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ABSTRACT

Reported is a study of the use of instructional blocks in college freshman mathematics courses. The first year of the project, one course was organized to have three large group lectures per week and one small group tutorial session per week. In addition, help sessions were made available. Problem materials were developed in connection with text materials and a computer laboratory was used. During the second year, four courses were involved; one with programed texts and which met every day; three with the same format as before. Evaluation of results indicates significant reduction in staff cost per student and discovery of topics and media for future study. The instructional pattern will receive further evaluation. Appendices include: a mathematics diagnostic test, sample tutorial problems, sample of a computer program, teacher evaluation questionnaire, and sample journal articles for tutorials. (JG)

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BLOCK FORMAT INSTRUCTION

Morton R. Kenner
Stephens College
Columbia, Missouri 65201

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U. S. Department of
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U.S. DEPARTMENT OF

HEALTH, EDUCATION, AND WELFARE

Office of Education

Bureau of Research

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Section 1. Summary

The investigation, reported here under the title, Block Format Instruction, was a pilot study which sought to determine methods (and consequent problems for further investigation and refinement) of partitioning the freshman undergraduate mathematics program into instructional blocks which maximized the efficiency of human and physical resources deployed for that purpose. The pilot study lasted two academic years: 1968-1969 and 1969-1970.

During the 1968-1969 academic year different instructional blocks were used in a single course: Freshman Finite Mathematics. The instructional blocks used were Large Lectures, Tutorials (6-10 students), Programmed Materials, Help Sessions, Computer Laboratory.

During the 1969-1970 academic year different instructional blocks were used in three courses: Freshman Finite Mathematics, Freshman Mathematics for Elementary Teachers, and Freshman Mathematics for General Education. The instructional blocks used were Large Lectures, Tutorials (6-10 students), Programmed Materials, Help Sessions, Computer Laboratory.

Evaluation of results was of two kinds. Firstly, the staff cost of instruction per student was considered. This analysis showed a decrease in the instructional cost per student as well as a decrease in fringe staff load. The latter resulted primarily from an almost total drop-off of individual students seeking help from staff outside of class scheduled times. Secondly, the evaluation of the effectiveness of the instructional methods was done subjectively. That is to say, that such evaluation was based on judgments of students, mathematics staff involved in teaching, and judgments of other professional persons including consultants. The consensus indicated that varying instructional formats was beneficial to the learning process. Particularly noteworthy elements are variety of instructional format enhances the learning process, individual differences and needs were able to be considered more easily and effectively, a greater variety of material was able to be covered, and ad hoc changes in the course could be made more easily increasing course flexibility.

Significant problems remain to be studied: the matching of content material and instructional format, the mechanical organization of selection for tutorial section, the more selective use of media to make more effective the lectures, the role of computer instruction in Freshman mathematics programs, the utilization of staff with varying competencies.

The instructional pattern will be continued at Stephens College and in addition will be introduced, experimentally, at Northwest Missouri State College for the 1970-1971 academic year.

Section 2. Introduction

The central concern of the original proposal for an experiment in Block Format Instruction was to initiate, as a pilot study, institutional instructional changes leading to a more efficient deployment of human and physical resources in the learning process. A non-trivial side effect was also to be a more efficient use of time committed to specific learning tasks by students.

The constraints imposed on the pilot study were such as to insure a search for means of varying the learning patterns by using different instructional formats which did not require the permanent commitment of significant amounts of additional human or physical resources by the college. That is to say that instruction can clearly be improved and individualized if an exceptionally competent staff member is assigned to each student with learning to proceed on a one-to-one basis. The effort here was to develop an instructional methodology whereby--apart from the initial developmental phase--significant reduction in instructional costs could be obtained while at the same time improving the learning situation.

The problem is, of course, a central one in all phases of higher education. Increasing enrollments accompanied by an increased reluctance of the public to maintain even the same (purchasing power) cost outlays per student demand attention to the dual problem of increasing learning efficiency while decreasing learning costs per student. The freshman mathematics course was chosen as a basis for the pilot study in part because of the establishment of both a new four-year and a new freshman mathematics course. With no traditions of instructional patterns for the specific course, it seemed plausible to assume that this course might be a good point of departure for initiating investigation into the general problem.

A second seeming advantage of initiating the investigation in the context of a freshman mathematics course was that the content of a mathematics course seemed most appropriate and accessible to the task of partitioning of material for different learning patterns. Thus, the easy ability to focus on problem solving and discovery and generalization in tutorial sections, or the easy ability to focus on remedial skills in help sessions or programmed materials was utilized. This is not to suggest that other subjects and/or inter-disciplinary courses do not have similarly definable chunks. They are especially apparent in the expected outcomes of almost any freshman mathematics program.

Given the reluctance to move toward the utilization of any learning devices committing the college to long-range costs, minimum attention was given, in this pilot study, to the use of specialized media. Thus, the use of closed-circuit television, for example, while offering many ostensible learning advantages might clearly result in increased, rather than decreased costs. Assuming the costs of production and the cost of continuous updating and changes which should be done for the material to remain both fresh and topical might well be a commitment to increased cost. As noted in our conclusion, this is an area deserving of continued study. Similar comments can be made about initiating now the use of computer-assisted instruction. Again, with computer-assisted instruction, there appears to be significant cost requirement for experimentation in the development of instructional formats not now able to be realized as effectively by other less expensive patterns. The basis for cost estimates may be radically altered when and if both closed-circuit television and computer-assisted instruction become more commonly used. Should such a time arise, costs would then be borne by more instructional units, decreasing the cost to all.

There is an additional factor to be considered in the development of innovative learning formats which relates less to their cost than to the likelihood of their being adopted at all. Whenever instructional formats differ too greatly from accepted instructional patterns, their integrity --and hence real value--is often undermined by staff untrained and/or unwilling. Educational innovation which is to be truly permanent must take place slowly and not disenfranchise large numbers of staff from participating in improved learning situations. Staff directly involved in development are far more motivated and excited and, as a result, spend significant amounts of their own time on what to them becomes and is an act of creation. Similar expectancies should not be assumed from staff not so involved.

Thus, from the point of view of many experiments now being carried on in the United States and elsewhere, the pilot study reported here is very modest. It attempted to make no demands on resources not already at hand nor easily and inexpensively obtainable. It attempted to utilize those skills and resources available: varying degrees of mathematical competencies of the staff, staff skills in lecturing, staff skills in working with small groups of students, staff skills in determining abilities and needs of individual students, staff ability to develop problem materials, staff ability to select readings, and the abilities of exceptionally able students to help other students.

Section 3. Methods

The course chosen to begin our pilot study in the fall of 1968 was the new freshman mathematics course, Finite Mathematics. At that time the mathematics department offered three (3) courses available to beginning mathematics students: Mathematics 101 (for students unprepared for Finite Mathematics, for general education students, and for elementary education majors), Mathematics 111, Finite Mathematics (for students who intended either to major in mathematics or allied fields, and for students who might be using significant amounts of mathematics in their future work, i.e., students intending to major in the social sciences), and Mathematics 122, Calculus (for advanced students who planned to continue their work in mathematics).

In August 1968, Mr. Stockton and Dr. Kenner began work on developing a combination placement-diagnostic test for incoming freshmen. The main purpose of the test, at that time, was to discriminate among students in placing them either in Mathematics 101 or in Mathematics 111. As we note below, the use of the test and its subsequent refinement was altered for incoming freshmen for 1969-1970. Appendix I is a copy of the final diagnostic test used. On the basis of the first draft of the placement-diagnostic test, College Board scores, and high school records, students were enrolled.

In the fall of 1968, approximately 100 students were enrolled in Mathematics 111, Finite Mathematics. The 100 students were organized into two basic sections of 50 each. (The lecture room assigned had a maximum capacity of 60. As will be noted in our conclusion, the number in each basic section could have been 75-100 without loss of effectiveness.) Each of the basic sections was organized into 5 tutorials of 10 students each. The tutorial time had not been pre-scheduled. Thus a questionnaire was distributed to each student to determine free periods and professional interest. In addition, the diagnostic test and College Board scores were used to attempt to define tutorial sections which as nearly as possible were filled by students of similar ability and professional interest. One tutorial was specifically set aside for students displaying specific weaknesses on the diagnostic test. Although the tutorial section was not called a remedial section, part of its function was remedial in nature. The instructor of each tutorial section was given a copy of the student's test.

The basic organizational framework of each basic section in the fall of 1968 was

3 lectures per week - required
1 tutorial per week - required.

The course carried four (4) semester hours of credit. It will be noted that we adhered to the traditional formula of one credit for each required period. The lecturers were Mr. Stockton and Dr. Kenner. The tutorial instructors were Mrs. Kenner, Mr. McDonald, Mr. Stockton, and Dr. Kenner.

In addition to the required periods, help sessions--both group and individual--were established. Three 2-hour group help sessions were made available and students were also able to see a student assistant individually on a regular basis. That is, a student did not need to make an appointment to receive individual help from the student assistant. The help sessions were staffed by two exceptionally able undergraduate mathematics majors, Miss Pamela Comer and Miss Deborah Dunphy.

A two-week period was set aside (at different times for the two sections) to concentrate on developing a minimal operational acquaintance with computer language. Mrs. Ellen Scheer, of our staff, assisted by Mr. Tony Evers of the Computing Center Staff of the University of Missouri, supervised this two-week period. Stephens College facilities consist of key punches. Programs were batch processed at the University of Missouri Computing Center with four (4) pick-ups and deliveries each day. Programming problems were developed to aid the students during this brief introduction to FORTRAN IV. (Appendices V, VI, VII are samples of the material developed for this purpose.)

Problem materials were also developed in connection with text materials, to be used with special effectiveness in the tutorial sections. (See Appendices II, III and IV for samples of these.) There was, of course, great variation in the tutorial sections since sorting had been done in part on the basis of ability. In some tutorials, for example, extra readings, such as found in Appendix X, were read and discussed. These readings may not have been suitable for all tutorial groups. The lecturers for the lecture sections coordinated their efforts and were never more than one day apart during the year, with the exception, of course, of the two-week alternated time devoted to computer instruction.

During the year, Professor Ralph Lee, Director of the Computer Center, University of Missouri at Rolla, and Professor A. M. Mark, Associate Chairman, Department of Mathematics, Southern Illinois University at Carbondale, served as consultants. Dr. Lee considered and analyzed our work in computer education and Dr. Mark helped us define and prepare our tutorial section problem materials. Suggestions from both of these valuable consultants (noted in results) were incorporated in our planning for the 1969-1970 academic year.

At the end of the year we administered to each student a Teacher Evaluation Form (Appendix VIII) and a Course Format Evaluation Form (Appendix IX). No statistical analysis was made from these subjective evaluation forms. They did provide, however, significant amounts of information about student activities, about what we were doing, and about how successful the students thought we were.

Following the 1968-1969 academic year, significant revisions in procedure were made for the 1969-1970 academic yaar. On the basis of a comparison of College Board scores, our own placement test scores, and high school records, we found little gained by using our placement scores. In fact, high school record, alone, seemed to be a fairly good indicator of correct level of entry into the freshman mathematics program. We had found, too, that preparing for Mathematics 111 by enrolling first in Mathematics 101 was not usually necessary or successful. As a result, the freshman mathematics entry points were changed for the 1969-1970 year. To accommodate these changes in entry points, some course revision was also established. The courses and the basis for entry are given on the next page.

Mathematics 101: Only for students (principally in the Arts and Humanities) who plan no further study of mathematics--a course in General Education. Entry: one and one-half units of algebra or more. No placement test.

Mathematics 101e: Only for students who plan to satisfy requirements for state certification in Elementary Education. Entry: one and one-half units of algebra or more. No placement test.

Mathematics 111R: For students who plan further work in or need for mathematics (a remedial section of Finite Mathematics). Entry: a score below 15-20 on the diagnostic test.

Mathematics 111: For students who plan further work in or need for mathematics. Entry: Three or four years of satisfactory high school mathematics.

Mathematics 211: For students who plan further work in mathematics. Entry: Advanced work in high school for 4 years and College Board score greater than 650.

By a total partitioning of the freshman mathematics offerings, we were able to move toward the design of course curricula which were unicursal in objective. The revision of the placement-diagnostic test (see Appendix I) enabled us to identify students wishing to enter the non-terminal course sequence and to determine on what mathematical topics their specific weaknesses were. The remedial section of Finite Mathematics had a limited enrollment--no more than 15--and it met 5 periods a week, even though no additional credit was given for the extra meeting. The conjecture--which turned out to be true--was that by intensive attention to weaknesses, when needed, the same course coverage might be obtained with the additional meetings and small class size. As we shall note in the conclusion, after the first semester of the 1969-1970 academic year, students from the remedial section were able to join the regular sections for the second semester.

