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ABSTRACT

This curriculum is an intermediate result of a research program exploring the application of detailed behavior analysis procedures to the problem of designing sequences of learning objectives. The research program, part of the Primary Education Project (PEP), is an attempt to develop a systematic method of specifying and validating learning hierarchies so that instructional programs can be designed which provide an optimal match for a child's natural sequence of acquisition. An operational definition of the number concept is proposed in the form of a set of eight behavioral objectives which, taken together, permit the inference that the child has an abstract concept of "number." Each behavioral objective is analyzed to identify both its component behaviors or skills and the prerequisite behaviors necessary for learning those components. Specific sequences of learning objectives are then proposed which are hypothesized to facilitate optimal learning by maximizing transfer from earlier to later objectives. A formalized "mastery" model is described in which children are tested to determine their entering level in the curriculum and are then passed on to higher level objectives on the basis of their demonstrated mastery of lower level ones. (JY)

UNIVERSITY OF PITTSBURGH - LEARNING R & D CENTER

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TECHNICAL REPORT

A HIERARCHICALLY SEQUENCED INTRODUCTORY MATHEMATICS CURRICULUM

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13. ABSTRACT

A method of systematic behavior analysis is applied to the problem of designing a sequence of learning objectives to provide an optimal match for the child's natural sequence of acquisition of mathematical skills and concepts. The authors begin by proposing an operational definition of the number concept in the form of a set of behaviors (the curriculum "objectives") which, taken together, permit the inference that the child has an abstract concept of "number". Each behavior is then analyzed to identify hypothesized components of skilled performance and pre-requisites for learning these components. On the basis of these analyses, specific sequences of learning objectives are proposed which are hypothesized to facilitate optimal learning by maximizing transfer from earlier to later objectives. Relevant literature on early learning and cognitive development is considered in conjunction with the behavior analyses and the resulting sequences. A discussion of the ways in which a hierarchically sequenced early learning curriculum can be used in schools includes a description of a formalized "mastery" model, in which children are tested to determine entering level and in which they pass to higher level objectives on the basis of demonstrated mastery of lower-level ones. Alternative models are considered briefly.

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A HIERARCHICALLY SEQUENCED

INTRODUCTORY MATHEMATICS CURRICULUM

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Preface

This report describes further work concerned with subject-matter structures and instructional design. A previous report (Glaser, 1970) pointed out that a basic requirement for instructional design is the development of methodology that can be used (a) to analyze the properties of behavior to be learned and (b) to specify the sequence of component tasks involved in the course of learning. The analysis of the component tasks involves identifying a hierarchy of prerequisite skills and knowledge which facilitate the transfer of learning from a subordinate set of tasks to more complex tasks. A hierarchical analysis can provide ordered sets of tasks for inclusion in a training program and also specifies the skills a student needs to successfully enter the program.

Present techniques employed for generating and establishing hierarchical structures for training essentially provide hypotheses about how learning can proceed, and they require empirical validation. A component analysis may suggest several possible curriculum sequences; it may suggest task sequences which do not maximize transfer effects; and it may fail to identify certain necessary prerequisite behaviors. With this in mind, a second report in this project has described procedures for the validation of learning hierarchies (Resnick & Wang, 1969). Psychometric scaling procedures can be used in validation work, and the results obtained can lead to the redesign of instructional sequences and to the identification of questions for the experimental analysis of the transfer relationships between hierarchical components in a curriculum.

The methodological problems in generating effective instructional sequences on the basis of hierarchical component task analysis

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are complex, and techniques need to be worked out in simple situations and then applied to more difficult subject matters. In the effort to develop operational procedures for the hierarchical analysis of instructional tasks, the present project has analyzed the relatively simple structures involved in young children learning elementary mathematics concepts. The principles and procedures involved might then be applied to more complex learning situations.

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INTRODUCTORY MATHEMATICS CURRICULUM

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The curriculum to be presented in this paper is an intermediate result of a research program exploring application of detailed behavior analysis procedures to the problem of designing sequences of learning objectives. The aim of this research program is to develop a systematic method of specifying and validating learning hierarchies so that instructional programs can be designed which provide an optimal match for a child's natural sequence of acquisition. It is assumed that curricula which closely parallel this sequence will facilitate learning under a wide variety of specific teaching methods.

The basic rationale for the methods employed here has been presented in papers by Resnick (1967) and by Resnick & Wang (1969). Briefly, the strategy is to develop hierarchies of learning objectives such that mastery of objectives lower in the hierarchy (simpler tasks) facilitates learning of higher objectives (more complex tasks), and ability to perform higher level tasks reliably predicts ability to perform lower level tasks. This involves a process of task and behavior analysis similar to that proposed and elaborated by Gagné (1962, 1968). Detailed procedures of analysis will be explicated in the course of this paper.

Exploration of this hierarchical approach to curriculum design is a major component of the Primary Education Project (PEP). PEP is a research and development project engaged in the development and testing of an individualized educational program for young children. It operates as a joint public school-university project, with major responsibility for pre-school and primary grade programs in an urban elementary school, and combines research in early learning processes and motivation with developmental work ranging from curriculum design to teacher training and classroom management. The present mathematics curriculum is one of several introductory curriculum sequences currently in use and under study in PEP classrooms.

Content of an Introductory Mathematics Curriculum

The PEP introductory mathematics curriculum is intended to to provide a basis for the child's continuing experience in mathematics. To serve this function the curriculum must present the fundamental concepts of mathematics, or operations leading to them, in forms simple enough to be learned by very young children yet broad enough to serve as a conceptual foundation for later work. Methodologically, this requires that target concepts be identified, and that hierarchies of specific objectives then be constructed to guide the child from naivete to competence in understanding and using these concepts.

The Concept of Number

One of the main goals of the mathematics curriculum reform movement during the past decade has been to present mathematics as a body of knowledge which obeys well-defined principles or laws. Emphasis on the inherent structure of mathematics can be seen throughout the curricula and writings of various groups of reformers (e.g., Cambridge Conference on School Mathematics, 1963; Devault & Kriewall, 1969).

At the heart of the structures present in school mathematics are the concepts of sets, relations, and numbers. In the early years of a child's mathematical education, the newer curricula emphasize experiences designed to foster the concept of number. With the acquisition of the number concept, the child is prepared to advance to the operations on natural numbers, and to study the properties of these operations. The structure of the natural numbers, then, is one of the central concerns of mathematics curricula throughout elementary school.

To a mathematician, the concept of natural number is the common property shared by all sets which are in a one-to-one correspondence with each other. Thus, the concept of the natural (or cardinal) number "two" is derived from the (only) property which is shared by all sets in a one-to-one correspondence with, for instance, the set $\{a, b\}$. This property is called the number "two;" as a generalization, it is the concept "two." Other natural numbers are defined in a similar manner.

While the concept of number is clearly defined mathematically, it is not at all clear how a child attains the concept, or even what kinds of performance signify such attainment. Traditional arithmetic has stressed the learning of such skills as counting objects, using written numerals, and, later, calculating. Both Piaget-oriented researchers in mathematics learning (e.g., Dienes, 1966, 1967; Lovell, 1966) and developmental psychologists (e.g., Flavell, 1963; Kohlberg, 1968; Wohlwill, 1960) focus instead on processes that reflect more directly the mathematical definition of the number concept. Mathematicians stress logical relations among ordered sets, and particularly the notion of one-to-one correspondence among sets. New math curricula reflect these concerns and are intended to provide the child with the experiences with sets and logic which will directly develop these concepts. Piaget adds to the mathematicians' concern a special emphasis on seriation, on the child's recognition of invariance of number across spatial transformations (conservation), and on the correspondence of ordinal and cardinal number (Piaget, 1965).

The basic goal of the PEP mathematics curriculum is the development in children of a stable concept of number. Many developmental psychologists are skeptical of the possibility of directly teaching these concepts, stressing instead the role of "general experience" in inducing the stage of "concrete operations," which includes mathematical operations along with classificatory logic and related concepts (Kohlberg, 1968). PEP, however, operates from a broad assumption that operational number concepts can be taught, believing that "general experience" is in fact composed of a multiplicity of specific experiences, certain ones of which are critical in the acquisition of an operational number concept. The problem, both for psychological research and educational design, is to discover which experiences are the crucial ones; that is, which early behaviors from the building blocks of the higher level competence one seeks to establish.

Behavioral Definition of the Number Concept

The first step in developing a hierarchy of curriculum objectives leading to an operational concept of number was to specify in behavioral terms a number of specific components of the number concept. The behaviors thus specified comprise an operational definition of the number concept in the form of concrete performances, which, taken together, permit the inference that the child has an abstract concept of "number." Some of the behaviors relate directly to the mathematical-psychological definition of number; some are linked to pragmatic uses of number such as counting and comparing; and others are associated with common symbols for numbers. These behaviors comprise the actual objectives of the curriculum. They appear in a hierarchically sequenced form in Figures 1 through 8. Each figure represents a unit of the curriculum.

Insert Figures 1 - 8 about here

Each box in these figures defines a terminal objective of the curriculum--an objective deemed important enough to be subjected to direct measurement in assessment of a child's progress through the curriculum. In each box, the entry above the line describes the stimulus situation with which the child will be presented, and the entry below the line describes the child's response. Thus, in Unit 1 (Figure 1), box B should be read as, "Given a set of zero to five moveable objects; the child can count the objects, moving them out of the set as he counts." Box E would be read, "Given a numeral, stated (to 5), and a set of objects (to 5), the child can count out a subset of the size indicated by the numeral." This convention is followed throughout, except where a box is used merely to refer to another unit or task that is described elsewhere (e.g., bottom box of Figure 2, which specifies that Unit 1 is a prerequisite for beginning Unit 2).

In determining possible teaching sequences, the charts are read from the bottom up. The simplest objectives in a given unit appear at the bottom and are considered prerequisite to those appearing above and connected by a line. In Unit 1, for example, B is prerequisite to C and E; and C is prerequisite to D. C and E, however, have no prerequisite relation to each other and can be taught in either order. F has two prerequisites, D and E, and would not normally be taught until both of these skills were acquired.

There are eight units in the introductory curriculum (see Figures 1 - 8). Units 1 and 2 cover counting skills to ten and simple comparison of sets by one-to-one correspondence. Units 3 and 4 cover the use of numerals. Units 5 and 6 include more complex processes of comparing and ordering sets. Unit 7 introduces the processes of addition and subtraction, while Unit 8 uses equations to establish more sophisticated understanding of partition and combination of sets. The specific objectives for each unit are discussed in the sections below. The complete PEP early learning curriculum includes a heavy emphasis on classification skills and concepts (including multiple relations, sorting, intersection of sets, etc.). Such skills and concepts are recognized as likely prerequisites for full mathematical understanding, but have not been included directly in the mathematics curriculum. Instead, they appear in separate "classification and language" sequences which can be implemented prior to or simultaneously with the mathematics curriculum.

The division of the curriculum into units was based on considerations of educational practice rather than on mathematical theory or behavior analysis. In general, the aim was to establish units that would maximize the child's experience of success and also make for relative ease of administration in an individualized classroom. These criteria explain, for example, the decision to break the initial introduction of counting skills into two units, one for sets up to five (Unit 1), and the second for sets up to ten (Unit 2). The use of written numerals (Units 3 and 4) is treated as a separate group of objectives, largely because of classroom and experimental evidence that counting is learned earlier than written numeral presentation and that learning the numerals is easier once counting is well established (Wang, Resnick & Boozer, 1970). The numbering of the units if for reference purposes, and does

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does not imply a linear order of instruction. Figure 9 shows the pattern of hierarchical relationships among the units and the order in which they can be presented without skipping prerequisites.

Insert Figure 9 about here.

Behavioral Analysis and Sequencing of the Objectives

The ordering of objectives within each unit is based on detailed analyses of each task. These analyses are designed to reveal component and prerequisite behaviors for each terminal objective, both as a basis for sequencing the objectives and to provide suggestions for teaching a given objective to children who are experiencing difficulty. The detailed analyses identify many behaviors that are not part of the formal curriculum, but which underlie the stated objectives and may need to be taught explicitly to some children. Often, two superficially similar tasks differ with respect to their demands on some basic function such as memory or perceptual organization. These differences between tasks provide the basis for ordering tasks according to complexity and thus for predicting optimal instructional sequences.

Behavior analyses for individual objectives appear in Figures 10 - 43. In each of these analysis charts the top box contains a statement of the objective being analyzed. This box as well as all others in the chart follows the "Given . . . the child can . . ." convention described above. Adherence to this convention assures that each box in the analysis will contain a behaviorally defined task, one that can be tested by direct observation.

The first step in performing a behavior analysis is to describe in as much detail as possible the actual steps involved in skilled performance of the task. The procedure is similar to, although less formalized than, the technique of "protocol analysis" developed by Newell & Simon (Newell, 1968) in connection with studies in computer simulation of thinking.

The results of this "component analysis" are shown in level II of each chart. The double lines around the boxes indicate that these behaviors are components of the terminal behavior; it is hypothesized that the skilled person actually performs these steps (although sometimes very quickly and covertly) as he performs the terminal task. The arrows between the boxes indicate that the component behaviors are performed in a temporal sequence. Sometimes (e.g. Figure 10) there are "loops" in the chain, indicating that it is necessary to recycle through some of the steps several times to complete the task. Where a box is divided vertically, a choice or decision point in the task is indicated. For example, in Figure 14, box IId shows a point at which either of two different responses might be appropriate, depending on whether two numbers are found to be the same or different.

Once the components are identified, a second stage of analysis begins. Each component that has been specified is now considered separately, and the following question asked: "In order to perform this behavior, which simpler behavior(s) must a person be able to perform?" Here, the aim is to specify prerequisites for each of the behaviors. Prerequisite behaviors, in contrast to component behaviors, are not actually performed in the course of the terminal performance. However, they are thought to facilitate learning of the higher level skill. More precisely, if A is prerequisite to B, then learning A first should result in positive transfer when B is learned, and anyone able to perform B should be able to perform A as well. The first set of prerequisites appears in level III of each chart.

Continuing the analysis, identified prerequisites are themselves further analyzed to determine still simpler prerequisite behaviors. This can result in charts showing several levels of prerequisites, with complex interrelationships among the behaviors (e.g. Figure 29). Analysis stops when a level of behavior is reached which can be assumed in most of the student population in question, or when another terminal behavior in the set under analysis appears as a prerequisite. In the latter case, reference is made to the analysis of that behavior (e. g. Figure 12, box IIIa). Sometimes a single behavior is prerequisite to more than one higher-level behavior. Conversely, a given component or prerequisite can have more than a single prerequisite. In reading the charts it is necessary to remember simply that a given behavior is prerequisite to all behaviors above it and connected with a line.

The interrelations among objectives revealed by these analyses form the basis for sequencing objectives within units of the curriculum. The detailed rationale for such sequencing will be described in the following sections, which discuss each of the units in some detail.

Counting: Units 1 and 2

Units 1 and 2 each specify several different kinds of counting behavior (Figure 1 and 2, Objectives A - F). Analyses of these behaviors (Figures 10 - 14) suggest that each type of counting task has certain unique components and prerequisites. Because the tasks are behaviorally different they have been included as separate objectives in the curriculum.

Figure 10 shows the analysis for Objective 1 - 2:B, counting a set of moveable objects. The key component is moving an object out of the set while saying a numeral (boxes IIa and IIb). This behavior has two prerequisites: synchronizing touches with counts (box IIIa) and re-

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citing the numerals in order (box IIIb). Because he can move objects out of the set as he counts them, the child has no problem of remembering which objects have been counted. In counting a fixed set (Objective C; Figure 11), on the other hand, the child must touch the objects in a fixed pattern in order not to miss any objects nor touch any of them twice (cf. Potter & Levy, 1968). This additional prerequisite is shown in Figure 11 in box IIIc. Since Objective C has all the prerequisites of B plus an additional one, C was placed above B in the unit hierarchy (see Figures 1 and 2). This indicates a hypothesis that learning B first will facilitate the learning of C.

Insert Figures 10 and 11 about here.

Objective D (Figure 12) adds still another new component. When the objects to be counted are physically scattered (unordered) rather than lined up in a row or other recognizable pattern, the task of keeping track of which objects have been touched is considerably more difficult. Beckwith & Restle (1966) have presented data suggesting that this problem is typically solved by first visually grouping or patterning the objects and then counting as if the set had been ordered to begin with. Figure 12 (box IIa) shows this behavior of visual grouping as a component of counting unordered sets. Box IIb on this chart describes a behavior equivalent to counting an ordered set, and the reader is referred to Objective 1 - 2:C for further analysis. Since C appears as a prerequisite to D in the behavior analysis, Objective D appears above C in Units 1 and 2.

Insert Figure 12 about here.

Objective E (Figure 13), counting out a subset from a larger set, returns to the use of moveable objects, as in Objective B. However, whereas in B the child simply continues counting until the set is exhausted, in E he must remember the number of the subset he has been asked for (box IIa) and stop when he reaches that number (IIc). Figure 13, therefore, shows Objective 1 - 2:B as a prerequisite to E (box IIa), and this dependency is reflected in the unit hierarchies. Counting out a subset does not share with counting fixed arrays the component of keeping track of which objects have been counted. For this reason, the unit charts show E as independent of C and D. Objective F (Figure 14), on the other hand, has both the memory component (boxes IIa and IIc) similar to that in E, and the component of counting fixed arrays (box IIb). as in C and D. For this reason the unit hierarchies suggest that Objective F not be introduced until both the C - D sequence and E have been learned.

Insert Figures 13 and 14 about here.

At the same time as he is learning to count the child can be working on another basic aspect of the number concept, one-to-one correspondence. In Objective G, H and I (Figures 15, 16, and 17) he learns to pair objects from two sets to determine whether the sets are equivalent or which set has more (or less) objects. The analyses of Objectives G ("equivalent") and H ("more") show nearly identical components (see Figures 15 and 16). The only difference appears in the third component (box IIc in both Figures): To determine <u>which</u> set has more objects the child must correctly select the set with extra objects, while to decide whether the sets are equivalent he need only determine whether there are extra objects in <u>either</u> set. On the basis of this slight additional complexity for Objective H, H was placed above G in the unit hierarchies.

Insert Figures 15 and 16 about here.

To determine which of two sets has less objects (Objective I), it is necessary to determine which set has extra objects and then choose the other set (Figure 17, boxes IIc and IIIb). This is behaviorally analogous to using negative information (see box IIIb), which is known to be difficult for young children. Thus the behavior analysis suggests that the concept "less" should be more difficult to learn than the concept "more." For this reason, Objective I was placed above H in the unit hierarchy, yielding a predicted learning sequence for one-to-one correspondence tasks in which "equivalent" (G) is prerequisite to "more" (H), which is in turn prerequisite to "less" (I).

Insert Figure 17 about here.

The sequence G-H-I is supported empirically in a study by Uprichard (1970) in which "equivalent to," "greater than," and "less than" was shown to be the optimal order for teaching these three concepts. On the other hand, data from a scaling study by Wang (1970) suggest that preschool children normally learn the concept "more" before they learn "equivalent." Thus there is some doubt as to the appropriate sequence for Objectives G and H; it may, in fact, be likely that both objectives will be learned most easily when taught simultaneously, as "contrast" cases for one another. The Uprichard and the Wang, et al. findings are in agreement concerning the dependency of the concept of "less than" on "more" and "equivalent." In addition, Donaldson (1968) has found that children at about age four typically respond to the term "less" as if it were synonymous with "more." Thus, for this concept, existing empirical data support the predictions derived from behavior analysis.

Numerals: Units 3 and 4

Units 3 and 4 introduce written numerals. Objectives A, B, and C in each unit establish the basic skills of recognizing and reading numerals. The sequence of matching (A), identifying (B), and naming (C) numerals is a basic sequence for teaching the names of a set of objects. It is used elsewhere in PEP for teaching labels such as color names, geometric shapes, names of common objects, etc. This sequence has been empirically validated in two separate studies (Wang, 1970; Wang, Resnick, & Boozer, 1970).

Objectives D through F are intended to insure that the child attaches meaning to the written symbols. In D (Figure 18), he matches sets with numerals, thus combining counting and numeration skills. In E (Figure 19) the child compares numerals for size. The analysis of this objective shows as prerequisites counting out a set of the size indicated by a numeral (box IIIa) and comparing sets by one-to-one correspondence (box IIb). Neither of these behaviors is a component in the sense that skilled persons would actually perform them in the process of comparing numerals. However, they are the processes which logically underlie the assignment of relative value to numerals, and therefore represent prerequisites to performing the terminal task with comprehension rather than purely algorithmically. They are also prerequisites in the sense that a skilled person undertaking to explain the process to a novice would probably demonstrate these behaviors.

Insert Figures 18 and 19 about here.

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Objective F requires ordering a set of numerals. Two different methods of performing this task are shown in Figures 20 and 21. The first method (Figure 20) involves placing the lowest numeral first, then the next lowest, and the next, until the set of numerals is exhausted. The critical component in this sequence is selecting the lowest numeral (boxes IIa and IIc), and this component, in turn, can be performed by either of two methods. The method described in box IIa involves reciting the numeral chain and selecting the numerals as they are named. The second method of selecting the lowest numeral in a set (boxes IIb and IIc) is slightly more complicated, involving comparison of successive pairs of numerals. This process may well be a precursor of operational transitivity (Murray & Youniss, 1968; Smedslund, 1963) in that an ordering of several elements is achieved without explicitly comparing all possible pairs.

Insert Figure 20 about here.

A second analysis of objective F appears in Figure 21. Here the method is to order two numerals, then arrange a third numeral with respect to the first two, and continuing inserting new numerals into the series by a process of successive comparison. An elementary form of transitivity seems to be involved in this process as well, since a numeral is placed as soon as a single higher numeral is found (boxes IIe, first half; and IIf, first half). Comparison with the rest of the numerals higher in the series is not required. This method appears more complicated with respect to maintaining a spatial arrangement and keeping track of which positions have been tested (see box IIIa) than the method shown in Figure 20. However, with respect to prerequisites involving the concept of number or the logic of seriation itself, the two methods

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may be equivalent. This is a question of some theoretical interest, which will be encountered again in Unit 6 when seriation of length and of sets of objects appear.

Insert Figure 21 about here.

Comparison of Sets: Unit 5

Units 5 and 6 are the points at which the child begins to combine his skills in counting, one-to-one correspondence, and numeration into an integrated, operational number concept. In Objectives A & B of Unit 5, he learns a new method of comparing set size, this time by counting the sets and comparing the numerals stated. Analyses of these objectives, in Figures 22 and 23, show comparison of sets by one-to-one correspondence as a prerequisite (boxes IVa and IVb in both figures). While it would probably be possible for a child to learn to count and compare without being able to perform one-to-one correspondence operations, his comprehension of the nature of number comparison would be in doubt in such a case. By specifying one-to-one correspondence as a prerequisite, the curriculum insures that children will relate their counting operations to the basic mathematical definition of number. Thus, as was the case for Objective E of Units 3 and 4, specification of the process that logically underlies the performance being learned as a prequisite helps to assure that the new performance will not be learned purely as an algorithm.

Insert Figures 22 and 23 about here.

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Objectives 5:C and 5:D (Figures 24 and 25) require the comparison of a set with a numeral. This represents a consolidation of numeration skills taught in Units 3 and 4 and their integration with the concepts of set size and set comparison. As is shown in Figure 24, these objectives have as prerequisites reading numerals (3 - 4:C), counting sets (1 - 2:D), comparison of sets (5:A and 5:B) and comparison of numerals (3 - 4:E). Since comparison of sets and of numerals are combined in a single objective, the child's performance of Objectives C and D can give some assurance that the numerals the child works with are tied to a basic concept of number and set size.

Insert Figures 24 and 25 about here.

