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ABSTRACT

GRADES OR AGES: K-6. SUBJECT MATTER: Mathematics. ORGANIZATION AND PHYSICAL APPEARANCE: The guide is divided into several chapters. The central and longest chapter, which outlines course content, is further subdivided into 15 units, one for each of 15 content objectives. This chapter is in list form. The guide is offset printed and staple-bound with a paper cover. OBJECTIVES AND ACTIVITIES: The central chapter lists 15 mathematical concepts such as numeration systems, ratio and proportion, size and shape, or measurement. A list of related behavioral objectives for each concept at each grade level is then presented. No specific activities are mentioned, although one short chapter gives general guidelines on developing problem-solving situations. INSTRUCTIONAL MATERIALS: A short bibliography lists several references for each chapter. STUDENT ASSESSMENT: A chapter on evaluation and testing gives guidelines for developing tests and choosing appropriate methods for different grade levels. (RT)

Guidelines to Mathematics K-6

Key Content Objectives, Student Behavioral Objectives, and Other Topics Related to Elementary School Mathematics

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Foreword

In a dynamic society such as ours, change is constant. People accept this fact more readily in many fields than in the area of mathematics — particularly at the elementary school level. And yet, in recent years new projects, new courses, and new materials remind us that a great potential exists for improved changes in the elementary mathematics program.

The primary purpose of *Guidelines to Mathematics, K-6* is to help those who have responsibility in local school districts for providing a well conceived elementary mathematics program. Each of the sections of this guide has been designed to meet specific objectives. Although no section represents the final word on the topic being discussed, it is intended that all sections represent definitive statements which can serve as a sound foundation for further study and investigation.

The main section of this guide has been devoted to a careful development of the major concepts of elementary school mathematics and associated behavioral objectives. In addition, several other sections have been devoted to issues concerning the effective teaching of mathematics at the elementary school level.

The Wisconsin State Department of Public Instruction trusts that the efforts of all who helped make this publication possible will result in improved mathematical experiences for all Wisconsin youth.

William C. Kahl
State Superintendent

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Mathematics Instruction: A Point of View

The variety of mathematics programs now available for the elementary grades promises to benefit both teachers and pupils alike. Having been freed from the fixed form that characterized the traditional program, teachers now have a greater opportunity than ever to select the materials and teaching techniques best suited to a child's needs. The basis used for selecting methods and materials is important and is dependent upon the viewpoint of the teacher in regard to his instructional task. Just as mathematics programs have changed, so might certain aspects of the traditional view of instruction be changed.

It has been common practice heretofore to consider certain mathematics skills and concepts as being "all-or-nothing" attainments. That is, mastery of a certain body of facts or skills has been traditionally regarded as the proper province of the teacher at a particular grade level. The failure of some children to achieve the expected level of proficiency has forced the teacher of the next grade to "reteach" these areas, a task that has long been a source of irritation to many teachers. This type of compartmentalization of a course of study should be minimized, if not eliminated. Instead of expecting all children at a given grade level to master in a rather final sense a certain portion of the mathematical training, educators should develop an approach which recognizes the existence of individual growth rates and stresses the continuity of the instructional process.

Teaching for Growth

Perhaps the one factor most essential to the success of the mathematics curriculum is an emphasis on understanding, that is, understanding of mathematics. This emphasis represents a marked shift in the focus of educators' attention from overt behavioral skills and social applications to understandings developed as a child organizes and codifies mathematical ideas. It is no longer deemed sufficient for a teacher to be satisfied with competent computational performance by a child, in spite of the obvious necessity for such skill. The need to find a more efficient, a more enjoyable, and a more illuminating method of instruction has led to the consensus that clear and penetrating understanding of certain essential mathematics must precede, but certainly not supplant the traditional point of emphasis, computation. How is this to be achieved?

Experiencing the Physical World

Certain principles of instruction deserve renewed emphasis. It is agreed that one should traverse from the known to the unknown, from the particular to the general; from the concrete to the abstract. Clearly, children should be encouraged to develop their first concepts of mathematics from their experiences with physical objects.

This approach implies, in many cases, that prior to the introduction of a concept, time should be provided for each child to experiment with physical objects appropriate to the objective. For instance, every child should have the experience of actually combining sets or groups of things as a prelude to the introduction of addition; he should work on an abacus or place-value device prior to any contact with sophisticated positional notation; he should have "pre-number" experiences, such as learning to compare the numerousness of groups through the process of one-to-one matching, rather than counting in the traditional fashion.

For example, the classroom situation in which children are seated at their desks provides a natural one-to-one correspondence between children and occupied chairs. The number of chairs can be compared to the number of children by simply noting whether any chairs are empty, rather than counting the members of each group. Many other similar situations arise naturally in which the child's interest readily leads to non-verbal awareness of such concepts as equal to, less than, and more than. By such purposeful "playing," important concepts that previously have been left largely to chance development will be given appropriate attention and will be established on a firm foundation.

Useful Unifying Concepts

Because understanding is an ambiguous term and because a teacher must make as clear as possible the relationship between former objectives and present points of emphasis, it may be well to consider just a few of the many concepts that are especially important to clarify and develop from the earliest stages.

For the foreseeable future, the study of number systems will continue to be the essence of the mathematics program. Involved in this study are the collections of various kinds of number ideas, such as natural numbers, *together with* the operations defined on these number ideas and the properties of these operations (for example, commutative, associative, distributive properties). Obviously, the development of an understanding of such ideas must take place over an extended period of time and at considerably different rates for different children. There is no such thing as complete understanding of any particular number system by a child. Rather, teachers at each level should ascertain that the instruction is directed toward deepening and extending the broad mathematical streams cited in later sections of this report. How can this be done?

As a child's grasp of mathematics grows, he must be guided toward the acquisition of the broad and essential concepts which tie together seemingly disconnected particulars into a coherent general structure. The same operations are encountered in several peculiar and different contexts in the grades. Each particular use really represents a different operation.

For instance, the multiplication of two natural numbers such as 3×5 may be properly regarded as the process of starting with nothing and adding 5 objects to a set 3 times. But can $1/2 \times 1/3$ be similarly interpreted as the process of repeatedly adding 1/3 objects for 1/2 times? Or what meaning can be ascribed to $(-3)(-5)$? Certainly not -5 things repeatedly added for -3 times! Such statements require different interpretations because the operation called *multiplication* has several meanings and definitions depending upon the kinds of numbers or set elements to which it is applied.

Natural numbers, integers, and *rational numbers* each combine somewhat differently under a given operation. But rather than invent new symbols to represent the changing definitions of an operation as it applies to the various number systems, one uses the same symbol throughout. In this situation, as in many similar ones, children must be taught to interpret symbols in context. Only in this way will mathematics shorthand serve the important purpose of clarifying and improving communication. Furthermore, the child should be led to understand that although the specific interpretation of an operation differs from one system to the next, a structural unity still remains which is founded on the commutative, associative, and distributive properties as they pertain to the operations. Because of this structural unity, one can rely on the context in which a symbol is used to determine its meaning, rather than create new symbols for each particular variation of a concept.

Again, the need for techniques which lead the child to discover ideas through his experiences with physical objects must be stressed. Furthermore, attention must be given to rather subtle ideas that may have been overlooked or thought too obvious to mention in the past. For instance, when a child combines two sets of objects into a single collection, he must come to the realization that the abstract property of numerousness is conserved: that there are exactly as many objects after as there were before the two groups were joined. As fundamental as such an assumption is to understanding the addition operation, it is by no means obvious to all beginners. This fact is substantiated by the work of Piaget and others.

Another matter often treated too lightly involves the freedoms or restrictions which may or may not exist for a given operation. When one matches the elements of a set with the word names of the ordinal numbers, he must know that the order in which the objects are matched to the numerals will not affect the count. On the other hand, cases exist in which the order of presentation of information is important. One cannot mix up numerator and denominator with impunity or name the coordinates of a point by whim. The basis for free choice, where it exists, and the reasons for restrictions, as they occur, must be made clear if understanding is to be achieved. Such matters are rarely as obvious to children as they seem to adults.

To sum up, children should come to realize that mathematics is a logical system *which exists as a human invention*, formulated, enriched, extended, and revised in response to the twin needs to perfect it as a logical structure and to use it as a convenient method of describing certain aspects of nature as seen by man.

Undue Emphasis on Particulars

Educators should avoid giving undue emphasis to any single aspect of the total mathematics program. The idea of sets, for instance, should not be glorified beyond its usefulness in contributing to the attainment of the broad goals of the program. Similarly, it is unwise to go to the extreme of downgrading the importance of computational proficiency to the point where long range goals are placed in jeopardy. Furthermore, big words or impressive terms and symbols must never interfere with a child's understanding of the concepts which the words or symbols represent. Clarity of communication is the chief purpose of mathematical language and symbolism. If a particular term or symbol does not serve the cause of clarity, then it should not be used. Children, of course, must eventually learn how to use the language effectively and understand it in context. However, teachers should exercise great care in working toward such a goal and should avoid any proliferation of symbols or premature verbalizations that will hinder goal attainment.

Problem Solving

One of the greatest challenges for the elementary teacher is the development of appropriate, real-life problems which meaningfully involve learners. Classroom teachers recognize the limitations of textbooks in providing such verbal problems. Many problems fail to challenge children; they fail to represent a world which is real to children; and too often they fail to contribute to the child's skill in problem solving situations outside the textbook. Notwithstanding these limitations, textbook verbal problems will continue to be prominent in most elementary classrooms until individual teachers identify other ways of meeting the problem solving obligation.

There is increasing evidence that through experimental programs and through the efforts of individual classroom teachers improved ways of meeting this obligation are being developed. Much is being heard about the importance of relating mathematics to science, of units of study in mathematics, of enrichment activities in mathematics, of the study of the history of mathematics, and of children creating their own mathematics problems. Instead of stressing the social development of the child, these efforts emphasize the structure of mathematics through problem solving experiences. These activities are hopeful signs pointing to a greater stress being placed, in the near future, on the importance of meaningfully-structured problem solving experiences as a part of the elementary mathematics program.

Individual Differences

If mathematics instruction is viewed as a process of initiating understanding and of carefully nurturing this understanding as the child matures, it will then be necessary to discover effective techniques for accommodating the widely differing rates at which children develop. Implicit in this statement is the need for schools and teachers to recognize that some children may not be able to grow substantially or may seem to terminate their potential for growth at some point along the way. In such cases, provision should be made for experiences most appropriate to the child's welfare.

To date, no satisfactory method has been developed to cope with the vast range of individual differences. Flexible grouping procedures, nongraded classrooms, individualized instruction, team-teaching, and television teaching are receiving extensive testing and evaluation. Perhaps the experiments currently underway will yield effective techniques once more is learned about the problem. In the meantime, each teacher must exercise professional judgment and common sense in adopting an optimal arrangement that is compatible with his own abilities, with the characteristics of pupils in his class, and with the physical facilities and administrative policies of the school.

In summary, mathematics instruction should be viewed as a continuous effort to develop in the child a knowledge of mathematics that is characterized by its depth and connectedness. To the extent possible, the child should be encouraged to experiment with the objects of his environment. Thus prepared, he may be led to the invention or discovery of those ideas which provide both a broad basis for further exploration and a sense of delight in a well-founded mastery of the subject. The task is a challenging one and deserves much effort. To this end, the teacher should do everything possible to see that instruction is well-planned and is provided on a regular basis. As an additional, but essential measure, cooperative action such as inservice education should be taken by school personnel to develop in the teaching staff a view of instruction appropriate to present day needs and opportunities.

Key Mathematical Content Objectives and Related Student Behavioral Objectives

In the following outline, an attempt has been made to point out where key content objectives of mathematics might be introduced and developed in grades K-6. The corresponding behavioral objectives suggest a sequence of development of these mathematical ideas which provides for their reinforcement and continuity throughout the K-6 mathematics program. By means of a sequence of development, key ideas can be extended to conform with the maturity and background experience of the student.

The suggested content objectives have been organized under fifteen main topics: Sets and Numbers; Numeration Systems; Order; Number Systems, Operations and Their Properties; Ratio and Proportion; Computation; Size and Shape; Sets of Points; Symmetry; Congruence; Similarity; Coordinate Systems and Graphs; Constructions; Measurement; and Mathematical Sentences.

