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ABSTRACT

This booklet is one of the ten in the series "Experiences in Mathematical Discovery", produced by the General Mathematics Writing Project of the NCTM. Each is designed for use by students of ninth-grade general mathematics. The discussion and problems are designed to guide the student through an understanding of: 1) fractions, 2) equivalent fractions, 3) rational numbers on the number line, and 4) the arithmetic of rational numbers. (RS)

7 Rational Numbers

EXPERIENCES IN MATHEMATICAL DISCOVERY

National Council of
Teachers of Mathematics

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Decimals, Ratios, Percent*

Unit 9: *Positive and Negative Numbers*

Unit 10: *Measurement*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimulate the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

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Finally, a word of grateful thanks is extended to the NCTM headquarters staff for their assistance, and in particular to Clare Stifft and Gladys Majette for their careful attention to production editing details.

EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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Experiences in Mathematical Discovery

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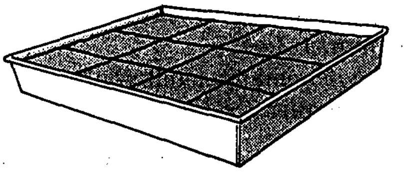
Rational Numbers

Fractions

Jane made some fudge for her friend Bill. She poured the fudge into a rectangular-shaped pan and cut it into twelve pieces of the same size. Bill ate five of the pieces.



Jane wants to know *what part* of the pan of fudge Bill ate. It is easy to answer Jane's question by using a *fraction*. In this case the fraction is $\frac{5}{12}$ (read "five-twelfths"). But what is a fraction?

A fraction is an *ordered pair of whole numbers*. (Recall that a whole number is a number like 0, 1, 2, 3, and so on.) An ordered pair of numbers consists of a *first* number and a *second* number. The order is important. To "write a fraction" you simply write the first number above a bar and the second number below the bar. The first number can be any whole number. The second number can be any whole number except zero.

In the beginning of this unit you will have experience in forming fractions and seeing how certain fractions are related. Then you will see how fractions can be used to indicate *rational numbers*. Although there are negative as well as positive rational numbers, negative numbers are not considered in this unit.

Class Discussion

1. In the fraction $\frac{5}{12}$, the whole number 5 is called the *numerator*, and the whole number 12 is called the *denominator*.

a. Does the denominator tell you into how many pieces of the same size Jane cut the pan of fudge?

b. What does the numerator tell you?

2. Suppose Jane had cut the pan of fudge into 10 pieces of the same size and Bill had eaten 3 of them. What fraction would you use to describe the part of the fudge that Bill ate?

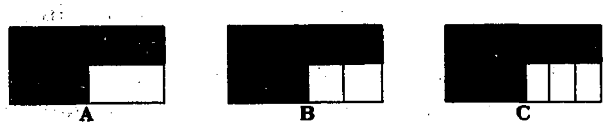
a. What is the denominator of this fraction?

b. What is the numerator?

c. What does the denominator tell you?

d. What does the numerator tell you?

3. Diagrams A, B, and C show three different ways in which Jane can cut up the same pan of fudge. Suppose that the colored portion in each diagram represents the amount of fudge that Bill can eat in one evening. For each diagram give a fraction that describes the part of the fudge that Bill can eat in one evening.



4. Do the three fractions you gave in exercise 3 have different denominators? Do they have different numerators? Are the fractions all different?

5. Do you think Bill would have eaten as much fudge if he had eaten $\frac{3}{4}$ of the pan of fudge as he would if he had eaten $\frac{6}{8}$ of it?

   Do you think $\frac{9}{12}$ of the pan of fudge is the same as $\frac{3}{4}$ of it?

6. In your answer to exercise 3 you should have given three different fractions to describe the part of the fudge that Bill can eat in one evening. But, since the same portion of each diagram is shaded, the three fractions you gave should all represent the *same amount* of fudge. Fractions like these are *equivalent fractions.*

   Give two fractions that are equivalent to $\frac{3}{4}$.

[redacted]

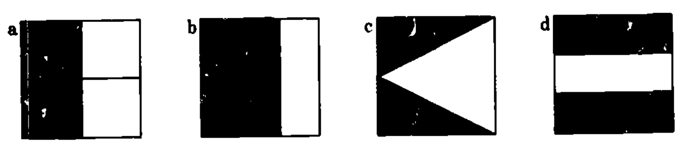
Suppose you are to think about a certain number of pieces in a pan of fudge that has been cut up into pieces of the same size. You can use a fraction to describe the part of the fudge about which you are thinking. To form the fraction you need to use two whole numbers. The denominator of the fraction will be the number of pieces of the same size into which the pan of fudge has been divided. The numerator of the fraction will be the number of pieces about which you are thinking.

By cutting up a pan of fudge in various ways, you can show that more than one fraction can be used to refer to the same amount of fudge. Different fractions that represent the same amount are equivalent.

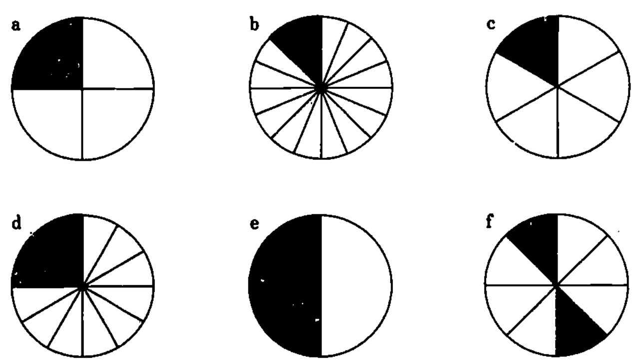
[redacted]

1. Pictured below are four square regions that have the *same size and shape.* Two regions that have the same size and shape

are said to be *congruent*. Give a fraction that tells what part of each region is shown in color.



2. Pictured below are six congruent circular regions. For each region give a fraction that indicates the part of the region that is shown in color.



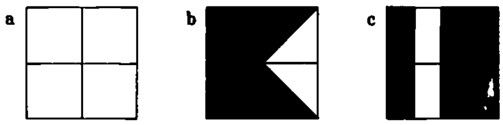
3. Look again at the diagrams in exercise 2. Notice that each circular region is divided into *subregions* of the same size and shape. This means that the subregions in each circular region are congruent. In which circular regions is the same amount shown in color?

4. Use the diagrams in exercise 2 and your answers to exercise 3 to help you decide which pairs of fractions listed below are equivalent.

a. $\frac{1}{4}$, $\frac{2}{6}$ c. $\frac{3}{12}$, $\frac{2}{6}$ e. $\frac{4}{16}$, $\frac{1}{4}$

b. $\frac{2}{8}$, $\frac{1}{4}$ d. $\frac{1}{4}$, $\frac{3}{12}$ f. $\frac{1}{2}$, $\frac{2}{6}$

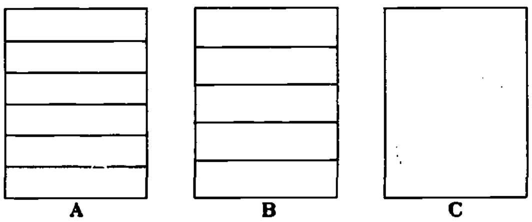
5. In which of the regions pictured below is the same amount shown in color? (Be careful!)



Class Discussion

In each fraction considered thus far the numerator was less than the denominator. In the exercises that follow you will see that the numerator of a fraction can also be equal to or greater than the denominator.

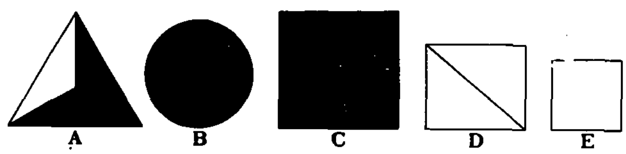
1. Diagrams A, B, and C show three rectangular regions.



a. How many congruent subregions are there in diagram A?
b. How many congruent subregions are there in diagram B?
c. How many congruent subregions are there in diagram C?

You can think of a region that is NOT divided by lines as having ONE subregion that is congruent with itself.

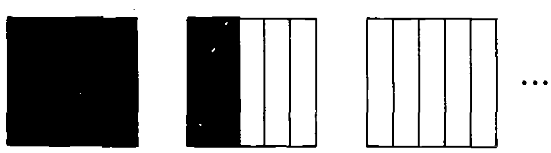
2. For each diagram write a fraction that represents the part of the region that is shown in color. The denominator should be the number of congruent subregions in the given region. The numerator should be the number of subregions that are shown in color.



a. For which diagrams is the numerator of the fraction you obtained the same as the denominator?
b. For which diagrams did you obtain a fraction with a denominator of 1?
c. For which diagrams did you obtain a fraction with a numerator of 0?

3. Why would it not make sense to have 0 as the denominator of a fraction? What would this mean in terms of subregions of a region?

4. Each square region pictured below is an exact copy of each of the others. The three dots at the right tell you that the set of copies can be continued.

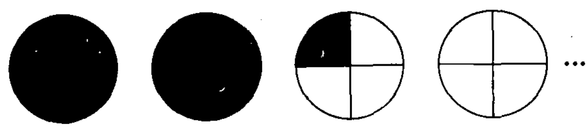


In a set of copies like this we think of each copy as representing *one whole region.*

a. How many congruent subregions are there in one whole region?
b. What is the total number of colored subregions in all the whole regions that are pictured?
c. Write a fraction in which the denominator is the number of congruent subregions in one whole region and the numerator is the total number of shaded subregions in all the whole regions.
d. In the fraction you just gave is the numerator greater than

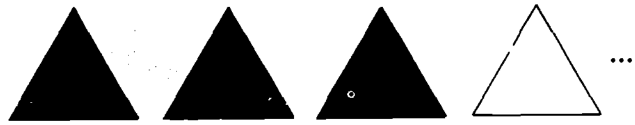
the denominator? Does this fraction represent more than one whole region?

5. Each circular region pictured below is an exact copy of each of the others. The three dots at the right tell you that the set of copies can be continued. As in exercise 4, think of each copy as representing one whole region.



a. How many congruent subregions are there in one whole region?
b. What is the total number of colored subregions in all the whole regions that are pictured?
c. Write a fraction to indicate the amount that is shown in color.
d. In the fraction you just wrote, which is greater, the numerator or the denominator?

6. Each triangular region pictured below is an exact copy of each of the others. The three dots at the right tell you that the set of copies can be continued. Think of each copy as representing one whole region.



a. How many congruent subregions are there in one whole region?
b. What is the total number of shaded subregions in all the whole regions that are pictured?
c. What fraction indicates the amount that is shown in color?
d. In the fraction you just gave, which is greater, the numerator or the denominator?

[redacted]

Any pair of whole numbers may be used to form a fraction, except that the denominator cannot be 0.

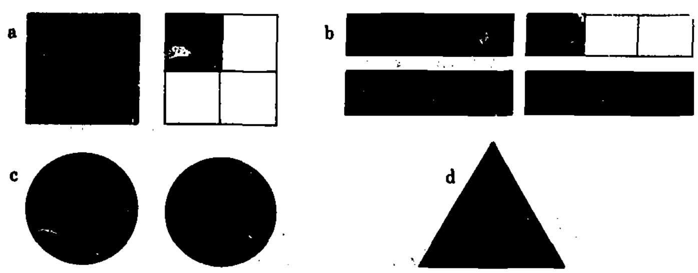
If you list the fractions in the answers to the exercises in Class Discussion 1b, your list should include the following:

$$\frac{2}{3}, \frac{1}{1}, \frac{0}{1}, \frac{4}{4}, \frac{0}{2}, \frac{7}{5}, \frac{9}{4}, \frac{3}{1}.$$

By looking at the list you can see that the numerator of a fraction may be greater than, less than, or equal to the denominator.

[redacted]

1. For each exercise write a fraction that can be used to describe the amount that is shown in color.



2. Make a diagram for each fraction. Use a square to represent one whole region.

a. $\frac{3}{5}$ **c.** $\frac{0}{5}$ **e.** $\frac{3}{1}$

b. $\frac{5}{5}$ **d.** $\frac{9}{5}$ **f.** $\frac{0}{1}$

3. Which fraction below refers to an amount that is smaller than one whole region? More than one whole region? The same as one whole region?

a. $\frac{4}{7}$ **b.** $\frac{7}{7}$ **c.** $\frac{9}{7}$

4. Complete each exercise so that the resulting sentence is true.

a. $\frac{10}{5}$ of a region is the same amount as ___ whole regions.

b. $\frac{9}{3}$ of a region is the same amount as ___ whole regions.

c. $\frac{4}{2}$ of a region is the same amount as ___ whole regions.

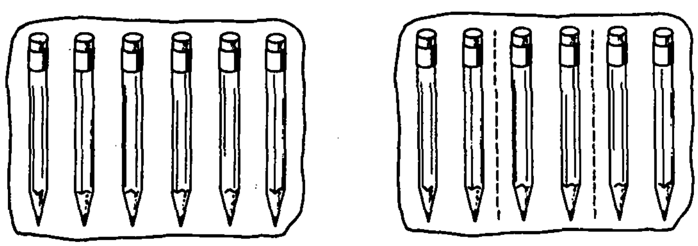
5. For each exercise, decide whether or not the two fractions are equivalent.

a. $\frac{10}{5}, \frac{4}{2}$ **b.** $\frac{4}{2}, \frac{9}{3}$ **c.** $\frac{10}{5}, \frac{9}{3}$ **d.** $\frac{9}{3}, \frac{3}{1}$

Class Discussion

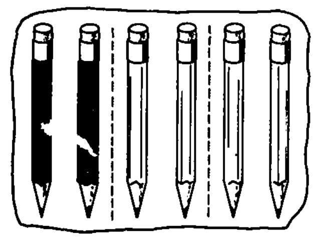
Jane wondered if fractions could be used in situations that involve grouping. She had six pencils, and Bill borrowed two of them. How can Jane use a fraction to tell what part of the pencils Bill borrowed?

1. Think of the six pencils Jane had as a *group*. The group of six pencils is shown in diagram *A*. Diagram *B* shows Jane's group of pencils divided into *subgroups*. Does each subgroup contain the same number of pencils? How many subgroups are there?



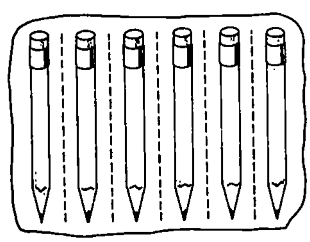
A B

2. In the diagram below, the pencils that Bill borrowed are shown in color. How many subgroups did he borrow?



3. Jane can say that Bill borrowed $\frac{1}{3}$ of the pencils. What does the denominator of this fraction represent? What does the numerator represent?

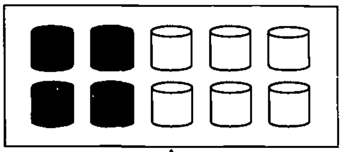
4. Jane can use a different fraction to tell what part of the group of pencils Bill borrowed. The diagram below shows another way to divide Jane's group of pencils into subgroups. How many pencils are there in each subgroup this time? How many subgroups are there?



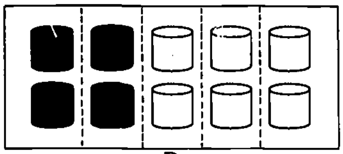
5. How many subgroups would you shade in the diagram above to show the pencils Bill borrowed?

6. Explain why the fraction $\frac{2}{6}$ can be used to describe the part of Jane's group of pencils that Bill borrowed. Since the fractions $\frac{1}{3}$ and $\frac{2}{6}$ can both be used to describe the same part of Jane's group of pencils, the two fractions are equivalent.

7. Think of the ten tanks shown in diagram *A* as a group. Imagine that the four tanks shown in color are filled with water.



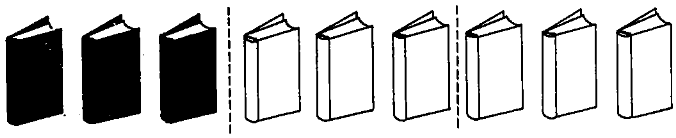
A



B

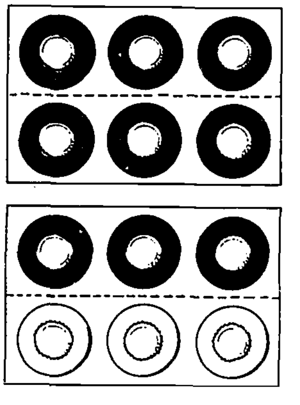
   a. Diagram *B* shows the group of tanks divided into subgroups each containing the same number of tanks. How many subgroups are there?
   b. How many subgroups of tanks are filled with water?
   c. Use your answers to exercises 7a and 7b to form a fraction that can be used to tell what part of the group of tanks is filled with water.
   d. Make a diagram to show that the fraction $\frac{4}{10}$ can also be used to tell what part of the group of tanks is filled with water.
   e. How are the fractions $\frac{2}{5}$ and $\frac{4}{10}$ related?

8. The picture below shows a group of nine books divided into subgroups of three books each. Some of the books have blue covers and the rest have white covers.



   a. Explain why you can say that $\frac{1}{3}$ of the books have blue covers.
   b. Can you also say that $\frac{3}{9}$ of the books have blue covers? Explain your answer.

9. Pictured at the right are two cartons of pop bottles. The bottles shown in color represent full bottles. Think of each carton of six bottles as one whole group.
   a. Into how many subgroups is each whole group divided?
   b. Of all the subgroups that are pictured how many are shown in color?
   c. Why can you say that the full bottles make up $\frac{3}{2}$ of a carton?
   d. Can you also say that the full bottles make up $\frac{9}{6}$ of a carton? Explain your answer.