Objective 5:E requires the comparison of rows of objects deliberately arranged so that length and number are uncorrelated. For example, in successive test items for this objective, the longer row might have fewer objects, the longer row more objects, two rows of equal length might have different numbers of objects, and two rows of unequal length might have an equal number of objects. Successful performance of this task requires that the child attend to number as a dimension independent of length. Thus, the objective constitutes a somewhat unorthodox test of conservation of number (Piaget, 1965).

A more usual test of conservation is to present two sets of objects, paired in one-to-one correspondence, and obtain agreement from the child that the sets are equal in number. One of the rows is then contracted, expanded, or otherwise rearranged, with the child watching, and the child is asked whether the sets still have the same number. Non-conserving children do not recognize that equivalence of number is maintained despite spatial transformation.

This test, along with most tests developed for laboratory study of conservation behavior, can be easily invalidated by teaching.[2] With enough rehearsal, the child will undoubtedly learn to state, "They still have the same number," after rearrangement; but there is every chance that he will merely be saying what he knows the teacher wants to hear. Although a minor problem in the laboratory, where rehearsal is usually deliberately avoided, this would be a serious weakness were the laboratory task to be used directly in an educational curriculum, particularly a "mastery" curriculum in which teachers are encouraged to directly "teach for" each specified objective.

The task specified in Objective 5:E is not subject to this problem. A large number of different test and practice items for the objective can be prepared, and each new item presented will require that the child figure out for himself which row has more objects. If he believes that longer (or denser) rows always have more, the teacher will surely discover it. This particular test of number conservation was chosen because in a pilot experiment it showed a strong correlation ($r = .77$) with the standard test of number conservation described above. More formal experiments to validate this finding are now underway.

Figure 26 shows the analysis of Objective 5:E. There are two alternative methods by which the child can solve the problem posed by this task. In the "counting method" (box IIa) he counts each set separately and then compares the stated numbers. This is equivalent to Objective 5:A, to which the reader is referred (box IVa). The "one-to-one correspondence method" (box IIb) requires that the child visually "pair" the objects in the two rows and then determine whether there are "extra" items in either set. With the exception of the components of visually pairing the objects (box IIIb) and remembering which have been paired (box IVb), this process is the equivalent of Objectives G and H in Units 1 and 2, which are therefore referenced in box Va. However, it should be recognized that the process of visual pairing, with its concomitant memory demand (box IVc) substantially increases the difficulty of the task and may be one of the reasons that young children tend strongly to respond to the physical shape of the array in conservation tests.

Insert Figure 26 about here.

In Objective 5:F the child must compare several sets, selecting the one with the most (or least) objects. The behavior analysis for this objective (Figure 27) shows a process of successive comparison. Two sets are compared and the largest selected; then the selected set is compared with the third set, and the largest of these two selected. The process is analogous to the one already described as a component of ordering numerals (Figure 20, boxes IIIb and IIIc). This primitive form of transitivity will also reappear in connection with seriating objects and sets in Unit 6.

Insert Figure 27 about here.

Seriation: Unit 6

A child's ability to seriate sets according to numerosity (Objective 6:C) demonstrates his comprehension of the ordered relationship among sets of different numbers, and thus is yet another indicator of the child's possession of an operational number concept. Seriation by size (Objective 6:B) and by numerosity jointly provide the basis for eventually establishing correspondence between ordinal and cardinal number.

This ability is treated as an important aspect of the number concept by Piaget (1965), although in America it has been almost completely overshadowed by conservation as a topic of interest to developmental psychologists.

There are at least two different methods of performing the seriation task. One method is to select the largest (or smallest) of the array, then the largest (or smallest) of those remaining, and continue until all items have been selected and placed. This is the method of "operational seriation" described by Inhelder & Piaget (1964). Figure 29 shows the analysis of this method for seriating objects; Figure 31 shows the analysis for seriating sets. The two objectives share a common set of prerequisites concerning the performance of sequential operations (boxes IIIb, IVb, and IVc in each figure). An additional hypothesized prerequisite for size seriation is the ability to simply recognize a misordering (box IIIc). According to our informal observations during attempts to directly teach seriation, many children who cannot seriate also lack this ability. The sharpest difference between size and set seriation seems to lie in the process of selecting the largest in the array. Selection of the largest size object can be accomplished by direct perceptual inspection, which permits comparison of several objects virtually simultaneously. Selection of the more numerous set, however, requires successive comparisons of pairs of sets (see Figure 27; Objective 5:F). Successive rather than simultaneous comparison is also required for size seriation when the task is performed tactually rather than visually, or when the differences between adjacent sizes are so slight as to require direct measurement. Tactual seriation is more difficult than visual seriation (Inhelder & Piaget, 1964). By analogy, it is reasonable to expect set seriation to be more difficult than visual size seriation. In addition, selection of the more numerous set requires operations of counting and of remembering numbers while counting, neither of which is required for size seriation. Thus, a reasonable prediction is that learning size seriation first will facilitate, but not directly produce, learning to seriate sets.

Insert Figures 28 - 32 about here.

Figures 30 and 32 show analyses of a second method of seriation. Using this method, the child orders two objects or sets, then places a third item with respect to the first two. He continues placing new items until all items have been ordered. A primitive form of transitivity operates in this solution in that the child need not directly compare each new set with all sets already ordered. As shown in box IIe of each figure, he stops as soon as he finds a set smaller than the new set he is trying to place, assuming that all subsequent sets will also be smaller. Of course, at an early stage in learning the child might indeed make many logically unnecessary direct comparisons. However, in skilled performance of the seriation task, the extra comparisons should drop out.

As in the first method, the size and set seriation tasks share prerequisites concerned with spatial organization and maintenance of sequence. However, set seriation requires, in addition, counting and memory functions (see boxes IIIa and IIIb of Figure 32), and thus should be the more difficult skill to acquire.

The two methods of seriations described here for ordering according to size and numerosity are directly analogous to the two methods identified earlier for ordering numerals (Objective 3-4:F: Figures 20 and 21). The same methods could be applied to problems

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of ordering weights, color intensities or other dimensions. Thus, the logical operations of seriation are not restricted to size or numerosity, and considerable positive transfer from one seriation task to another can be expected. There is some reason to believe that the second method, which requires successive comparisons, is the more generalizable, since, logically, it would not need to be modified to apply to problems (such as tactual seriation or weight seriation) in which simultaneous perceptual comparisons of several objects were impossible. This hypothesis, however, is in need of a direct empirical test.

Addition and Subtraction: Units 7 and 8

Unit 7 introduces the concepts of union and partition of sets, in the form of addition and subtraction. These concepts are included in the introductory part of the PEP curriculum, in order to round out and stabilize the child's concept of set and number and to prepare him for a more abstract stage of mathematical understanding. Children who learn to count reliably under various conditions, as in Units 1 and 2, and who learn the relation of counting to other components of the number system, as in Units 5 and 6, often seem to move naturally into addition and subtraction. For these children, an expanded definition of "four" can include the fact that it can be made of two "two's," or of a "three" and a "one," and later, that two "fours" can be combined to make an "eight." The aim of this unit is to develop these basic concepts rather than to have the child memorize the addition and subtraction combinations.

To implement this goal Unit 7 contains objectives that specify two different methods of adding and subtracting. In Objectives A and B (Figures 33 and 34) the child learns to use "counters" (these could be tally marks as well as counting blocks, chips, or other objects) to establish sets and then unite (A) or partition (B) them. In Objectives C

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and D (Figures 35 and 36) number is translated into length as the child uses a number line in his calculations. The behavior analyses of these skills suggest that using a number line is a more complex task than using counters. As shown in Figures 35 and 36, the number line requires basic spatial organization skills (box IIIc) in addition to appropriate use of the "zero" position, and the reading of numerals. None of these behaviors are directly called for in adding or subtracting with counters. It is likely, therefore, that Objectives A and B will be learned more easily than C and D. However, since the two processes seem quite independent, in the sense of having few common prerequisites, they have been treated as separate branches within the unit. Should later studies of hierarchical relationships among these objectives suggest that learning A and B first would strongly facilitate learning C and D, these objectives would be combined into a single linear sequence.

Insert Figures 33 - 36 about here.

Only after the basic concepts of addition and subtraction are established does the curriculum introduce word problems and written formats (Objectives E, F, and G) as specific objectives. Objectives F and G require a straight forward reading of symbols and have not been separately analyzed. Solving "word problems" (Objective E), however, is frequently quite difficult even for children who can solve symbolically presented addition and subtraction problems. These children have difficulty in translating the verbal statements into a familiar and solvable addition or subtraction problem. Figures 37 and 38 present preliminary analyses of the process of translation. Further analyses of this kind are now being undertaken, preparatory to experiments in teaching children to solve verbally presented mathematics problems.

Insert Figures 37 and 38 about here.

For many children the written equation or word problems may be the best way of giving instruction in Objectives A through D. These children will pass Objectives E, F, or G simultaneously with A - D. However, the separation of concept from symbolization in the formal curriculum permits children who need to work on one problem at a time to do so, and to experience measurable success at an early stage.

The expansion of equation formats in Unit 8 is not simply a matter of algebraic virtuosity. Rather, each step in the sequence is designed to direct the child's attention to some basic mathematical concept. It is assumed that counters or a number line will continue to be used, both as an aid to calculation and as a means of highlighting the number concept underlying the algebraic processes. Objectives A and B (Figures 39 and 40), for example, are intended to show the child that there are many ways of composing a given number. They also provide occasion for demonstrating the fact that $x + y$ is always equivalent to $y + x$, the rule of "commutativity," although this rule need not be formally learned at this stage. Objective D (with C as a transition) requires the child to complete an equation with one addend plus the sum given. This is very difficult for young children and requires considerable flexibility in the manipulation of addition concepts. One way of performing the task, as shown in Figure 42, is to treat it as a subtraction problem (box IIb and below). To highlight the addition-subtraction complementarity, Objective E has been placed at the same level as D, suggesting that the two objectives be taught simultaneously. E requires

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the child to construct subtraction equations that are complementary to a given addition equation. Figure 43 shows both "counters" and "number line" methods for demonstrating the relationship. In Objective F the child is freed from pre-set problems; he now composes equations in all the formats he has experienced. With this objective, the child can be assumed to have developed a self-monitored control over number operations.

Insert Figures 39 - 43 about here.

Use of the Curriculum by Schools

The curriculum presented here provides an organized set of learning objectives around which instructional programs of many types can be organized. The particular form of instruction--group versus individual; "programmed" versus "discovery," etc.--is not specified. This omission is deliberate. The important question in a mastery curriculum is not how an objective is taught but whether it is learned by each child. On this view, the school's job is to assure that all children do learn, regardless of time needed or specific teaching method. In this work, a carefully sequenced curriculum is one of the essential tools.

In practice, implementation of a mastery curriculum implies that children will be permitted to proceed through the curriculum at varied rates and in various styles, skipping formal instruction altogether in skills or concepts they are able to master in other ways. This demand for individualization, in turn, requires that there be some method of assessing mastery of the various objectives in the curriculum. If children are to work only on objectives in which they need instruction and for which they are "ready," in the sense of having mastered major prerequisites, then teachers need to feel considerable assurance that mastery has in fact occurred.

In PEP classrooms, the need for assessment is met through frequent testing and systematic record keeping. A brief test for each objective in the curriculum has been written (Wang, 1969). These tests directly sample the behavior described in the objective. If the objective is counting objects, for example, the child is given sets of objects to count. If the objective involves seriating rods, he is given rods to place in order. The test informs the teacher of the presence or absence of the behavior in question. Thus the test items are a direct reflection of the curriculum objectives and define very explicitly what the child is expected to learn.

After a child is socially comfortable in the classroom and routines are well established, the teacher or aide takes him aside and begins the testing program. The first task is to find his "entering level." This is normally done by administering a special "placement test," composed of a sampling of items from the units. Children can be rated as passing or failing each unit on the basis of this test. For units failed, tests on the individual objectives may then be administered to determine exactly which objectives the child needs to work on. The placement testing procedure is an efficient one in terms of testing time, especially for groups in which the entering levels of individual children are expected to spread over a wide range. An alternate procedure is to administer the unit tests themselves, beginning with Unit 1 and moving through subsequent units until the child stops passing tests. This is the point in the curriculum in which instruction should begin.

When a child does not pass a test, indicating that he needs work on a given objective, he is given one or several "prescriptions", or assignments, of activities relevant to learning that objective. Prescriptions in the mathematics curriculum are extremely varied. For independent work by children, they range from interactive games for two or more children to formal written worksheets. Small group and individual "tutorials" with the teacher are also prescribed when needed. Conceptual mathematics teaching materials such as those developed by Montessori, Dienes, and Cuisenaire are used, along with material from virtually every major educational supply house in America. Audio-visual devices such as the Language Master and Audio Flashcard machines are used, and other devices are being investigated. Each teacher also continues to develop many materials on her own to meet specific needs.

PEP has a basic bias in favor of manipulative materials for early mathematics experiences. Even with 6-year-olds, teachers are asked to use pencil and paper methods sparingly at first, to begin work on a new objective using manipulative materials, and to keep those materials available in support of more symbolic performance for as long as the child wants them. Except for general guidelines of this kind, teachers choose among the various materials according to their own judgements of the child's need. Although the objectives are carefully sequenced, there is currently no fixed sequence of lessons for a given objective.

In this process the testing program serves the teacher as a constant check on her success. When a child has completed prescribed work on an objective, he is retested, and if necessary further instruction is provided until mastery is demonstrated. A child may work on several different objectives during a given instruction period, working up independent branches of the curriculum sequence. As the child moves through the curriculum, a pre-test on each new objective assures that he will be allowed to skip over objectives he has been able to learn on his own.[3]

It is important to indicate that the testing experience is generally pleasurable for the child. For one thing, he is getting individual attention from a teacher. Equally as important, the testing strategy assures that his dominant experience will be one of success, for he begins with the simplest tests and stops as soon as he begins to have difficulty. Furthermore, the PEP teaching staff makes a special point of praising and otherwise rewarding good test performance (and not commenting on poor performance). Nevertheless, many schools may find the heavy emphasis on formal testing too unwieldy, too costly, or simply incompatible with a preferred style of teaching. For such schools, the testing program can be modified in various ways while still retaining the benefit of the structured sequence of curriculum objectives.

The most radical such modification would be to do away with formal testing altogether and to use the curriculum sequence itself as a guide to the kinds of learning experiences to be provided to children at different points in their intellectual development. Such a use of the curriculum would, we believe, be compatible with the "free" organization of classrooms following the English infant school or "Leicestershire" model of early education (Plowden, 1966). Its success would depend on the ability of the teacher to make accurate judgements of children's capabilities on the basis of informal observations. Thus, it demands a highly skilled teaching staff.

A less demanding modification would be to retain the tests, but to administer them only at well spaced intervals, rather than on the nearly continuous schedule used in PEP classrooms. This would provide periodic "checks" on the teacher's intuitive judgement of progress. A related modification would use only the placement test items. This would determine the unit on which the child needed work,

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but leave judgements as to exactly where within the unit he should begin up to the teacher. The success of such a procedure, of course, would depend upon how well chosen the placement-test items were--i.e., to what extent they accurately predicted the child's ability to perform all objectives in the unit from which they were drawn. Accurate selection of items, in turn, depends upon validation of the hierarchical sequence within each curriculum unit (cf., Cox & Graham, 1966; Resnick & Wang, 1969). A series of hierarchy validation studies for the PEP introductory mathematics curriculum is currently underway. The results of these studies will be used in designing a shortened testing procedure for use in PEP classrooms.

Continuing validation studies of this kind, together with regular data from the classroom testing program, will also provide the basis for revision of the curriculum objectives over an extended period of time. This is a crucial aspect of the project's strategy of curriculum design, and is one reason for the PEP program's heavy emphasis on testing. The tests provide a form of continuous "feedback" on the strengths and weaknesses of the curriculum. From these data specific sections needing revision can be identified. Such revisions can include modifying, adding, dropping or reordering objectives to maximize ease and reliability of learning.

Given this approach to curriculum design, implementation of the curriculum in a school does not mark the conclusion of a research or curriculum writing program, but the creation of a "laboratory" in which empirical study of the curriculum can proceed while at the same time children's immediate needs are being met. Thus, the curriculum outlined here should be regarded as still under study and development. By reporting it at this intermediate stage, we hope to provide both a practical guide for educators seeking to develop a systematic early learning program and a basis for continuing exchange among researchers interested in questions of early mathematics learning.

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Footnotes

[1]Inquiries and requests for reprints may be directed to Lauren B. Resnick, Learning Research and Development Center, University of Pittsburgh, Pittsburgh, Pennsylvania 15213.

[2]For a critique of experimental tests of conservation, see Rothenberg (1969).

[3]The effectiveness of the general procedure can be estimated from data on the results of the first year of the PEP program at Frick Elementary School (Wang, Resnick, & Schuetz, 1970). Kindergarten children from a predominantly black and poor neighborhood learned, on the average, 23 mathematics objectives between November and June. Most of the children had mastered the equivalent of the present Units 1 through 4 by the end of the year, and were working on counting and numerals to 20 as well as simple addition and subtraction problems. On the Wide Range Achievement Test, the median percentile rank in arithmetic for these children was 73. The same children had a median percentile rank of 39 in reading, a subject in which no special instruction had been offered.

Figure Captions

Figure 1: Unit 1. Counting One-to-One Correspondence, to 5.

Figure 2: Unit 2. Counting One-to-One Correspondence, to 10.

Figure 3: Unit 3. Numerals to 5.

Figure 4: Unit 4. Numerals to 10.

Figure 5: Unit 5. Comparison of Sets.

Figure 6: Unit 6. Seriation and Ordinal Position.

Figure 7: Unit 7. Addition and Subtraction.

Figure 8: Unit 8. Addition and Subtraction Equations.

Figure 9: Sequence of Introductory Mathematics Units.

Figure 10: Behavior Analysis of Objective B, Units 1 and 2.

Figure 11: Behavior Analysis of Objective C, Units 1 and 2.

Figure 12: Behavior Analysis of Objective D, Units 1 and 2.

Figure 13: Behavior Analysis of Objective E, Units 1 and 2.

Figure 14: Behavior Analysis of Objective F, Units 1 and 2.

Figure 15: Behavior Analysis of Objective G, Units 1 and 2.

Figure 16: Behavior Analysis of Objective H, Units 1 and 2.

Figure 17: Behavior Analysis of Objective I, Units 1 and 2.

Figure 18: Behavior Analysis of Objective D, Units 3 and 4.

Figure 19: Behavior Analysis of Objective E, Units 3 and 4.

Figure 20: Behavior Analysis of Objective F, Units 3 - 4 (Alternate 1).

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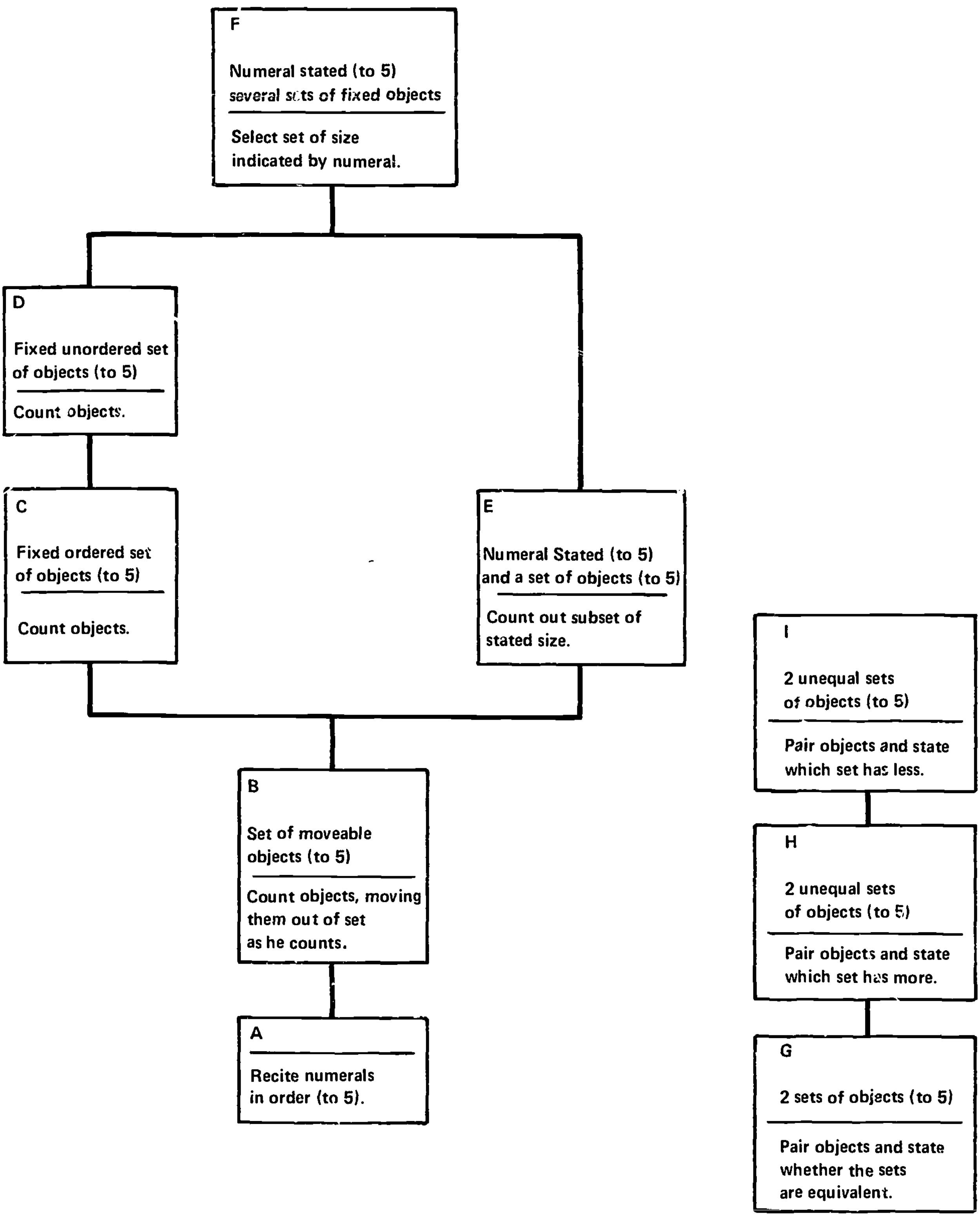

Figure 1: Unit 1. Counting One-to-One Correspondence, to 5.