It is not intended that the placement of topics in this outline be considered as the only correct arrangement or that all of the topics necessarily be taught at every grade level as presented here. The ideas listed for each grade level should be regarded as a *suggested* guide for introducing various topics: the outline is not intended to be all-inclusive. The teacher will find it necessary to alter the order of topics to meet the needs of the children or the needs of a particular group in a given classroom.

Furthermore, no fixed amount of time or emphasis has been suggested for any objective in this outline. A disproportionate amount of space has been devoted to the geometric objectives due to their recent introduction in elementary mathematics courses. Many of the geometric objectives can be attained in a relatively short time as compared with the time and emphasis necessary to attain those objectives of elementary school arithmetic which are central to an understanding and use of the rational number system.

This outline should prove useful to the school administration, the mathematics curriculum committee, and the teachers of a school district in one or more of the following ways:

- As an orientation to the key content and behavioral objectives of the elementary school mathematics curriculum.
- As a source of knowledge of the related objectives of elementary school mathematics, their development and reinforcement throughout the entire K-6 mathematics program.
- As a means of evaluating the content and behavioral objectives of the present mathematics program of the school on the basis of the needs of the school community.
- As a guide for the preparation of a mathematics curriculum guide, K-6, for use by the school district.
- As an aid in the evaluation and selection of textbooks.
- As a guide in determining the concepts that need to be highlighted in inservice activities for teachers.
- As an aid in choosing standardized tests.

Throughout this outline, an asterisk (*) appearing under any behavioral objective indicates that the understanding and skills listed for that objective in previous grades are to be expanded and developed.

Mathematical Concepts

Arithmetic Concepts

Sets and Numbers

A set is a collection of things. Sets may have many properties, but a property peculiar to a collection or set is that of *number*. When an orderly arrangement of these numbers has been made, one has the tools needed for counting. Numbers can be used in the cardinal sense, that is, to indicate numerousness. For example, a triangle has *three* vertices. They can also be used in the ordinal sense, that is, to indicate the arrangement of numbers by order. For example, a student in *fifth* grade is studying page *100* (the *hundredth* page) of his textbook. Sometimes a number is used in the nominal sense, that is, as a name. A social security number is used this way.

In this guide, the concept of number is extended to include positive rational numbers and then positive and negative integers. These number concepts will be more fully developed in the junior and senior high school mathematics programs.

Students should be able to:

Numeration Systems

Names are given to the cardinal numbers. These names are numerals and are used to convey the idea of number. Numerals are symbolized forms used for communication. A given number has many different names which can be used depending on the idea one wishes to express. Organized methods of arranging the numerals represent the numeration systems of the past and present. Study of different numeration systems will give the student insight and appreciation of the more efficient ones.

Students should be able to:

Order

The concept of order is a generalization made as a result of experience in comparing unmatched sets of objects to determine which has more members. Order is expressed in sentences by means of the symbols $>$ (is greater than) and $<$ (is less than). The concepts of equality and of inequality (numbers that are not equal) are considered here for convenience.

Students should be able to:

Number Systems, Operations and Their Properties

A number system consists of three parts: a set of numbers, two operations defined on the numbers in the set, and the properties (laws or rules) of the operations.

Two number systems, the whole numbers and the positive rational numbers, are studied in detail in the elementary school. A third system, the integers, is introduced in the elementary school when the physical situation demands that the number system be extended to include negative numbers. The integers and a fourth number system, the real numbers, are studied in more detail in junior and senior high school. An understanding of the integer and real number systems and their properties depends on careful and complete development of the whole number and (positive) rational number systems in the elementary school.

The operations on numbers studied in the elementary school are addition and multiplication, and their respective inverses, subtraction and division.

The important and useful properties of the operations on all number systems studied in arithmetic are given here. They are:

Addition

Commutative: $a + b = b + a$
Associative: $(a + b) + c = a + (b + c)$
Identity element: $a + 0 = a$
Inverse element: $a + (-a) = 0$

Multiplication

Commutative: $a \times b = b \times a$
Associative: $(a \times b) \times c = a \times (b \times c)$
Identity element: $a \times 1 = a$
Inverse element: $a \times 1/a = 1, a \neq 0$
Property of zero: $a \times 0 = 0$
Distributive: $a \times (b + c) = (a \times b) + (a \times c)$

Students should be able to:

Behavioral Objectives

Kindergarten

Sets and Numbers

1. Identify two equivalent sets by placing the members of the set in one-to-one correspondence.
2. Select the set of objects associated with a given number.
3. Count orally by matching numerals with sets having a given number of objects.
4. Identify, without counting, the number of sets with two, three, or four objects.
5. Use such terms as *more than, as many as, fewer than* when comparing sets of objects.

Numeration Systems

1. Identify the numerals 0 through 9.

Order

1. Determine whether two sets are equivalent (can be matched or placed in a one-to-one correspondence).
2. Compare two non-matching sets of fewer than 10 objects and decide which set has more members and which set has fewer members.
3. Determine that 3 is greater than 2 and that 2 is less than 3 by comparing appropriate sets of objects and do this for any two numbers less than 6.
4. Utilize the idea "one more than" in organizing sets in the natural order.

Number Systems, Operations and Their Properties

1. Rearrange sets of objects to demonstrate the joining and separating of sets, and thereby develop a readiness for addition and subtraction.
2. Tell a story about a problem represented by the above activities.
3. Use objects to represent the "action" or conditions of a problem.

Behavioral Objectives

First Grade

Sets and Numbers

1. Count the members of a set containing one hundred or fewer members.
2. Demonstrate one-half, one-fourth of a physical unit.
3. Use the ordinal numbers through fifth.
4. Use "0" as the symbol for the number of elements in the empty set.
5. Insert missing sets, such as a set of 3 elements between a set of 2 elements and a set of 4 elements.

Numeration Systems

1. Read and write any numeral from 0 through 100.
2. Give different numerals for a given number such as 6 + 2, 10 − 2, and 8 for eight.
3. Read number words through *ten*.
4. Interpret the place-value concept for writing whole numbers to one hundred; such as, 89 is the same as 8 tens, 9 ones.

Order

1. Compare two non-matching sets of less than 100 objects to decide which set has fewer (more) members.
2. Determine that 8 is greater than 5 and that 5 is less than 8 by comparing appropriate sets of objects and do this for any two numbers less than 10.

Number Systems, Operations and Their Properties

1. Identify the process of addition through experience with joining two sets of objects.
2. Identify the process of subtraction through experience with separating a subset from a set of objects.
3. Recognize examples of the commutative property for addition in the set of whole numbers.
4. Demonstrate with sets of objects the relationship between such sentences as $4 + 2 = 6$, $6 - 2 = 4$, and $6 - 4 = 2$.
5. Use the symbols +, −, and = to form sentences such as $3 + 6 = \square$.

Behavioral Objectives

Second Grade

Sets and Numbers

1. Determine the cardinality of a set to one thousand when the objects are already grouped. For example, 4 bundles of 100 sticks, 9 bundles of 10 sticks, and 4 sticks are the same as 494 sticks.
2. Use ordinal numbers through tenth.
3. Identify 1/2, 1/4, 1/3 of a whole by using physical objects.

Numeration Systems

1. Read and write any numeral through 999.
2. Write many symbols for the same number; for example, $6 + 3$, $5 + 4$, $17 - 8$, and 9 for nine.
3. Count by 5's, 10's, and 100's.
4. Use bundles of sticks to demonstrate place value through 999; for example, $234 = 200 + 30 + 4$.
5. Write three-digit numerals in expanded notation; for example, $765 = 700 + 60 + 5$.
6. Use physical objects to demonstrate regrouping; for example, 2 bundles of 10 sticks and 16 sticks have the same number of sticks as 3 bundles of 10 sticks and 6 sticks. Also 4 bundles of 10 sticks and 8 sticks have the same number of sticks as 3 bundles of 10 sticks and 18 sticks.

Order

1. Use the terms *greater than* and *less than*, and *equals* in sentences.
2. Use symbols $>$, $<$, and $=$ in mathematical sentences.
3. Insert missing numbers, such as 3 between 2 and 4.
4. Name successors and predecessors of each number through 99.
5. Determine the order of numbers through 99.

Number Systems, Operations and Their Properties

1. Recognize that there is no largest whole number.
2. Use the associative property of addition in the set of whole numbers, for example, $(3 + 4) + 5 = 3 + (4 + 5)$.
3. Recognize that subtraction is not commutative
4. Identify the process of multiplication through experience with joining several equivalent sets of objects.
5. Discover from the addition table number patterns through the sum 18.
6. Recognize zero as the identity element for addition in the set of whole numbers and its special role in subtraction.

Behavioral Objectives

Third Grade

Sets and Numbers

1. Determine the cardinality of a set to 10,000 through appropriate experiences.
2. Use ordinal numbers beyond tenth.
3. Identify 2/3 and 3/4 of a whole.
4. Show 2/4 = 1/2, etc., by the use of physical objects or pictures.

Numeration Systems

1. Read and write any numeral to 10,000.
2. Write many symbols for the same number, such as 7 + 5, 4 × 3, 10 + 2, and 12 for twelve.
3. Interpret place value to 10,000.
4. Write four-digit numerals in expanded notation; for example, 4567 = 4000 + 500 + 60 + 7.
5. Recognize that numerals such as 57 can be expressed as 40 + 17.
6. Read and write roman numerals through X (10).

Order

1. Determine betweenness, greater than, or less than for numbers through 999.
2. Recognize greater than or less than for the fractions 1/4, 1/3, 1/2 with physical objects. (Note that *fractions* is being used here for *rational numbers.*)

Number Systems, Operations and Their Properties

1. Use the commutative and associative properties of multiplication in the set of whole numbers; for example, 4 × 3 = 3 × 4 and (4 × 3) × 2 = 4 × (3 × 2).
2. Recognize the role of 1 as the identity element for multiplication in the set of whole numbers.
3. Recognize the distributive property of multiplication over addition in the set of whole numbers; for example, 3 × (2 + 4) = (3 × 2) + (3 × 4).
4. Discover number patterns from the addition and multiplication tables.
5. Demonstrate the concept of division by partitioning a set into several equivalent subsets; for example, separate a set of 12 objects into sets of 3 objects.
6. Demonstrate with sets of objects the relationship between such sentences as 4 × 7 = 28, 28 ÷ 4 = 7, and 28 ÷ 7 = 4.

Behavioral Objectives

Fourth Grade

Sets and Numbers

1. Determine the factors of a counting number (a whole number other than zero).
2. Determine common factors of two counting numbers.

Numeration Systems

1. Read and write numerals as needed.
2. Interpret place value for large numbers.
3. Use roman numerals through XXV.

Order*

Number Systems, Operations and Their Properties

1. Recognize the inverse relation between addition sentences and two subtractions sentences, such as $725 + 342 = 1067$ and $1067 - 725 = 342$ and $1067 - 342 = 725$.
2. Recognize the special role of 1 as a divisor.
3. Use parentheses to show order of operation; for example, $2 + 4 \times 3 = 2 + (4 \times 3) = 14$ and $(2 + 4) \times 3 = 6 \times 3 = 18$.

Behavioral Objectives

Fifth Grade

Sets and Numbers

1. Identify prime numbers such as 2, 3, 5, 7, 11, 13, 17, ...
2. Find the prime factors of numbers through 100.
3. Identify composite numbers.
4. Determine the least common multiple of two counting numbers.
5. Determine the greatest common factor of two counting numbers.
6. Construct sets of equivalent fractions through working with sets of objects. An example of such a set is {2/3, 4/6, 6/9, 8/12 . . .}.

Numeration Systems

1. Write many names for the same rational number.
2. Work with bases, such as 3, 4, 5, 6, and 7, to demonstrate an understanding of the base of a numeration system.
3. Use simple exponents such as $10^2 = 100$, $3^2 = 9$ and express 300 as 3×10^2.
4. Read and write simple decimals.
5. Read and write roman numerals.

