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AUTHOR	Schremmer, A. G.
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ABSTRACT

This experiment attempted to teach abstract mathematics fo college freshmen with A.C.T. scores less than 15 in a three semester terminal course sequence. The course content included a formal mathematical language, set theory, Boolean Algebra, relations and functions, operations, cardinals and ordinals, the rational numbers, and college algebra. The tests for this course consisted of problems not previously encountered in class as well as "open questions" of the "prove or disprove" nature. The students in this experimental course were compared to students in the traditional three semester terminal sequence with respect to passing and failing rates. The results indicated that the students in the experimental course fared consistently better and caused the author to conclude that "abstract mathematics can be taught to almost anybody willing to try, at no other cost than time and rigor." (CT)

A Preliminary Attempt at Teaching Abstract Mathematics to Freshmen with an A.C.T. Score of Less Than 15.

by A.G. Schremmer (*)

"Good" students may be defined to be those who "succeed" in spite of their teaching. They are the ones who have the enormous amount of confidence in the world necessary to retain knowledge which they do not understand, but which they accept because "that is the way it is supposed to be."

In contrast, the student with an ACT score of less than 15 in mathematics can be characterized by his lack of mastery of even that which he thinks he knows. This occurs primarily in three areas: i) language, - his ACT score in English is practically always below 12, ii) logical consequence, and iii) abstraction-generalization.

i. The language. By taking advantage of the instructor's natural tendency to select in a student's answer the most favorable meaning, the student's profound inability to express himself protects him to some extent. It also rules out completely the possibility of any clear thinking: for a precise definition, for instance, the student will substitute a vague description of some hazy mental image, whence his constant use of such expressions as "like if you have ...". Also, his desire to get it over with often leads him to abbreviate to the point of saying something completely different from what he may have wanted to say.

(*) The author wishes to express here his indebtedness to the Community College of Philadelphia, where this attempt was made and which supported it in many ways. His gratitude particularly goes to Dr. Mamelak, chairman of the mathematics department, whose comments and criticisms were of great value in the course of this experiment.

ii. The concept of logical consequence. As the student is "answer oriented" rather than "problem oriented", his aptitude to jump to conclusions leads him to confuse "if p, then q" with "p, then q" - i.e. with "p and q". Often, he will even reduce it to the simple affirmation of "q". Quantification is another difficulty: 9 students out of 10 will say that the negation of "always" is "never", even after 'hearing' several corrections of this.

iii. The abstraction-generalization process. (1) The so-called approach "from the particular to the general" is the abstraction process by which in a red apple we perceive only "redness" (2). That is, it is the process by which we abstract, i.e. ignore, all concomitant but irrelevant properties which a red apple may have. It is an extremely difficult process. The usual approach consists in showing a red apple, saying "red", and hoping hard that the student won't understand "round" (3). On the other hand, generalization, which is often confused with abstraction and nearly always with extrapolation, consists, in its simplest mathematical form, in embedding one structure into another - as, for example, $\mathbb{N} \hookrightarrow \mathbb{Z}$ or $\mathbb{Q} \hookrightarrow \mathbb{R}$. On the basis of their previous learning, it was not even remotely conceivable that these students could learn these processes, without spending an unlikely amount of time.

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(1) Z.P. Dienes, "On abstraction and generalization", Harvard Educational review 31, No. 3, Summer 1961, and later publications of same author.

(2) It is by the way debatable whether "perceive" is the right word.

(3) For example: in a "Fundamentals of Mathematics" one can read: "The real numbers will be denoted by the letters a,b,c, etc. It will be assumed that the relation = (equal) has the following properties; Reflexive: For any real number a, a=a; Symmetric: For..." The question was then put to the students: "Write a sentence using the word 'reflexive'." The unanimous answer was: "a is reflexive with a".

