

DOCUMENT RESUME

ED 049 307

TM 000 507

AUTHOR Tucker, Ledyard R.; And Others  
TITLE Topics in Factor Analysis II.  
INSTITUTION Illinois Univ., Urbana. Dept. of Educational Psychology.  
SPONS AGENCY Office of Naval Research, Washington, D.C. Psychological Sciences Div.  
PUB DATE Sep 70  
NOTE 67p.  
EDRS PRICE EDRS Price MF-\$0.65 HC-\$3.29  
DESCRIPTORS Correlation, \*Factor Analysis, Hypothesis Testing, Individual Differences, \*Learning Characteristics, Mathematical Models, Probability Theory, Reliability, \*Research Methodology, \*Serial Learning, \*Statistical Analysis  
IDENTIFIERS \*Maximum Likelihood Factor Analysis

ABSTRACT

Three topics in factor analysis are covered: a) a reliability coefficient for assessing the quality of a maximum likelihood factor analysis, b) an application of three-mode factor analysis to serial learning data, showing variations in learning curves over stages of learning and individuals, and c) the use of personal probability functions to define and fit simple structure hyperplanes in factor rotation. (Author)

ED049307

TOPICS IN FACTOR ANALYSIS .II .

# A RELIABILITY COEFFICIENT FOR MAXIMUM LIKELIHOOD FACTOR ANALYSIS

Ledyard R Tucker  
Charles Lewis

# A THREE-MODE FACTOR ANALYSIS OF SERIAL LEARNING

William D. Love  
Ledyard R Tucker

# FITTING OF FACTOR ANALYTIC HYPERPLANES BY A PERSONAL PROBABILITY FUNCTION

Ledyard R Tucker

September, 1970

The research of this report was supported by the Personnel and Training Research Programs Office of the Office of Naval Research under contract US NAVY/00014-67-A-0303-0003, NR 150-304.

Project on Techniques for Investigation of Structure of Individual Differences in Psychological Phenomena  
Ledyard R Tucker, Principal Investigator

Department of Psychology  
University of Illinois at Urbana-Champaign  
Champaign, Illinois



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2000502



Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified

1. ORIGINATING ACTIVITY (Corporate author)  Psychology Department University of Illinois Champaign, Illinois		2a. REPORT SECURITY CLASSIFICATION  Unclassified	
2b. GROUP			
3. REPORT TITLE  TOPICS IN FACTOR ANALYSIS II			
4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)			
5. AUTHOR(S) (First name, middle initial, last name)  Ledyard R Tucker William D. Love Charles Lewis			
6. REPORT DATE  September, 1970		7a. TOTAL NO. OF PAGES  57	7b. NO. OF REFS  38
8a. CONTRACT OR GRANT NO.  US NAVY/00014-67-A-0305-0003		9a. ORIGINATOR'S REPORT NUMBER(S)	
b. PROJECT NO.  NR 150-304		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
c.			
d.			
10. DISTRIBUTION STATEMENT  This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY  Office of Naval Research Personnel and Training Branch	
13. ABSTRACT  Three topics in factor analysis are covered: a) a reliability coefficient for assessing the quality of a maximum likelihood factor analysis, b) an application of three-mode factor analysis to serial learning data, showing variations in learning curves over stages of learning and individuals, and c) the use of personal probability functions to define and fit simple structure hyperplanes in factor rotation.			

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Factor Analysis Hyperplane Individual differences Learning curves Likelihood ratio Maximum likelihood Person Space Personal probabilities Reliability Rotation Serial learning						

A RELIABILITY COEFFICIENT FOR  
MAXIMUM LIKELIHOOD FACTOR ANALYSIS

Ledyard R Tucker

and

Charles Lewis

University of Illinois

ABSTRACT

Maximum likelihood factor analysis provides an effective method for estimation of factor matrices and a useful test statistic in the likelihood ratio for rejection of overly simple factor models. A reliability coefficient is proposed to indicate quality of representation of interrelations among attributes in a battery by a maximum likelihood factor analysis. Usually, for a large sample of individuals or objects, the likelihood ratio statistic could indicate that an otherwise acceptable factor model does not exactly represent the interrelations among the attributes for a population. The reliability coefficient could indicate a very close representation in this case and be a better indication as to whether to accept or reject the factor solution.

Maximum likelihood factor analysis offers effective procedures for statistical estimation of factor matrices and for statistical tests as to whether a factor analysis model represents the interrelations of attributes in a battery for a population of objects or individuals. In practical use of these methods, however, there is a problem in judging the quality of a factor analytic study. While the factor analytic approach may be quite profitable in establishing latent traits which account for essential interrelations among observations in a domain of phenomena, the factor analytic model involving a limited number of common factors almost surely will not represent exactly the phenomena for a population of objects. This proposition raises questions as to the use of the likelihood ratio test associated with maximum likelihood factor analysis. When a study is conducted with a very large sample of individuals the statistical test may indicate that the factor analytic model with a scientifically desirable number of common factors would not represent data for a population of objects. In these cases a measure of goodness of fit of the model to the phenomena is needed.

Lawley's (1940) initial solution of maximum likelihood factor analysis appeared to offer an elegant procedure for estimation of factor matrices and the associated likelihood ratio statistic seemed to promise a solution for the long standing number of factors problem. Due to the extensive calculations involved these procedures were little used but were discussed in the theoretic literature. Rao (1955) derived canonical factor analysis and demonstrated the equivalence to maximum likelihood

factor analysis. Lord (1956) provided the first application of maximum likelihood factor analysis to a large battery of measures using the Whirlwind computer at the Massachusetts Institute of Technology. With the further developments of high speed digital computers and of effective computer programs by Jöreskog (1967) maximum likelihood factor analysis has become quite feasible for application. Experience with maximum likelihood factor analysis has been developing with these applications. This experience indicates a dilemma in the application of the likelihood ratio statistic to decisions concerning the factor analyses.

The problem with the use of the likelihood ratio statistic, or any other similar statistic, involves the form of the decision procedure. The statistical hypothesis is that the factor analytic model with a restricted number of common factors applies strictly for a population of objects. Rejection of this hypothesis most surely will occur for a very large sample of objects at any usual level of significance. This rejects the scientific hypothesis. A reversal as to role of the statistical hypothesis and alternate hypothesis has occurred from the common use of decision procedures in scientific investigations for which the scientific hypothesis is the alternate hypothesis and is to be accepted when the statistical hypothesis has been rejected.

Consider a case involving a well developed battery of attribute measures such that with an extremely large sample of objects there would be common agreement that  $r$  important common factors are involved and that any further factors are trivial and uninteresting. Tucker, Koopman, and Linn (1969) proposed a system for producing correlation matrices based on the conception of a major factor domain and a minor factor domain to simulate observed correlation matrices. Browne

(1969) pointed out that any correlation matrix may be perfectly reproduced from a factor matrix when a large enough number of factors was permitted and that the different methods of factoring differed in the definition of a limited number of factors accepted and in the factors not accepted. The failure of the factor analytic model with a limited number of common factors to reproduce the matrix of correlations or covariances can be transformed to the existence of additional common factors which are to be rejected. In the case being considered there is common agreement as to the number of major factors in the common factor space and that the remaining common factors derive from a minor factor space and are to be discarded. The likelihood ratio and usual decision process would be quite appropriate in rejecting fewer than  $r$  factors. The problem is that this statistic and decision procedure probably would reject also  $r$  common factors. This would occur with a large enough sample of objects even for very trivial and meaningless minor factors. For example, Harman (1967, see page 229) states in reference to the maximum likelihood solutions for his 8 physical variables example:

"This example illustrates the general principle that one tends to underestimate the number of factors that are statistically significant. For twenty years, two factors had been considered adequate, but statistically two factors do *not* adequately account for the observed correlations based on a random sample of 305 girls. However, the third factor (whose total contribution to the variance ranges from 2 per cent to 5 per cent for the different solutions) has little "practical significance," and certainly a fourth factor would have no practical value."

As shown in Table 1 both the 2 factor and 3 factor models would be rejected at high levels of significance,  $p$  less than .001 and .01, respectively.



This situation is quite analogous to that of paired comparison scaling for which Mosteller (1951) provided a statistical test concerned with whether the model might or might not be rejected. Gulliksen and Tukey (1958) provided a reliability type coefficient for measuring goodness of fit of the model to data. They contrasted two examples: Mosteller's baseball data, for which the significance test did not reject the model while the reliability was low, with quality of handwriting data, for which the significance test indicated a decision to reject the model while the reliability was high. This contrast was due in part to quite different numbers of cases on which each proportion used was based: 22 for the baseball data versus 200 for the total sample for the handwriting data. An analogous reliability type coefficient is needed for factor analysis.

In developing a reliability coefficient for maximum likelihood factor analysis an asymptotic identity developed by Lawley (1940) for large  $N$  and several analogies are used. ( $N$  is the number of objects in a sample.) Jöreskog (1967) utilizes a derived function,  $F_m$  for  $m$  common factors, which is minimized to maximize the likelihood function. He indicates that the likelihood ratio is  $(N-1)F_m$ . Let  $C$  be the observed covariance matrix for a battery of  $n$  attributes,  $\hat{A}_m$  be the estimated factor matrix for  $m$  common factors,  $\hat{U}_m$  be the estimated unique factor loadings for an  $m$  common factor model, and  $G_m$  be an  $n \times n$ , symmetric matrix defined by

$$G_m = \hat{U}_m^{-1}(C - \hat{A}_m \hat{A}_m') \hat{U}_m^{-1} \quad (1)$$

In the maximum likelihood solution

$$\hat{U}_m^2 = \text{Diag} (C - \hat{A}_m \hat{A}_m') \quad (2)$$

Consequently

$$\text{Diag} (G_m) = I \quad (3)$$

The matrix  $G_m$  may be considered to contain the partial inter-correlations of the attributes partialling out the estimated common factors. Lawley's identity may be combined with Jöreskog's function  $F_m$  to yield

$$F_m = \sum_{j=1}^{n-1} \sum_{j'=j+1}^n g_{mjj'}^2 \quad (4)$$

where  $g_{mjj'}$  are the entries in  $G_m$ . Thus,  $F_m$  may be considered as approximately the sum of squares of the partial correlations on one side of the diagonal in  $G_m$ .