Order

1. Determine greater than, less than, and betweenness for rational numbers.

Number Systems, Operations and Their Properties

1. Recognize the set of positive rational numbers (fractions) as an extension of the set of whole numbers.
2. Recognize that zero is the identity element for addition in the set of positive rational numbers as well as in the set of whole numbers.
3. Recognize that subtraction is not always possible in the set of positive rational numbers and in the set of whole numbers.
4. Use the commutative and associative properties for addition in the set of positive rational numbers.
5. Recognize that there is no smallest positive rational number.
6. Add, subtract, and multiply simple rational numbers by use of physical objects, diagrams, etc.

Behavioral Objectives

Sixth Grade

Sets and Numbers

1. Use negative numbers in many different situations.

Numeration Systems

1. Represent rational numbers by decimals and fractions.
2. Express large numbers by using scientific notation, such as the distance from earth to the sun as 9.3×10^7 miles.
3. Use exponential notation in representing numbers; for example, $2345 = 2 \times 10^3 + 3 \times 10^2 + 4 \times 10 + 5$.
4. Demonstrate an understanding of the relationship between decimals and common fractions.

Order

1. Determine greater than, less than, and betweenness for (positive, negative, and zero) integers.

Number Systems, Operations and Their Properties

1. Recognize that 1/1 or 1 is an identity element for multiplication in the set of rational numbers.
2. Recognize the multiplicative inverse (reciprocal) for every positive rational number except zero and use it in the division of rational numbers. For example, $1/2 \div 3/4 = 1/2 \times 4/3$.
3. Recognize that the operation of division is the inverse of multiplication in the set of positive rational numbers. For example, the sentences $3/4 \times 2/3 = 1/2$, $1/2 \div 3/4 = 2/3$, and $1/2 \div 2/3 = 3/4$ have this relationship.
4. Recognize that there is no smallest or largest rational number between two positive integers.
5. Recognize that the integers (positive and negative whole numbers and zero) are an extension of the whole numbers.
6. Find the additive inverse (opposite) for each integer by using the number line.
7. Recognize that the rational numbers (positive and negative whole numbers, positive and negative fractions, and zero) are an extension of the integers.
8. Recognize that finding an integral power of a number involves repeated multiplication of the same number. For example, $(2/3)^3 = 2/3 \times 2/3 \times 2/3$.
9. Use the commutative and associative properties of multiplication for rational numbers.
10. Use the distributive property of multiplication with respect to addition of rational numbers.
11. Use the commutative property of addition for integers.
12. Recognize that the rational number system is dense; that is, between each two different rational numbers, there is a rational number.

Mathematical Concepts

Arithmetic Concepts

Ratio and Proportion

A ratio is a pair of numbers used to compare quantities or to express a rate.

Symbols commonly used for a ratio are (a,b), a:b, and a/b. Note that the symbol (a,b) is also used to denote the coordinates of a point in a plane and that the symbol a/b is generally used to name a rational number, when a and b are integers and b is not zero.

A proportion is a statement that two ratios are equivalent (that two pairs of numbers express the same rate). A proportion is written in the form a/b = c/d, when $a \times d = c \times b$.

Students should be able to:

Computation

A certain amount of proficiency in the use of the various algorithms of arithmetic is necessary. The grade placement used for these computational skills in this guide is only an approximation. The expected use and recall of certain addition, subtraction, multiplication, and division facts for the various grade levels are listed under the behavioral objectives.

Students should be able to:

Geometric Concepts

Size and Shape

The classification of and distinguishing characteristics of simple two and three dimensional figures are determined by their size and shape. Informal examination of the size and shape of geometric figures develops an awareness of geometric patterns.

Students should be able to:

Sets of Points

Space can be considered as a set of points. Intuitively, a point represents and is represented by a position or location in space. Lines and planes are subsets of space. Each line is a set of points of space and each plane is a set of points of space. These subsets and many of their properties should be intuitively presented.

Students should be able to:

Symmetry

Many geometric figures and designs have a kind of balance called symmetry. If a figure can be folded so that corresponding parts coincide, then it has a line of symmetry. A figure has a center of symmetry if for every point of the figure there is a second point such that a line segment joining them is bisected by this center.

Students should be able to:

Congruence

Intuitively two geometric figures are congruent if they "fit" each other exactly, that is, if they have the same size and shape. More precisely, two sets of points are congruent if there is a one-to-one correspondence between them which preserves distances; that is, if two points of one set are one inch apart, the corresponding points of a congruent set are also one inch apart.

Students should be able to:

Behavioral Objectives

Kindergarten

Ratio and Proportion

Computation

Size and Shape

1. Recognize squares, rectangles, circles, and triangles.
2. Use the terms *inside*, *outside*, and *on* as related to these figures.

Sets of Points

Symmetry

Congruence

Behavioral Objectives

First Grade

Ratio and Proportion

Computation

1. Use the addition facts through the sum of 10 and the corresponding subtraction facts.

Size and Shape

1. Observe distinguishing features of spheres, rectangular prisms (boxes), cylinders, and other objects.
2. Use the terms *round, face, edge, corner,* and *surface.*

Sets of Points

1. Recognize physical representations of points, line segments, and portions of a plane (flat surfaces).
2. Recognize that squares, rectangles, triangles, and circles are closed curves and tell whether a point is inside, outside, or on such a curve.

Symmetry

Congruence

Behavioral Objectives

Second Grade

Ratio and Proportion

Computation

1. Recall the addition facts through the sum of 18 and the corresponding subtraction facts.
2. Use the vertical algorithm in the addition of three addends with one-place numerals; for example:

$$\begin{array}{r} 3 \\ 4 \\ +\ 5 \\ \hline \end{array}$$

3. Use the multiplication facts through the product 18.

Size and Shape*

Sets of Points

1. Recognize a point as a position and a line segment or curve as a set of points.
2. Recognize a straight line as a set of points with no beginning and no end.
3. Recognize a simple curve (in a plane) as one that does not cross itself.
4. Recognize closed and open simple curves.
5. Recognize the inside and outside of simple closed curves.

Symmetry

Congruence

1. Recognize congruent, plane figures as figures which fit on one another.
2. Recognize congruent segments as segments having the same length.

Behavioral Objectives

Third Grade

Ratio and Proportion

1. Interpret simple ratio situations, such as 2 apples for 15¢, written $\frac{2 \text{ (apples)}}{15 \text{ (cents)}}$.
2. Recognize that ratios such as $\frac{2 \text{ (pencils)}}{5 \text{ (cents)}}$ and $\frac{4 \text{ (pencils)}}{10 \text{ (cents)}}$ are equivalent ratios (represent the same rate).

Computation

1. Use the multiplication facts with products through 45 and the corresponding division facts.
2. Use the vertical algorithm in addition and subtraction with two- and three-place numerals when regrouping may be necessary.
3. Use the vertical algorithm to carry out multiplication with multipliers less than 10 when regrouping may be involved.
4. Multiply mentally by 10 and by 100.
5. Estimate the sum of two numbers. For example, 287 + 520 is approximately 300 + 500 or 800.

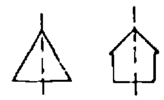
Size and Shape*

Sets of Points

1. Recognize rays and angles.
2. Recognize that there is only one line through two points and that two lines can intersect at only one point.
3. Recognize that many lines may pass through a point.

Symmetry

1. Recognize symmetry with respect to a line by folding paper containing symmetrical figures such as



along their vertical axes of symmetry.

Congruence

1. Recognize congruent angles.
2. Recognize that a rectangular sheet of paper can be divided into two or more congruent parts through folding.

Behavioral Objectives

Fourth Grade

Ratio and Proportion

1. Make up sets of equivalent ratios for given physical situations, such as 1/2, 2/4, 3/6, 4/8, ...
2. Determine if two ratios are equivalent by using the property of proportions commonly called *cross multiplication.* For example, $3/4 = 9/12$ because $3 \times 12 = 4 \times 9$, whereas $6/7 \neq 7/8$ because $6 \times 8 \neq 7 \times 7$.
3. Find the missing whole number in two equivalent ratios like $2/3 = \square/9$ or $5/\square = 25/70$.
4. Use equivalent ratios to convert units of measure, such as $\frac{\text{(gallons) } 1}{\text{(pints) } 8} = \frac{3 \text{ (gallons)}}{\square \text{ (pints)}}$ to find how many pints there are in 3 gallons.

Computation

1. Recall the multiplication facts through 10×10.
2. Do column addition with several four-place or five-place addends.
3. Subtract using three-place numerals and four-place numerals.
4. Multiply a number by multiples of 10, by multiples of 100.
5. Use the multiplication algorithm with two-place multipliers.
6. Use the subtractive division algorithm with two-place divisors ending in 1, 2, 3, 4.
7. Estimate the product of two numbers and the quotient of two numbers. For example, 21×88 is approximately 20×80 or 1800 and $795 \div 23$ is approximately $800 \div 20$ or 40.

Size and Shape

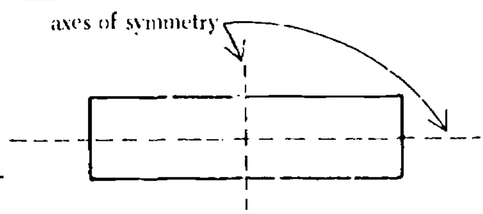
1. Recognize isosceles and equilateral triangles and parallelograms.

Sets of Points

1. Interpret space as the set of all points.
2. Recognize a plane as flat surface which contains lines and points.
3. Describe lines as intersections of planes.
4. Interpret a circle as the set of all points in a plane that are at the same distance from a fixed point.
5. Recognize parallel lines as lines in a plane which do not intersect.

Symmetry

1. Recognize that some figures have two or more axes of symmetry through paper folding. For example, two axes of symmetry are indicated for the rectangle shown at right.



Congruence

1. Recognize congruent angles.

Behavioral Objectives

Fifth Grade

Ratio and Proportion

1. Use the ideas of ratio and equivalent ratio with problems that include fractions as terms. For example, find the missing number in $\frac{2/3}{5} = \square/20$.
2. Find the missing term in a proportion such as $2/5 = \square/9$ by using the cross multiplication property and find the solution of the mathematical sentence $18 = 5 \times \square$.
3. Use members of sets of equivalent ratios with the same first term or the same second term to compare different ratios. For example, to compare 5/9 and 3/4, show $20/36 < 27/36$, and to compare 5/11 and 3/7, show $15/33 > 15/35$. Also, $5/9 < 3/4$ because $5 \times 4 < 9 \times 3$. Likewise $5/11 > 3/7$ because $5 \times 7 > 11 \times 3$.

Computation

1. Add and subtract rational numbers.
2. Use the multiplication algorithm with numerals up to four places.
3. Use the subtractive division algorithm.
4. Express the quotient of integers as a mixed numeral; for example, $24 \div 5 = 4\ 4/5$.
5. Find many ways to express a rational number; for example, $2/3 = \frac{2 \times 4}{3 \times 4} = 8/12$ and $8/12 = \frac{8 \div 4}{12 \div 4} = 2/3$.

Size and Shape

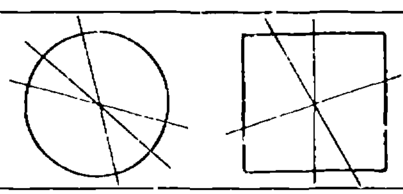
1. Recognize common polyhedra, such as a tetrahedron, a cube, rectangular prism.
2. Identify faces, edges, vertices, and diagonals of common polyhedra.
3. Use Euler's formula, namely, $V + F = E + 2$ where V is the number of vertices, F is the number of faces, and E is the number of edges of any polyhedron.

Sets of Points

1. Recognize acute, right, and obtuse angles.
2. Recognize parallel planes.
3. Recognize perpendicular lines.
4. Recognize the radius and diameter of a circle.
5. Recognize that a plane is determined by three points not all on one line.

Symmetry

1. Recognize symmetry with respect to a point by folding a paper along a line through the center of such geometric figures as a circle and a square.



Congruence

1. Recognize that triangles are congruent if corresponding sides are congruent and corresponding angles are congruent.

Behavioral Objectives

Sixth Grade

Ratio and Proportion

1. Interpret percent as a ratio in which the second number is always 100.
2. Solve all three cases of percentage problems as problems in which they find the missing term of two equivalent ratios. For example, 20% of 30 and: 20/100 = □/30; 30 is what percent of 55 and: □/100 = 30/55; 25 is 40% of what number and: 40/100 = 25/□.
3. Use equivalent ratios to convert fractions to decimals and conversely; for example, to write 3/5 as hundredths, solve for n in 3/5 = n/100; to write 44 hundredths as 25ths, solve for n in n/25 = 44/100.
4. Solve ratio problems where some or all of the terms of the ratios are written as decimals.
5. Use proportions in problems about the lengths of sides of similar triangles.