All these deficiencies really proceed from a single cause: the student's fear of becoming involved in a situation where he will never be in a position to be in control of what he is doing. He is therefore prepared to claim that he knows all there is to know about a given subject rather than doing anything other than perform certain manipulations he is familiar with. This is psychologically destructive, since the student reads his basic insecurity as an inborn lack of ability. This attitude begins as early as elementary school: "Not only does (incompetence) reduce what others expect and demand of you, it reduces what you expect or even hope for yourself. When you set out to fail, one thing is certain - you can't be disappointed." [1]

II

The underlying idea consisted mainly in trying to find out if the situation was really hopeless, as some maintained, or if any <u>real</u> mathematics at all could be taught to these students by acknowledging their characteristics, rather than by ignoring or bypassing them. It was decided to propose to the students "a precise presentation of easier materials (rather) than vague intuitive descriptions of deeper results" [2]. This seemed <u>a priori</u> to have some advantages:

1. The constant reconsideration of any given concepts from as many viewpoints as possible together with the analysis of the various ways in which they may be interconnected ought to provide sufficient reinforcement. Then, apart from the psychological soothing effect, the familiarization obtained with such basic concepts as equivalences, operations other than numerical, morphisms, congruences, etc. ought to provide a firm foundation for potential further studies in mathematics.

(1) J. Holt, How Children Fail, Dell Pub. Co. 1964. p. 59
(2) S.T.Hu, Introduction to Contemporary Mathematics Holden Day 1966, which was used as a textbook.

ii. Careful attention could then be given to the order in which the concepts would be introduced, thus respecting mathematical and psychological imperatives about the "filiation of structures" ([1]).

iii. Limiting the material would allow a decreasing use of formalized language and semi-formalized proofs which would, at first, prevent the student from "arm waving", and then would progressively introduce him to the usual mathematical vernacular ([2]).

iv. Inasmuch as it is akin to playing games like chess, dealing with abstract concepts ought to be much easier than abstracting as a process ([3]); It might require only a careful linguistic preparation.

The following topics were therefore treated:

i. A formal language was progressively derived from a study of ambiguity in the natural language ([4]). The theory of deduction, however, was a semantic one and while a syntactic treatment with a natural deductive system _à la_ Gentzen such as J. Corcoran's ([5]) should have been tried, it was not, for no better reason than the author's essentially conservative mind.

([1]) Etudes d'epistemologie genetique, ed. by J. Piaget Vol. 14,15,16. Presses Universitaires de France.

([2]) Proofs like: "$A \cap B = B \cap A$, since any element which is in A and B clearly is also in B and A" in the best case leave the student wondering whether anything has been proved and, if so, from what. In the worst case, he accepts it as such. In any case a serious disservice has been perpetrated on the embryonic student.

([3]) Then there is the confusion of terms by which we call "abstract" that which we ought to call "non-familiar". This is particularly the case in mathematics: not so long ago, negative numbers were held to be absurd and complex numbers rejected as imaginary. Today one would probably dismiss their analogs in an introductory course as abstract!

([4]) But also in mathematics: frequent mistakes were for instance, found to arise from the fact that students tend to see _two_ numbers in 2 + 3.

([5]) J. Corcoran: "Discourse Grammars and the Structure of Logical Reasoning" in _Structural and Mathematical Learning_ ed. by J. Scandura, Prentice Hall (1969)

ii. Axiomatic Set Theory thus becoming possible, the axioms for equality, the regulative axioms, the axiom of extension, the constructive axioms and the axiom of separation were discussed in this order.

iii. A limited amount of Boolean Algebra was thoroughly treated. Boolean lattices were mentioned and the group structure of the symmetric difference was checked; this, to reinforce sentential logic but also to provide examples for later on.

iv. Relations and functions were studied from both local and global viewpoints; for instance, the transitivity of R was looked at as the universalization of the local aRb & $bRc \Rightarrow aRc$, and globally as $R^2 \subseteq R$.

v. Operations were studied similarly, for instance the associativity of an operation $S \times S \xrightarrow{*} S$ was looked at as the universalization of the local $a*(b*c) = (a*b)*c$, and, categorically, as the commutativity of the diagram

$$\begin{tikzcd} SxSxS \arrow[r, "*x1"] \arrow[d, "1x*"] & SxS \arrow[d, "*"] \\ SxS \arrow[r, "*"] & S \end{tikzcd}$$

v. Cardinals and ordinals were briefly discussed and some of Peano's axioms were verified to hold in these models. $\mathbb{N}$ was checked to be a abelian cancellation semi-group, so that:

vi. $\mathbb{Z}$ and Q were obtained from $\mathbb{N}$ by the Grothendieck construction.