The preceding suggests an analogy with components of variation in analysis of variance. In this interpretation let  $M_m$  be a mean square corresponding to  $F_m$ .

$$M_m = F_m / df_m \quad (5)$$

where  $df_m$  is the degrees of freedom associated with  $F_m$  in the maximum likelihood solution. For variance components let  $\alpha_m$  be a variance associated with a model having  $m$  common factors,  $\delta_m$  be

a variance representing the deviation of the model from actuality, and  $\epsilon_m$  be a variance associated with sampling. For this component of variance model consider

$$E(M_Q) = \alpha_m + \delta_m + \epsilon_m \quad , \quad (6)$$

$$E(M_m) = \delta_m + \epsilon_m \quad , \quad (7)$$

where  $M_Q$  is the mean square for a model having zero common factors. A value for  $\epsilon_m$  may be obtained for the case when  $\delta_m$  is zero, that is when the model fits exactly for a population of objects. Then  $(N-1)F_m$  is distributed as chi square with  $df_m$  degrees of freedom and has an expected value of  $df_m$ . From this and equation (5) the expected value of  $(N-1)M_m$  is unity and the expected value of  $M_m$  is  $1/(N-1)$ . Using this result as a value for  $\epsilon_m$  equations (6) and (7) become

$$E(M_Q) = \alpha_m + \delta_m + \frac{1}{(N-1)} \quad , \quad (8)$$

$$E(M_m) = \delta_m + \frac{1}{(N-1)} \quad . \quad (9)$$

A reliability coefficient may be defined by

$$\rho_{in} = \frac{\alpha_m}{\alpha_m + \delta_m} \quad . \quad (10)$$

This is analogous to an intraclass correlation. It represents a ratio of the proportion of variance associated with the model to total

variance. An estimate may be obtained by substitution of observed values of  $M_0$  and  $M_m$  for the expected values in equations (8) and (9). Then

$$\rho_m = \frac{M_0 - M_m}{M_0 - 1 / (N - 1)} \quad (11)$$

This reliability coefficient may be interpreted as indicating how well a factor model with  $m$  common factors represents the covariances among the attributes for a population of objects. Lack of fit would indicate that the relations among the attributes are more complex than can be represented by  $m$  common factors.

Several examples are given in the tables for application of the reliability coefficient to correlation matrices taken from the literature. Table 1 presents results for Harman's (1967) 8 physical measures example. These measures were selected from a battery of 17 measures used by Mullen (1939) and the correlations based on an  $N$  of 305 were taken from her study. As indicated previously, a two common factor model is rejected by the likelihood ratio statistic at a significance level of .001. The reliability,  $\rho$ , was .934 for the two factor solution which has been accepted for years. A three common factor model may be rejected according to the likelihood statistic for which the  $p$  was less than .01. The reliability had risen to .975. This three common factor structure has two very highly correlated factors after rotation; one for height and length of lower leg, and one for arm span and length of forearm. These four measures loaded on a single rotated factor in the two factor solution; thus, the three factor solution is providing a differentiation between

length of leg bones and length of arm bones which may be of scientific interest. The two and three factor solutions had similar factors for the last four measures involving weight and girths. A four common factor model cannot be rejected by the likelihood ratio statistic and the reliability has risen to .994. However, the four factor solution does not add a meaningful factor in our judgment to the rotated solution for the three factor solution. We suggest that the three factor solution should be accepted in that the reliability is high and in that the four factor solution does not add a meaningful factor beyond the three in the three factor solution. This suggestion disregards the likelihood ratio result which indicates that the three factor solution would not be exact for a population of girls.

The square roots of the  $M$ 's for the various numbers of factors are listed also in Table 1. These values may be interpreted as root mean squares of the partial correlations among the attributes after the given number of factors have been extracted. One point to remember is that the approximate sum of squares,  $F$ , has been divided by the number of degrees of freedom remaining rather than by the number of partial correlations. Thus, even though  $F$  dropped from .074 for 3 factors to .014 for 4 factors,  $M$  dropped only from .0093 to .0047 since the degrees of freedom decreased markedly from 8 to 3. The corresponding decrease in the root mean square was only from .096 to .069. With consideration of this point, the values of  $M^{1/2}$  may be considered as measures of the sizes of the partial correlations. Both .096 and .069 are quite small.

A second example is presented in Table 2. This is the eleven test combined battery selected by Tucker (1958) from the larger battery

studied by Thurstone and Thurstone (1941). Tucker used this battery to illustrate his inter-battery factor analytic method and selected it to have two common factors. The two factor solution may be rejected at a high level of significance,  $p < .001$ ; however, the two factor solution has a reliability of .981 indicating a very good fit of the model to the interrelations among the scores on these eleven tests. The three factor solution, which may not be rejected and for which the reliability is 1.000, does not add a third meaningful factor. Loadings on the third dimension are moderately small. In consequence, the two factor solution which does not fit the data by the likelihood ratio test appears justified to represent the relations among the scores on these eleven tests.

Results for Harman's 24 psychological test example, which he obtained from a study by Holzinger and Swineford (1939), are shown in Table 3. Four factor solutions have been used in past analyses. This size model may be rejected by the likelihood ratio test at a value of  $p = .02$ . Reliability of the four factor model is relatively high at .952. Rotation of axes for the five factor solution presents some problems whereas the four factor solution has four rather nice rotated factors. Consequently, the four factor solution appears to be appropriate.

The number of individuals in the 24 psychological test example is less than the numbers of individuals in the preceding two examples. A conjecture may be made that if the study were repeated on a larger sample, the four factor model could be rejected by the likelihood ratio statistic at a higher level of significance. If our development of the reliability coefficient is justified, it should not change in a

systematic fashion.

Table 4 presents results for the eighteen special tests in Lord's (1956) study of speed factors. Six tests were constructed in each of three ability factors. Two of the tests for each factor were power tests, one test was moderately speeded, and three were speed tests. A three factor model may be rejected at a very extreme level of significance but this model has a moderately high reliability of .958 and the three rotated factors represent the three ability factors. Some psychologists might wish to accept this representation of the relations among the scores on these tests. A four factor solution may also be rejected at an extreme level of significance but it has a quite high reliability of .988. This solution adds a small general speed factor to the three ability factors. Again, some psychologists might wish to accept this solution. Rejection of a five factor model on the basis of the likelihood ratio is problematic. By this number of factors the reliability has become extremely high. However, the five factor solution adds only an indication of some differentiation among the types of speeded tests. Otherwise the results appear very similar to the four factor solution.

The preceding examples utilized data from studies involving measures and performances of real people. To gain further experience with the reliability coefficient, maximum likelihood solutions were obtained for twelve of the correlation matrices in the study by Tucker, Koopman, and Linn (1969) on simulated correlation matrices. These matrices were constructed for 20 attributes and for populations of individuals. A domain of major common factors was combined with minor common factors and unique factors. The minor common factors numbered

180 and had random factor loadings in decreasing magnitude with progression of factors. These twelve correlation matrices differ on three experimental design variables: number of factors in the major domain, 3 or 7 ; proportion of variance in the measures deriving from the major domain (range of  $B$  ), high ( .6 to .8 ) or low ( .2 to .4 ) ; and form of derivation model: "formal" involving only major domain common factors and unique factors, "middle" involving all three types of factors; and "simulation" involving only major domain and minor domain common factors. A point to be considered is that maximum likelihood factor solutions are quite feasible for these matrices but that the likelihood ratio statistics are not appropriate. There is no sampling of individuals problem. Differences between the matrices represent differences in how well the theoretic factor model with a limited number of common factors represents the interrelations of the attributes.

Results for these twelve matrices are given in Table 5 which presents the reliabilities for factor models having numbers of factors equal to the number of factors in the major domain. In all four cases the fit was exact for the formal model and the reliabilities were unity. Results for the middle model were higher than for the simulation model. Reliabilities, except for the formal model were higher for three factors in the major domain than for seven factors in the major domain. The combination of high range of  $B$  and middle model yielded quite acceptable reliabilities in the middle nineties while combinations of low range of  $B$  and simulation model yielded quite unacceptable reliabilities around .5 . These results indicate the relation of the reliability coefficient to the quality of the data



entered into a factor analysis. For high reliability, the minor common factors should be held to a low level of influence. Higher reliabilities are obtained when higher proportions of the variances of measures on attributes are derived from the major factor domains. It is better to have a higher ratio of number of attributes to number of factors in the major domain.

The proposed reliability coefficient for maximum likelihood factoring appears to summarize the quality of representation of the interrelation of attributes in a battery by a factor analytic model having a limited number of common factors. It does not appear to provide a criterion as to how many common factors to accept. However, as pointed out previously, the likelihood ratio test also does not provide such a criterion. The number of factors to accept appears to depend on size of loadings and meaningfulness of factoring results. In conducting a factor analytic study, a large enough sample of individuals or objects should be used to yield stable results. The likelihood ratio statistic should indicate that all models with fewer common factors than acceptable on other grounds should be rejected at an extreme level of confidence. This statistic might indicate that the accepted model would be rejected as not exactly representing the interrelations for a population. Any accepted solution should have a high coefficient of reliability.

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Table 1  
 Maximum Likelihood Factoring Reliability  
 Harman's 8 Physical Measures Example  
 N = 305 , n = 8

<u>Number of Factors</u>	<u>F</u>	<u>df</u>	<u>p</u>	<u>M</u>	<u>M<sup>1/2</sup></u>	<u>p</u>
0	6.861	28	***	.2450	.495	---
1	2.011	20	***	.1006	.317	.598
2	.249	13	***	.0192	.138	.934
3	.074	8	**	.0093	.096	.975
4	.014	3	.23	.0047	.069	.994

\*\* p < .01

\*\*\* p < .001

Table 2  
 Maximum Likelihood Factoring Reliability  
 Selected Battery from Thurstone & Thurstone  
 N = 710 , n = 11

<u>Number of Factors</u>	<u>F</u>	<u>df</u>	<u>p</u>	<u>M</u>	<u>M<sup>1/2</sup></u>	<u>p</u>
0	4.465	36	***	.1240	.352	---
1	1.300	27	***	.0481	.219	.623
2	.071	19	***	.0037	.061	.981
3	.017	12	.46	.0014	.038	1.000

\*\*\* p < .001

Table 3  
 Maximum Likelihood Factoring Reliability  
 Harman's 24 Psychological Tests Example  
 N = 145 , n = 24

<u>Number of Factors</u>	<u>F</u>	<u>df</u>	<u>p</u>	<u>M</u>	<u>M<sup>1/2</sup></u>	<u>p</u>
0	10.735	276	***	.0389	.197	---
1	4.326	252	***	.0172	.131	.680
2	2.918	229	***	.0127	.113	.818
3	2.053	207	***	.0099	.100	.907
4	1.574	186	.02	.0085	.092	.952
5	1.297	166	.13	.0078	.088	.973

\*\*\* p < .001

Table 4  
 Maximum Likelihood Factoring Reliability  
 Example From Lord's Speed Factor Study  
 N = 649 , n = 18

<u>Number of Factors</u>	<u>F</u>	<u>df</u>	<u>p</u>	<u>M</u>	<u>M<sup>1/2</sup></u>	<u>ρ</u>
0	13.984	153	***	.0914	.302	---
1	7.489	135	***	.0555	.236	.400
2	2.220	118	***	.0188	.137	.808
3	.540	102	***	.0053	.073	.958
4	.231	87	***	.0027	.052	.988
5	.140	73	.08	.0019	.044	.996
6	.098	61	.39	.0016	.040	.999

\*\*\* p < .001

Table 5  
 Maximum Likelihood Factoring Reliability  
 Simulated Correlation Matrices

$N = \infty$ ,  $n = 20$

Three Factors in Major Domain, Reliabilities for Three Common  
 Factor Models.