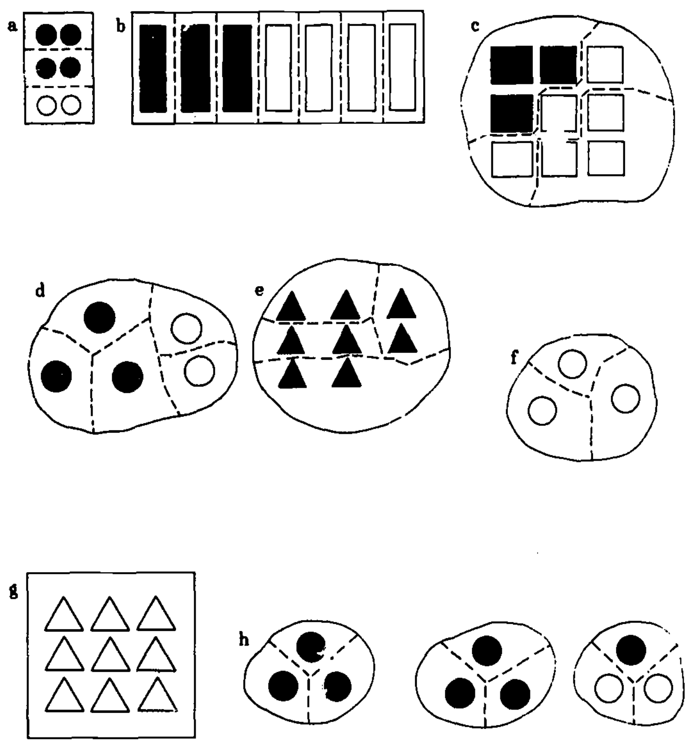


[redacted]

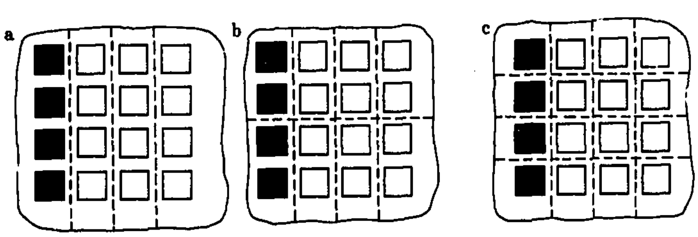
You have seen that a group of objects can be regarded as a "whole," just as a region can be considered as a "whole." Furthermore, a group of objects can be divided into subgroups that have the same number of objects, just as a region can be divided into congruent subregions. Therefore, a fraction can be used to describe a part of a group in the same way that a fraction can be used to describe a part of a region.

[redacted]

1. For each diagram, use a fraction to tell what part of a whole group is shown in color. The dotted lines tell you what to consider as a subgroup in each case.



2. The three diagrams below show three different ways of dividing a group of objects into subgroups. For each diagram give a fraction that indicates the part of the group that is shown in color.

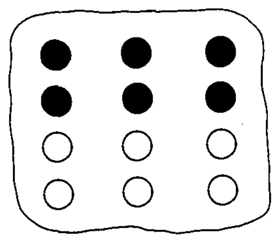


3. Make diagrams with different subgroups to show that the objects shown in color are

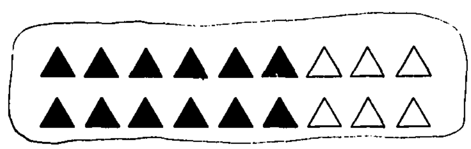
a. $\frac{1}{2}$ of the group.

b. $\frac{2}{4}$ of the group.

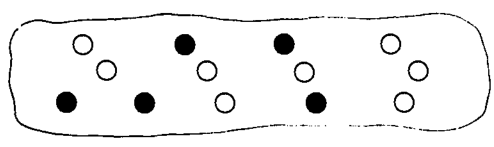
c. $\frac{6}{12}$ of the group.



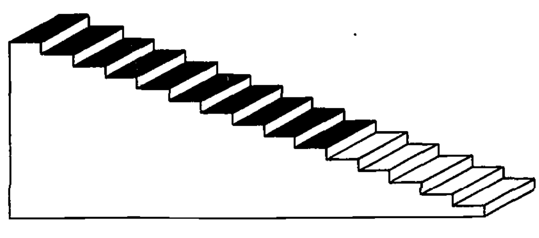
4. Choose subgroups in four different ways to find four equivalent fractions. Each fraction is to indicate the part of the group that is shown in color.



5. What part of the group is shown in color?



6. Bill had the task of painting a staircase. After painting the steps shown in blue, he decided to figure out what part of the staircase he had painted and what part remained to be painted. He counted to find the number of steps in the whole staircase and also the number he had painted.



a. What fraction can Bill use to describe the part of the staircase he has painted?

b. What part of the staircase does he have left to paint?

c. If Bill thinks of the staircase as made up of subgroups, each consisting of five steps, what fraction can he use to describe the part he has painted? The part he has left to paint? Is either of these fractions equivalent to the fraction you gave in exercise 6a?

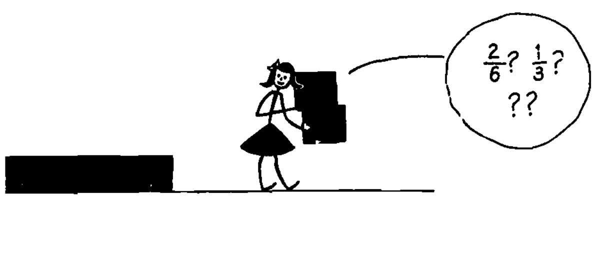
7. For each fraction make a diagram that can be used to explain the meaning of the fraction. Use 12 circles as a whole group.

a. $\frac{7}{12}$ **c.** $\frac{5}{6}$ **e.** $\frac{6}{6}$

b. $\frac{1}{4}$ **d.** $\frac{2}{3}$ **f.** $\frac{0}{3}$

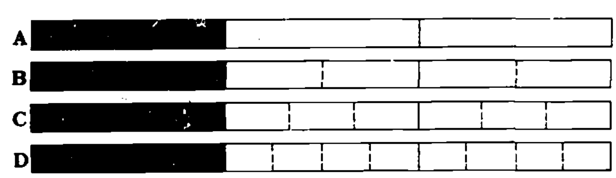
Lists of Equivalent Fractions

Jane has seen that more than one fraction can be used to describe the same part of a pan of fudge, the same part of a region, and the same part of a group. She knows that fractions that represent the same part of something are equivalent. She would like to know how many different fractions are equivalent to a given fraction.



Class Discussion

1. The rectangular region shown in diagram A is divided into three subregions, one of which is shaded. What fraction represents the part of the region that is shown in color?



2. The rectangular region in diagram A is shown again in diagrams B, C, and D. Is the same amount shaded in each diagram?

3. Notice that there are twice as many subregions in diagram B as in diagram A. This means that there are 3×2 subregions in diagram B, and that 1×2 subregions are shaded. Thus, $\frac{1 \times 2}{3 \times 2}$ indicates the part of the region that is shaded in diagram B.

   a. Explain why $\frac{1 \times 3}{3 \times 3}$ indicates the part of the region that is shaded in diagram C.

   b. Why does $\frac{1 \times 4}{3 \times 4}$ indicate the part of the region that is shaded in diagram D?

4. A simpler name for $\frac{1 \times 2}{3 \times 2}$ is $\frac{2}{6}$. Tell how $\frac{2}{6}$ is obtained from $\frac{1 \times 2}{3 \times 2}$. Give simpler names for each of the following:

   a. $\frac{1 \times 1}{3 \times 1}$ b. $\frac{1 \times 2}{3 \times 2}$ c. $\frac{1 \times 3}{3 \times 3}$ d. $\frac{1 \times 4}{3 \times 4}$.

5. Explain why each fraction listed in exercise 4 is equivalent to each of the others.

6. Make diagrams like those in exercise 1 to show that $\frac{1}{3}$ is

equivalent to $\frac{1 \times 5}{3 \times 5}$. Write a simpler name for $\frac{1 \times 5}{3 \times 5}$.

7. In exercises 1 through 6 you saw that each of the fractions $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, and $\frac{5}{15}$ is equivalent to $\frac{1}{3}$. You also saw how each of the fractions $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, and $\frac{5}{15}$ can be obtained from $\frac{1}{3}$.

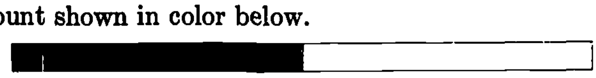
a. Is $\frac{1 \times 6}{3 \times 6}$, or $\frac{6}{18}$, equivalent to $\frac{1}{3}$?

b. Is $\frac{1 \times 7}{3 \times 7}$, or $\frac{7}{21}$, equivalent to $\frac{1}{3}$?

c. Is $\frac{1 \times 100}{3 \times 100}$, or $\frac{100}{300}$, equivalent to $\frac{1}{3}$?

8. How many fractions equivalent to $\frac{1}{3}$ do you think there are? Explain how you would obtain them.

9. Use the method suggested in the exercises above to get three fractions that are equivalent to $\frac{1}{2}$. Make three diagrams that show that the three fractions all indicate the same amount. Each diagram should have the same amount shaded as the amount shown in color below.



10. **a.** Can you obtain $\frac{1}{2}$ from $\frac{1 \times 0}{2 \times 0}$?

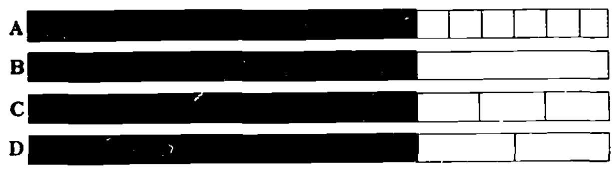
b. Can you get a fraction that is equivalent to $\frac{1}{2}$ if you multiply both the numerator and denominator by zero?

c. Can you ever get a fraction that is equivalent to a given fraction if you multiply both the numerator and the denominator of the given fraction by zero?

d. Do you obtain a fraction if you multiply both the numerator and denominator of a given fraction by zero?

11. The diagrams below illustrate another way of finding fractions that are equivalent to a given fraction.

a. Look at diagram *A*. Explain why $\frac{12}{18}$ indicates the part of the region that is shaded.



b. Compare diagrams *A* and *B*. Explain why $\frac{12 \div 6}{18 \div 6}$ indicates the part of the region that is shaded in diagram *B*.

c. Compare diagrams *A* and *C*. Explain why $\frac{12 \div 2}{18 \div 2}$ indicates the part of the region that is shaded in diagram *C*.

d. Compare diagrams *A* and *D*. Does $\frac{12 \div 3}{18 \div 3}$ indicate the part of the region that is shaded in diagram *D*?

12. How do you know that the fractions listed below are equivalent? Give simpler names for the fractions in exercises b, c, and d.

a. $\frac{12}{18}$ **b.** $\frac{12 \div 6}{18 \div 6}$ **c.** $\frac{12 \div 2}{18 \div 2}$ **d.** $\frac{12 \div 3}{18 \div 3}$

13. Do you think that $\frac{4}{6}$ is equivalent to $\frac{4 \div 2}{6 \div 2}$?

14. a. Is $\frac{18}{9}$ equivalent to $\frac{2}{1}$?

b. By what number would you divide the numerator and denominator of $\frac{18}{9}$ to get $\frac{2}{1}$?

c. Divide the numerator and denominator of $\frac{18}{9}$ by another number to get a second equivalent fraction.

[redacted]

You can find indefinitely many fractions that are equivalent to a given fraction. One way is to multiply the numerator and denominator of the given fraction by the same whole number, not zero. Suppose, for example, that you start with the fraction $\frac{10}{12}$. If you multiply both the numerator and denominator in turn by 1, 2, 3, 4, $\cdots$, you get the list of fractions below.

$$\frac{10 \times 1}{12 \times 1}, \quad \text{or} \quad \frac{10}{12}$$

$$\frac{10 \times 2}{12 \times 2}, \quad \text{or} \quad \frac{20}{24}$$

$$\frac{10 \times 3}{12 \times 3}, \quad \text{or} \quad \frac{30}{36}$$

$$\frac{10 \times 4}{12 \times 4}, \quad \text{or} \quad \frac{40}{48}$$

$$\cdots$$

Each fraction in the list is equivalent to $\frac{10}{12}$. Notice that $\frac{10}{12}$ is equivalent to itself.

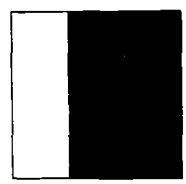
You can also obtain a fraction that is equivalent to a given fraction by dividing both the numerator and denominator of the given fraction by the same whole number, not zero. For example,

$$\frac{10 \div 2}{12 \div 2}, \quad \text{or} \quad \frac{5}{6}, \quad \text{is equivalent to} \quad \frac{10}{12}.$$

Unfortunately, you cannot divide the numerator and denominator of a given fraction by *any* whole number that comes into your mind. For example, you cannot divide both the numerator and denominator of $\frac{10}{12}$ by 3. The reason is that dividing 10 by 3 does not give you a whole number.

Exercises—2a

1. Pictured at the right is a square region that is divided into three congruent subregions. Two of the subregions are shown in color. Draw two more square regions, each congruent to the given region. Use the three square regions to show that $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent fractions.



2. Jane invited 6 people to a party. She made a cake and sliced it into 6 pieces of the same size so that each person might have $\frac{1}{6}$ of the cake. Then Jane decided that each person should have 2 small pieces rather than one large piece. Draw a picture to show how the cake looked after Jane made the second set of cuts. Draw a picture to show how the cake would look if Jane sliced it so that each person would get 4 small pieces of the same size. Give two fractions different from $\frac{1}{6}$ that can be used to represent each person's share of the cake.

3. In the sentences below, you are to replace a, b, c, d, e, f, g, and h by whole numbers. For each sentence, choose whole numbers that make the sentence true.

   a. $\frac{9}{12}$ is equivalent to $\frac{9 \div 3}{12 \div 3}$, which is $\frac{a}{b}$.

   b. $\frac{2}{3}$ is equivalent to $\frac{4 \times 2}{4 \times 3}$, which is $\frac{c}{d}$.

   c. $\frac{6}{8}$ is equivalent to $\frac{e \times 6}{f \times 8}$, which is $\frac{24}{32}$.

   d. $\frac{5}{10}$ is equivalent to $\frac{5 \div g}{10 \div h}$, which is $\frac{1}{2}$.

4. In the sentences below, you are to replace a, b, c, d, e, f, g, and h

by whole numbers. For each sentence, choose whole numbers that make the sentence true.

a. $\frac{1}{2}$ is equivalent to $\frac{a}{8}$.

b. $\frac{15}{10}$ is equivalent to $\frac{3}{b}$.

c. $\frac{12}{18}$ is equivalent to $\frac{2}{c}$.

d. $\frac{3}{8}$ is equivalent to $\frac{d}{24}$.

e. $\frac{1}{4}$ is equivalent to $\frac{12}{e}$.

f. $\frac{16}{24}$ is equivalent to $\frac{f}{3}$.

g. $\frac{21}{7}$ is equivalent to $\frac{g}{49}$.

h. $\frac{13}{13}$ is equivalent to $\frac{h}{1}$.

5. Find five fractions that are equivalent to $\frac{4}{7}$. (Hint: Multiply both the numerator and the denominator of $\frac{4}{7}$ in turn by each of five different whole numbers, not zero.) Using this method, is it possible to get more than five different fractions that are equivalent to $\frac{4}{7}$?

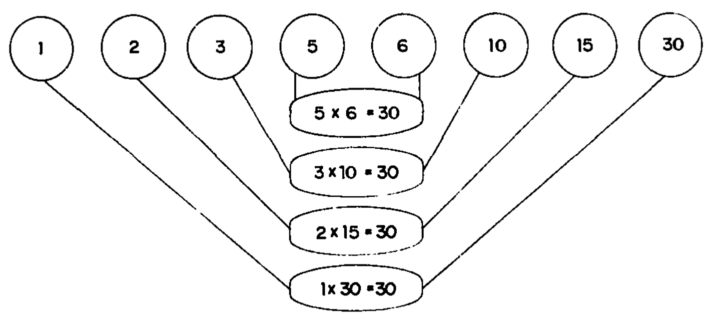
6. Obtain five different fractions that are equivalent to $\frac{18}{36}$. (Hint: Divide both the numerator and the denominator of $\frac{18}{36}$ in turn by each of five different whole numbers, not zero.) Using this method, can you get more than five fractions that are equivalent to $\frac{18}{36}$? Keep in mind that both the numerator and denominator of a fraction are whole numbers.

Class Discussion

There are only certain numbers by which you can divide the numerator and denominator of a given fraction to get an equivalent fraction. This fact will become clearer as you learn about the *factors* of a whole number.

Since $2 \times 4 = 8$, we say that 2 and 4 are factors of 8. Also, since $1 \times 8 = 8$, we say that 1 and 8 are factors of 8.

As another example, the numbers 1, 2, 3, 5, 6, 10, 15, and 30 are all factors of 30. Note that $2 \times 15 = 30$; $3 \times 10 = 30$; $5 \times 6 = 30$; and $1 \times 30 = 30$. See the diagram below.



1. For each sentence, name the factors of 12 that are indicated by the given sentence.

   a. $2 \times 6 = 12$. b. $1 \times 12 = 12$. c. $3 \times 4 = 12$.