The third semester was to have been devoted to the construction of $\mathbb{R}$ by completion of the order and perhaps also by completion of the metric, proving the results to be isomorphic. The Chairman of the Department, however, insisted that, for comparison purposes, it be devoted to College Algebra. Thus computations were carried in $\mathbb{Z}$, but also in $\mathbb{Z}_n$. Then a little bit of the geometry on Q was discussed. The incompleteness of Q was then proven not investigated for lack of time.

III

89 incoming students with an ACT score below 15 were enrolled who had pre-registered for the "remedial" course in the three semesters terminal sequence ([1]). Since their mathematical background, from previous experience, was known to be a hindrance rather than an asset, not even an elementary knowledge of arithmetic was required or assumed and the only way in which these scores were acknowledged was that the material was covered only as fast as the majority of the students could take it, very, very slowly. Everything else was kept as close as possible to ordinary college conditions. Essentially, no attempt was made at any mode of teaching other than ex cathedra exposition; and thus, the classes were traditional ones, i.e. with practically no student participation. The instructor, however, was available 3 or 4 hours a week.

Since the textbook was not really readable by the students, who on the other hand were notoriously unable to use their own notes, approximately 6 single spaced pages of lecture notes per lecture were mimeographed ([2]). Unfortunately, most of these notes were distributed post facto, thus greatly reducing their effect.

Except for a few weeks during the second semester, no exercise session was held. At that time, however, three different hours were held open where the students could come in to work as they pleased. Only about half of the students ever became interested in questions raised by the instructor. They were interested mostly in duplicating questions asked at previous tests. The students could and eventually did work in small groups. The instructor kept moving, watching what each group was doing, but did not interfere unless requested. Even then, though, questions were answered mostly by other questions; the instructor made it a point never to say "this is ok" or "this is false". As usual, students were very reluctant to refer back to definitions.

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[1] This remedial course is a non credit course
[2] The usual ratio is about 2:1.

A special hour, different from class time, was set up each week so that all students could take the test simultaneously after having had exactly the same number of lectures. The tests were graded but not corrected. Instead, answer sheets were made with red carbon from the original blue dittomasters. A student who failed to take a test for any reason whatsoever got zero for that test: it was felt that individual make-up tests might alter the homogeneity of the results. Instead, each series of 6 tests was followed by two make-up tests open to everybody and which could be substituted for any regular test. The tests could involve anything covered to that point and had progressively increasing weights to correspond to this accumulation effect. The scale for each semester was: 20, 30, 50, 60, 70, 70: 80, 100, 120, 130 140. The make-up tests were of course substituted with the proper weight. The grade scales were about those used by the department in the regular terminal sequence. A(100-85), B(85-70), C(70-55), D(55-40), F(40-0).

The questions asked were not particularly difficult, but were rather disconcerting for this type of "answer-oriented" students. One kind of question was aimed at testing their ability to prove or disprove, but were as often as possible phrased as an "open question":

- Prove or disprove: $A \subseteq B \Rightarrow B - (B - A) = A$,

Another way to avoid routine exercises was to provide the student with some previously unknown information and ask him to prove something.

- Given in addition to the inductive definitions of + and . in $\mathbb{N}$ that:

  $\forall a \ulcorner a^1 = a \urcorner$ and $\forall a \forall m \ulcorner a^{m+1} = a^m \, . \, a \urcorner$

  Prove by induction on n that : $\forall a \forall m \forall n \ulcorner a^m \, . \, a^n = a^{m+n} \urcorner$ (1)

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(1) Whatever exposure to exponents the students had had previously hadn't left any trace.

