4th derivatives are equal, then the 3rd derivatives differ by at most a constant. For those of you who have not studied calculus, this means nothing. For those of you who have, the above is not meant to be a clear cut proof; let's see you observe, explore and discover. One hint: try the converse first.

At any rate, since Pascal preceded the Calculus, we can see why he didn't use these techniques that we're developing here.

Now in step 3 we have assumed that

$\sum_{i=1}^{n} (2i-1)^3 = an^4 + bn^3 + cn^2 + dn + e$. If we do indeed believe the validity of the above outlined proof, we only have one task left. And that's to find the values for a, b, c, d and e. That's all.

Trivial. Absolutely trivial. Just watch. Since the relationship is to hold for all n, certainly it will hold for two "specifics", say k and k+1. Plug k+1 in first and then k, obtaining the following two conditional relationships:

1.) $\sum_{i=1}^{k+1} (2i-1)^3 = a(k+1)^4 + b(k+1)^3 + c(k+1)^2 + d(k+1) + e$

2.) $\sum_{i=1}^{k} (2i-1)^3 = ak^4 + bk^3 + ck^2 + dk + e$.

Now, if we subtract 2.) from 1.), the LHS becomes $(2(k+1)-1)^3$ (pleeeease check it out ---- carefully), while the RHS becomes a mess. First simplify and expand the LHS getting $(2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$; and then take a closer look at the RHS. Because we are so familiar with the binomial expansion, there is really no mess at all. For instance we immediately see $a(k+1)^4$ as $a(k^4 + 4k^3 + 6k^2 + 4k + 1)$, or $ak^4 + 4ak^3 + 6ak^2 + 4ak + a$. Likewise, $b(k+1)^3 = bk^3 + 3bk^2 + 3bk + b$, $c(k+1)^2 = ck^2 + 2ck + c$ and $d(k+1) = dk + d$. If as you perform the subtraction on the RHS you also gather like terms, the RHS will look like this:

$4ak^3 + (6a + 3b)k^2 + (4a + 3b + 2c)k + (a + b + c + d)$.

Therefore, when 2.) is subtracted from 1.), the result is:

3.) $8k^3 + 12k^2 + 6k + 1 = 4ak^3 + (6a+3b)k^2 + (4a+3b+2c)k + (a+b+c+d)$.

It is extremely necessary to keep in mind that this is a conditional statement. We are trying to find values for a,b,c and d such that 3.) will be true for all k, and therefore 2.) and 1.) also. We must now manufacture values for a,b,c and d to make 3.) true; and in case you haven't noticed, we have the perfect set-up.

Let $4a = 8$; i.e., set the coefficients of the k^3-terms equal. This means that a must have a value of 2. Now what happens if we equate the coefficients of the k^2-terms? We would have $12 = 6a + 3b$; but we already have $a = 2$; and therefore we should choose $b = 0$ so that equality will hold. Do you see the recursiveness involved? It's beautiful!

For instance, to find c we merely equate the coefficients of the k-terms; $6 = 4a + 3b + 2c$. Since we already know a and b, $c = -1$ pops right out. And equating the constant terms, we obtain $d = 0$ since $1 = a + b + c + d$, and $a = 2$, $b = 0$ and $c = -1$ are already known. We have therefore manufactured values for a,b,c and d which make 3.) true; but these values will make 2.) true also, and we therefore have obtained an expression for the LHS of conjecture 6:

$\sum_{i=1}^{n} (2i-1)^3 = an^4 + bn^3 + cn^2 + dn + e$ can be specifically written as $\sum_{i=1}^{n} (2i-1)^3 = 2n^4 - n^2$. You have already verified this, so you know it is correct. (What happened to the e? It didn't bother you, so you don't bother it!)

This process, outlined below, was used to find all of the seven relationships of CG III, section 1.2. Make sure you try this process or you will have wasted all the time you spent reading this! Here's what I did:

Step 1: Set up a sequence of partial sums.

Step 2: Find the order m of the sequence of partial sums.

Step 3: Assume the existence of a polynomial of degree m.

Step 4: Evaluate the conditional relationship between the summation formula and the polynomial at k+1 and k.

Step 5: Subtract: simplify, expand and gather like terms.

Step 6: Equate coefficients of like terms and solve for the constants recursively.