Computation

1. Multiply and divide rational numbers.
2. Use the conventional division algorithm.
3. Add integers.

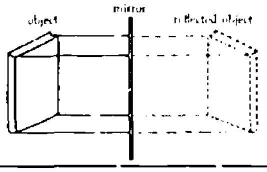
Size and Shape*

Sets of Points

1. Recognize the properties of isosceles triangles, equilateral triangles, and scalene triangles, such as the fact that the longest side of a triangle is opposite the angle of greatest measure.
2. Recognize that a line (one-dimensional space) is a subset of a plane (two-dimensional space) and that both are subsets of space (three-dimensional space).
3. Recognize the relationship between the circumference and the diameter of a circle.

Symmetry

1. Recognize the reflection of a plane figure in a mirror and draw diagrams such as the figure at right.



Congruence*

Mathematical Concepts

Geometric Concepts

Similarity

Two geometric figures that have the same shape, though not necessarily the same size, are said to be similar. The measures of corresponding dimensions of these figures will, thus, have the same ratio. Ideas of similarity are needed for the interpretation and drawing of building plans, road maps, and scale models.

Students should be able to:

Coordinate Systems and Graphs

A line is called a number line if a one-to-one correspondence exists between a given set of numbers and a subset of points on the line and if, by means of this correspondence, the points are kept in the same order as their corresponding numbers. The number corresponding to a point is the coordinate of that point. The idea of assigning numbers to points can be extended to points in a plane, that is, a one-to-one correspondence between ordered pairs of numbers and points in a plane.

Students should be able to:

Construction

Students can gain insight to geometry by using a variety of materials and instruments to construct models of geometric figures. Some useful materials and instruments are paper (for folding), wire, string, coins, rulers, compasses, and protractors.

Students should be able to:

Measurement

The process of measuring associates a number with a property of an object. Measuring an object is done either directly or indirectly. In direct measurement the number assigned to an object is determined by its direct comparison to a selected unit of measure of the same nature as the object being measured. When a measuring instrument cannot be applied directly to the object to be measured, indirect measurement is employed.

Measure of physical objects is approximate. The accuracy of the measure obtained is restricted by the unevenness of the object measured, by the limitations of the measuring instrument used, and by human inabilities. The object to be measured must be measured by any unit with the same characteristics: a unit segment to measure segments; a unit angle to measure angles; a unit closed region to measure closed regions; and a unit solid to measure solids.

Students should be able to:

Mathematical Sentences

Mathematical Sentences

Mathematical sentences and ordinary linguistic sentences have the following common characteristics:

1. Both types of sentences use words (symbols) to communicate ideas.
2. The construction of both sentences follows a predetermined set of rules.
3. Both sentences are true statements or false statements depending on the words or symbols used and the contexts in which they are used.

Mathematical sentences like $2 + 3 = 5$ (an equation) and $3 + 4 < 6$ (an inequality) are either true or false. Thus the first sentence is true whereas the second one is false. Mathematical sentences like $\square + 3 = 7$ and $12 - N > 5$ are true for some replacements of $\square$ and N and are false for others. If the set of positive integers is used as replacements for $\square$ and N in the above sentences, then the respective solution sets are $\{4\}$ and $\{1, 2, 3, 4, 5, 6\}$.

Behavioral Objectives

Kindergarten

Similarity

Coordinate Systems and Graphs

Construction

Measurement

1. Recognize pennies, nickels, dimes.
2. Make comparisons in time and count whole units of time (day, week, month, year).
3. Use appropriately such words as *longer, shorter, heavier, lighter, higher, lower, larger, smaller.*

Mathematical Sentences

Behavioral Objectives

First Grade

Similarity

Coordinate Systems and Graphs

1. Use the number line to illustrate addition and subtraction problems.

Construction

Measurement

1. Determine which of two line segments is the longer or the shorter, or whether they are the same length.
2. Recognize the comparative value of coins (pennies, nickels, dimes) and use them in making change.
3. Tell time to the nearest half-hour.
4. Identify various instruments of measurement of time, temperature, weight, and length, such as clocks, thermometers, scales, rulers.
5. Use non-standard units of linear measure and liquid measure, such as a pencil or book for length, and a paper cup for liquid measure.

Mathematical Sentences

1. Write an appropriate mathematical sentence like $3 + 4 = \square$ or $5 - 2 = \square$ for a physical situation or a story problem where the "action" of the problem suggests the operation of addition or subtraction.
2. Find the "solution" for sentences like $3 + 4 = \square$ and $5 - 2 = \square$.
3. Find solutions for sentences like $\square + \triangle = 7$ in which many correct solutions are possible.
4. Make up a problem situation to fit a given mathematical sentence involving addition or subtraction.

Behavioral Objectives

Second Grade

Similarity

Coordinate Systems and Graphs

1. Use the number line to illustrate counting by fives and by tens.

Construction

Measurement

1. Make a ruler with divisions showing half units.
2. Use standard units to the nearest whole unit for linear measure (inches and feet), for weight (pounds), and for liquid measure (pints and quarts).
3. Identify proper instruments for measuring different objects.
4. Tell time to the nearest quarter hour.
5. Make change correctly for quantities up to 25¢.

Mathematical Sentences

1. Use sentences like $5 + \square = 12$, $\square + 6 = 8$, $12 - \square = 8$, and $\square - 5 = 6$ to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Find solutions for sentences like $3 + 2 = 8 - \square$, $\square + 5 = 8 + 7$, $8 + \square < 12$, and $4 + 9 > \square + 5$, with the aid of sets of objects or the number line.
4. Place the correct symbol (+ or −) in the placeholder in sentences like $13 \triangle 5 = 8$ and $4 \square 3 < 5$.
5. Use sentences like $27 = 20 + 7$ and $43 = 30 + 13$ to indicate regrouping or the use of different symbols for the same number.
6. Use equivalent sentences like $3 + \square = 7$ and $7 - \square = 3$ to show subtraction as the inverse of addition.

Behavioral Objectives

Third Grade

Similarity

1. Recognize that figures are similar if they have the same shape. For example, all squares are similar.

Coordinate Systems and Graphs

1. Recognize that a point on a line can be described by a number (coordinate).
2. Recognize that different (uniform) scales can be applied to the same line.
3. Use the number line to illustrate multiplication problems.

Construction

Measurement

1. Use common standard units such as inches, feet, yards, in determining the measure of a distance.
2. Use standard units of measure, such as cups, gallons, ounces, in determining capacity and weight.
3. Find the perimeter of a rectangle or parallelogram.
4. Make change for any purchase up to $1.00.

Mathematical Sentences

1. Use sentences like $3 \times 5 = \square$, $\square \times 7 = 14$, and $4 \times \square = 12$ to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Place the correct symbol ($<$, $>$, $=$) in the place holder in sentences such as $3 \times 5 \; \square \; 7 + 8$, $25 + 42 \; \square \; 87 - 28$, and $75 - 39 \; \square \; 5 \times 7$.
4. Demonstrate understanding of grouping and regrouping by completing sentences such as $458 = \square + 50 + 8$ and $394 = 3$ hundreds $+ \square$ tens $+ 4$ ones, by means of tally boxes or other devices.
5. Find solutions for sentences like $\square + 239 = 239 + \square$ and $1987 + (\square + 548) = (1987 + \square) + 548$ to generalize the idea of the commutative and associative properties for addition.
6. Use many different kinds of placeholders like $\square$, $\triangle$, N, X in mathematical sentences.

Behavioral Objectives

Fourth Grade

Similarity

1. Recognize that all congruent figures are similar, but not all similar figures are congruent.

Coordinate Systems and Graphs

1. Recognize that a line segment is a set of points.
2. Recognize that points in a plane (the first quadrant) can be represented by (ordered) pairs of numbers (coordinates).
3. Use the number line to illustrate division problems.

Construction

1. Demonstrate through paper folding an understanding of a line as an intersection of two planes.
2. Reproduce a line segment by using a compass and straightedge.
3. Bisect a line segment by using a compass and straightedge.

Measurement

1. Compare measures such as: 25 inches and 2 feet; 31 ounces and 2 pounds; 75 seconds and 1 minute; and 15 pints and 2 gallons.
2. Express different names for the same measure.
3. Measure perimeters of triangles and quadrilaterals.
4. Find areas of simple regions informally. For example, a rectangular region with dimensions 2″ by 3″ can be covered by six one-inch squares (regions).

Mathematical Sentences

1. Use sentences like $36 \div 4 = \square$ and $\square \times 3 = 12$ to represent physical situations and find solutions for the sentences.
2. Make up story problems to fit sentences like those shown above.
3. Use sentences like $\square \times 5 = 45$ and $45 \div 5 = \square$ to show division as the inverse of multiplication.
4. Find solutions for sentences like $723 \times \square = \square \times 723$ and $(\square \times 176) \times 19 = \square \times (176 \times 19)$ to generalize the idea of the commutative and associative properties of multiplication.
5. Find solutions for sentences like $3 \times 13 = (\square \times 10) + (\square \times 3)$, $\square \times \triangle = (5 \times 7) + (5 \times 8)$, and $\square \times 16 = (\square \times 10) + (\square \times 6)$ to generalize the distributive property of multiplication over addition.
6. Find solutions for sentences like $\square \times \triangle = 36$.
7. Recognize that $3 \times \square = 7$ has no whole number solution.
8. Find solutions for mathematical sentences involving more than one operation such as $(2 \times 5) + 4 = \square$ and $(3 \times 2) + \square = 10$.
9. Make up problem situations to fit mathematical sentences involving more than one operation. For example, make up a story to fit the sentence $(3 \times 4) + 2 = \square$.

Behavioral Objectives

Fifth Grade

Similarity

1. Recognize the similarity of maps made with different scales.

Coordinate Systems and Graphs

1. Construct simple picture, bar, and line graphs
2. Use the number line to represent positive rational integers.
3. Use the number line to represent negative integers.

Construction

1. Demonstrate an understanding of various polyhedra by making appropriate paper models.
2. Bisect an angle. (Students may discover several different constructions.)
3. Reconstruct an angle and a triangle by using a compass and a straightedge.

Measurement

1. Measure an angle by using a protractor.
2. Demonstrate that the sum of the measures of the angles of a triangle is 180° by tearing off and matching corners of a triangular piece of paper.
3. Find the area of a plane region, such as a rectangular region.
4. Recognize informal concepts of volume; for example, a box with dimensions 2″ by 3″ by 4″ contains 24 one inch cubes.
5. Recognize that a right angle has the measure 90°.
6. Estimate distances to the nearest unit.
7. Recognize that all measurement involves approximation.

Mathematical Sentences

1. Use all of the previously introduced sentence forms with fractions.
2. Write sentences using fractions to represent physical situations.
3. Use previously described sentence forms to generalize the commutative property of addition, the associative property of addition, and the distributive property of multiplication over addition for rational numbers in fractional form.

Behavioral Objectives

Sixth Grade

Similarity*

Coordinate Systems and Graphs*

Construction

1. Construct a line perpendicular to a given line.
2. Construct parallel lines.
3. Make models of various prisms and find their surface areas.

Measurement

1. Find the volume of a rectangular prism.
2. Estimate and compare perimeters of polygons, such as rectangles, triangles, and parallelograms.
3. Estimate the area of an irregular plane region by use of a grid where an approximation to the area is the average of the inner and outer areas.
4. Use formulas for the areas of rectangles, parallelograms, and triangles.
5. Use the formula for the circumference of a circle.
6. Use the metric system of measure for length.
7. Use formulas of volume for common solids.
8. Work with approximate numbers. For example, know that the area of a square whose sides measure 6.5 and 3.6 inches to the nearest tenth of an inch has an area between 6.4×3.5 and 6.6×3.7 square inches.
9. Solve problems involving the measurement of inaccessible heights and distances indirectly by using the properties of similar triangles.