The other questions required the analysis of some set of specifications and the possible construction of an object(s) to meet them.

- Suppose that $(\mathcal{A} \& \mathcal{B}) \Rightarrow \mathcal{C}$ and $\mathcal{A} \& \mathcal{B}$ are both true, what can you say about the truth values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$?
- Given that $u = \{a,b,c\}$, $v = \{m,a\}$, $u = v$ and $a \neq c$. What are then the possibilities as to which member of u is (identical with) which member of v?
- Construct a 1-1 function from $\{a,b,c,d,e\}$ to $\{m,n,q,r\}$. Could it be also onto? Could there be a function which would be just onto?
- Given the following relation (1), construct the smallest equivalence which contains it.
- Construct on a finite set an operation which is right cancellable but where not every equation $a*x = b$ is solvable. (2)

Of course, these exercises have nothing particularly new in them but it must again be remembered who the students were. They had NEVER seen any test where they had anything other to do than duplicate a procedure in a very familiar situation (e.g. add two fractions after having added fractions for quite a while (3)).

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(1) A relation was given by a graph <u>à la</u> G. Papy on some very small set.

(2) This one was a disaster. The trouble was with the notion of cancellability which the students found very hard to grasp.

(3) And even then....

III

As already mentioned, the author's concern was the response to abstract mathematics of the "underdeveloped" students who normally take a three-course terminal sequence [1]. The department, however, was more interested in knowing whether this would provide a more efficient terminal sequence in terms of failing and passing rates. Although not readily obtained, the answer to this last question was fairly simple to obtain and will be dealt with first.

About the only precise data available for the regular sequence were on a per course basis. Thus, to make any meaningful comparison, it was first necessary to obtain the attrition rate and grade distribution throughout the whole sequence for a given group of freshmen. A list was obtained of 120 students with an ACT score of below 15, who had enrolled at the same time as those in the experimental sequence. It seemed then to be only a matter of counting the survivors after three semesters in each sequence. However, two problems complicated matters.

The first problem was to decide which students could be considered as having significantly tried, and not just as having registered. The problem was further complicated by the necessity to evaluate the normal drop-out rate, which, in a two-year college with an open door policy, is fairly high, and is not necessarily due to poor scholarship. The following lists were established:

	EXPER.	CONTROL
ORIGINAL list (preregistered students)	89	120
BASIC list (Original list less those who never showed up)	75	107
REDUCED list (Students still attending at mid-term of the first semester)	61	98 [2]

[1] of which the first is "remedial" and therefore not a credit course.

[2] This is only an estimate.

The second problem was then to constitute equivalence classes modulo success. The corresponding classes in the two groups had moreover to be reasonably comparable in terms of achievement, even though what had been achieved in the two groups was probably not comparable. The following nine equivalence classes were finally constituted, counting as usual 4 points for an A, 3 for a B, etc.

		EXPER.	CONTROL
I	Students with at least 8 points	5	5
II	Students with 6,7 pts., 5 if D for 1st sem. (1)	12	16
III	Drop-outs with 7,8 pts. for first two sem. (2)	3	2
IV	Drop-outs with 5,6 pts. for first two sem. (2)	4	5
V	Drop-outs with 3,4 pts. for first two sem. (2)	1	6
VI	Students who quit sequence after having failed the first two sem. (1)	13	24
VII	Students who quit sequence after having failed the first sem.	6	7
VIII	Drop-outs after having failed one or two sem. (2)	7	20
IX	Students who completed but failed to pass sequence: less than 6 pts.	10	13

A point worth noting is that in the regular sequence students are allowed to take the second semester even if they have failed the first one. This practice had then to be extended to the experimental sequence.