The remedial section of the Finite Mathematics was taught as a regular class, with small enrollment. It met daily. We purchased a number of different program texts of high school topics and when and if the need arose for individual remedial work on an individual topic, the instructor, Mr. B. A. Chavies, was able to direct students to appropriate sections of appropriate texts. The basic text for the course, however, was the same as that used in the regular Finite Mathematics section. The remedial section had the time to be both more leisurely and to stop when deficiencies in competencies were discovered.

The other two sections of Finite Mathematics were organized as during the previous year: three lectures and one tutorial section required. Tutorials were not, however, sectioned by ability due to a severe scheduling problem. (See comment in recommendations.) We also did not repeat the experiment in the introduction to computer language as a regular part of the course. We comment on this, too, in our conclusion. Otherwise, the pattern followed for 1969-1970 was the same as that followed for 1968-1969 in the Finite Mathematics course.

As a supplement to our earlier experience, we scheduled in 1969-1970 one section each of Mathematics 101 and Mathematics 101e in the lecture-tutorial format. Both sections had enrollments of about 50 students. The intent here was to determine the ability to use the format as a regular pattern without extra staff planning, development and materials pre-preparation.

We have thus established in all of our freshman mathematics programs a lecture-tutorial format, and have begun to gain the requisite experience to adapt it to specialized needs and abilities of students.

Section 4. Results

In our original proposal we divided the purpose of the pilot study into two main categories. One category dealt with the problem of optimum deployment of competent staff. The second category dealt with the problem of optimum use of available media. In our discussion, here, of the results of the developmental work, it is useful to be faithful to these earlier defined categories. As was noted in our Progress Report of March 1969, the initial delays in starting have had the unexpectedly happy result of yielding more experience and hopefully more insight into these two problems than we had originally expected.

The problem of optimum deployment of competent staff:

From a purely cost basis, the instructional format suggests significant reduction in staff cost per student. In the spring semester 1968 there were 185 students enrolled in mathematics courses with four full-time staff members, with a ratio of approximately 46 students per staff member. In the spring semester 1970, there were 324 students enrolled in mathematics courses with five full-time equivalent staff members, a ratio of approximately 66 students per staff member -- the latter despite increased offerings by the department. We omit the spring semester 1969 since part of the staff funding was borne under the grant supporting the pilot study reported here.

The above approximate data deals with the entire department enrollments. From one point of view this is justifiable in a small department since all staff carry part of the load of all kinds of instruction. During the two-year period all staff members participated in the instructional team for the Finite Mathematics program as lecturer or tutorial instructor. The cost efficiency, however, survives the test of a more narrowly conceived assignment pattern.

In 1967-1968 there were four sections of College Algebra, 3 hours credit, the course which was replaced by Finite Mathematics. These four sections were at least a full load for one staff member. (We use here an estimate of load which is at least weighted against favorable comparison with the format of the pilot study.) Despite an increase of credit hours from 3 to 4, and despite an increase in enrollment to approximately 100 in 1969-1970, the staff time alloted to the two sections was 2/3 of a position. Assuming a staff compensation level of $12,000, this means that in 1967-1968, $12,000 was committed by the college for the instruction of 80 students. In 1969-1970 the college committed $8,000 staff compensation for 25 percent more students and 25 percent more earned credit hours. Even if one includes the full cost of two departmental student assistants at $500 annually each, the comparison becomes $12,000 vs. $9,000 with the 25 percent increase in both students and student credit hours.

It thus seems clear that, assuming educational feasibility and desirability, the lecture-tutorial format offers promise of a significant decrease in cost per student per credit hour. To return to the use of competent staff in a small department, it seems clear that one especially competent and able staff member can assume direction of a team for such a course, be given some reduced load for this effort, and still offer the college promise of more efficient use of funds. The experience gained in the pilot study would seem to confirm the possibility and desirability, yielding as it does the ability to be more flexible in staff assignments.

<u>The problem of optimum use of available media:</u>

As was suggested in our introduction, our definition of available media is intended to be persuasive. That is to say, we have not, in this study attempted to consider media such as computer-assisted instruction or closed-circuit television since both of these clearly require the commitment of resources which would enable cost efficiencies only if used in a large variety of learning situations stretching far beyond the single department in a small college. For this study, the problem of optimum use of available media meant more realistically the discovery of the kinds of topics, the specialized learnings, the individual student needs, best served by large lectures or best served by small tutorial sections or best served by programmed texts or best served by individual help. It is clear that we have only begun to study this problem. Yet certain features begin to emerge which suggest that it is correct to assume that the best learning design for classes as heterogeneous as entering freshman mathematics classes, will be learning formats which vary their patterns and attempt to build-in flexibility. For the purposes of reporting here, we shall, obviously, artificially treat the individual components of the learning formats separately.

Large Lectures

It is clear that for material which an instructor chooses to present by lecture, there is little difference between an audience of 25 and an audience of up to 100. Beyond 100, the room size begins to draw the lecturer totally away from his class. A classroom capable of holding up to 100 students preserves the dimensions enabling an instructor to "reach out" regularly to maintain connections. It is possible, too, that a class of at least 50 makes lecturing easier and more effective. (We assume, here, of course, the opportunities for teacher-pupil dialogue exist in other contexts.) The larger class discourages interruption, thus enabling a lecturer--who is truly prepared--to develop continuously his scenario--to maintain a continuous drama which is often destroyed by interruption--to allow and encourage identification by all students without interruption from any. The task is not to destroy the lecture but to revifify it by determining what is best developed in a continuous sweep. That material which is not accessible to lecturing is generally not suitable to a class of 25 either. Thus, in one very important sense, the Block Format helps return integrity to the prepared lecture as an educational experience.

Tutorial Sections

If the large lecture is suitable for material to be developed and presented continuously and with drama, the tutorial is for material which must come as a result of dialogue and joint discovery--material which, so to speak, must be teased from the student rather than from the lecturer.

Tutorial sections, when possible, were limited to no more than ten students. This enabled a seminar-type atmosphere. We tried, too, not to hold tutorials in classrooms. (Remodeled facilities enable us now to hold tutorials in a small room with a few large tables and a very large blackboard.) It was possible to vary the purpose of given tutorials. For particularly able students, tutorials were often the occasion for going significantly beyond the regular lecture topics. Optimally this was done without notes or references and with leisure. Often supplemental readings (see Appendix X) were distributed and reported on. For many sections it was an opportunity to consider carefully connections between the lectures and the problem materials. For some sections it was an opportunity to detect specific weaknesses in background, facilitating referral for extra study. For all sections it was an opportunity for student and teacher to interact and engage in real dialogue without fear of "holding up the class." The success of the tutorial sections was felt by both staff and students. On the format evaluation instrument (see Appendix IX) a majority of the students called for more than one tutorial section a week.

Help Sessions and Programmed Materials

In many ways this was the least successfully used instructional facility by the students. In part, no doubt, this was due to the fact that, unlike lectures and tutorials which were required, these were optional. It seems possible, however, that there is a more profound reason since help sessions, in particular, were very well attended the week before a test. Superficially this says that tests motivate students to extra work. But more importantly it also says that before tests most students found help sessions useful but at other times most students did not find help sessions useful. It would thus seem to follow that it is possible that the focus of the help sessions was only on preparation for learnings to be tested. As a result there is clearly here a challenge to uncover ways of motivating students to use all learning facilities more readily. Perhaps a procedure worth investigating is a deliberate assignment of certain topics to the help sessions and the accompanying avoidance of them in either tutorial or lecture. This procedure would be more useful early in the course so that students could develop early satisfactory learning relationships with the help session assistants.

Section 5. Conclusions and Recommendations

Since this pilot study had, as one of its central objectives, the defining of problems for further study and action, it is appropriate that we combine our conclusions and recommendations into a single section.

It is important, however, to assert first, the collective judgments of the mathematics staff that the pilot study was successful. It was successful in that it enabled us to free ourselves from established patterns of what instruction is and what it is not. It freed us from arbitrary and a-logical equations of credit and class time and homework time. Establishing this freedom may, in fact, be the single most important outcome of the pilot study.

The evidence of its success exists, too, in that we are expanding the use of the format to other classes and hope to begin experimentation at a larger institution, Northwest Missouri State College, in 1970-1971. Staff and students have also become operationally aware of the fact that all learnings are not able to be dealt with effectively by the same learning methodologies. The staff has noted that student commitments to learning are more easily sustained and nourished by varying the learning formats. We think noteworthy, too, the development of the diagnostic test leading to the organization of a remedial section of the same course, rather than, in the traditional format, moving the student one step down the ladder.

There was one notable failure. The attempt to introduce computer facility in a two-week block during the 1968-1969 year was clearly far from successful despite the imaginative development of quasi-programmed programs (see Appendices V, VI, VII.) Dr. Ralph Lee confirmed our evaluation that two weeks is not a long enough time for significant learnings by all students without significant follow up. The course outline does not allow time for such significant follow up. It is our recommendation that a short, but more prolonged, computer course be offered, on the side, and only then develop inputs for the more elementary mathematics courses. This entire problem needs significant study. We should like, too, to see attention given to the utilization of a computer as a simulater as a teacher aid, simulating such patterns as a random walk as part of a lecture.

In the context of the two main problems of our pilot study, as described in Section 4, we would recommend that the following questions be given further study and investigation.

1. Assuming that it is possible to define learnings more effectively handled by a lecture format, what are the optimum lecture class sizes which preserve the potential for student identification with the lecturer?

2. How can students, themselves, be brought more fully into participating as staff in the learning situation? And how can their experiences by students become themselves significant learnings?

3. What are promising strategies for identifying material best suited to differing learning formats? And how can they be used to insure continuous feedback to course designers?

4. What are promising strategies for incorporating more varied instructional media whose cost requirements dictate simultaneous introduction in more than one disciplinary area?

5. How can one effectively determine if optimum format for material may also be a function of different learning responses by different students to such formats?

6. How important is the fact of exposure to a variety of learning formats as contrasted with its learning effectiveness?

7. How effectively independent of learning formats are our instruments of evaluation?

It should be clear that the above set of questions recommended for further study give evidence that our pilot study has significantly deepened our awareness of the problems connected with course format design--problems, hopefully, not totally intractable.

Appendix I

<u>Mathematics Diagnostic Test</u>

1. $(a \cdot b^{-2}) \cdot (a \cdot b)^3 = (?)$

(a) $\frac{1}{a \cdot b}$ (b) $\frac{a}{b^2}$

(c) $\frac{a}{b}$ (d) $\frac{1}{ab^2}$

2. ($(x^3y) \cdot (x \cdot y^2) \cdot (x^{-1} \cdot y^{-3}) = (?)$

(a) $x^{-3}y^{-6}$ (b) 0

(c) x^3 (d) y^2

3. $(m^2 \cdot n)^3 = (?)$

(a) m^6n^3 (b) m^5n^4

(c) $(m \cdot n)^6$ (d) m^5n^3

4. $(3) \cdot (2)^{-4} = (?)$

(a) $-(6^4)$ (b) $\frac{1}{6^{-4}}$

(c) $\frac{1}{3^4 \cdot 2^4}$ (d) $\frac{3}{2^4}$

5. $(2^{-3})^2 = (?)$

(a) 2^{-5} (b) 2^{-6}

(c) 2^{-1} (d) 2

6. $31.7 \times 10^{-3} = (?)$

(a) 31700 (b) 3.17 (c) .317

(d) .0317

7. $(1.1)(.02) = (?)$

(a) 2.2 (b) .22 (c) .022

(d) 1.02

8. $2.17 \times 10^{2} = (?)$

(a) 217 (b) .217 (c) .0217

(d) 21.7

9. $(-3.2) \times (.4) = (?)$

(a) -1.28 (b) -12.8 (c) -.128

(d) -128

10. $(.001) \times (.2) = (?)$

(a) .002 (b) .0002 (c) .02

(d) .2

11. $\sqrt{18} + \sqrt{2} + 3 = (?)$

(a) $\sqrt{20} + 3$ (b) 9

(c) $3 + 4\sqrt{2}$ (d) $7\sqrt{2}$

12. $\sqrt[3]{3}\,\sqrt[6]{4} = (?)$

(a) $\sqrt[9]{7}$ (b) $\sqrt[6]{12}$ (c) $\sqrt[3]{6}$

(d) $\sqrt[18]{12}$

13. $\frac{3}{2 - 3\sqrt{2}} = (?)$

(a) $\frac{1}{2 - \sqrt{2}}$ (b) $\frac{2 + \sqrt{2}}{2}$

(c) $\frac{3(2 + 3\sqrt{2})}{-14}$ (d) $\frac{3}{2} - \frac{1}{\sqrt{2}}$

14. $\sqrt{6} \div \sqrt{3} = (?)$

(a) $\sqrt{2}$ (b) 3 (c) $\frac{1}{2}$

(d) $\frac{1}{\sqrt{3}}$

15. $\left(\frac{1}{\sqrt{12}}\right) \cdot \left(\frac{\sqrt{3}}{\sqrt{2}}\right) = (?)$

(a) $\frac{\sqrt{3}}{\sqrt{12}}$ (b) $\frac{1}{\sqrt{4}}$ (c) $\frac{\sqrt{2}}{4}$

(d) $\frac{1}{\sqrt{2}}$

16. Which of the following relations does x satisfy if $-5x + 3 < x - 4$?

(a) $x < -\frac{7}{6}$ (b) $x > \frac{7}{6}$

(c) $x < -\frac{1}{6}$ (d) $x < \frac{7}{6}$

17. Which of the following implies that $2x + 1 > 2$?

(a) $x > \frac{3}{2}$ (b) $x > \frac{1}{2}$

(c) $x > -\frac{1}{2}$ (d) $x < \frac{1}{2}$

18. If $3x < 7$ and $y < 0$ then which of the following statements is always true?

(a) $3xy < 7y$ (b) $3xy < 0$

(c) $3xy > 7y$ (d) $3xy > 0$

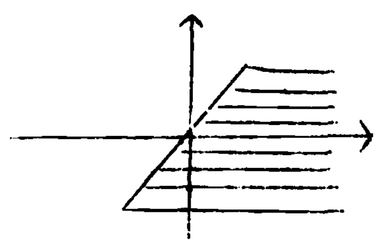
19. If $3x^2 > 2x$ then $3x > 2$ provided which of the following holcs?