2. To decide whether or not 5 is a factor of 12, you should ask yourself this question: Is there a whole number n for which it is true that $n \times 5 = 12$?
   a. Is 5 a factor of 12?
   b. Is 7 a factor of 12?
   c. Is 11 a factor of 12?
3. List all the factors of 12.
4. Write 20 as a product of two whole numbers in as many different ways as you can. Do not list products such as 4×5 and 5×4 separately.
5. List all the factors of 20 in order from the least to the greatest.
6. a. Is 3 a factor of 9?
   b. Is 5 a factor of 55?
   c. Is 16 a factor of 16?
   d. Is 1 a factor of 7?
7. a. Is 1 a factor of every whole number?
   b. Is every whole number other than zero a factor of itself?
   c. Is every whole number a factor of zero?
8. a. List all the factors of 16.

- **b.** List all the factors of 4.
- **c.** List all the numbers that appear in both of the lists that you made. These numbers are the *common factors* of 16 and 4.
- **d.** A number that is a factor of each of two numbers is a common factor of the two numbers. Find all the common factors of 6 and 9.
- **e.** Is 1 a common factor of every pair of whole numbers?

9. Use the idea of common factors to find a fraction that is equivalent to a given fraction.

- **a.** Is 2 a common factor of 4 and 16?
- **b.** Is the fraction $\frac{4 \div 2}{16 \div 2}$ equivalent to $\frac{4}{16}$?

10. **a.** Is 4 a common factor of 4 and 16?

- **b.** What fraction do you get when you divide both the numerator and denominator of $\frac{4}{16}$ by 4? Is the fraction that you get equivalent to $\frac{4}{16}$?

11. Is there a whole number, other than 2, 4, or 1, by which you can divide both 4 and 16 to obtain a fraction that is equivalent to $\frac{4}{16}$?

12. **a.** List all whole numbers different from 1 that are common factors of 10 and 20.

- **b.** Use these common factors to get fractions that are equivalent to $\frac{10}{20}$.

[redacted]

Suppose a, b, and n are whole numbers (b not zero, and n not zero). Then if n is a common factor of a and b, the fraction $\frac{a \div n}{b \div n}$ is equivalent to $\frac{a}{b}$.

1. List all the factors of each number in order from the least to the greatest.
   a. 24 b. 54 c. 10 d. 27
2. List all the common factors of each pair of whole numbers.
   a. 24 and 27 c. 54 and 10
   b. 24 and 54 d. 27 and 54
3. For each fraction, obtain as many equivalent fractions as you can by dividing both the numerator and denominator by a common factor different from 1.
   a. $\frac{24}{27}$ c. $\frac{54}{10}$
   b. $\frac{24}{54}$ d. $\frac{27}{54}$
4. In the list of equivalent fractions $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots$ is there one fraction that you would consider the simplest fraction in the list?

Class Discussion

1. a. What are all the common factors of the numerator and the denominator of $\frac{10}{15}$?
   b. What are all the common factors of the numerator and denominator of $\frac{8}{12}$?
   c. What are all the common factors of the numerator and denominator of $\frac{6}{9}$? Of $\frac{4}{6}$?
   d. What is the only number that is a common factor of the numerator and denominator of $\frac{2}{3}$?

We say that $\frac{2}{3}$ is the *simplest fraction* in the list of equivalent fractions $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots$. We say this because 2 and 3 have no common factor different from 1.

2. Consider the three equivalent fractions $\frac{1}{4}, \frac{5}{20}$, and $\frac{10}{40}$.

a. What are the common factors of 1 and 4?
b. What are the common factors of 5 and 20?
c. What are the common factors of 10 and 40?
d. Which of the fractions $\frac{1}{4}, \frac{5}{20}$, or $\frac{10}{40}$ is the simplest fraction?

3. For each fraction given below find the simplest fraction that is equivalent to the given fraction. Start by dividing the numerator and the denominator of the given fraction by a common factor. If the numerator and denominator of the new fraction have a common factor different from 1, divide the numerator and denominator of the new fraction by this common factor. Continue this process until you get a fraction in which the numerator and denominator have only the number 1 as a common factor.

a. $\frac{6}{8}$ **b.** $\frac{20}{200}$ **c.** $\frac{12}{24}$
d. $\frac{30}{100}$ **e.** $\frac{50}{25}$ **f.** $\frac{25}{10}$

4. Consider the fraction $\frac{12}{18}$.

a. What are the common factors of 12 and 18?
b. If you divide both the numerator and the denominator of $\frac{12}{18}$ by 2, what fraction do you get? Is this fraction equivalent to $\frac{12}{18}$?
c. If you divide both the numerator and the denominator of $\frac{12}{18}$ by 3, what fraction do you get? Is this fraction equivalent to $\frac{12}{18}$?

d. If you divide both the numerator and the denominator of $\frac{12}{18}$ by 6, what fraction do you get? Is this fraction equivalent to $\frac{12}{18}$?

e. Which of the fractions you obtained in exercises b, c, and d is the simplest fraction? By what common factor did you divide 12 and 18 to obtain this fraction?

5. Now consider the fraction $\frac{8}{16}$.

a. What are the common factors of 8 and 16?

b. Divide both the numerator and the denominator of $\frac{8}{16}$ by each of the common factors of 8 and 16. How many fractions do you get? Is each fraction that you get equivalent to $\frac{8}{16}$?

c. Which fraction in the list of equivalent fractions that you obtained is the simplest fraction? By what common factor did you divide both 8 and 16 to obtain this fraction?

6. The common factors of 9 and 27 are 1, 3, and 9. By which of these factors should you divide both 9 and 27 to find the simplest fraction that is equivalent to $\frac{9}{27}$?

In a list of equivalent fractions, $\frac{a}{b}$ is the simplest fraction in the list if a and b have exactly one common factor and this factor is 1. To get the simplest fraction that is equivalent to a given fraction, divide both numerator and denominator by common factors until you obtain a fraction in which the numerator and denominator have no common factor other than 1. Given any fraction, if you divide both the numerator and denominator by their *greatest common factor* you get the simplest fraction in one step.

[redacted]

1. Listed in each exercise are three equivalent fractions. In each exercise decide which fraction is the simplest.

   a. $\frac{10}{12}$, $\frac{25}{30}$, $\frac{5}{6}$ c. $\frac{7}{1}$, $\frac{21}{3}$, $\frac{28}{4}$

   b. $\frac{8}{14}$, $\frac{4}{7}$, $\frac{16}{28}$ d. $\frac{2}{2}$, $\frac{5}{5}$, $\frac{1}{1}$

2. For each fraction find the simplest equivalent fraction.

   a. $\frac{75}{100}$ b. $\frac{7}{7}$ c. $\frac{21}{7}$ d. $\frac{13}{19}$

3. For each fraction, find the greatest common factor of the numerator and denominator. Then find the simplest fraction that is equivalent to the given fraction.

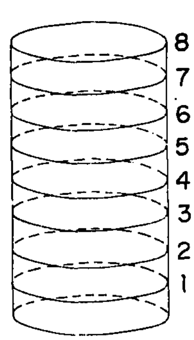
   a. $\frac{24}{8}$ b. $\frac{12}{16}$ c. $\frac{18}{45}$ d. $\frac{400}{400}$

4. How do you know that any fraction in which the numerator and denominator are even numbers is not a simplest fraction?

5. For each fraction, find the simplest fraction that is equivalent to the given fraction. Use any method you like.

   a. $\frac{96}{24}$ b. $\frac{48}{72}$ c. $\frac{96}{32}$ d. $\frac{256}{64}$

6. Show that $\frac{12}{48}$ is equivalent to $\frac{6}{24}$ by finding the simplest fraction that is equivalent to $\frac{12}{48}$ and the simplest fraction that is equivalent to $\frac{6}{24}$.

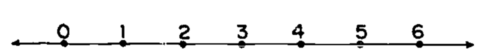


7. What is the simplest fraction that indicates the part of the tank that is filled if water reaches the sixth mark; the second mark; the third mark; the eighth mark? If the tank is empty?

The Number Line

Pictured below is a line that is divided into congruent segments.

Suppose we start at any division point that we please and assign the whole numbers *in order* (0, 1, 2, 3, 4, 5, . . .) to successive division points with the greater of two numbers always on the right. Such a *matching* of numbers with points in a line is called a *number line.* The diagram below illustrates the idea.



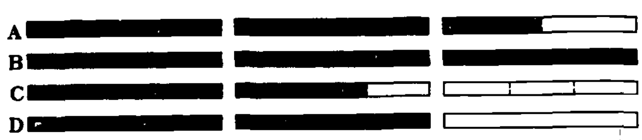
As you can see, the whole numbers correspond to only *some* of the points in a line. Knowing about fractions enables you to assign numbers to many more points in a line.

Recall that a fraction is an ordered pair of whole numbers (second number not zero). This means that a fraction is not a number! But fractions can be used to *indicate* numbers, and such numbers can be assigned to points in a line. A number that can be indicated by a fraction is called a *rational number.*

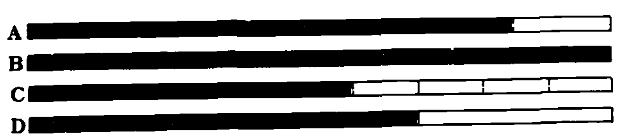
Suppose you wish to assign a number to the point that is midway between point 0 and point 1. This number would certainly not be a whole number. It would be a rational number. You can use the fraction $\frac{1}{2}$ to indicate this number. You can also use each of the fractions $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots$ to indicate this number. Do you think every fraction in a list of equivalent fractions can be used to indicate the same rational number? As you proceed in this unit you will find out that the answer to the last question is yes. In the discussion that follows you will see how to assign rational numbers to points in a line.

Class Discussion

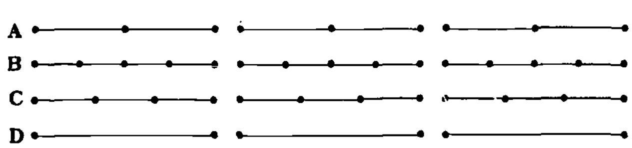
1. Three whole regions are shown in each diagram below. For each diagram, give a fraction that can be used to describe the amount that is shown in color.



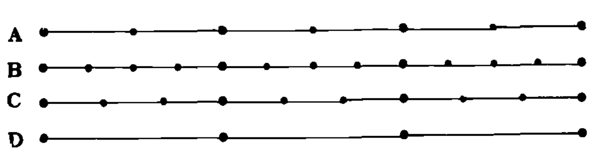
2. Again, three whole regions are shown in each diagram, but this time the regions are placed end to end. For each diagram, give a fraction that can be used to describe the amount that is shown in color.



3. Three whole segments are shown in each diagram below. For each diagram give a fraction that can be used to describe the amount that is shown in color.



4. Three whole segments placed end to end are shown in each diagram. For each diagram, give a fraction for the amount that is shown in color.



5. Think of a line as made up of whole segments that are placed end to end. You already know that you can assign whole numbers to the endpoints of these segments. It also makes sense to assign rational numbers to these endpoints.

   a. Explain why the fraction $\frac{1}{1}$ can be used to describe the amount shown in color in diagram *E*.

   b. The fraction $\frac{1}{1}$ can also be used to indicate the rational number to be assigned to the right-hand endpoint of the segment shown in color. Is this the same point to which the whole number 1 is assigned?

   c. Explain why the fraction $\frac{2}{1}$ can be used to describe the amount shown in color in diagram *F*.

   d. The fraction $\frac{2}{1}$ can also be used to indicate the rational number to be assigned to the right-hand endpoint of the second whole segment. Is this the same point to which the whole number 2 is assigned?

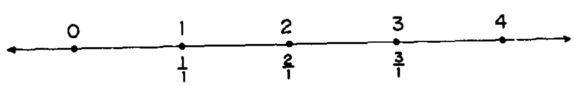
   e. Do you think the fraction $\frac{3}{1}$ can be used to indicate the rational number to be assigned to the point matched with the whole number 3?

6. In mathematics we agree to assign exactly *one number* to a point in a number line. This means that *some rational numbers are whole numbers.* Since the rational number indicated by the fraction $\frac{1}{1}$ is assigned to the same point as the whole number 1, we can write

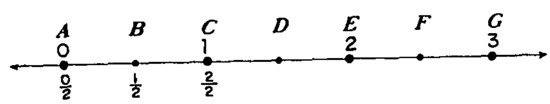
$$\frac{1}{1} = 1.$$

For the same reason, we can write $\frac{2}{1} = 2$; $\frac{3}{1} = 3$; and so on. Do you think that every whole number is a rational number?

7. Write a fraction that indicates the rational number that should be assigned to the point matched with 0; with 4; with 50.



8. The whole segments in a number line are usually called *unit segments*. Suppose each unit segment is divided into two congruent subsegments, as shown below.



a. Why can the fraction $\frac{0}{2}$ be used to indicate the rational number assigned to point A?

b. Why can the fraction $\frac{1}{2}$ be used to indicate the rational number assigned to point B?

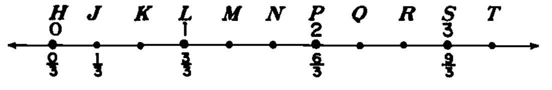
c. Why can the fraction $\frac{2}{2}$ be used to indicate the rational number assigned to point C?

d. What whole number is equal to the rational number indicated by the fraction $\frac{0}{2}$?

e. What whole number is equal to the rational number indicated by the fraction $\frac{2}{2}$?

f. What fraction should be used to indicate the rational number to be assigned to point D; point E; point F; point G?

9. In the number line below, each unit segment is divided into three congruent subsegments.

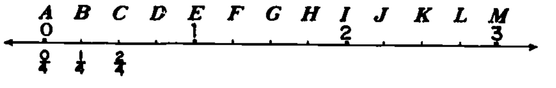


a. Why can the fraction $\frac{0}{3}$ be used to indicate the rational number assigned to point H?

b. Explain why the fractions $\frac{1}{3}$, $\frac{3}{3}$, $\frac{6}{3}$, and $\frac{9}{3}$ can be used to indicate the rational numbers assigned to points J, L, P, and S.

c. Write fractions to indicate the rational numbers that can be assigned to points K, M, N, Q, R, and T.

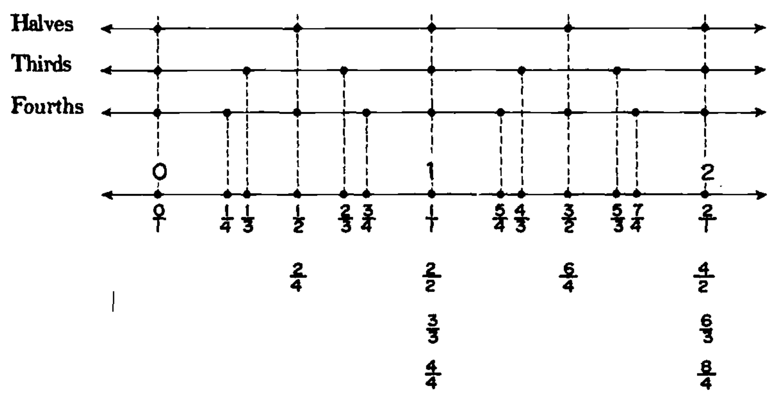
10. In the number line below, each unit segment is divided into how many congruent subsegments? What fractions should be used to indicate the rational numbers to be assigned to points D, E, F, G, H, I, J, K, L, and M?



11. You have used more than one fraction to indicate the rational number that corresponds to a particular point. For example, you have used the fractions $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$, and $\frac{4}{4}$ to indicate the rational number that corresponds to the same point as 1. Are all these fractions equivalent? How do you know?

12. Give four fractions that can be used to indicate the rational number that corresponds to the same point as 2. Are the four fractions you gave all equivalent?

13. The diagram below shows all the rational numbers that have been assigned to points in a line thus far. You can see that every time you subdivide the unit segment into smaller subsegments you can locate more rational numbers. Also, each new subdivision suggests more equivalent fractions for some of the rational numbers already located.



Find the point to which the rational number indicated by the fraction $\frac{1}{2}$ has been assigned. What other fractions have you used to indicate this rational number? What subdivision of the unit segment is suggested by the last fraction you just gave?

14. Find the point that corresponds to the rational number indicated by the fraction $\frac{1}{3}$. What subdivision of the unit segment is suggested by the fraction $\frac{1}{3}$? How many fractions are there that indicate the same rational number as the fraction $\frac{1}{3}$?

[redacted]

Rational numbers may be described as numbers that *can be indicated by fractions.* Rational numbers can be assigned to points in a line. Think of a line as made up of connected unit segments. By dividing each unit segment into two congruent subsegments you can locate points to which you can assign rational numbers indicated by the fractions $\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots$. By dividing each unit segment into four congruent subsegments, you can locate points to which you can assign rational numbers indicated by the fractions $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \ldots$.

There are indefinitely many fractions that indicate the same rational number. For example, every fraction in the list below indicates the same rational number.

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots, \frac{100}{200}, \ldots$$

The list is said to define the rational number indicated by the fraction $\frac{1}{2}$, or $\frac{2}{4}$, or $\ldots$. Only one fraction in the list is needed to indicate the rational number. Also, only one fraction in the list is needed to label the point in a line to which the rational number is assigned. All fractions that indicate the same rational number are equivalent.

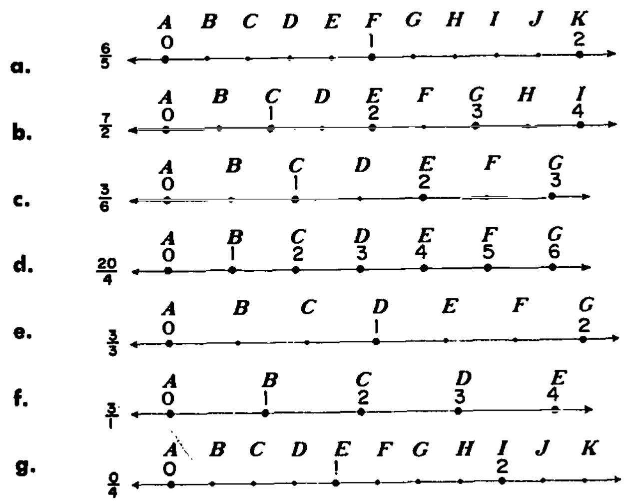
Some rational numbers are whole numbers. For example, the rational number defined by the list of fractions below is the same as the whole number 3.

$$\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots$$

So we can write $\frac{3}{1} = 3$; $\frac{6}{2} = 3$; and so on.