<u>Range of B</u>	Derivation Model		
	<u>Formal</u>	<u>Middle</u>	<u>Simulation</u>
High	1.000	.962	.832
Low	1.000	.851	.549

Seven Factors in Major Domain, Reliabilities for Seven Common  
 Factor Models.

<u>Range of B</u>	Derivation Model		
	<u>Formal</u>	<u>Middle</u>	<u>Simulation</u>
High	1.000	.941	.73
Low	1.000	.741	.483

A THREE-MODE FACTOR ANALYSIS OF SERIAL LEARNING

William D. Love

and

Ledyard R Tucker

University of Illinois

ABSTRACT

Variations in serial position learning curves over stages of learning and individuals were studied by means of a three-mode factor analysis of data for a list of 20 CVC trigrams. Ten scores, including partial credit for pairs of trigrams, were obtained on each of 19 trials for 33 subjects. A space of four dimensions described the trials mode, the transformed components representing successive stages of learning. Three of the four transformed components for the serial position space represented segments of serial position and the fourth component was the familiar U-shaped function of serial position curves. There were two dimensions in the person space, the first having nearly equal scores for all subjects and the second representing individual differences in performance. Individuals having positive scores on this second person dimension tended to show less of a U-shaped function and learned the earlier portions of the list faster than individuals with negative scores.

Serial learning tasks have been viewed from many angles. Usually, the investigator is interested in the shape of the learning curve over the serial positions. To get a descriptive curve, the investigator averages over trials and individuals for each position and plots his results. The familiar U-shaped curve slightly higher over the first positions is a product of such efforts (Ward, 1937; Hall, 1966, pp. 347-353).

Another investigator may be interested in looking at serial learning from another angle, that of trial by trial learning. He will average over positions and individuals to produce what usually is some form of a monotonically increase curve (eg., Osgood, 1953, p. 330).

More recently, individual differences have provided yet a third angle to view serial learning tasks. One notable result of this type of investigation has been by Duncan (1960). He points out that in a series of serial learning tasks slow learners have quite a different serial learning curve over positions in the early part of the series of tasks than do fast learners. This difference disappears in later learning tasks.

Two questions may be asked at this point. First, the descriptive techniques used are averaging techniques. The question then, how well does an average represent the data. For example, for a given individual there is a serial learning curve for each trial in the experiment. Some of these serial learning curves may show learning only on certain positions while other serial learning curves may show learning on other positions. In averaging over trials a representation of these serial



learning curves is produced which may not accurately represent either. An accurate representation for learning over trials may also be distorted by averaging. In both averaging procedures individual differences, if any, are totally lost.

One method to overcome this problem has been developed by Tucker (1966a). This method determines the minimum number of dimensions along which learning curves vary, and provides a specific type of learning curve for each dimension. These "generalized" learning curves can be combined linearly with weights to produce approximations to learning curves of different individuals and different trials and positions. Thus this representation of the data can capture the differences as well as the similarities in the learning curves of a serial learning task.

The second question about serial learning tasks has to do with relationships among variations over trials, positions, and individuals. Previous experiments have not been able to answer adequately such questions as "do certain types of individuals have different types of learning curves over trials and positions?" Tucker (1966b) has developed a method, known as the three-mode factor analysis technique, which provides the possibility of considering simultaneously all three angles of a serial learning task and develops an index which represents the relations among dimensions of variation of each. Each way of viewing the experiment, referred to as a mode, is probed for its structure and the interrelations among the structure of the modes is determined.

The combination of the procedure of "generalized" learning curves and the three-mode analysis, may provide information about a serial

learning task heretofore unknown. The following experiment does just that. It replicates a typical verbal serial learning task and compares the information obtained from averaging techniques to the factor analytic techniques.

#### METHOD AND PROCEDURES

##### Materials:

Twenty CVC trigrams were presented for 19 trials. Table 1 presents the list of CVC trigrams. The CVC trigrams were chosen from a list of all possible CVC trigrams listed by Archer (1960). Each vowel of the alphabet was represented five times in the list, and each consonant was represented at least once but not more than four times in the list.

To overcome the effects of variation in meaningfulness of the trigrams, CVC trigrams of approximate equal association value were used (Archer, 1960). The association values are shown in Table 1. Three random orderings of the list were prepared.

##### Procedure:

The serial anticipation method was utilized (see Andreas, 1960, p. 374). The list for each subject was presented for 19 trials using a Kodak Carousel slide projector automatically timed to present a new trigram every 5 seconds until the list of 20 trigrams had been shown. There was a 20 second interval after each trial while the slides were being readied for the next trial.

At the beginning of the list a slide with two pluses appeared indicating the start of a new trial. This was followed by a slide with

three pluses at which time the subjects were asked to anticipate the first CVC trigram. Then, as each item was presented, the subject anticipated the one which followed it in the list. A subject responded by writing his anticipation on an answer sheet. There was only one anticipation written on a page and one set of answer sheets to a trial. The subjects used the time between trials to change answer sheets and to write their identification number on the sheets.

In scoring, partial credit was given for partially correct anticipations. The three-mode factor analysis technique does not differentiate adequately when there are possible scores only of zero or one, which is the usual method for scoring serial learning trigrams. With partial credit and grouping of trigrams into pairs, a possible score of zero to ten was obtained for each of the ten pairs. In partial scoring two points were given for each consonant correctly positioned and one point for each vowel correctly positioned. This allowed a non-zero score for any anticipation that was not wholly incorrect.

The subjects were assigned at random to one of three groups corresponding to which ordering of the list was to be used.

#### Subjects:

The experiment used 33 students from a beginning course in psychology. Participation in experiments of this kind is one requirement for the course.

## RESULTS

The scores of the subjects were arranged in a three-mode matrix, subjects x trials x serial position, and analysed by Tucker's method I (1966b, see pages 297-298). Figure 1 presents the series of roots for factoring each of the three modes. Four dimensions were chosen for each of the trials and serial positions spaces and two dimensions were chosen for the person space. Transformations in these spaces were chosen to optimize interpretability of the results. Figure 2 presents the loadings on the transformed trials dimensions which were termed "transformed trials components". Each of these components has non-trivial loadings for a segment of the series of trials and has a maximum loading of approximately unity. These segments overlap between the components which have been ordered according to trials affected by the components. Trials affected by the components are: component 1 - early trials, component 2 - middle early trials, component 3 - middle late trials, component 4 - very late trials.

Figure 3 presents the loadings on the transformed serial position dimensions which were termed "transformed serial position components". The first three components affect a segment of serial positions from early in the list through middle late in the list to late in the list. The fourth component involves the first two serial positions and the last serial position, thus representing the familiar U-shaped serial position curve.

Figure 4 presents the transformed person space. Each point represents one of the subjects, having coordinates equal to the subject's scores on the transformed dimension. This transformation was determined

such that the sum of scores on the second dimension was zero and the correlation between scores on the two dimensions was zero. These dimensions were scaled such that the mean square score on each dimension was unity. The circles are for conceptual individuals chosen to represent the variety of performances of the actual subjects.

Investigation of possible effect of using different orderings of the list for three groups of the subjects was conducted for the person space. There was no significant effect, each of the three groups appeared to spread over the same area in the space.

Table 2 presents the transformed, three-mode core matrix. This matrix completes Tucker's (1966) three-mode factor analysis model:

$$\hat{x}_{ijk} = \sum_{mpq} a_{im} b_{jp} c_{kq} g_{mpq} \quad (1)$$

where  $\hat{x}_{ijk}$  is the fitted score of individual  $i$  on trial  $j$  at serial position  $k$ ,  $a_{im}$  is the score of individual  $i$  on person dimension  $m$ ,  $b_{jp}$  is the loading of trial  $j$  on trials component  $p$ ,  $c_{kq}$  is the loading of serial position  $k$  on serial position component  $q$ , and  $g_{mpq}$  is the entry in the three-mode core matrix. A two-mode core matrix may be defined for each individual by:

$$(h_i)_{pq} = \sum_m a_{im} g_{mpq} \quad (2)$$

so that the fitted score for this individual may be expressed as:

$$\hat{x}_{ijk} = \sum_{pq} b_{jp} (h_i)_{pq} c_{kq} \quad (3)$$

The matrices  $B$  of  $b_{jp}$ 's and  $C$  of  $c_{kq}$ 's form a constant framework for all subjects. The matrices  $H_i$  of  $(h_i)_{pq}$  contain the individual information for the different subjects. Table 3 presents the two-mode core matrices  $H_c$  for the three conceptual individuals,  $c$ , indicated by circles in Figure 4. Coordinates on the person dimensions for these conceptual individuals are given as vectors  $\underline{a}_c$  in Table 3.

Two groups of four subjects each were selected by scores on the second person dimension: a positive group having the highest positive scores and a negative group having the most negative scores. Mean scores for each group were computed at the ten serial positions for each of trials 3, 8, 13, and 18. These trials were chosen to represent the four trials components. Figure 5 presents these mean scores.

#### DISCUSSION

Several interesting effects appear in the results. The trials components give a basis for describing changes in the serial position curve for an individual subject associated with successive trials. These trials components might be conceived as reflecting stages of learning with the overlap representing transitions from one stage to the next. These trials components also enter into the differences between subjects by providing a basis for describing differences in performance of different subjects at the several stages of learning.

The serial position components provide an analysis of the learning of the list into several possibly more basic aspects of the learning process. At a minimum they provide a basis for describing changes

in the form of performances of individuals across the list at different stages of learning. An individual who learns the list from beginning toward the end would have higher entries in his two-mode core matrix for the earlier serial position components on the early trials components. In later stages of learning the larger entries in the core matrix would spread to the later serial position components. An individual whose learning was characterized by a strong U-shaped curve would have large entries in the fourth serial position component in his core matrix.