Step 7: Substitute the manufactured values of Step 6 into the assumed polynomial of Step 3 and check out the relationship.

The above process is very general and can be used again and again in many areas! Did you notice the use of the binomial expansion and the recursiveness concept? It's a good thing we know all about them, right?

The next conjecture group (CG IV) of section 1.2 is made up of all unique relationships. Number 3 merely depends on the fact that $1/((i)(i+1)) = 1/i - 1/(i+1)$; with that hint the RHS is easy to find. The basis for number 5 is found on page 522 of (15), and is pretty unique. Polya does some work on number 1 in Chapter 3 of his book; but I haven't got the faintest idea how to attack #2 or #6. But I'm working on them.

Of course the relationship exhibited in the one problem that constitutes CG V is not derivable because it is incorrect!!! If you verified that it does hold, it's because you failed to check out the $n = 1$ case. No, 2 does not equal 4; the $n = k$ case does indeed imply the $n = k+1$ case, but you've got "no leg to stand on" (the $n = 1$ case), so the entire verification crashes to the ground!

Section 2.2 From Divisibility to Factoring

In the previous section I showed you some techniques which enable you to find summation formulae for any arithmetic sequence of order m. Now we are going to find out where those 42 divisibility conjectures of section 1.3 sprung from.

The conjectures making up CG I, section 1.3, have two sources: the one is trivial, the other is unique. For the trivial cases all I've done is go to the summation problems of section 1.2; on the RHS of these relationships are found polynomials in n being divided by integers. Now certainly since the LHS of these expressions were merely sums and products of integers, and therefore themselves some integer, so the RHS had to be an integer for all values of n. And so any number found in the denominator must divide the numerator.

Examples of this type include number 7, which is just number 4 of CG I, section 1.2, revisited. Likewise, number 9 is just number 5 of CG III, section 1.2, revisited. And other problems of section 1.2 have been revisited in section 1.3 too.

Problem 20 of CG I, section 1.3, is an ald friend too, but was not encountered in Section 1.2. The expression $(n)(n-1)(n^2-5n+18)$ was encountered in the answer to problem 2.5.1 of Chapter T1. (That was where I gave the example of "too quick" induction; the number of points p on the circumference of a circle seemed to yield 2^{p-1} regions inside the circle. The derivation of the formula $1+(p(p-1)(p^2-5p+18))(1/24)$ can be done by following the steps outlined in section 2.1, Chapter T2, above.)

You were asked to verify that $24 \mid (n)(n-1)(n^2-5n+18)$; a very difficult problem to say the least. There are a couple of ways (at least) of doing that problem, but I use what I call the Principle of Triple Math Induction; you just keep going until you get where you want to get.

Now I know that what I just said is more than a bit mystifying (it's maddening, I'm sure), but see what you can do with those hints anyway.

There is another clever way of showing that $24 \mid (n)(n-1)(n^2-5n+18)$ however. This procedure is due to the initiative exhibited by the father of Gary Squire, a Syosset student. Look at the expression $n^2 - 5n + 18$: it doesn't factor (over the integral domain). But suppose we re-write it as $n^2 - 5n + 6 + 12$ and now look at only the first three terms which do factor; namely, $(n-2)(n-3) + 12$. Now multiply the revised $n^2-5n+18$ expression by n and n-1. The result will consist of two terms, the first consisting of 4 factors, the second of 3: $(n)(n-1)(n-2)(n-3) + (n)(n-1)(12)$. For n = 1, 2, 3 and 4, this expression is obviously divisible by 24 (check it out!), but what happens after that? Well, the first term consists of the product of 4 <u>consecutive</u> integers; and 4 consecutive integers always contain factors of 2, 3 and 4. (This can be readily proven by the division algorithm within the topic called congruences of numbers.) Likewise, the first two factors of the second term are consecutive integers and therefore one of them must be even. And an even number times 12 must be divisible by 24. Thus the entire expression is divisible by 24.

This technique by the way was what I essentially used in making up most of the conjectures of CG I, section 1.3.

For instance, the product of any 3 consecutive integers, say $(n-1)(n)(n+1)$, must be divisible by 3. Therefore n^3-n is divisible by 3. But so also is n^3-n+3n^2 divisible by 3, since the term $3n^2$ will always be divisible by 3. And so also will $n^3-n+3n = n^3+2n$ be divisible by 3; the first two terms are divisible by three and the last also, and therefore when combined the expression will still be divisible by 3.

