DOCUMENT RESUME

ED 052 021 | SE 011 325

AUTHOR	Helwig, G. Alfred; Shepperd, Anna G.
TITLE	Review of Academic Mathematics, A Tentative Guide.
INSTITUTION	Baltimore County Public Schools, Towson, Md.
PUB DATE	Jan 67
NOTE	95p.
EDRS PRICE	EDRS Price MF-$0.65 HC-$3.29
DESCRIPTORS	*Algebra, *Course Content, *Geometry, Instructional Materials, Mathematical Concepts, Mathematics Education, Number Concepts, Secondary School Mathematics, *Teaching Guides

ABSTRACT

This teaching guide outlines a semester course for those students who need review work in concepts from algebra and geometry. Successful completion of this material would serve as a prerequisite to the study of trigonometry. Sequence, textbook references, and time allotments are suggested. Units studied are: the real numbers; operations on the real numbers; relations, functions, and graphs; first-degree equations; inequalities; quadratic equations; and logarithms. Appendices provide instruction in set concepts, properties of right triangles, relations, functions, and graphs. (RS)

BALTIMORE COUNTY PUBLIC SCHOOLS

A Tentative Guide

REVIEW OF ACADEMIC MATHEMATICS

TOWSON, MARYLAND

JANUARY, 1967

BALTIMORE COUNTY PUBLIC SCHOOLS

A Tentative Guide

REVIEW OF ACADEMIC MATHEMATICS

Prepared under the direction of

G. Alfred Helwig
Director of Curriculum

Anna G. Shepperd
Assistant Superintendent of Instruction

Committee Members

Robert Erey, Chairman
Eva High
W. Edwin Freeny

Stanley A. Smith
Supervisor of Senior High School Mathematics

Vincent Brant
Coordinator of Mathematics

William S. Sartorius
Superintendent

Towson, Maryland
1967

TABLE OF CONTENTS

FOREWORD

Review of Academic Mathematics is a first semester course in the RAM-Trig. sequence designed for those students who have had Algebra II and Geometry, but whose background is still not adequate for Trigonometry. This course emphasizes those concepts and skills prerequisite to the study of Trigonometry. Topics such as operations on algebraic quantities, functions, geometry, and logarithms are included.

The time allotment and sequence of topics are suggestive. The teacher may adjust these according to the needs of the students. Some topics require supplementary material which may be obtained from your department chairman.

Materials

CODE:

WKP — Basic Text: Welchons, Krichenberger, Pearson
Algebra, Book Two. New York: Ginn and Co. 1957

Supplementary Materials
(For distribution to students)

App I — Appendix I: Special Right Triangles

App II — Review of Elementary Set Concepts

SMSG — SMSG: Inequalities

VB — Relations, Functions and Graphs

References

A O — Allendoerfer and Oakley. Principles of Mathematics

VGF — Vanatta, et al. Algebra Two, A Modern Course, new edition

I A — Adler, Irving. The New Mathematics

D — Dolciani, Mary P. Modern Algebra Book I

DBW — Dolciani, Berman and Wooton. Modern Algebra Book II

SMSG: Geometry with Coordinates

JDD — Jurgensen, Donnelly and Dolciani. Modern Geometry: Structure and Method

R — Rosenbach, et al., Plane Trigonometry

H — Heineman. Plane Trigonometry With Tables

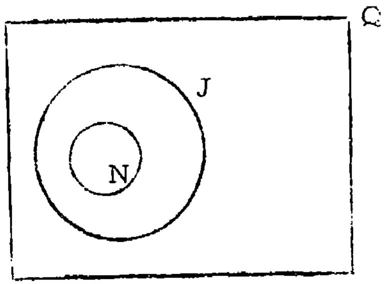
SCOPE	REFERENCES CODE	PAGES
UNIT I		
The Set of Real Numbers and its Subsets		
(Time Allotment 3 days)		
I. Introduction		
A. Historical background		
1. Development of numbers	A O	p. 39
2. Numerals	VGF	P. 5-6
II. Natural Numbers $N = \{1, 2, 3 \ldots\}$		
A. Subsets of natural numbers		
1. Unit set $\{1\}$		
2. Set of prime numbers $\{2, 3, 5, 7 \ldots\}$		
3. Set of composite numbers $\{4, 6, 8, 9, 10 \ldots\}$		
B. Symbols of grouping: $(\ )$, $[\ ]$, $\{\ \}$, $\overline{\quad}$		
$5 + \{3 + [8 + \overline{4(10) - 9}]\}$	VGF	p. 7
$5 + \{3 + [8 + \overline{40-9}]\}$	WKP	p. 26
$5 + \{3 + [8 + 31]\}$		
$5 + \{3 + 39\}$		
$5 + 42 = 47$		

SCOPE	REFERENCES CODE	PAGES
C. Order of Operations Example: Compute $4(3 + 5) - 24 \div 6 + 3 \cdot 7$ $= 4 \cdot 8 - 24 \div 6 + 3 \cdot 7$ $= 32 - 4 + 21$ $= 49$	WKP	p. 14
D. Properties of Natural Numbers (The teacher should use numerical examples to illustrate these properties.) 1. Order For different a and $b \in N$, $a > b$ or $b > a$ 2. Closure property of addition If a and b are natural numbers, then $a + b$ is a unique natural number 3. Closure property of multiplication If a and b are natural numbers, then ab is a unique natural number 4. Multiplicative identity property There is an identity element 1 for multiplication such that For all $a \in N$, $a \cdot 1 = 1 \cdot a = a$ 5. Commutative property of addition For all $a, b \in N$, $a + b = b + a$ 6. Commutative property of multiplication For all $a, b \in N$, $ab = ba$	VGF	p. 8-9

SCOPE	REFERENCES CODE	PAGES
7. Associative property of addition		
For all a , b , c $\in$ N , (a + b) + c = a + (b + c)		
8. Associative property of multiplication		
For all a , b , c $\in$ N , (ab)c = a(bc)		
9. Distributive property of multiplication over addition		
For all a , b , c $\in$ N , a(b+c) = ab + ac		
This property connects the two operations of multiplication and addition.		
III. Integers J = {. . . , -3, -2, -1, 0, 1, 2, 3, . . . }		
A. Properties of Integers (The teacher should use numerical examples to illustrate these properties.)	VGF	p. 10-11
1. Order	A O	p. 51
2. Closure property of addition		
3. Closure property of multiplication		
4. Additive identity property		
There is an identity element 0 for addition such that		
For all a $\in$ J , a + 0 = 0 + a = a		
5. Multiplicative identity property		
6. Commutative property of addition		
7. Commutative property of multiplication		

SCOPE	REFERENCES CODE	REFERENCES PAGES
8. Associative property of addition		
9. Associative property of multiplication		
10. Distributive property of multiplication over addition		
11. Additive inverse property		
For all $a \in J$ there is an additive inverse $(-a) \in J$ such that		
$a + (-a) = (-a) + a = 0$		
B. Special properties of zero		
1 For all $a \in J$, $a \cdot 0 = 0 \cdot a = 0$		
2. $ab = 0$ if and only if $a = 0$ or $b = 0$		
3. For all $a \in J$, $\frac{a}{0}$ is not defined		
IV. Rational Numbers. Let Q = set of rational numbers		
A. Definition		
1. A rational number is a number which may be expressed in the form		
$\frac{a}{b}$ where $a, b \in J$ and $b \neq 0$.	VGF	p. 22-25
2. Any rational number may be expressed as	WKP	p. 213-214
a. A terminating decimal; $\frac{1}{4} = .25$; $\frac{3}{8} = .375$; or		
b. An infinite repeating decimal; $\frac{1}{3} = .333\ldots$; $\frac{1}{3} = 0.\overline{3}$		
B. Summary of Properties of Rational Numbers		
1. All the properties of addition and multiplication of integers hold for the rational numbers.		

SCOPE	REFERENCES CODE | PAGES
2. Multiplicative inverse property For all $a \in Q$, $a \neq 0$, there is a multiplicative inverse $\frac{1}{a} \in Q$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$	
C. Venn Diagram For Natural Numbers, Rational Numbers, and Integers	



SCOPE	REFERENCES CODE | PAGES
V. Irrational Numbers. Let T = set of irrational numbers	WKP p. 214 VGF p. 30-33
A. Definition An irrational number is a number which may be expressed as a non-terminating, non-repeating decimal.	
B. Discovery credited to Pythagoras (circa 540 B. C.)	
C. Examples: $\sqrt{2}$, $\sqrt[3]{4}$, π, .12123123412345...	
VI. Real Numbers. Let R = the set of real numbers	
A. $R = Q \cup T$ and $Q \cap T = \emptyset$	

SCOPE	REFERENCES CODE	PAGES
B. Geometric interpretation of real numbers		
1. The number line	I A	p. 14
2. One to one correspondence between the set of real numbers and the set of points on a line.	VGF	p. 35
C. Classification of numbers (chart)	A O	p. 56, 65
D. Venn Diagrams	VGF	p. 36-37
E. Properties of Real Numbers (The teacher should use numerical examples to illustrate properties.) All the properties of addition and multiplication of rational numbers also hold for the real numbers.	JDD	p. 99-100
VII. Review		
VIII. Test		

SCOPE	REFERENCES CODE	PAGES
UNIT II Fundamental Operations of Real Numbers (Time Allotment 15 days)		
I. Addition and Subtraction The teacher should review the addition and subtraction of whole numbers, integers, and polynomials through daily drills.	WKP	p. 17-18 p. 35 Ex. 10-15
II. Laws of Rational Exponents for Real Numbers (The teacher should develop these laws using numerical exponents.) A. $x^m \cdot x^n = x^{m+n}$	WKP	p. 203
B. $\frac{x^m}{x^n} = x^{m-n}$, where $x \neq 0$		
C. $x^0 = 1$, where $x \neq 0$. Emphasize that this statement is a definition.		
D. $x^{-m} = \frac{1}{x^m}$, where $x \neq 0$. Emphasize that this statement is a definition.		
E. $(x^m)^n = x^{mn}$		
F. $(xy)^n = x^n y^n$		
G. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$, where $y \neq 0$	WKP	p. 204

SCOPE	REFERENCES CODE : PAGES
H. For all $x \in R$ and odd positive $b \in J$ For all real $x \geq 0$ and even positive $b \in J$ } $x^{\frac{1}{b}} = \sqrt[b]{x}$ For all real $x < 0$ and even positive $b \in J$, $x^{\frac{1}{b}}$ is not defined (does not exist)	
III. Laws of Radicals for Real Numbers x and y and Positive Integers m and n	WKP p. 216-223
A. For all $x \geq 0$, $(\sqrt[n]{x})^n = x$	
B. For all $x \geq 0$, $\sqrt[n]{x^n} = x$ For all $x < 0$, and positive even integer n, $\sqrt[n]{x^n} = \|x\|$ In particular $\sqrt{x^2} = \|x\|$. Stress that the symbol "$\sqrt[n]{\ }$" (where n is even) means a non-negative integer. For all $x < 0$ and positive odd integer n, $\sqrt[n]{x^n} = x$ $\sqrt[3]{(-5)^3} = -5$	
C. For all x, $y \geq 0$, $(\sqrt[n]{x^m}) = (\sqrt[n]{x})^m$	
D. For all x, $y \geq 0$, $\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$	
E. For all x, $y \geq 0$, $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$	

SCOPE	REFERENCES CODE	PAGES
F. For all $x \geq 0$, $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$		
IV. Products of Polynomials (The teacher should obtain student copies of the PRODUCTS AND FACTORING sheet from the department chairman.)	WKP	p. 19-23
A. Product of two monomials		
B. Product of a monomial and a polynomial		
C. Products of two polynomials		
V. Division	WKP	p. 23
A. Division of a polynomial by a monomial		
1. Include problems with integral and rational exponents.		
2. Emphasize the distributive principle in simplifying division problems.		
3. Stress the need for stating the restrictions on the values of the variable in the denominator in order that the rational expression may be properly defined.		
4. Avoid the use of the term "cancel." Instead refer to "$\frac{2a}{2a}$, $a \neq 0$" as just "another name for 1, the multiplicative identity." $\frac{2a^2 - 8a}{4a}$, $a \neq 0$ $= \frac{2a(a-4)}{2a \cdot a} = \frac{a-4}{a}$		

SCOPE	REFERENCES CODE	PAGES
5. Stress the reason for "inverting and multiplying"	WKP	p. 25
$\frac{3}{4} \div \frac{7}{8} = \frac{\frac{3}{4}}{\frac{7}{8}} = \frac{\frac{3}{4}}{\frac{7}{8}} \cdot \frac{\frac{8}{7}}{\frac{8}{7}} = \frac{3}{4} \cdot \frac{8}{7} = \frac{6}{7}$		
B. Division of a polynomial by a polynomial	WKP	p. 25
C. Synthetic division	WKP	p. 524
D. Rationalizing a radical denominator	WKP	p. 222
VI. Factoring	WKP	p. 71-84
Students should be able to identify the types of polynomials as a necessary condition for factoring. Use the student sheets on PRODUCTS AND FACTORING.		
A. Common monomial factoring		
B. Common binomial factoring		
C. General trinomial factoring		
D. Perfect square trinomial		
E. Difference of two squares factoring		
F. Sum and difference of cubes factoring		
G. Polynomial factoring involving the Remainder and Factor Theorems		

SCOPE	REFERENCES	
	CODE	PAGES
VII. Operations on Fractions Involving Factoring	WKP	p. 104-110
VIII. Complex Fractions	WKP	p. 111
IX. Complex Numbers - an extension of the real number system	WKP	p. 228-233
A. The need for extending the real numbers in order to satisfy the equation $x^2 = -1$		
B. The imaginary unit i and its powers		
C. Graphing imaginary numbers		
D. Operations with imaginary numbers		
E. Complex numbers		
F. Graphing complex numbers		
G. Operations with complex numbers		
H. Graphic addition and subtraction of complex numbers		
X. Review		
XI. Test		

SCOPE	REFERENCES CODE	PAGES
UNIT III Relations, Functions, and Graphs (Time Allotment 10 days)		
Teachers should obtain student copies of this unit from their Department Chairman.		
I. Introduction	VB	p. 1-3
II. The Use of Sentences in Mathematics		
III. Truth Values of Sentences and Relations		
IV. Binary Relations		
V. Procedure in Investigating Relations	VB	p. 3-6
A. Test question for determining true or false sentences		
B. Necessity for clearly defined sentences		
C. Abbreviation of defining sentence		
D. Order of elements in a pair		
E. Summary of the features of binary relations		
VI. Relations and Sets	VB	p. 6-9
A. The link between relations and sets		
B. The elements of a truth set		
VII. The Cartesian Product of Two Sets	VB	p. 9-13
A. Introduction		
B. Definition of Cartesian product of two sets		

SCOPE	REFERENCES CODE	PAGES
VIII. Graphs of the Cartesian Products to be Used in Sentences	VB	p. 13-16
A. Introduction		
B. Natural numbers		
C. Integers		
D. Rational numbers		
E. Real numbers		
IX. Sentences in Two Variables	VB	p. 17-21
A. Solution set of sentences in two variables		
B. Domain and range of a relation		
C. Functions		
X. Sentences Expressing Equality	VB	p. 22-26
XI. Inequalities in Sentences in Two Variables	VB	p. 26-35
A. Introduction		
B. Example 1. To find the solution set and graph of: $\{(x,y) \mid y > x\}$ when $U = \{1, 2, 3, 4\}$		
C. Example 2. To find the solution set and graph of: $\{(x,y) \mid y > x\}$ when $U = \{-2, -1, 0, +1, +2\}$		
D. Example 3. To find the solution set and graph of: $\{(x,y) \mid y > x\}$ when U = the set of real numbers.		