(a) $x = 0$ (b) $x > 0$ (c) $x < 0$

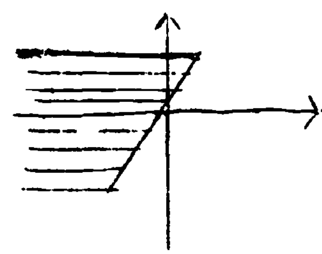
(d) $x \neq 0$

20. The solutions set for the inequality $y < x$ is represented by which of the following shaded regions?

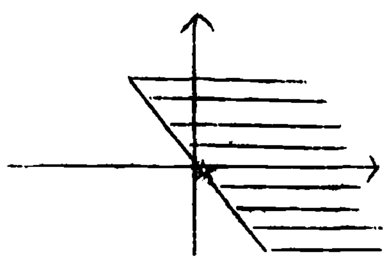
(a)



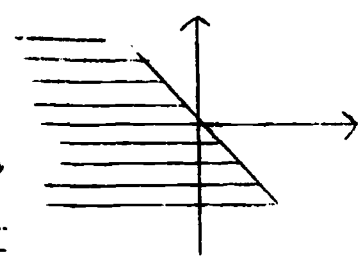
(b)



(c)



(d)



21. Under which of the following conditions is the quantity $|x - y|$ negative?

(a) $x > y$ (b) $x < y$ (c) never

(d) always

22. If $-1 < x < 0$ then $|x - 2x^2| =$ (?)

(a) $-x - 2x^2$ (b) $x + 2x^2$

(c) $-x + 2x^2$ (d) $x - 2x^2$

23. If $x < 2$ then $|x - 4| =$ (?)

(a) $-x + 4$ (b) $-x - 4$

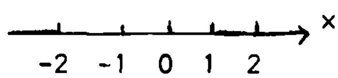
(c) $x + 4$ (c) $x - 4$

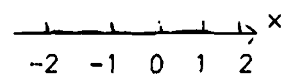
24. If $|2x - 5| = 1$ then $x =$ (?)

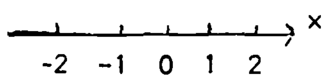
(a) 3 (b) $\frac{7}{2}$ (c) -2

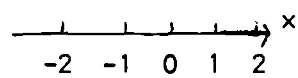
(d) either 2 or 3

25. Which of the following shaded regions is the correct solution set for x if $(x - 1)(x + 2) \leq 0$?

(a) 

(b) 

(c) 

(d) 

26. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cap B =$ (?)

(a) {1, 4} (b) {1, 2, 3, 4}

(c) Ø (d) {2, 3}

27. If $A = \{x, y, z\}$ and $B = \{z, w, v\}$ then $A \cup B =$ (?)

(a) {z} (b) {x, y, z, w, v}

(c) {x, y, v} (d) {x, y, w, v}

28. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$ then $(A \cup B) \cap C =$ (?)

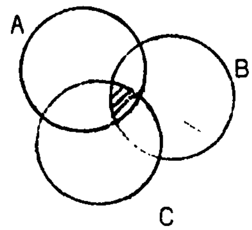
(a) {3} (b) {1, 2, 3, 4}

(c) {3, 4} (d) {3, 4, 5}

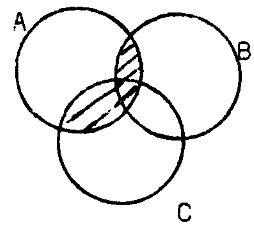
2

29. Which of the following shaded regions represents the set $A \cap (B \cup C)$?

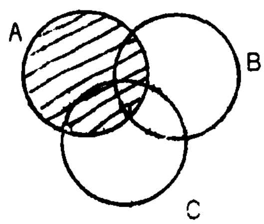
(a)



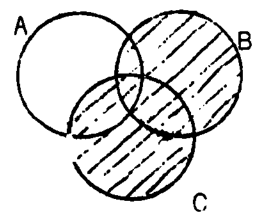
(b)



(c)

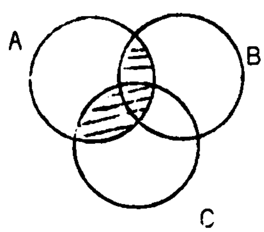


(d)

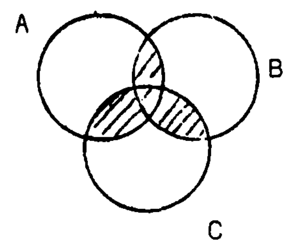


30. Which of the following shaded regions represents the set $(A \cup B) \cap (A \cup C)$?

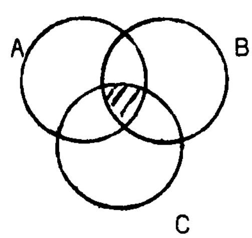
(a)



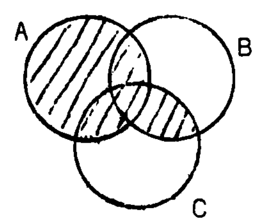
(b)

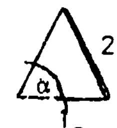


(c)



(d)



31. In the following triangle $\sin \alpha$ = (?) 

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{5}}$

32. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3}$ = (?)

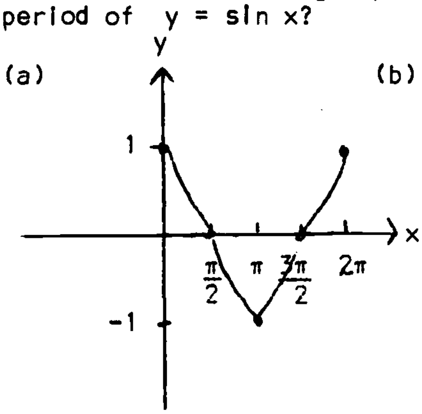
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $2\sqrt{2}$

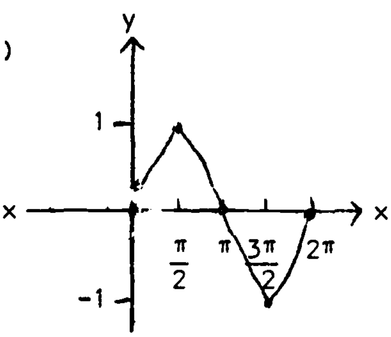
33. $\cos(\alpha + \beta)$ = (?)

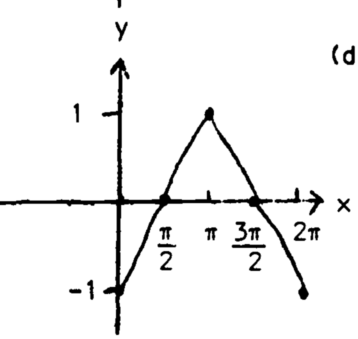
(a) $\cos \alpha + \cos \beta$ (b) $\cos \alpha \cos \beta - \sin \alpha \sin \beta$

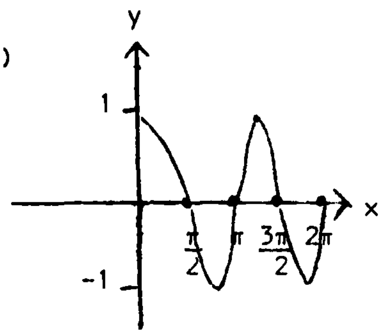
(c) $\cos \alpha - \cos \beta$ (d) $\sin(\alpha - \beta)$

34. Which of the following represents a graph of one period of $y = \sin x$?

(a)


(b)


(c)


(d)


35. If $\sin\alpha = \frac{3}{4}$ then $\cos\alpha =$ (?)

(a) $\frac{1}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{7}{16}$

(d) $\frac{\sqrt{7}}{4}$

36. Which of the following is a solution to the pair of linear equations $2x = 3y + 1$, $y = -4x + 2$?

(a) $x = -2$ and $y = \frac{1}{3}$ (b) $x = 1$ and $y = -2$

(c) $x = \frac{1}{2}$ and $y = 0$ (d) no solution exists.

37. The curve $y^2 = 3x + 1$ and the line $x = 5$ satisfy which of the following?

(a) intersect at exactly 1 point

(b) do not intersect

(c) intersect at exactly 2 points

(d) intersect at infinitely many points

38. Which of the following is a solutior to the pair of linear equations $3x + 2y - 1 = 0$, $2x - 3y + 1 = 0$?

(a) $x = \frac{1}{13}$ and $y = \frac{5}{13}$

(b) $x = \frac{1}{3}$ and $y = \frac{1}{2}$ (c) $x = \frac{1}{6}$ and $y = \frac{5}{6}$

(d) $x = \frac{2}{11}$ and $y = \frac{-3}{11}$

Appendix II

Tutorial Problems on Functions

1. Show that the following holds

   $\langle x, y, z\rangle = \langle r, s, t\rangle$ iff $x = r, y = s, z = t$

   by identifying an ordered triplet as a mapping and then using the criterion for the equality of two mappings.

2. Consider $f: A \rightarrow B$ where A, B are finite sets with α the number of elements in A and β the number of elements in B. Which of the following cases are possible?

   a. $\alpha < \beta$ and f onto
   b. $\beta < \alpha$ and f one-to-one
   c. $\alpha \neq \beta$ and f onto and f one-to-one
   d. $\alpha = \beta$ and f onto and f not one-to-one
   e. $\alpha - \beta$ and f one-to-one and f not onto

3. Let f, g be defined for all $x \in \mathbb{R}$ by

   $f(x) = 3 - 5x \qquad f(x) = \frac{3 - x}{5}$

   a. Show that $f(g(x)) = x$ for all x
   b. Show that $g(f(x)) = x$ for all x
   c. What can you conclude about f and g from the results of a., b?

4. Which of the functions f of exercise 4, in the previous section are one-to one? For each such f, find f^{-1}

5. If $F:A \rightarrow B$ is defined as follows, determine whether f is onto, whether f is one-to-one and determine the pre-image(x) of each $y \in B$.

   a. $f(x) = x^2$, $A = \mathbb{R}$, $B = \mathbb{R}$
   b. $f(x) = x^2$, $A = \mathbb{R}$, $B = \mathbb{R}^0$
   c. $f(x) = x^2$, $A = \mathbb{R}^0$, $B = \mathbb{R}^0$
   d. $f(x) = 2x$, $A = \mathbb{R}$, $B = \mathbb{R}$
   e. $f(x) = 2x$, $A = \mathbb{Z}$, $B = \mathbb{Z}$.

6. Relate the following statements by implications. (For example: (b) $\Longrightarrow$ (a))

   a. $f: A \longrightarrow B$
   b. $f: A \xrightarrow{\text{onto}} B$
   c. $f: A \xrightarrow{1-1} B$
   d. $f: A \xrightarrow[\text{onto}]{1-1} B$

7. Show that the function f of example 3.1.14 in the text is one-to-one and find its inverse function.

8. Let f be defined by $f(x) = \frac{1}{x}$ for $x \in \mathbb{R}^+$.

   a. Find the range of f.
   b. Show that f is one-to-one
   c. Find the inverse function f^{-1}.

9. This problem is a continuation of example 3.1.12 in the text.

   Let $S = \{a, b, c, d\}$ and define the mapping Q by

   $Q(x) = P(P(x))$ for $x \in S$

   a. Show that $D_Q = R_Q = S$

   b. Show that Q is one-to-one

   c. Determine the inverse function Q^{-1}.