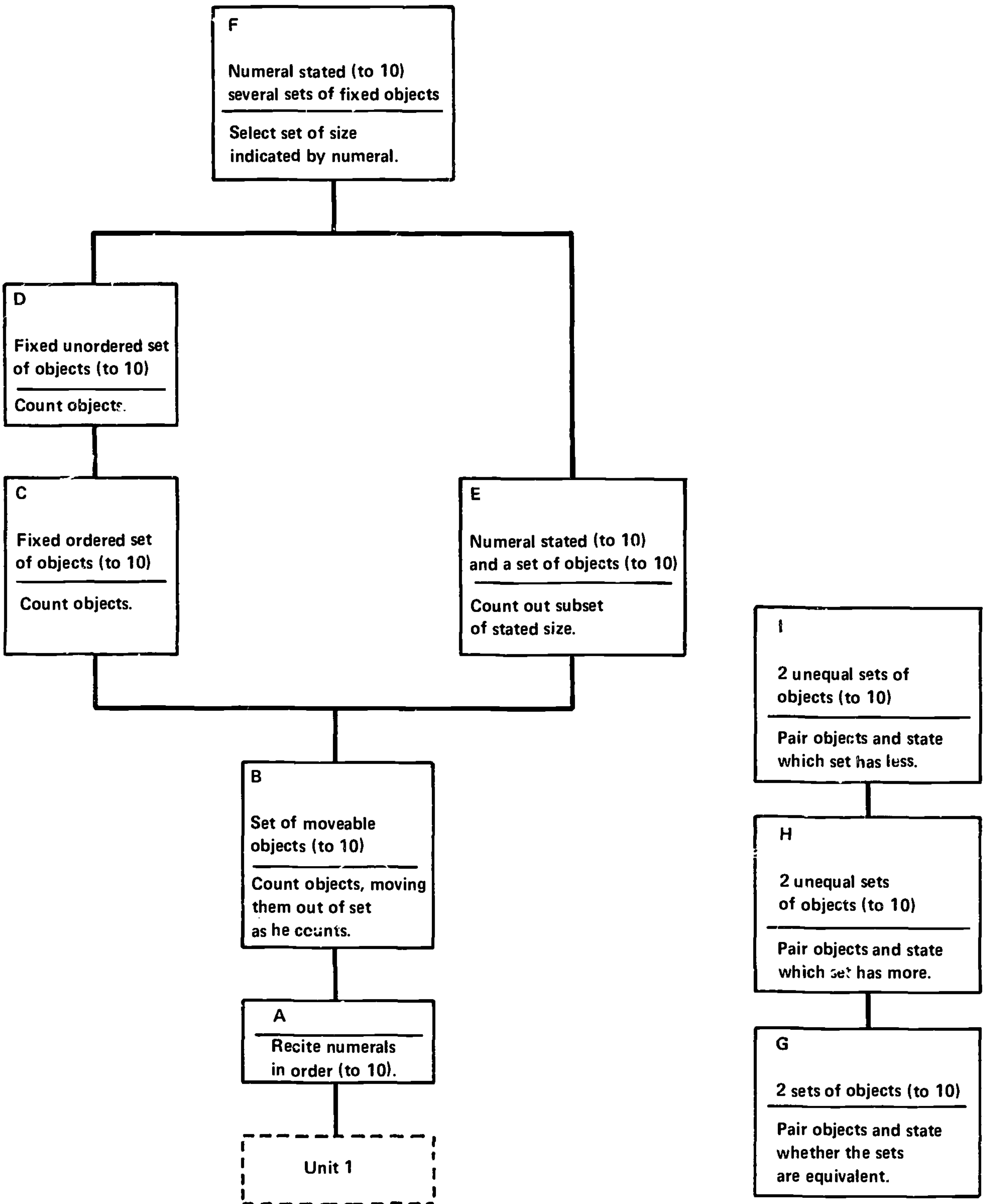

Figure 2: Unit 2. Counting One-to-One Correspondence, to 10.

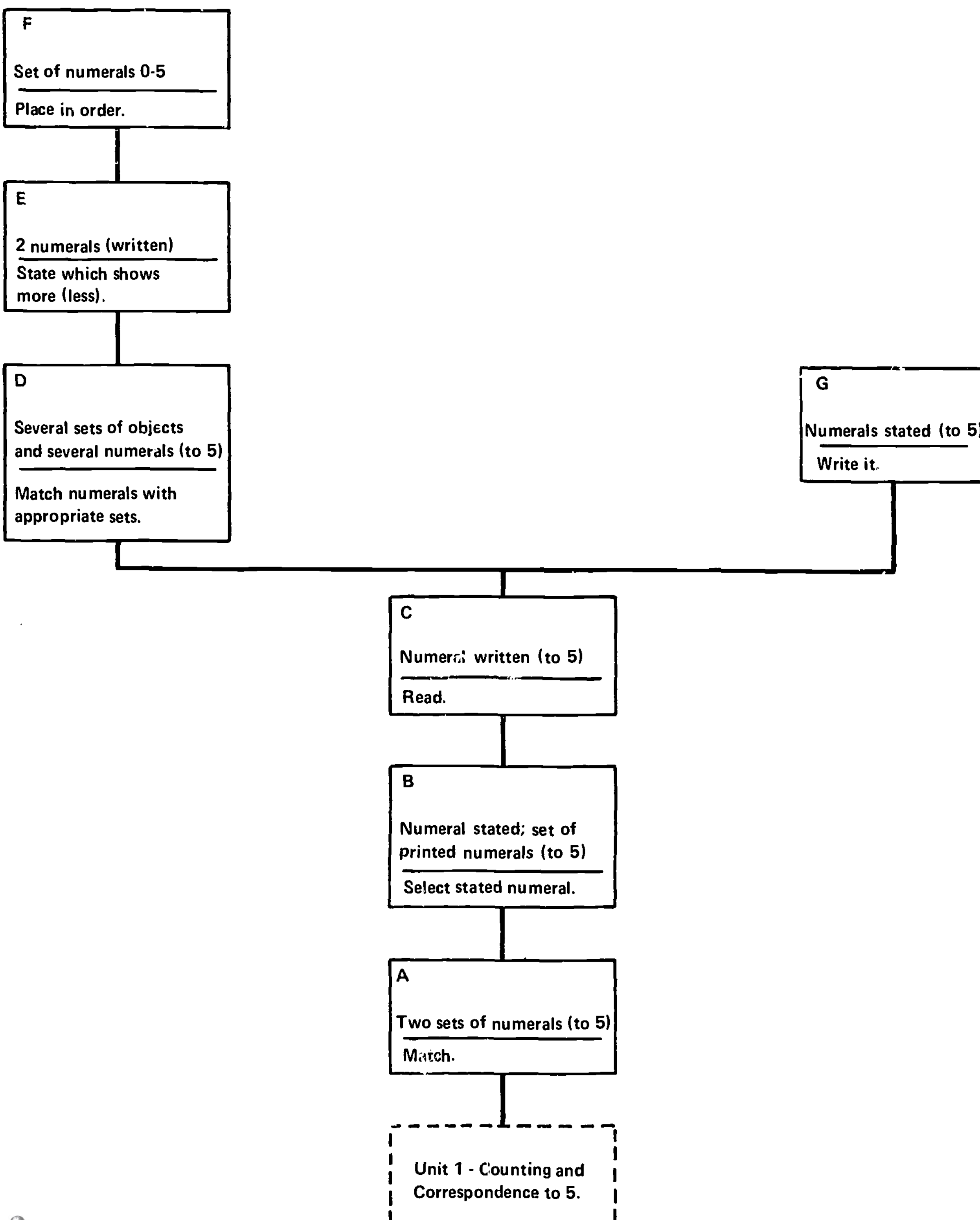

Figure 3: Unit 3. Numerals to 5.

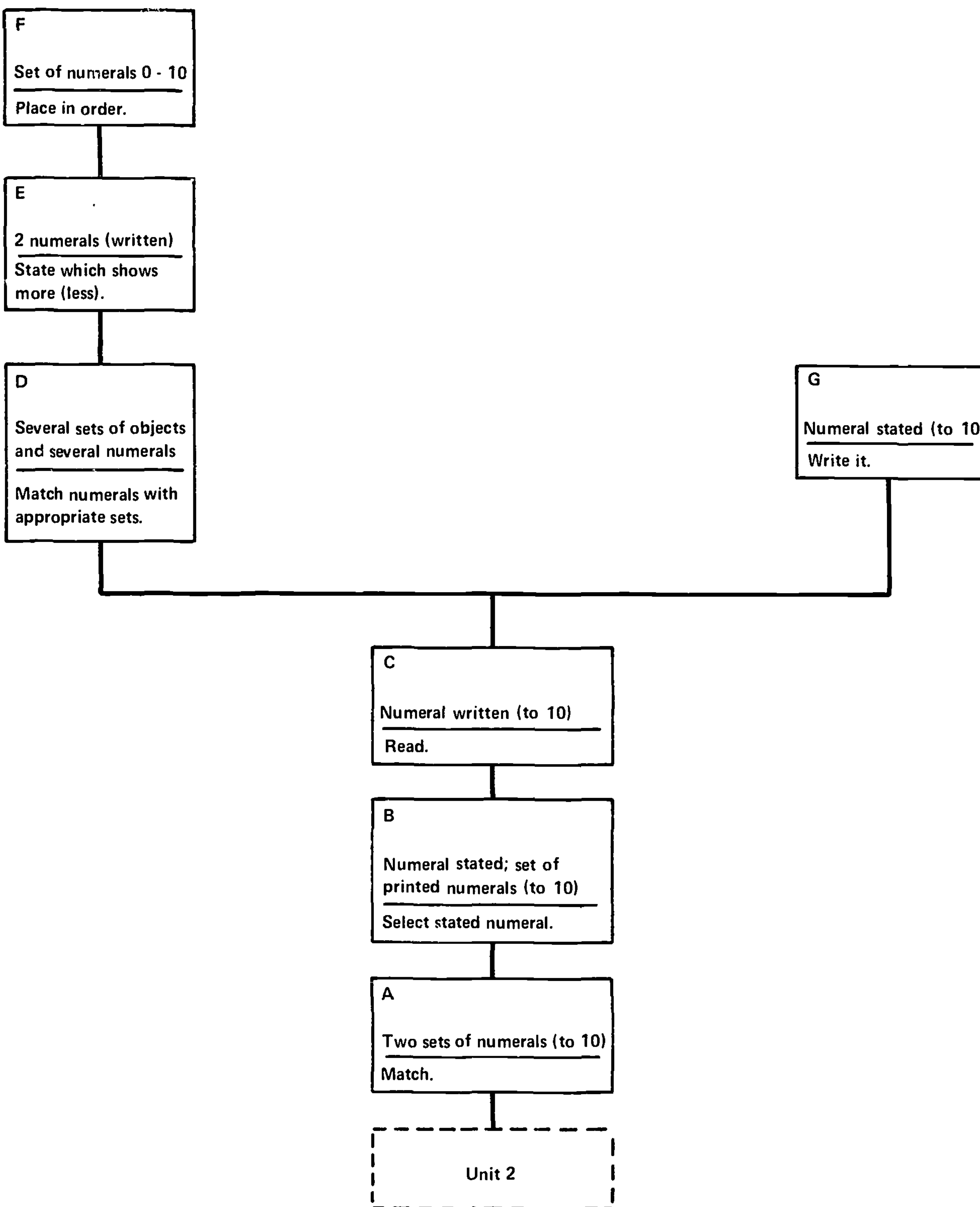

Figure 4: Unit 4. Numerals to 10.

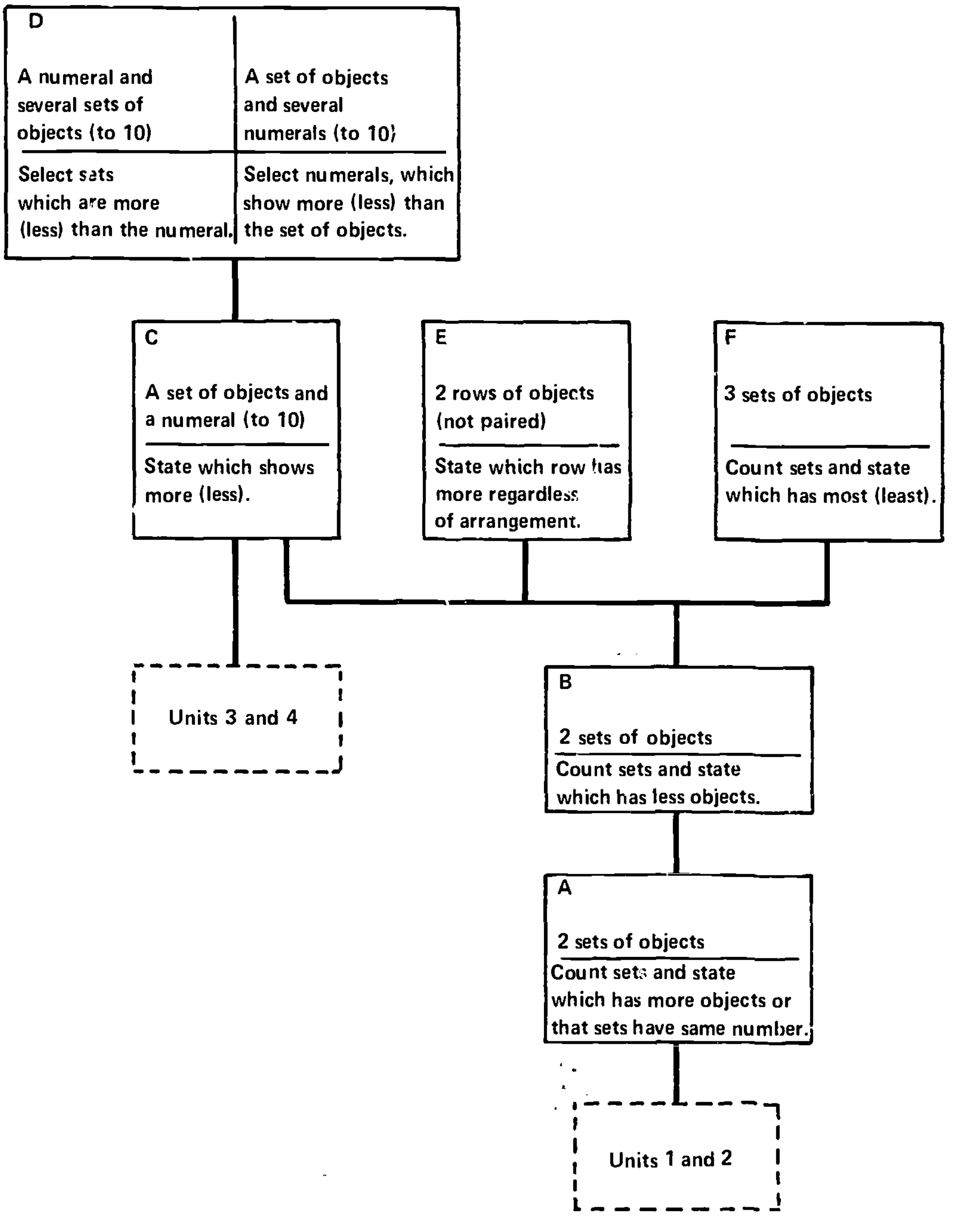

Figure 5: Unit 5. Comparison of Sets.

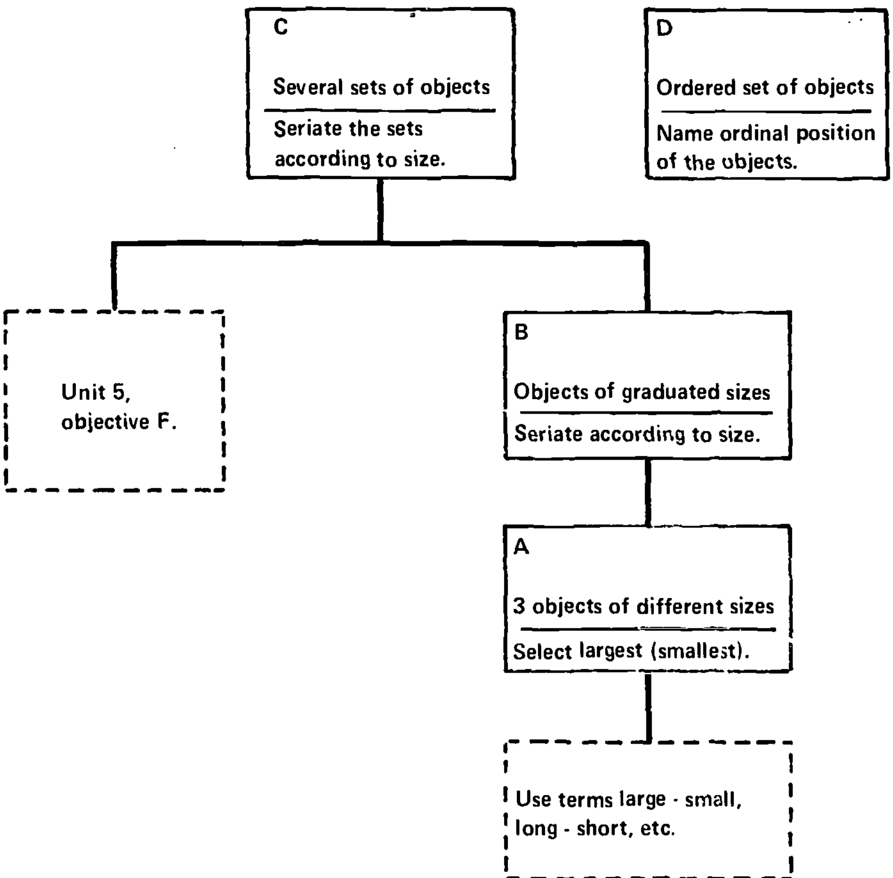

Figure 6: Unit 6. Seriation and Ordinal Position.

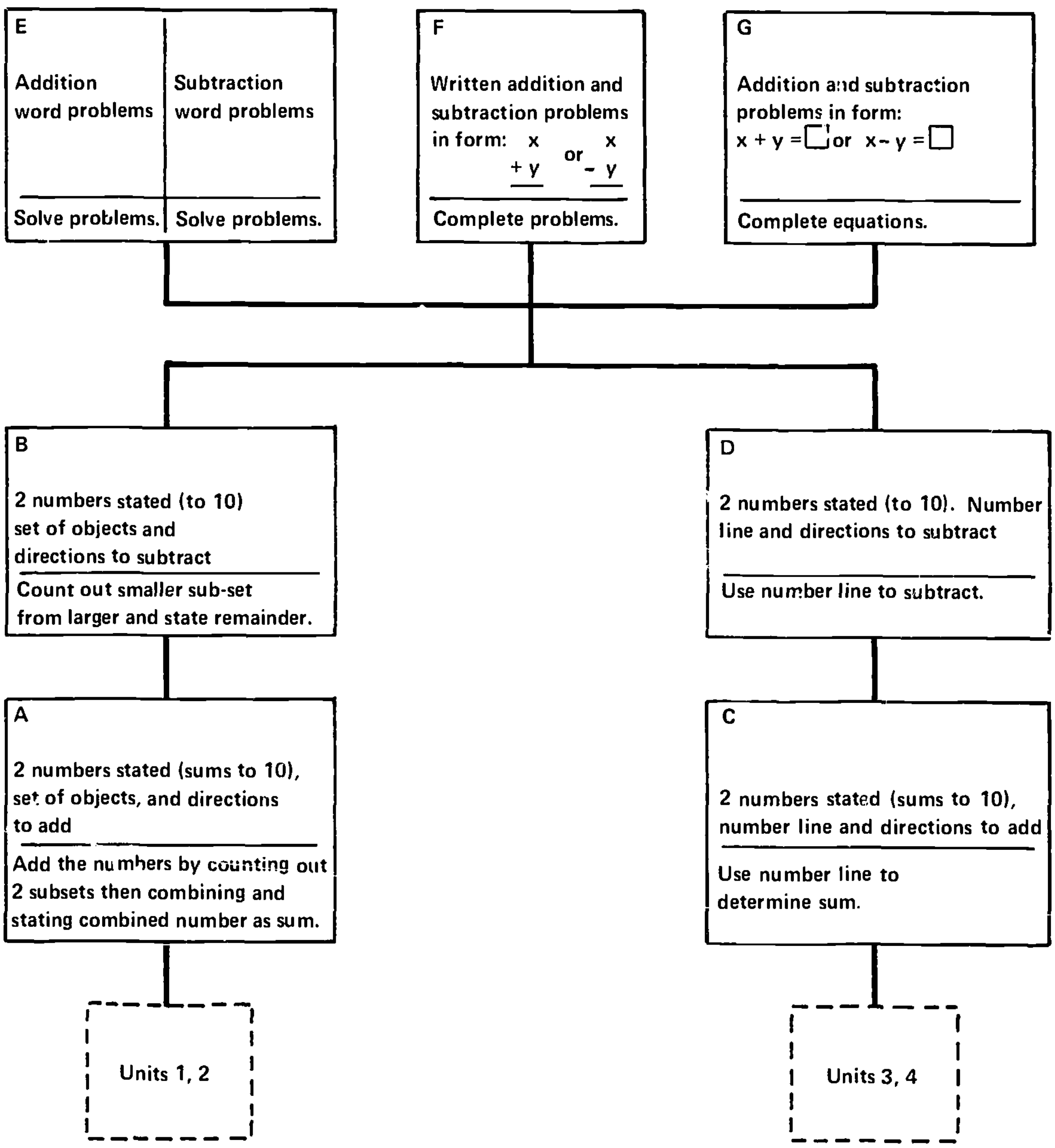

Figure 7: Unit 7. Addition and Subtraction.

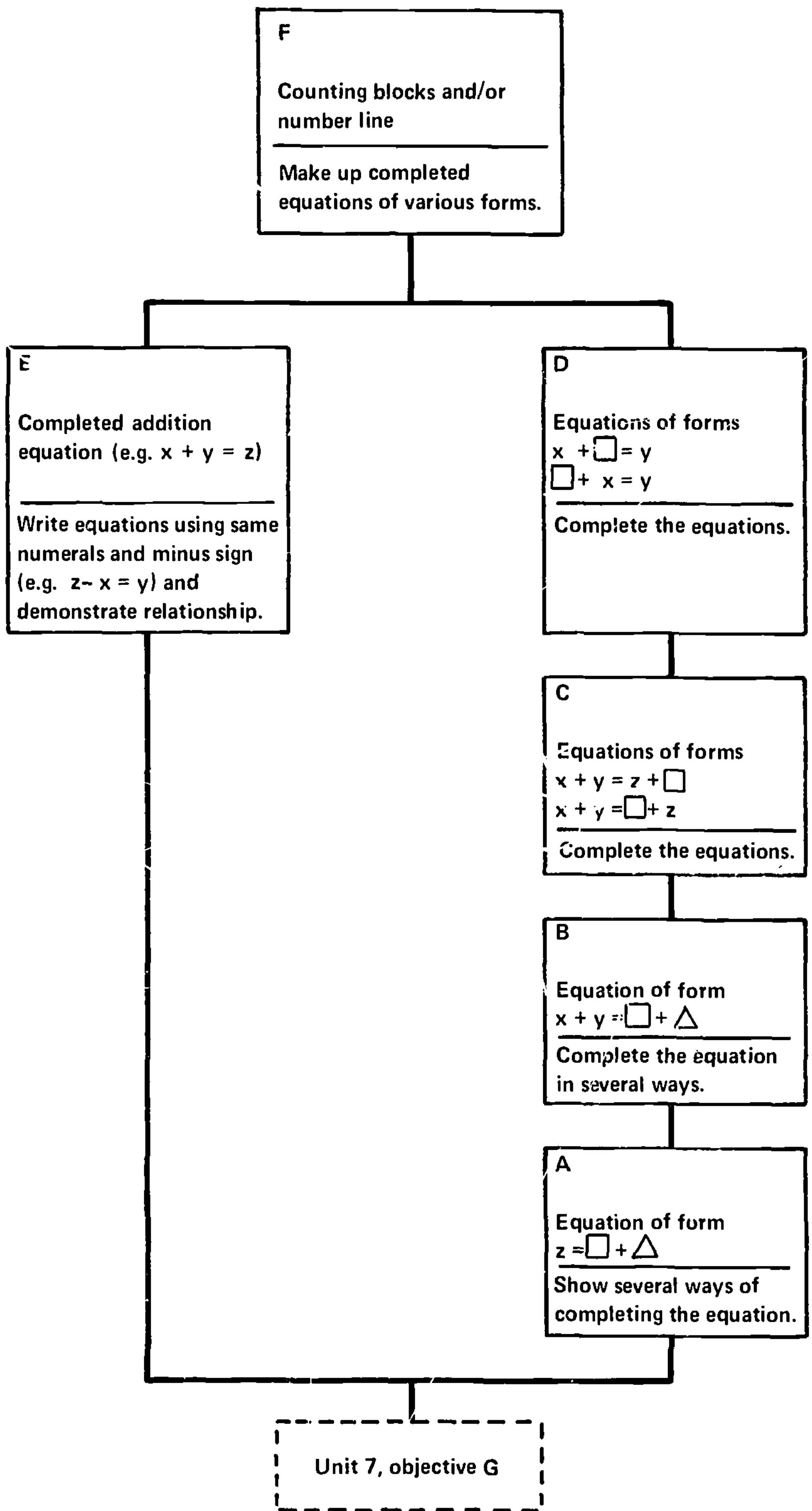

Figure 8: Unit 8. Addition and Subtraction Equations.