The effects of these influences can be seen in the two-mode core matrices for the conceptual individuals given in Table 3. The first of these conceptual individuals, having scores of 1.0 and 1.5 on the two person space dimensions, starts out with some learning in the early and middle portions of the list but is characterized more on trials components 1 and 2 by large entries for the fourth serial position component, the U-shaped component. For trials component 2 this individual also has a high entry for the second serial position component which indicates that he would have a bump in the middle of his serial position curve at this stage. This bump is very evident in Figure 5 for the positive group. As the trials continue to stages 3 and 4 of the trials components, the performance of this individual becomes more characterized by high entries on the first two serial position components and then on the first three serial position components. The effect of the fourth serial position component has diminished. This pattern indicates a serial position curve for this individual at the final stage which is relatively flat with a possible slope from the beginning of the list toward the end. The mean performance of

the positive group on trial 18 in Figure 5 illustrates this form of the serial learning curve.

Individuals having negative scores on the second person dimension appear to be influenced more by the U-shaped serial position component 4 after the first trials component as indicated in Table 3 for the third conceptual individual for which the person dimension scores are 1.0 and -1.5 . This same effect may be seen in Figure 5 for the negative group. This conceptual individual appears to learn the list from the back toward the first part of the list with entries becoming larger on serial position component 3 than on the first two serial position components. This effect tends to give a slope upward toward the end of the list as seen for the negative group in Figure 5.

The forms of the serial position curves at different stages of learning for these two kinds of people are quite different. The curve forms for individuals between these extreme individuals will be between the curve forms illustrated. These different forms might indicate different learning abilities of the subjects or different approaches to the learning task, maybe both. One speculation is that the individuals characterized by positive scores on the second person dimension worked harder at the learning task than those having negative scores by actively reviewing the list from beginning toward the end. Individuals with negative scores on the second person factor may have followed a strategy of allowing the list to be absorbed into their memory. Such a strategy might emphasize the effects of primacy and especially of recency. This possibility should be checked in further experimentation.



The foregoing experiment and analysis appears to indicate changing forms of the serial position curve for individuals as they progress from one stage of learning to the next and different forms for different individuals. Several influences appeared in the analysis in terms of the serial position components. The effects of these influences differed at different stages of learning and for different individuals. These effects, however, are not chaotic but are limited or constrained according to the parameters in the three-mode factor analytic model.

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Table 1

The 20 CVC Trigrams and Their Associative Value (Archer, 1960)

Trigrams	Associative Value
GAC -----	17%
QAS -----	17%
XAT -----	14%
LAJ -----	14%
XER -----	12%
VEF -----	12%
DEJ -----	13%
FEP -----	14%
RIW -----	15%
KIH -----	15%
YIB -----	16%
JIY -----	14%
GOQ -----	13%
YOX -----	16%
MOJ -----	13%
ZOS -----	13%
NUQ -----	13%
BUV -----	15%
WUC -----	16%
ZUX -----	14%

Table 2  
The Transformed, Three-Mode Core Matrix

Person Dimension,  $m = 1$

Trials Components	Serial Position Components			
	1	2	3	4
1	.59	.43	-.07	4.85
2	2.17	3.54	2.76	7.38
3	3.94	4.01	3.98	4.55
4	7.60	6.31	7.85	4.60

Person Dimension,  $m = 2$

Trials Components	Serial Position Components			
	1	2	3	4
1	.27	.82	.20	.66
2	.96	2.05	.72	-.18
3	2.09	2.00	.74	-1.95
4	1.78	2.10	.88	-1.70

Table 3

## Two-Mode Core Matrices for Conceptual Individuals

$$\underline{a}_c = (1.0, 1.5)$$

Trials Components	Serial Position Components			
	1	2	3	4
1	1.00	1.67	.22	5.85
2	3.61	6.61	3.83	7.10
3	7.08	7.00	5.09	1.62
4	10.28	9.45	9.17	2.05

$$\underline{a}_c = (1.0, 0.0)$$

Trials Components	Serial Position Components			
	1	2	3	4
1	.59	.43	-.07	4.85
2	2.17	3.54	2.76	7.38
3	3.94	4.01	3.98	4.55
4	7.60	6.31	7.85	4.60

$$\underline{a}_c = (1.0, -1.5)$$

Trials Components	Serial Position Components			
	1	2	3	4
1	.19	-.80	-.37	3.86
2	.73	.46	1.69	7.65
3	.79	1.01	2.86	7.48
4	4.92	3.16	6.52	7.15