SCOPE	REFERENCES CODE	PAGES
E. The absolute value of real numbers		
F. Sentences involving inequalities with absolute values		
G. Sentences involving variables of the second degree		
XII. Functional Notation	VB	p. 36-38
A. $F = \{(x, y) \mid y = x - 1\}$ or $G = \{(x, y) \mid y = x^2\}$ $f = \{(x, y) \mid y = 2x\}$ or $g = \{(x, y) \mid y = x^2\}$		
B. $f(x)$ or $g(x)$		
C. The function can therefore be represented in different forms: $f = \{(x, y) \mid y = x^2\}$ $f = \{(x, x^2)\}$ $f = \{(x, f(x)) \mid f(x) = x^2\}$		
D. Example: Given the function f defined by $\{(x, y) \mid y = x - 1\}$, find $f(1)$, $f(0)$, $f(-12)$		
E. Example: Given the function G defined by $\{(x, y) \mid y = 3x^2\}$ for the set $U = \{-2, -1, 0, +1, +2\}$ find $G(-2)$, $G(1)$		
F. Example: Given the function f defined by the sentence or rule $f(r) = \frac{4}{3} \pi r^3$, find $f(-5)$, $f(a)$, $f(3t)$		

SCOPE	REFERENCES CODE	PAGES
XIII. Systems of Sentences Involving Equations and Inequalities	VB	p. 38-40
A. Introduction		
B. Example 1: Graph the following sentences and state the solution common to both sentences where U = the set of real numbers $\begin{cases} x + 3y = 6 \\ 4x - y = 11 \end{cases}$		
C. Example 2: Find the solution common to both of the following sentences where U = the set of real numbers $\begin{cases} y \geq x \\ x + y \geq 6 \end{cases}$		
D. Example 3: Find the solution common to both sentences where U = the set of real numbers $\begin{cases} x^2 + y^2 < 9 \\ y \leq x \end{cases}$		
XIV. Summary	VB	p. 41
XV. Review		
XVI. Test		

SCOPE	REFERENCES CODE	PAGES
UNIT IV First-Degree Equations (Time Allotment 15 days)		
The teacher should stress a set theoretic approach using the ideas in Unit III.		
I. Sets and Variables (These topics concerning sets should be a brief review. Obtain student copies of "Review of Elementary Set Concepts" from your Department Chairman.)	B	p. 1-23
A. Concept of set		
B. Basic terms		
member or element ($\in$) — equal sets		
non-member ($\notin$) — replacement set		
rule method — constant		
roster method — open sentence		
finite set — solution set or truth set		
infinite set — root		
the empty or null set		
II. Use of Sets		
A. Types of subsets		
1. Proper subset		

SCOPE	REFERENCES CODE	PAGES
2. Improper subset		
(NOTE: The empty set is a proper subset of every set except itself.)		
B. Operations on sets		
1. Intersection $(A \cap B)$		
2. Union $(A \cup B)$		
3. Complement $(\overline{A})$		
C. Venn diagrams		
1. Universal set		
2. Diagrams for intersection, union, and complement		
III. The Solution of Open Sentences	DBF	p. 80-83
A. Transforming equations with properties of equality	WKP	p. 41
1. Addition property of equality		
For all real numbers a, b, c, if a = b, then a + c = b + c		
2. Multiplication property of equality		
For all real numbers a, b, c, if a = b, then ac = bc		
B. Steps in solving (Follow outline as listed in text.)		
The teacher should emphasize that the value found for the variable is a root only when it satisfies the original equation.	WKP	p. 42-44

SCOPE	REFERENCES CODE	PAGES
C. Literal equations and formulas	WKP	p. 45
D. Evaluation of formulas	WKP	p. 46
IV. Using Variables to Solve Verbal Problems		
A. Interpreting algebraic expressions and sentences	WKP	p. 47-48
(The key to solving "verbal problems" is skill in translating	DBF	p. 49-54
English phrases and sentences about numbers into corresponding	DBW	p. 60-61
algebraic expressions and sentences.)		
B. Steps in solving problems		
(Use steps as listed in text.)		
C. Types of problems	WKP	p. 49
1. Geometry	WKP	p. 51-58
2. Motion		
3. Lever		
4. Mixture		
V. Fractional Equations		
A. Techniques employed in solving fractional equations	WKP	p. 113
1. Finding the least common multiple of the denominators	WKP	p. 114-117
2. Transforming the given equation into an equivalent equation.	DBW	p. 27 (Teacher's manual)
The student should identify at the outset the values of the variable		

SCOPE	REFERENCES CODE	PAGES
for which one or both members of the equation are not defined. This means that the replacement set should be meaningful for the problem. Multiplying an equation by a variable expression which may represent 0 will frequently produce an equation which is not equivalent to the given equation.		
B. Verbal problems involving fractional equations		
1. Work problems		
2. Motion problems		
3. Problems dealing with numbers		
VI. Systems of Linear Equations		
A. Linear combination method ("elimination by addition or subtraction") (Emphasize that the root, if it exists, is an ordered pair of numbers, not a pair of equations.)	WKP DBW	p. 164-165 p. 95-98
B. Substitution method	WKP	p. 166
C. Determinant method	WKP	p. 419-425
1. Definition of determinant		
2. Solution of two linear equations in two variables		
3. Solution of three linear equations in three variables (Expansion by diagonals only.)		

SCOPE	REFERENCES CODE	PAGES
VII. Ratio, Proportion, and Variation		
A. Definitions	WKP	p. 189
B. Expressing ratios and solving problems dealing with ratios	WKP	p. 190
C. Expressing proportions and solving problems with proportions	WKP	p. 191-192
1. Geometry problems (Stress proportional sides of similar triangles.)	WKP	p. 193
D. Direct variation	WKP DBW	p. 194 p. 211-212
E. Inverse variation	WKP	p. 195
F. Joint variation (optional)	WKP DBW	p. 196 p. 313-315
VIII. Review		
A. Chapter reviews		
B. Chapter tests		
C. Checking your understanding (end of each chapter in text)		
IX. Test		

SCOPE	REFERENCES CODE / PAGES
UNIT V Inequalities (Time Allotment 10 days)	
I. Order on the Number Line	
A. The number line	SMSG Inequalities p. 1-13
B. Sentences involving inequalities	
C. Open sentences	
D. Graph of truth sets	
E. Compound inequalities: <u>Or</u>	
F. Compound inequalities: <u>And</u>	
G. Graph of truth sets of compound open sentences	
II. Properties of Order	
A. Comparison property	SMSG Inequalities p. 15-37
B. Transitive property of order	
C. Order of opposites	
D. Order relation for real numbers	
E. Addition property of order	
F. Multiplication property of order	

SCOPE	REFERENCES CODE | PAGES
III. Solution of Inequalities	
A. Equivalent inequalities	SMSG Inequalities p. 39-47
B. Polynomial inequalities	
C. Rational inequalities	
IV. Graphs of Open Sentences in Two Variables	
A. Inequalities in two variables	SMSG Inequalities p. 49-63
B. Graphs of open sentences involving integers only	
C. Systems of inequalities	
V. Review	
VI. Test	

SCOPE	REFERENCES CODE	PAGES
UNIT VI Quadratic Functions and Equations (Time Allotment 10 days)		
I. Quadratic Function defined by $y = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$	WKP	p. 247-253
A. Terminology to be stressed: parabola, turning point, maximum value, minimum value, axis of symmetry		
B. Graph		
C. Maximum and minimum values		
D. Axis of symmetry		
II. Quadratic Equations set notation to be used in stating equation	WKP	p. 255
A. Graphic solution		
B. Algebraic solutions	WKP	p. 257-264
1. Solution of pure or incomplete quadratic Solved by using principle: roots of equals are equal.		
2. Solution of complete quadratics		
a. Solution by factoring		
b. Solution by completing the square		
3. Derivation of quadratic formula by solving the general quadratic $ax^2 + bx + c = 0$ by completing the square	WKP	p. 265
a. Use in solving any type of quadratic		

SCOPE	REFERENCES	
	CODE	PAGES
4. Solution of fractional equations which become quadratic equations upon clearing of denominators	WKP	p. 269
5. Solution of radical equations which result in quadratic equations upon squaring both members	WKP	p. 271
6. Solution of verbal problems	WKP	p. 274-275
III. Advanced Topics in Quadratic Equations		
A. Character of nature of roots of a quadratic equation	WKP	p. 407
1. Use of the discriminant of a quadratic equation	WKP	p. 410
B. The sum and product of the roots of a quadratic equation		
1. Use in checking the roots of a quadratic equation		
2. Use in forming a quadratic when the roots are known		
IV. Review		
V. Test		

SCOPE	REFERENCES CODE	REFERENCES PAGES
UNIT VII		
Logarithms		
(Time Allotment 10 days)		
I. The Exponential and Logarithmic Functions		
A. Definitions	DBW	p. 338-340
1. The exponential function and its graph		
2. The logarithmic function and its graph		
B. The exponential and logarithmic forms of equivalent statements	WKP	p. 289
Practice in converting forms	R	p. 141
	H	p. 126-127
II. Properties of Logarithms		
A. Proofs of theorems		
1. $\log_b MN = \log_b M + \log_b N$	R	p. 142-148
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$	WKP	p. 303
	H	p. 128-130
3. $\log_b M^p = p \log_b M$		
III. Common Logarithms		
A. The characteristic and mantissa	WKP	p. 290-303
1. Definitions	R	p. 149-153
	H	p. 131-134
2. Methods of determining characteristics		

SCOPE	REFERENCES CODE	PAGES
B. Use of tables		
1. Given N, to find log N	R	p. 153-159
2. Given log N, to find N	H	p. 135-136
C. Linear interpolation		
1. Given N, to find log N	DBW	p. 343-345
2. Given log N, to find N	R	p. 159-165
	H	p. 137-139
IV. Logarithmic Computation		
A. Approximations and significant figures	WKP	p. 286-287
B. Products	R	p. 165-175
	H	p. 139-143
C. Quotients		
D. Roots and powers		
E. Combined operations		
V. Equations		
A. Solving logarithmic equations	WKP	p. 310-313
B. Solving exponential equations	H	p. 144-148
VI. Change of Base of Logarithms		
VII. Review		
VIII. Test		

BALTIMORE COUNTY PUBLIC SCHOOLS

Classification for Products and Factoring

IDENTIFICATION OF PRODUCT	PRODUCT	FACTORS	IDENTIFICATION OF FACTORED FORM
Case I			Case I
A. Polynomials with a common monomial factor	1. $3x+3y$ ⟷	1. $3(x+y)$	A. The product of a monomial and a polynomial
	2. $9d^3-12d^2+3d$ ⟷	2. $3d(3d^2-4d+1)$	
	3. $20a^{2n}b-5a^{3n}b^2-15a^{2n}b^3$ ⟷	3. $5a^{2n}b(4-a^nb-3b^2)$	
	4. $\frac{1}{4}x^3+2x^2y-\frac{1}{2}xy^2$ ⟷	4. $\frac{1}{4}x(x^2+8xy-2y^2)$	
B. Polynomials with a common binomial factor	1. $ax-ay+bx-by$ ⟷	1. $(x-y)(a+b)$	B. The product of two polynomials
	2. $na-nb-3b+3a$ ⟷	2. $(a-b)(n+3)$	
	3. $2ax-4bx-3ay+6by-5az+10bz$ ⟷	3. $(a-2b)(2x-3y-5z)$	
Case II			Case II
Perfect square trinomial	1. $x^2-2xy+y^2$ ⟷	1. $(x-y)^2$	Square of a binomial
	2. $9x^2+12xy+4y^2$ ⟷	2. $(3x+2y)^2$	
	3. $\frac{9}{16}x^{2b}-9x^by^2+36y^4$ ⟷	3. $(\frac{3}{4}x^b-6y^2)^2$	
	4. $(a-b)^2+12c(a-b)+36c^2$ ⟷	4. $(a-b+6c)^2$	

IDENTIFICATION OF PRODUCT	PRODUCT	FACTORS	IDENTIFICATION OF FACTORED FORM
Case III Difference of two squares	1. $x^2 - y^2$ ⟷ 2. $25a^2 - 9b^2$ ⟷ 3. $x^4 - y^4$ ⟷ 4. $x^8 - y^8$ ⟷ 5. $36m^{2x} - 25n^{2y}$ ⟷ 6. $81x^2 - 162xy + 81y^2 - 121z^2$ ⟷	1. $(x+y)(x-y)$ 2. $(5a+3b)(5a-3b)$ 3. $(x^2+y^2)(x+y)(x-y)$ 4. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$ 5. $(6m^x+5n^y)(6m^x-5n^y)$ 6. $(9x-9y+11z)(9x-9y-11z)$	Case III Product of the sum and difference of the same two terms
Case IV General trinomial	1. x^2+5x+6 ⟷ 2. x^2-x-6 ⟷ 3. x^2-6x+8 ⟷ 4. $6x^2+7x-20$ ⟷ 5. $10a^{2x}-11a^x b^y-6b^{2y}$ ⟷	1. $(x+3)(x+2)$ 2. $(x-3)(x+2)$ 3. $(x-2)(x-4)$ 4. $(3x-4)(2x+5)$ 5. $(2a^x-3b^y)(5a^x+2b^y)$	Case IV Product of two different binomials

IDENTIFICATION OF PRODUCT	PRODUCT	FACTORS	IDENTIFICATION OF FACTORED FORM
Case V Sum or difference or two cubes	1. a^3+b^3 ⟷ 2. a^3-b^3 ⟷ 3. $.027x^3-64y^3$ ⟷ 4. $\frac{1}{8}a^3+\frac{1}{125}b^3$ ⟷	1. $(a+b)(a^2-ab+b^2)$ 2. $(a-b)(a^2+ab+b^2)$ 3. $(.3x-4y)(.09x^2+1.2xy+16y^2)$ 4. $(\frac{1}{2}a+\frac{1}{5}b)(\frac{1}{4}a^2-\frac{1}{10}ab+\frac{1}{25}b^2)$	Case V The product of a binomial and a trinomial
Case VI Polynomials factorable by the Factor Theorem (Use synthetic division):	x^3+x^2+x+6 Testing x + 1 as a factor - 1 | 1 + 1 + 1 + 6 - 1 0 - 1 1 + 0 + 1 || +5 remainder ⟶ Testing x - 1 as a factor + 1 | 1 + 1 + 1 + 6 + 1 + 2 + 3 1 + 2 + 3 || +9 remainder ⟶ Testing x + 2 as a factor - 2 | 1 + 1 + 1 + 6 - 2 + 2 - 6 1 - 1 + 3 || 0 remainder ⟶	∴ x+1 is <u>not</u> a factor ∴ x-1 is <u>not</u> a factor ∴ x+2 is a factor The complete factors are: $(x+2)(x^2-x+3)$	Case VI The product of two polynomials

IDENTIFICATION OF PRODUCT	PRODUCT	FACTORS	IDENTIFICATION OF FACTORED FORM
Case VII Sum or difference of like powers $(a^n \pm b^n)$	1. $x^5 - y^5$ ⟷	1. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$	Case VII Product of two polynomials
	2. $x^7 + y^7$ ⟷	2. $(x+y)(x^6-x^5y+x^4y^2-x^3y^3$ $+x^2y^4-xy^5+y^6)$	
	3. $x^{10}+y^{10}=(x^2)^5+(y^2)^5$ ⟷	3. $(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8)$	
	4. x^4+y^4 ⟷	4. Not factorable by real numbers. Cannot be represented as odd powers.	
	5. x^4+y^4 ⟷	5. Factorable by complex numbers. $x^4 - (-y^4) = (x^2)^2 - (y^2i)^2 =$ $(x^2+y^2i)(x^2-y^2i)$	

BALTIMORE COUNTY PUBLIC SCHOOLS

APPENDIX II

RE IEW OF ELEMENTARY SET CONCEPTS

by

Vincent Brant
Sup rvisor of Senior High School Mathematics

July 1965

REVIEW OF ELEMENTARY SET CONCEPTS

1\. Member — Each object in a set is called a member (or element) of the set.

2\. — The symbol " $\in$ " means "belongs to" or "is a member of" or "is an element of."

3\. Set Notation

(1) The phrase method

Example: the set of the (names of the) vertices of $\triangle ABC$

(2) The roster (listing) method

Example: Using the same set as in (1) above: $\{A, B, C\}$

(3) The rule method

Example: Using the same set as in (1) and (2) above:

$\{ x \mid x \text{ is a vertex of } \triangle ABC \}$

This is read as "the set of all x such that x is a vertex of triangle ABC."

4\. Infinite set — A set which is unending is called an infinite set.

5\. Finite set — A set in which the members can be listed and the listing terminates is called a finite set.

6\. The empty set — The set which contains no members is called the empty set or the null set.

The symbol for the empty set is " ϕ "

7\. Equal sets — Two sets are equal if and only if they have the same members.

8\. Subset — Set A is a subset of set B (denoted "$A \subset B$") if and only if each member of A is also a member of B.

Every set is a subset of itself; that is, if A is a set, then $A \subset A$.

The empty set is a subset of every set; that is, if A is a set, then $\phi \subset A$.

Set A = set B if and only if $A \subset B$ and $B \subset A$.

Set A is a proper subset of set B if and only if there is at least one member of B which is not a member of A.

Examples: Let $A = \{ 1, 3, 4 \}$
$B = \{ 1, 2, 3, 4, 5 \}$
$C = \{ 4 \}$
$D = \{ 2, 5, 4, 3, 1 \}$
$E = \{ 3, 5, 6 \}$

Then, $A \subset B$; $C \subset B$
$A \subset D$ and $D \subset A$; hence $A = D$
A and C are proper subsets of B
$A \subset A$; $B \subset B$; $C \subset C$; $D \subset D$; $E \subset E$
ϕ is a subset of every set A, B, C, D, E.

But $E \not\subset B$ (E is not a subset of B)

9. Disjoint sets

Two sets which have no members in common are called disjoint sets.

10. Intersection

The intersection (denoted by "$\cap$") of two sets A and B is the set consisting of all the members common to A and B.

Symbolically, $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

If A and B are disjoint sets, then $A \cap B = \phi$.

If two sets are said to intersect (verb form), then their intersection has at least one member; that is, their intersection is non-empty.

Examples: Let $A = \{ 1, 2, 3, 4, 5 \}$
$B = \{ 3, 4, 6, 7, \}$
$C = \{ 1, 3, 4, 8 \}$
$D = \{ 10, 11, 12 \}$

Then, $B \cap C = \{ 3, 4 \}$
$A \cap C = \{ 1, 3, 4 \}$
$B \cap D = \phi$

11. Union

The union (denoted by "$\cup$") of two sets A and B is the set consisting of all the members which are in at least one of the two given sets A and B.

Symbolically, $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

Observe that the connective "or" in mathematics is used in the inclusive sense; that is, x is a member of A or x is a member of B or x is a member of both A and B.

Examples: Let $A = \{ 1, 2, 3, 4, 5 \}$
$B = \{ 3, 4, 6, 7 \}$
$C = \{ 1, 3, 4, 8 \}$

Then, $A \cup B = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$B \cup C = \{ 1, 3, 4, 6, 7, 8 \}$

12. Cartesian Product

The Cartesian Product of two sets A and B (not necessarily different) is the set of all ordered pairs in which the first coordinate belongs to A and the second coordinate belongs to B.

The Cartesian Product of A and B is denoted as:

$A \times B$ (read "A cross B")

Example: Let $A = \{ a, b, c, \}$ and $B = \{ 1, 2 \}$

Then, $A \times B = \{ (a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2) \}$

13. Relation

A relation is a set of ordered pairs.

or

A relation from A to B is a subset of $A \times B$.

Example: Let $A = \{ a, b, c \}$; let $B = \{ 1, 2 \}$

Some relations from A to B are:

Relation $R = \{ (a, 1), (b, 2), (c, 1) \}$
Relation $Q = \{ (a, 2) \}$
Relation $T = \{ (a, 1), (b, 1), (c, 1) \}$

Of course, $A \times B$ is also a relation.

Also, the empty set, ϕ, is a relation.

14. Domain of a relation

The domain of a relation is the set of all the first coordinates of the ordered pairs of the relation.

15. Range of a relation

The range of a relation is the set of all the second coordinates of the ordered pairs of the relation.

Example: Let Relation $T = \{ (a, 2), (b, 3), (a, 5), (c, 2) \}$

Then, Domain of $T = \{ a, b, c \}$
Range of $T = \{ 2, 3, 5 \}$

16. Function : A function from A to B is a special kind of relation such that for each first coordinate there is one and only one second coordinate.