10. Under what conditions (if any) is the mapping P_1 of example 3.1.20 in the text a one-to-one mapping?

11. Let f be defined by $f(x) = x^2 + 1$ for $x \in \mathbb{R}$ and g by $g(x) = \sqrt{x - 1}$ for $x \in \mathbb{R}$, $x \geq 1$.

    a. Show that $f(g(x)) = x$ for all $x \in D_g$.
    b. Disprove: $g(f(x)) = x$ for all $x \in D_f$.

12. This problem is a continuation of example 3.1.13. in the text. Let A and B be subsets of U.

a. Show that A = B iff $X_A = X_B$

b. Show that $X_{A'}(t) = 1 - X_A(t) \cdot X_B(t)$ for all $t \in U$.

c. Show that $X_{A \cap B}(t) = X_A(t) \cdot X_B(t)$ for all $t \in U$.

d. Show that $X_{A \cup B}(t) = X_A(t) + X_B(t) - X_A(t)X_B(t)$ for all $t \in U$.

e. Show that the association $A \rightarrow X_A$ for each subset A of U defines a one-to-one mapping with domain P(U).

13. a. Prove: If f is a function and $g \subset f$, then g is a function.

b. Prove: If f is one-to-one and $g \subset f$, then g is one-to-one.

Context: If f is a function and $g \subset f$, we call g a restriction of f and we call , an extension of g. If $D_g = C$, we call g the restriction of f to C. (Note that we must have $D_g \subset D_f$.) Alternatively we may say that g is the restriction of f to C iff g has domain C and $g(x) = f(x)$ for all $x \in C$.

In examples 3.2.2 of the text we considered functions f, g such that g was the restriction of f to R^0. The example shows that we may have g one-to-one, $g \subset f$, and f not one-to-one.

14. This problem is a continuation of example 3.1.22 in the text.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

we call A^T, A-Transpose. Now let Θ_T be the mapping with domain $\mathcal{M}$ defined by $\Theta(A) = A^T$.

Appendix III

Tutorial Problems on Vectors

1. Let $\alpha = \langle 1, -3\rangle$, $\beta = \langle 3, -2\rangle$, $\gamma = \langle 0, -1\rangle$. Find

   a. $2\alpha + 3\beta - 5\gamma$

   b. $|2\alpha + 3\beta - 5\gamma|$

   c. $(2\alpha) * (\gamma)$

   d. $(\alpha) * (3\beta + 4\gamma)$

   e. $(3\alpha - \beta) * (3\alpha - \beta)$

   f. $\frac{|\alpha|}{|\beta|}$

   g. $\frac{\alpha * \alpha}{|\alpha|}$

   h. $\frac{\alpha * \alpha + \beta * \beta}{|\alpha + \beta|}$

   i. $|\alpha + \beta| (\alpha * \beta)$

   j. $|\alpha - \beta| \alpha - |\alpha - \beta| \beta$

2. Let $\alpha = \langle -2, 1\rangle$, $\beta = \langle 5, -1\rangle$, $\gamma = \langle 2, 7\rangle$. Find

   a. $\alpha + 2\beta - 2\gamma$

   b. $5(\alpha + \beta) - 2(\beta - \gamma)$

   c. $|\alpha - \beta + \gamma|$

   d. $\frac{|\alpha| - |\beta| + |\gamma|}{|\alpha - \beta + \gamma|}$

   e. $\alpha * (\beta - \gamma)$

   f. $(2\beta) * (3\gamma - \alpha)$

   g. $\frac{\alpha * \alpha + \beta * \beta}{|\alpha| + |\beta|}$

   h. $|\alpha|\alpha + |\beta|\beta + |\gamma|\gamma$

   i. $(|\alpha|\alpha) * \alpha$

   j. $(|\alpha + \beta|)(\alpha + \beta) * (\alpha + \beta)$

3. Show that the following sets of vectors in $V_2(\mathbb{R})$ are linearly independent.

   a. $\{\langle 1, 3\rangle, \langle 2, -1\rangle\}$

   b. $\{\langle 1, 0\rangle, \langle 2, 3\rangle\}$

   c. $\{\langle 4, -1\rangle, \langle 0, 2\rangle\}$

   d. $\{\langle 0, 2\rangle, \langle 3, 1\rangle\}$

   e. $\{\langle -\frac{1}{2}, 0\rangle, \langle 0, \frac{2}{3}\rangle\}$

4. Let $\alpha = \langle 0, 2\rangle$, $\beta = \langle -1, 1\rangle$. Find values of k_1 and k_2 (possibly different for each problem) such that

   a. $\langle 2, 3\rangle = k_1\alpha + k_2\beta$

   b. $\langle -1, 2\rangle = k_1\alpha + k_2\beta$

   c. $\langle 3, -4\rangle = k_1\alpha + k_2\beta$

   d. $\langle 2, -\frac{1}{2}\rangle = k_1\alpha + k_2\beta$

5. a. Let $\alpha = \langle 1, 0\rangle$, $\beta = \langle 1, -1\rangle$. Show that the set $\{\alpha, \beta\}$ is a linearly independent set.

   b. Show that $V_2(R) = \{k_1\alpha + k_2\beta \colon k_1, k_2 \quad R\}$

6. a. Let $\alpha = \langle -1, 1\rangle$, $\beta = \langle -1, -1\rangle$. Show that the set $\{\alpha, \beta\}$ is a linearly independent set.

   b. Show that $V_2(R) = k_1\alpha + k_2\beta : k_1, k_2 \quad R$

7. Show that the set $\{\langle 1, 0\rangle, \langle 0, -1\rangle, \langle 2, 1\rangle\}$ is a linearly dependent set.

8. Show that the following relationships hold:

   a. $|k\alpha| = |k|\,|\alpha|$

   b. $\frac{\alpha}{k} = \frac{|\alpha|}{|k|}$ $k \neq 0$

   c. $|\alpha + \beta| \leq |\alpha| + |\beta|$

   d. $|\alpha - \beta| \geq |\alpha| - |\beta|$

9. Give an interpretation of the following operations or relationships in the representation of vectors in E_2, $\alpha, \beta, \gamma \in V_2$.

a. $\alpha + \beta$

b. $\alpha + \beta = \gamma$

c. $|\alpha + \beta| < |\alpha| + |\beta|$

d. $|\alpha - \beta| \geq |\alpha| - |\beta|$

e. $\frac{1}{2}\alpha - \frac{1}{2}\beta = \frac{1}{2}(\alpha - \beta)$

f. $\frac{1}{k}\alpha - \frac{1}{k}\beta = \frac{1}{k}(\alpha - \beta) \quad k \neq 0$

g. $\alpha = 2\beta$, $\alpha = k\beta \quad k \neq 0$

h. $\alpha * \beta \neq 0$

i. $\alpha * \beta = 0$, $\alpha * \gamma = 0$
$\beta = k\gamma$

Answers for Problems on Vectors

1. a. $<11, -7>$
   b. $\sqrt{170}$
   c. 6
   d. 39
   e. 49
   f. $\frac{\sqrt{10}}{\sqrt{13}}$ or $\frac{\sqrt{130}}{13}$
   g. $\frac{10}{\sqrt{10}}$ or $\sqrt{10}$
   h. $\frac{23}{\sqrt{41}}$
   i. $9\sqrt{41}$
   j. $<-2\sqrt{5}, -\sqrt{5}>$

2. a. $<4, -15>$
   b. $<9, 16>$
   c. $\sqrt{106}$
   d. $\frac{\sqrt{5} - \sqrt{26} + \sqrt{53}}{\sqrt{106}}$
   e. -14
   f. 40
   g. $\frac{31}{\sqrt{5} + \sqrt{26}}$
   h. $<-2\sqrt{5} + 5\sqrt{26} + 2\sqrt{53}, \sqrt{5} - \sqrt{26} + 7\sqrt{53}>$
   i. $5\sqrt{5}$
   j. 27

4. a. $k_1 = \frac{5}{2}$, $k_2 + -2$

b. $k_1 = \frac{1}{2}$, $k_2 = 1$

c. $k_1 = -\frac{1}{2}$, $k_2 = -3$

d. $k_1 = \frac{3}{4}$, $k_2 = -2$

6. b. Hint: $\langle a_1, a_2 \rangle = \frac{a_2 - a}{2} \langle -1, 1 \rangle + \left(\frac{a_1 + a_2}{-2} \right) \langle -1, -1 \rangle$

9. Hints

a. diagonals of parallelogram

c. lengths of sides of triangle

e. the length of the line joining midpoints of 2 sides of a triangle equals one-half the 3rd side.

i. If 2 line segments are perpendicular to a 3rd line segment, they are parallel to each other.

Appendix IV

Tutorial Problems on Probability

1. Let S be a sample space of 4 elements: $S = \{a_1, a_2, a_3, a_4\}$.

   a. Let P be a function of $S \longrightarrow R$ such that $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = \frac{1}{5}$. Does P define a Probability Function?

   b. Let P be a probability function. Find $P(a_1)$ if $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{6}$, $P(a_4) = \frac{1}{9}$.

   c. Let P be a probability function. Find $P(a_1)$ and $P(a_2)$ if $P(a_3) = P(a_4) = \frac{1}{4}$ and $P(a_1) = 2P(a_2)$.

   d. Let P be a probaility function. Find $P(a_1)$ if

   $$P(\{a_2, a_3\}) = \frac{2}{3}$$

   $$P(\{a_2, a_3\}) = \frac{1}{2}$$

   $$P(a_2) = \frac{1}{3}$$

2. Two men m_1 and m_2 and three women w_1, w_2, w_3 are in a chess tournament. Those of the same sex have equal probabilities of winning, but each man is likely to win as any woman.

   a. Find the probability that a woman wins the tournament.
   b. If m_1 and w_1 are married, find the probability that one of them wins the tournament.

3. Let a die be weighted so that the probability of a number appearing when the die is tossed is proportional to the given number (e.g. 6 has twice the probability of appearing as 3). Let A = {even number} , B = {prime number}, C = {odd number}.

   a. Describe the probability space, i.e. find the probability of each sample point.
   b. Find P(A), P(B), and P(C).
   c. Find the probability that

      i. An even or prime number occurs.
      ii. An odd prime number occurs.
      iii. A but not B occurs.

4. Determine the probability of each event:

   a. An even number appears in the toss of a fair die.
   b. A king appears in drawing a single card from an ordinary deck of 52 cards.
   c. At least one tail appears in the toss of three fair coins.
   d. A white marble appears in drawing a single marble from an urn containing 4 white, 3 red and 5 blue marbles.

5. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{4}$. Find

   a. $P(A \cup B)$
   b. $P(A')$ and $P(B')$
   c. $P(A' \cap B')$
   d. $P(A' \cup B')$
   e. $P(A \cap B')$
   f. $P(B \cap A')$

6. Let A and B be events with $P(A \cup B) = \frac{3}{4}$, $P(A') = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find

   a. $P(A)$
   b. $P(B)$
   c. $P(A \cap B')$

7. A die is tossed 100 times. The following table lists the six numbers and frequency with which each number appeared:

Number	1	2	3	4	5	6
Frequency	14	17	20	18	15	16

Find the relative frequency of the event.

a. A 3 appears.
b. A 5 appears
c. An even number appears
d. A prime appears.

8. Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability that:

a. None of the three chosen is defective.
b. Exactly one of the three chosen is defective.
c. At least one of the three chosen is defective.

9. Let A and B be events with $P(A \cup B) = 7/8$, $P(A \cap B) = 1/4$ and $P(A') = 5/8$. Find $P(A)$, $P(B)$, and $P(A \cap B')$.

10. Let A and B be events with $P(A) = 1/2$, $P(A \cup B) = 3/4$ and $P(B') = 5/8$. Find $P(A \cap B)$, $P(A' \cap B')$, $P(A' \cup B')$ and $P(B \cap A')$.

11. Let $S = \{a_1, a_2, \ldots, a_s\}$ and $T = \{b_1, b_2, \ldots, b_t\}$ be finite probability spaces. Let the number $p_{ij} = P(a_i)P(b_j)$ be assigned to the ordered pair (a_i, b_j) in the product set

$$S \times T = \{(s,t): s \in S,\ t \in T\}.$$

Show that the p_{ij} define a probability space on SXT, i.e. that the p_{ij} are non-negative and add up to one.

12. A pair of fair dice is thrown. Find the probability that the sum is 10 or greater if:

    a. A 5 appears on the first die.
    b. A 5 appears on at least one of the dice.

13. Three fair coins are tossed. Find the probability that they are all heads if

    a. The first coin is heads.
    b. One of the coins is heads.

14. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability p that

    a. The sum is six
    b. An ace appears
    c. The sum is 4 or less

15. An urn contains 7 red marbles and 3 white marbles. Three marbles are drawn from the urn one after the other. Find the probability p that the first two are red and the third is white.

16. The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if

    a. the class consists of 4 boys and 3 girls.
    b. the class consists of 3 boys and 3 girls.