Even after the students had been so classified, and assuming that the classification made sense, it remained quite hard to decide who was to be considered a success and who a failure. Moreover, the question remained:

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(1) The first semester being a non credit course, a passing grade is not required to register for the next course.
(2) By drop-out are meant students who left college, not just the sequence.

which students could be considered as having minimally tried, i.e. which list to take as a basis for percentages. They were finally computed in the following ten different ways to see if some regularity could be observed.

	EXPERIMENTAL				CONTROL			
		%ORIG	%BAS	%RED		%ORIG	%BAS	%RED
ORIGINAL Class List	89				120			
BASIC Class List	75	84.2			107			
REDUCED Class List	61	68.3	81.3		98	81.6	91.5	
PASSING								
I	5	5.6	6.6	8.2	5	4.1	4.7	5.1
I + II	17	19.1	22.6	27.8	21	17.5	19.6	21.4
I + II + III	20	22.4	26.6	32.7	23	19.1	21.5	23.4
I + II + III + IV	24	26.9	32.0	39.3	28	23.3	26.1	28.5
V + I + II + III + IV	25	28.0	33.3	41.0	34	28.3	31.7	34.7
FAILING								
VI	13	14.6	17.3	21.3	24	20.0	22.4	24.4
VI + VII	19	21.3	23.3	31.1	31	25.8	28.9	31.6
VI + VII + VIII	26	29.2	34.6	42.6	51	42.5	47.6	52
VI + VII + VIII + IX	36	40.4	48	59.0	64	53.3	59.8	65.3
V + VI + VII + VIII + IX	37	41.5	49.3	60.7	70	58.3	65.4	71.5

The passing rate and the failing rate were therefore consistently in favor of the experimental sequence, in which ever way they were computed. The differences are admittedly not very significant, but the consistancy is. This essentially proves that <u>(Abstract) Mathematics can be taught to almost anybody willing to try, at no other cost than time and rigor</u>.

As regards the author's concern, the response of the students seems to [have] been sufficiently encouraging to warrant some more research along these lines. The author, however, had never really believed that there was any ground for watering down mathematics to the level usually encountered in the type of course offered under the various names of "Survey of Mathematics", "Fundamentals of Mathematics", and "Introduction to Mathematics". The argument usually given in favor of such courses is that nobody requires that a TV salesman know anything about TV. Similarly, all that is required of a liberal arts student is to refrain from gaping when the word "polynomial" or "determinant" arises in a conversation. Such a course is supposed to have attained its goal if the utterance of a word such as "matrix" evokes a vague image of a three by three arrangement of numbers. The author's position is that if a student is required to take three semesters of mathematics, he can in turn require to have three semesters of MATHEMATICS. This of course does not mean that every student ought to become a mathematics major, but he ought to be given at least the opportunity. Now, the regular sequence is a terminal one in the fullest sense of the term: it cannot under any condition lead to any further study of mathematics and the successful student can either forget mathematics or start all over again in another sequence. Even for the liberal arts student, it is totally without value inasmuch as it is a loose fabric of dead ends which are furthermore kept totally disconnected, thus leaving the student with a rather strange picture of mathematics (1). Because of this, and in view of the situation described in the introduction, it seems to be of much greater importance to reconstruct the student's confidence in his own ability to come to grips with problems, rather than to equip him with a few half memorized recipes. Furthermore, the possibility then remains open to him of further mathematical studies.

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(1) As an example, the textbook currently in use in the regular sequence introduces separately plane rotations, cos and sin, matrices as arrays of numbers and complex numbers as ordered pairs, but it carefully refrains from mentioning any relationship, not even in the case of a clockwise, quarter-turn rotation!

By a coincidence, while this attempt was coming to an end, an article appeared in which the pervasive belief that "democracy should mean equal opportunity for competition among people who are genetically unequal" was challenged [1]. The article contended that "...the best-supported general genetic or psychological theory does not validate the conclusion that individual intellectual capacity (for learning) is innately unequal". The opinion was furthermore expressed there that "some variation (of innate ability potential) may be possible (among people), but since all of the ability potential is well beyond the normal (intellectual) demand level, the variation makes virtually no operational difference." Thus the observed variation in intelligence was deemed to be a variation in "output", as resulting from environmental differences. It is suggested that the present attempt, if it does anything, supports the above thesis.