As you can see I could have made up more than a million such problems instead of just 21, but as you already know I'm a very nice guy. But look at problems 15, 16 and 17 of CG I, section 1.3. They are "divisibility by 5" problems. Problem 15 was obtained by multiplying $(n-2)(n-1)(n)(n+1)(n+2)$ together; do you see how the problems 16 and 17 were found? If you do, make up one of your own and verify it using the PMI.

In CG II, section 1.3, problems 1 - 13 (exclusive of #11 which is a steal from problem 4, CG II, section 1.2) and 19 - 21 are all based on the same principle. And that principle is essentially the factorization of the expression $x^n - y^n$ and related factorizations.

Look at $x^5 - y^5$: this can be factored into

A.) $(x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$. To verify this factorization we'll use the hiccup distributive law. Referring to A.), it goes like this:

1. First multiply x times x^4 getting x^5, and that takes care of that.
2. Now for the hiccups: multiply x times x^3y getting x^4y, and then hiccup back to the $-y$ and mulciply it times the x^4 term; this gives $-x^4y$ and the sum total of before and after the hiccup is zero.

3. Multiply x times x^2y^2 getting x^3y^2; hiccup back and multiply $-y$ times x^3y getting $-x^3y^2$; sum total is zero.

4. Every x multiplication followed by a hiccup $-y$ multiplication gives a zero except for the last hiccup; that would be $-y$ times y^4, giving a product of $-y^5$, and Q.E.D. (Or la-de-da, take your choice.)

Likewise, the factorization of $x^n - y^n$ can succinctly be defined as: $x^n - y^n = (x-y)(\sum_{i=0}^{n-1} x^{n-1-i}y^i)$. (Boy, that sigma-sign notation is handy to have around!)

Problem 2 of CG II, section 1.3, says that 4 5^n-1. Well if x = 5 and y = 1 in the above factorization, then certainly (5-1) (5^n-1^n). Likewise, 3 $(37^n - 34^n)$ since x = 37, y = 34 and x-y = 3.

Now the above factorization holds for all n, and therefore we might expect the expression $6^{2n} - 1$ to be divisible by 5 even though the exponent has been restricted to even integers. And indeed it is by previous observations. But furthermore, $6^{2n} - 1$ is divisible by 7, as you've already verified in problem 5 of CG II, section 1.3. This would seem to indicate that $x^{2n} - y^{2n}$ has a factor of x+y as well as a factor of x-y. Let's check this out.

First let's look at the case where n is odd and observe, explore and discover. For instance, $x^5-y^5 = (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$. The second factor has 5 terms, and that's that as far as factorization over the rationals is concerned! (There is an extremely complex (?) technique for factoring the second factor over the complex field.) If the second factor had an even number of terms (when will this occur?), then it uld be factored due to the symmetry of the exponents. Let's look at one: $x^6-y^6 = (x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)$. Lurking within those six terms is a common (binomial) factor; we will use the key idea of partial factoring (again it makes an appearance!) to ferret out the lurking (binomial) factor. Inspect the first two terms, the middle two and the last two terms; $\underline{x^5+x^4y} + \underline{x^3y^2+x^2y^3} + \underline{xy^4+y^5}$. Partial factor, getting: $x^4(x+y) + x^2y^2(x+y) + y^4(x+y)$. And now we can see the binomial factor that had been lurking there all the time; namely, $(x+y)$. Finally, x^6-y^6 can be factored thusly:

1.) $x^6-y^6 = (x-y)(x+y)(x^4 + x^2y^2 + y^4)$.

Did I say finally? Why there's nothing final about that factorization at all! By the use of a propitious zero we can readily factor $x^4 + x^2y^2 + y^4$; just introduce the terms $x^2y^2 - x^2y^2$. Then $x^4 + x^2y^2 + y^4$ becomes

2.) $x^4 + 2x^2y^2 + y^4 - x^2y^2$. But what do you recognize about the first three terms of that expression? See the 1, 2, 1? Therefore 2.) can be written as $(x^2 + y^2)^2 - (xy)^2$. But now we have the difference of two squares; and 2.) can be factored into $(x^2+y^2-xy)(x^2+y^2+xy)$. And finally (over the rationals, anyway) $x^6 - y^6 = (x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$.