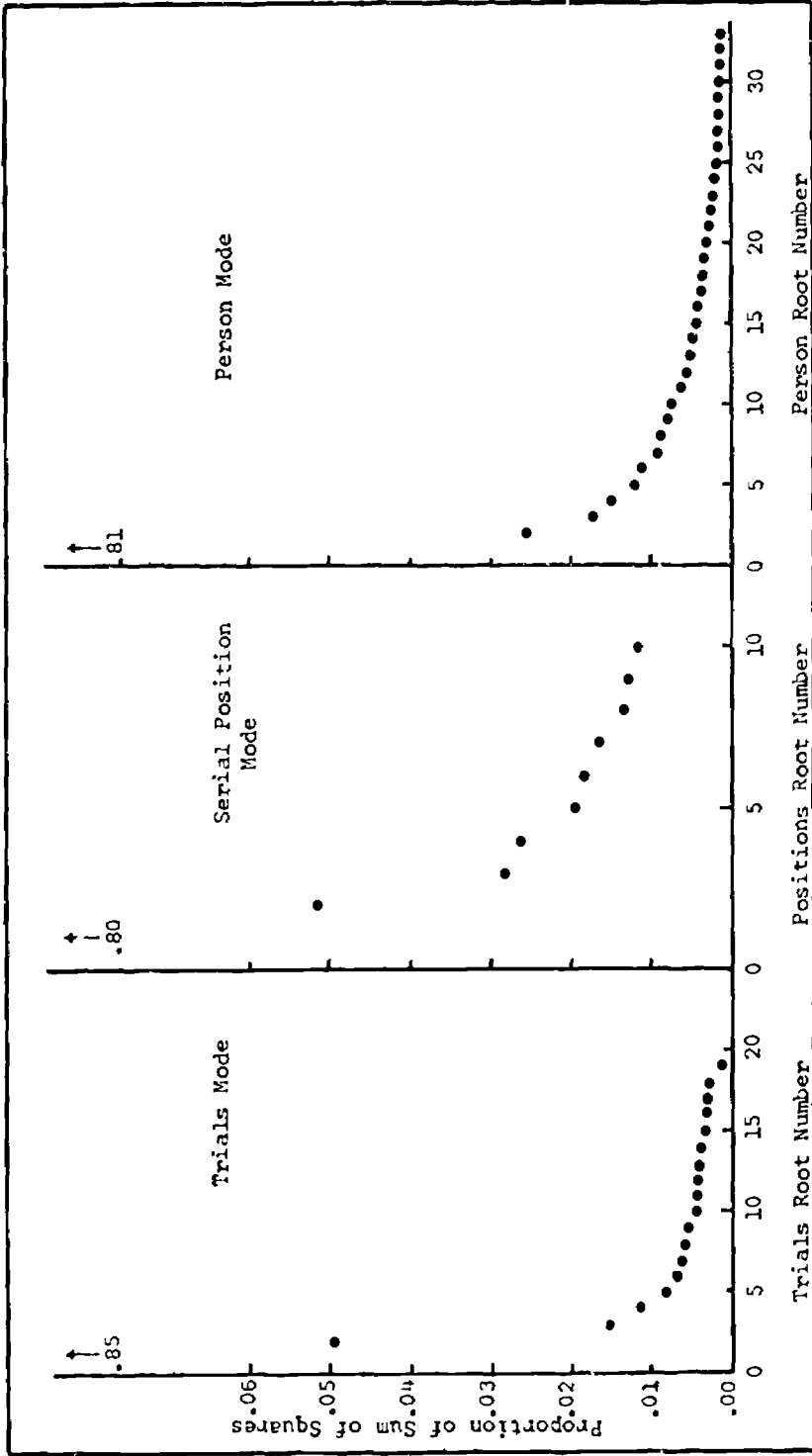


Figure 1

Series of Roots for the Three Modes

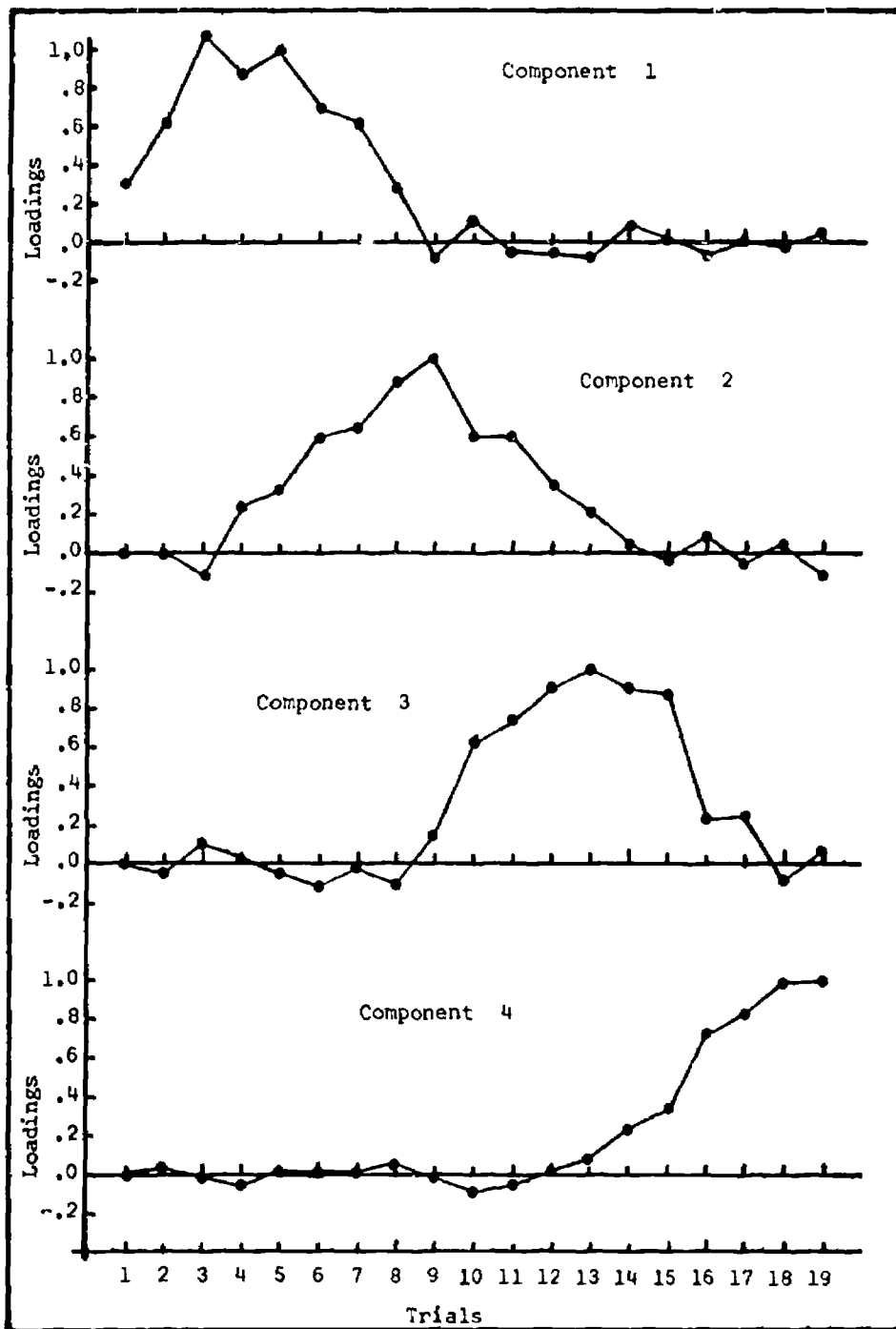


Figure 2

Transformed Trials Components

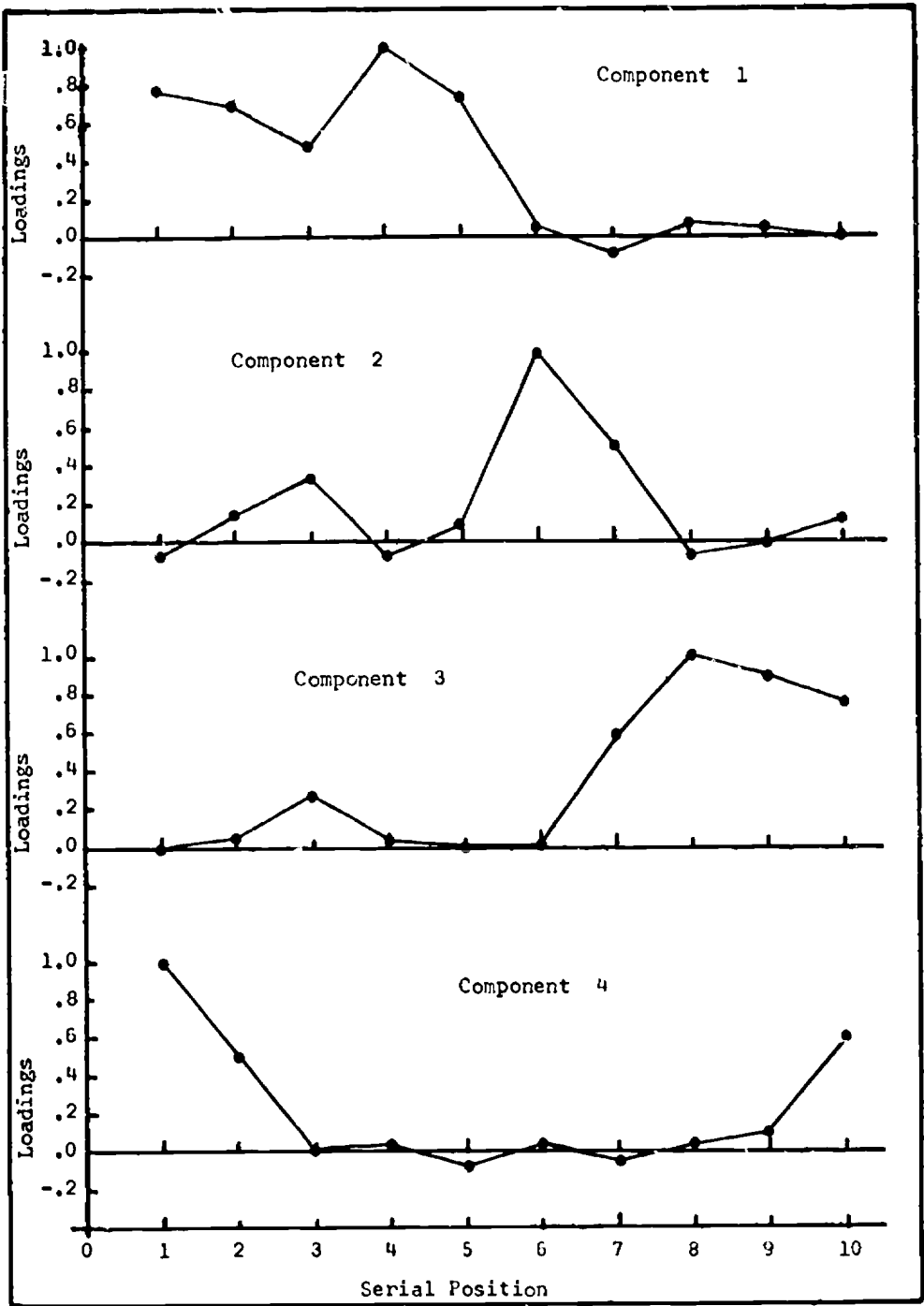


Figure 3

Serial Position Transformed Components



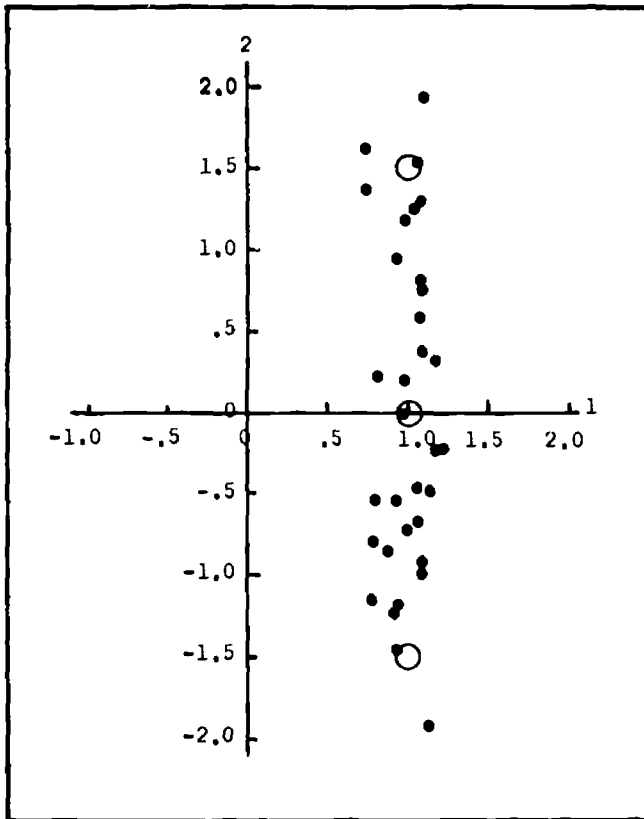


Figure 4

Transformed Person Space

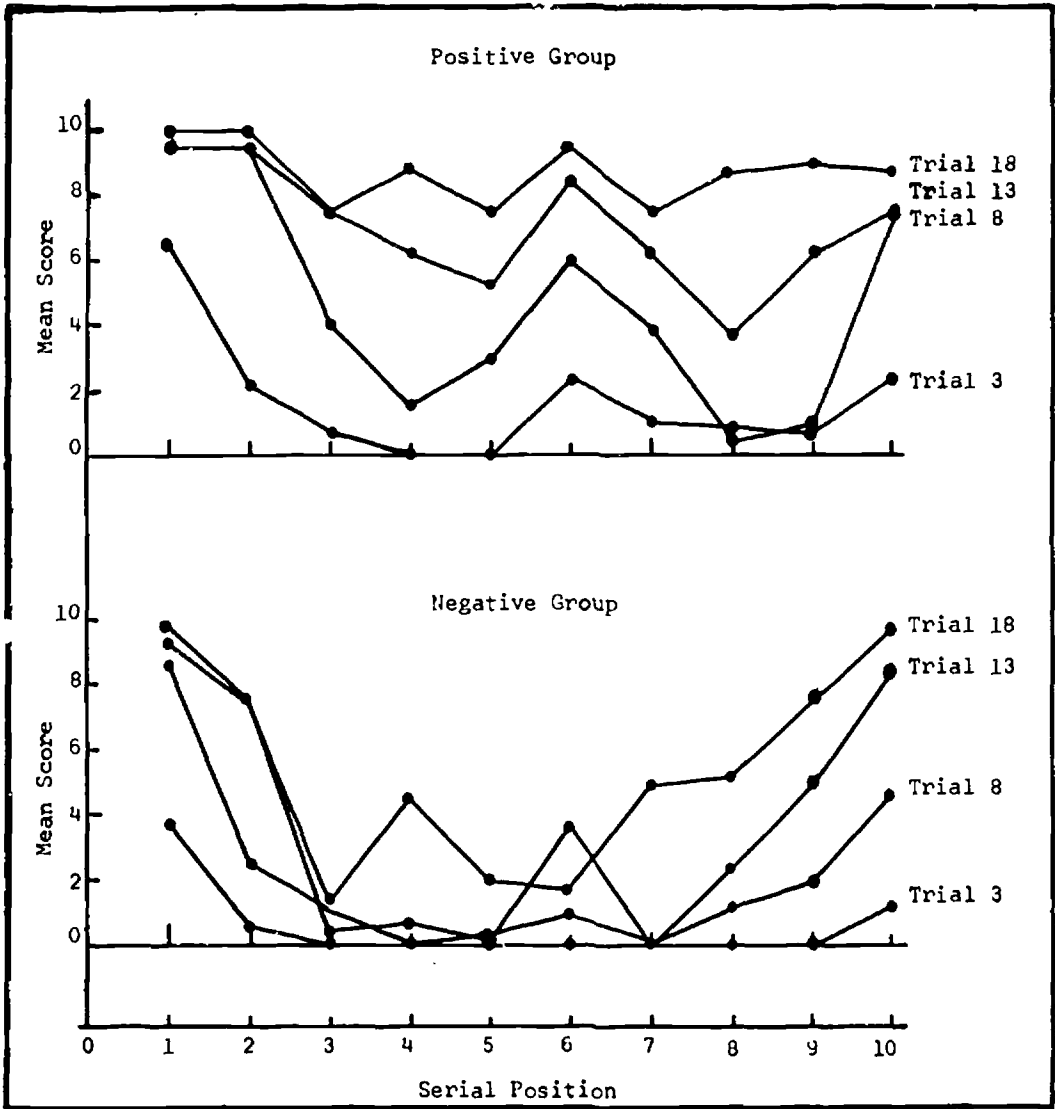


Figure 5

Mean Performances for Selected Groups of Subjects  
on Selected Trials

# FITTING OF FACTOR ANALYTIC HYPERPLANES

BY A PERSONAL PROBABILITY FUNCTION

Ledyard R Tucker

University of Illinois

## Abstract

Possible use of personal probability functions is proposed to define and fit simple structure hyperplanes in factor analysis. This development would automatize many of the subjective judgments in graphical rotation of axes and would replace judgments as to which attribute vectors are to be considered "in a hyperplane." The personal probability of an attribute vector being in a hyperplane is written as a function of the projection of the vector on the normal to the hyperplane and these personal probabilities for the vectors in a study are taken as weights. The hyperplane is fitted so as to minimize the weighted mean square projection of the attribute vectors on the normal to the hyperplane. A symmetric, or two-sided function could be used when interpretable attribute projections might be either sign or a one-sided function could be used when a positive manifold is expected. Use of a one-sided function is illustrated on three factor matrices.