Mathematical Sentences

1. Use all of the previously introduced sentence forms with decimal numerals.
2. Write sentences using decimal numerals to represent physical situations.
3. Use previously described sentence forms to generalize the commutative property of multiplication, the associative property of multiplication, and the distributive property of multiplication over addition for rational numbers in any form.

Adjusting Mathematics Programs to Student Abilities

Preliminary Considerations

The fact that individual differences do exist among students is not only one of the most consistently documented findings of educational research, but also one characteristic of any group of children obvious to every teacher, experienced or novice. Furthermore, this classroom reality fairly demands that educators do something about it. Teachers feel great pressure to adjust the program of instruction so it corresponds to the varying abilities of their students. This fact is especially true in mathematics instruction where the range of abilities encountered is extraordinarily large. Unfortunately, the techniques for gearing programs to individual differences are not as obvious or readily available as is the need for it. *It is fair to say that the problem of individualizing instruction is one of the major unsolved problems in contemporary education.* In order to understand this situation, it is necessary to examine the kinds of individual differences that one must take into account in attempting to solve this problem.

In addition to differences in levels of mathematical achievement, other kinds of individual differences, such as those of mental age, attitudes, beliefs, maturity, motivation, sex, talent, and vocational aspirations, exist among children. These differences have a definite impact on the effectiveness of instruction. Although each of these factors may be an important consideration in the construction of instructional programs, they are nevertheless variables which represent only one dimension of the problem. Differences *in schools, in school systems,* in curricula, and in teachers must be taken into consideration in planning any serious program of individualized instruction.

Teachers, as a group, differ in probably as many ways as do their pupils. From the standpoint of mathematics instruction, the main differences in teachers that need to be considered are those of subject matter interest and knowledge. In mathematics instruction, the teacher's attitude toward the subject may be as influential in determining his effectiveness as his subject matter knowledge. In any case, both are important, and no effective program of mathematics instruction is possible with serious deficiencies of this type.

Individual differences in schools and curricula form yet a third dimension of the problem. Schools and school systems are as different as the people who comprise them. There are inner city schools and suburban schools, each with distinct problems. The curriculum and philosophy of education of these schools also vary greatly. Some systems have multi-tracked curricula; others have single-tracked. Although most schools establish a common pace and instructional content for all pupils, some schools experiment with individualized pacing and even content selection. Some schools tend to rely on the self-contained classroom taught by one teacher. Others use semi-departmentalized concepts of organization. Non-graded and team teaching patterns of organization are becoming increasingly popular. Increased utilization of such devices as closed circuit TV, educational network TV, programmed instruction, and even computer-assisted instruction is evident. Although the influence and effects of these variables are often subtle and difficult to analyze, they are real and, in fact, may quite possibly become the controlling factors in efforts to individualize mathematics instruction.

Classroom Management and Grouping Devices

A primary obstacle to individualized instruction is related to problems of classroom management. When children are all lock-stepped in the curriculum, a condition of minimal administrative stress is established. It is easy to lecture the group en masse, to test them, to sense who is competent and who is not, and so on. But as soon as the lock step is broken, the teacher must try to keep track of what each individual in the class is doing. With different pupils doing different work, the conventional base of comparison is lost, and evaluation of the pupils' progress becomes increasingly complicated. Finally, the demands of a classroom of individuals for individual attention can easily exceed the limits of physical endurance of any teacher.

To cope with these management problems, a variety of organizational schemes has been utilized. One of the simplest is permanent intra-class groups. This technique customarily involves the use of an ability criterion to separate the class into, say, the A, B, and C sections. In theory, the idea is to construct a number of nearly homogeneous subgroups in a heterogeneous class. This compromise preserves the class structure within the subgroups. The teacher can assign work on the basis of the ability of the group, maintain contact with the sections much in the same manner as he would with the whole class, and continue to evaluate students' progress in a conventional manner. Some problems occur in ranking and grading students, of course. For example, should the best student in the least talented group be given a higher grade than say the least able person in the most talented group? Some teachers determine the maximum grade to be given to any one group, e.g., A, B, or C, respectively. However, this grading policy tends to restrict students to certain performance levels. One apparent remedy is to make the intra-class groups flexible, thereby allowing a student to shift from one group to another depending on his performance in a given group.

Another grouping technique is that of departmentalizing the school. In a departmentalized school, a single student could be placed in groups of various ability levels depending on the subject and his level of performance in that subject. This type of arrangement, though meeting some of the desirable democratic principles of classroom organization, begins to produce considerable stress on the teacher or the team of teachers. Keeping track of the problems and progress of individuals, preserving good articulation between groups, and setting fair and consistent criteria for intra-class group transition are not easy tasks.

A somewhat more involved technique of grouping is employed in non-graded schools and in schools with inter-class grouping. In these cases, the option exists of fixing grouping arrangements for a given period of time or allowing them to remain flexible. In a flexible arrangement, a child is placed in a particular class if he meets certain criteria levels. The simplest criteria to use are those of performance on achievement tests; however, a variety of sociometric and psychometric measures may be used in addition to purely intellectual parameters or subjective judgment by the teacher. The particular combination of criteria chosen depends on many factors: the school — its organization and administration, the teacher or team, the objectives of the curriculum, the type of children, and so on.

There is no one grouping arrangement that is best for all children, all teachers, all curricula, and all schools. The design used in meeting individual differences must be highly specific to each of these considerations. It is a poor strategy to adopt totally and uncritically someone else's scheme. The chances of success in such a situation are small.

The Instructional Variables

Once a grouping arrangement has been selected, decisions have to be made with respect to the kind of instructional differences one wishes to provide for groups of pupils. There are three basic methods of adjusting the mathematics curriculum; these methods, incidentally, are not mutually exclusive, but are mutually consistent.

The first and most obvious way is that of pacing. In general, one might expect brighter groups of students to learn faster. This assumption, however, does not always prove to be valid. The consideration basic to the notion of pacing is that the child's behavior be adaptive. In other words, ideal pacing for each group of students should vary over time in such a way as to provide optimal progress toward educational goals. Some children work effectively at rather high rates of speed, whereas others benefit from more contemplative rates. In either case, the ideal rate does vary with time and is not a simple function of intelligence or achievement. Thus, one would expect considerable variation among individuals within groups.

Pacing may be externally controlled by the demands of the curriculum or by the teacher. On the other hand, it may be self-determined by the pupil. The former method maintains teacher control and minimizes management problems. However, it runs the risk of not producing optimum learning effects for all concerned. Self-pacing, on the other hand, may be effective with students who have some insight into their own learning mechanism and optimum pacing. Other students who, for reasons of motivation or maturity, are not so gifted may work too fast for their own good (or not at all). Recent experiments with primary grade children have indicated that considerable acceleration is possible for gifted children. Even in such a select group, enormous differences in learning periods are encountered. Whether acceleration is *desirable* or not is a decision that each school and its staff must make. If acceleration is encouraged, a school should take into consideration articulation problems from grade level to grade level, if these exist, or from school to school. Care must be exercised in designing the long range components of an individualized program.

A second instructional variable that can be adjusted to the individual's needs is that of level of difficulty. In some mathematics textbooks, exercises have been arranged according to presumed levels of difficulty for this purpose. A problem that often occurs when this method is used by schools is that the factors determining difficulty are poorly understood. In addition, students are often pressured into working toward "top-level" groups by their own drives or by their parents for reasons not always consistent with effective learning.

The third instructional variable, closely related to variations in pacing and in level of difficulty, is enrichment activities. If a child works effectively at a high speed and completes even difficult exercises with dispatch, he is often led to study other areas of mathematics not normally a part of the standard curriculum. Enrichment activities have been used more frequently in recent years, not only for the mathematically talented, but also, to some degree, for students anywhere on the spectrum of individual differences.

An important influence on enrichment activities in many elementary classes has been the "arithmetic corner" or "arithmetic learning center." Many elementary teachers have found that individual differences and needs are detected and best met if a rich variety of instructional materials is available to individual learners. Learning centers facilitate enrichment activities when they include an abundance of such objects as counters, counting frames, measuring instruments, place value charts, geometric objects, and number lines.

In a broad sense, the adjustment of these instructional variables presents the possibility of an individualized curriculum for intra-class groups, inter-class groups, and, in the extreme, for single individuals. Little is known at this time about how such a program could be administered or, in fact, whether children could be expected to select topics for study as they are expected to pace themselves or to select fitting levels of difficulty at which to work. However, trends in mathematics education are breaking down the uniformity and conformity which characterized traditional programs. Hand-in-hand with this development has come the necessity of rethinking the nature and objectives of the mathematics curriculum. Experiments with individualized curricula will hopefully yield a theory of individualization which will subsume the special cases of enrichment and acceleration.

Summary

The adjustment of mathematics programs to individual abilities is a challenging task. Success requires the effective cooperation of all concerned with the instructional task. The school must meet its responsibilities with imaginative administration, especially in developing a well-integrated and articulated curriculum, in providing assistance in classroom management, in utilizing teachers effectively, and in devising grouping schemes. Teachers of mathematics must meet their responsibilities by improving their subject matter competence so as to be more capable of evaluating pupil progress, of assigning students to groups, of advising students on enrichment materials, and of monitoring effective working speeds of pupils.

School systems should do their best to provide the material assistance necessary for the program. Individualization may be fostered through the use of adequately supplied mathematics learning centers in each classroom. Audio-visual aids are frequently useful in programs designed to meet individual differences as is the use of instructional TV or closed circuit TV. In some cases programmed instructional materials may be found beneficial. Good library resources are necessary, both for enrichment activities and for accelerated students. In large-scale programs it may be necessary to turn to modern technology for assistance. Efforts have been made in the last few years to provide computer assistance for the classroom management aspects of programs that are adjusted to abilities of individual students.

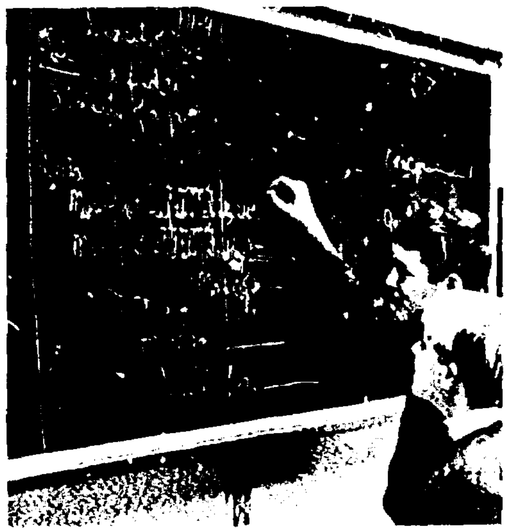
Problem Solving

A situation becomes a problem for a student only if the following conditions exist:

1. A question is posed (either implicitly by the materials being studied or explicitly by the student) for which an answer is not immediately available.
2. The student is sufficiently interested to feel some intrinsic need to find a solution.
3. The student has enough confidence in himself to believe a solution is possible.
4. The student is required to use more than immediate recall or previously established patterns of action to find a method for arriving at a conclusion.

Not all pupils will consider a given situation a problem. If a student can see a "method of solution" immediately, then the situation is not a problem for him, but simply an exercise or an application of some process which he has already mastered. For example, an exercise such as 3×14 is not a problem for a student who has already mastered the idea of multiplication with two place numerals. However, this example could very well be a problem for a student who has studied only a few elementary multiplication combinations. In order for this student to obtain a correct solution, he would have to think of some method of renaming 14 such as $10 + 4$ and find some method of reducing the problem to simpler stages with which he could work, such as $3 \times 14 = 3 \times (10 + 4) = (3 \times 10) + (3 \times 4)$. Real problem solving is not recalling a numerical fact or fitting a situation into a memorized pattern, but is discovering one's own method for extending previous learnings to new situations.

It is not the intent of this guide to consider all of the general aspects of problem solving. Only some of the problem situations that involve the use of known mathematical ideas by students will be discussed. The verbal problems found in textbooks make up a large portion of the problem situations considered in mathematics programs. When such problems are presented, the mathematical ideas necessary to solve them are generally known to the student, but the contexts in which they are presented require the student to find the methods of solution or to decide which of the ideas he already knows will help him find the solution.