[redacted]

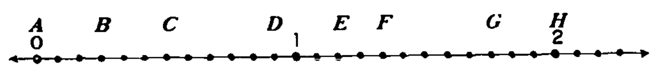
1. A rational number is given beside each number line. With what point can the given rational number be matched?



2. Make a number line and locate the following points:

a. Point $\frac{4}{5}$ **b.** Point $\frac{2}{2}$ **c.** Point $\frac{4}{3}$ **d.** Point $\frac{12}{4}$

3. For each point labeled by a letter, name the rational number. In each case give the simplest fraction for the number.

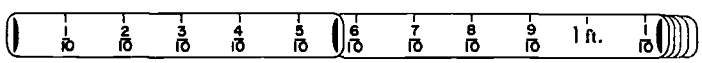


4. Which of the points listed below can be matched with whole numbers?

a. Point $\frac{2}{3}$ **c.** Point $\frac{2}{1}$ **e.** Point $\frac{14}{3}$ **g.** Point $\frac{27}{1}$

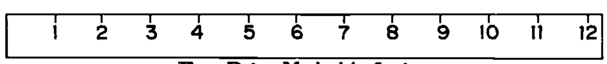
b. Point $\frac{4}{1}$ **d.** Point $\frac{12}{6}$ **f.** Point $\frac{0}{4}$ **h.** Point $\frac{100}{1}$

Comparing Rational Numbers



Engineer's Folding Ruler

Pictured at the top is an engineer's folding ruler marked off in tenths of a foot. Pictured below is an ordinary ruler marked off in inches. Can you tell by looking at the two pictures if $\frac{6}{10}$ of a foot is the same as 7 inches? Think of 7 inches as $\frac{7}{12}$ of a foot. Is the rational number $\frac{6}{10}$ equal to the rational number $\frac{7}{12}$?



Foot Ruler Marked in Inches

Class Discussion

1. Two rational numbers are equal if the fractions for the numbers are equivalent.

   a. Is the fraction $\frac{6}{10}$ equivalent to the fraction $\frac{36}{60}$?

   b. Is the fraction $\frac{7}{12}$ equivalent to the fraction $\frac{35}{60}$?

   c. Is the rational number $\frac{36}{60}$ equal to the rational number $\frac{35}{60}$?

   d. Explain why the rational number $\frac{6}{10}$ is *not equal* to the rational number $\frac{7}{12}$.

2. If two numbers are *not equal*, then one number is *greater than* the other. One way to decide which of two *unequal* numbers is greater is by using a number line. On a number line, the greater of two unequal numbers is always on the right.

   a. The unit segment of the number line on the following page is divided into 60 subsegments. Is this the smallest number of subsegments needed to assign both $\frac{6}{10}$ and $\frac{7}{12}$ to division points?

   b. Is $\frac{36}{60}$ to the right of $\frac{35}{60}$?

   c. Is $\frac{6}{10}$ greater than $\frac{7}{12}$? Explain.

3. If the first of two numbers is greater than the second, then the second number is *less than* the first.

   Let's compare the rational numbers $\frac{2}{3}$ and $\frac{7}{10}$.

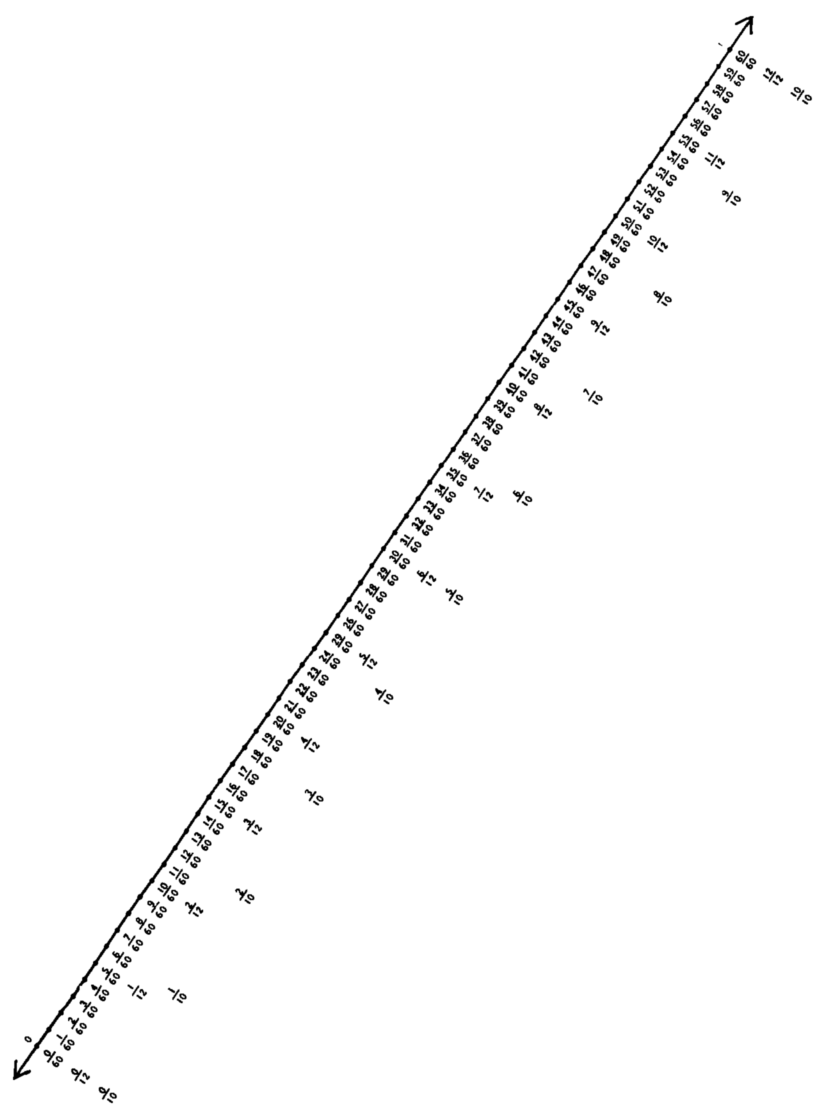
   a. Into how many congruent subsegments must the unit segment be divided to assign both $\frac{2}{3}$ and $\frac{7}{10}$ to division points?

   b. Is the fraction $\frac{2}{3}$ equivalent to the fraction $\frac{20}{30}$? Is the fraction $\frac{7}{10}$ equivalent to the fraction $\frac{21}{30}$?

   c. Explain why the rational number $\frac{20}{30}$ is less than the rational number $\frac{21}{30}$.

   d. Explain why the rational number $\frac{2}{3}$ is less than the rational number $\frac{7}{10}$.

4. Comparing two rational numbers is easy if the fractions for the numbers have the *same denominator* (also called the common denominator).



a. How do the rational numbers represented by $\frac{a}{b}$ and $\frac{c}{b}$ compare if a is greater than c? (Assume a, b, c are whole numbers and b is not zero.)

b. How do the rational numbers compare if a is less than c? If a equals c?

5. Suppose you want to find fractions with a common denominator in order to compare the rational numbers $\frac{3}{5}$ and $\frac{2}{3}$. Do you think you can use 15 as a common denominator? Explain your answer.

6. Replace a, b, c, and d by whole numbers that make the resulting sentences true.

   a. $\frac{3}{5}$ is equivalent to $\frac{3 \times a}{5 \times a}$, which is $\frac{b}{15}$.

   b. $\frac{2}{3}$ is equivalent to $\frac{2 \times c}{3 \times c}$, which is $\frac{d}{15}$.

   c. Which number is greater, $\frac{3}{5}$ or $\frac{2}{3}$?

7. a. What number would you use as a common denominator if you wanted to compare the rational numbers $\frac{5}{11}$ and $\frac{4}{9}$?

   b. Which is the greater number, $\frac{5}{11}$ or $\frac{4}{9}$?

[redacted]

One way to compare two rational numbers is by locating the two numbers on a number line. If the numbers correspond to the same point they are equal. If the numbers correspond to different points, the number on the right is the greater.

Another way to compare two rational numbers is by finding fractions with a common denominator for the two numbers. If the numerators of the fractions are equal, then the two rational numbers are equal. If the numerators are not equal, then the fraction with the greater numerator indicates the greater rational number.

[redacted]

1. The symbol > means "is greater than" and the symbol < means "is less than." The first sentence below is read "Two-thirds is greater than one-third." How should the second sentence be read?

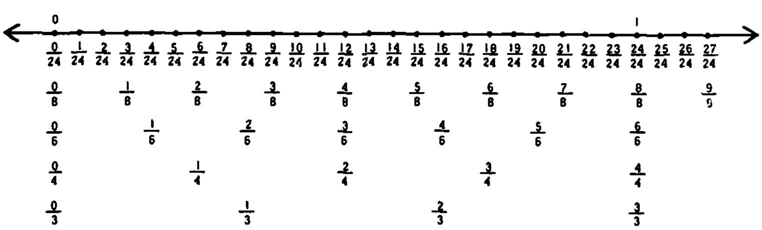
$$\frac{2}{3} > \frac{1}{3}.$$
$$\frac{1}{3} < \frac{2}{3}.$$

2. In each exercise, locate the two given rational numbers on the number line below. Then complete the sentence using whichever symbol, >, <, or =, makes the sentence true.

   a. $\frac{2}{3}$ ___ $\frac{5}{8}$. d. $\frac{5}{6}$ ___ $\frac{6}{8}$.

   b. $\frac{4}{6}$ ___ $\frac{2}{3}$. e. $\frac{3}{4}$ ___ $\frac{7}{8}$.

   c. $\frac{3}{8}$ ___ $\frac{1}{3}$. f. $\frac{3}{6}$ ___ $\frac{5}{8}$.



3. In each exercise decide which symbol, >, <, or =, makes the sentence true.

   a. $\frac{1}{5}$ ___ $\frac{3}{5}$. c. $\frac{18}{11}$ ___ $\frac{17}{11}$.

   b. $\frac{9}{12}$ ___ $\frac{9}{12}$. d. $\frac{10}{4}$ ___ $\frac{16}{4}$.

4. In each exercise find fractions with a common denominator for the two given rational numbers. Then use one of the symbols,

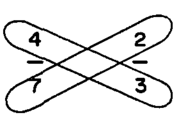
>, <, or =, to write a true sentence about the two rational numbers.

a. $\frac{3}{10}, \frac{2}{3}$ b. $\frac{3}{7}, \frac{6}{14}$ c. $\frac{16}{9}, \frac{3}{2}$ d. $\frac{5}{6}, \frac{9}{11}$

Have you discovered a shortcut for comparing two rational numbers? If not, then be on the lookout for a shortcut in the discussion that follows.

Class Discussion

1. You can compare the rational numbers $\frac{4}{7}$ and $\frac{2}{3}$ by using fractions with a common denominator.
   a. What is the product of the denominators of $\frac{4}{7}$ and $\frac{2}{3}$?
   b. Can you use 21 as a common denominator?
   c. Are the fractions $\frac{4}{7}$ and $\frac{12}{21}$ equivalent?
   d. Are the fractions $\frac{2}{3}$ and $\frac{14}{21}$ equivalent?
2. a. Use the diagram below to explain how the numerators of $\frac{12}{21}$ and $\frac{14}{21}$ can be obtained from $\frac{4}{7}$ and $\frac{2}{3}$.



   b. Is the product of the numerator of $\frac{4}{7}$ and the denominator of $\frac{2}{3}$ equal to the numerator of $\frac{12}{21}$? Does $4 \times 3 = 12$?
   c. Is the product of the denominator of $\frac{4}{7}$ and the numerator of $\frac{2}{3}$ equal to the numerator of $\frac{14}{21}$? Does $7 \times 2 = 14$?

d. Is it true that $\frac{12}{21} < \frac{14}{21}$?

e. Is it true that $\frac{4}{7} < \frac{2}{3}$? Give a reason for your answer.

3. Make a diagram like the one in exercise 2 to compare the rational numbers $\frac{7}{6}$ and $\frac{4}{3}$.

   a. Is $7 \times 3 < 6 \times 4$?

   b. Is it true that $\frac{7}{6} < \frac{4}{3}$?

4. Make a diagram like the one in exercise 2 to compare the rational numbers $\frac{6}{7}$ and $\frac{4}{5}$.

   a. Is $6 \times 5 > 7 \times 4$?

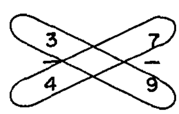
   b. Is it true that $\frac{6}{7} > \frac{4}{5}$?

5. Make a diagram like the one in exercise 2 to compare the rational numbers $\frac{6}{8}$ and $\frac{9}{12}$.

   a. Does $6 \times 12 = 8 \times 9$?

   b. Are the rational numbers $\frac{6}{8}$ and $\frac{9}{12}$ equal?

6. Use the diagram below to decide whether $\frac{3}{4}$ is less than, equal to, or greater than $\frac{7}{9}$.



[redacted]

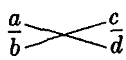
Suppose $\frac{a}{b}$ and $\frac{c}{d}$ represent two rational numbers.

If $a \times d = b \times c$, then $\frac{a}{b} = \frac{c}{d}$.

If $a \times d < b \times c$, then $\frac{a}{b} < \frac{c}{d}$.

If $a \times d > b \times c$, then $\frac{a}{b} > \frac{c}{d}$.

The above method of comparing two rational numbers is known as the *cross-product method.* The diagram below will help you remember it.



1. Use the cross-product method to decide whether or not each sentence is true.

a. $\frac{3}{12} = \frac{5}{20}$. **c.** $\frac{17}{51} = \frac{2}{6}$.

b. $\frac{19}{2} = \frac{54}{3}$. **d.** $\frac{11}{19} = \frac{121}{209}$.

2. Use the cross-product method to compare the two rational numbers in each exercise. Decide whether the first number is less than, greater than, or equal to the second number.

a. $\frac{2}{3}, \frac{3}{5}$ **c.** $\frac{3}{8}, \frac{4}{9}$ **e.** $\frac{11}{13}, \frac{2}{3}$

b. $\frac{4}{7}, \frac{3}{5}$ **d.** $\frac{5}{6}, \frac{7}{9}$ **f.** $\frac{9}{5}, \frac{7}{4}$

3. a. Is $\frac{1}{8} < \frac{3}{8}$? Is $\frac{3}{8} < \frac{5}{8}$? Is $\frac{5}{8} < \frac{7}{8}$?

b. Are $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ listed in order, from the least to the greatest?

4. In each exercise arrange the numbers in order, from the least to the greatest.

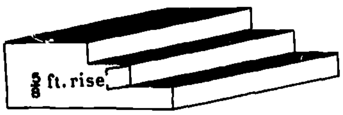
a. $\frac{3}{5}, \frac{7}{5}, \frac{0}{5}, \frac{2}{5}, \frac{8}{5}$

b. $\frac{3}{2}$, $\frac{1}{4}$, $\frac{7}{8}$, $\frac{3}{4}$, $\frac{4}{8}$

c. $\frac{14}{13}$, $\frac{11}{8}$, $\frac{19}{15}$

5. There are 25 students in Becky's and Jim's class. Becky says that the girls make up $\frac{10}{25}$ of the class. Jim says that $\frac{3}{5}$ of the students are girls. Use the cross-product method to determine if Becky and Jim are saying the same thing.

6. Joe and Bill built some steps in the school shop. Joe's steps had a $\frac{5}{8}$ foot rise. Bill's steps had a $\frac{2}{3}$ foot rise. Which steps had the greater rise?

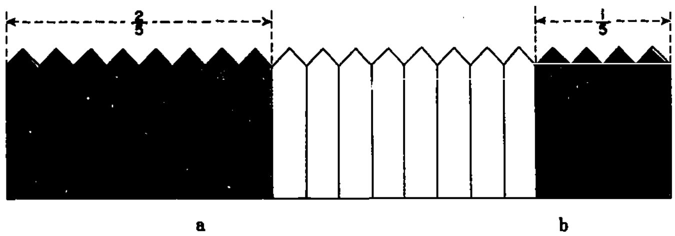


Joe's Steps

7. Mary wanted to buy $\frac{3}{4}$ of a yard of material. The clerk told her that there was $\frac{5}{8}$ of a yard in stock. Was this enough material for Mary?

Adding Rational Numbers

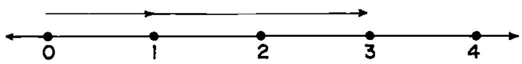
Joe and Bill started to paint a picket fence. Joe started at one end and Bill started at the other. After Joe had painted $\frac{2}{5}$ of the fence and Bill had painted $\frac{1}{5}$ of the fence. Joe wanted to know what part of the fence was painted.



Joe's question can be answered by finding the sum of the rational numbers $\frac{2}{5}$ and $\frac{1}{5}$.

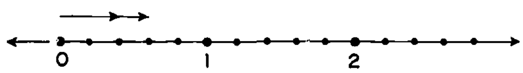
Class Discussion

1. To help you see how to find the sum of two rational numbers let us first look at how addition of whole numbers can be explained by using a number line. To get to the point that corresponds to $1 + 2$, begin at point 0 and make two moves to the right.



   a. How many units long is the first move? How many units long is the second move?
   b. The two moves take you from point 0 to what point?
   c. What point on the number line corresponds to $1 + 2$?