17. An urn contains 3 red marbles and 7 white marbles. A marble is drawn from the urn and a marble of the other color is then put into the urn. A second marble is drawn from the urn.

    a. Find the probability p that the second marble is red.
    b. If both marbles were of the same color, what is the probability p that they were both white.

18. We are given two urns as follows:

    Urn A contains 3 red and 2 white marbles.

    Urn B contains 2 red and 5 white marbles.

An urn is selected at random: a marble is drawn and put into the other urn; then a marble is drawn from the second urn. Find the probability p that both marbles drawn are of the same color.

19. The probability that a man will live 10 more years is 1/4, and the probability that his wife will live 10 more years is 1/3, Find the probability that:

    a. Both will be alive in 10 years.
    b. At least one will be alive in 10 years.
    c. Neither will be alive in 10 years.
    d. Only the wife will be alive in 10 years.

20. We are given two urns as follows:

    Urn A contains 5 red marbles, 3 white marbles and 8 blue marbles.
    Urn B contains 3 red marbles and 5 white marbles.

    A fair die is tossed; if 3 or 6 appears, a marble is chosen from B, otherwise a marble is chosen from A. Find the probability that (a) a red marble is chosen, (b) a white marble is chosen, (c) a blue marble is chosen.

Answers to Probability Problems

1. (a) No (b) 7/18

   (c) $P(a_1) = 1/3$, $P(a_2) = 1/6$ (d) 1/6

2. (a) 3/7 (b) 3/7

3. (a) Let P(1) = p. The P(2) = 2p, P(3) = 3p, P(4) = 4p, P(5) = 5p, and P(6) = 6p.

   $P(1) = \frac{1}{21}$, $P(2) = \frac{2}{21}$, $P(3) = \frac{1}{7}$, $P(4) = \frac{4}{21}$,

   $P(5) = \frac{5}{21}$, $P(6) = \frac{2}{7}$

   (b) $\frac{4}{7}$, $\frac{10}{21}$, $\frac{3}{7}$

   (c) $\frac{20}{21}$, $\frac{8}{21}$, $\frac{10}{21}$

4. (a) 1/2, (b) 1/13 (c) 7/8 (d) 1/3

5. (a) 5/8 (b) 5/8 and 1/2 (c) 3/8

   (d) 3/4 (e) 1/8 (f) 1/4

6. (a) 1/3 (b) 2/3 (c) 1/12

7. (a) .20 (b) .15 (c) .51 (d) .52

8. There are 455 ways to choose 3 bulbs from the 15 bulbs.

   (a) $\frac{120}{455}$ or $\frac{24}{91}$ $\frac{226}{455}$ or $\frac{24}{91}$

   (c) $1 - \frac{24}{91} = \frac{67}{91}$

9. 3/8, 3/4, 1/8

10. 1/8, 1/4, 7/8, 1/4

12. (a) 1/3 (b) 3/11

13. (a) 1/4 (b) 1/7

15. 7/40

16. (a) 1/35 (b) 1/10 [Two mutually exclusive cases--first student a boy or first student a girl.]

17. (a) 17/50 (b) Probability that both were white is 21/50. Probability that both were same color is 12/25.

$$p = \frac{21}{50} / \frac{12}{25} = \frac{7}{8}$$

18. Construct a tree diagram. There are four paths which lead to two marbles of the same color.

$$p = \frac{1}{3}\cdot\frac{3}{5}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{5}{7}\cdot\frac{1}{2} = \frac{901}{1680}$$

19. (a) 1/12 (b) 1/2 (c) 1/2

(d) 1/4

20. (a) 1/3 (b) 1/3 (c) 1/3

Appendix V

Sample of the Computer Segment Program

Write a program to calculate the average of the following numbers.

9.6, 87.2, 33.49, .987, 8.34

```
C       PROGRAM NUMBER 1 FINDS THE AVERAGE
C       OF THE NUMBERS (9.6, 87.2, 33.49, 937, 8.34)
        A=9.6
        B=87.2
        C=33.49
        S=.987
        P=8.34
        AVE=(A+C+S+P+B)/5
        WRITE (6,100) AVE
    100 FORMAT (2X,8HAVERAGE=, E18.7)
        STOP
        END
```

Appendix VI

Sample of the Computer Segment Program

Write a program which will compare two values, A and B, and perform the following:

If	A > B	then write A
If	A < B	then write B
If	A = B	then write a comment stating that they were equal.

Construct the program so that it will perform the above on n sets of A and B.

In particular let n = 5 and let the 5 sets of A and B be:

A	B
3	5
8	4
7	7
5	2
3	3

The program will first be written as if there were only 1 set of A and B. When this has been accomplished we will extend the program for n sets of A and B.

```
C       PROGRAM NUMBER 5                              S1
        READ(5,100) A,B                               S2
    100 FORMAT(2F10.0)                                S3
        IF(A-B)2,3,4                                  S4
      3 WRITE(6,101)                                  S5
    101 FORMAT(14X,21HTHE VALUES WERE EQUAL)          S6
        GO TO 5                                       S7
      4 WRITE(6,102) A                                S8
    102 FORMAT(14X,2HA=, E17.8)                       S9
        GO TO 5                                       S10
      2 WRITE(6,103) B                                S11
    103 FORMAT(14X,2HB=,E17.8)                        S12
      5 STOP                                          S13
        END                                           S14
```

```
$ENTRY

    Col. 1-10          Col. 11-20
         3.                 5.
```

The symbol Sn (n=1,2,14) to the right of each statement is not part of the language, but a method of labeling each statement for reference in the text that follows.

For the values A=3. and B=5. only the statements S2, S4, S11, and S13 would be executed. They were executed in that order. This is what we wanted.

If A=8. and B=4. then statements S2, S4, S8, S10 and S13 would be the only ones executed. They would be executed in that order. Exactly what is wanted. If A=7. and B=7 what statements would be executed? Thus the program works properly for any set of values A and B.

The program will be rewritten to work for n sets of values A and B.

```
C     PROGRAM NUMBER 5 (GENERALIZED)
      ICOUNT=0
      READ(5,100) NSETS
  100 FORMAT (I10)
    6 READ(5,101) A,B
  101 FORMAT(2F10.0)
      IF(A-B)2,3,4
    3 WRITE(6,104)
  104 FORMAT(14X,21HTHE VALUES WERE EQUAL)
      GO TO 5
    4 WRITE(6,102)
  102 FORMAT(14X,2HA=, E17.8)
      GO TO 5
    2 WRITE(6,103) B
  103 FORMAT(14X,2HB=, E17.8)
    5 ICOUNT=ICOUNT+1
      IF(NSETS.EQ.ICOUNT)  STOP
      GO TO 6
      END
$ENTRY
        Col.  10
               5
        Col.  1-10          Col.  11-20
               3.                   5.
               8.                   4.
               7.                   7.
               5.                   2.
               3.                   3.
```

Appendix VII

Sample of the Computer Segment Program

Generalize program 9 to find the sum of the even integers starting with n and extending through m. n and m are even integers.
i.e. $SUM = n + (n + 1) + (n + 2) \div + m$

From program 9 we know that
$2 + 4 \div 6 + + k = \frac{k}{4}(k + 2)$

Applying this fact twice we have

$n + (n - 1) + (n + 2) + + m = \frac{m}{4}(m + 2) -$

$\frac{(n - 2)}{4}(n - 2 + 2)$

$= \frac{m}{4}(m + 2) - \frac{n}{4}(n - 2)$

In particular find the following sums.

2 + 4 + 6 + + 100	n = 2, m = 100
8 + 10 + 12 + + 20	n = 8, m = 20
42 + 44 + 46 + + 96	n = 42, m = 96
100 + 102 + + 200	n = 100, m = 200

Let the variable nstart be the value n.

Let the variable nfinal be the value m.

Let the variable numtim be the number of sums, which we want the program to find

```
C       PROGRAM NUMBER 10
        ICOUNT=0
        READ(5,100) NUMTIM
    100 FORMAT(2I10)
      8 READ(5,100) NSTART, NFINAL
        NSAVE=NSTART
        NSUM=0
      5 NSUM=NSUM+NSTART
        NSTART=NSTART+2
        IF(NFINAL-NSTART)6,5,5
      6 FNFINL=NFINAL
        FNSTRT=NSAVE
        CHECK=(FNFINL/4.)*(FNFINL+2.) - (FNSTRT/4.)*(FNSTRT-2.)
        WRITE(6,104)NSAVE, NFINAL, NSUM, CHECK
    104 FORMAT(2X,2HN=,I10,2X,2HM=,I10,2X,8HSUMMING=,I10,2X
       110HBYFORMULA,E17.8
        ICOUNT=ICOUNT+1
        IF(NUMTIM-ICOUNT)9,9,8
      9 STOP
        END
$ENTRY
```

Col. 10	Col. 20
4	
2	100
8	20
42	96
100	200

Appendix VIII

Teacher Evaluation Questionaire

Stephens College

Major ________________ Sex____________________

Class ________________ Grade Point Average ______

Directions: It is the desire of your instructor to achieve the best possible instruction in this course. To help accomplish the purpose, this evaluation sheet was devised to obtain a systematic poll of student opinion. Carefully consider each question, then record your judgment by encircling one of the letters A, B, C, D, E for each item. A blank space has been provided at the end for adding comments you wish to make.

---

1. Were important objectives met?

A	B	C	D	E
The course is an important contribution to my college education.		Contributes about as much as the average college course.		This course doesn't seem worthwhile to me.

---

2. Does instructor's presentation of subject matter enhance learning?

A	B	C	D	E
Presentation very meaningful and facilitates learning.		Presentation not unusually good or bad, about average.		Presentation often confusing; seldom helpful.

---

3. Is instructor's speech effective?

A	B	C	D	E
Instructor's speaking skill concentrates my attention on subject.		Speech sometimes invites attention on speaker rather than subject.		Speech usually distracting, concentration very difficult.

---

4. How well does the instructor work with students?

A	B	C	D	E
I feel welcome to seek extra help as often as needed.		I feel hesitant to ask for extra help.		I would avoid asking this instructor for extra help unless absolutely necessary.

5. Does the instructor stimulate independent thinking?

A	B	C	D	E
Instructor continually inspires me to extra effort and thought beyond course requirements.		In general, I do only the usual thinking involved in the assignments.		I seldom do more than rote memory work and cramming.

6. Do grading procedures give valid results?

A	B	C	D	E
Instructor's estimate of my over all accomplishments has been quite accurate to date.		Instructor's estimate of my accomplishments is of average accuracy.		I feel that the instructor's estimate is quite inaccurate.

7. How does this instructor rank with others you have had?

One of the best instructors I have ever had.	Satisfactory or above average.	One of the poorest instructors I have ever had.

Comments;

(favorable)

(unfavorable)

Appendix IX

Format Evaluation Sheet

As you all know this course is in the stage of development. We would therefore appreciate your filling out the following questionaire. You will note that your response is anonymous. You should thus find little difficulty in being quite frank. In each question please check the appropriate box.

1. As a final grade in this course I expect to receive an:

   A B C D E

2. I found the text materials:

   Very Useful Of Average Use

   Of Little Value

3. I fourd the lectures:

   Very Useful Of Average Use

   Of Little Value

4. I found the Tutorials:

   Very Useful Of Average Use

   Of Little Value

5. I use the Problem Sessions:

   Frequently Occassionally

   Not at all

6. Compared with other Mathematics Courses I have had, I found the Course:

   Challenging Of Average Difiicuity

   Easy

7. Compared with other Mathematics Courses I have had I found the Course:

   Very Interesting   Fairly Interesting

   Dull

8. On the basis of my background in Mathematics, I think the course was:

   Too Hard   About Right   Too Easy

9. I found the Computer Work

   Very Interesting   Fairly Interesting

   Dull

10. I found the Computer Work:

    Too Hard   About Right

    Too Easy

11. I found the notes on the use of Computer:

    Very Helpful   Fairly Helpful

    Not Helpful

12. Compared with other courses I am taking <u>at Stephens</u>, I found the course:

    Very Challenging   Fairly Challenging

    Not Challenging

13. It is my intent to take:

    Additional Mathematics Courses

    No more Math Courses

14. I felt that the lectures and tutorials were:

Sufficiently Co-ordinated

Insufficiently Co-ordinated

15. I would suggest the following changes to improve the Course. (Please feel free to make any suggestion you wish.)

Appendix X

Reprinted on the following pages are two sample journal articles used for collateral reading and discussion in the Tutorial Sections of the Finite Mathematics Course.

Reprinted from: Mathematics Teaching, summer 1964.

Tesselations

AN ADVENTURE IN MATHEMATICS

GEOFFREY GILES
Strathallan School, Perth.

The greatest handicap a pupil can have in the study of mathematics is the belief that it is primarily logical. These are strong words; but strong words are necessary.