A number of attempts have been made to reduce or eliminate subjective judgments in rotation of axes in exploratory factor analysis. And, while several objective criteria have been developed, the final judgments as to acceptability of results produced by these objective criteria appear to have been made partially on subjective bases. Two classes of questions are relevant. First, to the evaluator of proposed objective techniques: to what extent do the results of application of an objective criterion to a number of factor matrices conform to a simple structure evident to a perceptive factor analyst? Second, to an experimenter conducting an exploratory factor analysis: are the results of an objective rotation of axes for his study an adequate approximation to a rotation to simple structure or should he undertake further rotations guided by subjective judgments. Thurstone (1938a, 1938b, 1947) described graphical procedures for rotation of axes involving many detailed subjective judgments. Answers to the two areas of questions have involved not only the types of subjective judgments inherent in graphical rotation of axes but also some actual graphical rotations. The present development is an attempt to translate some of the features of the subjective judgments in graphical rotations into mathematical or computational form, thus providing an objective basis for operations involving these types of judgments.

The present attempt to translate the detailed judgments of graphical rotation of axes into more formal statements and procedures may be contrasted with attempts to develop objective criteria for rotation of axes. Many of the objective criteria have replaced the concept of simple structure by a mathematical form which emphasizes some feature of simple structure. Thurstone (1935, 1947) proposed an equation for simple structure which has theoretical interest, but is not a complete specification, even for constructed factor matrices, and has questionable applicability to matrices obtained from real world observations. The equation states that the product of loadings for each attribute on all common factors should equal zero. This specification is insufficient in that it may be satisfied by the occurrence of one zero loading for each attribute, therefore being insensitive to cases when there might be two or more zero loadings for an attribute. Carroll (1953, 1957) modified this equation to involve products of squared loadings for pairs of factors for each attribute and summed over all possible pairs and attributes. This sum was minimized to produce a best fitting simple structure in the sense of this criterion. Saunders (1953), Neuhaus and Wrigley (1954), and Pinzka and Saunders (1954) interpreted the desirable feature of many zero loadings into statistical distribution theory. They emphasized the kurtosis of distributions of loadings across factors for the attributes by summing the fourth power of the loadings and maximizing this sum. Kaiser (1958) in the varimax criterion emphasized the distribution of loadings by factors rather than by attributes. Tucker (1955) and Cattell and Muerle (1960) have emphasized counting the number of loadings within an interval about zero and maximizing

this count. In contrast to the foregoing attempts to set up satisfactory objective criteria the present development proposes to use a personal probability function which may reflect the subjective judgments for detailed decisions in the rotation of axes. It is an attempt to translate the judgments involved in graphical rotation of axes rather than an attempt to translate the principle of simple structure into operational form.

Use of a personal probability function, as proposed here, to determine a simple structure hyperplane is a direct descendant of a proposed definition of a simple structure hyperplane as a least squares fit to a subgroup of attribute vectors considered to be "in the hyperplane", that is, having small projections on the normal to the hyperplane. Thurstone (1936) proposed a procedure that was based on this definition of the best fitting hyperplane. A major problem has been in ways to decide on the subgroups of attributes. Tucker (1940, 1944) considered procedures for making these decisions, his 1944 procedure involving inspection of the inter-factor graphs and making subjective judgments quite analogous to the subjective judgments in graphical rotation of axes. This latter procedure eliminated judgments as to angles of rotation in the graphical methods. Subsequently, Tucker (1955) proposed an automatic procedure for decisions as to inclusion of attributes in the subgroups. This procedure also emphasized the maximization of the number of attributes in the subgroups. One common feature of the decisions concerning inclusion of attributes in the subgroups was that these decisions were based on the data being analysed rather than a priori hypotheses. A second common feature was that the total

group of attributes was partitioned into two subgroups on the basis of projections on the normal to the hyperplane, one subgroup with projections in an interval including zero and the other subgroup composed of remaining attributes. This interval could be symmetric about zero, an interval of  $\pm .10$  being common, or it could have a limit only on the positive side in case a positive manifold were desired. The partitioning of the attributes amounts to a step function based on the projections of the attributes on the normal to the hyperplane. An alternative conception of the procedure is that a weighted least squares fit of the hyperplane to the attributes is obtained for which the weights are unity or zero, the weights being a step function of the projections.

A major difficulty with the preceding proposals has been the use of the step function. Not only is it non-algebraic, which leads to operational difficulties, but it raises questions such as: why should an attribute having a projection just less than the limit of the interval be given a full weight in the determination of the hyperplane while another attribute having only a slightly larger projection, but a projection just larger than the limit, be given zero weight in the determination of the hyperplane. A continuous function would tend to eliminate both of these difficulties. A possible conception of such a continuous function is a personal probability function, the personal probability that an attribute should be considered in the subgroup defining the hyperplane. This personal probability would be a function of the projection of the attribute vector on the normal to the hyperplane. Further, the hyperplane could be defined so as to

minimize the mean weighted projection when the weights are the personal probabilities. A computer procedure has been developed based on this conception and has been tried out on several factor matrices.

Let  $A$  be the coordinate factor matrix for  $J$  attributes in an  $M$  dimensional common factor space. Entries in  $A$  will be designated  $a_{jm}$ . The computing procedure starts with a trial normal to a hyperplane. Let  $N_{kt}$  be a row vector containing the direction cosines for the  $t$ 'th trial normal for hyperplane  $k$ . Projections of the attribute vectors on this trial normal are contained in a column  $B_{kt}$ , the entries being  $b_{jkt}$ .

$$A N_{kt}' = B_{kt} \quad (1)$$

Let  $z_{jkt}$  be the trial personal probability that the vector for attribute  $j$  is to be considered to be in the hyperplane  $k$ . This trial personal probability is to be a function of the projection of the attribute vector on the trial normal, that is a function of  $b_{jkt}$ . The precise nature of this function will be discussed in subsequent paragraphs. Let  $\phi_{k(t+1)}$  be defined by

$$\phi_{k(t+1)} = \frac{[\sum_j z_{jkt} b_{jk(t+1)}^2]}{[\sum_j z_{jkt}]} \quad (2)$$

where  $b_{jk(t+1)}$  is the projection of attribute vector  $j$  on the next trial normal for hyperplane  $k$ ,  $N_{k(t+1)}$ . Note that  $\phi_{k(t+1)}$  is the weighted mean square of the projections of the attribute vectors on the next trial normal when the weights are determined from the



projections of the attribute vectors on the present trial vector.

The next trial normal  $N_{k(t+1)}$  is to be determined so as to minimize  $\phi_{k(t+1)}$ . In order to accomplish this let  $Z_{kt}$  be a diagonal matrix containing the trial personal probabilities  $z_{jkt}$  and define a matrix  $P_{kt}$  by

$$P_{kt} = A' Z_{kt} A \quad (3)$$

To minimize  $\phi_{k(t+1)}$  the next trial normal,  $N_{k(t+1)}$ , is the characteristic vector of  $P_{kt}$  corresponding to the least root.

When the trial personal probabilities  $z_{jkt}$  are an algebraic function of the projections  $b_{jkt}$  the preceding operations can be automated and a series of trials can be carried out on a computer. Experimental trials have indicated that such a procedure is quite feasible and a stable state is obtained in a very few trials, where the stable state is defined by very small change in the trial normal from one trial to the next. At such a stable point the weighted mean square projection of the attribute vectors on the normal is a minimum for the weights equal to the personal probabilities corresponding to these projections. This appears to be the desired result.

Two questions remain. First, what function might be used for the personal probability function? This question will be discussed subsequently. Second, what are reasonable first trial normals to the hyperplanes? In the experimental try outs of the procedure, Kaiser's (1958) normal varimax procedure has given very satisfactory results.

Selection of a personal probability function posed a problem. Consideration of functions derived from a priori assumed distributions of projections was abandoned due to an unattractiveness of any particular distribution of non-zero projections. These considerations did lead to a division between two types of functions, one for cases when non-zero projections might have either algebraic sign and the other for cases when a positive manifold is to be assumed. These two cases are termed here the two sided personal probability function and the one sided personal probability function. A two sided personal probability function should be symmetric, bell shaped with a maximum for a zero projection. A one sided personal probability function should be asymmetric with personal probabilities near unity corresponding to negative projections and approaching zero for increasing positive values of projections. Such a function is illustrated in Figure 1. Since selection of a function having an appropriate form seemed as arbitrary as the selection of a priori distributions of projections, consideration was given to simple functions that possessed desirable features.

A one sided personal probability function was developed by use of two functions, one stating values of  $z$ , the personal probability value, as a function of an artificial variable  $y$ , and the other stating values of  $y$  as a function of projections  $b$ . These two functions are given below:

$$z = \frac{1}{2} \left[ 1 - \frac{(1+c)y}{1+c|y|} \right] \quad (4)$$

where  $c$  is a parameter affecting the slope of this function, and

$$y = \frac{-d}{(1-d^2)} + b + \frac{d}{(1-d^2)} b^2 \quad (5)$$

where  $d$  is a parameter affecting the value of  $b$  corresponding to a value of  $z$  equalling one half. For  $b$  equal to  $-1$ ,  $y$  equals  $-1$  and  $z$  equals  $+1$ ; for  $b$  equal to  $+1$ ,  $y$  equals  $+1$  and  $z$  equals  $0$ . The function for  $z$  in terms of  $b$  has a negative slope in the range for  $b$  equal to  $-1$  to  $+1$  for values of  $d$  between  $0$  and  $.414$ . Study of this function indicated that a value of  $c$  equal to  $10/d$  yielded a desirable single parameter family of functions. Figure 1 presents three functions from this family. The parameter  $d$  can be thought of as a stringency of definition of the simple structure parameter. For a small value of  $d$  only attributes having small positive projections or having negative projections would have a high personal probability of being in the hyperplane. Larger values of  $d$  would correspond to a more lenient view as to inclusion of attributes in the hyperplane for which the projections are not as small as for a small value of  $d$ .