The solution of a typical textbook verbal problem involves a series of four interrelated steps. The complexity of each of the steps varies with the problem, but the method of solution follows this general pattern.

Step 1 — Recognizing the question being asked.

Step 2 — Translating the verbal problem into a mathematical sentence.

Step 3 — Finding a solution for the mathematical sentence.

Step 4 — Analyzing the solution for the sentence to see if it provides a reasonable solution for the original verbal problem.

Each step of the solution can involve many facets. Although all steps in the solution of a verbal problem are important, Steps 1 and 2, the abstraction or translation phase, can be considered the "heart" of the process.

The student may have to decide many things before he is able to write the mathematical sentence which represents the problem. If the question is not explicitly stated in the verbal problem, the student must formulate his own question or questions. He will have to decide if all of the necessary data for the solution is given in the statement, and he will have to select the pertinent data from the given information. He will have to decide which mathematical operation is suggested by the "action" of the problem. He must determine the order in which the data of the problem will appear in the corresponding sentence. The student must also decide what relationships, if any, are involved that will enable him to use previously solved problems as "models" and what approach is the most efficient or easiest for him to use in obtaining a correct solution. Steps 1 and 2 could also involve some trial and error activity in which different sentences are tested as to which one best fits the given situation. These steps include problem solving techniques commonly referred to in many textbooks and teacher guides as "Reading and understanding the problem" and "Restating the problem in your own words."

Step 3 involves finding a solution to the mathematical sentence. This procedure can be very simple or very complex depending on the given situation. In general this step involves the application of knowledge that the student has already attained through practice work with similar sentence forms.

Step 4 is the familiar "check" in which the student determines if the solution is acceptable for the given problem situation. The student must decide if the solution is reasonable or "makes sense." If it is apparent that the solution is not correct, the student must then "rework" his problem, check his computations, examine his sentence to see if it actually symbolizes the "story" of the problem, and look for careless errors or possible misinterpretations or misrepresentations of the data.

Some examples of typical verbal problems considered in the elementary school are given below.

In the first grade, simple "picture" or oral problems are presented which can be represented by sentences like $5 + 2 = \square$ where the sum does not exceed ten. The problem might be presented to the students in this form: "John has 5 toy cars. His mother gave him two more toy cars for his birthday. How many toy cars does he have now?" The "action" of the story is the joining of two sets of toy cars, and the mathematical sentence which represents this problem is $5 + 2 = \square$. Through such presentations, the student will learn to associate the notion of "joining" with the operation of addition. He will thus have a better understanding of what addition means and will develop techniques for solving similar problems.

In the second and third grades, problem situations are extended to include verbal problems represented by sentences like $3 + \square = 8$, $\square + 5 = 9$, $7 - 2 = \square$, $8 - \square = 5$, and $\square - 2 = 6$. A typical verbal problem at these grade levels is: "Mary had 15 doll dresses. After she gave some of the dresses to her sister, she found that she had 8 dresses left. How many doll dresses did she give to her sister?" This problem is represented by the sentence $15 - \square = 8$. The "action" of the problem is "taking away"; therefore, the corresponding operation of the sentence must be subtraction. The 15 represents the number of dresses Mary had in the beginning, the □ represents the number of dresses she gave to her sister, and the 8 represents the number of dresses she had left. Students should realize that although $15 - \square = 8$, $8 + \square = 15$, and $15 - 8 = \square$ can be represented by different physical situations, the computation involved in solving each of the sentences is the same.

Similar "joining" and "taking away" problem situations can be represented by the other addition and subtraction sentences mentioned above.

Comparative problem situations can also be described mathematically. For example, the problem: "Jack has 45 baseball cards and Tom has 62 cards.

How many more baseball cards does Tom have than Jack?" can be represented by the sentence $62 - 45 = \square$.

In the later elementary grades, problems are presented that involve larger numbers as in sentences such as $48 + \square = 327$ or $239 - \square = 76$ and that use the same principles as do the problems first introduced in the primary grades.

In the third and fourth grades, problem situations represented by sentences such as $3 \times 7 = \square$, $5 \times \square = 20$, and $\square \times 6 = 30$ should be introduced. A sample problem for this grade level is: "Tom found that he needed sixteen small cartons to cover the bottom of a packing case. If he needed four layers of cartons to fill the case, how many small cartons were in the case?" The "action" of this problem is repeated addition or multiplication. Thus the sentence used to represent the story could be $16 + 16 + 16 + 16 = \square$ or $4 \times 16 = \square$. The student should be allowed to use either sentence to represent the problem, but should be led to realize that the sentence $4 \times 16 = \square$ is the shorter way of writing a sentence for this type of problem. In the latter sentence, the 4 represents the number of sets, the 16 represents the number of objects in each set, and the $\square$ represents the total number of objects in all of the sets.

In the sentence $5 \times \square = 30$, the 5 represents the number of sets, the $\square$ represents the number of objects in each set, and the 30, the number of objects in all. Likewise the sentence $\square \times 6 = 30$ represents a problem situation in which the total number of objects (30) is given, the number of objects in each set (6) is given, and the number of sets, represented by the $\square$, is the solution to the problem. The sentences $5 \times \square = 30$ and $\square \times 6 = 30$ represent different physical situations, but both types of sentences can be solved by the same computational method of repeated subtraction if the multiplication fact that makes the sentence true is not known.

Problem situations introduced at the fourth and fifth grade levels are similar to the problem: "If John separates 12 marbles into small groups with 4 marbles in each group, how many groups of marbles will he have?" This problem can be represented by the sentence $\square \times 4 = 12$, for the question of the problem can be restated: "How many sets of 4 are there in 12?" The problem can also be interpreted as the division of a set of 12 objects into sets of 4 objects. With the latter interpretation, the problem can be represented by the division sentence $12 \div 4 = \square$. Both sentences, $\square \times 4 = 12$ and $12 \div 4 = \square$, represent the same physical situation and implicity ask the same question, "How many sets of 4 are there in 12?" Both sentences are solved by repeated subtraction if the number which makes the sentences true is not known.

As fractions, decimal numerals, and integers are introduced, problem situations involving their use in sentence forms similar to those considered previously should be presented.

The verbal "story" type problems are not the only source of problem situations for use in the elementary school mathematics program. Many other mathematical problems arise from physical situations, from social or mathematical applications of mathematical ideas, or from situations made up by the teacher or the pupils. These varied situations can be real, imagined, or in the nature of a puzzle.

Many of these problems fall into a category which might be regarded as a "higher order" of problem solving than the typical textbook "story" problem. It is not possible to outline a regular progression of steps to be followed in solving all such problem situations as the process will vary with the nature of the problem presented and the ability and experience of the pupil. The four steps outlined for the solution of verbal problems do not necessarily apply in general problem solving situations, for it is not always possible to translate such a problem into a mathematical sentence.

However, it is possible to list a few of the activities that are essential in the general problem solving process. These activities include: translating the problem into a simpler form, critically examining the given data, forming hypotheses or conjectures, reasoning on a trial and error basis, analyzing or evaluating results on the basis of past experience, and forming generalizations from similar problem situations. These activities are not the only ones that may be involved, and not every problem situation will require the student to become engaged in all of the activities listed above. It is important to note that the order in which the student performs these activities may vary for different problem situations.

A few examples of "higher order" problems that can be included in the elementary mathematics program are given below. The types of problems presented at various grade levels should vary only in the complexity of the thinking required, not in the method of thinking or the kinds of mental activities necessary to solve the problems.

Example 1 — I have 80 cents in coins in my pocket. I have the same number of pennies, nickels, and dimes. How many coins of each kind do I have in my pocket?

Answer — Five coins of each kind.

Example 2 — Fill in the boxes so that the numbers in each column, each row, and each diagonal have a sum of 18. (Do not use any number more than once.)

□	□	7
11	□	□
□	3	□

Answer —

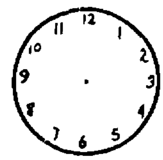
2	9	7
11	6	1
5	3	10

Other answers are possible.

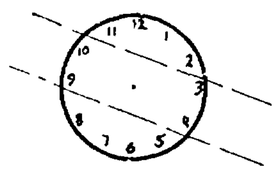
Example 3 — Is the point P inside or outside the closed curve? Tell the class the method you used to find your answer.

Answer — The point P is inside the closed curve. If we place another point outside the curve, we *must* cross the curve in order to draw a continuous path between the two points. Also note that any straight line connecting P to an outside point cuts the figure an odd number of times, if you count a point of tangency as two times.

Example 4 — Draw two straight lines through the clock face, dividing it into three parts, so that the numbers represented in each of the three parts have the same sum.



Answer —



Example 5 — If 2, 3, 5, and 7 are examples of prime numbers, is 103 then a prime number?

Answer — 103 is a prime number as it can be divided evenly only by 1 and itself. (1 and 103 are the only whole number factors of 103.)

Example 6 — Points A, B, and C all lie on the same circle. Find the center of the circle that contains points A, B, and C.

A
.

.B

C
.

Answer — Construct the perpendicular bisectors of two of the segments joining the three given points. The point where the perpendicular bisectors meet is the center.

Example 7 — What number can be divided by 11 so that the remainder is zero, but when divided by 2, 3, 4, 5, or 6 leaves a remainder of 1?

Answer — 121

Example 8 -- Roy had four cards with *1/2* written on the first card and three other numerals written on the other three cards. Roy claimed that he could form all of the combinations from 1/2 to 20, counting by 1/2 (that is, 1/2, 1, 1 1/2, and so on), by using a single card or by adding or subtracting the numbers represented on two or more of the cards. What were the numerals Roy had written on the other three cards?

Answer -- 1 1/2, 4 1/2, 13 1/2.

The teacher should provide some problem situations similar to the examples above in addition to the typical "story" type problems.

Teachers can do much to help students improve their problem solving abilities for all types of mathematical problems. The teacher's function should be to create a questioning, challenging atmosphere; to introduce problem situations; and to guide and encourage students to develop their own problem solving techniques.

A few suggestions that can help the teacher promote good problem solving techniques are listed below.

- The teacher should present problem situations that involve many basic mathematical principles. He should be certain that these situations are related to the kinds of experience the pupil has already had so that the pupil is able to apply the principles.
- The teacher should let his students use their own methods. Many problems have no single, best method of solution. Insistence on one "favorite" method of solution often destroys enthusiasm and original thinking.
- Whenever the problem permits, the teacher should emphasize the writing of a mathematical sentence that shows the "action" of the problem. If children are to acquire good problem solving habits, they must be able to describe the action of the problem in terms of mathematical symbols whenever applicable. Writing the mathematical sentence for a problem is as important mathematically as finding the solution for the sentence.
- The teacher should encourage his students to use diagrams, estimates, dramatizations, or other techniques that help them understand the problem.
- The teacher should suggest that students try various approaches to a problem when they are not certain of the correct method. Students can evaluate the method used by ascertaining that the solution is reasonable. Students should realize that the trial and error method can be an effective approach to difficult problems.
- The teacher should confront students with some situations in which they must formulate their own questions. This type of presentation is similar to the kinds of problems they are apt to face in their future vocations.

Evaluation and Testing

Evaluation is a continuous and integral part of the successful elementary school mathematics program. It has many facets and necessarily involves all staff members who are associated with the teaching of mathematics. Certain phases of evaluation may be objective in nature, but the process must accommodate the many value judgments that are both necessary and desirable. The techniques of evaluation must be both formal and informal and must include procedures for appraising the development of interests and attitudes, as well as skills and understandings.

Evaluating the Mathematics Curriculum

The purposes of evaluating the mathematics curriculum are two-fold, but are very intimately related. The first is to measure the extent to which the experience and behavior of the children reflect the content and behavioral objectives of the mathematics curriculum. The second is to use this information in making adjustments in the instructional program that it will provide for even greater realization of the objectives. As curricula are revised and the objectives are perhaps modified, the evaluation procedures must be adjusted so as to measure the intended outcomes of the revised curricula.

Published or standardized objective tests continue to be an important source of information about the effectiveness of the mathematics program. Tests must be carefully chosen. They should measure only those abilities that the curriculum is designed to develop, if the test scores are to have validity. As the objectives of the mathematics curriculum are broadened, the task of selecting appropriate standardized tests becomes even more important and increasingly difficult. The once popular test with major emphasis on computational skills is of little value when used alone to evaluate today's mathematics curriculum with its varied and extended objectives. Test scores take on significance only as they are considered along with other measurements of the effectiveness of the curriculum.