2. In a similar way it is possible to find a point that corresponds to $\frac{2}{5} + \frac{1}{5}$ units.



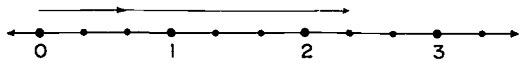
   a. The first move shown above is $\frac{2}{5}$ of a unit long. The second move is $\frac{1}{5}$ of a unit long. The two moves take you from point 0 to what point?

b. What point corresponds to $\frac{2}{5} + \frac{1}{5}$?

c. Do you see that $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$?

Since $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$ is a true sentence, either $\frac{3}{5}$ or the expression $\frac{2}{5} + \frac{1}{5}$ can be thought of as the sum of $\frac{2}{5}$ and $\frac{1}{5}$. However, when you are asked to "find the sum of two rational numbers," or to "add two rational numbers," then you are to find a single fraction for the sum.

3. Does the diagram show that $\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$?



4. Use a number line to find the sum of $\frac{5}{4}$ and $\frac{6}{4}$.

a. Divide each unit segment into 4 congruent subsegments.

b. Show a move of $\frac{5}{4}$ units followed by a move of $\frac{6}{4}$ units.

c. $\frac{5}{4} + \frac{6}{4} = ?$

5. For each exercise make a number line and use it to complete the sentence.

a. $\frac{6}{3} + \frac{5}{3} =$ ___.

b. $\frac{0}{6} + \frac{5}{6} =$ ___.

c. $\frac{1}{2} + \frac{4}{2} =$ ___.

6. Collected below are all examples of addition of rational numbers considered thus far.

$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$. $\frac{6}{3} + \frac{5}{3} = \frac{11}{3}$.

$\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$. $\frac{0}{6} + \frac{5}{6} = \frac{5}{6}$.

$\frac{5}{4} + \frac{6}{4} = \frac{11}{4}$. $\frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.

a. In each example, explain how you could obtain the denominator for the sum without using a number line.

b. How could you obtain the numerator for the sum without using a number line?

7. Suppose that $\frac{a}{c}$ and $\frac{b}{c}$ represent two rational numbers. Notice that the fractions have the same denominator. What does $\frac{a+b}{c}$ represent?

8. Replace m, n, w, r, s, and t by rational numbers that make the resulting sentences true.

a. $\frac{6}{7} + \frac{0}{7} = m$.

b. $\frac{1}{4} + \frac{1}{4} = n$.

c. $\frac{5}{50} + \frac{16}{50} = w$.

d. $\frac{7}{100} + \frac{160}{100} = r$.

e. $\frac{3}{5} + \frac{5}{5} = s$.

f. $\frac{10}{200} + \frac{91}{200} = t$.

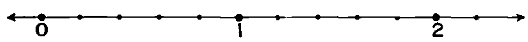
If you look back you will notice that the fractions for every pair of rational numbers you added thus far had the same denominator. When two fractions have the same denominator, we say they have a *common denominator*.

9. Suppose you wish to find the sum of $\frac{2}{5}$ and $\frac{1}{4}$.

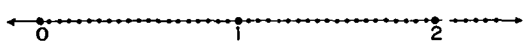
a. Do the fractions for the numbers have a common denominator? Can you find the sum by using the method suggested in exercise 7? Perhaps using a number line will work.

b. You need to find a point in a number line that corresponds to $\frac{2}{5} + \frac{1}{4}$. Can you find such a point if you divide each unit segment into 4 congruent subsegments?

If you divide each unit segment into 5 congruent subsegments?



20 congruent subsegments. Locate the points corresponding to $\frac{2}{5}$ and $\frac{1}{4}$ on this number line.



Does $\frac{2}{5}$ equal $\frac{8}{20}$? Does $\frac{1}{4}$ equal $\frac{5}{20}$? Locate the point that corresponds to $\frac{8}{20} + \frac{5}{20}$. Do you think it is true that $\frac{2}{5} + \frac{1}{4} = \frac{8}{20} + \frac{5}{20}$?

d. Does $\frac{2}{5} + \frac{1}{4} = \frac{13}{20}$? Explain.

10. If Joe drank $\frac{1}{5}$ of a bottle of pop and Bill drank $\frac{3}{4}$ of a bottle, how much pop did the boys drink?

a. Is $\frac{4}{20}$ of a bottle the same as $\frac{1}{5}$ of a bottle?

b. Is $\frac{15}{20}$ of a bottle the same as $\frac{3}{4}$ of a bottle?

c. Do you think it is true that $\frac{1}{5} + \frac{3}{4} = \frac{4}{20} + \frac{15}{20}$?

d. $\frac{4}{20} + \frac{15}{20} = \frac{?}{20}$.

e. What replacement for n makes the following sentence true?

$$\frac{1}{5} + \frac{3}{4} = n.$$

11. When finding the sum of two rational numbers that have fractions with different denominators, what should you do first? Add the numerators, or replace the given fractions with equivalent fractions that have a common denominator?

12. Find the sum of $\frac{1}{8}$ and $\frac{1}{9}$ by using equivalent fractions that have a common denominator.
    - a. Can 72 be used as a common denominator?
    - b. Can 144 be used as a common denominator?
    - c. Can 216 be used as a common denominator?

[redacted]

Finding the sum of two rational numbers is easy if the fractions for the numbers have a common denominator. Simply keep the denominator and add the numerators. If the fractions for the two rational numbers have different denominators, first find equivalent fractions with a common denominator. Then use the same method as before.

[redacted]

1. Replace w, k, m, t, c, and x by rational numbers that make the sentences true.

   a. $\frac{1}{2} + \frac{4}{2} = w.$ c. $\frac{2}{6} + \frac{11}{6} = m.$ e. $\frac{3}{47} + \frac{18}{47} = c.$

   b. $\frac{7}{3} + \frac{2}{3} = k.$ d. $\frac{8}{9} + \frac{1}{9} = t.$ f. $\frac{2}{500} + \frac{3}{500} = x.$

2. In each exercise find the simplest fraction for the sum. (Hint: First find two fractions that are equivalent to the given fractions and that have a common denominator.)

   a. $\frac{1}{8} + \frac{1}{5}$ c. $\frac{2}{7} + \frac{1}{6}$ e. $\frac{7}{80} + \frac{3}{50}$

   b. $\frac{2}{3} + \frac{3}{5}$ d. $\frac{3}{4} + \frac{1}{5}$ f. $\frac{2}{19} + \frac{4}{7}$

3. If Joe had painted $\frac{2}{5}$ of the fence and Bill had painted $\frac{1}{2}$ of it, how much of the fence would the boys have painted?

4. Could Joe have painted $\frac{3}{5}$ of the fence if Bill had painted $\frac{1}{2}$ of it? How much of the fence would this be?

5. After Joe had painted $\frac{2}{5}$ of the fence and Bill had painted $\frac{1}{2}$ of it, Bill took some time off. While Bill was gone, Joe painted another $\frac{1}{10}$ of the fence. Was the whole fence painted when Bill returned?

When the fence was finally painted, Bill and Joe rested. While resting, the boys decided to "brush up" on addition of rational numbers.

Class Discussion

1. Bill asked Joe to find the sum $\frac{2}{5} + \frac{3}{10}$.
   a. Can Joe use 50 as a common denominator?
   b. Is it true that $\frac{2}{5} + \frac{3}{10} = \frac{20}{50} + \frac{15}{50}$?
   c. Is it true that $\frac{2}{5} + \frac{3}{10} = \frac{35}{50}$?
   d. What is the simplest fraction for the rational number $\frac{35}{50}$?
2. Perhaps you can use another denominator to find the sum $\frac{2}{5} + \frac{3}{10}$.
   a. Replace a and b by whole numbers that make the following sentences true.

   $$\frac{2}{5} = \frac{a}{20}. \qquad \frac{3}{10} = \frac{b}{20}.$$

   b. How do you know that $\frac{2}{5} + \frac{3}{10} = \frac{14}{20}$?

c. What is the simplest fraction for the rational number $\frac{14}{20}$?

3. So far you have used two different common denominators to find the sum of $\frac{2}{5}$ and $\frac{3}{10}$. You thought of the sum as $\frac{20}{50} + \frac{15}{50}$ and as $\frac{8}{20} + \frac{6}{20}$. Did you get $\frac{7}{10}$ as the simplest fraction for the sum in each case?

4. Use 10 as a common denominator to find the sum of $\frac{2}{5}$ and $\frac{3}{10}$. Do you think 10 is the least number that can be used as a common denominator?

The least number that is a common denominator of two fractions is called the *least common denominator* of the two fractions. When adding two rational numbers it is convenient to use the least common denominator of the fractions that indicate the numbers.

5. See if you can discover a method of finding the least common denominator of the fractions that indicate two rational numbers.

   a. To add $\frac{2}{5}$ and $\frac{3}{10}$, you used each of the numbers 50, 20, and 10 as a common denominator. Must any number that is a common denominator of $\frac{2}{5}$ and $\frac{3}{10}$ have both 5 and 10 as factors?

   b. If the list below were continued indefinitely would it include every non-zero whole number that has 5 as a factor? That has 10 as a factor?

      5, 10, 15, 20, 25, 30 . . .

   c. Explain why the list below is the same as the list above. How would you continue this list?

      $5 \times 1, 5 \times 2, 5 \times 3, 5 \times 4, 5 \times 5, 5 \times 6, \ldots$

   d. In the list above, what is the least number that has both 5 as a factor and 10 as a factor? Is this number the least common denominator of $\frac{2}{5}$ and $\frac{3}{10}$?

6. Suppose you want to find fractions that are equivalent to $\frac{7}{12}$ and $\frac{5}{8}$, respectively, and you want the fractions you find to have the least common denominator.

   a. You know that any common denominator must have both 12 as a factor and 8 as a factor. Does the first list below include every non-zero whole number that has 12 as a factor? Does the second list include every non-zero whole number that has 8 as a factor?

      12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, . . .
      8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, . . .

   b. Start at the left and copy every number that is in both lists. Stop when you get to 96. Does each number you copied have both 12 as a factor and 8 as a factor?

   c. What is the least non-zero whole number that has both 12 as a factor and 8 as a factor? Using this number as the common denominator, find fractions that are equivalent to $\frac{7}{12}$ and $\frac{5}{8}$, respectively. Find the sum $\frac{7}{12} + \frac{5}{8}$.

7. Now use 8×12, or 96, as the common denominator to find the sum $\frac{7}{12} + \frac{5}{8}$. Do you get the same number that you got when you used the least common denominator? Explain your answer.

The least common denominator of two fractions is the least non-zero whole number that has the denominator of each fraction as a factor.

Using the least common denominator often shortens the work of adding two rational numbers. In the last two exercises, you probably found that it was easier to think of $\frac{7}{12} + \frac{5}{8}$ as $\frac{14}{24} + \frac{15}{24}$ than as $\frac{56}{96} + \frac{60}{96}$.

[redacted]

1. Find the sum of the two rational numbers in each exercise. First find equivalent fractions with the least common denominator.

   a. $\frac{5}{24}, \frac{1}{12}$ c. $\frac{1}{8}, \frac{1}{6}$ e. $\frac{1}{7}, \frac{3}{5}$

   b. $\frac{3}{7}, \frac{2}{14}$ d. $\frac{2}{9}, \frac{4}{3}$ f. $\frac{3}{4}, \frac{5}{15}$

2. In Class Discussion 5b you saw one way of finding the least common denominator of $\frac{7}{12}$ and $\frac{5}{8}$. Here is another way.

   a. List all whole numbers that are factors of 12.
   b. Then list all whole numbers that are factors of 8.
   c. Study the two lists. Is 4 the greatest common factor of 12 and 8?
   d. Write 12 and 8 as products of their greatest common factor and another whole number. Complete the sentences below.

   $12 = \_\_\_ \times 3. \qquad 8 = \_\_\_ \times 2.$

   e. Does $3 \times 4 \times 2$ represent the least common denominator of $\frac{7}{12}$ and $\frac{5}{8}$?
   f. Is $3 \times 4 \times 2$ another way to write 24?
   g. If you write $3 \times 4 \times 2$ as $(3 \times 4) \times 2$ you can see that 24 has 12 as a factor. How would you write $3 \times 4 \times 2$ to show that 24 has 8 as a factor?

3. To find the least common denominator of $\frac{7}{15}$ and $\frac{1}{6}$, follow the steps below.

   a. List all whole numbers that are factors of 15.
   b. List all whole numbers that are factors of 6.
   c. Look at the two lists of factors. Pick out the greatest common factor of 15 and 6.
   d. Write 15 and 6 as products of their greatest common factor

and another whole number. Replace a and b by whole numbers that make the following sentences true.

$$15 = 3 \times a. \qquad 6 = 3 \times b.$$

e. In the expression below, replace a and b by the same whole numbers as above.

$$a \times 3 \times b$$

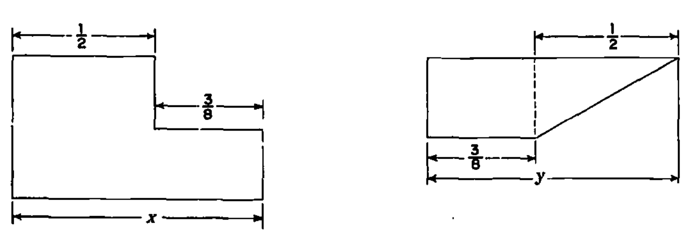
If the proper replacements are made for a and b, the resulting expression represents the least common denominator of $\frac{7}{15}$ and $\frac{1}{6}$.

4. Use the method suggested in exercise 3 to find the least common denominator of $\frac{1}{8}$ and $\frac{1}{10}$.

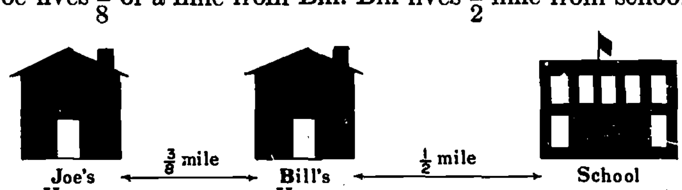
5. In each exercise find the simplest fraction for the sum.

a. $\frac{8}{8} + \frac{9}{10}$ b. $\frac{3}{10} + \frac{1}{2}$ c. $\frac{0}{5} + \frac{10}{5}$ d. $\frac{4}{15} + \frac{7}{20}$

6. Find the distance represented by x and the distance represented by y. Are the two distances equal?



7. On the way to school each day, Joe stops at Bill's house. From Bill's house, the two boys walk the rest of the way together. Joe lives $\frac{3}{8}$ of a mile from Bill. Bill lives $\frac{1}{2}$ mile from school.



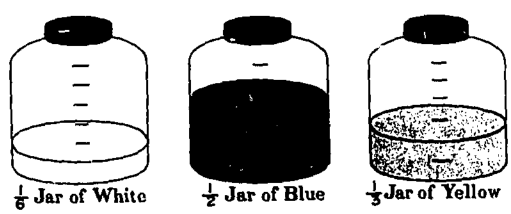
a. How far does Joe walk on the way to school? Find the sum $\frac{3}{8} + \frac{1}{2}$.

b. On the way home from school, Joe and Bill walk together as far as Bill's house. Then Joe walks the rest of the way home alone. Without doing any arithmetic, can you tell how far Joe walks on the way home? Find the sum $\frac{1}{2} + \frac{3}{8}$.

c. Does $\frac{3}{8} + \frac{1}{2} = \frac{1}{2} + \frac{3}{8}$?

d. Do you think the order in which two rational numbers are added affects the answer?

8. Jane and Betty needed some light green paint to decorate scenery for a school play. The teacher said they could get the right shade by mixing $\frac{1}{6}$ jar of white paint, $\frac{1}{2}$ jar of blue, and $\frac{1}{3}$ jar of yellow. The girls decided to mix two batches.



a. Jane mixed the first batch. She poured $\frac{1}{6}$ jar of white paint into an empty jar. With this she mixed $\frac{1}{2}$ jar of blue. Finally she added $\frac{1}{3}$ jar of yellow paint. Does the expression below indicate how Jane combined different amounts? Does this expression represent the total amount of paint Jane mixed? How much paint did Jane mix?

$$\left(\frac{1}{6} + \frac{1}{2}\right) + \frac{1}{3}$$

b. Betty mixed the second batch. First she poured $\frac{1}{6}$ jar of white into an empty jar and set it aside. Next she poured $\frac{1}{2}$ jar of blue into an empty jar and stirred in $\frac{1}{3}$ jar of yellow. Finally, Betty added the blue and yellow mixture to the white paint she had set aside. Look at the expression below. Does it indicate how Betty combined different amounts? Does this expression represent the total amount of paint Betty mixed? How much paint did Betty mix?

$$\frac{1}{6} + \left(\frac{1}{2} + \frac{1}{3}\right)$$

c. Did Betty mix the same amount of paint as Jane? Is the following sentence true?

$$\frac{1}{6} + \left(\frac{1}{2} + \frac{1}{3}\right) = \left(\frac{1}{6} + \frac{1}{2}\right) + \frac{1}{3}.$$

d. When you add three rational numbers, does the way that you group them affect the result?

Class Discussion

The recipe that Jane used to make fudge for her friend Bill is shown at the right. By now, all numbers used in the recipe should be familiar to you, except perhaps $2\frac{1}{3}$ (read "two and one-third"). What kind of number is represented by $2\frac{1}{3}$? Do you think it is a rational number? The exercises that follow will help you decide.