What distinguishes a mathematician? Surely it is his insight, his intuitive grasp of the mathematical aspects of situations, and his ability to choose and to use appropriate techniques. Only in the use of these techniques are we unquestionably concerned with logic.

A surgeon tends to be judged by his mastery of technique and his skill at manipulation. But before he starts any operation he must know exactly what he is trying to do, and he must decide how he is going to do it. The same pattern is evident in any task you care to consider – in building an airliner, in keeping hens, in teaching a class. There is a great temptation to ignore these two essential preliminaries as they usually involve only thought. Indeed, in repetitive work they are relegated to the unconscious – we are said to have "experience" and know what to do without thinking. But without the *motivation* and *direction* supplied by the first two stages the third stage of *action* becomes as meaningless as a computer without an operator.

Throw a problem at a mathematician. What does he do? He *thinks* about it. If necessary he will translate it into his own language—and this may cause him the greatest trouble. Then he will look for a pattern and try to relate it to what he knows. Suddenly he jumps up with an exclamation, grabs a pencil and scribbles furiously. If his intuition was right and his manipulation is sound he will soon produce an answer. But note that the bulk of his work was done before he started writing.

Now look at any exam paper in mathematics. Ignore the computation and what is left? Possibly a couple of ludicrous, stereotyped problems. Is this a decent preparation for life? Is the ability to pass such exams any indication of ability to use mathematics in real life problems? What chance do candidates have to develop their mathematical intuition? What we teach should mix intuition and reason in quantities suited to the age and ability of those being taught. Throughout the whole of school mathematics a primary aim should be to highlight the aid that reason can give to intuition.

The Quadrant Problem

The following problem which arose quite naturally and unexpectedly started me off on an investigation which can be best described as an adventure. Some boys were making wall lamps. These were to have conical shades which would be made out of quarter circles of parchment. How much parchment should I order? Once sizes and numbers were known this was a straight forward matter. But it hid a far more interesting question: if a manufacturer had to supply a large number of quadrants all stamped out of sheet parchment how small could he keep his wastage?

Clearly the relative size of the sheets of parchment and the required quadrants is important. If the quadrants were larger than the sheets the wastage would be 100%. On the other hand the larger the sheets the smaller the edge effects will be. (Does one appreciate this by reason or by intuition?) Let us avoid this difficulty by supposing a continuous sheet of parchment.

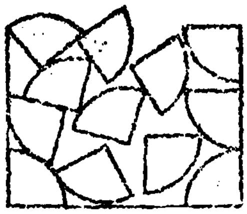


Fig. 1 - Random packing

Now we have to consider how we should cut out the quadrants if we wish to minimise waste. We could work empirically, packing them in one by one like groceries in a box. (Fig. 1) But it seems reasonable to suppose that the best method will form a regular pattern. (At least it seems so to me. But I certainly would not like to *prove* that it is so.)

Assuming this regularity, we can find the percentage wastage by considering one "unit" of the pattern, as every other unit will be exactly the same. In other words: *we are looking for a shape which must circumscribe the required quadrant with as little extra area as possible - but which must also be suitable for tesselloting a plane.* For example the plane could be tessellated with squares, and one quadrant could be cut out of each square. (Fig. 2). In this case the wastage is 21·5%. Not very good.

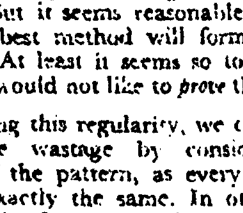


Fig. 2—Ordered packing

Now, this restatement of the problem is a big step forward. We know this by the number of ideas and possibilities that are now jumping into our minds. There are many ways of proceeding and it is difficult to know which to choose. Let us have a look at a couple of them. (If you want the fun of working it out for yourself stop reading now —and don't look at the diagrams!).

Path 1

Let us try fitting quadrants together methodically. Knowing that four quadrants can make a circle we try packing circles (Figs. 3, 4), marking in the background tessellation. That is as far as we can go with circles. But wait a minute! Fig. 3 uses the same tessellation as Fig. 2, and using the same one again we could arrange the quadrants as in Fig. 5 in which the waste can be obviously decreased by "squeezing". This raises a nice problem of a more traditional type: what is the percentage wastage now? (But this still involves real thought and not just manipulation).

What about other ways of fitting quadrants together methodically? Now it is not so easy to visualise possible arrangements. The best way to proceed would be to cut some out and try . . . (*Memo: must collect a few circular beer mats on next visit to pub.*)

Path 2

What tessellations are there? Can we adapt them to suit our quadrant problem? Let us go through some of them: square (we've considered this already!), rectangle (have a look at Fig. 4); parallelogram (any ideas? I haven't!); triangle (Fig. 6 —this can be adapted as shown later to give a solution).

What about the quadrilateral? We try it doubtfully. It works! (Fig. 7) Let us use it. But how do we circumscribe a quadrilateral of minimum area round our quadrant? (Fig. 8, Is it A, B, or C? After some thought our intuition tells us that it will be a kite composed of two isosceles triangles. (Or do we arrive at this conclusion logically?) On gazing at it a few minutes longer we realise that it is a quarter of a regular octagon. Interesting! On looking back at Fig. 4 we see that the basic quadrilateral is a quarter of a regular hexagon. More interesting!

Fig. 9 shows part of the tessellation we get. Might we not decrease the wastage by squeezing it a bit more so that the right angles came through to meet the circumferences? What does that make the wastage now?

Where do we go after quadrilaterals? Pentagons? The regular pentagon will not tessellate. What pentagons will? This is more difficult. It is part of the general problem that we must now tackle: *what shapes are suitable for forming tessellations?*

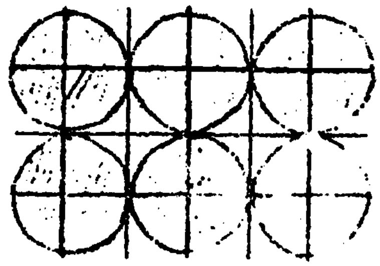


Fig. 3—Square packing of circles

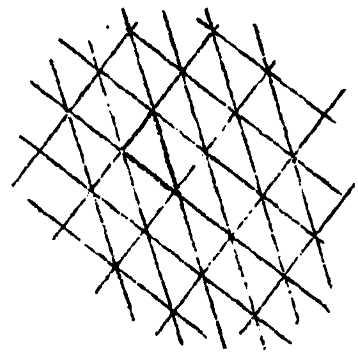


Fig. 6—Triangle-based tessellation

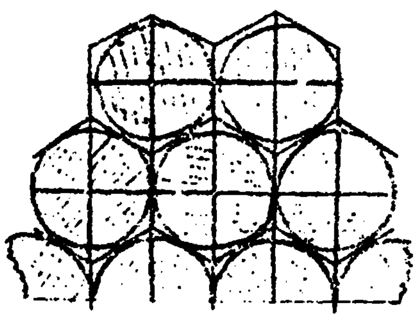


Fig. 4—Triangular packing of circles

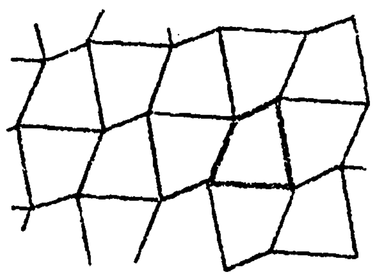


Fig. 7—Quadrilateral-based tessellation

Fig. 8—Which quadrilateral has minimum area?

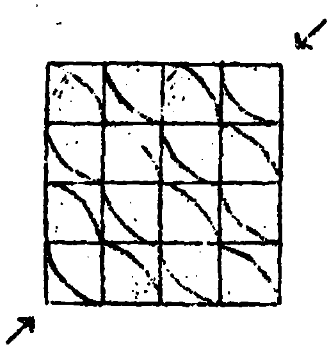


Fig. 5—Another square packing of quadrants

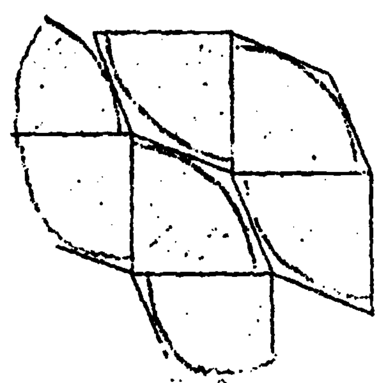


Fig. 9

Before starting this let us review our progress so far. Having tidied up our problem we used our experience and intuition to express it in a different way. We knew that the new form was better because of the thoughts and ideas it provoked. Now, apart from having found a number of ways of tessellating the parchment, we have gained experience with tessellations on which we will draw later.

But is this mathematics? Although it looks like geometry we have not invoked Euclid, and no standard techniques have been used — except in calculating the percentage waste. Does this type of work benefit a pupil in any way? Or would he be better off doing another fifty examples on factorisation?

I suggest that this kind of investigation helps to develop an attitude of mind which incorporates (a) critical awareness of the problem: (b) a directed intuition, (c) a flexibility of approach. What better preparation can we give to pupils who will have to face so many unforeseeable problems in the last quarter of this century? This attitude of mind is of far greater importance even than mathematics itself.

Nor is it only the minority who require this preparation. No one can expect to be bypassed by the rapid advance of technology. The centuries-old demand for unskilled labour is rapidly diminishing. Few of our pupils will find themselves in jobs where they are not expected to think. The vast majority will be involved in new techniques and modern developments whatever career they choose. And there will be a premium on clear thinking.

Tessellations

For convenience let us call any shape which can form a plane tessellation a **tess**. Let us approach the problem of what shapes are tesses by considering how a tessellation is formed. Look at one of the quadrilaterals in Fig. 7. How is it related to its neighbours? Sooner or later we see that a rotation of 180° is involved about the mid-point of a side. This explains how a tessellation is built up.

The same is true of the triangular tessellation. But here our experience and reasoning help us to see further. We can think of a triangle as a quadrilateral with one side of negligible length. As a rotation of the quadrilateral about the mid-point of this side is equivalent to the rotation of the triangle about a vertex, we see that the rotation of any triangle in the tessellation through 180° about the mid-point of any side or about any vertex gives the position of another triangle of the tessellation. Indeed the whole tessellation is symmetrical about these points.

Leisurely pondering these ideas, we suddenly see an important advance we can make. The sides of the triangle or quadrilateral do not have to be straight for it to be a tess — *so long as they are symmetrical about their mid-points.* (Fig. 10) Let us call these modified triangles and quadrilaterals **trisides** and **quadrisides.**

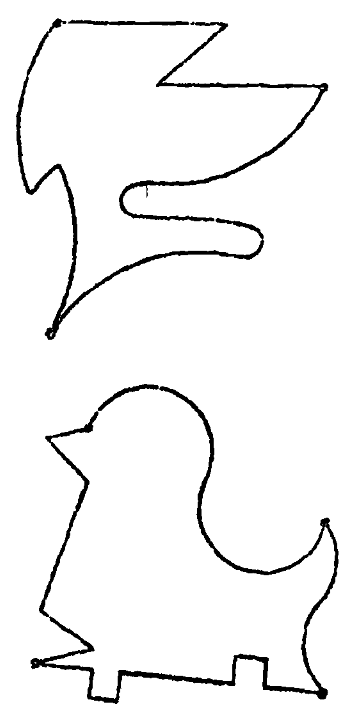


Fig. 10—A triside and a quadriside

We can now give a sufficient condition for a shape to be a tess. *Any figure will be a tess if its perimeter can be divided by three or four "vertices" into sides each of which is symmetrical about its mid-point.* This unfolds an enormous range of possibilities. To avoid the danger of bewilderment amongst such proliferation let us take refuge in squared paper. Fig. 11 shows some straightforward quadrisides.

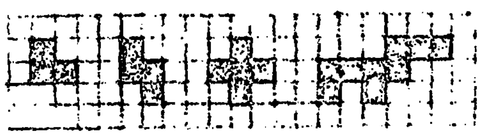


Fig. 11

On trying to draw a triside in this way we make a most interesting discovery. The vertices of a tess need not lie on the edge of its area (Fig. 12). This increases once more the range of shapes we can show to be tesses. Fig. 13 shows the twelve ways in which a shape can be made by simply joining five squares. S. W. Golomb calls them "pentominoes" ("polyomino" being the complete generalisation of domino).* All the pentominoes can be shown to be trisides. And if you find that straightforward could you check that the same is true of the 35 hexominoes?

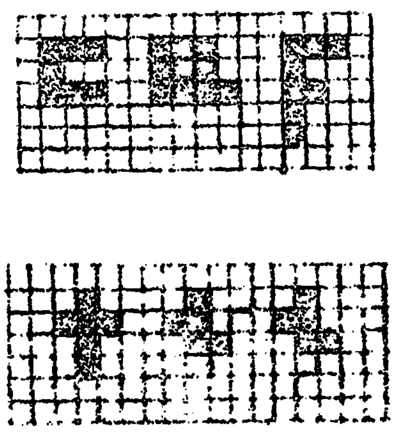


Fig. 12

* See Gardner: Mathematical Puzzles and Diversions from Scientific American (Bell).