Of course if you try the propitious zero stunt on either of the last two factors you'll run into quick trouble (even though they do indeed look ripe!). However, by all means give it a try. In case you haven't figured it out yet, the propitious zero stunt consists of introducing a zero such that the given expression can be "made into" the difference of two perfect squares. Here's another example: factor x^4+10x^2+49. Merely add the propitious zero $4x^2-4x^2$, obtaining the expression $x^4+14x^2+49 - 4x^2$. Since $x^4+14x^2+49 = (x^2+7)^2$, the factorization is immediate. Namely, $(x^2-2x+7)(x^2+2x+7)$.

I've introduced two ideas here: lurking factors and propitious zeros. How good a lurker are you (it usually takes minutes of hard training)? Let's go back and look at 1.): $x^6 - y^6 = (x-y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)$. Above we found the factor $(x+y)$ lurking within those 6 terms. But there is another larger factor (tri-nomial) lurking within those six terms; to find it we need only change our point of view.

Look at the first three terms and the second three terms of $\underline{x^5+x^4y+x^3y^2} + \underline{x^2y^3+xy^4+y^5}$. Use partial factoring on each of these sets of 3 terms; you should get $x^3(x^2+xy+y^2)+y^3(x^2+xy+y^2)$. But this yields the common (tri-nomial) factor x^2+xy+y^2; and $x^6 - y^6 = (x-y)(x^2+xy+y^2)(x^3+y^3)$. But we know from above that $x+y$ must be a factor of x^6-y^6; and from what we just found it must be a factor of x^3+y^3. But we don't know how to factor the sum of two terms!

Let's not panic, however, as we do have some experience in this area. Suppose we give a look-see at the hiccup distributive law; it could help us. We know that x^3+y^3 has a factor of $x+y$ by virtue of all that we did above. We also know its second factor; namely, x^2-xy+y^2. Does the hiccup distributive law verify this factorization? Does $(x+y)(x^2-xy+y^2) = x^3+y^3$? (It must! —— or I'm in dire trouble.) Well, x times x^2 yields the x^3 term; now for the hiccups. x times $-xy$ gives $-x^2y$, and right behind that we have y times x^2 or x^2y; the sum is zero. Likewise, x times y^2 gives xy^2 but this is hiccupped by the product of y and $-xy$. And then y times y^2 yields the y^3 term. Beautiful!

The above is a short outline of when $x^n + y^n$ will have a factor of x+y; you should observe that $x^n + y^n$ will only have a factor of x+y when n is an odd positive integer. (You should be able to verify why this is so upon inspection of the number of terms in the second factor of $x^{2n+1} + y^{2n+1}$; remember, 2n+1 represents an odd integer). Therefore, we write the factorization thusly:

$x^{2n+1} + y^{2n+1} = (x+y)(\sum_{i=0}^{2n} (-1)^i x^{2n-i} y^i)$. Please verify this also.

And now we can see why 8 $(3^{2n+1} + 5^{2n+1})$ (problem 12, CG II, section 1.3); since (x+y) $x^{2n+1} + y^{2n+1}$, for x = 3 and y = 5, the problem is immediate.

I have therefore justified all the conjectures of CG II, section 1.3, except problems 14 - 18. Two of these were found in (15), but no reason was given. I've guessed the generating process and you've verified all 5, so we know that the expression $x^{n+2} + (x+1)^{2n+1}$ has a factor of x^2+x+1, but I don't know how to justify or produce that factorization. But I'm working on it! Care to join me?

Before I give you some factoring problems to look at, I want to investigate two additional uses for the factorization of $x^n - y^n$.

The standard derivation found in all the textbooks for the sum of the terms of a geometric progression can be replaced by a simple observation relative to the factorization of $x^n - y^n$. The sum of a geometric progression is expressed as $a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}$ or $a(1+r+r^2+\ldots+r^{n-2}+r^{n-1})$. The expression in parentheses however is nothing more than the second factor of $x^n - y^n$ with x = 1 and y = r; i.e., $x^n-y^n = (x-y)(x^{n-1}+x^{n-2}y+\ldots+xy^{n-2}+y^{n-1})$ becomes upon substitution of 1 for x and r for y, $1-r^n = (1-r)(1+r+\ldots+r^{n-2}+r^{n-1})$. Therefore, the sum of a geometric progression is readily seen to be $(a)((1-r^n)/(1-r))$, and where the a is thrown in just for good luck.