An operational point for computations involving a one sided personal probability function is that the trial normals should be directed such that the larger projections are positive. In the experimental try outs each trial normal was considered in a preliminary and in a final direction. The sum of projections of attributes on the normal in its preliminary direction was obtained. If this sum was positive, the normal in its preliminary direction was accepted as the

normal in its final direction. If the sum of projections was negative, the normal was reversed in direction from its preliminary direction to its final direction. In the first case, the projections on the normal were retained as computed; in the second case, the projections on the normal were reversed in sign.

No satisfying two sided personal probability function has been developed.

A by-product of the consideration of a personal probability approach to attributes being in hyperplanes is the possible use of the complement, the personal probability of attributes not being in hyperplanes, in interpretation of the factors. In an analogy to decision processes, significance levels might be established for interpretation that a factor had an effect on the measures of an attribute. The personal probability significance of the projection of an attribute could be entered into the inductive inferential process involved in the interpretation of a factor.

Three analyses will be presented to illustrate the application of the foregoing development of factor matrices. The first illustration is for Thurstone's (1947, see page 194 for the centroid factor matrix used here as the coordinate factor matrix) 20 attribute box problem. Results are given in Table 1. A first step in the analysis for the example was to establish an "ideal" solution. This was done by hypothesizing, on an a priori basis, that there should be a factor for each dimension of the boxes and that the measures, or attributes, not involving that dimension should have zero projections, or be in the hyperplane. The least squares solution for

these hypothesized zero projections is given at the left of Table 1. Two personal probability function solutions are given, one for  $d$  equal to .15 and one for  $d$  equal to .05. Kaiser's (1958) normal varimax solution was used to obtain initial trial normals for the three factors in each of the personal probability function solutions. Both of these solutions are very close to the ideal solution at the left. There is a difference in the complementary personal probability of attributes not being in the hyperplanes. Consider attribute 19,  $e^y$ , and its projections of .06 and .08\* for the two solutions. Interpretation that this attribute depends on the first factor, which appears to be an  $x$  dimension factor, would be an error analogous to a type I error in decision processes. This situation may be considered as indicating that a  $d$  of .05 is too stringent for the box problem. A similar effect occurs for attribute 14,  $\log y$ , on the third factor. A point of strategy might be that as stringent a definition of the hyperplane, low value of  $d$ , should be used as does not yield nonsensical interpretations.

Table 2 presents results for the second example, Harman's (1967) 24 psychological tests example. These data were derived from the study by Holzinger and Swineford (1939), a revised four factor MINRES solution supplied by Harman (personal communication) being used as the coordinate factor matrix. Harman also supplied hypotheses as to which tests should have zero loadings on each of four factors. The least squares solution for these hypotheses is given on the left of Table 2. A personal probability solution with a  $d$  of .15 is given on the right of Table 2. As in the previous example, Kaiser's

(1958) normal varimax solution was used to obtain initial trial normals. Again, the two solutions correspond very closely. The largest discrepancies are for factor 3 for which the projections tend to be more positive. This effect is greatest for test 10 with a shift in projection from -21 to -10, a more acceptable projection for a positive manifold. There are a few cases for which the interpretation of effects of factors differ. Factor 1 appears to be a verbal factor. Harman hypothesized that test 24, arithmetic problems, should have a zero projection while the personal probability solution indicated, with a probability greater than .90, that this test is not in the hyperplane. The indication from the personal probability solution is that the verbal factor has some small effect on scores on the test of arithmetic problems, a not unreasonable result considering a number of other findings with verbally stated arithmetic problems. Similar shifts from hypotheses of zero projections to projections for which the personal probability would indicate that the tests were not in the hyperplane occur on factor 3 for tests 12, counting dots, and 18, number-figure which is a memory test. Factor 3 appears to be a visual perception factor and it is not unreasonable that the tests 12 and 18 should be affected to an appreciable, but small extent by this factor.

The third factor matrix used in the experimental try outs was one suggested by Horst (personal communication) as a very tough example for automated rotational procedures. This is a specially constructed factor matrix having vectors for nine attributes in a

three dimensional space. There are three unit length vectors in each plane of a right spherical triangle. The vectors in each plane are  $22\frac{1}{2}$  degrees apart centered in the space between two corners of the spherical triangle. This configuration is presented schematically in Figure 2 in which an equilateral triangle is used to represent the spherical triangle. There are no vectors at the corners of this triangle. A number of series of trials were run for a personal probability solution starting from randomly directed initial trial normals. Figure 2 illustrates results obtained with one initial trial normal. Several values of  $d$  were employed. In series 1 of trials a value of  $d$  of .05 was employed. The stable position of the plane was across a corner of the spherical triangle with vectors 4 and 7 nearly zero. This is a very undesirable solution. In series 2, a value of  $d$  of .15 was used first and after a stable position was obtained the value of  $d$  was reduced to .05. The solution with  $d$  equal to .15 went beyond the corner but when  $d$  was reduced to .05 the trial vector returned to the corner as a stable position. In series 3 values of  $d$  of .25, .15, and .05 were used in succession, a stable position being obtained with one value of  $d$  before the next smaller value was used. The stable position for a  $d$  of .25 was nearer one of the planes of the spherical triangle than had been obtained in the previous series using smaller values of  $d$  and the starting position of the series as the initial trial normal. In series 3, when  $d$  was reduced to .15 and the stable normal from the  $d$  equal to .25

solution was used as an initial trial normal, a stable position was found very close to the plane of the spherical triangle. Reduction of  $d$  to a value of .05 lead to a stable position even closer to the plane of the spherical triangle.

This third example illustrates several points. First, there may be a number of stable positions of the normal in a personal probability function solution. The existence of these many solutions may depend on the incompleteness of the definition of hyperplanes by the vectors in a factor matrix. Horst's example is characterized by this incompleteness of definition since the vectors are all some distance from the corners of the spherical triangle. Secondly, a satisfactory solution may be obtained by sequential use of values of  $d$  in descending order as to size. Thirdly, and this is a relatively technical point, the solution for a constructed example such as Horst's can be made to approach the theoretically correct solution, the plane of the spherical triangle, as closely as desired by using successively smaller and smaller values of  $d$ .

In conclusion, the foregoing use of a personal probability function in fitting hyperplanes in factor analysis appears to replace subjective judgments to a considerable extent and to yield very satisfying results. Further, it yields guides to the interpretation of the factors with the probabilities of attribute vectors not being in the hyperplanes. This procedure does not eliminate completely judgment by the experimenter and analyst.



Appropriate values of  $d$  must be chosen and there must be a constant awareness that extra solutions may exist so that a series of trials may end at one of these solutions. These areas of judgment and precaution are no more, however, than should be involved in any careful scientific study.

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TABLE 1

THURSTONE'S 20 ATTRIBUTE BOX PROBLEM

Least Squares Solution for  
Attributes Hypothesized  
to be in Hyperplanes

Personal Probability Function Solutions

d = .15

d = .05

Direction Cosines of Normals

Dimension	Direction Cosines of Normals			Dimen- sion	Direction Cosines of Normals			Dimen- sion	Direction Cosines of Normals		
	1	2	3		1	2	3		1	2	3
1	47	46	48	1	47	45	48	1	48	46	49
2	-84	25	55	2	-84	25	55	2	-84	25	56
3	28	-85	68	3	28	-86	69	3	26	-85	67

Projections on Normals

Attribute	Projections on Normals			Attri- bute	Projections on Normals			Attri- bute	Projections on Normals		
	1	2	3		1	2	3		1	2	3
1 $x^2$	97	00#	01#	1	96***	00	00	1	97***	00	00
2 $y^2$	01#	94	03#	2	01	94***	00	2	03	94***	02
3 $z^2$	00#	01#	96	3	00	01	96***	3	00	01	96***
4 xy	46	72	00#	4	46**	72***	00	4	48***	72***	01
5 xz	38	-01#	85	5	38**	-01	85***	5	38***	-01	85***
6 yz	00#	38	79	6	00	38**	79***	6	00	38***	80***
7 $\sqrt{x^2+y^2}$	71	51	-02#	7	71***	51**	-03	7	72***	51***	-02
8 $\sqrt{x^2+z^2}$	35	01#	39	8	84***	00	33**	8	85***	01	39***
9 $\sqrt{y^2+z^2}$	-02#	75	44	9	-03	75***	44**	9	-01	75***	45***
10 $2x+2y$	60	62	-01#	10	60***	62***	-01	10	62***	62***	00
11 $2x+2z$	66	00#	65	11	66***	-01	64***	11	66***	00	64***
12 $2y+2z$	-02#	60	62	12	-02	60***	62***	12	-01	60***	64***
13 log x	95	03#	00#	13	95***	03	-01	13	96***	03	-01
14 log y	-04#	92	06#	14	-04	92***	05	14	-02	92***	06#
15 log z	02#	-03#	94	15	02	-04	94***	15	02	-03	94***
16 xyz	27	35	70	16	27**	35**	70***	16	28***	35***	71***
17 $\sqrt{x^2+y^2+z^2}$	63	45	28	17	63***	45**	27**	17	64***	45***	26***
18 $e^x$	94	-04#	02#	18	94***	-04	02	18	95***	-01	01
19 $e^y$	06#	90	-05#	19	06	90***	-05	19	05#	90***	-03
20 $e^z$	00#	02#	95	20	-01	02	95***	20	-01	01	95***

# = Hypothesized zero projections.

\* = Personal probability of attribute not being in the hyperplane > .90.

\*\* = " " " " " " " " " " " " > .95.

\*\*\* = " " " " " " " " " " " " > .99.