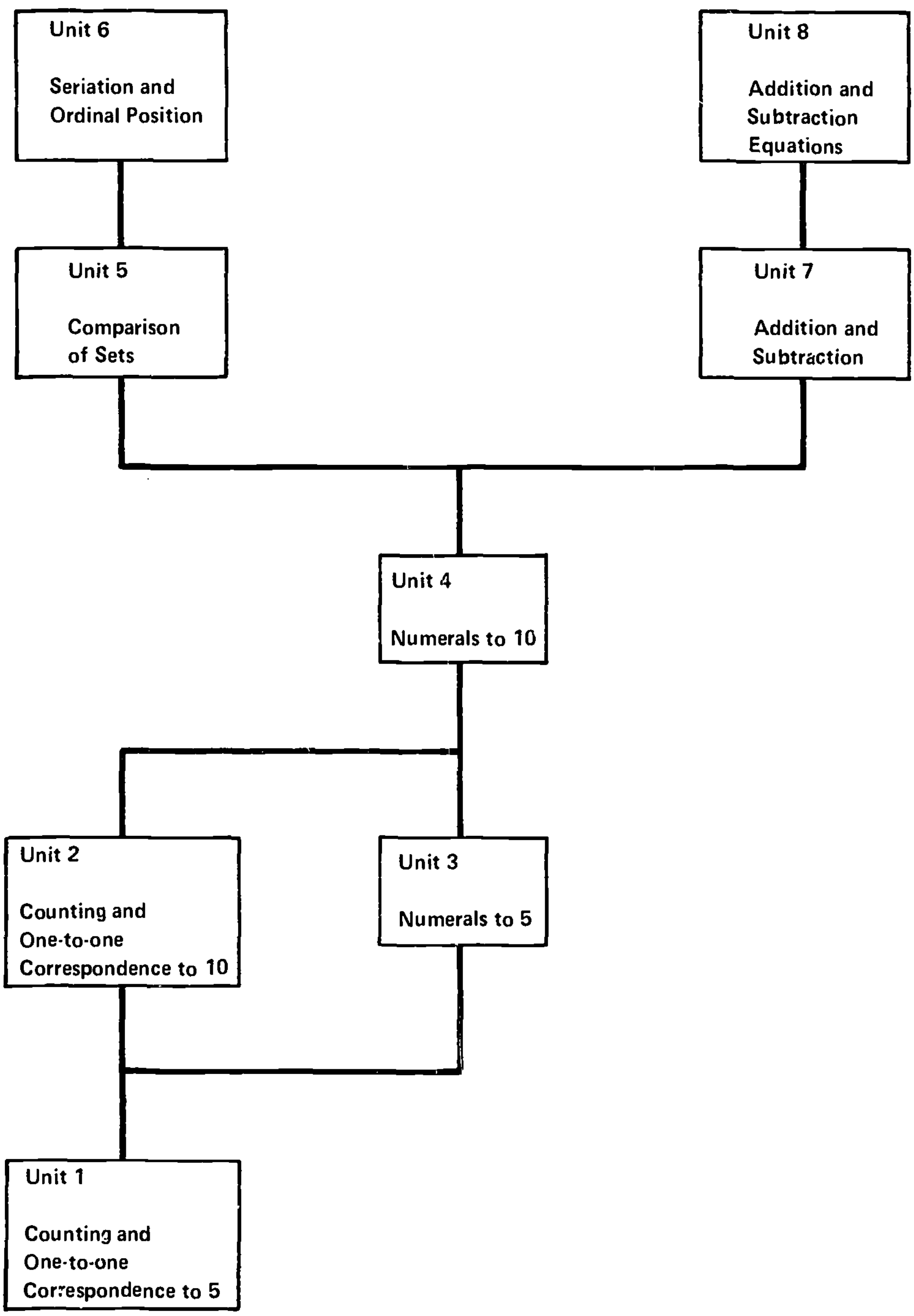

Figure 9: Sequence of Introductory Mathematics Units

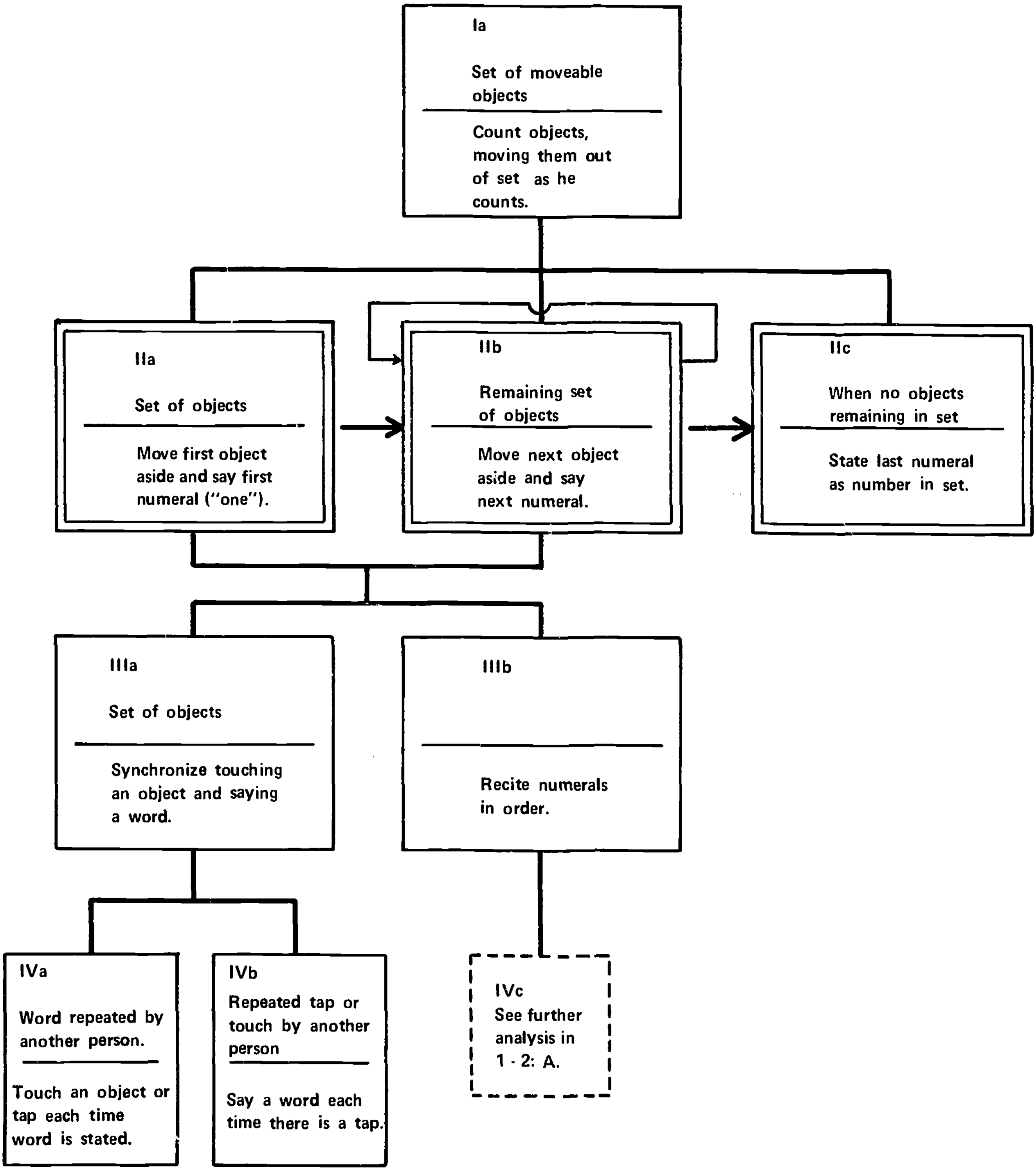

Figure 10: Behavior Analysis of Objective B, Units 1 and 2.

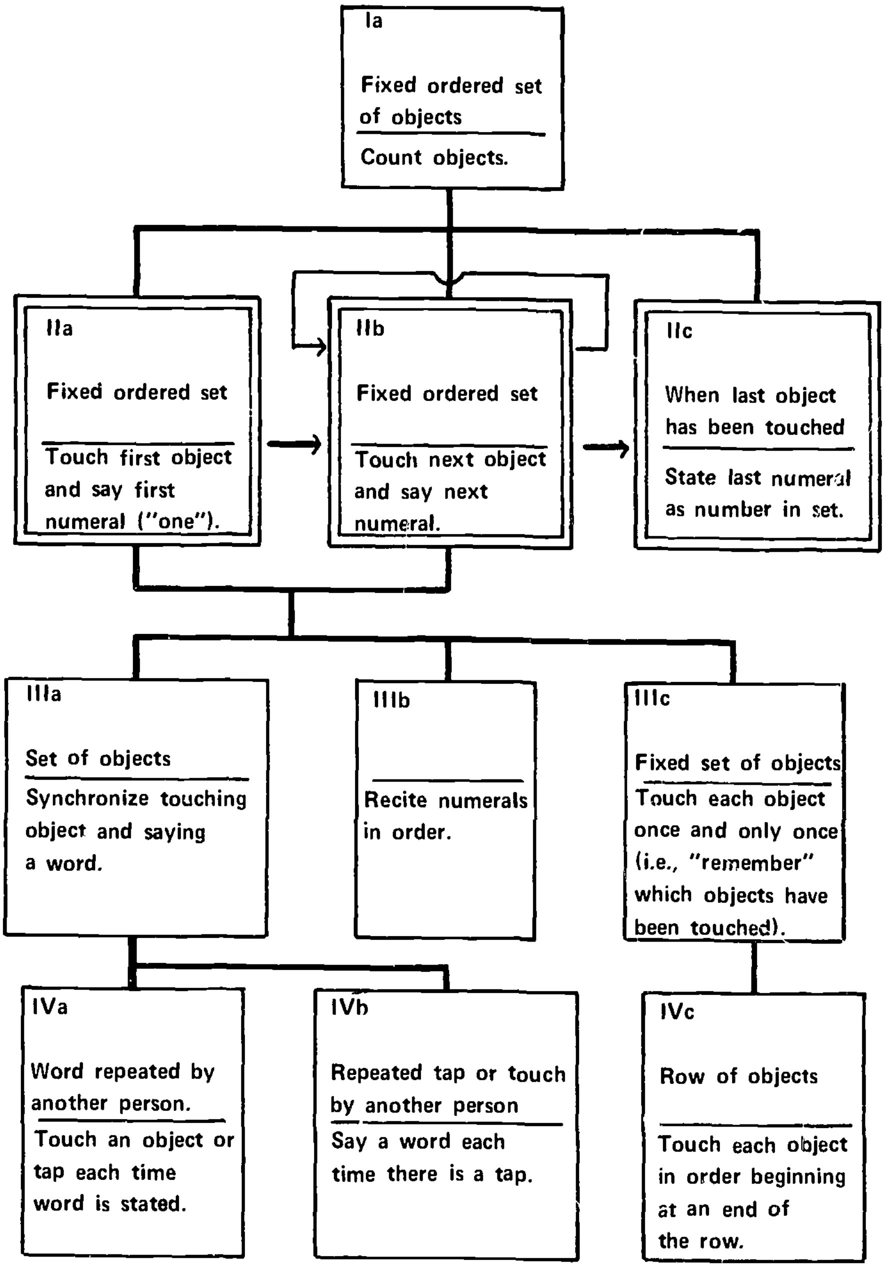

Figure 11: Behavior Analysis of Objective C, Units 1 and 2.

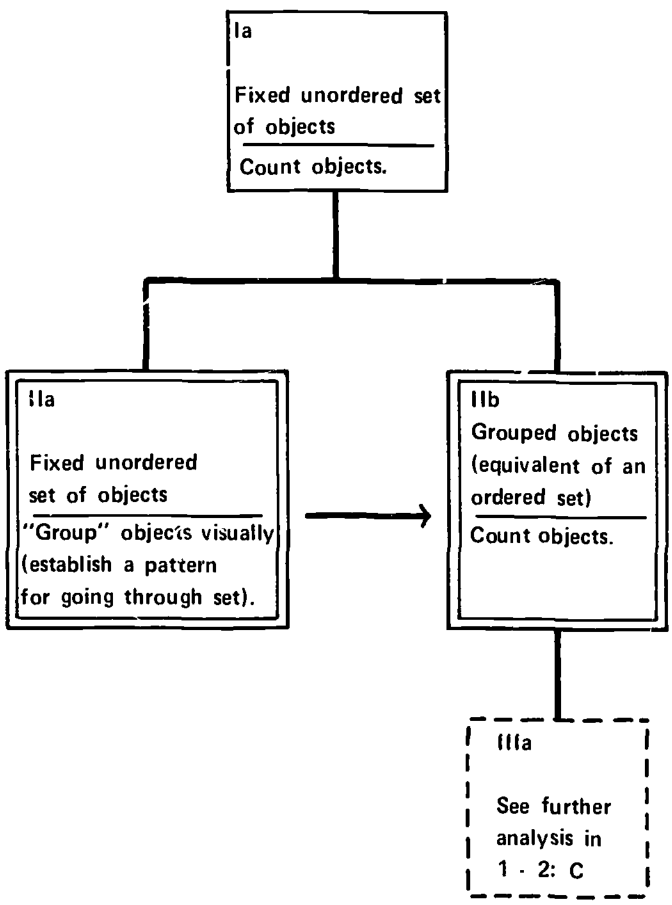

Figure 12: Behavior Analysis of Objective D, Units 1 and 2.

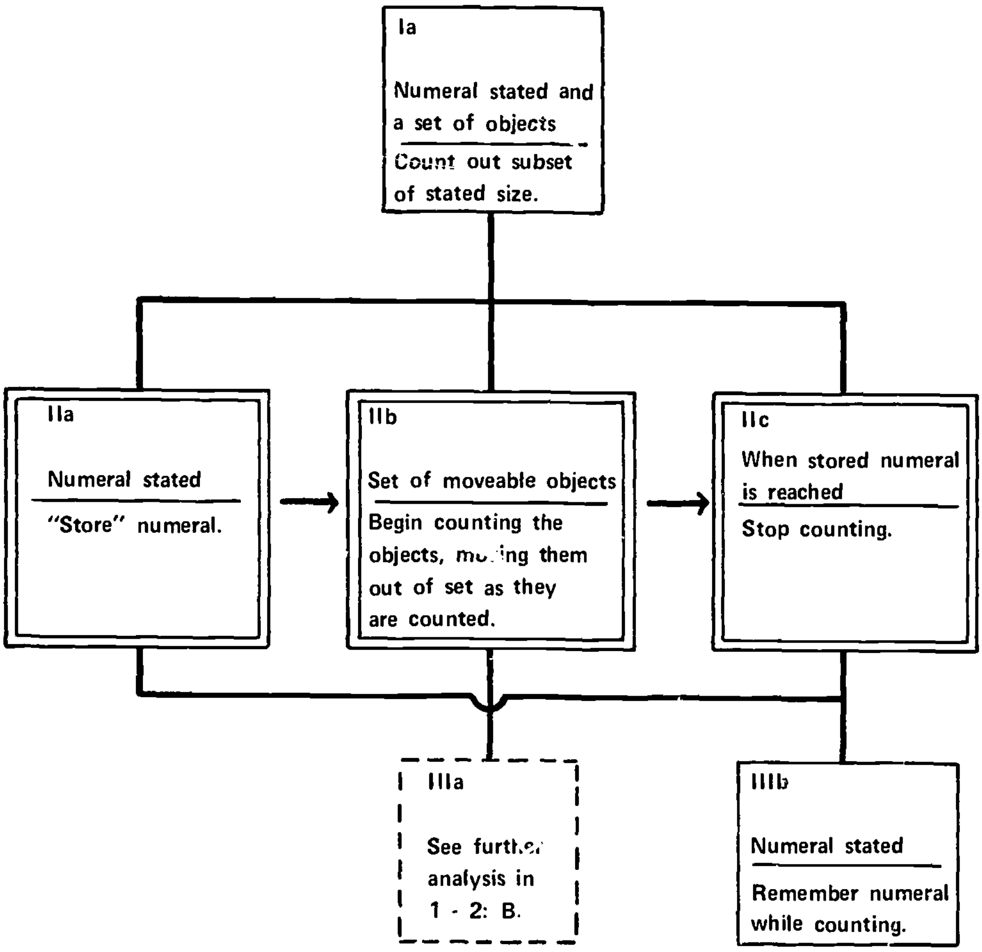

Figure 13: Behavior Analysis of Objective E, Units 1 and 2.

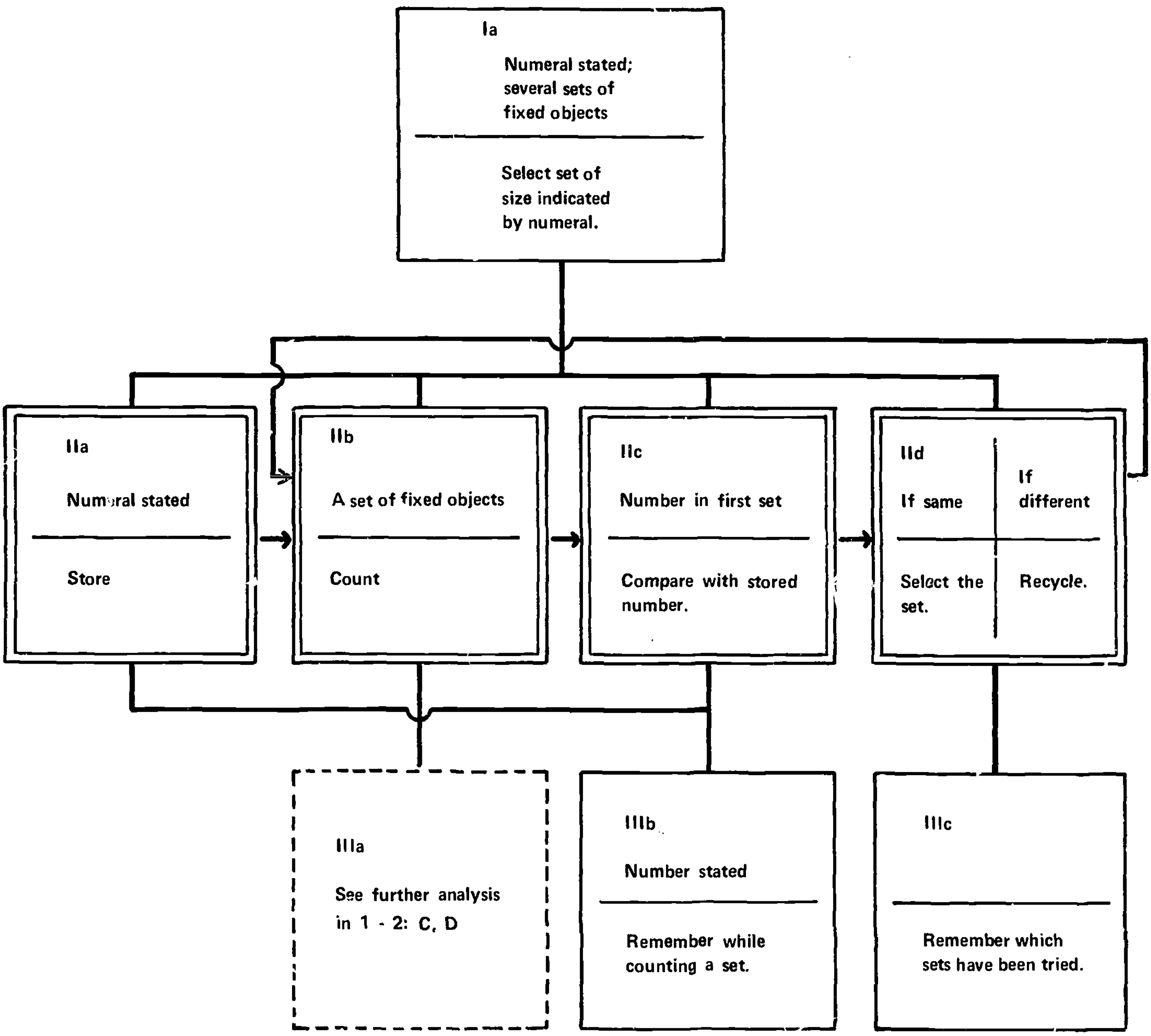

Figure 14: Behavior Analysis of Objective F, Units 1 and 2.

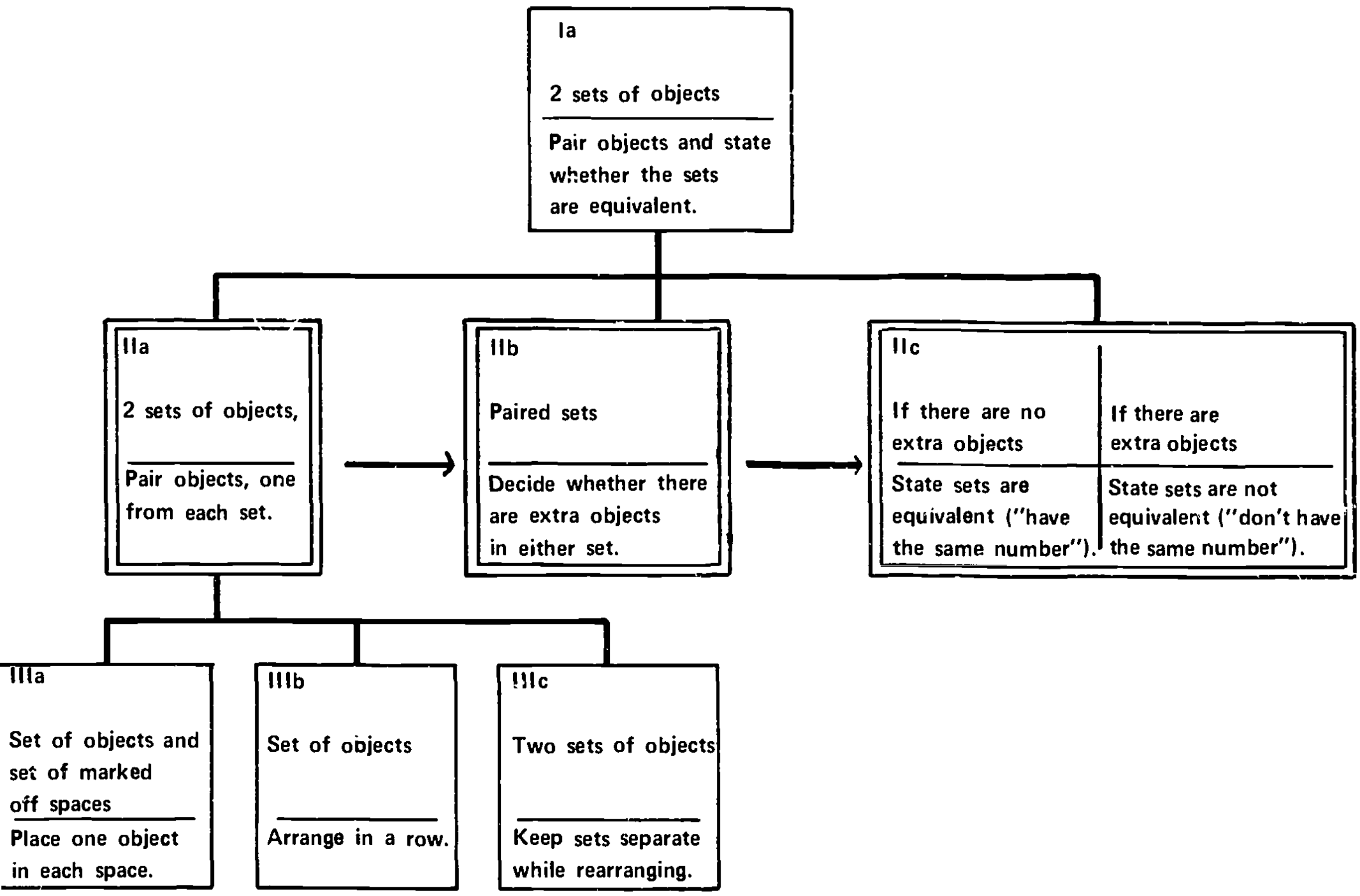

Figure 15: Behavior Analysis of Objective G, Units 1 and 2.