The majority of standardized tests are of the objective, short answer type, but the nature of these tests does not necessarily limit them to measuring only discrete, factual information. Properly designed objective tests can measure mathematical insight and originality, as well as levels of learning of difficult concepts.

Regardless of how appropriate a standardized test may be in evaluating the mathematics curriculum, its use is justified only if the results can be employed in improving instruction. System-wide analysis of scores is certainly one legitimate way of using test scores for evaluating and ultimately improving the mathematics instruction. When tests are used for this purpose, the question always arises as to what standards or norms should be used. Many school systems are less interested in comparing their children with other groups than they are in studying the performance of the children within their own system over a period of years. In this case, school systems should be encouraged to develop local norms. The use of a national or regional norm is not necessarily to be discouraged; however, it should be used only in the light of complete information with regard to its development.

It is also important that standardized test results be used to improve instruction within each teacher's classroom. Certainly these results embody significant information that will help increase the effectiveness of any teacher's instruction. The standardized test results may also be of some value in analyzing the instructional needs of the *individual child*.

Whatever use be made of standardized tests, schools must be careful that the material in the test does not dictate what is taught in the classroom. The pitfall of teaching to the test must be avoided at all costs! Some of the motivation for teaching to the test can be eliminated if everyone recognizes that a given test, when administered, does not measure recent achievement alone. Tests measure achievement and understandings developed by a child over several years. All teachers that a child has had must share both in his success and in his lack of success as reflected in a given test result.

Few questions associated with the evaluation process have definite "yes" or "no" answers. Contemplation of questions such as the following may, however, give some insight into the evaluation of the mathematics curriculum.

a) Does the curriculum differ markedly from that of most school systems?
b) Does the curriculum develop and maintain interest in mathematics?
c) Is the curriculum flexible enough to meet the needs of each child?
d) Does the curriculum encourage creativity, originality, and mathematical experimentation by the children?
e) Does the curriculum develop adequate facility with computational skills?
f) Does the curriculum give proper emphasis to the unifying concepts and structure of mathematics?
g) Does the curriculum prepare the children to use their mathematical skills and knowledge outside the mathematics class?
h) Does the curriculum present mathematics as an enjoyable and fascinatingly alive subject?

The process of evaluating the curriculum must not focus on simply whether the children are learning what is being taught. Attention must be given to evaluating whether that which is being taught is that which should be taught. It must also be recognized that due to the varied backgrounds and abilities of the children, flexibility in the expected minimum levels of achievement is necessary.

Evaluation of the Child's Mathematical Development

Although evaluation of the effectiveness of the entire elementary school mathematics curriculum is of prime importance, the evaluation of the learning experiences and achievement of each child as an *individual* is of at least equal significance. In reality, such evaluation is the final test of the effectiveness of a mathematics curriculum in a society which places ultimate value on the worth of every individual.

The evaluation of a child's growth in developing mathematical concepts and abilities, like the evaluation of the entire mathematics curriculum, necessarily involves a variety of techniques and procedures. This appraisal must include the many aspects of learning beyond the acquisition of facts and skills. The expanded content and behavioral objectives of today's mathematics curriculum are making the evaluation of each child's progress an increasingly difficult task. Not only must educators know whether the pupil is succeeding or failing, but they must also strive to ascertain the level of success he is experiencing with respect to each of the several objectives of the mathematics program.

The evaluation of a child's progress must be deliberate and systematic, yet it must allow for the measurement of growth (or lack of growth) at any time when it can be observed. It matters little whether the evaluative procedure is the interview, a conversation or exchange of ideas between teacher and child, a standardized or teacher-made test, or the observation of the behavior of the child in a mathematical situation. The relevance of an evaluation is determined by the degree to which it measures the pupil's growth toward realization of specific objectives of the mathematics curriculum.

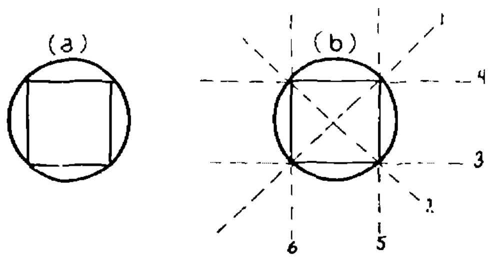
With reasonable care and a small amount of ingenuity, the teacher can devise test questions or activities to evaluate progress toward the realization of any of the content or behavioral objectives of the mathematics program. The following are offered as illustrations of possible evaluative activities.

1. In the lower elementary grades (even at the prenumber level), the child's appreciation for number properties can be appraised by confronting him with two collections of objects and asking him whether there are as many objects in one collection as in the other. If the collections of objects are so large that he cannot mentally match the objects, his understanding of the role of one-to-one correspondence between sets with the same number property can be appraised.
2. The appreciation of the kindergarten child for the number property of a set can be observed by asking him which of two sets of objects has the greater number of objects, when one set has a large number of small objects and the other has a smaller number of much larger objects.
3. The ability of the second or third grade child to generalize relationships can be evaluated by reminding him that 5 + 7 is 12, 15 + 7 is 22, and 25 + 7 is 32, and by then asking him what he thinks 85 + 7 equals. If his reaction is immediately "92," he has probably made the desired generalization. If his response is slower, or if he gives evidence of counting or actually attempts to add the two numbers, he has probably not been able to make the generalization.
4. This example is one way of presenting the second or third grade child with an opportunity of showing his understanding of the distributive property. State, perhaps orally, that John has seven marbles and Ted has three marbles. Then ask how many marbles they would have altogether if they each had two times as many marbles. The child who understands that the correct answer may be found by computing $(2 \times 7) + (2 \times 3)$ or by computing $2 \times (7 + 3)$ has demonstrated his understanding of the distributive law. He understands the property even if he is not able to verbalize the relationship expressed by it.
5. In the middle grades, evidence of the child's ability to generalize may be obtained in the following way. Show him that: $(12)^2 = 144$ and $(13)^2 = 144 + 24 + 1 = 169$; and that $(20)^2 = 400$ and $(21)^2 = 400 + 40 + 1 = 441$. Then tell him $(40)^2 = 1600$ and ask him what $(41)^2$ equals. A further generalization would be required for him to know what $(39)^2$ equals.
6. In the middle grades, the child's feeling for the associative property of multiplication can be appraised by observing how he computes $(13 \times 5) \times 2$. The student who understands the associative property will probably recognize that $(13 \times 5) \times 2 = 13 \times (5 \times 2)$ and will then be able to carry out the operation mentally by using 10×13. Such problems when given as mental exercises can be used both to develop and to test an understanding of this basic property of multiplication.
7. The ability to translate an English statement into a mathematical sentence is a vital part of solving some problems. This aspect of problem solving can be evaluated in the middle grades by giving the child the following exercise.

   Choose the mathematical phrase or sentence that best expresses the English phrase or sentence and write the letter of your choice in the blank preceding the English phrase or sentence.

—five times the number	A. 5 × □ = 5
—the number is five	B. □ + 5
—five less than the number	C. 5 × □ 5
—five more than the number	D. □ − 5
	E. 5 − □
	F. 5 + □
	G. 5 × □ + □
	H. □ = 5

8. The following is an example of the type of simple problem situation that can be presented to children in grades 4 to 6. Provide the child with a piece of waxed paper on which a circle has been marked. Then challenge him to form, by folding the paper, a square that just fits inside the circle as in the manner shown in (a) below.



Tell him that he can fold the paper as many times as he desires and in any way. Sketch (b) indicates at least one way in which successive folds can be made so as to carry out the assignment. Observing the child in the process of solving such a problem, even if a simple one, provides the teacher with real insight into a child's ability to bring a variety of knowledge to bear on a new or strange situation.

9. The following type of performance exercise could be used in grade five or six to assess the child's appreciation for the relationship between similar figures. Give the child a triangular piece of paper and ask him to cut it so as to form a triangular piece of paper similar in shape to the original triangle, but with one-half the perimeter. The child who proceeds to cut the original triangular shape from the midpoint of one side to the midpoint of another side will be displaying an understanding of the relationship between similar triangles.

10. A multiple choice exercise such as the following might be useful in appraising a sixth grader's understanding of why division by zero is not possible.

Which of the following is a correct statement about the problem 6 ÷ 0?

A. The answer to 6 ÷ 0 is 6.
B. The answer to 6 ÷ 0 is 0.
C. There is no answer to 6 ÷ 0 because there is no number that when multiplied by 0 gives 6 as a result.
D. Either 0 or 6 is correct so there is no one answer to the problem 6 ÷ 0. It depends on the answer you want to use.

The above examples of evaluation procedures are not intended to be taken as models, but only as illustrations of the wide variety of techniques that can be used. It is frequently possible to present, with slight modification, the same basic question or test situation to the pupil in the form of an oral, a written, or a performance experience. It may be necessary to adjust the evaluation procedure to the pupil, for not all pupils can be expected to achieve the same level of understanding of mathematical ideas. In order to determine the level of understanding a child has reached with respect to a given concept, one may have to devise a series of questions, each intended to measure a different level of understanding. One may also employ a more lengthy question that calls for several immediate responses, each requiring a different level of understanding. In any event, the child's background and potential must be considered, both when deciding which evaluation procedures to use and when interpreting the degree of success the pupil has experienced.

An increasing number of rating scales and check lists are available to help the teacher in evaluating the pupil's attitudes toward mathematics as well as his interest in the subject. For the most part, these instruments are of unproven reliability and validity. Consequently, the teacher's value judgment remains a significant source of evaluation in this area. These factors along with the pupil's enjoyment of mathematics may be appraised by observing his behavior when confronted with mathematical situations. Is the child number conscious in his recreational reading or in his reading of other school subjects? Is he interested in recreational mathematics? Is he fascinated by mathematical puzzles? Does he display an inquiring attitude toward mathematics by raising questions about numbers, operations, or their properties that indicate he is thinking about concepts beyond those which he has studied?

Children in the lower grades can be asked questions about what they like best or least about numbers, adding, circles, or counting. In the upper grades, the child can be asked to write a short essay or paragraph on what he thinks about when someone says *mathematics* or on what he likes best or least about mathematics. At times the teacher should request that his pupils leave their essays or paragraphs unsigned so as to encourage a freer response and thus to obtain a more accurate picture of the attitude of the class as a whole.

Today's elementary school mathematics program continues to have as an objective, the development of computational skills. Both commercial and teacher-made objective tests are effective instruments in evaluating the degree to which this objective is achieved. Tests should be used which have a wide variety of items, each designed to test a particular skill. In this respect, there is some advantage in using commercial tests in that more care and perhaps greater and more varied test writing experience are employed in designing a given test.

Care must be exercised in interpreting a test score when it is being used to evaluate a child's progress. A single test score has meaning only when it is viewed in relation to other measures of his achievement. Patterns formed over a period of time by test scores are also useful in assessing a child's mathematical development.

Evaluation of the mathematics curriculum or the mathematics achievement of an individual child must not be separated from the teaching of mathematics. Evaluation is an integral part of teaching and must be continuously brought into play in the process of directing and re-directing learning experiences. It is a vital part of the ongoing program of curriculum improvement. Evaluation in elementary school mathematics must not be viewed as a process of appraising a finished product or a spent program. Instead, it must be recognized as the process by which a continuous record is maintained of the pulse rate of every aspect of both the mathematics curriculum and the experiences of the children under the curriculum.

Inservice Education

Need

Current demands that elementary mathematics instruction be based on both an understanding of the learning process and an understanding of mathematical content present a challenge to all teachers. For many decades the importance of mathematics in the elementary school was minimized; little attention was given to it in teacher education. The need for more thorough teacher preparation in this area is now being recognized. Several professional groups have proposed raising the minimum requirements in the training of elementary teachers; e.g., the Committee on Undergraduate Programs in Mathematics (CUPM) recommends a minimum of twelve hours of mathematics for pre-service teachers. Gradually teacher training institutions are increasing their requirements for graduation and are offering courses specifically designed for prospective elementary teachers.