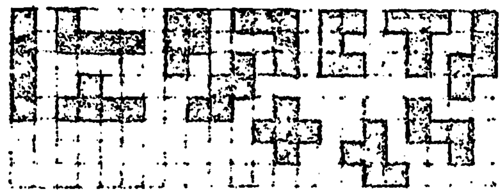


Fig. 13—These pentominoes are all trisides

Where does all this get us? Let us go back to our quadrant problem and see how we can apply what we have found out—especially the idea of "virtual" vertices. Thinking again of the pentagon, we soon realise that we can state: *any pentagon is a tess if two of its sides are parallel* (Fig. 14). This leads to a better solution. But once again how do we draw the pentagon round the quadrant? I am satisfied that the area will be a minimum

Fig. 14

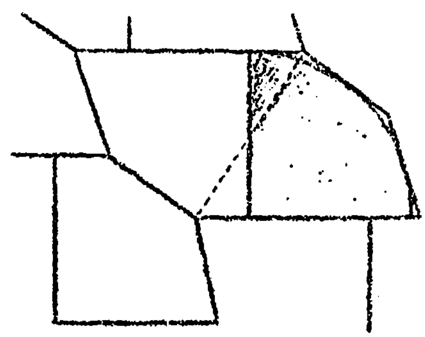


Fig. 15

when the pentagon is a quarter of a regular decagon (Fig. 15). If you wish to prove it go ahead. For me a formal proof would be irrelevant as it would not add to my conviction.

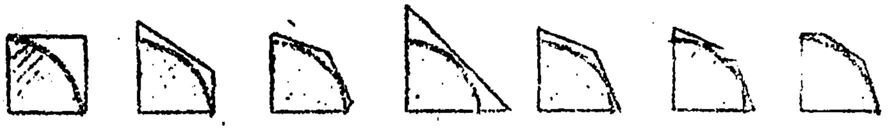


Fig. 16

What about the percentage wastage? Fig. 16 shows the seven solutions in the order in which we found them. The decrease in wastage between A and G is quite remarkable and could represent a big saving in cost of material in large scale production. The actual calculation of these wastages is of interest itself as it stresses the need for finding a route to the goal rather than the application of a technique.

Now we sit back and heave a sigh. We have reached what is surely the best solution. We cannot imagine anyone improving on 3·3%. What is more, we think smugly, we know all about tessellations. At a time like this it is fitting that one's complacency should be shaken. We try putting two of our final pentagons back to back (Fig. 15). Idly we draw in a diagonal—and find that what we have is essentially two quadrilaterals! If we had known what to look for we could have seen this solution right back at Fig. 7 !

Then we get another rude awakening when we realise, possibly on studying solution C, that we can dislocate our basic quadrilateral tessellation and get another type of tessellation (Fig. 17).

Finally, it was at about this stage that I received *Mathematics Teaching* No. 21 containing some fascinating examples of the work of M. C. Escher, including the one below. Quite clearly it was about time I studied tessellations.

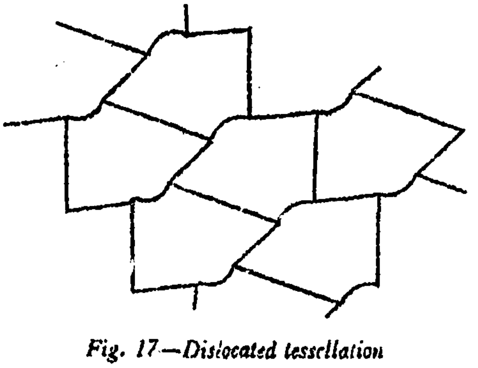


Fig. 17—Dislocated tessellation

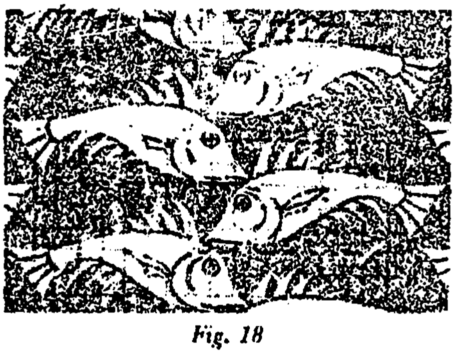


Fig. 18

Fig. 18 is reproduced from the book "The Graphic Work of M. C. Escher" by permission of the publishers, Oldbourne Press.

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THE ARCHITECTURE OF MATHEMATICS*

NICHOLAS BOURBAKI†

1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares, it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment.

It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?

Although this question is perhaps of greater urgency now than ever before, it is by no means a new one; it has been asked almost from the very beginning of mathematical science. Indeed, quite apart from applied mathematics, there has

* Authorized translation by Arnold Dresden of a chapter in "Les grands courants de la pensée mathématique," edited by F. Le Lionnais (Cahiers du Sud, 1948).

† "Professor N. Bourbaki, formerly of the Royal Poldavian Academy, now residing in Nancy, France, is the author of a comprehensive treatise of modern mathematics, in course of publication under the title *Éléments de Mathématique* (Hermann et Cie, Paris 1939–), of which ten volumes have appeared so far."

always existed a dualism between the origins of geometry and of arithmetic (certainly in their elementary aspects), since the latter was at the start a science of discrete magnitude, while the former has always been a science of continuous extent; these two aspects have brought about two points of view which have been in opposition to each other since the discovery of irrationals. Indeed, it is exactly this discovery which defeated the first attempt to unify the science, *viz.*, the arithmetization of the Pythagoreans ("everything is number").

It would carry us too far if we were to attempt to follow the vicissitudes of the unitary conception of mathematics from the period of Pythagoras to the present time. Moreover this task would suit a philosopher better than a mathematician; for it is a common characteristic of the various attempts to integrate the whole of mathematics into a coherent whole—whether we think of Plato, of Descartes or of Leibnitz, of arithmetization, or of the logistics of the 19th century—that they have all been made in connection with a philosophical system, more or less wide in scope; always starting from *a priori* views concerning the relations of mathematics with the twofold universe of the external world and the world of thought. We can do no better on this point than to refer the reader to the historical and critical study of L. Brunschvicg [1]. Our task is a more modest and a less extensive one; we shall not undertake to examine the relations of mathematics to reality or to the great categories of thought; we intend to remain within the field of mathematics and we shall look for an answer to the question which we have raised, by analyzing the procedures of mathematics themselves.

2. **Logical formalism and axiomatic method.** After the more or less evident bankruptcy of the different systems, to which we have referred above, it looked, at the beginning of the present century as if the attempt had just about been abandoned to conceive of mathematics as a science characterized by a definitely specified purpose and method; instead there was a tendency to look upon mathematics as "a collection of disciplines based on particular, exactly specified concepts," interrelated by "a thousand roads of communication," allowing the methods of any one of these disciplines to fertilize one or more of the others [1, page 447]. Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical theories, and which has led to what is generally known as the "axiomatic method."

The words "formalism" and "formalistic method" are also often used; but it is important to be on one's guard from the start against the confusion which may be caused by the use of these ill-defined words, and which is but too frequently made use of by the opponents of the axiomatic method. Everyone knows that superficially mathematics appears as this "long chain of reasons" of which Descartes spoke; every mathematical theory is a concatenation of propositions, each one derived from the preceding ones in conformity with the rules of a logical system, which is essentially the one codified, since the time of Aristotle, under the name of "formal logic," conveniently adapted to the particular aims of the mathematician. It is therefore a meaningless truism to say that this "deductive reasoning" is a unifying principle for mathematics. So superficial a remark can certainly not account for the evident complexity of different mathematical theories, not any more than one could, for example, unite physics and biology into a single science on the ground that both use the experimental method. The method of reasoning by means of chains of syllogisms is nothing but a transforming mechanism, applicable just as well to one set of premises as to another; it could not serve therefore to characterize these premises. In other words, it is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others,* in short, the language suited to mathematics; this is all, no further significance should be attached to it. To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful; indeed this constitutes one aspect of the axiomatic method, the one that can properly be called logical formalism (or "logistics" as it is sometimes called). But we emphasize that it is but one aspect of this method, indeed the least interesting one.

What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" [1, page 446] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light.

3. The notion of structure. In what form can this be done? It is here that the axiomatic method comes closest to the experimental method. Like the latter drawing its strength from the source of Cartesianism, it will "divide the difficulties in order to overcome them better." It will try, in the demonstrations of a theory, to separate out the principal mainsprings of its arguments; then, taking each of these separately and formulating it in abstract form, it will develop

* Indeed every mathematician knows that a proof has not really been "understood" if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one.

the consequences which follow from it alone. Returning after that to the theory under consideration, it will recombine the component elements, which had previously been separated out, and it will inquire how these different components influence one another. There is indeed nothing new in this classical going to-and-fro between analysis and synthesis; the originality of the method lies entirely in the way in which it is applied.

In order to illustrate the procedure which we have just sketched, by an example, we shall take one of the oldest (and also one of the simplest) of axiomatic theories, *viz.* that of the "abstract groups." Let us consider for example, the three following operations: 1. the addition of real numbers, their sum (positive negative or zero) being defined in the usual manner; 2. the multiplication of integers "modulo a prime number p," (where the elements under consideration are the whole numbers $1, 2, \cdots, p-1$) and the "product" of two of these numbers is, by agreement, defined as the remainder of the division of their usual product by p; 3 the "composition" of displacements in three-dimensional Euclidean space, the "resultant" (or "product") of two displacements S, T (taken in this order) being defined as the displacement obtained by carrying out first the displacement T and then the displacement S. In each of these three theories, one makes correspond, by means of a procedure defined for each theory, to two elements x, y (taken in that order) of the set under consideration (in the first case the set of real numbers, in the second the set of numbers $1, 2, \cdots, p-1$, in the third the set of all displacements) a well-determined third element; we shall agree to designate this third element in all three cases by $x\tau y$ (this will be the sum of x and y if x and y are real numbers, their product "modulo p" if they are integers $\leq p-1$, their resultant if they are displacements). If we now examine the various properties of this "operation" in each of the three theories, we discover a remarkable parallelism; but, in each of the separate theories, the properties are interconnected, and an analysis of their logical connections leads us to select a small number of them which are independent (*i.e.*, none of them is a logical consequence of all the others). For example,* one can take the three following, which we shall express by means of our symbolic notation, common to the three theories, but which it would be very easy to translate into the particular language of each of them:

(a) For all elements x, y, z, one has $x\tau(y\tau z)=(x\tau y)\tau z$ ("associativity" of the operation $x\tau y$);

(b) There exists an element e, such that for every element x, one has $e\tau x=x\tau e=x$ (for the addition of real numbers, it is the number 0; for multiplication "modulo p," it is the number 1; for the composition of displacements, it is the "identical" displacement, which leaves every point of space fixed);

(c) Corresponding to every element x, there exists an element x' such that $x\tau x'=x'\tau x=e$ (for the addition of real numbers x' is the number $-x$; for the

* There is nothing absolute in this choice; several systems of axioms are known which are "equivalent" to the one which we are stating explicitly, the axioms of each of these systems being logical consequences of the axioms of any other one.

composition of displacements, x' is the "inverse" displacement of x, i.e. the displacement which replaces each point that had been displaced by x to its original position; for multiplication "modulo p," the existence of x' follows from a very simple arithmetic argument.*

It follows then that the properties which can be expressed in the same way in the three theories, by means of the common notation, are consequences of the three preceding ones. Let us try to show, for example that from $x\tau y = x\tau z$ follows $y = z$; one could do this in each of the theories by a reasoning peculiar to it. But, we can proceed as follows by a method that is applicable in all cases: from the relation $x\tau y = x\tau z$ we deriv. (x' having the meaning which was defined above) $x'\tau(x\tau y) = x'\tau(x\tau z)$; thence by applying (a), $(x'\tau x)\tau y = (x'\tau x)\tau z$; by means of (c), this relation takes the form $e\tau y = e\tau z$, and finally, by applying (b), $y = z$, which was to be proved. In this reasoning the nature of the elements x, y, z under consideration has been left completely out of account; we have not been concerned to know whether they are real numbers, or integers $\leq p-1$, or displacements; the only premise that was of importance was that the operation $x\tau y$ on these elements has the properties (a), (b), and (c). Even if it were only to avoid irksome repetitions, it is readily seen that it would be convenient to develop once and for all the logical consequences of the three properties (a), (b), (c) only. For linguistic convenience, it is of course desirable to adopt a common terminology for the three sets. One says that a set in which an operation $x\tau y$ has been defined which has the three properties (a), (b), (c) is provided with a group structure (or, briefly, that it is a group); the properties (a), (b), (c) are called the axioms of** the group structures, and the development of their consequences constitutes setting up the axiomatic theory of groups.