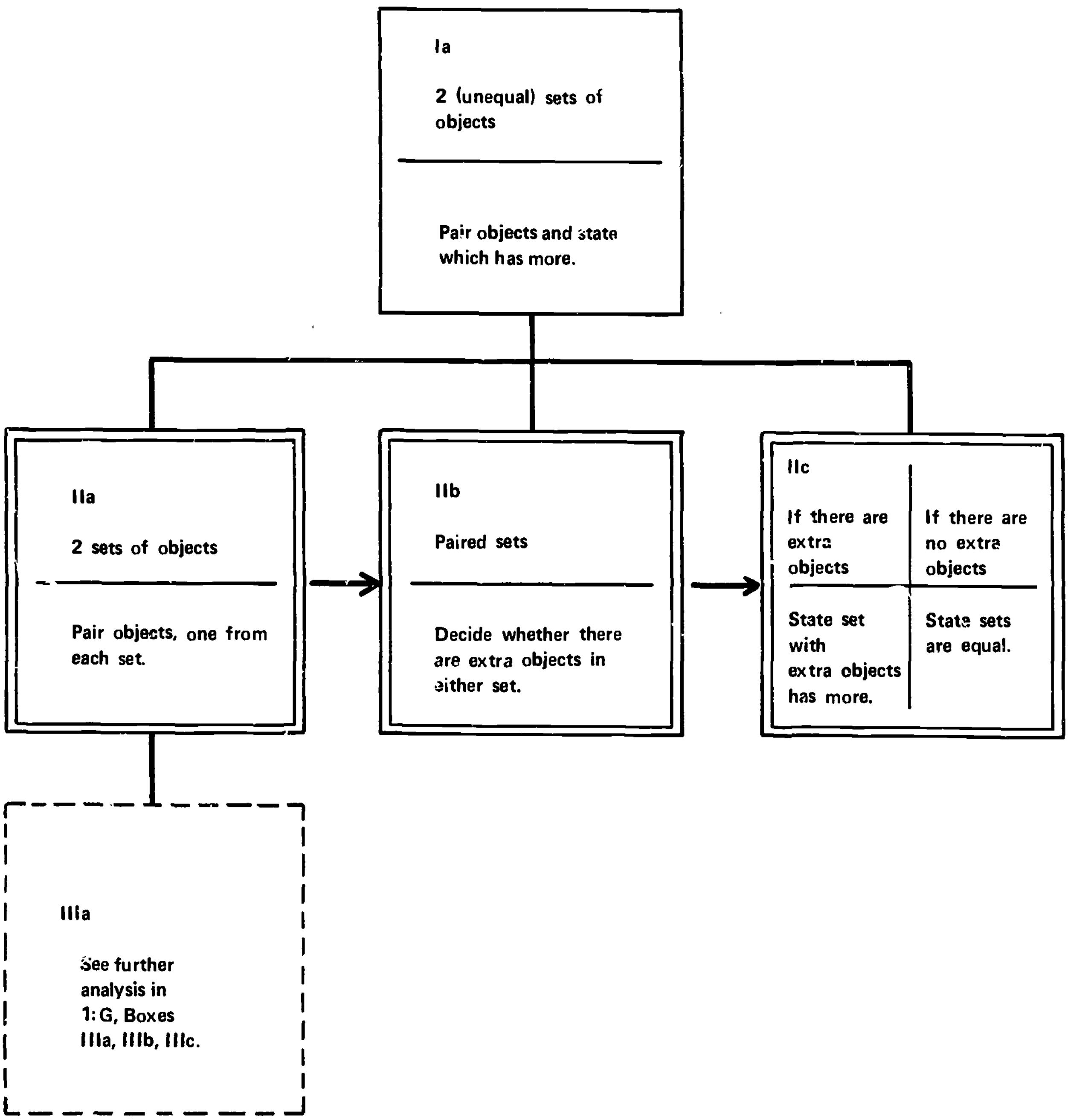

Figure 16: Behavior Analysis of Objective H, Units 1 and 2.

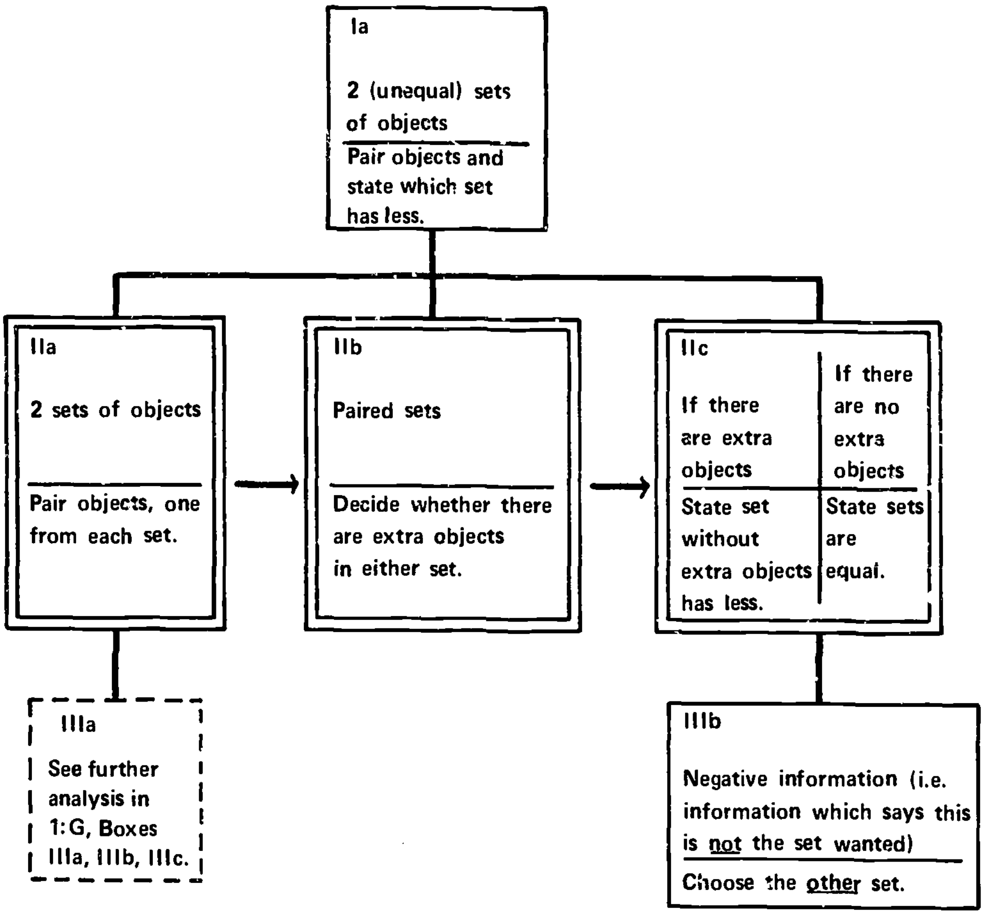

Figure 17: Behavior Analysis of Objective I, Units 1 and 2.

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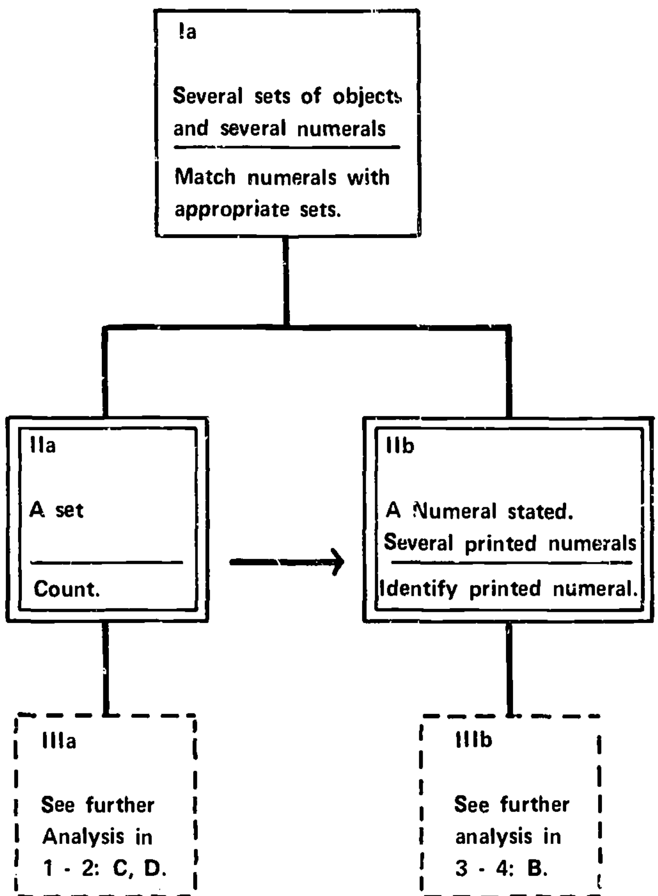

Figure 18: Behavior Analysis of Objective D, Units 3 and 4.

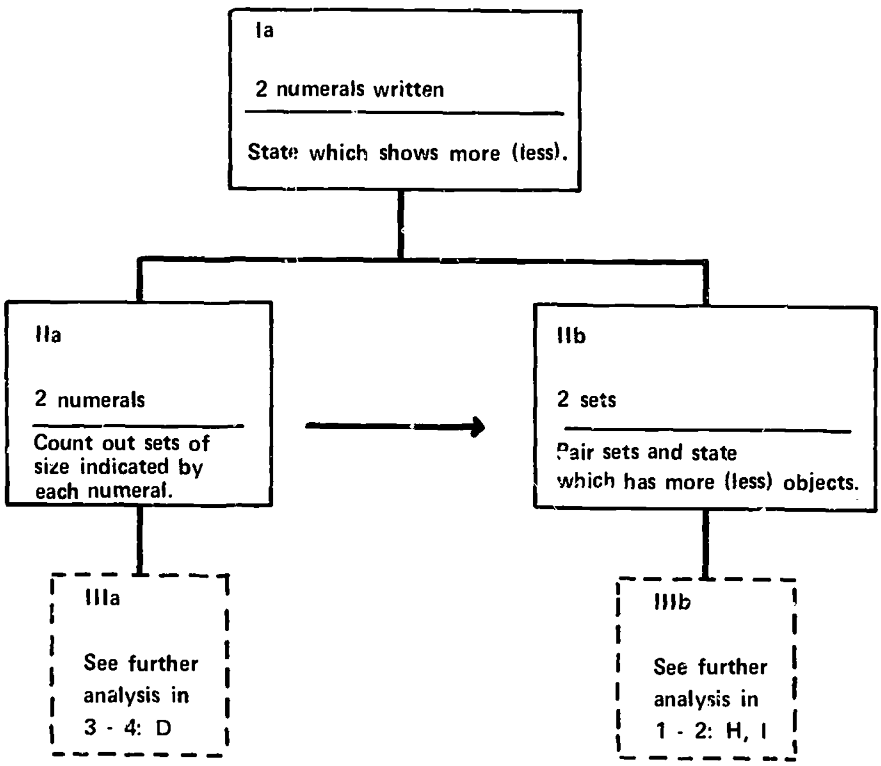

Figure 19: Behavior Analysis of Objective E, Units 3 and 4.

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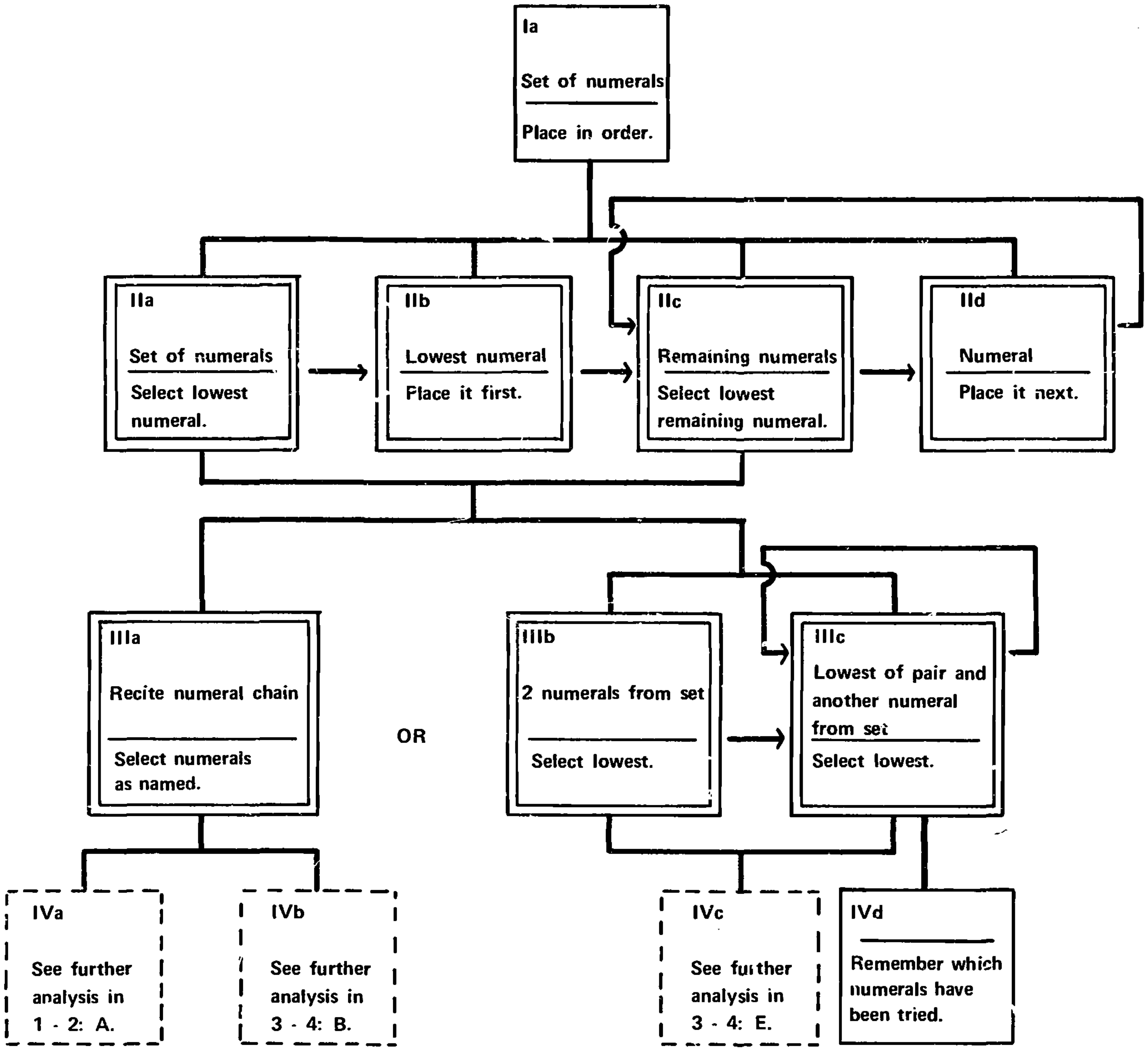

Figure 20: Behavior Analysis of Objective F, Units 3 - 4 (Alternate 1)

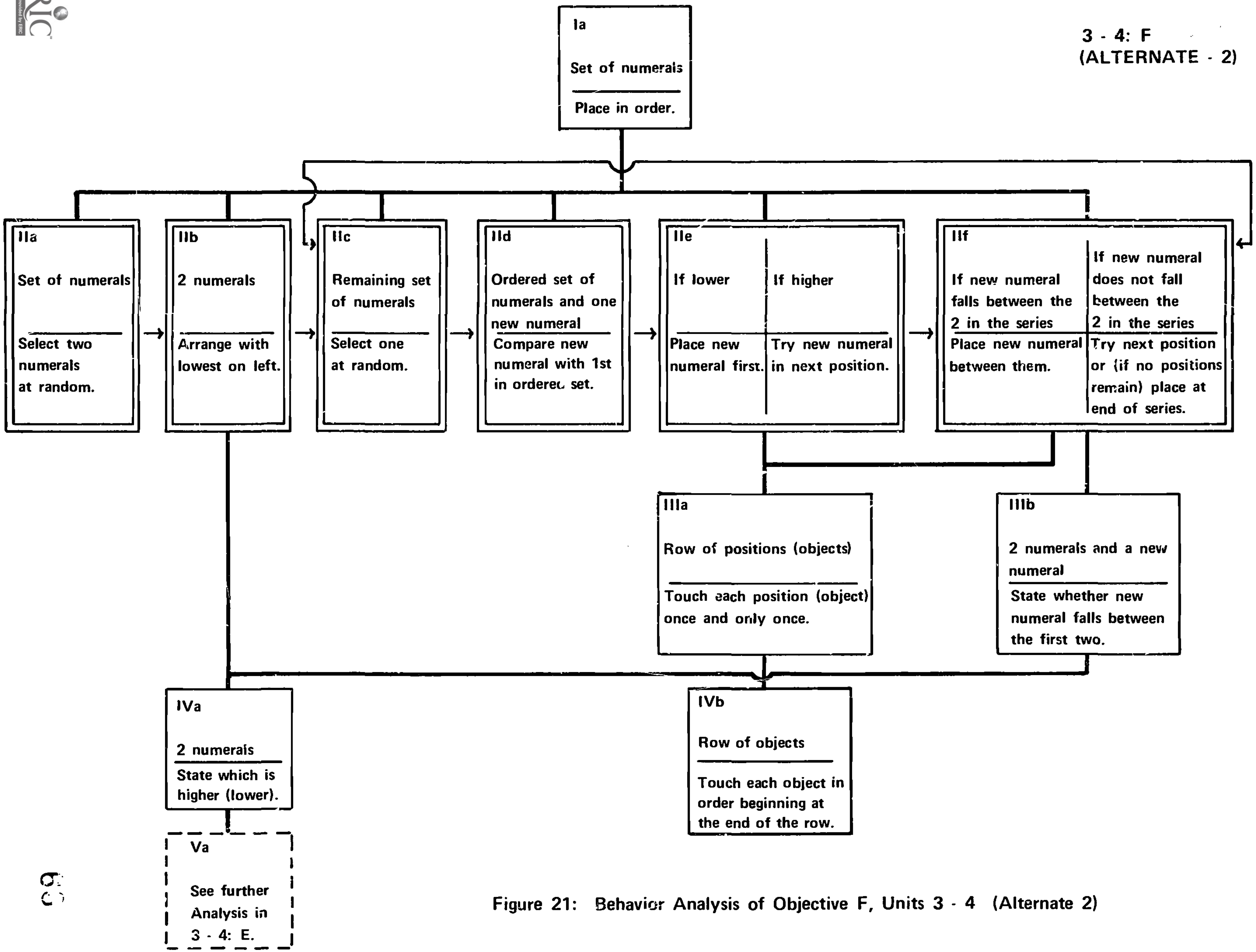

Figure 21: Behavior Analysis of Objective F, Units 3 - 4 (Alternate 2)

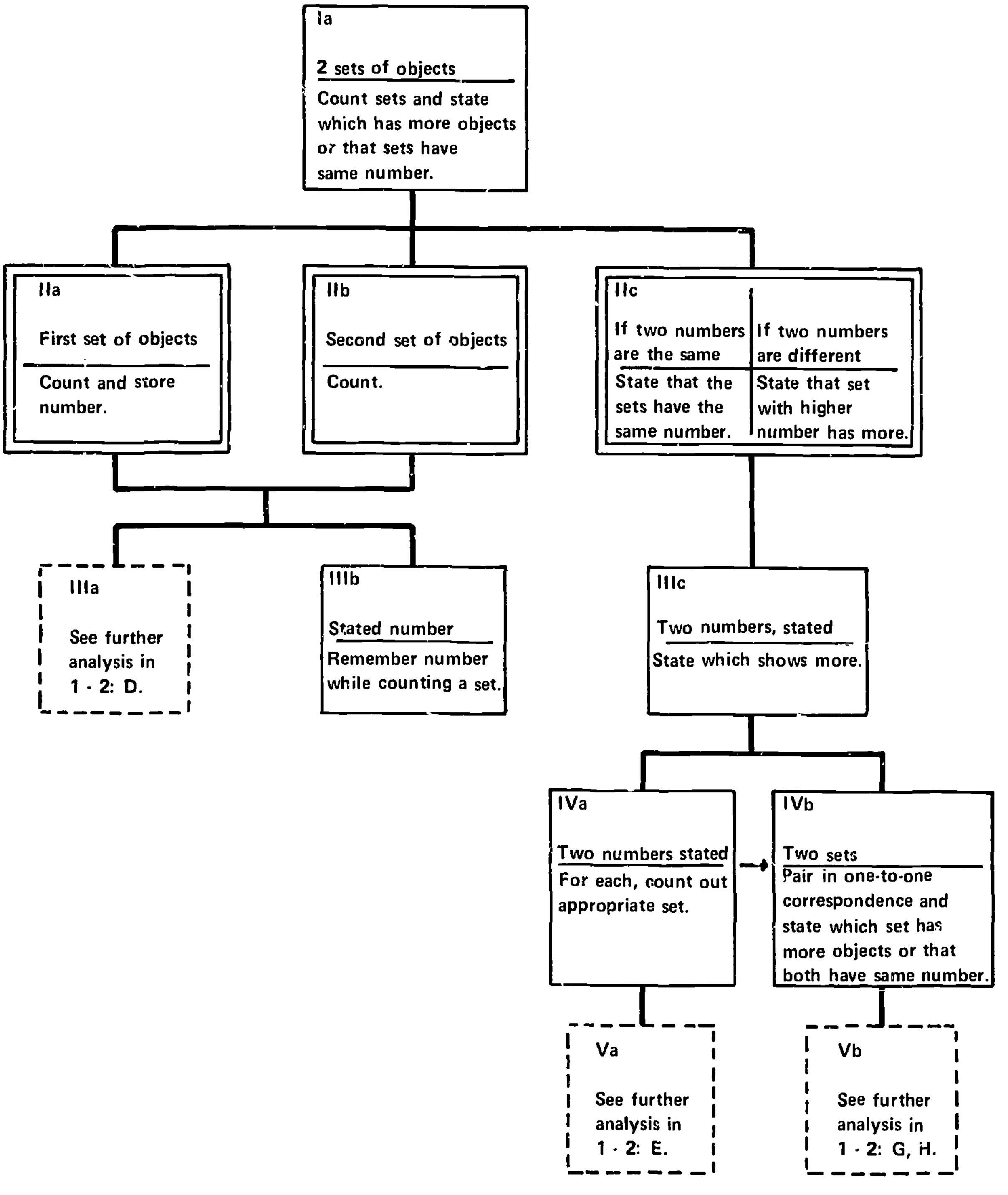

Figure 22: Behavior Analysis of Objective A, Unit 5.