There is also a greater realization that professional growth is a developmental process. Learning begun in pre-service education must be continued if a teacher is to keep pace with changes and their educational implications. The proliferation of knowledge resulting in an ever-changing curriculum makes the present-day problem more acute than ever before. With the introduction of new content and modern teaching strategies, even teachers with strong mathematical backgrounds feel a need for continuous education.

Planned programs of inservice education are one way of meeting the need for professional growth. However, these programs, essential as they are, do not automatically insure the implementation of new curriculum in the classroom. In the final analysis the effectiveness of inservice work can best be measured by what happens as a teacher works with children in a teaching-learning situation. Administrators and teachers share a joint responsibility for creating a learning situation which reflects the best current thinking in the field. Teachers with a commitment to the profession constantly seek to update their background of knowledge as well as their teaching techniques. They are open to change and avail themselves of professional growth opportunities. Administrators facilitate change by furnishing the necessary resources: time, instructional materials, professional books, consultant and supervisory services; and by offering support, encouragement, and understanding of problems encountered.

Objectives

The major purpose of inservice education is the improvement of professional competence through the development of attitudes, understandings, perceptions, and skills that enable teachers to provide students with a good mathematics program. Systematic, locally-sponsored teacher education programs should be designed to:

- develop a greater understanding of the structure of mathematics, and to insure a thorough understanding of the mathematical concepts taught.
- relate what is known about child growth and development and the psychology of learning to the teaching of mathematics.
- translate the results of research studies into classroom practices.
- familiarize teachers with many useful teaching materials.
- acquaint teachers with mathematical literature, current books, articles, and research reports.
- stimulate enthusiasm and interest in improving mathematics instruction.
- create an awareness of the need for continued study.

Guidelines

The approaches used to achieve the above objectives will vary from district to district. However, past experience indicates that inservice programs are most effective when:

- a systematic program of instruction is concentrated in a series of meetings or workshop sessions.
- the content of the program is related to the

specific local program that is being developed and is, therefore, relevant to the teachers' immediate needs.

- time is spent not only on mathematical ideas, but also on the demonstration of appropriate teaching techniques for various grade levels.
- participants are actively involved in discussions, demonstrations, and work sessions.
- participants have an opportunity to "discover" mathematical ideas for themselves.
- inservice programs and classroom experimentation with new ideas are carried on concurrently.

Organization of Inservice Programs

No matter what type of organization is used, an inservice program needs to be developed in stages. The first step should be the identification of problems. The problems may be those which are obvious to teachers or those which are of concern to supervisors. After the problem has been agreed upon and studied, implementation of the results in classroom instruction must follow. The end result of a quality inservice program will be desirable behavioral changes in pupils.

1. *Workshop*

   The most common approach to inservice education has been the workshop. Workshops may be conducted solely by local districts or by a local district in conjunction with a university or college. In either case careful consideration should be given to the following factors.

   a. Planning—Conducting an effective, well-organized workshop requires careful planning. The objectives of the workshop should be clearly defined, so that the program can be designed to meet the particular needs of the participants. When visiting consultants are used, it is essential that local supervisors or administrators plan with the consultants prior to the workshop.

   b. Schedules—Various times may be chosen for workshop activities:
      - a series of regular afternoon or evening sessions, e.g., 8 to 12 two-hour sessions.
      - a series of Saturday sessions.
      - a series of half-day or full-day sessions during the school year.
      - Summer workshops of varying periods, e.g., one, two, or three-week sessions.

      As inservice activities are planned, consideration should be given to the possibility of allowing some released time for teachers.

   c. Leadership — Consultants whether local or visiting should be knowledgeable in mathematics and in teaching techniques. Both aspects of the program may be handled by the same person, or the responsibilities may be divided among a team of consultants. In some instances content may be presented by a university or college instructor, and methodology by local qualified supervisors or teachers.

   d. Format — Meetings may be divided into two parts: one of which is devoted to the discussion of mathematical content and the other, to classroom application.

      Frequently the content aspect of the workshop is presented to the total group of participants. However, in some cases, it may be desirable to arrange to have the total group divided into two sections for discussion of mathematical ideas: one for those who have had previous courses or workshops, and the other for those who have had no such experience.

      For work on classroom application, division of the group into primary and intermediate sections is desirable.

2. *Curriculum Committee*

   Committee work provides an excellent opportunity for teachers to become involved in the development of new programs and in the preparation of curriculum materials useful in the implementation of these programs. Members of such committees may be volunteers or selected representatives from schools within a district. Supervisory personnel or teachers with special skill in mathematics may provide leadership, and consultants may be brought in as needed.

   The content of this guide may be used as a source in identifying problems that can be studied in light of the needs of a particular school or school system. Committees might be organized to:
   - develop a scope and sequence plan for a K-6 program.

- prepare a guide which suggests varied learning experiences related to the behavioral objectives.
- prepare units of study for particular grade levels on such topics as geometry, measurement, ratio, problem solving, etc.
- evaluate instructional materials: textbooks, audio-visual aids, manipulative materials.
- evaluate standardized tests.
- prepare tests related to the objectives of the local curriculum.

3. *Television*

Television provides an effective means of reaching many teachers. Where educational television facilities are available, local school systems may either prepare their own series of telecasts or use series produced by professional organizations, educational institutions, or commercial concerns. Generally speaking, inservice education through television has proved most successful when provisions have been made for some type of "feedback." Feedback may be achieved through a combination of television lessons and group meetings.

4. *Other Common Inservice Activities*

The following activities may be used to augment the above-mentioned programs:

- faculty discussion groups.
- teacher visitation between schools.
- action research.
- demonstration lessons.

5. *Other Professional Growth Opportunities*

In addition to the programs sponsored by schools or school systems, teachers may improve their teaching competency by:

- enrolling in college or university courses.
- attending professional meetings and conferences.
- applying for grants for summer institutes.
- reading professional literature.

Criteria for Program Development

What are the characteristics of a good mathematics program? What teaching tools (books, films, visual aids, etc.) should be selected to assist the teaching staff in achieving the objectives of the program? These are two of the most important questions that every school must answer periodically. They are also among the more difficult questions for any school staff to answer. It is the aim of this guide to provide the schools of Wisconsin with means whereby they can arrive at more satisfactory answers to important curricular questions.

The foundation of any good program incorporates the ideas which society has found to be of value in the education of its youth. These ideas are the content objectives of a program. For the mathematics program, they consist of the properties of numbers, similarity, ratios, rational numbers, etc., as outlined in the previous sections of this bulletin. In addition to content objectives, behavioral objectives for a program must be established. These, too, are outlined in this bulletin. If to these objectives, one adds the goal of teaching pupils to apply mathematics through problem solving, then one has the sound framework needed to build a mathematics curriculum for the schools.

This bulletin suggests the type of objectives the school should consider in developing its curriculum. However, the same criteria that are used to judge the adequacy of a program can, in part, be used to judge the adequacy of the teaching tools selected for use in the school. These judgments must be made by the teachers or their representative committees. To assist the schools in making these judgments, a sample checklist of basic criteria is given in this section.

Note that the sample criteria checklist consists of five main sections. Section 1 focuses on general considerations. Is the mathematical content sound? Is the development spiral in nature? Are the objectives appropriate for the various grade levels?

Section 2 is a listing of the content and behavioral objectives. Section 3 consists of pedagogical considerations. Section 4 contains criteria on which to judge supplementary aids, and Section 5 briefly touches on the importance of format in making judgments as to the usefulness of a teaching tool.

Committees of teachers will want to modify this sample criteria checklist to fit the needs of the school for which a program is being formulated. Furthermore, they may wish to assign weights to the various sections. In the opinion of many people, a great deal of weight should be placed on the various subsections of Section 2 of the sample criteria checklist. It is in this section that one can find the spirit of the "new" mathematics programs for the schools of Wisconsin.

Sample Criteria Checklist

Name of Program	Publisher	Grade	Teacher

1. Philosophy of Program

The program	Low 1	2	Rating 3	4	High 5
a) Is consistent with the K-12 mathematics program of the school.					
b) Contains good mathematical development.					
c) Presents a continuous development of concepts and skills.					
d) Is teachable and appropriate for the particular level.					
e) Uses a spiral development.					
f) Develops concepts through numerous examples.					
g) Uses consistent and understandable mathematical language.					

2. Key Mathematical Content and Behavioral Objectives of Program

A. ARITHMETIC CONCEPTS

	Low 1	2	Rating 3	4	High 5
a) The program provides for an understanding of sets and numbers, including: Cardinal numbers.					
Ordinal numbers.					
Whole numbers					
Fractions					
Negative integers.					
b) The program provides for an understanding of numeration systems through: A variety of grouping activities.					
An emphasis on place values (base 10).					
An introduction to other numeration systems.					

	Rating				
	Low				High
c) The program provides for an understanding of the concept of order through: Experiences in determining "greater than," "less than," and "equals."	1	2	3	4	5
The use of the symbols <, >, =.					
The ordering of a set of numbers (from smallest to largest).					
d) The program provides for an understanding of number systems, operations and their properties through: Experiences with addition (and subtraction as its inverse).					
Experiences with multiplication (and division as its inverse).					
Experiences with the structure of the system of whole numbers.					
Experiences with the structure of the rational number system.					
e) The program provides for an understanding of the concepts of ratio and proportion through: Separate treatment of the concept of ratio from that of rational number.					
Many experiences entailing the use of ratio and proportion in problem solving.					
f) The program provides for an understanding of computation through: A developmental program in basic facts.					
Experiences developing proficiency in common computation procedures (algorithms).					
The use of non-drill activities to build computational skills.					

B. MATHEMATICAL SENTENCES

	Rating				
	Low				High
g) The program provides for an understanding of mathematical sentences through: Equations and inequalities developed as "translations" and "interpretations" of physical situations.	1	2	3	4	5
Intuitive methods suggested for the solution of equations and inequalities.					
Practice in the solution of equations and inequalities.					
Use of equations and inequalities to solve problems.					

C. GEOMETRY CONCEPTS

	Rating: Low 1	2	3	4	High 5
The program provides for an understanding of the concepts of:					
h) Size and shape through: Experiences in identifying plane and solid figures.					
i) Sets of points through: Manipulative and conceptual experiences with points, lines, and planes.					
j) Symmetry through: The development of symmetry concepts.					
k) Congruence through: The development of congruence concepts.					
l) Similarity through: The development of similarity concepts.					
m) Coordinate systems and graphs through: The use of number line activities and the development of coordinate systems and graphing.					
n) Constructions through: Construction problems.					
o) Measurement through: Experiences with money and time.					
The development of concepts of basic units for length, area, volume, angles.					
The development of approximate measures.					
Experiences with equivalent measures (and the reduction of measures).					
The development of area formulas for squares, rectangles, parallelograms, and triangles.					
The development of the linear metric system.					

D. PROBLEM SOLVING

	Rating: Low 1	2	3	4	High 5
p) The program provides for an understanding of problem solving through:					
Continuous emphasis on problem solving.					
Practical help for solving problems.					
A wide distribution and variety of problem solving practice.					

3. Methodology Used In and Suggested for Program

The program provides:

	Rating Low 1	2	3	4	High 5
a) Mathematical concepts correctly and consistently used in explanations.					
b) A variety of approaches in developing a topic.					
c) A basic skills development and maintenance program.					
d) Adequate material for written assignments.					
e) Sufficient material for oral exercises.					
f) An emphasis on estimation and "trial-and-error" techniques in problem solving.					
g) "Mental arithmetic" activities.					
h) Independent thinking activities.					
i) Exercises graduated in difficulty.					
j) Consideration of individual differences.					
k) Clarity of examples.					
l) Supplementary activities for high-achievers.					
m) Supplementary-remedial activities for low-achievers.					
n) Independent work in geometry or other topics.					

4. Supplementary Aids to Program

	Rating Low 1	2	3	4	High 5
a) The teacher's manual Contains mathematical background for content of program.					
Provides adequate suggestions.					
suggests numerous activities for enrichment (high and low).					
b) A suggested testing program for use in pupil evaluation is available.					
c) The publishing company provides consultants, films, or other means of inservice aids.					

5. Physical Format of Book

	Rating Low 1	2	3	4	High 5
a) The general format of the book is attractive to the pupil.					
b) The binding of the book is of good quality.					

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