Stated differently, no two ordered pairs of a function may have the same first coordinate. Hence, the first coordinates in the ordered pairs of a function must be different. However, the second coordinate may be the same.

Examples: D = { (1, a), (2, b), (3, c), } is a function.
E = { (1, a), (2, b), (1, c) is not a function; it is a relation.
K = { (1, a), (2, a), (3, a) } is a function--- in a particular, a constant function.

Note: Domain of K = { 1, 2, 3 }
Range of K = { a }

I = { (x, y) | y = x } is a function.

Note: Domain of I is the set of real numbers.
Range of I is the set of real numbers.

17. Tests for a function :

(a) The ordered pair test

A relation is a function provided every first coordinate is different.

(b) The vertical line test

A relation is a function provided every vertical line intersects the graph of the relation in exactly one point. If at least one vertical line intersects the graph in more than one point, then the graph does not represent a function, but a relation.

18. Comparison of f and f(x) :

In the function, f = { (x, y) | y = 5x }

(a) the function f is the entire set of ordered pairs.

(b) f(x) is not the function.

(c) "y", "f(x)", "5x" are different names for the second coordinate. Thus the function f might be written in different forms, as:

$$f = \{ (x, y) \mid y = 5x \}$$

or

$$f = \{ (x, f(x)) \mid f(x) = 5x \}$$

or

$$f = \{ (x, 5x) \mid x \text{ is a real number} \}$$

(d) The function is not "y = 5x". Instead, "y = 5x" is a sentence which defines the function. This sentence assigns to each real number used as first coordinate another real number 5 times as large for its corresponding second coordinate.

"f(x)" is read "f at x" or "f of x" or "the value of the function f at x".

The function, $f = \{ (x, y) \mid y = 5x \}$ is read:

the function f defined by y = 5x

HOMEWORK ASSIGNMENT

Study assignment: Study pages 1 - 5 in this Review of Elementary Set Concepts

Written assignment:

1. Construct the cartesian product of $A \times B$ where

   $A = \{ r, t, k \}$ and $B = \{ 6, 16, 1526 \}$

2. Construct the cartesian product of $R \times S$ where

   $R = \{ 0, 2, 4, 6 \}$ and $S = \{ 1, 3, 5 \}$

3. Construct the cartesian product of $E \times D$ where

   $D = \{ a \}$ and $E = \{ 1 \}$

<u>Directions for Examples 4 - 13:</u>

Each of the following relations is a relation from R to R where R is the set of real numbers. In each example, complete the following parts:

a. Draw the graph of the relation.
b. State whether the relation is a function.
c. State the domain of the relation.
d. State the range of the relation.

4. $R = \{ (a, y) \mid y = x \}$

5. $S = \{ (x, S(x)) \mid S(x) = \frac{1}{2} x \}$

6. $T = \{ (x, y) \mid y > 5x \}$

7. $U = \{ (x, y) \mid y = x^2 \}$

8. $V = \{ (x, V(x)) \mid V(x) = 2x - 5 \}$

9. $W = \{ (x, y) \mid y = x^3 \}$

10. $X = \{ (x, y) \mid x^2 + y^2 = 25 \}$

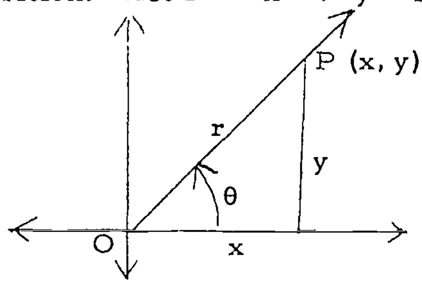
11. $Y = \{ (x, y) \mid xy = 12 \}$

12. $G = \{ (x, G(x)) \mid G(x) = 5 \}$

13. $H = \{ (x, y) \mid x = 3 \}$

Trigonometric Functions

Let P with coordinates (x, y) be a point on the terminal side of an angle with measure θ in standard position. Let $r = \sqrt{x^2 + y^2}$ represent distance of P from the origin.



1. The Sine Function

   Sine (or sin) = $\{ (\theta, \sin\theta) \mid \sin\theta = \frac{y}{r} \}$

   Observe that:

   (1) θ is a number----a number that measures the magnitude of the angle in degrees or radians.

   (2) $\sin\theta$ is a number----the number which is assigned as the second coordinate by the defining sentence

   $$\sin\theta = \frac{y}{r}$$

   (3) $\sin\theta$ is NOT the function.

   $\sin\theta$ is the name of the second coordinate in the sine function.

   $\sin\theta$ is the VALUE of the sine function at θ.

   (4) Each ordered pair of the sine function has real numbers for the first and second coordinates.

   These ideas apply as well to the remaining trigonometric functions.

2. The Cosine Function

   cosine (or cos) = $\{ (\theta, \cos\theta) \mid \cos\theta = \frac{x}{r} \}$

3. The Tangent Function

   tangent (or tan) = $\{ (\theta, \tan\theta) \mid \tan\theta = \frac{y}{x} \}$

4. The Cotangent Function

   cotangent (or cot) = $\{ (\theta, \cot\theta) \mid \cot\theta = \frac{x}{y} \}$

5. The Secant Function

   secant (or sec) = $\{ (\theta, \sec\theta) \mid \sec\theta = \frac{r}{x} \}$

6. The Cosecant Function

   cosecant (or csc) = $\{ (\theta, \csc\theta) \mid \csc\theta = \frac{r}{y} \}$

APPENDIX I

SPECIAL RIGHT TRIANGLES

Certain right triangles appear frequently in problems of the physical world such as engineering and in problems of related mathematics courses such as trigonometry. These special right triangles may be classified into sets of triangles, each set containing only triangles that are similar to one another. You should be able to recognize the special right triangles discussed in this unit, to understand the relationships that exist for each set, and to apply these relationships in the solution of problems.

I. The 3, 4, 5 right triangles.

Since $3^2 + 4^2 = 5^2$, the converse of the Pythagorean Theorem tells us that a triangle whose sides have measures 3, 4, 5 is a right triangle. In a similar manner, we can conclude that if any positive number k is used as a constant of proportionality, then a triangle whose sides measure 3k, 4k, 5k is a right triangle. For example, any triangle whose sides measure 6, 8, 10 or 150, 200, 250 is a member of this set of right triangles.

Example 1. Find the length of the hypotenuse of a right triangle if the length of the two legs of the triangle are 18 and 24.

You should note that the triangle is a 3, 4, 5 right triangle with 6 as the constant of proportionality since $18 = 6 \cdot 4$. Therefore, the length of the hypotenuse of the triangle is $6 \cdot 5$ or 30.

Example 2. Triangle ABC is a right triangle with $\angle C$ as the right angle. AB = 75 and BC = 45. Find the length of AC.

Since the length of AB is $15 \cdot 5$ and the length of BC is $15 \cdot 3$, the triangle is a 3, 4, 5 triangle with 15 as the constant of proportionality. Therefore, AC must equal $15 \cdot 4$ or 60.

II. The 5, 12, 13 right triangles.

Explain why a triangle whose sides have measures of 5, 12, 13 is a right angle. Explain why a triangle whose sides have measures of 5k, 12k, and 13k is a right triangle if k is some positive number. Explain why all these triangles are similar. Give other examples of three numbers which are the respective measures of the sides of triangles in this set of similar right triangles.

Example 1. Find the length of the hypotenuse of a right triangle if the lengths of the two legs of the triangle are 15 and 36.

You should note that the triangle is a 5, 12, 13 right triangle with 3 as the constant of proportionality since $15 = 3 \cdot 5$ and $36 = 3 \cdot 12$. Therefore, the length of the hypotenuse of the right triangle is $3 \cdot 13$ or 39.

Example 2. Triangle ABC is a right triangle with $\angle C$ as the right angle. If the hypotenuse of the triangle has a measure of 78 and one of the legs of the triangle has a measure of 72, find the measure of the other leg of the triangle.

Since $78 = 6 \cdot 13$ and $72 = 6 \cdot 12$, the triangle is a 5, 12, 13 triangle with 6 as the constant of proportionality. Therefore, the measure of the other leg is $6 \cdot 5$ or 30.

III. The 30, 60, 90 triangles.

Unlike the sets of triangles in (I) and (II), the triangles in this set are usually designated by measures of angles rather than lengths of sides. Suppose that in $\triangle ABC$, $m\angle C = 90$, $m\angle B = 60$, $m\angle A = 30$. We know by Theorem 5 · 7 on page 202 of your text, that the length of BC is one-half the length of AB.

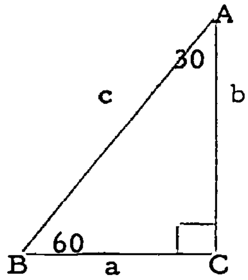
If we let $m(BC) = a$, then $m(AB) = c = 2a$, and by the Pythagorean Theorem

$$a^2 + b^2 = (2a)^2$$

$$a^2 + b^2 = 4a^2$$

$$b^2 = 3a^2$$

$$b = a\sqrt{3}$$



Thus we see that if $\triangle ABC$ is a 30, 60, 90 triangle $(a, b, c) \overset{=}{p} (1, \sqrt{3}, 2)$.

We can also prove the converse of this statement. If $\triangle A'B'C'$ is a triangle with $(B'C', C'A', A'B') \overset{=}{p} (1, \sqrt{3}, 2)$, then $\triangle A'B'C'$ is a 30, 60, 90 triangle.

We know that $\triangle A'B'C'$ is similar to $\triangle ABC$ by the SSS Similarity Theorem. Therefore, $m\angle A' = 30$, $m\angle B' = 60$, and $m\angle C' = 90$ because corresponding angles of similar triangles are equal in measure.

The above discussion can be summarized in the following theorem and corollaries:

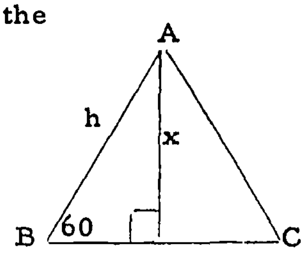
Theorem A. Triangle ABC is a right triangle with $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$ if and only if $(a, b, c) \overset{=}{p} (1, \sqrt{3}, 2)$.

Corollary 1. In a right triangle whose acute angles have measures of 30 and 60, the shorter leg is one-half the hypotenuse. $(s_{30} = \frac{h}{2})$

Corollary 2. In a right triangle whose acute angles have measures of 30 and 60, the longer leg is equal to one-half the hypotenuse multiplied by $\sqrt{3}$. $(s_{60} = \frac{h\sqrt{3}}{2})$

Example 1. Find the length of the altitude of an equilateral triangle if the length of one of the sides is 8.

The altitude of an equilateral triangle bisects the vertex angle making a 30, 60, 90 triangle.



$(4, x, 8) \stackrel{=}{p} (1, \sqrt{3}, 2)$.

The constant of proportionality is 4.

Therefore, $x = 4\sqrt{3}$.

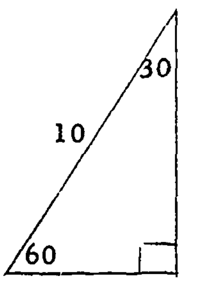
The problem can also be solved by using Corollary 2.

$$s_{60} = \frac{h\sqrt{3}}{2}$$

$$s_{60} = \frac{8 \cdot \sqrt{3}}{2}$$

$$s_{60} = 4\sqrt{3}.$$

Example 2. Find the measures of the sides of a 30, 60, 90 triangle if the length of the longer leg of the triangle is 10.



$(a, 10, c) \stackrel{=}{p} (1, \sqrt{3}, 2)$.

The constant of proportionality is $\frac{10}{\sqrt{3}}$.

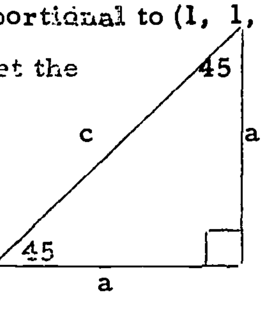
Therefore, $a = \frac{10}{\sqrt{3}}$ and $c = \frac{2 \cdot 10}{3}$, or

$$a = \frac{10\sqrt{3}}{3} \text{ and } c = \frac{20\sqrt{3}}{3}.$$

IV. The 45, 45, 90 triangles.

Since the triangles in this set have two angles that are equal in measure, they are sometimes called isosceles right triangles. We can prove that the lengths of the sides of any isosceles right triangle are proportional to $(1, 1, \sqrt{2})$.

In $\triangle ABC$. $m\angle C = 90$, $m\angle A = m\angle B = 45$. If we let the length of AC = a, then the length of BC = a. Why? By the Pythagorean Theorem



$$c^2 = a^2 + a^2$$

$$c^2 = 2a^2$$

$$c = a\sqrt{2}.$$

Therefore, we see that if $\triangle ABC$ is a 45, 45, 90 triangle, $(a, a, c) \stackrel{=}{p} (1, 1, \sqrt{2})$.

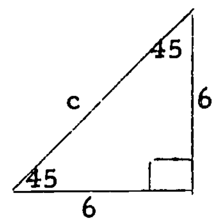
We shall now prove the converse of this statement. If $\triangle A'B'C'$ is a triangle with $(a', b', c') \stackrel{=}{p} (1, 1, \sqrt{2})$, then $\triangle A'B'C'$ is a 45, 45, 90 triangle.

We know that $\triangle A'B'C'$ is similar to $\triangle ABC$ by SSS Similarity Theorem. Therefore, $m \angle C = 90$, $m \angle A = 45$, $m \angle B = 45$ because corresponding angles of similar triangles have equal measure.

The above discussion can be summarized in the following theorem and corollary.

> <u>Theorem B.</u> Triangle ABC is a right triangle with $m \angle A = m \angle B = 45$ and $m \angle C = 90$ if and only if $(a, b, c) \stackrel{=}{p} (1, 1, \sqrt{2})$.
>
> <u>Corollary.</u> In an isosceles right triangle, either leg of the triangle is equal to one-half the hypotenuse multiplied by $\sqrt{2}$. ($s_{45} = \frac{h\sqrt{2}}{2}$)

<u>Example 1.</u> Find the length of the hypotenuse of a right isosceles triangle if each leg of the triangle has a measure of 6.

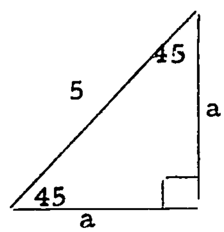


$(6, 6, c) \stackrel{=}{p} (1, 1, \sqrt{2})$

The constant of proportionality is 6.

Therefore, $c = 6\sqrt{2}$.

Example 2. Find the length of each leg of an isosceles right triangle if the length of the hypotenuse is 5.



$(a, a, 5) \stackrel{=}{p} (1, 1, \sqrt{2})$

The constant of proportionality is $\frac{5}{\sqrt{2}}$.

Therefore, $a = \frac{5}{\sqrt{2}}$, or $\frac{5\sqrt{2}}{2}$.

The problem can also be solved by using the corollary.

$$s_{45} = \frac{h\sqrt{2}}{2}$$

$$s_{45} = \frac{5\sqrt{2}}{2}.$$

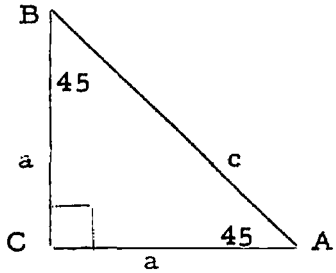
<u>Problem Set</u>

1. In each of the following exercises, the lengths of a leg and the hypotenuse of a right triangle are given. Which of the measures belong to a triangle similar to the 3, 4, 5 triangle? to the 5, 12, 13 triangle? to the 1, $\sqrt{3}$, 2 triangle? to the 1, 1, $\sqrt{2}$ triangle?

(a) 6, 10	(e) 3, 6	(i) 6, 6 · 5
(b) 12, 15	(f) 3, $2\sqrt{3}$	(j) 24, 26
(c) 24, 25	(g) 8, 10	(k) $3\sqrt{2}$, $5\sqrt{2}$
(d) 15, 39	(h) 1.5, 2.5	(l) $\sqrt{2}$, 2

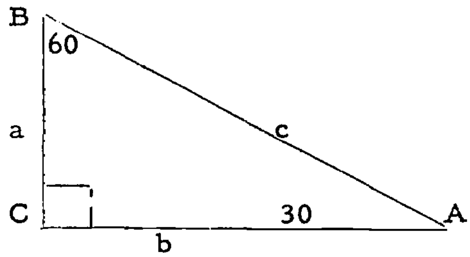
2. In each part of Problem 1, find the length of the side which is not given.

3. Complete the table for the right isosceles triangle in the diagram.



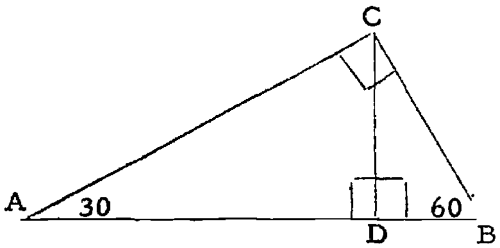
	a	c
(a)	10	
(b)	5	
(c)		9
(d)		6
(e)		$3\sqrt{2}$
(f)	$5\sqrt{2}$	

4. Complete the table for the 30-60-90 triangle in the diagram.



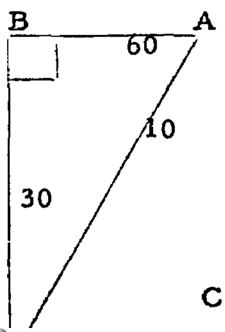
	a	b	c
(a)	10		
(b)	5		
(c)		18	
(d)		$9\sqrt{3}$	
(e)			20
(f)			$12\sqrt{3}$

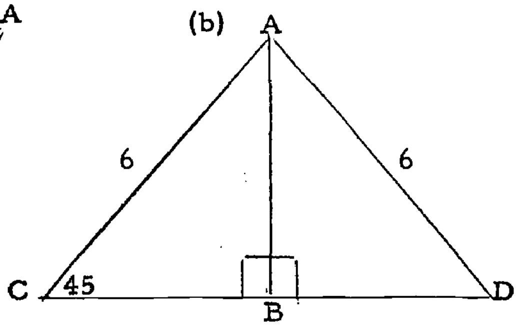
5. In each of the following exercises the length of one segment in the adjacent plane figure is given. Find the lengths of the remaining segments.