ERIC

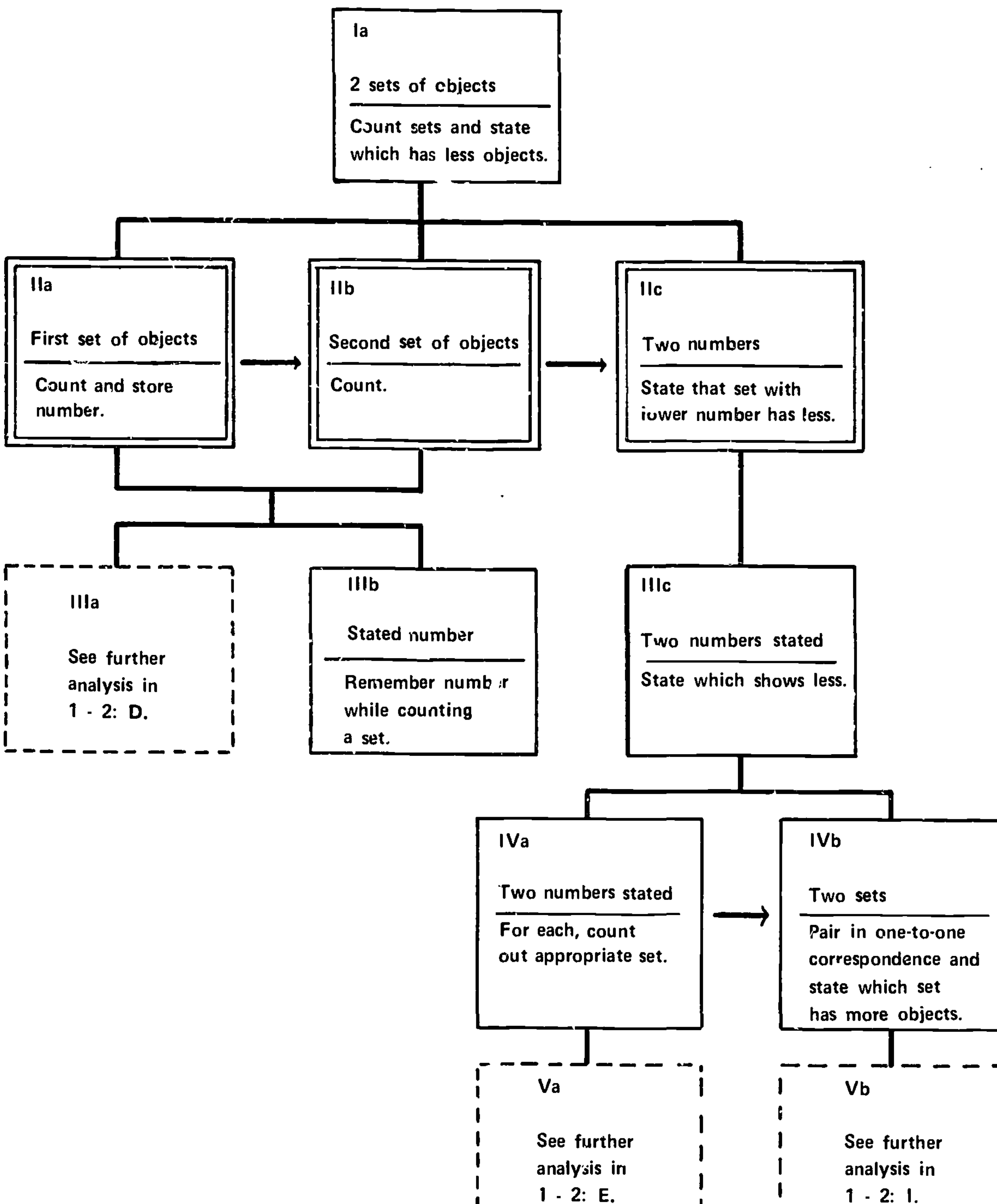

Figure 23: Behavior Analysis of Objective B, Unit 5.

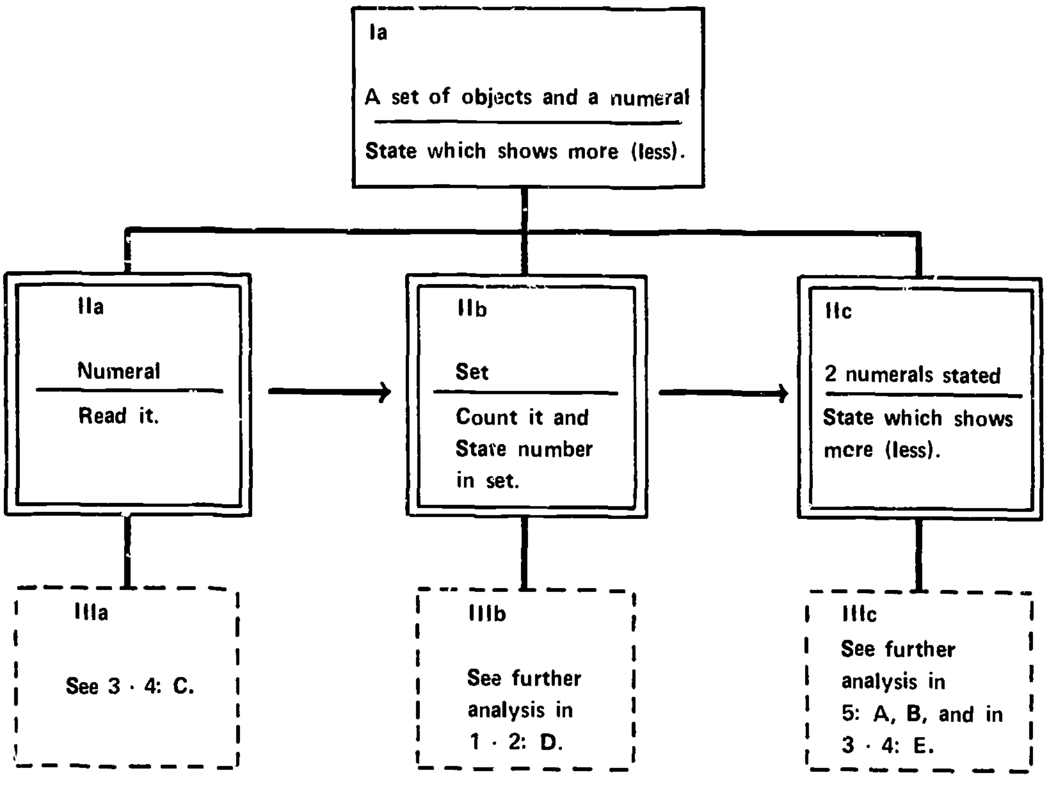

Figure 24: Behavior Analysis of Objective C, Unit 5.

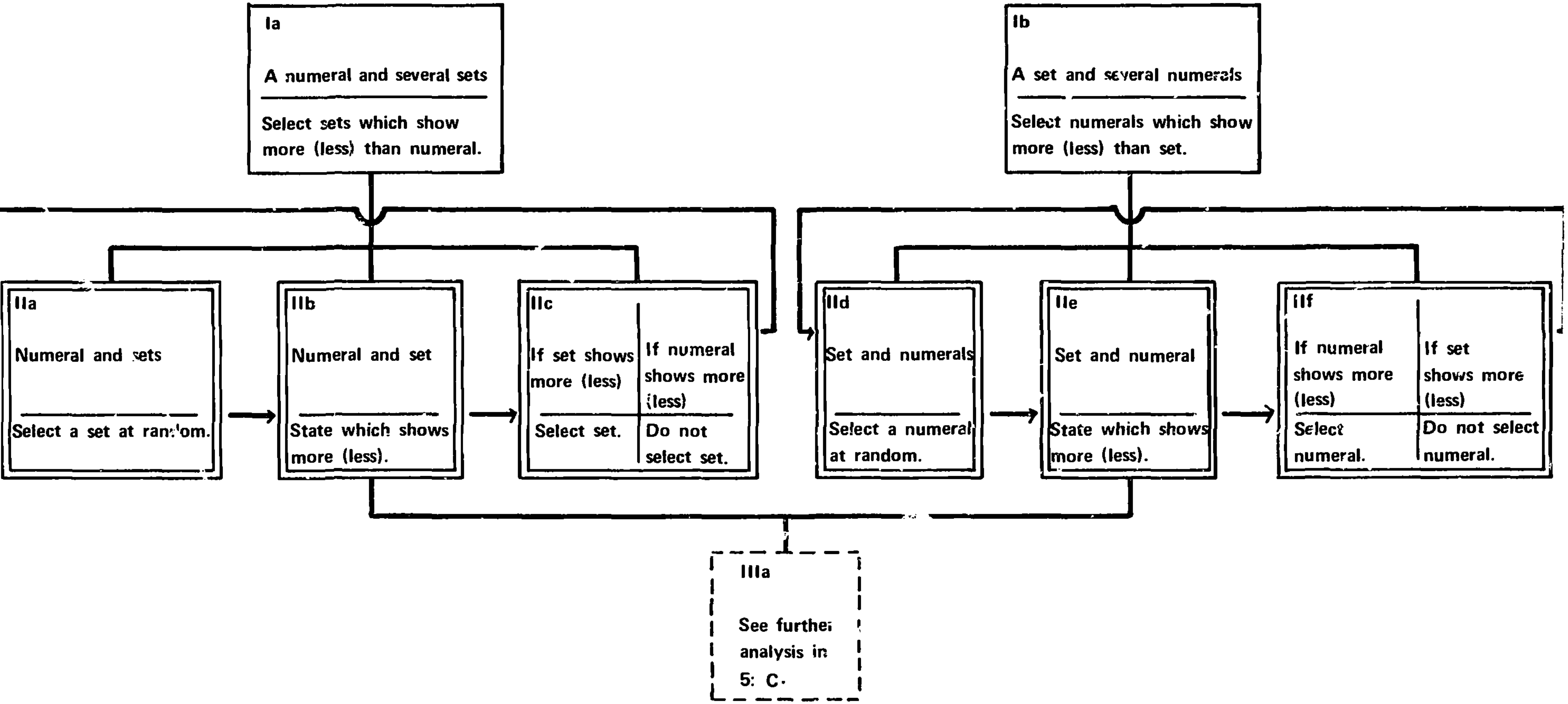

Figure 25: Behavior Analysis of Objective D, Unit 5.

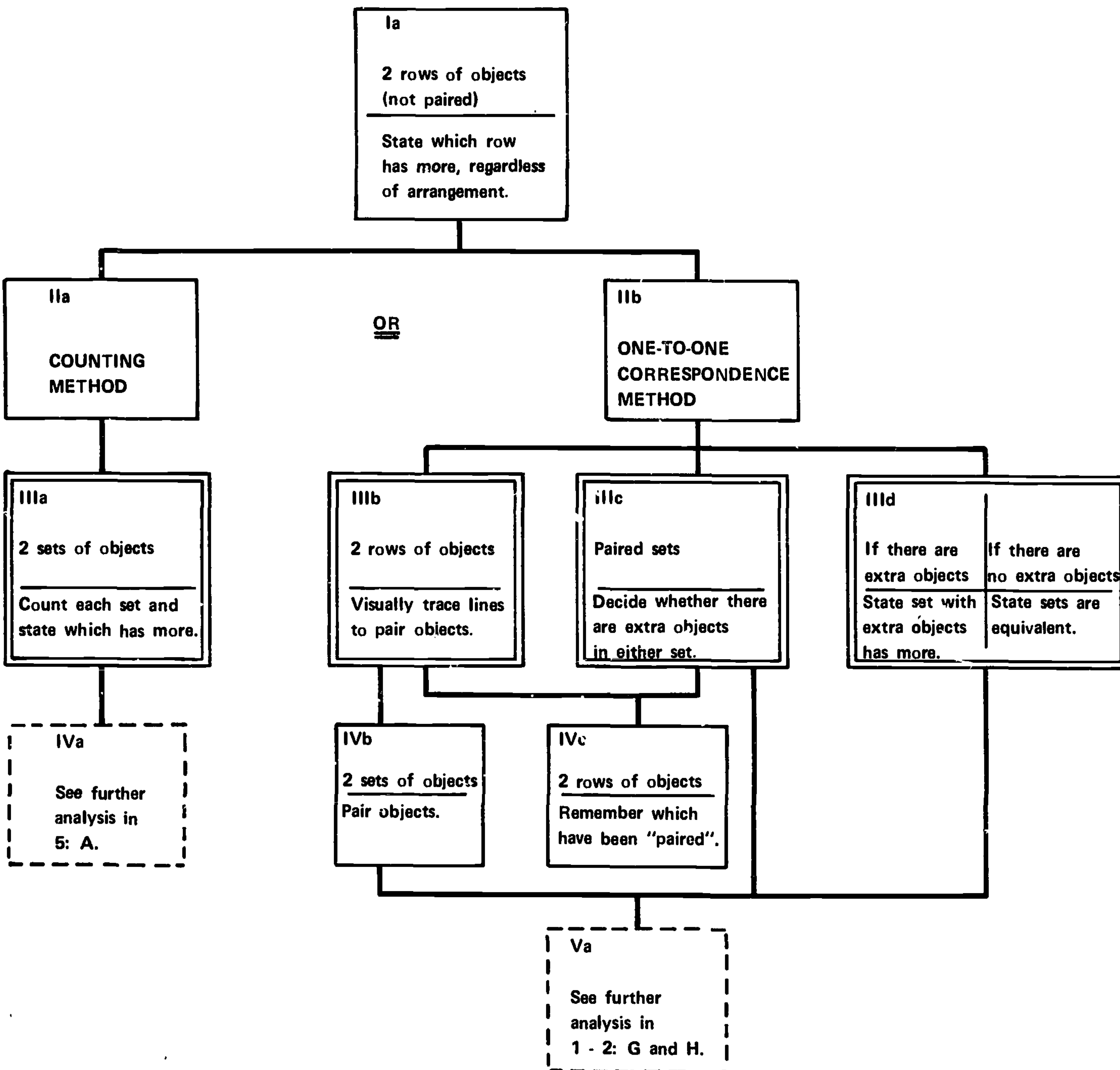

Figure 26: Behavior Analysis of Objective E, Unit 5.

ERIC

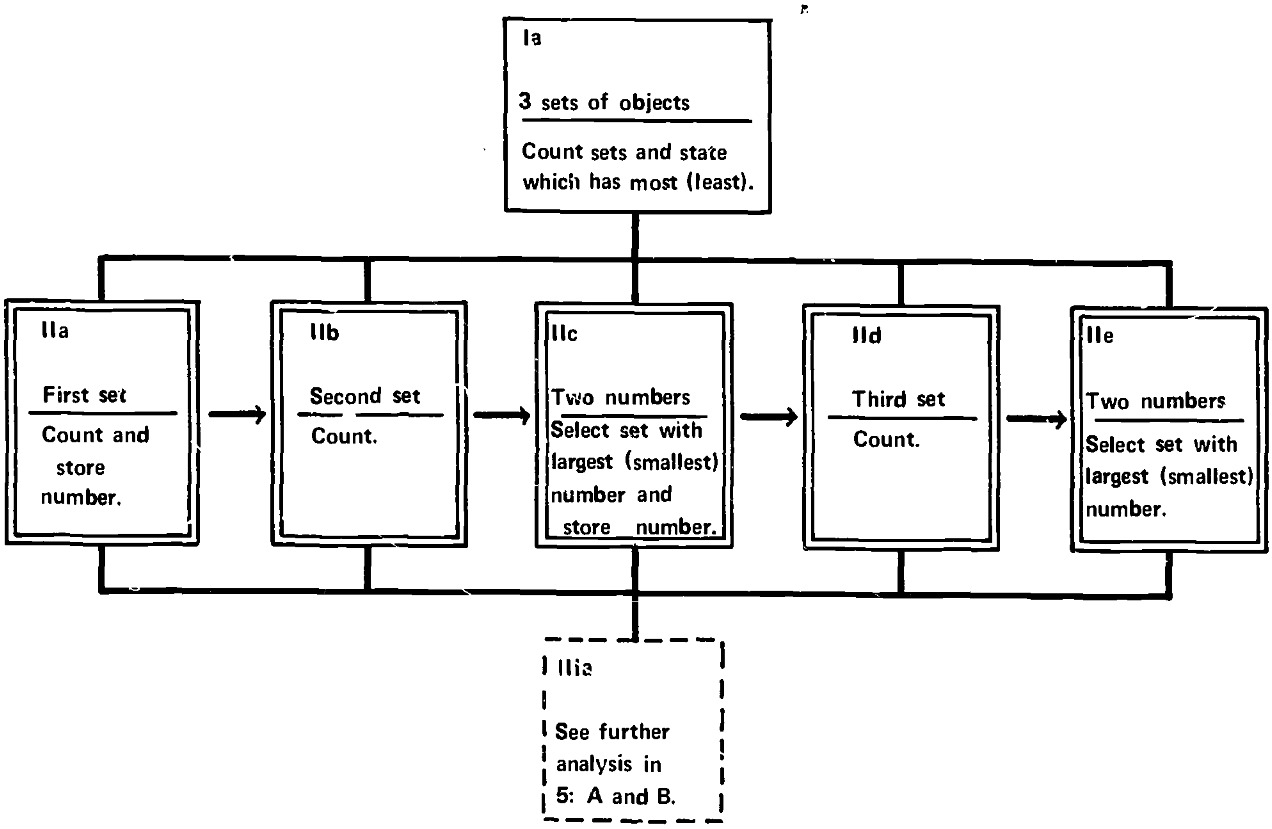

Figure 27: Behavior Analysis of Objective F, Unit 5.

Ia

3 objects of different sizes

Select largest (smallest).

IIa

2 objects of different sizes

Select larger (smaller).

IIIa

Several objects of different sizes

Select a large (small) object.

Figure 28: Behavior Analysis of Objective A, Unit 6.

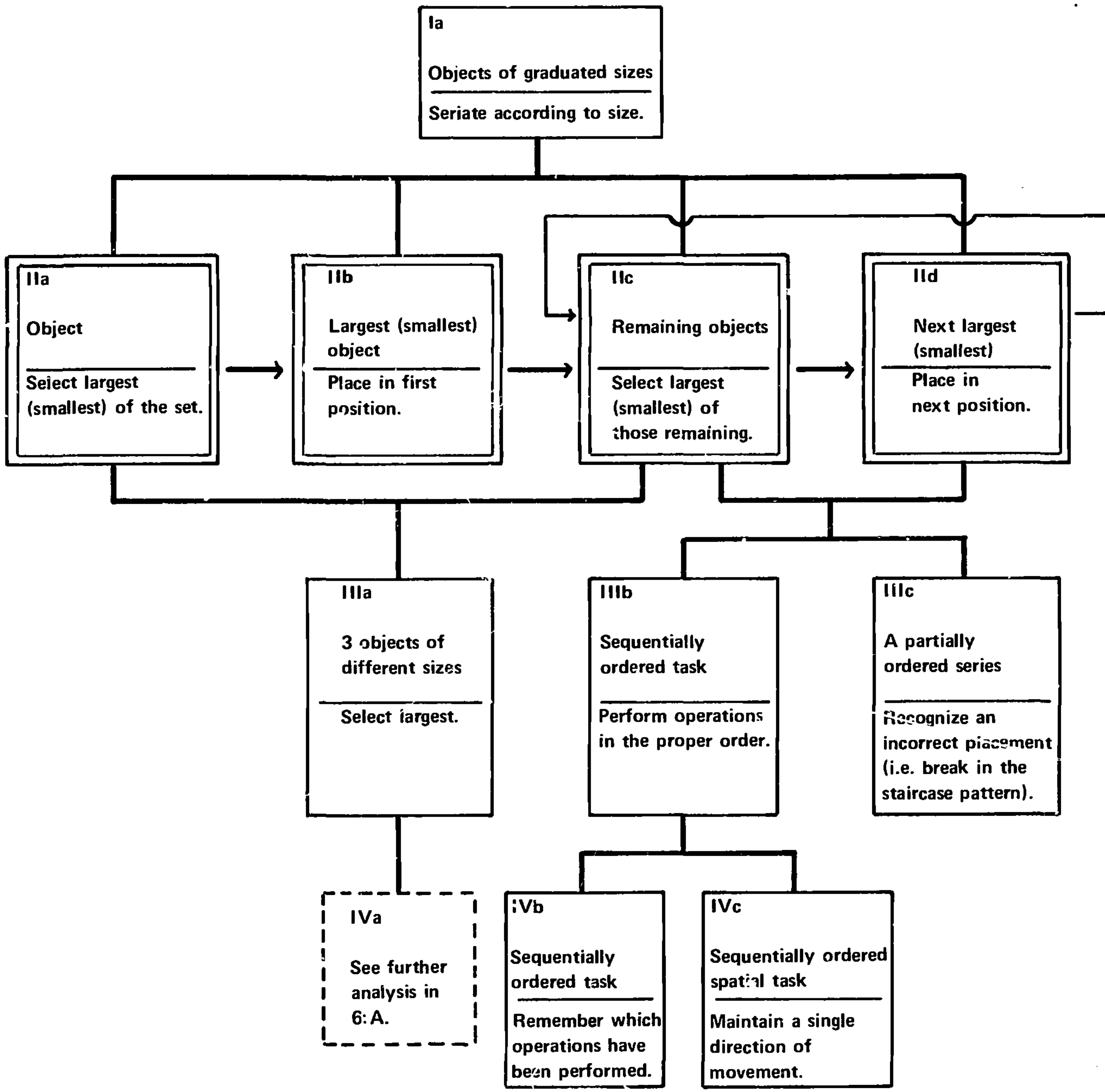

Figure 29: Behavior Analysis of Objective B, Unit 6 (alternate - 1).

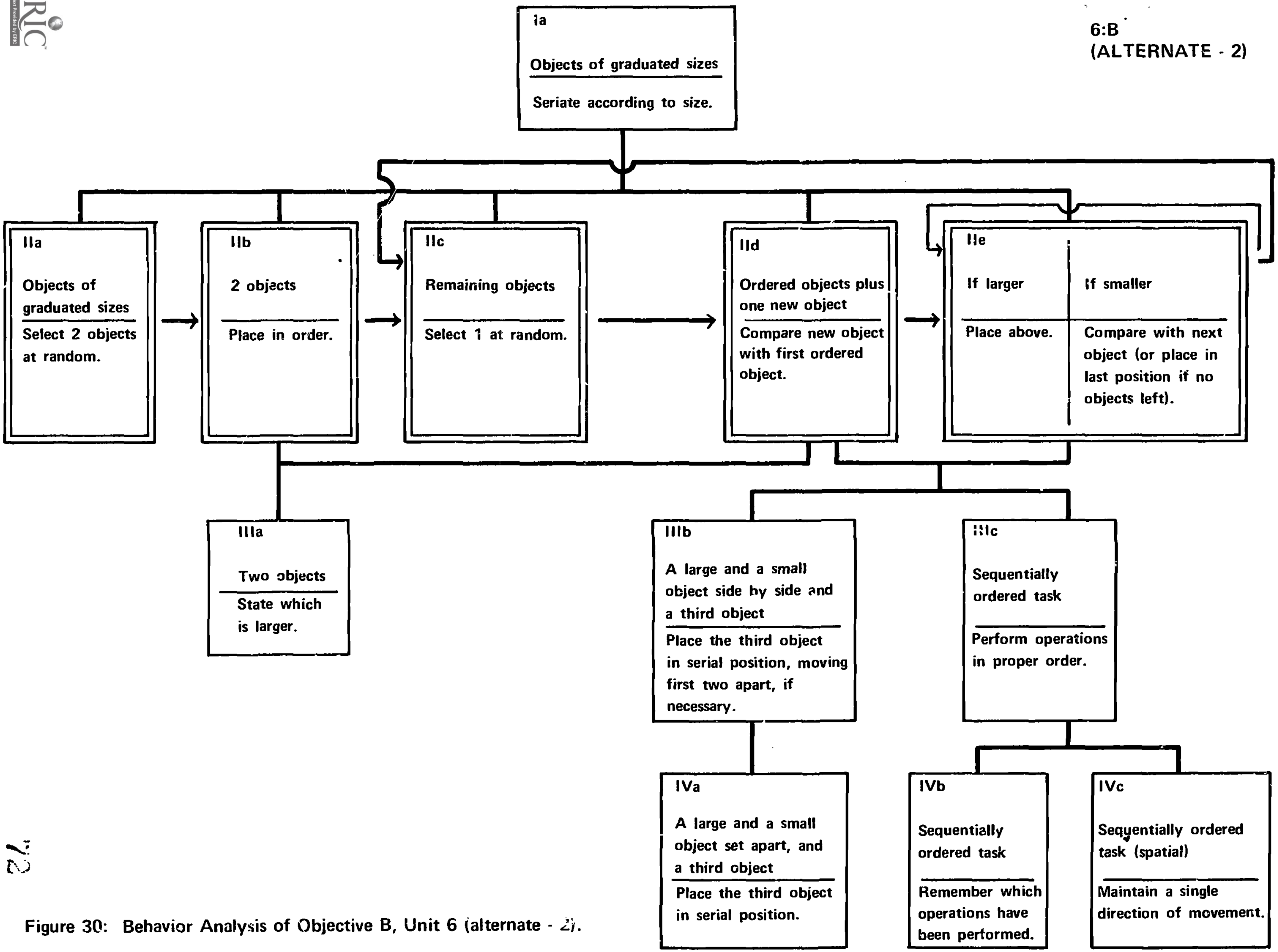

Figure 30: Behavior Analysis of Objective B, Unit 6 (alternate - 2).