Divinity Fudge*

$2\frac{1}{3}$ c. granulated sugar
$\frac{2}{3}$ c. white corn syrup
$\frac{1}{2}$ c. water $\quad \frac{1}{4}$ tsp. salt
2 egg whites
1 c. chopped nuts
$\frac{1}{2}$ tsp. vanilla

* To obtain directions for combining ingredients, see *The Good Housekeeping Cook Book* (New York: Farrar & Rinehart, 1944), p. 812. This recipe is not endorsed by the National Council of Teachers of Mathematics, but it can be used to make good fudge.

1. a. Look at the number line below. What number corresponds to point A?

b. If your answer to the last question is $\frac{7}{3}$, you are correct.

c. Think of $\frac{7}{3}$ as the number obtained when two rational numbers are added. Show that the following sentence is true.

$$\frac{2}{1} + \frac{1}{3} = \frac{7}{3}.$$

d. Use the number line in exercise 1a to explain why the following sentence is true.

$$\frac{2}{1} + \frac{1}{3} = 2 + \frac{1}{3}.$$

e. Why is it true that $2 + \frac{1}{3} = \frac{7}{3}$?

f. For convenience in writing $2 + \frac{1}{3}$ is often shortened to $2\frac{1}{3}$. This means that $2\frac{1}{3} = \frac{7}{3}$ is a true sentence. Does this tell you that $2\frac{1}{3}$ represents a rational number? Explain your answer.

2. From exercise 1f you know that $3\frac{1}{4}$ is a short way of writing $3 + \frac{1}{4}$.

a. Is it true that $3 + \frac{1}{4} = \frac{3}{1} + \frac{1}{4}$? Explain your answer.

b. Show that $\frac{3}{1} + \frac{1}{4} = \frac{13}{4}$.

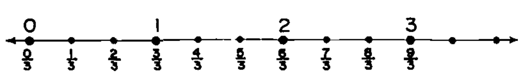
c. Is it true that $3\frac{1}{4} = \frac{13}{4}$?

3. Use the scheme suggested by exercise 2 to complete the sentence below.

$$2\frac{1}{6} = \frac{?}{6}.$$

4. Let us concentrate for a moment on the symbols that are used to represent numbers. A symbol like "4" is referred to as a *whole number numeral* and a symbol like "$\frac{2}{7}$" is referred to as a *fraction numeral.* Because of these facts, a symbol like "$4\frac{2}{7}$" is referred to as a *mixed numeral.* Do you see that there are three kinds of numerals, all of which may represent rational numbers? When a rational number can be represented either by a fraction numeral or by a mixed numeral, we will tell you if you are to use a mixed numeral.

5. Jane needed $\frac{8}{3}$ of a yard of fabric to make curtains for a window. Find a mixed numeral for $\frac{8}{3}$. Use the number line below. Notice that point $\frac{8}{3}$ is between point 2 and point 3.



a. What is the greatest whole number that is less than $\frac{8}{3}$?

b. Is it true that $\frac{8}{3} = \frac{6}{3} + \frac{2}{3}$?

c. Is it true that $\frac{6}{3} + \frac{2}{3} = 2 + \frac{2}{3}$? Explain your answer.

d. Give a mixed numeral for $\frac{8}{3}$.

6. Use the number line above to help you find a mixed numeral for $\frac{5}{3}$.

a. What is the greatest whole number that is less than $\frac{5}{3}$?

b. What replacement for x makes the following sentence true?

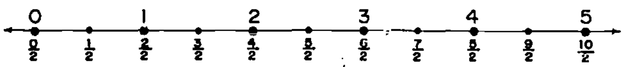
$$\frac{5}{3} = \frac{x}{3} + \frac{2}{3}.$$

c. What whole number is equal to $\frac{3}{3}$?

d. What is a mixed numeral for $\frac{5}{3}$?

7. Use the number line below to help you find a mixed numeral for each of the following:

a. $\frac{5}{2}$ **b.** $\frac{9}{2}$ **c.** $\frac{3}{2}$ **d.** $\frac{7}{2}$



8. Find a mixed numeral for $\frac{27}{6}$ without using a number line.

a. What whole number is equal to $\frac{6}{6}$; $\frac{12}{6}$; $\frac{18}{6}$?

b. What whole number is equal to $\frac{24}{6}$; $\frac{30}{6}$?

c. Is it true that $\frac{24}{6} < \frac{27}{6}$? Is it true that $\frac{27}{6} < \frac{30}{6}$?

d. $\frac{27}{6}$ is between what two whole numbers?

e. What replacement for x makes the following sentence true?

$$\frac{27}{6} = \frac{x}{6} + \frac{3}{6}.$$

f. Write a mixed numeral for $\frac{27}{6}$.

Numbers represented by *mixed numerals* are rational numbers. If a rational number is represented by a mixed numeral, it is always possible to find a fraction for the number.

Example: $2\frac{3}{4}$ can be rewritten as $2 + \frac{3}{4}$.

$$2 + \frac{3}{4} = \frac{2}{1} + \frac{3}{4}$$
$$= \frac{8}{4} + \frac{3}{4}$$
$$= \frac{11}{4}.$$

Therefore, $2\frac{3}{4} = \frac{11}{4}$.

Suppose a fraction for a rational number has a numerator that is equal to or greater than the denominator. Then the rational number either equals a whole number or is between two whole numbers.

Example: $\frac{5}{5} = 1$, and $\frac{10}{5} = 2$.

$\frac{7}{5}$ is between 1 and 2.

A rational number that is greater than 1 and is between two whole numbers can be represented by a mixed numeral.

Example: $\frac{7}{5} = \frac{5}{5} + \frac{2}{5}$.

$\frac{7}{5} = 1 + \frac{2}{5}$.

$1 + \frac{2}{5}$ can be rewritten as $1\frac{2}{5}$. Therefore, $\frac{7}{5} = 1\frac{2}{5}$.

1. Express each number using a mixed numeral.

   a. $\frac{8}{7}$ b. $\frac{23}{9}$ c. $\frac{14}{3}$ d. $\frac{50}{21}$ e. $\frac{31}{10}$

2. Replace r, s, t, v, w, x, y, z by whole numbers that make the sentences true.

a. $5\frac{2}{3} = \frac{x}{y}$. b. $7\frac{1}{8} = \frac{w}{z}$. c. $8\frac{1}{2} = \frac{r}{s}$. d. $6\frac{3}{7} = \frac{t}{v}$.

3. Suppose you want to find the sum of $2\frac{1}{6}$ and $1\frac{1}{3}$.

a. Explain why $2\frac{1}{6} + 1\frac{1}{3} = \frac{13}{6} + \frac{4}{3}$.

b. Find the simplest fraction for the sum $\frac{13}{6} + \frac{4}{3}$.

c. Write your answer to the last exercise, using a mixed numeral.

4. Perhaps you can think of another way to add $2\frac{1}{6}$ and $1\frac{1}{3}$.

a. Explain why $2\frac{1}{6} + 1\frac{1}{3} = \left(2 + \frac{1}{6}\right) + \left(1 + \frac{1}{3}\right)$.

b. Do you think it is true that

$$\left(2 + \frac{1}{6}\right) + \left(1 + \frac{1}{3}\right) = (2 + 1) + \left(\frac{1}{6} + \frac{1}{3}\right)?$$

Explain your answer.

c. Do you think it is true that

$$(2 + 1) + \left(\frac{1}{6} + \frac{1}{3}\right) = 3 + \frac{3}{6}?$$

Explain your answer.

d. Write a mixed numeral for $3 + \frac{3}{6}$.

5. a. Add $2\frac{2}{3}$ and $3\frac{1}{4}$ using the method suggested by exercise 4.

b. Do you get the same answer if you use the method of exercise 3?

6. Think of the sum $3\frac{1}{2} + 4\frac{2}{3}$ as $(3 + 4) + \left(\frac{1}{2} + \frac{2}{3}\right)$.

a. Show that $(3 + 4) + \left(\frac{1}{2} + \frac{2}{3}\right) = 7 + \frac{7}{6}$.

b. Do you think it is true that $7 + \frac{7}{6} = 7 + 1\frac{1}{6}$?

c. Do you think $7 + 1\frac{1}{6} = 8\frac{1}{6}$? Explain your answer.

d. $3\frac{1}{2} + 4\frac{2}{3} =$ ________.

7. Each sentence below is true. Explain why in each case.

a. $2\frac{3}{2} = 3\frac{1}{2}$.

b. $8\frac{9}{6} = 9\frac{3}{6}$.

c. $4\frac{3}{3} = 5$.

d. $6\frac{32}{17} = 7\frac{15}{17}$.

8. Each sum expressed below is between two whole numbers. What are the two whole numbers in each case?

a. $2\frac{3}{4} + 1\frac{1}{8}$.

b. $1\frac{5}{7} + \frac{6}{21}$.

c. $4\frac{2}{3} + 2\frac{1}{5}$.

d. $3\frac{4}{5} + 3\frac{4}{5}$.

Subtracting Rational Numbers

Jane wanted to make some pizza, but she had mislaid the recipe. So, in making the pizza mixture Jane used what she thought was the proper amount of each ingredient. Before she finished making the mixture, Jane found the stray recipe and discovered that she had put in $\frac{2}{3}$ of a cup of flour instead of $\frac{3}{4}$ of a cup. Jane knew that $\frac{2}{3}$ of a cup is less than $\frac{3}{4}$ of a cup. So she knew that she had to add more flour, but she did not know how much.

Class Discussion

1. Jane needs to add enough flour to a cup that is $\frac{2}{3}$ full to get a cup that is $\frac{3}{4}$ full. If you use n to represent the amount of flour to be added, does the sentence below express this idea?

$$\frac{2}{3} + n = \frac{3}{4}.$$

2. Jane thought of a way to measure the amount of flour that she needed to add. She filled $\frac{3}{4}$ of a measuring cup with flour and then poured enough of it out to fill $\frac{2}{3}$ of another cup. If you use n to represent the amount that was left in the first cup, does the following sentence represent what Jane did?

$$\frac{3}{4} - \frac{2}{3} = n.$$

3. Study the two mathematical sentences in exercises 1 and 2. How do you know that the same number should make both sentences true?

4. Practically, Jane's problem is solved. If she poured $\frac{2}{3}$ of a cup out of $\frac{3}{4}$ of a cup, the flour that was left is the amount she had to add to the pizza mixture. But what part of a cup did Jane need to add? You can find the answer to this question by solving either $\frac{2}{3} + n = \frac{3}{4}$ or $\frac{3}{4} - \frac{2}{3} = n$.

   a. Look at the first sentence. Why can you rewrite $\frac{2}{3} + n = \frac{3}{4}$ as $\frac{8}{12} + n = \frac{9}{12}$?

   b. How do you know that $\frac{1}{12}$ of a cup of flour is the amount Jane needed to add to the pizza mixture?

   c. What number makes $\frac{2}{3} + n = \frac{3}{4}$ true? What number makes $\frac{3}{4} - \frac{2}{3} = n$ true?

5. In a sum like $\frac{2}{3} + \frac{1}{12}$, the numbers $\frac{2}{3}$ and $\frac{1}{12}$ are referred to as *addends*. The "missing addend" in $\frac{2}{3} + n = \frac{3}{4}$ is the same as the

difference $\frac{3}{4} - \frac{2}{3}$. In other words, if you find the missing addend in $\frac{2}{3} + n = \frac{3}{4}$, you are at the same time solving the subtraction sentence $\frac{3}{4} - \frac{2}{3} = n$.

a. Is this similar to finding the difference of two whole numbers?

b. If you find the missing addend in $12 + n = 27$, what subtraction sentence are you solving?

6. For each addition sentence, write a subtraction sentence that has the same solution. Then give the number that makes both sentences true.

a. $\frac{4}{7} + n = \frac{7}{7}$.

b. $n + \frac{2}{3} = \frac{5}{3}$.

c. $n + \frac{2}{5} = \frac{11}{5}$.

d. $\frac{8}{9} + n = \frac{18}{9}$.

7. a. Find the simplest fraction for the difference $\frac{7}{7} - \frac{4}{7}$. How is the denominator of your answer related to the denominators of $\frac{7}{7}$ and $\frac{4}{7}$? Is the numerator equal to $7 - 4$?

b. What is the simplest fraction for the difference $\frac{4}{3} - \frac{2}{3}$? How is the denominator of your answer related to the denominators of $\frac{4}{3}$ and $\frac{2}{3}$? How is the numerator of your answer related to the numerators of $\frac{4}{3}$ and $\frac{2}{3}$?

c. Explain how you could use the numerators and denominators of $\frac{11}{5}$ and $\frac{2}{5}$ to find the numerator and denominator of a fraction for the difference $\frac{11}{5} - \frac{2}{5}$. Do the same for $\frac{18}{9} - \frac{8}{9}$.

8. Suppose $\frac{a}{b}$ and $\frac{c}{b}$ represent two rational numbers that are indicated by fractions with a common denominator. Suppose also that a is greater than or equal to c. (The case in which a

is less than c is not considered in this unit because negative numbers are not considered.)

a. Express the difference $\frac{a}{b} - \frac{c}{b}$ using a single fraction.

b. If a is greater than or equal to c, is subtraction always possible?

9. In Jane's subtraction problem, the fractions $\frac{3}{4}$ and $\frac{2}{3}$ did not have a common denominator. What did Jane need to do before she could find the difference $\frac{3}{4} - \frac{2}{3}$?

10. Rewrite the sentence $\frac{7}{8} - \frac{3}{4} = n$, using fractions with a common denominator. Then find a replacement for n that makes the sentence true. Give the simplest fraction.

11. Rewrite each sentence below, using fractions that have a common denominator. Then find the number that makes the sentence true.

a. $\frac{2}{3} - \frac{5}{12} = t.$ b. $\frac{4}{3} - \frac{8}{7} = k.$

Subtraction of rational numbers is related to addition, just as subtraction of whole numbers is related to addition. In each case, finding a difference is the same as finding a missing addend.

To subtract one rational number from another when the fractions for the numbers have a common denominator, you can use the mathematical sentence below.

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}.$$

The sentence can also be used if the fractions for the rational numbers have unequal denominators. However, in this case you must first replace the given fractions with equivalent fractions that have a common denominator.

[redacted]

1. For each sentence find a replacement for n that makes the sentence true.

a. $\frac{7}{10} - \frac{1}{10} = n.$

b. $n - \frac{4}{9} = \frac{9}{9}.$

c. $\frac{7}{6} - \frac{3}{6} = n.$

d. $n + \frac{5}{6} = \frac{15}{12}.$

e. $\frac{10}{12} - \frac{5}{6} = n.$

f. $\frac{7}{8} - \frac{2}{3} = n.$

g. $\frac{4}{7} - \frac{1}{3} = n.$

h. $\frac{3}{50} - \frac{1}{100} = n.$

2. Each sentence below is a step in finding a mixed numeral to express the difference $2\frac{1}{3} - 1\frac{1}{4}$. Explain why each sentence is true.

a. $2\frac{1}{3} - 1\frac{1}{4} = \frac{7}{3} - \frac{5}{4}.$

b. $\frac{7}{3} - \frac{5}{4} = \frac{28}{12} - \frac{15}{12}.$

c. $\frac{28}{12} - \frac{15}{12} = \frac{13}{12}.$

d. $2\frac{1}{3} - 1\frac{1}{4} = 1\frac{1}{12}.$

3. For each exercise use steps like those in exercise 2 to find a replacement for n that makes the sentence true. If an answer is greater than 1 and is between two whole numbers, express the answer with a mixed numeral.

a. $3\frac{2}{3} - 1\frac{1}{4} = n.$

b. $2\frac{5}{8} - 2 = n.$

c. $3 - \frac{11}{4} = n.$

d. $4\frac{1}{3} - \frac{2}{3} = n.$

e. $3\frac{1}{5} - 2\frac{2}{3} = n.$

f. $2\frac{2}{7} - \frac{5}{14} = n.$

4. Perhaps you can think of shortcuts for finding some of the answers in exercise 3. Two methods of finding a mixed numeral for the difference of $15\frac{1}{4}$ and $12\frac{3}{4}$ are shown below. Does method A give you the same answer as method B? Which method do you prefer?

METHOD A

$$15\frac{1}{4} - 12\frac{3}{4} = \frac{61}{4} - \frac{51}{4}$$
$$= \frac{10}{4}$$
$$= 2\frac{2}{4}.$$

METHOD B

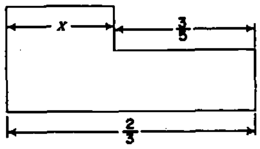
$$15\frac{1}{4} - 12\frac{3}{4} = 14\frac{5}{4} - 12\frac{3}{4}$$
$$= (14 - 12) + \left(\frac{5}{4} - \frac{3}{4}\right)$$
$$= 2 + \frac{2}{4}.$$
$$= 2\frac{2}{4}.$$

5. In the sentence below find the number n that makes the sentence true. First use method A. Then use method B.

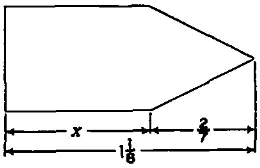
$$21\frac{3}{5} - 14\frac{4}{5} = n.$$

6. In each diagram find the distance represented by x.

a.



b.

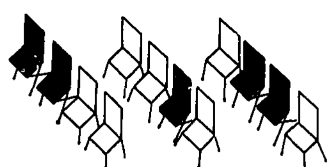


7. Which weighs more, $\frac{3}{4}$ of a pound of chocolates or $\frac{5}{8}$ of a pound of peppermints? How much more?

8. Bill's father thought his driveway was $60\frac{3}{10}$ feet long. When he measured it, he found that it was $59\frac{7}{10}$ feet long. The driveway was how much shorter than Bill's father thought it was?