It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature† has not been specified; to define a structure, one takes as given one or

* We observe that the remainders left when the numbers $x, x^2, \ldots, x^n, \ldots$ are divided by p, can not all be distinct; by expressing the fact that two of these remainders are equal, one shows easily that a power x^m of x exists which has a remainder equal to 1; if now x' is the remainder of the division of x^{m-1} by p, we conclude that the product "modulo p" of x and x' is equal to 1.

** It goes without saying that there is no longer any connection between this interpretation of the word "axiom" and its traditional meaning of "evident truth."

† We take here a naive point of view and do not deal with the thorny questions, half philosophical, half mathematical, raised by the problem of the "nature" of the mathematical "beings" or "objects." Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representation of these "beings"—thought of at first as ideal "abstractions" of sense experiences and retaining all their heterogeneity—by an unitary concept, gradually reducing all the mathematical notions, first to the concept of the natural number and then, in a second stage, to the notion of set. This latter concept, considered for a long time as "primitive" and "undefinable," has been the object of endless polemics, as a result of its extremely general character and on account of the very vague type of mental representation which it calls forth; the difficulties did not disappear until the notion of set itself disappeared (and with it all the metaphysical pseudo-problems concerning mathematical "beings" in the light of the recent work on logical formalism. From this new point of view, mathematical

several relations, into which these elements enter* (in the case of groups, this was the relation $z = x\tau y$ between three arbitrary elements); then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.)† To set up the axiomatic theory of a given structure, amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypotheses as to their own nature).

4. The great types of structures. *The relations which form the starting point for the definition of a structure can be of very different characters.* The one which occurs in the group structure is what one calls a "law of composition," *i.e.*, a relation between three elements which determines the third uniquely as a function of the first two. When the relations which enter the definition of a structure are "laws of composition," the corresponding structure is called an algebraic structure (for example, a field structure is defined by two laws of composition, with suitable axioms: the addition and multiplication of real numbers define a field structure on the set of these numbers).

Another important type is furnished by the structures defined by an order relation; this is a relation between two elements x, y which is expressed most frequently in the form "x is at most equal to y," and which we shall represent in general by xRy. It is not at all supposed here that it determines one of the two elements x, y uniquely as a function of the other; the axioms to which it is subjected are the following: (a) for every x we have xRx; (b) from the relations xRy and yRx follows $x = y$; (c) the relations xRy and yRz have as a consequence xRz. An obvious example of a set with a structure of this kind is the set of integers (or that of real numbers), when the symbol R is replaced by the symbol $\leq$. But it must be observed that we have not included among the axioms the following property, which seems to be inseparable from the popular notion of "order," "for every pair of elements x and y, either xRy or yRx holds." In other words, the case in which x and y are incomparable is not excluded. This may seem paradoxical at first sight, but it is easy to give examples of very important order structures, in which such a phenomenon appears. This is what happens when X and Y denote parts of the same set and the relation XRY is interpreted to mean "X is contained in Y"; again when x and y are positive integers and xRy means

structures become, properly speaking, the only "objects" of mathematics. The reader will find fuller developments of this point in articles by *J. Dieudonné* [2] and H. Cartan [3].

* In effect, this definition of structures is not sufficiently general for the needs of mathematics; it is also necessary to consider the case in which the relations which define a structure hold not between elements *of the set under consideration*, but also between parts of this set and even, more generally, between elements of *sets of still higher "degree"* in the terminology of the "hierarchy of types." For further details on this point, see [4].

† Strictly speaking, one should, in the case of groups, count among the axioms, besides properties (a), (b), (c) stated above, the fact that the relation $z = x\tau y$ determines one and only one z when x and y are given; one usually considers this property as tacitly implied by the form in which the relation is written.

"x divides y"; also if $f(x)$ and $g(x)$ are real-valued functions defined on an interval $a \leqq x \leqq b$, while $f(x)Rg(x)$ is interpreted to mean "for every x: $f(x) \leqq g(x)$." These examples also give an indication of the great variety of domains in which order structures appear and thus point to the interest attached to their study.

We want to say a few words about a third large type of structures, *viz.* topological structures (or topologies): they furnish an abstract mathematical formulation of the intuitive concepts of neighborhood, limit and continuity, to which we are led by our idea of space. The degree of abstraction required for the formulation of the axioms of such a structure is decidedly greater than it was in the preceding examples; the character of the present article makes it necessary to refer interested readers to special treatises. See, for example, [5].

5. **The standardization of mathematical technique.** We have probably said enough to enable the reader to form a fairly accurate idea of the axiomatic method. It should be clear from what precedes that its most striking feature is to effect a considerable economy of thought. The "structures" are tools for the mathematician; as soon as he has recognized among the elements, which he is studying, relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type. Previously, on the other hand, he was obliged to forge for himself the means of attack on his problems; their power depended on his personal talents and they were often loaded down with restrictive hypotheses, resulting from the peculiarities of the problem that was being studied. One could say that the axiomatic method is nothing but the "Taylor system" for mathematics.

This is however, a very poor analogy; the mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition,* which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior, which he seems to have the right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world. Now, each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis described above has derived the structure. And, for the research worker who suddenly discovers this structure in the phenomena which he is studying, it is like a sudden modulation which orients at one stroke in an unexpected direction the intuitive course of his thought and which illumines with a new light the mathematical landscape in which he is moving about. Let us think—to take an old example—of the progress made at the beginning of the nineteenth century by the geometric representation of imaginaries. From our point of view, this amounted to discovering in the set of complex numbers a well-known topological structure, that of the Euclidean plane, with all the possibilities for applications which this in-

* Like all intuitions, this one also is frequently wrong.

volved; in the hands of Gauss, Abel, Cauchy and Riemann, it gave new life to analysis in less than a century. Such examples have occurred repeatedly during the last fifty years: Hilbert space, and more generally, functional spaces, establishing topological structures in sets whose elements are no longer points, but functions; the theory of the Hensel p-adic numbers, where, in a still more astounding way, topology invades a region which had been until then the domain *par excellence* of the discrete, of the discontinuous, *viz.* the set of whole numbers; Haar measure, which enlarged enormously the field of application of the concept of integral, and made possible a very profound analysis of the properties of continuous groups;—all of these are decisive instances of mathematical progress, of turning points at which a stroke of genius brought about a new orientation of a theory, by revealing the existence in it of a structure which did not *a priori* seem to play a part in it.

What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state.

6. A general survey. Let us now try, guided by the axiomatic concept, to look over the whole of the mathematical universe. It is clear that we shall no longer recognize the traditional order of things, which, just like the first nomenclatures of animal species, restricted itself to placing side by side the theories which showed greatest external similarity. In place of the sharply bounded compartments of algebra, of analysis, of the theory of numbers, and of geometry, we shall see, for example, that the theory of prime numbers is a close neighbor of the theory of algebraic curves, or, that Euclidean geometry borders on the theory of integral equations. The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular.

At the center of our universe are found the great types of structures, of which the principal ones were mentioned above: they might be called the mother-structures. A considerable diversity exists in each of these types; one has to distinguish between the most general structure of the type under consideration, with the smallest number of axioms, and those which are obtained by enriching the type with supplementary axioms, from each of which comes a harvest of new consequences. Thus, the theory of groups contains, beyond the general conclusions valid for all groups and depending only on the axioms enunciated above, a particular theory of finite groups (obtained by adding the axiom that the number of elements of the group is finite), a particular theory of abelian groups (in which $x\tau y = y\tau x$ for every x and y), as well as a theory of finite abelian groups (where these two axioms are supposed to hold simultaneously). Similarly, in the theory of ordered sets, one notices in particular those sets (as for example, the set of integers, or of real numbers) in which any two elements are comparable, and which are called totally ordered. Among the latter, further attention is given to the sets which are called well-ordered (in which, as in the set of integers greater than 0, every subset has a "least element"). There is an analogous gradation among topological structures.

Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures simultaneously not in simple juxtaposition (which would not produce anything new), but combined organically by one or more axioms which set up a connection between them. Thus, one has topological algebra. This is a study of structures in which occur at the same time, one or more laws of composition and a topology, connected by the condition that the algebraic operations be (for the topology under consideration) continuous functions of the elements on which they operate. Not less important is algebraic topology, in which certain sets of points in space, defined by topological properties (simplexes, cycles, etc.) are themselves taken as elements on which laws of composition operate. The combination of order structures and algebraic structures is also fertile in results, leading, in one direction to the theory of divisibility and of ideals, and in another to integration and to the "spectral theory" or operators, in which topology also joins in.

Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which, in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. At this point we merge with the theories of classical mathematics, the analysis of functions of a real or complex variable, differential geometry, algebraic geometry, theory of numbers. But they have no longer their former autonomy; they have become crossroads, where several more general mathematical structures meet and react upon one another.

To maintain a correct perspective, we must at once add to this rapid sketch, the remark that it has to be looked upon as only a very rough approximation of the actual state of mathematics, as it exists; the sketch is schematic, and idealized as well as frozen.

Schematic—because in the actual procedures, things do not happen in as simple and as systematic a manner as has been described above. There occur, among other things, unexpected reverse movements, in which a specialized theory, such as the theory of real numbers, lends indispensable aid in the construction of a general theory like topology or integration.

Idealized—because it is far from true that in all fields of mathematics, the role of each of the great structures is clearly recognized and marked off; in certain theories (for example in the theory of numbers), there remain numerous isolated results, which it has thus far not been possible to classify, nor to connect in a satisfactory way with known structures.

Finally *frozen*,—for nothing is farther from the axiomatic method than a static conception of the science. We do not want to lead the reader to think that we claim to have traced out a definitive state of the science. The structures are not immutable, neither in number nor in their essential contents. It is quite possible that the future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms, or of new combinations of axioms. We can look forward to important progress from the invention of structures, by considering the progress which has resulted from actually known structures. On the other hand, these are by no means finished edifices; it would indeed be very surprising if all the essence had already been extracted from their principles. Thus, with these indispensable qualifications, we can become better aware of the internal life of mathematics, of its unity as well as of its diversity. It is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.

7. Return to the past and conclusion. The concept which we have tried to present in the above paragraphs, was not formed all at once; rather is it a stage in an evolution, which has been in progress for more than a half-century, and which has not escaped serious opposition, among philosophers as well as among mathematicians themselves. Many of the latter have been unwilling for a long time to see in axiomatics anything else than futile logical hairsplitting not capable of fructifying any theory whatever. This critical attitude can probably be accounted for by a purely historical accident. The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, *i.e.*, theories which are entirely determined by their complete system of axioms; for this reason they could not be applied to any theory except the one from which they had been extracted (quite contrary to what we have seen, for instance, for the theory of groups). If the same had been true for all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.* But the further development of the method has revealed its power; and the repugnance which it still meets here and there, can only be explained by the natural difficulty of the mind to admit, in dealing with a concrete problem, that a form of intuition, which is not suggested directly by the given elements (and which often can be arrived at only by a higher and frequently difficult stage of abstraction), can turn out to be equally fruitful.

As concerns the objections of the philosophers, they are related to a domain, on which for reasons of inadequate competence we must guard ourselves from

* There also occurred, especially at the beginning of axiomatics, a whole crop of monster-structures, entirely without applications; their sole merit was that of showing the exact bearing of each axiom, by observing what happened if one omitted or changed it. There was of course a temptation to conclude that these were the only results that could be expected from the axiomatic method.

entering; the great problem of the relations between the empirical world and the mathematical world.* That there is an intimate connection between experimental phenomena and mathematical structures, seems to be fully confirmed in the most unexpected manner by the recent discoveries of contemporary physics. But we are completely ignorant as to the underlying reasons for this fact (supposing that one could indeed attribute a meaning to these words) and we shall perhaps always remain ignorant of them. There certainly is one observation which might lead the philosophers to greater circumspection on this point in the future: before the revolutionary developments of modern physics, a great deal of effort was spent on trying to derive mathematics from experimental truths, especially from immediate space intuitions. But, on the one hand, quantum physics has shown that this macroscopic intuition of reality covered microscopic phenomena of a totally different nature, connected with fields of mathematics which had certainly not been thought of for the purpose of applications to experimental science. And, on the other hand, the axiomatic method has shown that the "truths" from which it was hoped to develop mathematics, were but special aspects of general concepts, whose significance was not limited to these domains. Hence it turned out, after all was said and done, that this intimate connection, of which we were asked to admire the harmonious inner necessity, was nothing more than a fortuitous contact of two disciplines whose real connections are much more deeply hidden than could have been supposed *a priori*.

From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism." The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations."

* We do not consider here the objections which have arisen from the application of the rules of formal logic to the reasoning in axiomatic theories; these are connected with logical difficulties encountered in the theory of sets. Suffice it to point out that these difficulties can be overcome in a way which leaves neither the slightest qualms nor any doubt as to the correctness of the reasoning; [2] and [3] are valuable references for this point.