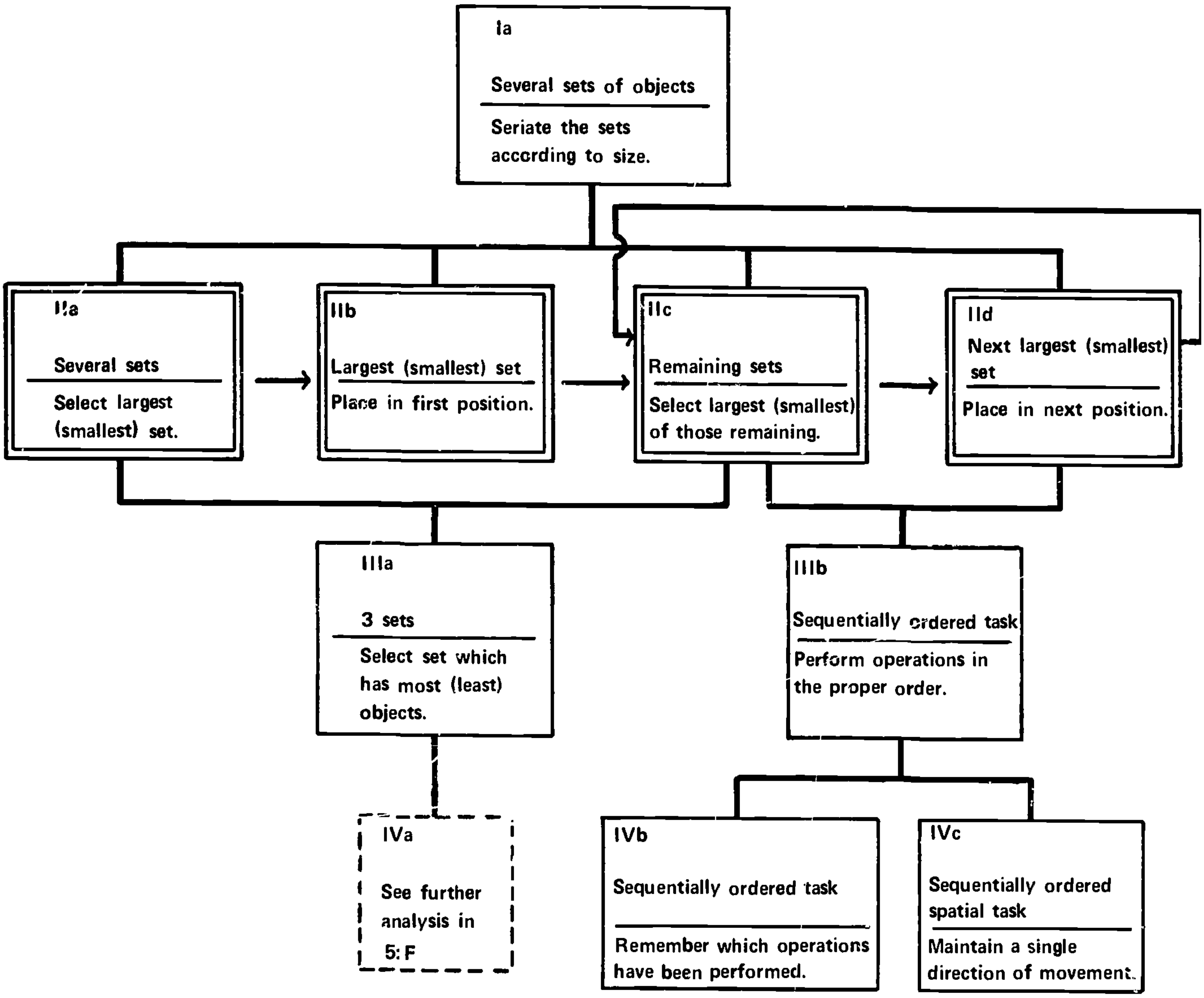

Figure 31: Behavior Analysis of Objective C, Unit 6 (alternate - 1).

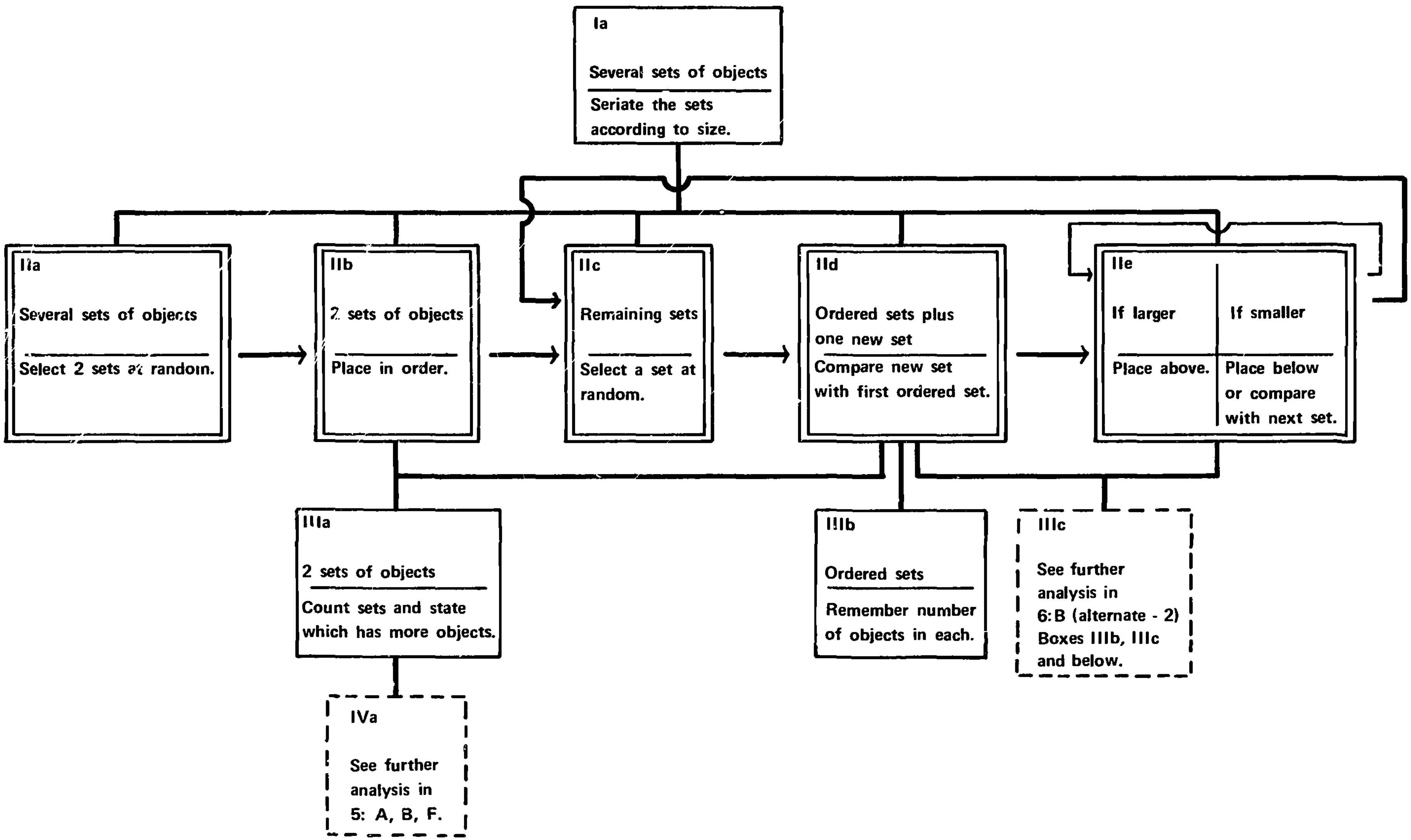

Figure 32: Behavior Analysis of Objective C, Unit 6 (alternate - 2).

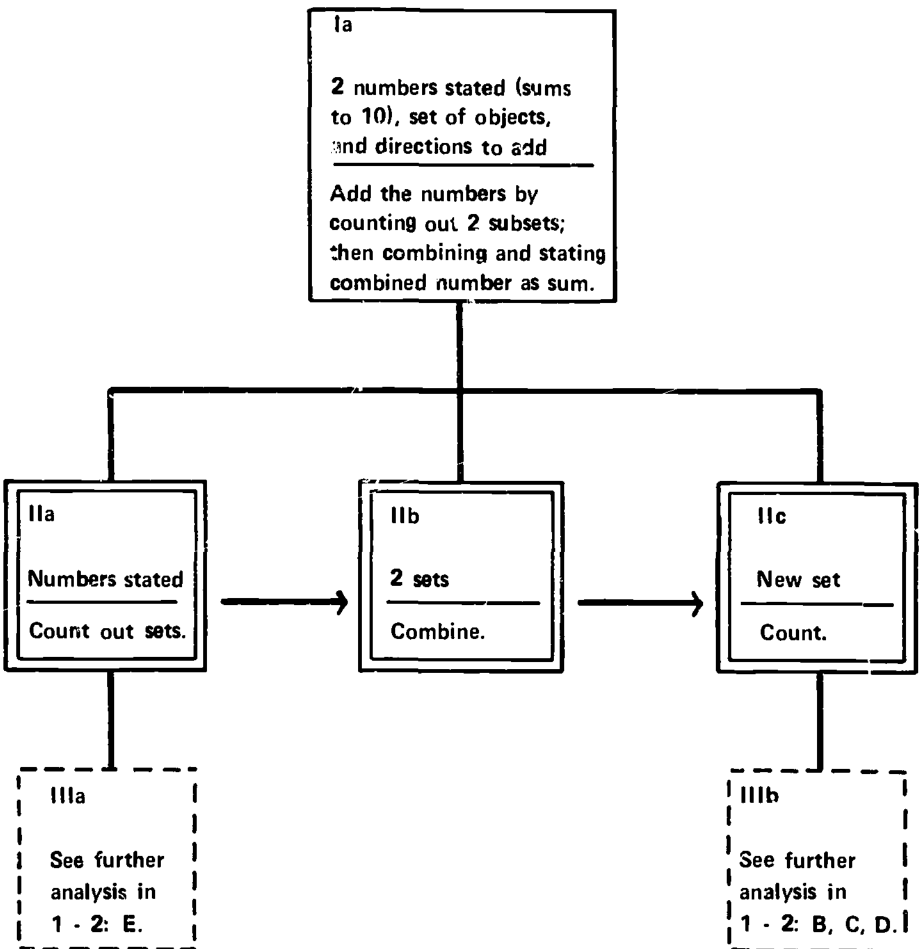

Figure 33: Behavior Analysis of Objective A, Unit 7.

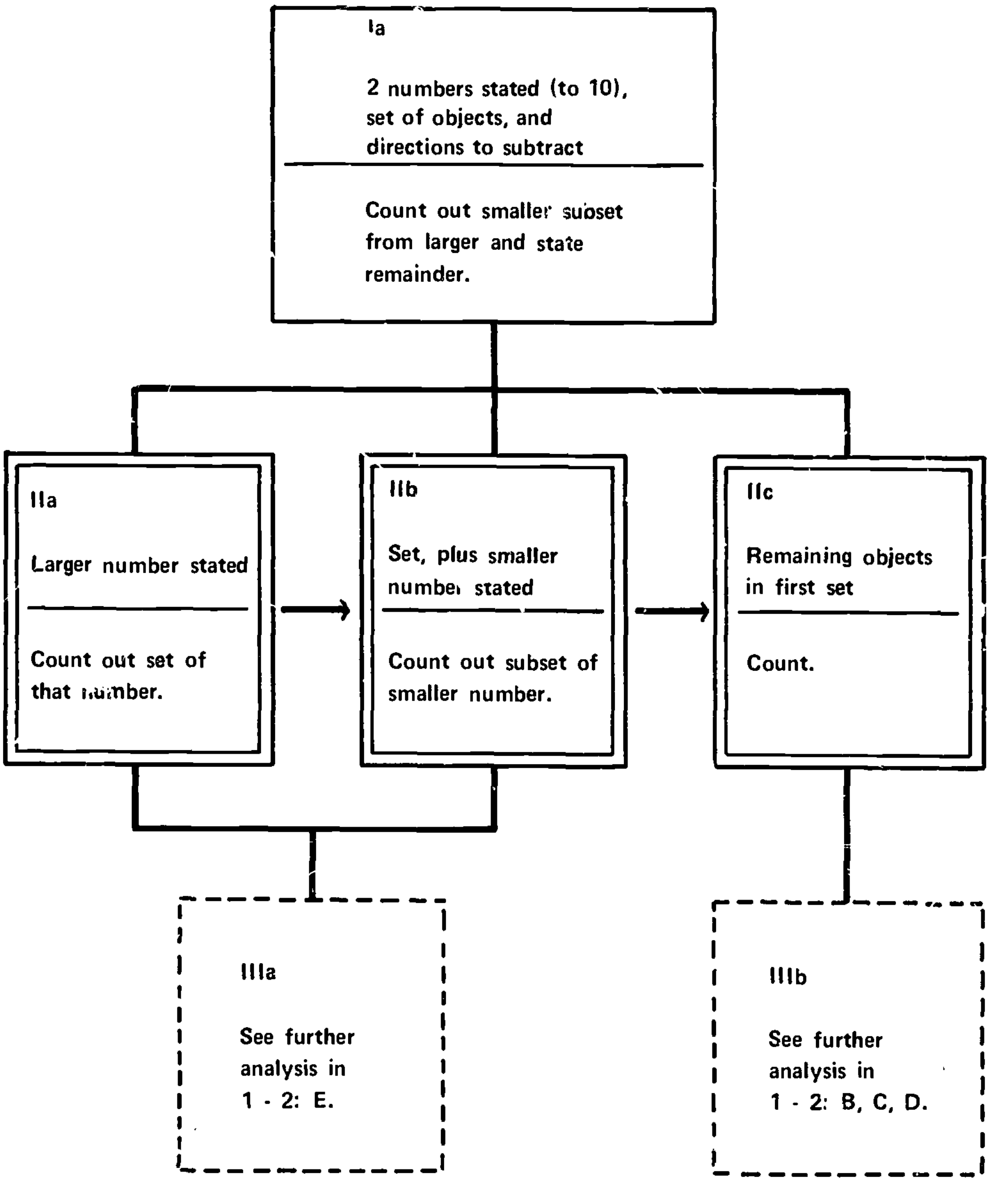

Figure 34: Behavior Analysis of Objective B, Unit 7.

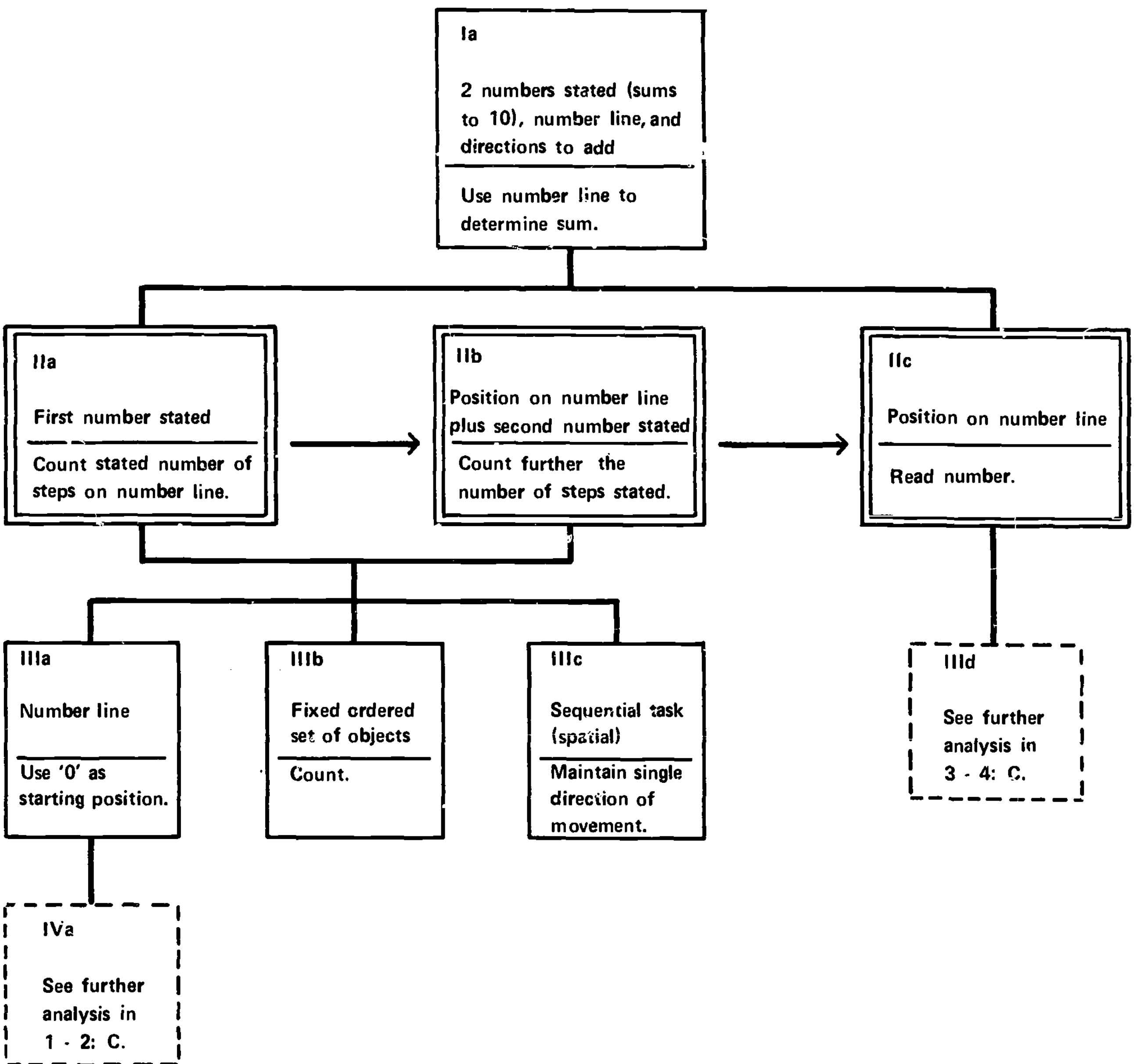

Figure 35: Behavior Analysis of Objective C, Unit 7.

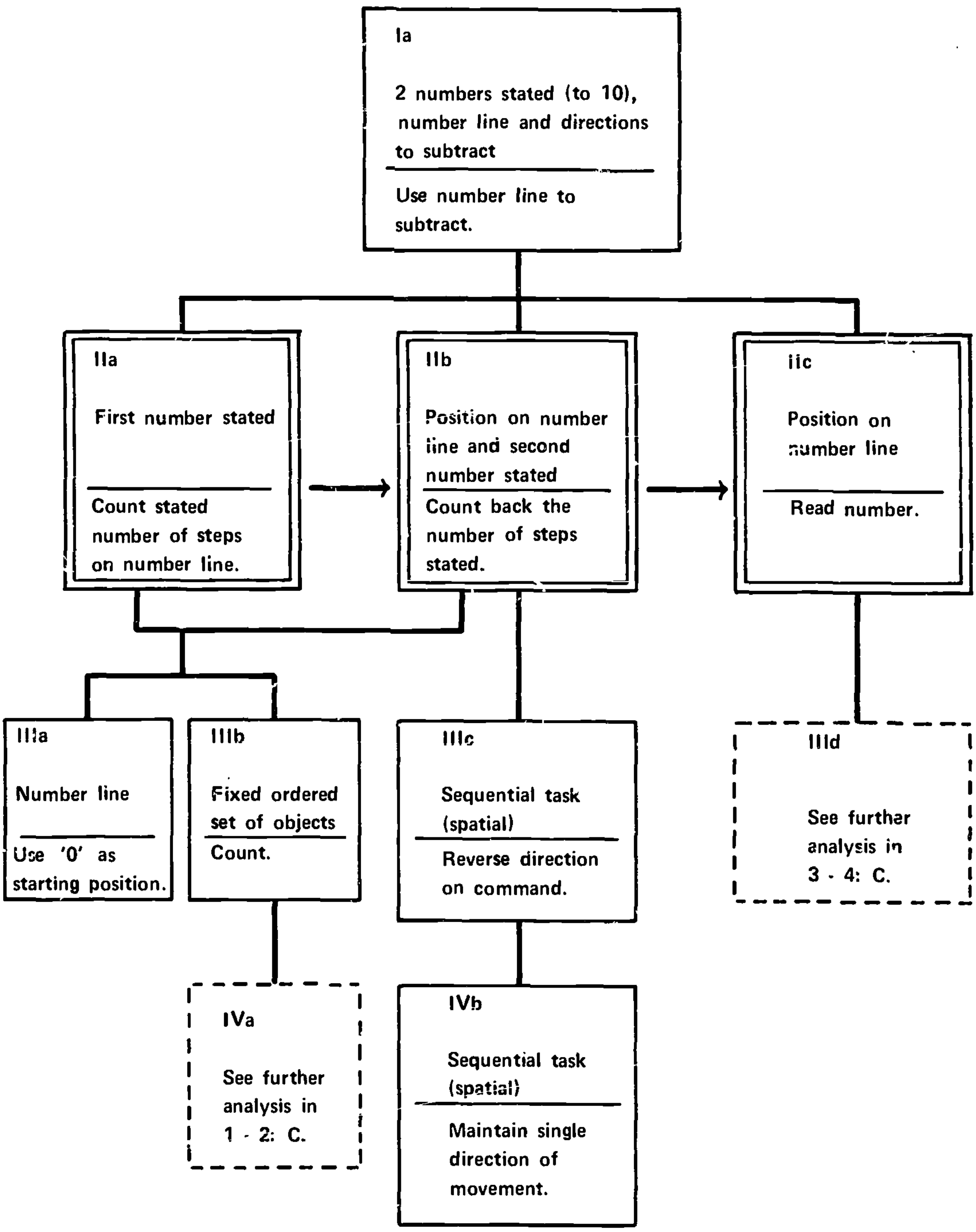

Figure 36: Behavior Analysis of Objective D, Unit 7.