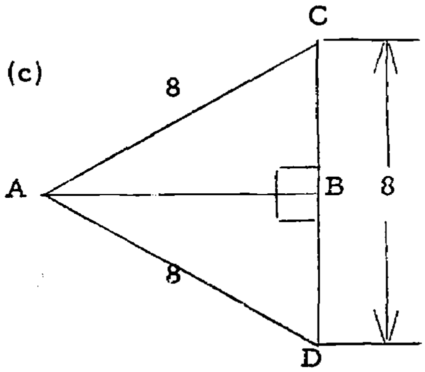


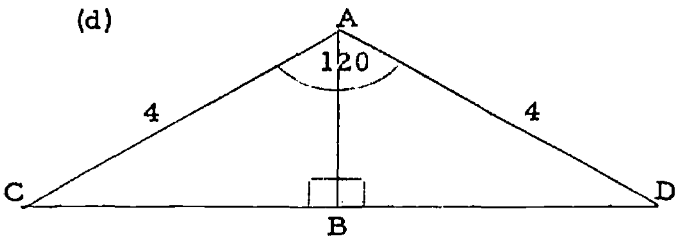
	m(AB)	m(BC)	m(CD)	m(AD)	m(DB)	m(AC)
(a)	8					
(b)		2				
(c)			$4\sqrt{3}$			
(d)				9		
(e)					$10\sqrt{3}$	
(f)						$8\sqrt{3}$

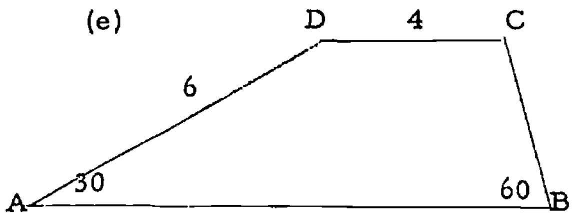
6. Use the three corollaries in this unit to find the length of AB and BC in each of the following plane figures. You should do all work mentally and write only the answer.

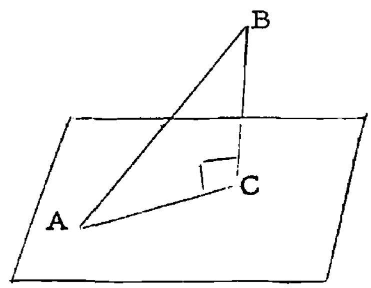
(a)


(b)


(c)








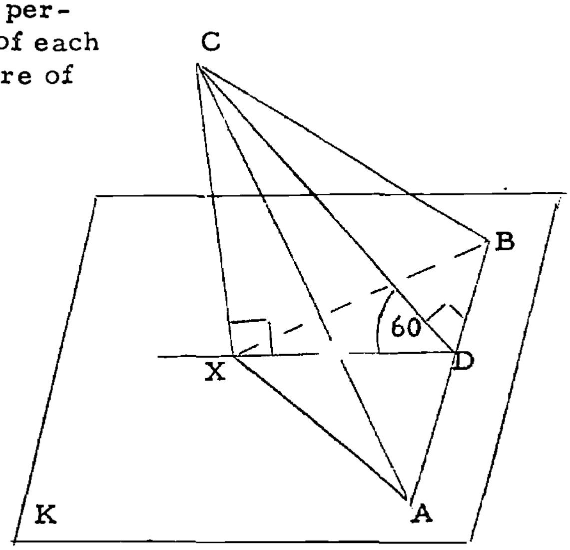
7. In the diagram, BC $\perp$ AC and m∠A = 30.

   (a) If m(AB) = 6, find m(AC).
   (b) If m(AB) = 4 $\sqrt{3}$, find m(AC).
   (c) If m(AC) = 9, find m(AB).
   (d) If m(AC) = 6 $\sqrt{3}$, find m(AB).

8. Repeat problem 7 if m∠A = 60.

9. In the diagram, △ABC is an equilateral triangle which is inclined at an angle to plane K. The measure of the dihedral angle C-AB-X is 60; CX is perpendicular to plane K; CD is perpendicular to AB. Find the measure of each of the following segments if the measure of AB = 6.

(a)	AC	(d)	CX
(b)	BC	(e)	AX
(c)	CD	(f)	BX



10. Repeat problem 9 if the measure of the dihedral angle is 30; (b) 45.

RELATIONS, FUNCTIONS and GRAPHS

by
Vincent Brant
Supervisor of Senior High School Mathematics
Board of Education of Baltimore County
Towson 4, Maryland

ERRATA SHEET ON THE UNIT, RELATIONS, FUNCTIONS, AND GRAPHS

1. Exercises labeling incorrectly:
   a. page 18: Exercise D should be Exercise E
   b. page 20: Exercise E should be Exercise F
   c. page 26: Exercise F should be Exercise G
   d. page 29: Exercise G should be Exercise H
   e. page 31: Exercise H should be Exercise I
   f. page 33: Exercise I should be Exercise J
   g. page 35: Exercise I should be Exercise K
   h. page 37: Exercise J should be Exercise L
   i. page 40: Exercise K should be Exercise M

2. page 4: Section D. Include also the following:

   S(6, 6): 6 is greater than 6

   S(2, 2): 2 is greater than 2

3. page 9: Exercise B, 1 a. x is greater than 5 and $x \in \{1, 2, 3, \ldots, 7\}$

4. page 10: Example 3: A X A, second column, first row should be (s, r)

5. page 22: Section X, A, 3, (c): Line 22

   All points not circled

6. page 23, line 10, Section B, 1 If $U = \{-2, -1, 0, +1, +2\}$,

7. page 26, line 1 Example 1: In problems, a - j,

8. page 32, line 8 Section F. Sentences involving equalities and inequalities

9. page 33, line 12, Exercise J, Directions: (1) Graph the relations in U if

   U = {integers}

10. page 34, Example 1 (b): Range: $\{0, 1, 4, 9 \ldots, n^2, \ldots\}$

11. page 37, line 1-9: Example

    Given the function G defined by $\{(x, y) \mid y = 2x^2\}$ for the Set U

    $\{-8, -7, \ldots, -1, 0, +1, +2, \ldots, +8\}$

    a. Describe the function G
    b. Find G(+ 2)
    c. Find G(-1)

Solution:

a. $G = \{x, y) \mid y = 2x^2\}$

$G = \{(-2, -8), (-1, +2), (0, 0), (+1, +2), (+2, +8)\}$

b. $G(+2) = +8$

c. $G(-1) = +2)$

12. Page 38, Example 11

$f(x + h) = x^2 + 2xh + h^2 + 4x + 4h - 1$

Relations, Functions, and Graphs

I. Introduction

We use the word "relation" in our everyday conversation to indicate that some association exists between two persons or objects under discussion. It is also possible that some association exists among three or more objects in our conversation. Such associations in normal usage are usually non-mathematical as can be seen from the following illustrations:

Tom is more handsome than George.
Mary loves Jim.
Panama has warmer weather than Canada.
Mary is the sister of Larry and Jim.
St. Louis is located between Los Angeles and New York City.

Such predicates as "is more handsome than," "loves," "has warmer weather than," "is the sister of," "is located between," are used to relate the objects under discussion. Thus, we convey the idea of association or relation between two objects or among three or more objects in our normal conversation. In mathematics, however, we attempt to be more precise about the language which we use in describing relations among objects or persons. The next sections will present a more precise description and definition of relations as used in mathematics.

II. The use of sentences in mathematics

Just as the grammarian uses sentences in English to communicate ideas, so does the mathematician use sentences in mathematics to convey mathematical ideas. In mathematics, however, a sentence is a declarative statement which can be judged true or false, <u>but not both.</u> For example, the following sentences are true:

Albany is located in the state of New York.
8 is greater than 3.
A rectangle is a parallelogram.
December 25 is called Christmas Day in the United States.
Light is faster than sound.

In contrast, the following sentences are false:

7 is less than 5.
Zero is a divisor of seven.
Black colors reflect more heat than white.

In mathematics many sentences contain a variable which is a symbol than can be replaced from some given set under discussion. These sentences containing a variable or variables are often called "open sentences." When the symbol is replaced by any symbol in the set, the sentence must be capable of being judged true or false but not both. Such sentences may use symbols as "x" or "y" for variables, in which case the sentences can be denoted symbolically as S(x) or S(y). In cases where there are two symbols (or more), such a "x" and "y", then the symbolic notation of the sentence is S(x, y). In all cases, replacements for these symbols from the specific set must make the sentence true or false, but not both. The following examples illustrate such sentences containing a variable or variables:

S(x) : 24 is a multiple of x.
S(x) : 6 is greater than x.
S(y) : y is greater than 7.
S(y) : Tom is taller than y.
S(x, y) : x is less than y.

III. Truth values of sentences and relations

In normal usage we often use sentences to imply that some association exists between two objects, or in other words, it is implied that the relation between the objects is true. It is possible, however, for a sentence to be classified as false because the phrase used to describe the relation between two objects is not true for the objects selected. The following sentences are given to illustrate that the choice of objects used in connection with the phrase describing the relation results in a false statement.

Mary loves Jim (and we know that she really doesn't).
Segment AB is perpendicular to segment AB.
6 is greater than 8.

However, for different objects, the modified sentences are true. For example,

Mary loves Tom (and she really does)
Segment AB is perpendicular to segment CD (when these lines are given as forming right angles)
8 is greater than 6.

Since these sentences are true, we say that the relation exists between these objects, or that the relation holds for the selected objects. In those cases where the sentences are judged as false, we state that the relation does not hold for the objects selected. The basic concept to be stressed is that the relation among objects exists when and only when the sentence describing the relation is true for the objects selected.

IV. Binary relations

In using sentences to describe relations among objects or persons, it should be emphasized that these objects are selected from specific set under discussion; for example, the set of males in Maryland, the set of natural numbers, the set of triangles, etc. Many relations in mathematics deal with pairs of objects from a given set. For example:

2 is less than 3 (using the set of natural numbers).
-105 is divisible by 5 (using the set of integers).
line a is a parallel to line b (using the set of lines in a plane).

When we consider relations between two objects from a given set, we refer to these relations as binary relations. However, it is possible to consider relations among three objects; such as,

Point A lies between point B and point C

Thus, this phrase "lies between" deals with three points. Whenever three members of a set are used in a relation we called this a ternary relation. However, we shall restrict our discussion in this unit to binary relations.

V. Procedure in investigating relations

A. Test question for determining true or false sentecnes

Binary relations express a method whereby some property described by the sentence can be used to show which pairs of objects chosen from the set make the sentence true in regard to that property, and which pairs do not make the sentence true. In order to sort out those pairs which make the sentence true from those pairs which make the sentence false, our first step must be to consider all possible pairs which can be formed from the given set. Our second step must be to separate those pairs making the sentence true and those pairs making the sentence false by asking the following test question of each and every pair which can be formed from the given test:

"Does this pair make the sentence true?"

All those pairs which receive a "yes" answer are in the relation. All those pairs which receive a "no" answer are not in the relation.

B. Necessity of clearly defined sentences

If a sentence which is used to describe a relation is so stated that there is ambiguity or doubt in our answer of "yes" or "no" to our test question, then we do not have a precise mathematical description for our relation, and cannot proceed until the defining sentence is modified. It follows from this that sentences describing relations must be meaningful, should not contain nonsense statements, contradictory statements or statements which do not apply.

C. Abbreviation of defining sentence

When two objects such as a and b from a given set are used in a sentence containing a predicate describing the relation (denoted by R), the resulting sentence may be symbolized as:

a R b.

This may be interpreted to read "a is in the R relation to b." Such a sentence or its abbreviation must be judged true or false depending upon the choice of the objects. If a pair makes the sentence true, that pair is in the relation. It must not be assumed that a R b means that the pair (a, b) makes the sentence true automatically. The abbreviation a R b simply means that it represents a sentence whose truth or falsity must be judged for the selected pairs.

D. Order of the elements in a pair

Our aim in achieving precise mathematical statements causes us to consider the following sentence, S (x, y) and the replacements for x and y in (x, y), from some set of natural numbers.

S(x, y) : x is greater than y.
N = {2, 6}

S(6, 2) : 6 is greater than 2
S(2, 6) : 2 is greater than 6.

It should be clear that the pair (6, 2) makes the sentence true and therefore belongs to the relation. On the other hand, the pair, (2, 6), makes the sentence false. We note further that the reason for this is the order of the two objects in the discussion. It is therefore necessary to consider the order of the objects in a pair in determining whether the pair belongs to the relation or does not belong to the relation.

E. Summary of the features of binary relations

1. Binary relations deal with the associations between two objects.

2. Pairs of objects selected from a specific set which make the sentence describing a relation true belong to the relation. Those pairs which make the sentence describing the relation false do not belong to the relation.

3. In order for sentences to define relations, they must avoid nonsense or contradictory statements for the objects under discussion.

4. The order of the elements in the pairs of objects selected from a given set is important.

5. A relation must be well-defined on a set. This means that the sentence describing the relations is capable of being judged true or false, but not both when suitable replacements are made in the given sentence.

Exercise A

1. Determine whether the following sentences are true or false.

   a. Air is heavier than water.

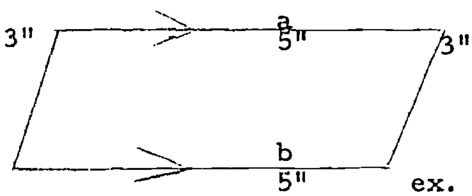
   b. Water is heavier than air.

   c. 2 is less than 3.

   d. 3 is less than 2.

   e. 2 is less than 2.

   f. 2 is less than or equal to 2.



ex. g-j

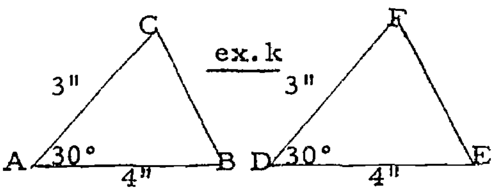
   g. Segment a is parallel to segment b.

   h. Segment b is parallel to segment a.

   i. Segment a is longer than segment b.

   j. Segment b is longer than segment a.

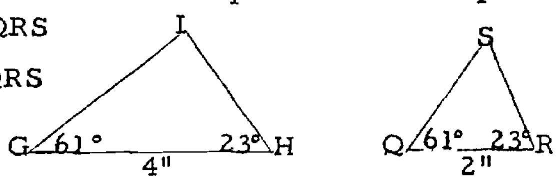
   k. Triangle ABC $\cong$ triangle DEF



ex. k

   l. Triangle GHI $\cong$ triangle QRS

   m. Triangle GHI $\sim$ triangle QRS

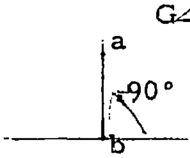


ex. l, m

   n. Line a $\perp$ line a

   o. Line a $\perp$ line b

   p. Line b $\perp$ line a



ex. n, o, p

2. Are the following relations well-defined on the set of ordered pairs formed from the given set. Remember that order is important in relations.

   a. a is parallel to b where a, b $\in$ N(set of natural numbers)

   b. a parallel to b where a, b are elements of the set of lines in a plane

   c. is a multiple of b where a, b $\in$ N (set of natural numers)

d. a is heavier than b where a, b $\in$ B (set of books in your school library.

e. A lives in the same house as b where a, b $\in$ P (set of planets of the solar system)

VI. Relations and sets

A. The link between relations and sets

1. So far we have attempted to show that a relation between pairs of elements from a given set simply means that there are pairs of objects which make the sentence describing the relation true. Our next objective is to show that a binary relation is a set of ordered pairs. In order to do this, let us review some basic concepts of elementary set theory.

One of the principle notions of sets is the intuitive principle of set construction. By agreement, a set is defined by a sentence S(x) when the replacements for x in the sentence from some given set make that sentence true. Therefore, if some object, m from a given set A, makes the defining sentence true when it replaces the x in S(x), then m is a member of the set satisfying the conditions of the sentence. If other objects also make the sentence true, they will also be members of the same set as m. It is for this reason that

$$\{ x \mid S(x) \}$$

is called the truth set. Furthermore, S(x) is often called the defining sentence of $\{x \mid S(x)\}$. The following example is given to illustrate the above concepts.

Given: $x \in N$; $N = \{1, 2, 3, 4, 5, 6, 7\}$

Defining sentence: S(x) ; x is greater than 5

Therefore, $\{ x \mid S(x) \} = \{6, 7\}$

Thus, $m \in \{x \mid S(x)\}$ if S (m) is true. It should be noted that each truth set will be a subset of the given set. In addition, when no elements of the given set satisfies the defining sentence, the resulting set is the empty set which is also a subset of the given set. Summarizing, we can say that the conditions implied in S(x) on a given set result in a set.

2. Let us continue the investigation of truth sets when we are concerned with sentences where the replacement of two objects from a given set is required to determine the truth or falsity of the sentence.

Given: $N = \{1, 2, 3\}; x, y \in N$

Defining sentence: $S(x, y)$: x is greater than y.

Replacing x and y with the elements of set N, we note the following:

S(x, y) : x is greater than y

S(1, 1) : 1 is greater than 1-------false

S(1, 2) : 1 is greater than 2 -------false

S(1, 3) : 1 is greater than 3-------false

S(2, 1) : 2 is greater than 1-------true

S(2, 2) : 2 is greater than 2-------false

S(2, 3) : 2 is greater than 3-------false

S(3, 1) : 3 is greater than 1-------true

S(3, 2) : 3 is greater than 2-------true

S(3, 3) : 3 is greater than 3-------false

Thus $\{(x, y) \mid S(x, y)\} = \{(2, 1), (3, 1), (3, 2)\}$.

Observe that the pairs for which S(x, y) is true form a set, -a truth set. While this set is not a subset of the given set N, we shall see in the next section that it is a subset of the set containing all possible pairs which can be formed from the given set. The main point of the above illustration is to show that a binary relation is a set of ordered pairs.