The totally uncontroversial evidence is rather small. It is only that out of 63 students declared in the name of the ACT tests to be of very low mathematical ability, 8 students turned out to be of such ability that the author has no doubt that they, with some more proper care, could have majored in mathematics in a four year institution. Of course, only a follow-up could have ascertained the point, and this only had some continued in mathematics.

The rest of the evidence is not as unarguable but certain a minima statements can still be made.

i. The students generally enjoyed abstract mathematics. They almost universally loathe the regular sequence.

ii. Given a problem of a non-familiar type, the students in the experimental group would be more likely to try thir , fiddle with them and even perhaps check on them [2]. It therefore seemed that the student's anxiety had been

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(1) "Are Children Born Unequal?" by W.H. Boyer and P. Walsh (U. of Hawaii) in Saturday Review (October 19, 1968).

(2) This however had been among the hardest things to obtain. Till the end, they remained afraid to find a mistake in what they had done.

at least partially alleviated and it would thus appear that there is no need to water down the contents for this reason, rather to the contrary.

iii. The computational ability of the students was tested independently of the author and on tests common with the regular sequence. In the author's view their ability to perform standard computations was rather low but turned out to be no lower than that of the students in the regular sequence. This was especially significant in the total absence of drilling in the experimental sequence, as opposed to the usual practice in the regular sequence.

These conclusions are harder to defend, mostly because they are intrinsically harder to substantiate, but also because the testing was inadequate.

IV

Rather than to say what a further experiment ought to be, it seems preferable to briefly review the difficulties encountered in this one. They essentially centered around the fact that to the author's knowledge, no textbook exists which would be sufficiently rigorous but still, detailed enough to be readable by this type of students (1). The consequence is that the instructor has to devote more time to writing lecture notes than to teaching. What of course compounds the problem is that this kind of text cannot be written beforehand. Only the classroom experience can tell how detailed a given treatment must be. For the same reason, it is quite difficult to create beforehand enough feasible interesting exercises.

Finally, although the author has been profoundly influenced by Piaget's genetic epistemology and by the works of Z.P. Dienes, the experiment lacked a proper psychological perspective on at least three counts.

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(1) The ratio: printed matter per lecture should be no less than 5, 6 printed pages per hour of instruction.

- While the testing of the comprehension of the contents was approximately appropriate, there was no psychological testing. This was a considerable lack. It would have been useful to measure the level of anxiety throughout both sequences, as well as the transferability to other fields of activity of the attitudes and behaviors acquired in each sequence.

- It is very well to speak about sets, about groups of transformations, etc... - but if the students could have lab work in which they would actually handle sets of attribute blocks, rotate and flip triangles, squares, etc... and then try to invent symbolisms for what they are doing, they would learn more. Most of these materials have already been developed for the elementary and high school by Z.P. Dienes and would probably require no more than a few adjustments. The main function of this lab work would be to provide the "concrete" basis so often spoken about. Its absence was certainly severely felt in this attempt. The time necessary for the students to familiarize themselves with the concepts dealt with in class could have certainly been drastically diminished.

- The author was also able to convince himself of the absolute necessity to devote one hour out of three to a period of <u>free</u> exercises. He realizes now that it would have allowed him to actually increase the overall pace.

The conditions in which this attempt was made were thus very far from being optimal, and could be vastly improved. Whether the results would then improve still remains to be seen. The author believes that they would, but hesitates as to by how much. On the other hand, the regular sequence was fairly "stabilized" and one does not see how its results could be improved in any way.

In conclusion, "we now create millions of people who think of themselves as failures - as social rejects". (1) This is particularly true with respect to mathematics, but it is hoped that this attempt will help mathematicians to convince themselves that almost anybody can be helped to learn mathematics, or at least, will incite them to investigate the possibility.

1) W.H. Boyer and P. Walsh - op. cit.