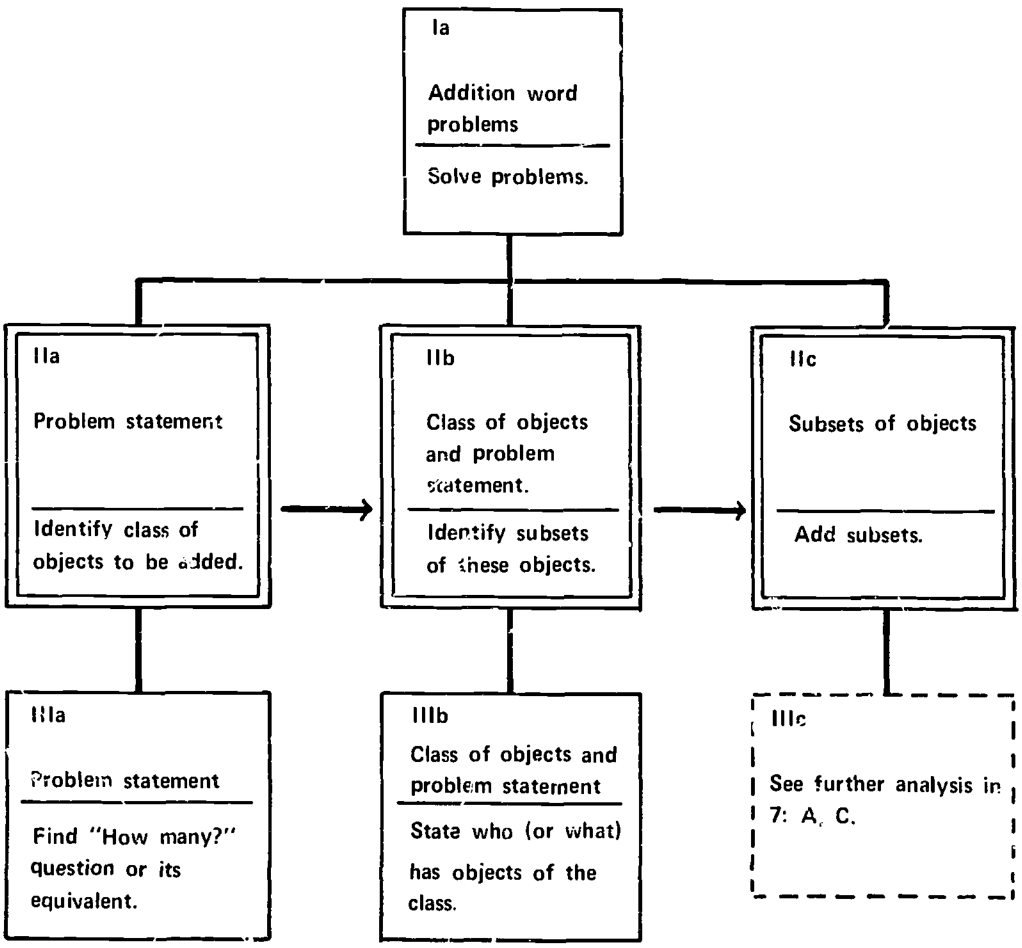

Figure 37: Behavior Analysis of Objective E, Unit 7 (Part 1).

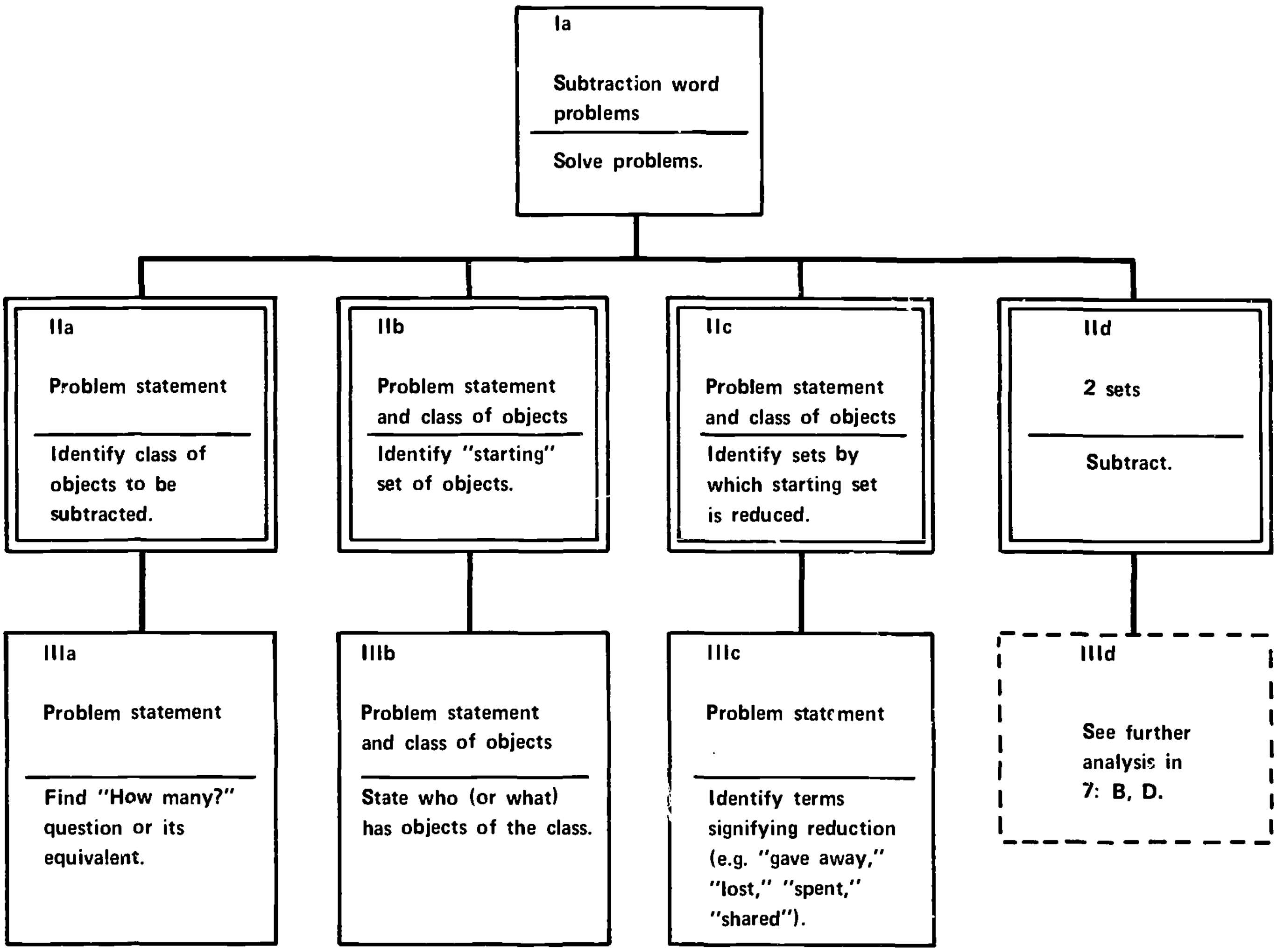

Figure 38: Behavior Analysis of Objective E, Unit 7 (part 2).

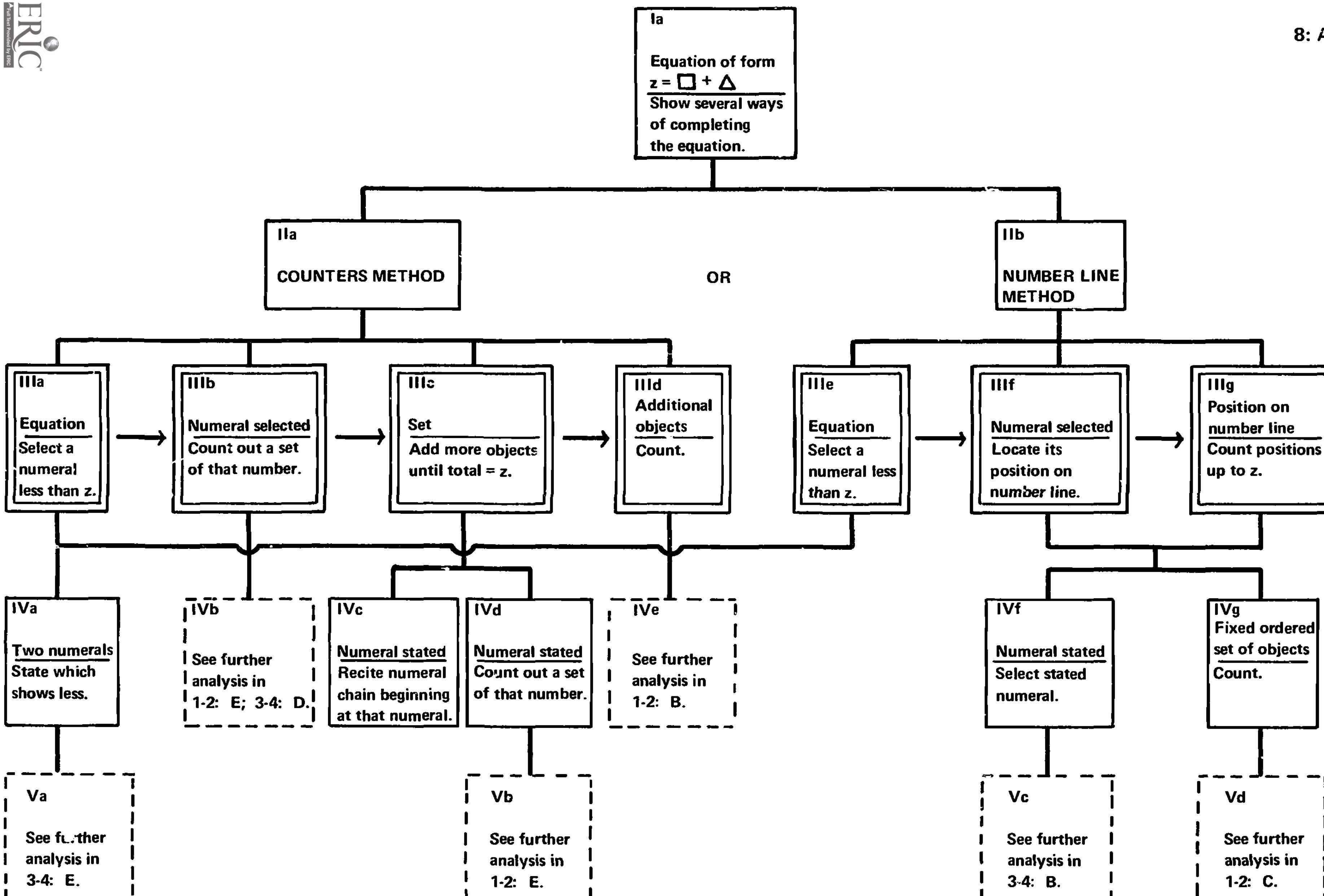

Figure 39: Behavior Analysis of Objective A, Unit 8.

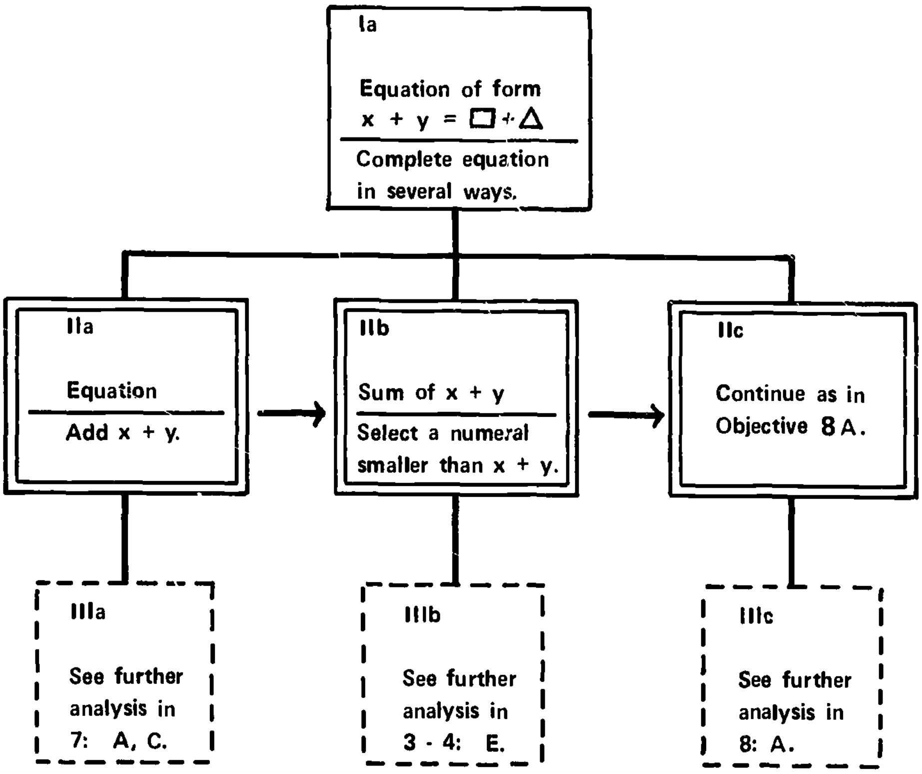

Figure 40: Behavior Analysis of Objective B, Unit 8.

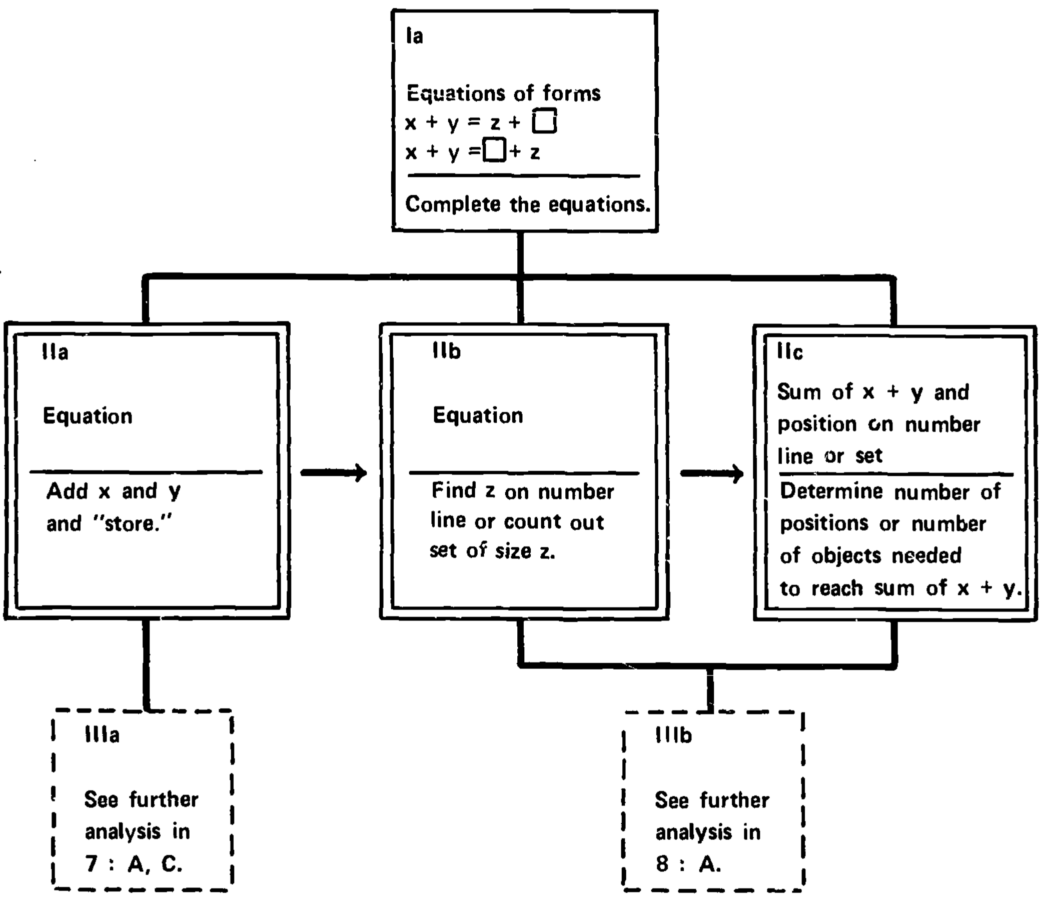

Figure 41: Behavior Analysis of Objective C, Unit 8.

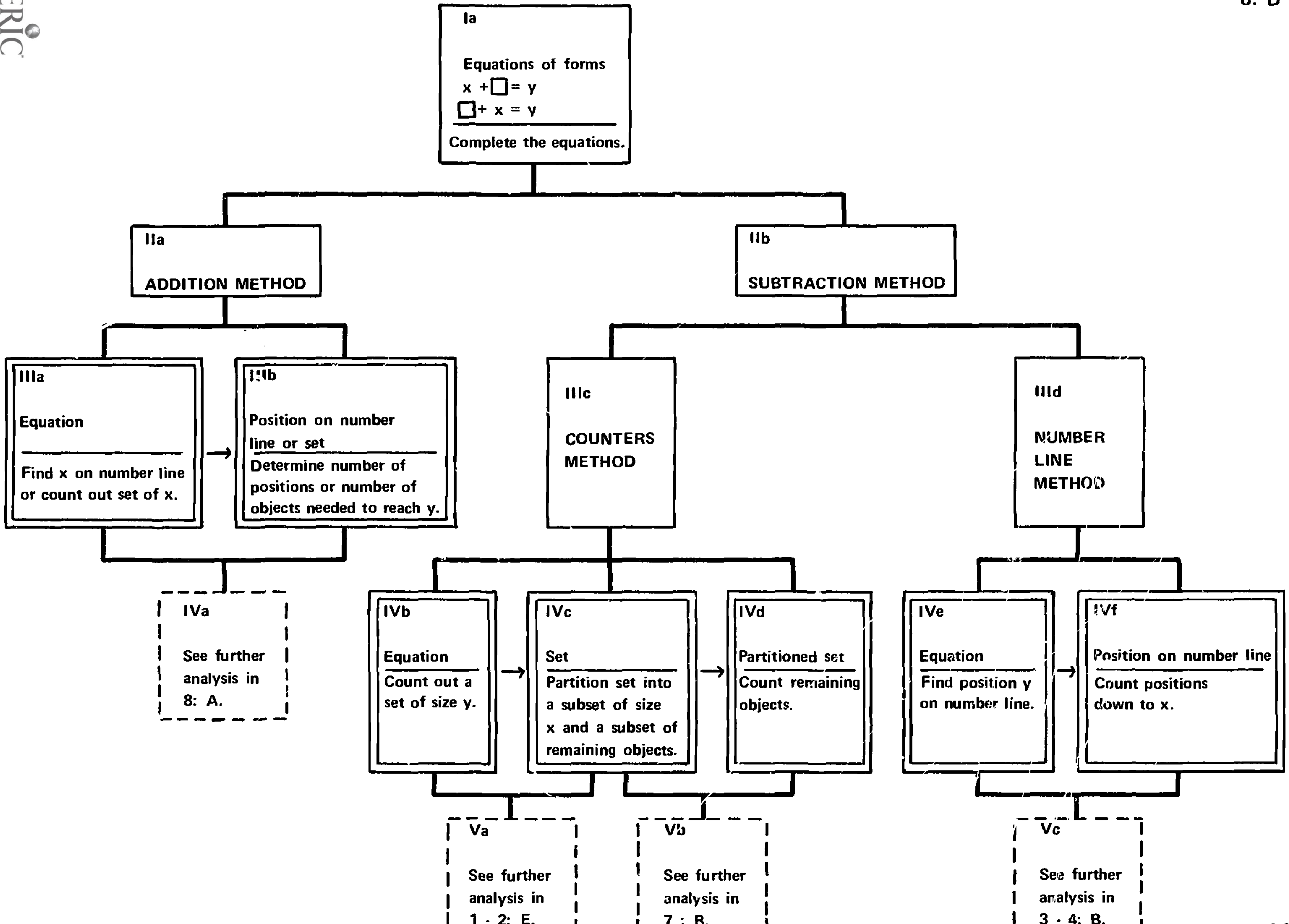

Figure 42: Behavior Analysis of Objective D, Unit 8.

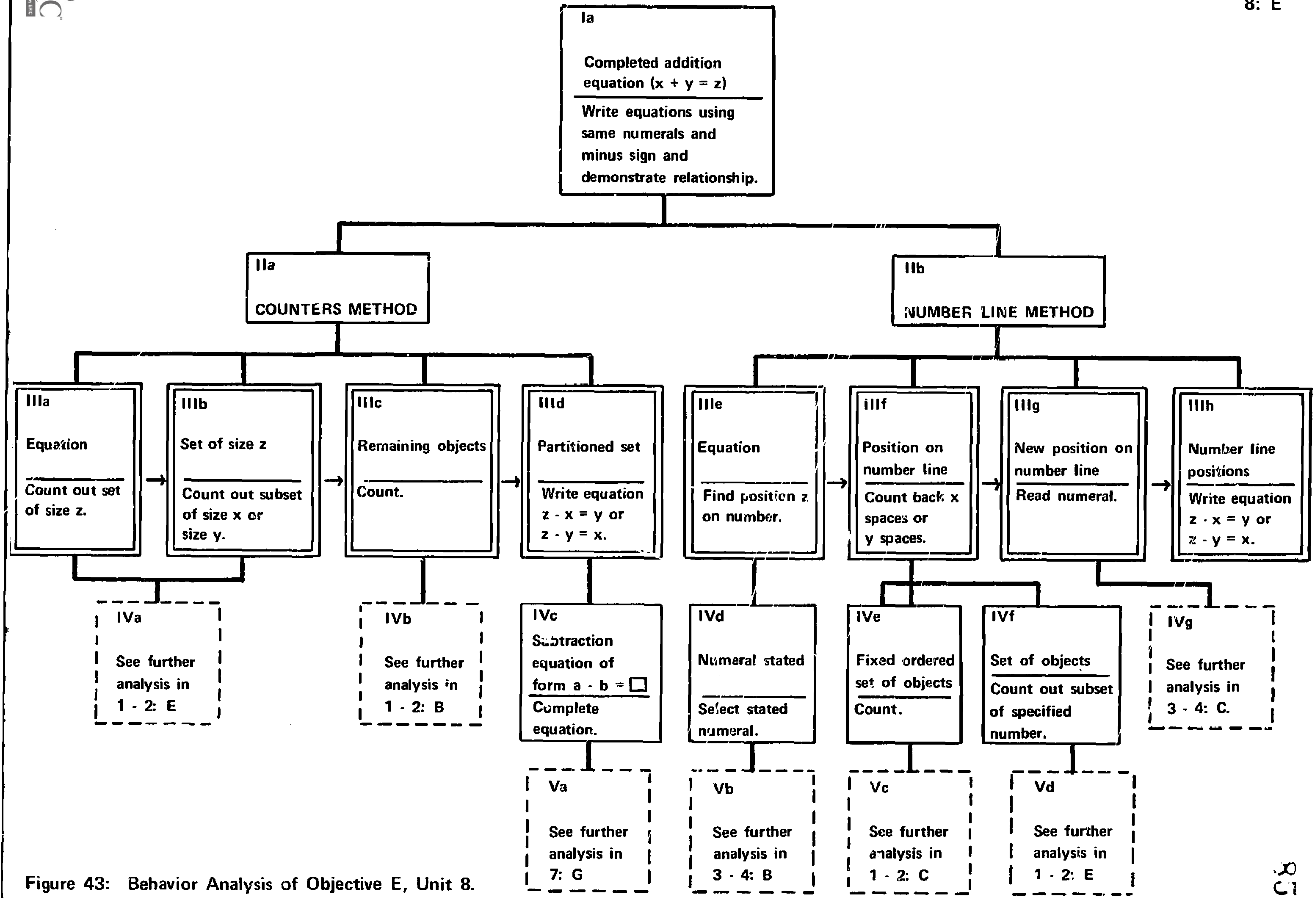

Figure 43: Behavior Analysis of Objective E, Unit 8.

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