We should be careful not to infer from the above statement that sets can only be formed by the use of defining sentences. This is not true since a perfectly good set might be {1, cow, red}. In this case, the only feature which they have in common is that they belong to the same set. In our discussion of mathematical sentences, however, we should realize that the resulting set is a truth set; that is, a set of objects which make a sentence such as S(x) or S(x, y) true upon proper replacement of x, or x and y.

B. The elements of a truth set

It should be noted that in $\{x \mid S(x)\}$, the elements of the resulting set were single objects. In the above illustration of $\{(x, y) \mid S(x, y)\}$, the elements of the resulting truth set were ordered paits.

Let us consider another example to emphasize the point that each of the ordered pairs in a truth set should be considered as a single element of that set. For example, let us toss a penny and a die simultaneously, and describe the result of each toss by an ordered pair in which the H (heads) or T (tails) of the tossed penny is written as the first coordinate of the pair, and the number of the die is written as the second coordinate of the pair. Thus the conditions of the set may be described as:

$Q = \{(x, y) \mid x$ is an H or T of the penny, and y is the number of the die$\}$

Then,

$$Q = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

Note that one element of the set is (H, 1). Another is (T, 5). Each pair is considered collectively as a single member of the set just as the word "pair" is considered as a single word. Thus, each element of the set actually is the result of a single toss of the penny and the die.

Another example might be given with ordered triples. For example, let us toss three different colored dice (red, green, white) into the air. The result of the toss could be indicated by writing the number of the red die first, the number of the green die second, and the number of the white die third, such as:

$$S = \{(1, 1, 1), (1, 1, 2), (1, 1, 3), \ldots, (2, 4, 5), (2, 4, 6), \ldots, (6, 6, 6)\}$$

Observe that each element of the set is the result of a single toss expressed by three symbols in a certain order.

This idea could be extended to order quadruples, quintuples, etc. For n items we use the term "n-tuples." This idea has great application in the study of vectors. However, we shall deal primarily with ordered pairs since our emphasis in this unit is on binary relations or relations between two objects. We now proceed to consider a more precise statement of the formation of these ordered pairs--the cartesian product of two sets.

Exercise B

1. Find the truth set in each of the following sentences:

   a. S(x) : is greater than 5, and $x \in \{1, 2, 3, \ldots, 7\}$

   b. S(x) : x is equal to or greater than 5, and $x \in \{1, 2, 3, \ldots, 7\}$

   c. S(x) : x is a divisor of 10, and $x \in \{1, 2, 3, \ldots, 10\}$

   d. S(x) : x is a multiple of 6, and $x \in \{1, 2, 3, \ldots, 48\}$

   e. S(x) : 5 is greater than x, and $x \in \{1, 2, 3, 4\}$

   f. S(x) : $x^2 = 2x$, and $x \in \{0, 1, 2, 3, 4, 5\}$

   g. S(x, y) : x is greater than y, and $x, y \in \{1, 2, 3, 4\}$

   h. S(x, y) : x is greater than or equal to y, and $x, y \in \{1, 2, 3, 4\}$

   i. S(x, y) : $y = x$ and $x, y \in \{1, 2, 3, 4\}$

   j. S(x, y) : $y = 2x$, and $x, y \in \{1, 2, 3, 4\}$

   k. S(x, y) : $y = 3x$ and $x, y \in \{1, 2, 3, \ldots, n, \ldots\}$

VII. The cartesian product of two sets

A. Introduction

If you refer to example 2 on page 7, you will recall that we must consider all possible ordered pairs which can be formed from the given set so that we may replace the variables in the sentences to determine those ordered pairs which make the sentence true. There is an operation on two sets which enables us to form all possible ordered pairs from two given sets. This operation is called the cartesian product of two sets. Keep in mind that the ordered pair (a, 1) is different from the ordered pair (1, a)

B. Definition of cartesian product of two sets

1. If R and S are two sets, the set of all ordered pairs, in which the first element belongs to set R and the second element belongs to set S is called the cartesian product of the two sets R and S. This entire set of ordered pairs is designated as "R X S" and is read as "R cross S."

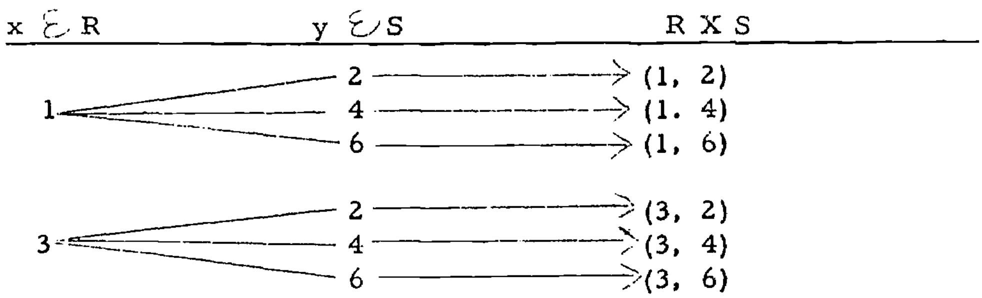
2. If the problem asks for the set of ordered pairs, S X R, then the elements of set S would be written first in the pair and the elements of set R would be written second.

3. The problem of finding the cartesian product of two finite sets with a small number of elements is easily accomplished by using the following tree diagram:

<u>Example 1:</u>

Given: $R = \{1, 3\}$
$S = \{2, 4, 6\}$

To find: R X S, or all ordered pairs (x, y) such that $x \in R$ and $y \in S$



∴ R X S = {(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6)}

4. <u>Example 2:</u>

Given: A = {m, n, o}

B = {s, t}, describe A X B

A X B = {(m, s), (m, t)
(n, s), (n, t)
(o, s), (o, t)}

5. <u>Example 3:</u>

Given: A = {r, s, t}, describe A X A

A X A = {(r, r), (s, t), (t, r)
(r, s), (s, s), (t, s)
(r, t), (s, t), (t, t)}

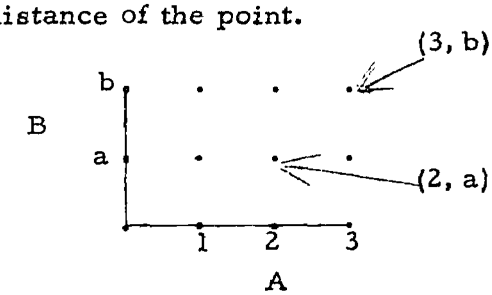
6. Example 4:

Given: $A = \{1, 2, 3\}$

$B = \{a, b\}$, describe A X B and construct a graph of A X B

$A \times B = \{(1, a), (2, a), (3, a)$
$(1, b), (2, b), (3, b)\}$

A graph or picture of A X B can be constructed easily by using points to represent the ordered pairs. First, draw orthogonal (perpendicular) axes and let the first coordinate of the ordered pairs represent the horizontal distance of the point, and let the second coordinate of the ordered pairs represent the vertical distance of the point.

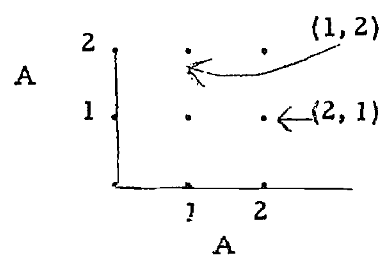


7. Example 5:

Given: $A = \{1, 2\}$

Required: Describe A X A and construct its graph.

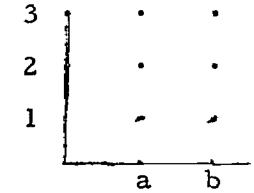
Solution: $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$



8. Example 6:

In Example 4 describe B X A and construct its graph

$B \times A = \{(a, 1), (b, 1)$
$(a, 2), (b, 2)$
$(a, 3), (b, 3)\}$



Exercise C

1. Given: A = {1, 2, 3} ; B = {r, s, t}

   a. Write the Cartesian product A X B ; graph A X B

   b. Write the Cartesian product B X A ; graph B X A

2. Given: R = {chocolate ice cream, vanilla ice cream, strawberry ice cream
   S = {chocolate syrup, marshmallow, whipped cream}

   Required:

   a) R X S ; b) graph of R X S

3. Given: Q = {spiced ham, liverwurst, bologna}
   T = {white bread, rye bread}

   Required:

   a) Q X T b) graph of Q X T

4. Two dice are used in an experiment, one green and the other red. The number of dots on each face of the green die are from 1 to 6 inclusive and are represented by the set

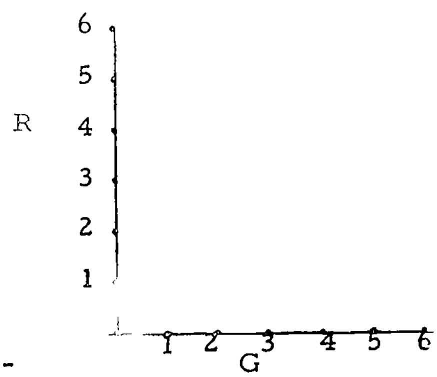
   G = {1, 2, 3, 4, 5, 6}

   The number of dots on each face of the red die are represented by the set

   R = {1, 2, 3, 4, 5, 6}

   a. Write the Cartesian product G X R

   b. Locate the points representing the ordered pairs of G X R on the following graph. G represents the horizontal axis, and R represents the vertical axis. The first number of the ordered pair indicates how many units to the right the point will be located. The second number of the ordered pair indicates how many units above the horizontal axis the point is to be placed. Both numbers are necessary for the location of the point. Plot all the points represented by the ordered pairs of G X R.



The result should be a figure in the form of a square, but made up of dots only. The inside of the square is also filled with evenly spaced dots. This figure is known as a lattice of points or a sample space. The latter concept is used in problems dealing with probability.

5. Can you describe what kind of a sample space would result if a red, a green, and a white die were used in the above experiment?

VIII. Graphs of the cartesian products to be used in mathematical sentences

A. Introduction

In the previous pages we have developed the following important ideas:

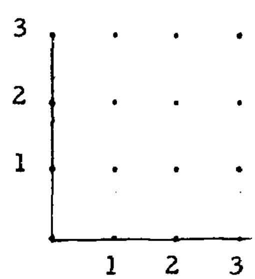
1. A relation is a set of ordered pairs
2. The given set from which all the possible ordered pairs can be formed to find the truth set must be clearly stated. Changing the given set often results in a different truth set for the very same sentence.

From this point on we shall refer to the given set as U and to U X U as the replacement set containing all the ordered pairs from which we will select those pairs which make the sentence true. In our subsequent work we shall be using sets of numbers for our given sets. Some of these will be finite, while others will be infinite (never-ending). It is necessary that we consider some of these number systems and note the types of graphs which will result in the graphing of the cartesian product of U X U.

B. Natural numbers

1. Example 1: A finite set U of natural numbers and the resulting graph of U X U Given: U = {1, 2, 3}

U X U = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

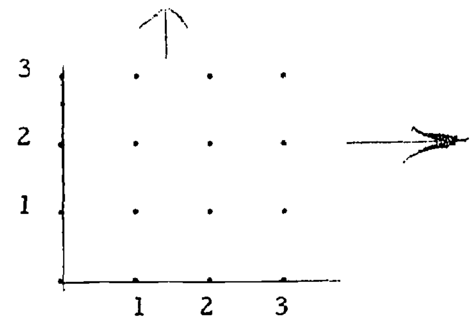


Observe that the set of natural numbers when graphed on a line do not account for all the points on the line. Thus, gaps appear between the points on both the axes and the graph of the Cartesian product.

2. <u>Example 2:</u> An infinite set U of natural numbers and the resulting graph of U X U

Given: U = {1, 2, 3, . . .}

U X U = {(1, 1), (1, 2), (1, 3), . . . , (2, 1), (2, 2), (2, 3), . . . , (3, 1), (3, 2), (3, 3), . . . , (n, n), (n, n+1), . . .}



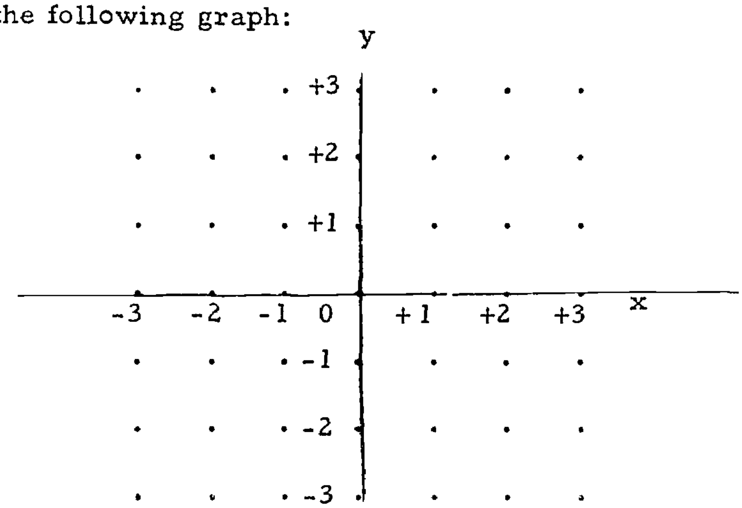
Note: Arrows are used to indicate that the graph of U X U is never-ending

C. Integers

1. <u>Example 1:</u> A finite set U of integers and the resulting graph of U X U

Given: U = {-3, -2, -1, 0, +1, +2, +3}

The set of ordered pairs of U X U can be represented by the points of the following graph:



2. Example 2: An infinite set U of integers and the resulting graph of U X U

Given: U = {... , -3, -2, -1, 0, +1, +2, +3, ...}

The graph of infinite set U X U in this case would be the same as the graph of the finite set given in example 1 above except that arrows would have to be inserted in the diagram to indicate a never-ending or infinite set of ordered pairs.

3. Remember that the graph of the integers on a line has gaps in the number line because the set of integers cannot account for all points on the line. Like the set of natural numbers, the graph of U X U for integers consists of isolated points.

D. Rational numbers

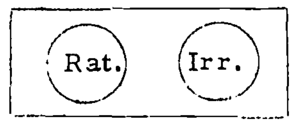
1. The graph of the cartesian product of the rational numbers resembles the graph of U X U for the integers. Although the rational numbers are closely packed, they also have gaps in the number line, and therefore the graph of U X U will also contain isolated points. Many students are mistaken when they think that the rational numbers "fill up" all the points on the line. For example, if we take the rational numbers between 0 and 1 and find the number midway between them. we obtain the number $\frac{1}{2}$. If we then find the number midway between 0 and $\frac{1}{2}$, we obtain the number $\frac{1}{4}$. If we continue this forever, we see that we shall never be able to account for all the points on the line. There will always be "holes" in the line for which no rational number can be assigned.

2. Because the set U of points representing the rational numbers is so closely packed, and the set U X U will also be closely packed, then all the dots used for points will appear to cover the whole plane. We know, however, that no matter how many points are used or how solidly packed the points are, these points are isolated and that gaps, however, tiny, will exist in the graph of U X U for rational numbers. For this reason, we shall not use the graph of U X U of rational numbers in our discussion.

E. Real numbers

1. The real numbers, which consist of all rational and irrational numbers, contain all the numbers necessary to account for every point on the number line. Thus, there are not gaps or "holes" in the number line. This also means that the graph of U X U for the set U of real numbers accounts for every point in the plane. One of the important ideas in analytic geometry is that to every

point on the plane there corresponds one and only one ordered pair of the set U X U where U is the set of real numbers and converselv.



2. In problems where we shall have to consider U as the set of real numbers and U X U as the replacement set, it is impossible to indicate U X U as we did in the case of the natural numbers or integers. We shall consider <u>mentally</u> that every point in the plane representing every ordered pair of U X U is to be used in considering those replacements which make the sentence true.

<u>Exercise D</u>

1. Directions:

   In the following problems a-e,
   A. write the set of ordered pairs of U X U
   B. construct a graph of U X U

   a. U = {-1, 0, 1}

   b. U = {2, 3}

   c. U = {-2, -1, 0, 1, 2}

   d. U = {-1, 0, 1, 2}

   e. U = {a. b} Hint: Let a and b represent the lengths of some arbitrary distance from the intersection of the axes.

2. If U = {integers}, construct a graph of U X U. Does this graph account for all points in the plane?

3. If U = {real numbers} and you are required to show each ordered pair on the graph by a dot made by your pencil point, what would you do to your graph so that all points would be accounted for?

4. If U has n elements how many members does U X U contain?

IX. Sentences in two variables

A. Solution set of sentences in two variables

1. When we select those ordered pairs of the universal set U X U which make a sentence in two variables true, we have obtained the truth set or the solution set. Thus, the sentence acts as a set selector because it divides the set of U X U into two subsets:

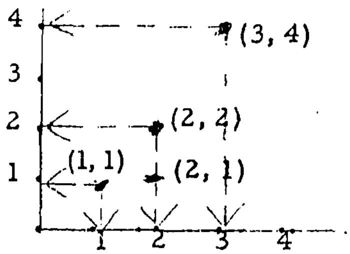
a. The subset of ordered pairs which makes the sentence true--the solution set or the truth set. Thus, a relation in U is a subset of U X U.

b. The subset of ordered pairs which makes the sentence false.

B. Domain and range of a relation

1. The domain of a relation is defined as the set of the first coordinates of the solution set. These first coordinates are sometimes referred to as the "pre-images" of the relation. The domain of a relation may be easily obtained from a graph by mentally projecting a vertical line from each and every point in the graph to the horizontal axis and reading the value of that point on the horizontal axis. The entire set of numbers obtained by such projections will make up the set known as the domain of the relation.

2. The range of the relation is defined as the set of the second coordinates of the solution set. The second coordinates are sometimes referred to as the "images" of the relation. Each image is associated with one or more pre-images of the relation. The range of a relation may be easily obtained from a graph by mentally projecting a horizontal line from each and every point in the graph of the relation until it meets the vertical axis. The entire set of numbers obtained by such projects will make up the set known as the range of the relation.