And now the second use: suppose I'd like to factor (x-y) into two factors which will have rational exponents. (The reverse problem is of primary importance in finding a derivative formula for rational exponents in the calculus.) For instance, I want $(x-y) = (x^{\frac{1}{4}} - y^{\frac{1}{4}})$ times "something else". The nature of the "something else" is immediate; namely, $x^{3/4} + x^{2/4}y^{1/4} + x^{1/4}y^{2/4} + y^{3/4}$. What I've done is factor $A^4 - B^4$ where $A = x^{1/4}$ and $B = y^{1/4}$; i.e., $(x^{1/4})^4 - (y^{1/4})^4 = (x^{1/4} - y^{1/4})((x^{1/4})^3 + (x^{1/4})^2(y^{1/4}) + (x^{1/4})(y^{1/4})^2 + (y^{1/4})^3)$. Neat, eh?

And now for some factoring problems. See your local Math teacher for answers to all of these problems.

Lurkers Plus

1. $x^2 + 3x + ax + 3a$
2. $x^4 + 3x^3 + 8x + 32$
3. $3ar + 3br - a - b$
4. $xm - xn + ym - yn$
5. $9x^3 + 9x^2 - x - 1$
6. $x^2y^2 - y^2 - x^2 + 1$

Hint: Keep going in #2 and #5.

$x^n - y^n$

1. $x^4 - y^4$
2. $(x-y)^4 - (x+y)^4$
3. $x^3 + (x-1)^3$

More $x^n - y^n$

4. $n^3 - (n+1)^3$
5. $x^6 - y^6$
6. $x^6 + y^6$
7. $x^9 - y^9$
8. $x^9 + y^9$
9. $x^{12} - y^{12}$

Propitious ZERO

1. $4a^2 - b^2 + 12ac + 9c^2$
2. $x^2 + y^2 - 25 + 2xy$
3. $x^4 + x^2 + 1$
4. $x^2 - 4y^2 + a^2 - 9b^2 - 2ax - 12by$

More Propitious Zero

5. $a^4 - 5a^2b^2 + 4b^2$

6. $x^4 - 14x^2 + 49$

7. $x^4 + 12x^2 + 64$

8. $x^4 - 18x^2 + 49$

Challenges

1. $x^4 - 27x$

2. $x^4 - x^3y + 64x - 64y$

3. $4(x^2-9) - 2x^2 - 5x + 3$

4. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$

5. $2a^2m^4 - 2abm^3 + 2abm - 2b^2$

Section 2.3 Summary and Enrichment

Just a few comments. Sections 2.1 and 2.2 are loaded with algebraic manipulations and a number of very sophisticated ideas. If you can master the calculations and concepts in those sections, there is very little algebra that will ever give you any trouble in the future. Those two sections contain the tools for handling many, many topics of mathematical analysis; master them and you will enjoy much of the mathematics in your future. On the contrary, you can't learn concepts if you keep tripping over the tools.

Two suggested enrichment topics: Liebnitz, the c0-discoverer of the calculus, invented another arithmetic triangle about fifty years after Pascal's death. It contains only fractions and has a "reverse" generation process relative to Pascal's Triangle. This triangle is presented in Polya's book

(book (6) of the T1 bibliography) on page 88, along with many questions and observations. Polya gives his answers to these questions on page 185; they will amaze you!

In this chapter we learned about the PMI. The processes we used were pretty standard; however, for a unique, clever and exciting variation of the usage of the PMI see Edwin Beckenbach and Richard Bellman's An Introduction to Inequalities, Random House - New Mathematical Library, 1961, pages 54 - 61. The authors name part of their variation of the PMI "Backward Induction", which they utilize to prove that the arithmetic mean is greater than or equal to the geometric mean for any number of values. Since most of you are quite backward, you should enjoy the topic very much.

Section 4.1 Bibliography

Same as that of Chapter T1, found on pages 101 and 102 of that chapter.