Multiplying Rational Numbers

Joe and Bill set up 12 chairs for a club meeting. They made 3 rows and put 4 chairs in each row. Among the chairs were 5 metal chairs and 7 wooden chairs.



Bill remarked that each row was part of the whole group of chairs and that part of each row was metal. He wondered if there were some relationship between a part of a row and a part of the whole group of chairs. In other words, he wondered if he could find "a part of a part of a whole." Bill was really hinting at multiplying rational numbers.

Class Discussion

1. The picture above shows the arrangement of the metal chairs and the wooden chairs.
   a. Why can Bill say that the two metal chairs are $\frac{1}{2}$ of row 1?
   b. The four chairs in row 1 are what part of the whole group of chairs?
   c. Why can Bill say that the metal chairs in row 1 are $\frac{1}{2}$ of $\frac{1}{3}$ of the whole group of chairs?
   d. Explain why 2 metal chairs, in a group of 12 chairs, are $\frac{1}{6}$ of the whole group of chairs.
   e. Is $\frac{1}{2}$ of $\frac{1}{3}$ the same as $\frac{1}{6}$?

f. The expression $\frac{1}{2}$ of $\frac{1}{3}$ means the same as the product $\frac{1}{2} \times \frac{1}{3}$.
Does $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$?

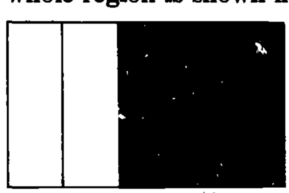
2. a. In row 2, how many chairs are wooden? What part of the row is this?

b. Does row 2 make up $\frac{1}{3}$ of the whole group of chairs? Can we say that the wooden chairs in row 2 are $\frac{3}{4}$ of $\frac{1}{3}$ of the whole group of chairs?

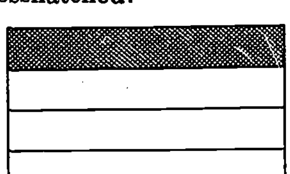
c. Write a product that means the same thing as $\frac{3}{4}$ of $\frac{1}{3}$.

d. From the picture, how can you tell that $\frac{3}{4}$ of $\frac{1}{3}$ is the same as $\frac{3}{12}$? Write a mathematical sentence that expresses this idea.

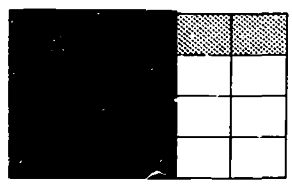
3. a. The diagram below shows a whole region divided into five congruent subregions, three of which are shown in color. What part of the whole region is shown in color?



b. The diagram below shows the same whole region that is shown in the diagram above. But this time the whole region is divided into four congruent subregions. What part of the whole region is crosshatched?



c. If the two diagrams above are combined, the resulting diagram looks like this:



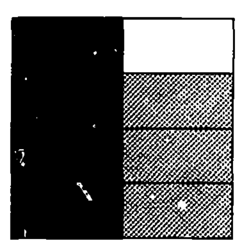
Can you say that $\frac{1}{4}$ of the colored part is crosshatched? Is this $\frac{1}{4}$ of $\frac{3}{5}$ of the whole region?

d. Use the above diagram to explain why $\frac{1}{4}$ of $\frac{3}{5}$ is the same as $\frac{3}{20}$.

e. Write a sentence containing a product which tells you that $\frac{1}{4}$ of $\frac{3}{5}$ is the same as $\frac{3}{20}$.

f. Look again at the diagram in exercise 3c. What part of the region is crosshatched? What part of the crosshatched region is colored? Is $\frac{3}{5}$ of $\frac{1}{4}$ of the whole region both colored and crosshatched? Does $\frac{3}{5}$ of $\frac{1}{4}$ of the region appear to be the same as $\frac{1}{4}$ of $\frac{3}{5}$ of the region? Does $\frac{3}{5} \times \frac{1}{4} = \frac{1}{4} \times \frac{3}{5}$? Do you think the order in which you multiply two rational numbers affects the answer?

4. The sentences below refer to the diagram at the right. Decide which sentences are true and which are false.



a. $\frac{1}{2}$ of the whole region is shown in color.

b. $\frac{3}{4}$ of the whole region is crosshatched.

c. $\frac{3}{4}$ of the portion shown in color is crosshatched.

d. $\frac{3}{4}$ of $\frac{1}{2}$ of the whole region is crosshatched and shown in color.

e. $\frac{3}{8}$ of the whole region is crosshatched and shown in color.

f. $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$.

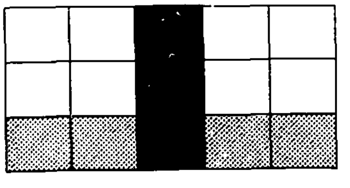
g. $\frac{1}{2}$ of the crosshatched region is shown in color.

h. $\frac{1}{2}$ of $\frac{3}{4}$ of the whole region is crosshatched and shown in color.

i. $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

j. $\frac{3}{4} \times \frac{1}{2} = \frac{1}{2} \times \frac{3}{4}$.

5. Make use of the diagram at the right to complete each sentence below.



a. ____ of the whole region is shown in color.

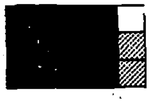
b. ____ of the whole region is crosshatched.

c. ____ of the whole region is crosshatched and shown in color.

d. $\frac{1}{3} \times \frac{1}{5} =$ ____.

6. In each exercise use the diagrams to complete the sentences.

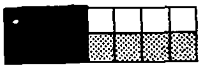
a.



$\frac{2}{3} \times \frac{3}{5} =$ ____.

$\frac{3}{5} \times \frac{2}{3} =$ ____.

b.



$\frac{1}{2} \times \frac{3}{7} =$ ____.

$\frac{3}{7} \times \frac{1}{2} =$ ____.

7. In exercise 1, you found that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. What is the product of the numerators of $\frac{1}{2}$ and $\frac{1}{3}$? What is the product of the denominators of $\frac{1}{2}$ and $\frac{1}{3}$?

8. In exercise 2, you found that $\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$. What is the product of the numerators of $\frac{3}{4}$ and $\frac{1}{3}$? What is the product of the denominators of $\frac{3}{4}$ and $\frac{1}{3}$?

9. In exercise 3 you found that $\frac{1}{4}$ of $\frac{3}{5}$ is $\frac{3}{20}$. How can you use the numerators of $\frac{1}{4}$ and $\frac{3}{5}$ to get the numerator of $\frac{3}{20}$? How can you use the denominators of $\frac{1}{4}$ and $\frac{3}{5}$ to get the denominator of $\frac{3}{20}$?

10. Use the method suggested in exercises 7, 8, and 9 to multiply the rational numbers in each exercise. Do your answers agree with those you found in exercises 5 and 6?

 a. $\frac{1}{3} \times \frac{1}{5}$ b. $\frac{2}{3} \times \frac{3}{5}$ c. $\frac{1}{2} \times \frac{3}{7}$

Finding "a part of a part of a whole" is the same as finding the product of two rational numbers. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers. To find the product, multiply the numerators and multiply the denominators. That is,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

1. In each exercise, replace n by a number that makes the sentence true.

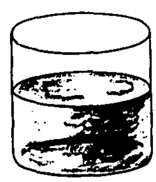
   a. $\frac{4}{5} \times \frac{7}{9} = n.$ c. $\frac{3}{1} \times \frac{6}{1} = n.$ e. $\frac{9}{7} \times \frac{3}{4} = n.$

   b. $\frac{2}{1} \times \frac{1}{7} = n.$ d. $\frac{5}{2} \times \frac{0}{2} = n.$ f. $\frac{1}{3} \times \frac{1}{3} = n.$

2. Sarah wants to bake a cake that will be one-half the size of the cake she would get if she followed a certain recipe. The recipe calls for $\frac{3}{4}$ of a cup of sugar. How much sugar should Sarah use?

3. If a metal bar weighs $\frac{7}{8}$ of a pound, what does $\frac{1}{3}$ of the bar weigh?

4. A glass container holds $\frac{3}{4}$ of a gallon of water when it is completely filled.



   What part of a gallon does the container hold when it is half full?

5. Peter had $\frac{3}{5}$ of a dollar. He spent $\frac{3}{4}$ of this amount. What part of a dollar did Peter have left? How many cents did Peter have left?

6. a. Find the product $\frac{1}{2} \times \frac{3}{5}$. What is $\frac{3}{4}$ of the result? What replacement for n makes the following sentence true?

$$\frac{3}{4} \times \left(\frac{1}{2} \times \frac{3}{5}\right) = n.$$

b. Find the product $\frac{3}{4} \times \frac{1}{2}$. Find the product of this answer and $\frac{3}{5}$. What replacement for p makes the following sentence true?

$$\left(\frac{3}{4} \times \frac{1}{2}\right) \times \frac{3}{5} = p.$$

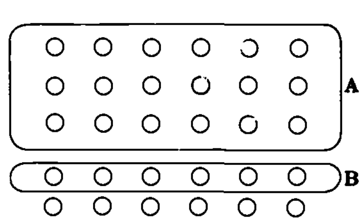
c. Do you think it is true that

$$\frac{3}{4} \times \left(\frac{1}{2} \times \frac{3}{5}\right) = \left(\frac{3}{4} \times \frac{1}{2}\right) \times \frac{3}{5}\ ?$$

d. Do you think the way in which three rational numbers are grouped when multiplying affects the result?

7. Bill and Joe were playing marbles. Bill started out with 30 marbles. In the first game Bill lost $\frac{3}{5}$ of the 30 marbles. In the second game he lost $\frac{1}{5}$ of the 30 marbles.

a. Make a diagram like the one below.



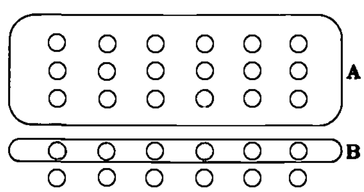
Bill's Marbles

b. Which group represents the marbles Bill lost in the first game? Which group represents the marbles Bill lost in the second game? What part of the 30 marbles did Bill lose in the two games? Find the sum $\frac{3}{5} + \frac{1}{5}$.

c. Joe was a good sport and agreed to give back $\frac{1}{3}$ of the marbles Bill lost. Draw a line around a group that represents $\frac{1}{3}$ of the marbles Bill lost. What part of Bill's 30 marbles did Joe give back?

$$\frac{1}{3} \times \left(\frac{3}{5} + \frac{1}{5}\right) = \_\_\_.$$

d. Now suppose Joe had agreed ahead of time to give back $\frac{1}{3}$ of the marbles Bill might lose in any game. Make a diagram like the one below.



Bill's Marbles

In the first game Bill lost $\frac{3}{5}$ of the 30 marbles. So Joe gave back $\frac{1}{3}$ of the marbles Bill lost. Circle a group that shows the marbles Joe gave back. What part of the 30 marbles did Joe give back?

$$\frac{1}{3} \times \frac{3}{5} = \_\_\_.$$

e. In the second game Bill lost $\frac{1}{5}$ of the 30 marbles. Circle a group that shows the marbles Joe gave back after the second game. What part of the 30 marbles did Joe give back after the second game?

$$\frac{1}{3} \times \frac{1}{5} = \_\_\_.$$

f. What part of the marbles did Joe give back altogether?

$$\left(\frac{1}{3} \times \frac{3}{5}\right) + \left(\frac{1}{3} \times \frac{1}{5}\right) = \_\_\_.$$

g. How do your answers to exercises 7c and 7f compare? Use the diagrams you made in exercises 7a and 7d to explain why the following sentence is true.

$$\frac{1}{3} \times \left(\frac{3}{5} + \frac{1}{5}\right) = \left(\frac{1}{3} \times \frac{3}{5}\right) + \left(\frac{1}{3} \times \frac{1}{5}\right).$$

8. a. Find the simplest fraction for

$$\left(\frac{3}{4} \times \frac{5}{8}\right) + \left(\frac{3}{4} \times \frac{3}{8}\right).$$

Multiply inside the parentheses first; then add the products.

b. Find the simplest fraction for

$$\frac{3}{4} \times \left(\frac{5}{8} + \frac{3}{8}\right).$$

This time add inside the parentheses first; then multiply by the number outside.

c. Compare the results for exercises 8a and 8b.

9. Find the simplest fraction for

$$\left(\frac{4}{5} \times \frac{3}{7}\right) + \left(\frac{4}{5} \times \frac{4}{7}\right).$$

using two different methods.

Class Discussion

The exercises below illustrate different ways of finding the simplest fraction for the product when multiplying two rational numbers.

1. You know that $\frac{4}{15} \times \frac{7}{2}$ equals $\frac{4 \times 7}{15 \times 2}$. Divide one factor of the numerator and one factor of the denominator by 2. Below is a short way of writing this.

$$\frac{\overset{2}{\cancel{4}} \times 7}{15 \times \underset{1}{\cancel{2}}}$$

2. Do you think $\frac{\overset{2}{\cancel{4}} \times 7}{15 \times \underset{1}{\cancel{2}}}$ equals $\frac{2 \times 7}{15 \times 1}$?

3. Does $\frac{2 \times 7}{15 \times 1} = \frac{14}{15}$?

4. Is $\frac{14}{15}$ the simplest fraction for the product $\frac{4}{15} \times \frac{7}{2}$?

5. Explain another way of finding the simplest fraction for the product $\frac{4}{15} \times \frac{7}{2}$.

6. You know that $\frac{3}{4} \times \frac{8}{15}$ equals $\frac{3 \times 8}{4 \times 15}$, which in turn equals $\frac{24}{60}$.

a. Is $\frac{24}{60}$ the simplest fraction for the product $\frac{3}{4} \times \frac{8}{15}$?

b. What fraction do you get when you divide both the numerator and denominator of $\frac{24}{60}$ by 4?

7. Is there a simpler fraction for the product $\frac{3}{4} \times \frac{8}{15}$ than the one you got in exercise 6b? Study A, B, and C below.

A	B	C
$\frac{3 \times 8}{4 \times 15}$	$\frac{3 \times \overset{2}{\cancel{8}}}{\underset{1}{\cancel{4}} \times 15}$	$\frac{\overset{1}{\cancel{3}} \times \overset{2}{\cancel{8}}}{\underset{1}{\cancel{4}} \times \underset{5}{\cancel{15}}}$

a. Does method B show that one factor of the numerator has been divided by 4 and that one factor of the denominator has been divided by 4?

b. What does method C show?

c. What is another way of writing $\frac{1 \times 2}{1 \times 5}$? Is $\frac{2}{5}$ the simplest fraction for the product $\frac{3}{4} \times \frac{8}{15}$?

8. Each sentence below represents a step in finding the simplest fraction for the product $\frac{2}{6} \times \frac{4}{5}$. Explain why each sentence is true.

a. $\frac{2}{6} \times \frac{4}{5} = \frac{2 \times 4}{6 \times 5}.$

b. $\frac{2 \times 4}{6 \times 5} = \frac{\overset{1}{\cancel{2}} \times 4}{\underset{3}{\cancel{6}} \times 5}.$

c. $\frac{\overset{1}{\cancel{2}} \times 4}{\underset{3}{\cancel{6}} \times 5} = \frac{4}{15}.$

9. Use the work shown below to explain why $\frac{6}{25} \times \frac{15}{16} = \frac{9}{40}.$

$$\frac{\overset{3}{\cancel{6}} \times \overset{3}{\cancel{15}}}{\underset{5}{\cancel{25}} \times \underset{8}{\cancel{16}}} = \frac{9}{40}.$$

Is $\frac{9}{40}$ the simplest fraction for the product $\frac{6}{25} \times \frac{15}{16}$?

10. Use the work shown below to explain why $\frac{8}{14} \times \frac{9}{21} = \frac{12}{49}.$

$$\frac{\overset{4}{\cancel{8}} \times \overset{3}{\cancel{9}}}{\underset{7}{\cancel{14}} \times \underset{7}{\cancel{21}}} = \frac{12}{49}.$$

Is $\frac{12}{49}$ the simplest fraction for the product $\frac{8}{14} \times \frac{9}{21}$?

[redacted]

You have seen two ways to get the simplest fraction for the product of two rational numbers. One way involves dividing both the numerator and denominator by common factors when the numerator and denominator are in factored form. The other way involves multiplying first, and then dividing both the numerator and the denominator by their greatest common factor.

[redacted]

1. For each exercise, find the simplest fraction for the rational number that makes the sentence true.

   a. $\frac{6}{12} \times \frac{25}{5} = n.$ **d.** $\frac{4}{21} \times \frac{14}{5} = n.$ **g.** $\frac{2}{7} \times \frac{3}{3} = n.$

   b. $\frac{10}{40} \times \frac{3}{12} = n.$ **e.** $\frac{28}{15} \times \frac{10}{21} = n.$ **h.** $\frac{5}{5} \times \frac{7}{9} = n.$

   c. $\frac{5}{9} \times \frac{6}{5} = n.$ **f.** $\frac{12}{15} \times \frac{21}{32} = n.$

2. **a.** $\frac{3}{3}$ equals what whole number?

   b. $\frac{5}{5}$ equals what whole number?

   c. Is it true that $\frac{2}{7} \times \frac{3}{3} = \frac{2}{7}$?

   d. Is it true that $\frac{5}{5} \times \frac{7}{9} = \frac{7}{9}$?

   e. What number n makes the following sentence true?

$$\frac{11}{17} \times n = \frac{11}{17}.$$

3. For each exercise, find the rational number that makes the sentence true. (Hint: First rewrite each sentence using fraction numerals.)

   a. $2\frac{1}{2} \times 5 = n.$ **c.** $\frac{5}{9} \times 3 = m.$

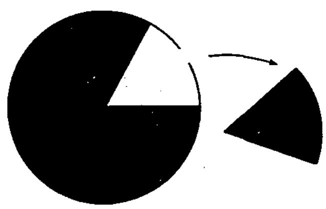
   b. $4\frac{1}{2} \times 3\frac{2}{3} = w.$ **d.** $7\frac{3}{6} \times 2\frac{2}{3} = k.$

4. Mrs. Swenson had $1\frac{3}{4}$ pounds of butter in her refrigerator. She used $\frac{1}{2}$ of the butter to make cookies. How much butter did she use?