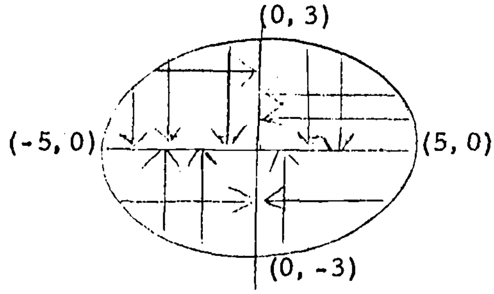


U = {1, 2, 3, 4}

Solution Set : {(1, 1), (2, 1), (2, 2), (3, 4)}

Domain: {1, 2, 3}

Range: {1, 2, 4}



U = {real numbers}

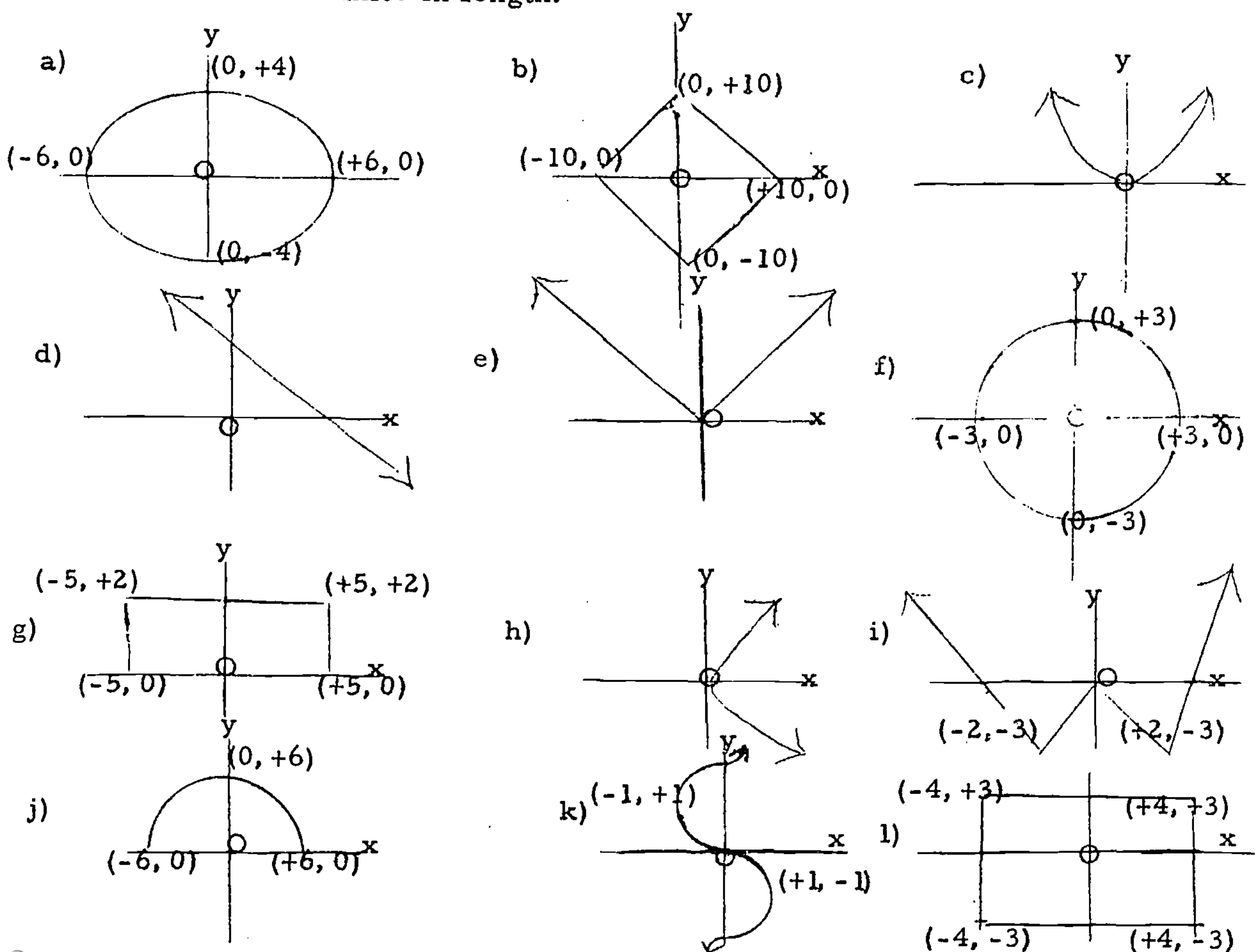
Solution Set ={infinite}; impossible to list

Domain: $\{-5 \leq x \leq +5\}$

Range: $\{-3 \leq y \leq +3\}$

Exercise D

1. The following graphs represent relations in U where U = {real numbers}. State in each case the domain and the range of the relation. Arrowheads are placed on lines to indicate that they are infinite in length.



2. The following ordered pairs in each problem represent a relation. State the domain and the range of the relation in each problem.

   a. $\{(1, 2), (2, 4), (3, 2), (4, 2)\}$

   b. $\{(1, 4), (1, 3), (1, 1), (2, 4)\}$

   c. $\{(-3, 8), (-5, 7), (-2, 6), (-1, -7), (3, 2)\}$

   d. $\{(7, 3), (6, 2), (5, 2), (4, 3), (2, 7)\}$

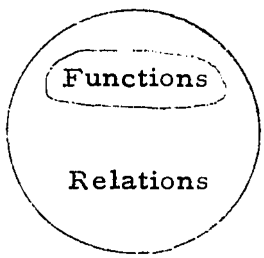
   e. $\{(-9, 2), (-8, 6), (4, 3), (7, 2), (-9, 1)\}$

   f. $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n), \ldots\}$

   g. $\{(3, 1), (3, 2), (3, 3), \ldots, (3, n), \ldots\}$

C. Functions

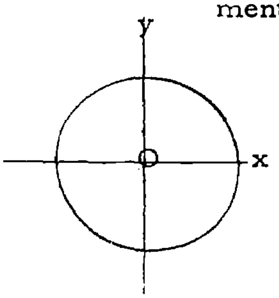
1. A function is a special kind of relation in U such that for each first coordinate there is one and only one second coordinate. A relation, however, may have one or more second coordinate for any first coordinate. It should be emphasized that every function is a relation, but that not every relation is a function. In other words, the set of functions are a subset of the set of relations as is represented by the following set diagram.



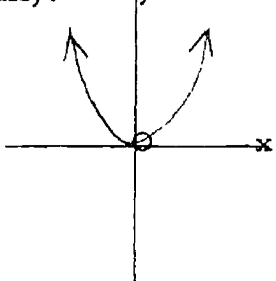
There are two methods by which we can determine whether a relation is a function. We shall discuss these methods in the next two sections.

2. Vertical line test of a function

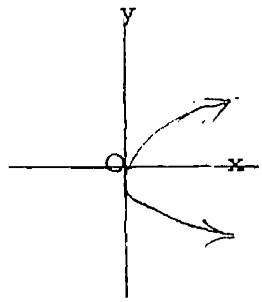
A simple method of determining whether a sentence in two variables represents a function is to draw the graph of the relation and then consider the vertical lines through each and every point of the graph. If every vertical line under consideration can intersect the graph in one and only one point, then the graph represents a function. If at least one vertical line intersects the graph in more than one point, then the graph does not represent a function, but a relation. Of course, the vertical lines do not actually have to be drawn, but thought of mentally.



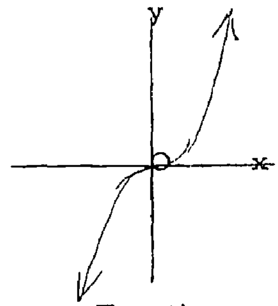
Relation, <u>not</u> a function



Function



Relation, <u>not</u> a function



Function

3. Ordered pair test

A second method to determine whether a relation is a function is to observe the ordered pairs of the relation. This method does not require a graph. If you remember the definition of function as a special relation or set of ordered pairs in which each first coordinate can have one and only one second coordinate, it should be clear that a function cannot have two ordered pairs with the same first coordinate. In other words, a function must have all ordered pairs with different first coordinates. However, the second coordinates may be different or can be the same. The test concerns the first coordinates of all the ordered pairs.

Example 1: $\{(1,2), (3,4), (5,4)\}$ This is a function.

Example 2: $\{(1.2), (1,3), (5,4)\}$ This is a relation, <u>not</u> a function. Observe that 1 is used as the first coordinate in two different pairs.

Example 3: $\{(a,b), (b,c), (c,d)\}$ This is a function.

Example 4: $\{(2,3), (3,3), (4,3), (5,3)\}$ This is a function since all first coordinates are different.

<u>Exercise E</u>

1. Refer to the graphs in problem 1 of Exercise D. Use the vertical line test to determine whether these graphs represent functions.

2. Refer to problem 2 of Exercise D. Use the ordered pair test to determine whether these relations are functions.

3. In problems a-f, the given graphs represent relations in U, where U = {1, 2, 3, 4} In each of the graphs

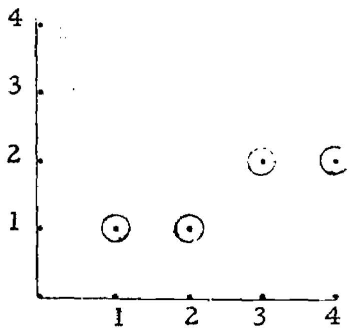
(1) Write the set of ordered pairs describing the relation represented by the circled points.

(2) Determine whether the relation is a function.

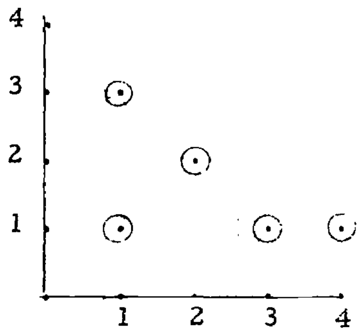
(3) State the domain of the relation

(4) State the range of the relation.

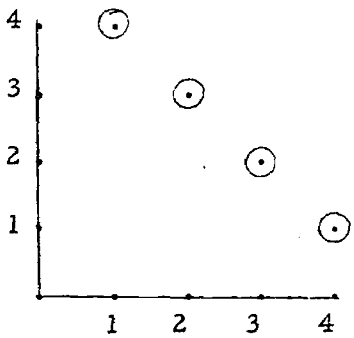
a)



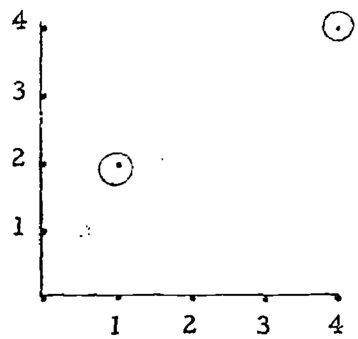
d)



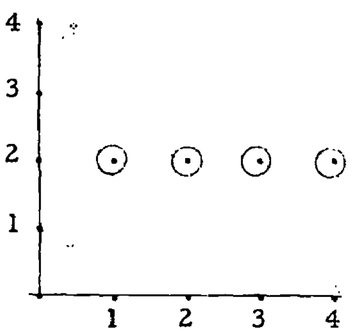
b)



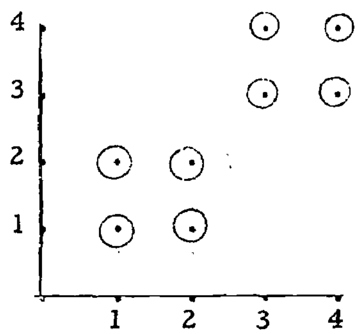
e)



c)



f)



X. Sentences expressing equality

A. Example 1 — Given: $\{(x, y) \mid y = x\}$ where $U = \{1, 2, 3, 4\}$

Required: a) to find the solution set
b) to graph the relation

1. If $U = \{1, 2, 3, 4\}$, then the universal set of ordered pairs of U X U is:

$$\{(1, 1), (2, 1), (3, 1), (4, 1)$$
$$(1, 2), (2, 2), (3, 2), (4, 2)$$
$$(1, 3), (2, 3), (3, 3), (4, 3)$$
$$(1, 4), (2, 4), (3, 4), (4, 4)\}$$

2. From this universal set, U X U, we are to select only those ordered pairs which make the sentence $y = x$ true. It should be clear that this solution set will consist of those ordered pairs whose first coordinate is equal to the second coordinate. Therefore, the solution set is:

$$\{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

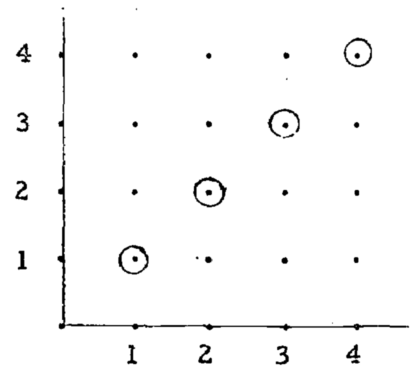
Note that the solution set is a subset of U X U.

3. The following graph reveals these ideas:

a. All the points in the graph represent the universal set.

b. Only those points which are circled belong to the solution set and make the sentence true.

c. All points not circled belong to the subset of the ordered pairs which make the sentence false.



4. Note that the graph of this sentence is <u>not</u> a continuous line, but is composed of isolated points.

5. Using the vertical line test, we see that vertical lines intersect the graph of the relation (the circled points) in one and only one point. Also by the ordered pair test, every first coordinate is different. Hence, this relation is a function.

6. Domain of the relation: $\{1, 2, 3, 4\}$

7. Range of the relation: $\{1, 2, 3, 4\}$

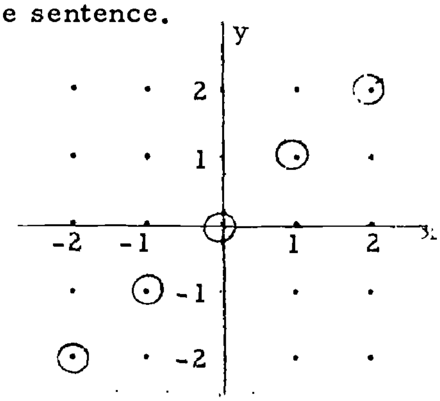
B. <u>Example 2</u> Given: $\{(x, y) \mid y = x\}$ where $U = \{-2, -1, 0, +1, +2\}$

Required: a. To find the solution set
b. To construct the graph of the relation

1. If $Y = \{-2, -1, 0, +1, +2\}$, the graph of U X U will consist of 25 points representing all the ordered pairs of U X U. From this universal set we are to select those ordered pairs which make the sentence y = x true. The solution set for this sentence is:

$$\{(-2, -2), (-1, -1), (0, 0), (+1, +1), (+2, +2)\}$$

2. The following graph illustrates the solution set by circling those points representing ordered pairs satisfying the requirements of the sentence.



3. Domain: $\{-2, -1, 0, +1, +2\}$

4. Range: $\{-2, -1, 0, +1, +2\}$

5. This relation is a function

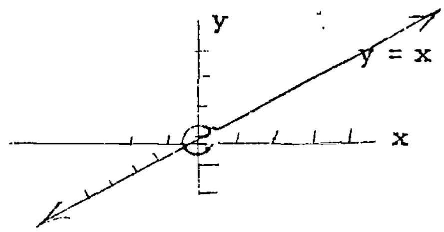
C. <u>Example 3</u> To find and graph: $\{(x, y) \mid y = x\}$
when U = {real numbers}

1. If we let U = {real numbers}, then the totality of ordered pairs under discussion in this problem will be represented by all the points of the plane. Note that in this case the

universal set is infinite. The solution set is also an infinite set consisting of such ordered pairs as:

(-3, -3), (+5, +5), (0, 0), ($\frac{1}{2}$, $\frac{1}{2}$), (.727272...., 727272...) $\sqrt{3}$, $\sqrt{3}$),
etc.

2. The graph of this sentence is the continuous line made up of points whose ordered pairs have equal components.



3. Domain: {real number}

4. Range: {real numbers}

5. This relation is a function

6. Observe that the same sentence was used in this problem and the two previous problems, yet the solution set was different in each case. The graph was therefore different. This difference in the solution set for the same sentence is due to the fact that different universal sets were used. This should emphasize the statement made previously that it is necessary to state precisely the universal set in each problem in order to avoid confusion and disagreement

D. Graphing mathematical sentences whose variables are of the first degree

1. Many of the equations which you will meet in your work will be of the first degree; that is, the exponent of each variable is 1 and only one variable can appear in each term. Thus, the following sentences are of the first degree.

$x + y = 5$

$y = 2$

$x = 3$

These sentences when graphed will have their points lying in a straight line. Those graphs using natural numbers of integers as the universal set will not have a continuous line, whereas the graph using the set of real numbers as the given set will result in a continuous straight line.

2. If the graph of a first degree equation is always a straight line, then we shall need three ordered pairs which shall be represented as points. Although it is true that only two points are needed to draw a line, it is wise to include a third point to check the accuracy of the first two points. If the three points are correct, only one line will pass through all three points. If the three points are not correct, then connecting the three points will result in a triangle.

3. To find any three ordered pairs which make the sentence true, it is convenient to transform the sentence so that the second coordinate is always expressed in terms of the first coordinate.