TABLE 2

HARMAN'S 24 PSYCHOLOGICAL TESTS EXAMPLE

Least Squares Solution For  
Attributes Hypothesized<sup>o</sup>  
to be in Hyperplanes

Personal Probability Function  
Solution with  $d = .15$

Dimensions	Direction Cosines of Normals				Dimen- sion	Personal Probability Function Solution with $d = .15$			
	1	2	3	4		1	2	3	4
1	-34	-23	-26	-16	1	-35	-22	-32	-16
2	-76	75	-01	26	2	-72	74	11	25
3	-56	-47	80	20	3	-59	-48	76	20
4	03	41	54	-93	4	03	41	55	-93

Test	Projections on Normals				Test	Personal Probability Function Solution with $d = .15$			
	1	2	3	4		1	2	3	4
1 Visual Perception	-03#	07#	58	-02#	1	-03	06	61***	-02
2 Cubes	01#	00#	39	-03#	2	00	00	40**	-03
3 Paper Form Board	03#	-12#	47	-01#	3	02	-12	47**	-01
4 Flags	10#	-02#	44	-08#	4	10	-02	45**	-07
5 General Information	61	08#	-02#	-06#	5	62***	08	00	-05
6 Paragraph Comprehension	64	-08#	-02#	04#	6	65***	-08	-02	04
7 Sentence Completion	71	02#	-02#	-13#	7	72***	02	-02	-13
8 Word Classification	43	09#	16#	-06#	8	43**	09	18	-06
9 Word Meaning	70	-10#	-04#	02#	9	70***	-10	-04	03
10 Addition	03#	77	-21#	03#	10	07	77***	-10	02
11 Code	01#	43	-04#	23	11	03	43***	05	23*
12 Counting Dots	-13#	65	14#	-05#	12	-11	65***	23*	-05
13 Straight-Curved Capitals	00#	44	32	-10#	13	02	44**	39**	-10
14 Word Recognition	05#	-04#	-12#	48	14	09	-04	-09	48**
15 Number Recognition	00#	-04#	-02#	45	15	00	-04	01	45**
16 Figure Recognition	-10#	-09#	28	40	16	-10	-09	31**	41**
17 Object-Number	-01#	10#	-10#	50	17	01	10	-05	50**
18 Number-Figure	-16#	27	18#	33	18	-15	22*	25*	33**
19 Figure-Word	01#	07#	12#	24	19	02	07	16	24*
20 Deduction	23	-04#	25	13#	20	23*	-04	27**	13
21 Numerical Puzzles	00#	32	29	04#	21	02	31**	35**	04
22 Problem Reasoning	22	-01#	24	12#	22	22*	-01	26*	13
23 Series Completion	20	08#	35	03#	23	20	08	39**	03
24 Arithmetic Problems	20#	36	02#	11#	24	22*	36**	08	11

<sup>o</sup>Hypotheses given by Harry H. Harman (personal communication).

# = Hypothesized zero projection.

\* = Personal probability of attribute not being in the hyperplane > .90 .

= " " " " " " " " " " " " > .95 .

= " " " " " " " " " " " " > .99 .

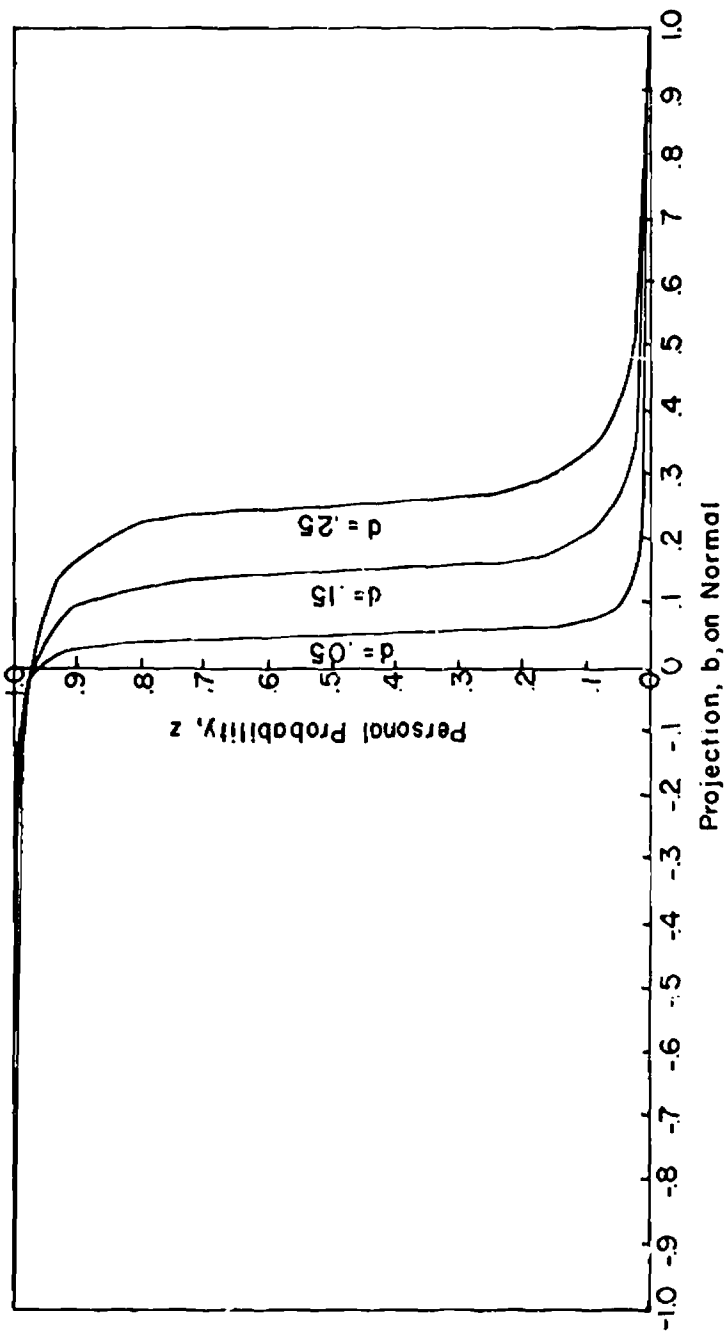


Figure 1  
A One Sided Personal Probability Function

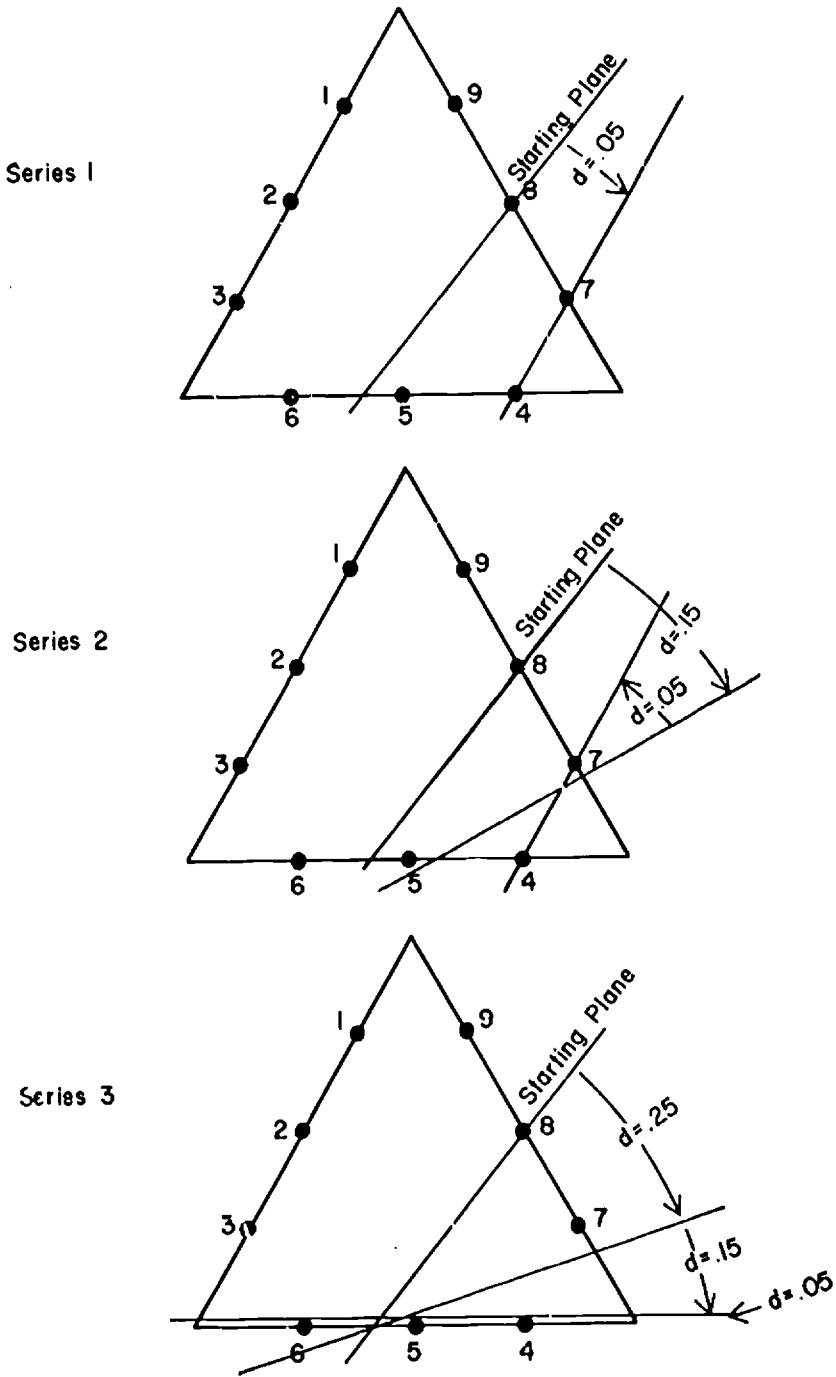


Figure 2

Results for Horst's 9 Vector Example

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St. Louis, Missouri 63130
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Elliot Hall  
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Human Sciences Research, Inc.  
Westgate Industrial Park  
7710 Old Springhouse Road  
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MISCELLANEOUS

-5-

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Department of Educational Psychology  
Graduate Center  
City University of New York  
33 West 42nd Street  
New York, New York 10036
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Department of Psychology  
University of Illinois  
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Human Factors Research, Inc.  
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Performance Research, Inc.  
919 Eighteenth Street, N.W., Suite 425  
Washington, D. C. 20036
- 1 Dr. Stanley M. Nealey  
Department of Psychology  
Colorado State University  
Fort Collins, Colorado 80521
- 1 Dr. Gabriel D. Ofiesh  
Center for Educational Technology  
Catholic University  
4001 Harewood Road, N. E.  
Washington, D. C. 20017
- 1 Mr. Luigi Petrullo  
2431 North Edgewood Street  
Arlington, Virginia 22207
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Psychology Department  
Montgomery College  
Rockville, Maryland 20852
- 1 Dr. Arthur I. Siegel  
Applied Psychological Services  
Science Center  
404 East Lancaster Avenue  
Wayne, Pennsylvania 19087
- 1 Dr. Paul Slovic  
Oregon Research Institute  
Post Office Box 3196  
Eugene, Oregon 97403
- 1 Dr. Ledyard R Tucker  
University of Illinois  
Psychology Building  
Urbana, Illinois 61820
- 1 Dr. John Annett  
Department of Psychology  
Hull University  
Hull  
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- 1 Dr. Joseph W. Rigney  
Behavioral Technology Laboratories  
University of Southern California  
University Park  
Los Angeles, California 90007
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Division of Psychological Studies  
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Princeton, New Jersey 08540
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Haddonfield, New Jersey 08033
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University of Umea  
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Umea 6, Sweden