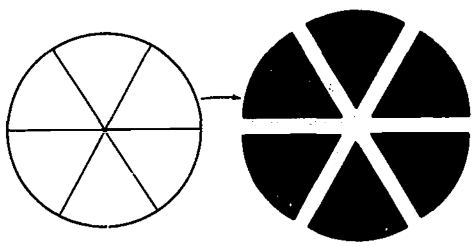
5. A plane travels at the rate of 400 miles per hour. How far does the plane travel in $2\frac{3}{4}$ hours?

Class Discussion

Jane made a pie and cut it into 6 pieces of the same size. She gave Joe one piece.



The pie was so delicious that Joe asked for another piece, then another, and another, until he had eaten the whole pie.



1. a. One piece of pie was what part of the whole pie?
   b. How many pieces did Joe eat altogether?
   c. Does $6 \times \frac{1}{6}$ express the amount of pie that Joe ate?
   d. Is $\frac{6}{1} \times \frac{1}{6}$ another way to write $6 \times \frac{1}{6}$?
   e. What is the simplest fraction for the product $\frac{6}{1} \times \frac{1}{6}$?
   f. What whole number equals $\frac{6}{1} \times \frac{1}{6}$?
2. a. Suppose each piece of pie had been $\frac{1}{5}$ of the whole pie. How many pieces would Joe have had to eat to finish the whole pie?

b. What number times $\frac{1}{5}$ equals $\frac{5}{5}$?

c. Does $\frac{5}{5} = 1$?

3. For each exercise, find the number n that makes the sentence true.

a. $\frac{5}{8} \times \frac{8}{5} = n.$ **c.** $\frac{22}{6} \times \frac{6}{22} = n.$

b. $\frac{11}{2} \times \frac{2}{11} = n.$ **d.** $4 \times \frac{1}{4} = n.$

4. Did you obtain 1 for each answer in exercise 3?

5. If the product of two numbers is 1, we say that each number is the *multiplicative inverse* of the other. For example, the multiplicative inverse of 6 is $\frac{1}{6}$, and the multiplicative inverse of $\frac{5}{8}$ is $\frac{8}{5}$. What is the multiplicative inverse of each of the following?

a. $\frac{11}{2}$ **b.** $\frac{22}{6}$ **c.** 4 **d.** $\frac{2}{7}$ **e.** $\frac{7}{2}$

6. Is there a replacement for n that makes the following sentence true?

$$\frac{0}{5} \times n = 1.$$

a. Does $\frac{0}{5}$ have a multiplicative inverse?

b. Do any of the numbers listed below have a multiplicative inverse? Are all these numbers equal?

$$\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \frac{0}{5}, \frac{0}{6}, \frac{0}{7}, \ldots$$

c. Is each number in the list equal to the whole number 0?

[redacted]

Two numbers are multiplicative inverses of each other if their product is 1. If $\frac{a}{b}$ represents a rational number and a is not 0, then

$\frac{b}{a}$ represents the multiplicative inverse of $\frac{a}{b}$. If $a = 0$, then $\frac{a}{b}$ does not have a multiplicative inverse.

1. Use the diagram below to explain why $8 \times \frac{1}{8} = 1$.

2. Make a diagram that can be used to show that $10 \times \frac{1}{10} = 1$.
3. By what number must each of the following be multiplied to obtain a product of 1?

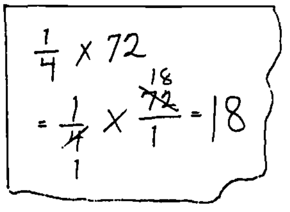
   a. $\frac{2}{5}$ c. 2 e. 1 g. $\frac{7}{4}$ i. $1\frac{1}{2}$

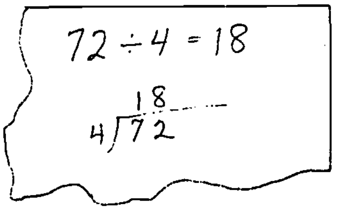
   b. $\frac{3}{4}$ d. 5 f. $\frac{5}{6}$ h. $\frac{3}{10}$
4. What is the multiplicative inverse of each of the following numbers?

   a. $\frac{5}{2}$ b. $\frac{4}{3}$ c. $\frac{1}{2}$ d. 1 e. 3 f. $3\frac{2}{3}$
5. Jeffrey had \$10. He gave $\frac{1}{10}$ of his money to charity. How much money did Jeffrey give to charity?

Dividing Rational Numbers

Bill and Joe had a strip of wood that was 72 inches long. They wanted to saw the strip into 4 pieces of the same length. Each boy made his own computation to find out how long each of the 4 pieces should be. Joe multiplied to find the answer, and Bill divided to find the answer. Their work is shown below. Notice that both boys got the same answer. Do you think every division problem can be solved by multiplying?





Class Discussion

1. a. 10 ÷ 2 equals what number?
   b. Now multiply 10 by 3 and also multiply 2 by 3. What answer do you get when you divide 30 by 6?
   c. Does (10 × 3) ÷ (2 × 3) equal the same number as 10 ÷ 2?
2. In the exercises below, the parentheses indicate that you should multiply first and then divide. What answer do you get in each case?
   a. (10 × 2) ÷ (2 × 2) = ____.
   b. (10 × 4) ÷ (2 × 4) = ____.
   c. (10 × 7) ÷ (2 × 7) = ____.
   d. (10 × 10) ÷ (2 × 10) = ____.
   e. (10 × 100) ÷ (2 × 100) = ____.
3. Do all exercises below have the same answer?
   a. 6 ÷ 3 = ____.
   b. (6 × 5) ÷ (3 × 5) = ____.
   c. (6 × 2) ÷ (3 × 2) = ____.
   d. (6 × 4) ÷ (3 × 4) = ____.

e. $(6 \times \frac{1}{3}) \div (3 \times \frac{1}{3}) =$ ____.

f. $(6 \times \frac{1}{2}) \div (3 \times \frac{1}{2}) =$ ____.

4. In the mathematical sentence $6 \div 3 = 2$, the number 6 is referred to as the *dividend;* 3 is referred to as the *divisor;* and 2 is called the *quotient.* If you multiply the dividend and the divisor by the same number, what effect does this have on the quotient?

5. Complete each sentence.

a. $7 \div 1 =$ ____.
b. $100 \div 1 =$ ____.
c. $43 \div 1 =$ ____.
d. $26 \div 1 =$ ____.
e. $376 \div 1 =$ ____.
f. $541 \div 1 =$ ____.

6. How does the quotient compare with the dividend whenever you divide a whole number by 1?

7. Do you think each of the following sentences is true?

a. $\frac{2}{3} \div 1 = \frac{2}{3}$.
b. $\frac{10}{7} \div 1 = \frac{10}{7}$.
c. $1\frac{1}{4} \div 1 = 1\frac{1}{4}$.
d. $\frac{7}{8} \div 1 = \frac{7}{8}$.

8. If you divide any rational number by 1, do you think the quotient will always be equal to the dividend?

9. Each sentence below is a step in finding the quotient of $\frac{2}{3}$ divided by $\frac{3}{4}$. Explain why each sentence is true.

a. $\frac{2}{3} \div \frac{3}{4} = \left(\frac{2}{3} \times \frac{4}{3}\right) \div \left(\frac{3}{4} \times \frac{4}{3}\right)$.

b. $\frac{2}{3} \div \frac{3}{4} = \left(\frac{2}{3} \times \frac{4}{3}\right) \div 1$.

c. $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$.

d. $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

10. Step c above shows that the quotient $\frac{2}{3} \div \frac{3}{4}$ equals the product $\frac{2}{3} \times \frac{4}{3}$. How are the numbers $\frac{3}{4}$ and $\frac{4}{3}$ related?

11. Explain why each sentence below is true.

a. $\frac{4}{7} \div \frac{3}{8} = \left(\frac{4}{7} \times \frac{8}{3}\right) \div \left(\frac{3}{8} \times \frac{8}{3}\right)$.

b. $\frac{4}{7} \div \frac{3}{8} = \left(\frac{4}{7} \times \frac{8}{3}\right) \div 1$.

c. $\frac{4}{7} \div \frac{3}{8} = \frac{4}{7} \times \frac{8}{3}$.

d. $\frac{4}{7} \div \frac{3}{8} = \frac{32}{21}$.

12. To find the quotient of $\frac{4}{7} \div \frac{3}{8}$, what two numbers should you multiply? How are $\frac{3}{8}$ and $\frac{8}{3}$ related?

13. Write steps like those in exercises 9 and 11 to show how to divide $\frac{4}{5}$ by $\frac{1}{2}$.

14. $\frac{72}{1} \div \frac{4}{1}$ equals what product? Do you see why Joe could multiply and Bill could divide to solve the problem about the strip of wood?

15. Complete each sentence.

a. $\frac{5}{2} \div \frac{2}{3} = \frac{5}{2} \times$ ——.

b. $\frac{2}{7} \div \frac{1}{4} = \frac{2}{7} \times$ ——.

c. $\frac{1}{9} \div \frac{4}{5} = \frac{1}{9} \times$ ——.

d. $\frac{1}{3} \div \frac{2}{5} = \frac{1}{3} \times$ ——.

16. a. Does $\frac{0}{5}$ have a multiplicative inverse? (See exercise 6a in Class Discussion 7c.)

b. The "sentence" below does not make sense. Why not?

$$\frac{2}{3} \div \frac{0}{5} = \frac{2}{3} \times \frac{5}{0}.$$

c. Explain why $\frac{2}{3} \div \frac{0}{5}$ does not equal a number.

d. Is it possible to divide by 0?

17. a. Is it true that $\frac{0}{4} \div \frac{2}{3} = \frac{0}{4} \times \frac{3}{2} = 0$?

b. Can 0 be used as a dividend?

c. If the dividend is 0, and the divisor is not 0, is the quotient always 0?

18. If you divide a rational number by any rational number except 0, do you think the quotient is always a rational number? Explain your answer.

[redacted]

To find the quotient of two rational numbers, multiply the dividend by the multiplicative inverse of the divisor. If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers (b, c, and d not zero), then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. Division, except by 0, is always possible.

[redacted]

1. In each exercise find the number that makes the sentence true.

a. $\frac{5}{8} \div \frac{3}{7} = n$.

b. $\frac{1}{2} \div \frac{1}{3} = m$.

c. $\frac{5}{8} \div \frac{1}{4} = c$.

d. $7 \div \frac{2}{3} = t$.

e. $\frac{3}{4} \div 5 = w$.

f. $3\frac{1}{3} \div 5 = y$.

g. $2\frac{1}{4} \div 1\frac{1}{2} = k$.

h. $2\frac{1}{2} \div 3\frac{1}{4} = b$.

2. If it takes $6\frac{1}{4}$ yards of material to make a man's suit, how many suits can be made with 100 yards of material?

3. If $2\frac{1}{2}$ pounds of meat cost \$2, what does 1 pound cost?

4. It takes a bricklayer $\frac{4}{5}$ of an hour to complete one section of wall. How many sections can the bricklayer complete in 8 hours?

5. Find the simplest fraction for each quotient.

   a. $\frac{5}{1} \div \frac{7}{1}$ b. $\frac{3}{1} \div \frac{8}{1}$ c. $\frac{7}{1} \div \frac{2}{1}$ d. $\frac{9}{1} \div \frac{4}{1}$

6. Look again at your answers to exercise 5. Can you explain why $\frac{5}{7}$ is sometimes thought of as "5 divided by 7"? Can you think of $\frac{3}{8}$, $\frac{7}{2}$, and $\frac{9}{4}$ in the same way?

General Summary

In this unit you learned that a rational number is a number that can be indicated by a fraction. A fraction was defined as an ordered pair of whole numbers with the second number not zero.

You learned that each rational number can be matched with exactly one point in the number line. You also learned that each rational number can be indicated by indefinitely many equivalent fractions. Given any fraction, you can find an equivalent fraction by multiplying the numerator and denominator of the given fraction by any whole number (not zero), or by dividing the numerator and denominator of the given fraction by a common factor.

There are three kinds of numerals that are used to express rational numbers. Ordinarily, fraction numerals and mixed numerals are used. Those rational numbers that are whole numbers can be expressed by whole-number numerals.

In this unit you learned to add, subtract, multiply, and divide rational numbers.

Listed below are statements of the properties of addition of rational numbers.

1. The sum of any two rational numbers is a rational number. [If s and t represent rational numbers, then $s + t$ represents a rational number.]
2. The order in which two rational numbers are added does not affect the sum. [If s and t represent rational numbers, then $s + t = t + s$.]
3. The way in which three rational numbers are grouped when adding does not affect the result. [If s, t, and w represent rational numbers, then $(s + t) + w = s + (t + w)$.]
4. The sum of any given rational number and zero is the given rational number. [If s represents a rational number, $s + 0 = 0 + s = s$.]

Listed below are statements of the properties of multiplication of rational numbers.

5. The product of any two rational numbers is a rational number. [If s and t represent rational numbers, then $s \times t$ represents a rational number.]
6. The order in which two rational numbers are multiplied does not affect the product. [If s and t represent rational numbers, then $s \times t = t \times s$.]
7. The way in which three rational numbers are grouped when multiplying does not affect the result. [If s, t, and w represent rational numbers, then $(s \times t) \times w = s \times (t \times w)$.]
8. Multiplication of rational numbers distributes over addition. [If s, t, and w represent rational numbers, then
$$s \times (t + w) = (s \times t) + (s \times w).]$$
9. The product of 1 and any given rational number is the given rational number. [If s represents a rational number, $s \times 1 = 1 \times s = s$.]
10. The product of zero and any rational number is zero. [If s represents a rational number, $s \times 0 = 0 \times s = 0$.]

There are two other important properties:

11. Every rational number except zero has a multiplicative inverse. [If s represents a rational number (not zero), then there is a rational number t such that $s \times t = 1$.]

12. Division of one rational number by another (except by zero) is always possible. [If s and t represent rational numbers and t is not zero, then $s \div t$ represents a rational number.]

Review Exercises

1. In a class of 36 students there are 16 boys and 20 girls. Use a fraction to tell what part of the class is made up of girls. What is the simplest fraction that is equivalent to this fraction?

2. For each number, write the letter of the point that corresponds to the given number.

   a. $\frac{3}{8}$ c. $\frac{3}{4}$ e. $\frac{5}{4}$

   b. $\frac{15}{8}$ d. $\frac{5}{2}$ f. 2

3. For each exercise, give a fraction that is equivalent to the given fraction but has a greater numerator.

   a. $\frac{5}{10}$ b. $\frac{12}{16}$ c. $\frac{72}{81}$

   How many such fractions are there for each of the given fractions?

4. For each fraction in exercise 3, give an equivalent fraction that has a smaller numerator. How many such fractions are there for each of the given fractions?

5. Give five fractions for the rational number that equals 8.

6. a. Find the simplest fraction for each of the rational numbers $\frac{15}{9}$ and $\frac{40}{24}$. Then decide if the two numbers are equal.
   b. Find fractions with a common denominator for the rational numbers $\frac{8}{10}$ and $\frac{13}{15}$. Are the two numbers equal?
   c. Use the cross-product rule to decide if it is true that $\frac{10}{15} = \frac{6}{9}$.

7. Complete each sentence by using whichever one of the symbols, >, <, or =, makes the sentence true.

   a. $\frac{3}{5}$ —— $\frac{2}{3}$.
   b. $\frac{5}{8}$ —— $\frac{6}{8}$.
   c. $\frac{15}{9}$ —— $\frac{20}{12}$.
   d. 1 —— $\frac{9}{8}$.
   e. $\frac{14}{16}$ —— $\frac{20}{24}$.

8. a. Find a mixed numeral for $\frac{14}{5}$.
   b. Find a fraction for $3\frac{2}{5}$.

9. In each exercise, find the number that makes the sentence true.

   a. $\frac{1}{2} + \frac{3}{2} = n$.
   b. $\frac{3}{4} + \frac{2}{3} = w$.
   c. $\frac{17}{10} \div \frac{6}{5} = m$.
   d. $3 + 1\frac{1}{4} = p$.
   e. $\frac{5}{2} \times \frac{2}{5} = t$.
   f. $\frac{20}{15} \times \frac{9}{8} = y$.
   g. $\frac{2}{3} - \frac{4}{5} = f$.
   h. $1\frac{1}{2} \div \frac{6}{7} = c$.

10. Bill had $16\frac{1}{2}$ feet of wire. Joe cut off a piece $2\frac{1}{4}$ feet long. How many feet of wire did Bill have left?

11. Jane made 40 cookies. Bill was hungry and ate $\frac{3}{4}$ of them. How many cookies did Bill eat?

12. In Jane's music class, $\frac{3}{8}$ of the students are girls and $\frac{2}{3}$ of the girls have blonde hair. What part of Jane's music class is made up of girls with blonde hair?

13. Jane has a job making fancy bows that are used for wrapping gifts. If it takes $1\frac{1}{4}$ yards of ribbon to make one bow, how many bows can she make with $12\frac{1}{2}$ yards?