Example: Given sentence: $x + y = 5$
Transformed sentence: $y = 5 - x$

Observe that y and 5-x are different names for the same number. If we choose any three values arbitrarily from the given set to represent the value of the first coordinate x and then replace these values in the transformed sentence, we shall obtain the corresponding second coordinate. Suppose our given set is the set of real numbers and we select arbitrarily $x = 3$, $x = -1$, and $x = 7$

<u>If $x = 3$</u>

$y = 5 - x$
$y = 5 - 3$ Therefore, this ordered paid is (3, 2)
$y = 2$

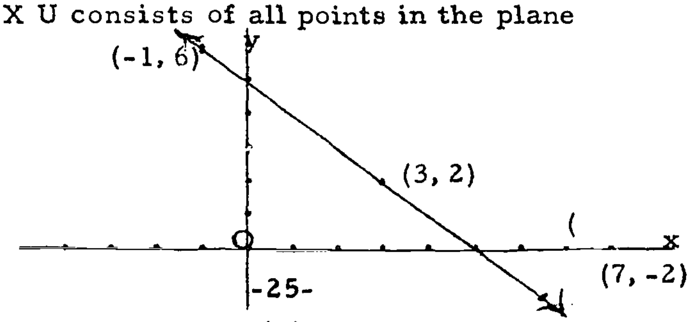
<u>If $x = -1$</u>

$y = 5 - x$
$y = 5 - (-1)$ Therefore, this ordered pair is (-1, 6)
$y = 6$

<u>If $x = 7$</u>

$y = 5 - x$
$y = 5 - 7$ Therefore, this ordered pair is (7, -2)
$y = -2$

With these three ordered pairs we may draw a continuous line since U X U consists of all points in the plane



Exercise F

1. In problems a - k, graph the relations in U if $U = \{1, 2, 3, 4, 5\}$. In addition, state:

   (1) the solution set
   (2) whether the relation is a function
   (3) the domain of the relation
   (4) the range of the relation

   a. $\{(x, y) \mid y = 2x\}$

   b. $\{(x, y) \mid y = \frac{1}{2}x\}$

   c. $\{(x, y) \mid x + y = 5\}$

   d. $\{(x, y) \mid y + 2x = 5\}$

   e. $\{(x, y) \mid y = x - 1\}$

   f. $\{(x, y) \mid x = 2\}$

   g. $\{(x, y) \mid y = -3\}$

   h. $\{(x, y) \mid x = -2\}$

   i. $\{(x, y) \mid y = 3\}$

   j. $\{(x, y) \mid 2x - 3y = 6\}$

2. Repeat problem 1 using $U = \{-2, -1, 0, +1, +2\}$

3. Repeat problem 1 using $U = \{\text{real numbers}\}$. Do not answer part (1) since the solution sets are infinite.

XI Inequalities in sentences in two variables

A. Introduction

1. The concept of ordered pairs as the solution set of sentences in two variables lends itself well to the solution of inequalities.

2. In considering inequalities we consider the universal set in exactly the same way which we used in considering equalities (equations). Recall that changing the universal set may often affect the solution set.

B. Example 1 To find the solution set and graph of:

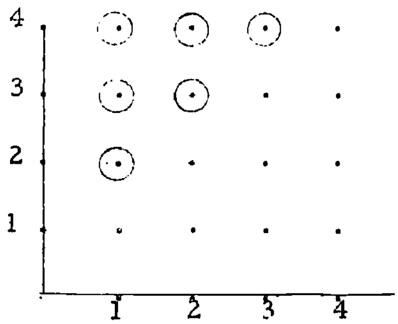
$$\{(x, y) \mid y > x\} \text{ when } U = \{1, 2, 3, 4\}$$

1. If we let $U = \{1, 2, 3, 4\}$, we have a universal set consisting of 16 ordered pairs and a graph of the universal set consisting of 16 points.

2. Note that the requirements of the sentence state that the second component must always be greater than the first component. It should be clear that the solution set must consist of the following ordered pairs:

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

3. The circled points of the following graph indicate those points representing ordered pairs which are solutions to the given sentence.



4. Domain: $\{1, 2, 3\}$

5. Range: $\{2, 3, 4\}$

6. This relation is not a function by the vertical line test.

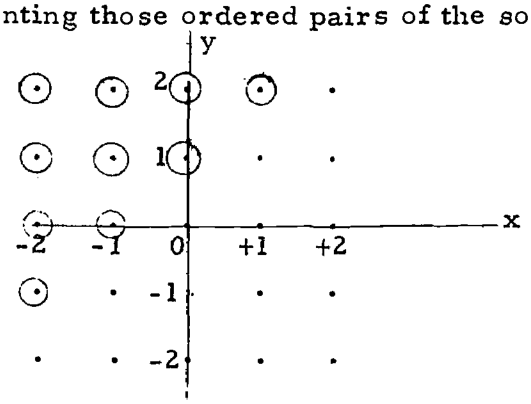
C. Example 2 To find the solution set and graph of:

$$\{(x, y) \mid y > x\} \text{ when } U = \{-2, -1, 0, +1, +2\}$$

1. Let $U = \{-2, -1, 0, +1, +2\}$. The universal set will consist of 25 ordered pairs represented by 25 points. Observe that the second component must always be greater than the first component.

Solution Set = {(-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, 0), (-1, 1), (-1, 2) (0, 1), (0, 2), (1, 2)}

2. The circled points of the following graph indicate those points representing those ordered pairs of the solution set.



3. Domain: $\{-2, -1, 0, +1,\}$

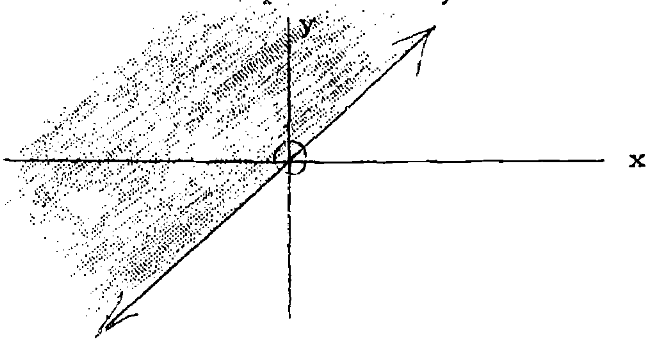
4. Range: $\{-1, 0, +1, +2\}$

5. This relation is not a function.

6. Note that the graph of the above solution set consists of those points which lie above the graph of $\{(x, y) \mid y = x\}$ having the same universal set.

D. Example 3 To find the solution set and graph of $\{(x, y) \mid y > x\}$ where $U = \{\text{real numbers}\}$.

1. If $U = \{\text{real numbers}\}$, then the totality of ordered pairs of $U \times U$ can be represented by all the points of the plane. Instead of circling those points which represent the solution set, we shall shade the entire region containing the points representing the solution set. In this case, the shaded region including all points lying above the graph of $y = x$ represents geometrically the solution set of the sentence. Note that the region should exclude all points of the line represented by the sentence $y = x$.



2. Domain: {real numbers}

3. Range: {real numbers}

4. This relation is not a function.

5. We usually think of a line as a geometric figure which connects two points on a plane. However, a line performs another task; that is, it divides the plane into three infinite sets of points. In this problem where $U = \{\text{real numbers}\}$, the line represented by the sentence $y = x$ divides the plane into the following sets:

   a. the set of points of the line represented by the sentence $y = x$.

   b. the set of points of the half-plane lying above the line represented by the sentence $y = x$; that is, the points of this region represent all the ordered pairs which make the sentence $y > x$ true.

   c. the set of points of the half-plane lying below the line represented by the sentence $y = x$; that is, the points of this region represent all the ordered pairs which make the sentence $y < x$ true.

6. Because lines can slant in many directions, confusion often arises as to which region contains those points whose ordered pairs make a sentence true. A simple test is to select an arbitrary point belonging to one of the regions, and determine whether the ordered pair representing that selected point makes the sentence true. If the sentence is thus judged true for this particular ordered pair, then the region must lie on the same side as the selected point. If the sentence is judged false for that particular ordered pair, then the region must lie on the other side of the line.

   The point which results in the least amount of calculation is the point at the origin represented by the ordered pair, (0, 0). If, however, the line passes through the origin, some other point must be selected which lies on either side of the line.

Exercise G

In the following problems you are given sentences in two variables with a specified set U. For each of the problems:

(1) Graph the relation in U. The only points to be used in this graph are those whose ordered pairs make the sentence true. DO NOT PLACE IN YOUR DIAGRAM ALL OF THE ORDERED PAIRS OF U X U.

(2) State the solution set in ordered pairs if U X U is a small finite set; in a descriptive statement, if possible, where U X U is an infinite set

(3) State the domain of the relation

(4) State the range of the relation

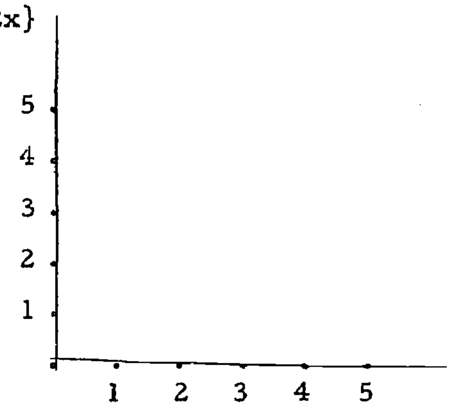
(5) State whether the relation is a function

1. If U = {1, 2, 3, 4, 5}

   a. $\{(x, y) \mid y = 2x\}$

   Solution:

   (1)



(2) Solution set: $\{(1, 2), (2, 4)\}$

(3) Domain: $\{1, 2\}$

(4) Range: $\{2, 4\}$

(5) This relation is a function

b. $\{(x, y) \mid y < x\}$

c. $\{(x, y \mid y = -x,$

d. $\{(x, y) \mid y \geq x\}$

e. $\{(x, y) \mid y = 2x\}$

f. $\{(x, y) \mid y < 2x\}$

g. $\{(x, y) \mid y = 6x\}$

h. $\{(x, y) \mid x + y = 5\}$

i. $\{(x, y) \mid x + y < 5\}$

j. $\{(x, y) \mid x + y \geq 5\}$

k. $\{(x, y) \mid x = 2\}$

l. $\{(x, y) \mid y = 3\}$

m. $\{(x, y) \mid x = -1\}$

n. $\{(x, y) \mid y = -2\}$

o. $\{(x, y) \mid y \leq -\frac{1}{2}x\}$

2. Repeat problem 1(a) to (o) if U = $\{-2, -1, 0, +1, +2\}$

3. Use the same sentences in problem 1 (a) to (o), but let U = {real numbers}. For each problem:

   (1) Graph the relation
   (2) State the domain of the relation
   (3) State the range of the relation
   (4) State whether the relation is a function

E. The absolute value of signed numbers

1. The absolute value of a positive number is that same positive number. The symbol for absolute value consists of two vertical bars on each side of the number.

   $|+5| = +5$

   $|+7| = +7$

   In many texts the above examples are written as

   $|+5| = 5$

   $|+7| = 7$

   There should be no confusion, however, because these texts use the numeral "5" as another name of "+5" since the set of natural numbers have the same structure as the positive integers. To avoid confusion, we shall agree to use the notation as given in the first example.

2. The absolute value of a negative number is the additive inverse of the negative number.

$$|-5| = -(-5) \text{ or } |-5| = +5$$

$$|-7| = -(-7) \text{ or } |-7| = +7$$

3. The absolute value of zero is zero

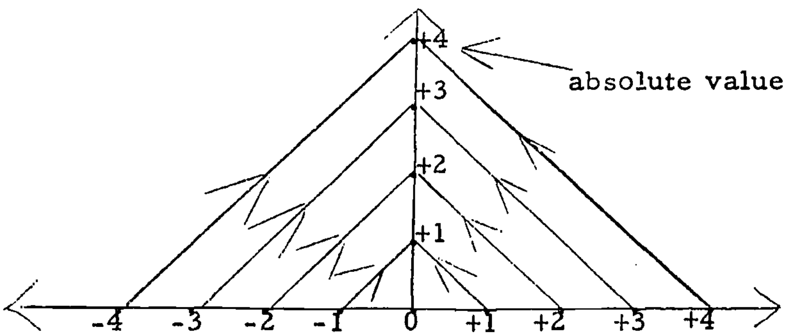
$$|0| = 0$$

4. Summary of absolute value of signed numbers

$$|x| = x \text{ where } x \geq 0$$

$$|x| = -x \text{ where } x < 0$$

The following diagram shows how the absolute value represents a relation whereby a positive or negative number is associated with itself or its additive inverse, and zero is associated with itself.



<u>Exercise H</u>

1. Determine the following:

   a. $|+18|$

   b. $|-36|$

   c. $|0|$

2. State whether the following statements are true or false:

a. $|-3| \neq |+3|$

b. $-3 \neq +3$

c. $|-4| > |-19|$

d. $|+4| < |+19|$

e. $|+4| < |-19|$

f. $|-4| > |+19|$

g. $|-10| < +36$

h. $|-101| < |+36|$

i. $|-5| > |-125|$

j. $|-26| = |+26|$

k. $0 > -5$

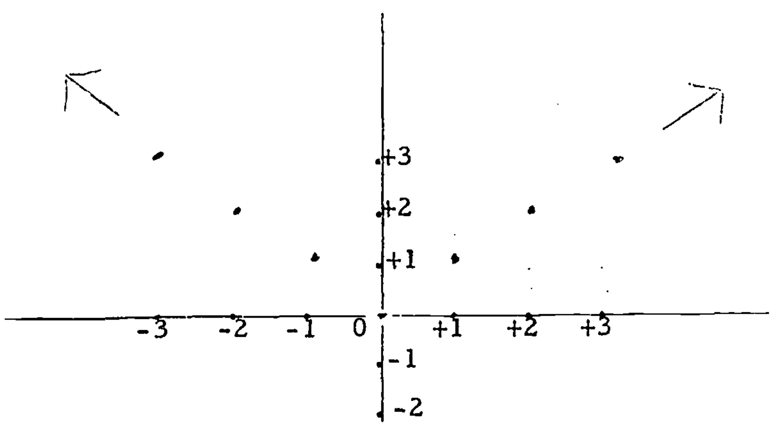
l. $|0| > |-5|$

F. Sentences involving inequalities with absolute values

1. Example 1 — To find the solution set and graph of $\{(x, y) \mid y = |x|\}$ where U = {integers}.

a. Observe that the sentence to be satisfied requires that the second coordinate always be equal to the absolute value of the first coordinate.

Solution set = {(0, 0), (+1, +1), (-1, +1), (+2, +2), (-2, +2), (+3, +3), (-3, +3), ... }



b. Domain: {integers}

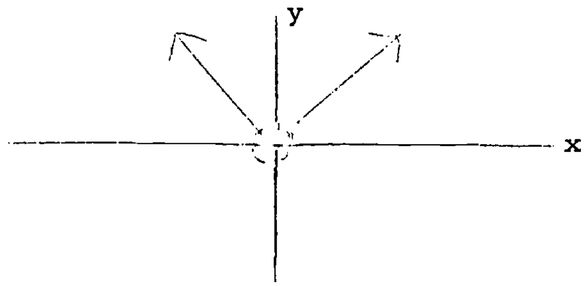
c. Range: {all non-negative integers} or {all positive integers and zero}

d. This relation is a function.

2. Example 2: To find the solution set and graph of

$\{(x, y) \mid y = |x|\}$ where U = { real numbers}

a. From previous discussions it should be clear that the solution set consists of all ordered pairs which make up the two half-lines as shown in the following graph. It is impossible to list all the ordered pairs which make the sentence true.



b. Domain: {all real numbers}

c. Range: {all non-negative real numbers}

d. This relation is a function.

Exercise I

Direction: In the following sentences

(1) Graph the relation in U if U = {real numbers}
(2) State the domain of the relation
(3) State the range of the relation
(4) State whether the relation is a function

1. $\{(x, y) \mid y \geq x\}$
2. $\{(x, y) \mid |x| = +3\}$
   (Note. y can take on any values and not affect the sentence)
3. $\{(x, y) \mid |x| > +3\}$
4. $\{(x, y) \mid |x| < 3\}$
   (Note: "3" is another name for "+3"
5. $\{(x, y) \mid |x| \neq 3\}$
6. $\{(x, y) \mid 3 < x < 5\}$
7. $\{(x, y) \mid 3 \leq x \leq 5\}$
8. $\{(x, y) \mid y = |2x|\}$
9. $\{(x, y) \mid |y| = 4\}$
10. $\{(x, y) \mid |y| \leq 2\}$
11. $\{(x, y) \mid |x| + |y| = 4\}$
    Since $|x| = +x$ if $x \geq 0$
    and $|x| = -x$ if $x < 0$
    then we really have four sentences to satisfy
    $+x + y = 4$ where $x \geq 0$ and $y \geq 0$
    $-x + y = 4$ where $x \leq 0$ and $y \geq 0$
    $+x - y = 4$ where $x \geq 0$ and $y \leq 0$
    $-x - y = 4$ where $x \leq 0$ and $y \leq 0$
12. $\{(x, y) \mid |x| + |y| < 4\}$
13. $\{(x, y) \mid |x| + |y| > 2\}$
14. $\{(x, y) \mid |x| + |y| \neq 3\}$
15. $\{(x, y) \mid |x| - |y| > 2\}$

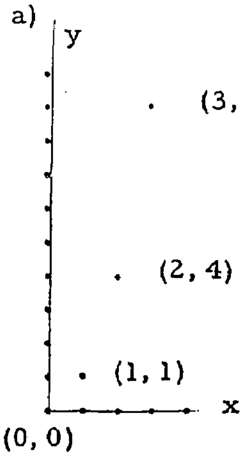
Problems 15-28: Repeat problems 1-14 but use U = {real numbers}.

G. Sentences involving variables of the second degree

1. Example 1 — Discuss the graph of $\{(x, y) \mid y = x^2\}$ when

   a. U = {0, 1, 2, 3, . . . , 9}
   b. U = {integers}
   c. U = {real numbers}

a)

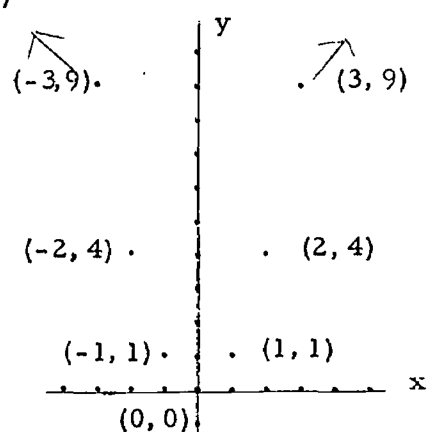


Domain: ={0, 1, 2, 3}

Range: ={0, 1, 4, 9}

This relation is a function

b)

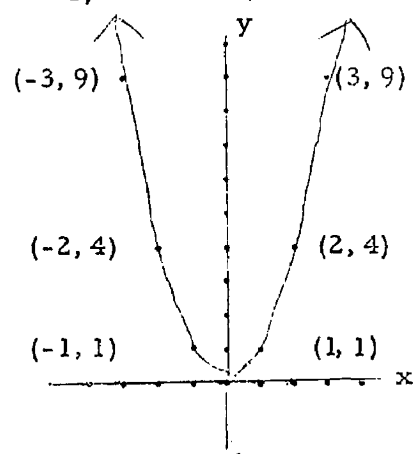


Domain: {integers}

Range: $\{0, 1, 4, 9, \ldots, n^2, \ldots\}$

This relation is a function

c)

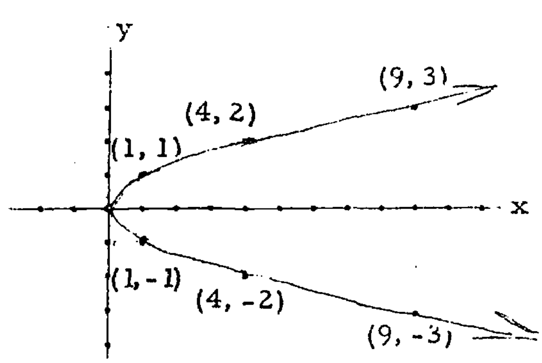


Domain: {real numbers}

Range: {non-negative real numbers}

This relation is a function

2. Example 2 — Discuss the graph of $\{(x, y) \mid x = y^2\}$ when U = {real numbers}

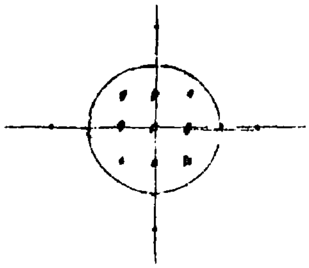


Domain: {non-negative real numbers}

Range: {real numbers}

This relation is not a function.

3. Example 3 Discuss the graph of $\{(x, y) \mid x^2 + y^2 < 4\}$ when $U = \{-2, -1, 0, +1, +2\}$



Domain: $\{-1, 0, +1\}$

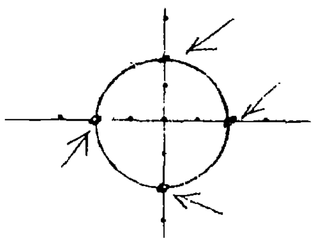
Range: $\{-1, 0, +1\}$

This relation is not a function

4. Example 4 Discuss the graph of $\{(x, y) \mid x^2 + y^2 = 4\}$ when

a. $U = \{-2, -1, 0, +1+2\}$
b. $U = \{\text{real numbers}\}$

a.

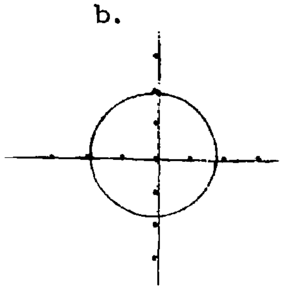


Domain: $\{-2, 0, +2\}$

Range: $\{-2, 0, +2\}$

This relation is not a function

b.



Domain: $\{-2 \leq x \leq +2\}$

Range: $\{-2 \leq y \leq +2\}$

This relation is not a function

Exercise I

Directions: Discuss the graphs of the following problems when

a. $U = \{\text{integers}\}$
b. $U = \{\text{real numbers}\}$

1. $\{(x, y) \mid x^2 + y^2 = 16$
2. $\{(x, y) \mid x^2 + y^2 > 9\}$
3. $\{(x, y) \mid x^2 + y^2 = 25\}$
4. $\{(x, y \mid x^2 + y^2 < 25\}$
5. $\{(x, y) \mid 9 < x^2 + y^2 < 25\}$
6. $\{(x, y) \mid 9 \leq x^2 + y^2 \leq 25\}$
8. $\{(x, y) \mid y = x^2 - 2\}$
9. $\{(x, y) \mid x = y^2 - 1\}$
10. $\{(x, y) \mid y = x^2 + 2\}$
11. $\{(x, y) \mid x = y^2 + 1\}$
12. $\{(x, y) \mid y \geq x^2\}$
13. $\{(x, y) \mid x \leq y^2\}$

XII. Functional notation

A. From our work in sets, we note that a set is simply designated by a capital letter, such as U or S or A, etc. Since we have developed the concept that a function is a special kind of relation or set of ordered pairs, we often designate the function by a capital letter, such as F or G, etc. Thus we may use the notation:

$$F = \{(x, y) \mid y = x - 1\} \text{ or } G = \{(x, y) \mid y = x^2\}$$

Frequently we find that a small (lower-case) letter is used to represent the function or set of ordered pairs such as:

$$f = \{(x,y) \mid y = 2x\} \text{ or } g = \{(x, y) \mid y = x^2\}$$

B. We have seen how equations or inequalities act as devices to assign a second coordinate to a first coordinate of the ordered pair. The second coordinate assigned to any first coordinate x is written:

$$f(x) \text{ or } g(x).$$

This is read as "f at x" or "g at x"; sometimes it is read "f of x" or g of x" Therefore, $y = f(x)$. y and f(x) are names for the same number. It should be emphasized that f and f(x) are not the same. The notation "f" means the entire set of ordered pairs which make the sentence true, whereas the notation "f(x)" simply means the second coordinate of the ordered pair which is assigned to some first coordinate x.

C. The function can therefore be represented in different forms:

$$f = \{(x, y) \mid y = x^2\}$$

$$f = \{(x, x^2)\}$$

$$f = \{(x, f(x) \mid f(x) = x^2\}$$

These forms are used interchangeably so that it is wise to become familiar with them.

D. Example: Given the function f defined by $\{(x, y) \mid y = x - 1\}$, find f(1), f(0), f(-12)

$f(1) = 1 - 1 = 0$; therefore $f(1) = 0$

$f(0) = 0 - 1 = -1$; therefore $f(0) = -1$

$f(-12) = 1 - (-12) = 1 + 12 = 13$; therefore $f(-12) = 13$

E. <u>Example:</u> Given the function G defined by $\{(x, y) \mid y = 3x^2\}$ for the set $U = \{-2, -1, 0, +1, +2\}$

a. Describe the function G
b. Find G (+2)
c. Find G (-1)

Solution: a. $G = \{(x, y) \mid y = 3x^2\}$

$G = \{(-2, +12), (-1, +3), (0, 0), (+1, +3), (+2, +12)\}$

b. G (+2) = +12
c. G (-1) = + 3

F. <u>Example:</u> Given the function f defined by the sentence or rule

$$f(r) = (4/3)\,\pi\, r^3$$

find: (1) f(-5)
(2) f(a)
(3) f(3t)

Solution: (1) $f(-5) = -500\pi/3$

(2) $f(a) = (4/3)\,\pi a^3$

(3) $f(3t) = 36\pi t^3$

<u>Exercise J</u>

Directions: Given the following sentences which define the function f in which the universal set is {real numbers} calculate the following for each problem from 1 - 8

a. f(-3)	d. f (0)	g. f(-3a)
b. $f(\frac{1}{2})$	e. f(2.13)	h. f(+100)
c. f(5)	f. f(4a)	

1. $y = 4x + 9$
2. $f(x) = 8x - 5$
3. $f(x) = 3x^2 - 4x + 5$
4. $y = x^2 + 3x - 2$
5. $y = 2.13/x$
6. $r = 3t^2 - 5t$
7. $y = 2$
8. $f(x) = -2$

*9. Given $G = \{(x, y) \mid G(x) = 4^x\}$, find $G(0)$; $G(-\frac{1}{2})$; $G(\frac{1}{2})$

*10. Given $F = \{(x, y) \mid F(x) = \frac{x+1}{x-1}\}$ defined for all values except $x = 1$.

Find $F(0)$; $F(2)$; $F(-1)$; $F(m + 1)$

*11. Given $f = \{(x, y) \mid f(x) = x^2 + 4x - 1$, show that

$f(x + h) = x^2 - 3x + 7 + 2xh - 3h + h^2$

*12. Given $g = \{(x, y) \mid g(x) = x^4 + 3x^2 - 4\}$, show that $g(a) = g(-a)$.

*13. Given $f = \{(x, y) \mid f(x) = 1/x$, show that $f(x + h) - f(x) = -h/x^2 + xh$

*14. Given $g = \{(x, y) \mid g(x) = mx + n$, show that $g(x + t) - g(x) = mt$

*15. Given $f = \{(x, y) \mid f(x) = x + \frac{1}{x}\}$ and $g = \{(x, y) \mid g(x) = x - \frac{1}{x}\}$
show that $f(x)\, g(x) = g(x^2)$

XIII. Systems of sentences involving equations and inequalities

A. Introduction

In the previous sections the required solution involved finding a set of ordered pairs which satisfied the conditions defined by a sentence of equality or inequality. There are many problems in mathematics which require a set of ordered pairs which must satisfy two or more sentences. Such a problem is referred to as a simultaneous system. The solution set which satisfies two or more sentences simultaneously contains the ordered pairs which make the sentences true. The number of ordered pairs which make up the solution set may be none, one, more than one, and in some cases, the solution set may be an infinite set. Such a set represents the intersection of the two sets because it contains the ordered pair or pairs common to both sentences.

The method of graphing the solution of such systems involves the graphing of the lines or regions as shown in the previous sections. Where two regions are involved, it is wise to indicate one region by shading one region with horizontal lines, and shading the second region by vertical lines. Then the region which shows the double shading or cross-hatching represents the region containing the points representing the ordered pairs making the sentences true. If more than two regions are involved, slanted shading may be used. Pencils of different colors are also helpful.

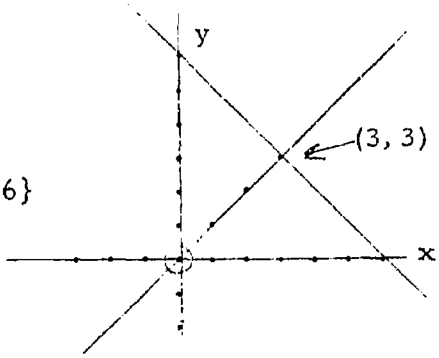
B. Example 1: Graph the following relations and state the solution set common to both sentences, where $U = \{\text{real numbers}\}$

$\{(x, y) \mid y = x\}$

$\{(x, y) \mid x + y = 6\}$

or

$\{(x, y) \mid y = x\} \cap \{(x, y) \mid x + y = 6\}$

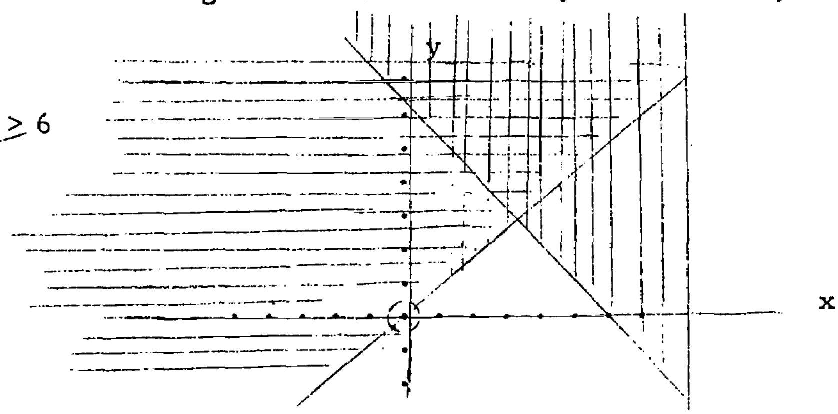


Solution set: $\{(3, 3)\}$ or $\begin{matrix} x = 3 \\ y = 3 \end{matrix}$

C. Example 2: Find the solution set common to both of the following sentences where $U = \{\text{real numbers}\}$

$y \geq x$

$x + y \geq 6$



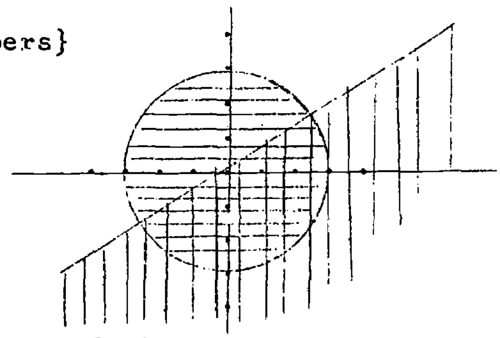
The region indicated by the cross-hatching contains those points representing those ordered pairs of the solution set which are common to both sentences. This solution set is infinite.

D. Example 3: Find the common solution set to both sentences where

$U = \{\text{real numbers}\}$

$x^2 + y^2 < 9$

$y \leq x$



Notice that the solution set common to both sentences is represented by those points lying inside the circle and lying on and below the straight line but not on the circumference of the circle.

Exercise K

Directions: In the following simultaneous systems of sentences make a graph of the sentences and state the solution set by a description of the graph of the solution set or listing the members of the solution set where it is small. In all cases consider $U = \{\text{real numbers}\}$

1. $y = x$
   $y = 3x$

2. $y = x$
   $y = x + 2$

3. $x + y = 5$
   $x = 3$

4. $x + y = 5$
   $x - y = 3$

5. $x + 2y = 8$
   $y = \frac{1}{2}x$

6. $2x + y = 5$
   $x - y = 1$

7. $x - y = 0$
   $x + y = 5$

8. $y - x = 1$
   $3y = 2x$

9. $x - y = 1$
   $x + y = 9$

10. $y = x + 5$
    $y = x + 3$

11. $y \geq x$
    $y \geq 3x$

12. $y > x$
    $y < x + 4$

13. $x + y \leq 5$
    $x \geq 3$

14. $y > x + 5$
    $y < x + 2$

15. $y < x + 5$
    $y > x + 2$

16. $x > 0$
    $y > 0$

17. $x < 0$
    $y > 0$

18. $x < 0$
    $y < 0$

19. $x > 0$
    $y < 0$

20. $x > 0$
    $y > 0$
    $x + y \geq 5$

21. $y > x^2$
    $y < x - 6$

22. $x > 3$
    $x < -3$

23. $x < 3$
    $x < -3$

24. $x > 3$
    $x > -3$

25. $y > 5$
    $y < 2$

26. $x^2 > 1$
    $y > x^2$

27. $x^2 + y^2 > 4$
    $y < x$

28. $y > x^2$
    $y < 3$

29. $y > x^2$
    $x < -2$

30. $x^2 + y^2 > 9$
    $x \geq 5$

31. $x^2 + y^2 \leq 25$
    $x^2 + y^2 \geq 4$

32. $x^2 + y^2 \geq 25$
    $x^2 + y^2 \leq 4$

33. $x^2 + y^2 \leq 9$
    $y > x^2$

34. $x > 4$
    $y < 2$

35. $x < 4$
    $y > 2$

36. $x > 4$
    $x > 4$

37. $x < 4$
    $y < 2$

38. $x + y \leq 5$
    $y \geq x - 6$
    $y \geq -3$

Honor Problems

Directions: Graph the solution set of the following sentences:

1. $xy = yx$
2. $\frac{x}{x} + \frac{y}{y} = 2$
3. $x + y = y + x$
4. $\frac{x}{y} = \frac{y}{x}$
5. $x^2 - y^2 = 0$
6. $| xy | > 0$
7. $x - y = y - x$
8. $xy = 0$

XIV. Summary

A. A relation is a set of ordered pairs.

B. A function is a special kind of relation such that no two ordered pairs have the same first coordinate.

C. The concept of function involves a domain, range, and a defining sentence (rule or formula) which assigns to the first coordinate of the ordered pair one and only one value for the second coordinate.

D. The domain of a relation is the set of the first coordinates of the ordered pairs which make up the relation.

E. The range of a relation is the set of the second coordinates of the ordered pairs which make up the relation.

F. The function is not the sentence, rule or formula. The function is the set of ordered pairs. When we point to a graph and state " this is the graph of the function", we always mean those points and only those points of the graph which represent the ordered pairs of the function.

G. f(x) is not the function nor does it represent the function. f(x) is second coordinate of the ordered pair whose first coordinate is x. The term, f(x) is often used loosely to mean the function. It is better to use the symbol "f" when we mean the function, and reserve the symbol "f(x) to mean the second coordinate of the ordered pair.

BIBLIOGRAPHY

Aiken, J. A., and Beseman, C. A. MODERN MATHEMATICS; TOPICS AND AND PROBLEMS. New York: McGraw-Hill Book Company, Inc. 1959.

Andree, Richard V. SELECTIONS FROM MODERN ALGEBRA. New York: Henry Holt and Company. 1958

Birkhoff G. and MacLane S. A SURVEY OF MODERN ALGEBRA. (Revised Edition) New York: The Macmillan Company. 1953

Breuer, Joseph. (translated by Howard Fehr). THE THEORY OF SETS. Englewood Cliff, N. J.: Prentice-Hall, Inc. 1958

Christian, Robert. INTRODUCTION TO LOGIC AND SETS. New York, N. Y.: Ginn and Company. 1958

College Entrance Examination Board. Princeton, N. J.: 1959.
APPENDICES, 1959
ELEMENTARY PROBABILITY AND STATISTICAL INFERENCE FOR SECONDARY SCHOOLS, AN EXPERIMENTAL COURSE. 1957

Freund, John E. A MODERN INTRODUCTION TO MATHEMATICS. Englewood Cliffs, N. J.: Prenctice-Hall, Inc. 1956

Johnson, Richard E. FIRST COURSE IN ABSTRACT ALGEBRA. Englewood Cliffs, N. J.: Prentice-Hall, Inc. 1953

Kelley, John L. INTRODUCTION TO MODERN ALGEBRA. Princeton, N. J.: D. Van Nostrand Company, Inc. 1960

Kershner, R. G. and Wilcox, L. R. THE ANATOMY OF MATHEMATICS. New York: The Ronald Press Company. 1950

May, Kenneth O. ELEMENTS OF MODERN MATHEMATICS. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc. 1959

McCoy, Neal H. INTRODUCTION TO MODERN ALGEBRA. Boston, Massachusetts: Allyn and Bacon, Inc. 1960

National Council of Teachers of Mathematics. THE GROWTH OF MATHEMATICAL IDEAS, Twenty Fourth Yearbook. Washington, D. C.: 1959

Woodward, Edith J., and McLenna, Roderick C. ELEMENTARY CONCEPTS OF SETS. New York: Henry Holt and Company. 1959.

University of Illinois Committee on School Mathematics. ORDERED PAIRS AND GRAPHS, UNIT 4. Urbana, Illinois: University of Illinois. 1959