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ABSTRACT

This commentary is designed for use with "Unified Modern Mathematics, Course I," Parts 1 and 2. Included in the commentary are statements of the specific purposes and goals of each section of every chapter, suggestions for teaching the concepts presented in each section, time estimates for each section, suggested instructional aids for presenting various concepts, and references for further study. Also, suggested chapter examinations are provided which constitute comprehensive tests for each chapter. [Not available in hardcopy due to marginal legibility of original document.] (FL)

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Secondary School Mathematics
Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

COURSE I

TEACHERS COMMENTARY

TEACHERS COLLEGE COLUMBIA UNIVERSITY

1

TEACHERS COMMENTARY

FOR

COURSE I

OF

UNIFIED MODERN MATHEMATICS

TEACHERS COLLEGE COLUMBIA UNIVERSITY

SECONDARY SCHOOL MATHEMATICS

CURRICULUM IMPROVEMENT STUDY

Secondary School Mathematics Curriculum Improvement Study

Course I - Teacher Commentary

HOW TO USE THIS COMMENTARY

1. Purposes. At the start of the commentary for each chapter, the overall purposes and goals for the chapter are stated. Often, specific sections within the chapter are identified here as they would relate to each purpose stated. Similarly, the commentary for every section within the chapter will begin with a statement of specific purposes.

2. Sections. There are two basic types of sections within each chapter. One type presents concepts; the second type consists of exercises. The sections have been ordered so that every section of exposition is immediately followed by a section of related exercises. Within various sections, the teacher will find: possible motivational devices; a variety of approaches; notations relative to difficult exercises; suggestions for placement of exercises as class work; homework or self-study; hints regarding difficulties that may occur; new vocabulary underscored; and some abstract background for the teacher.

3. Time Estimates. In terms of days, a time estimate will

be found at the beginning of each chapter commentary. This is the estimate for the chapter; it is based upon individual time estimates for sections within the chapter.

Time estimates are given only to those sections containing some form of exposition. It is assumed that each exercise section is to be grouped with the concept section immediately preceding it relative to time estimations.

The teacher should carefully note that the key chapters of this course are 1, 2, 3, 4, 6, 9, 10, and 12. If the class deviates widely from the time estimates given, the teacher should feel free to assign chapter 11 as a self-study unit and to pace his teaching so that emphasis is still placed on the chapters noted here.

4. Exercises. Certain exercises have proven to be more successful when discussed within the actual lesson rather than assigned as homework. Suggestions regarding the placement of exercises appear at various points within the commentary.

The teacher need not hold rigidly to the exercises as listed. He is free to choose, add or alter any exercises whatsoever. In instances stressing drill, the teacher may wish to select or limit exercises depending upon the particular skills of his class and/or individual students.

Difficult problems have been starred and may be considered as optional. However, these problems are the most rewarding as well as the most challenging, and the teacher should discuss some of these in the classroom and/or assign them to the better students as homework. In all instances, the teacher should study the exercises before assigning them, carefully noting the concepts involved and approximating the time required for those exercises chosen. To insure that the teacher's evaluation of time for an assignment is as accurate as possible, the teacher should occasionally ask students to time homework assignments, allowing him to compare the true mean time with his judgment.

5. Self-Study Units. At various points within each chapter, certain sections will be identified as "self-study" ones. In essence, these sections usually contain simple applications of concepts previously taught and such sections should be regarded as being within the scope of each student's ability.

6. Summary and Review Exercises. At the end of each chapter, the teacher will find a summary of the main concepts studied followed by a series of related review exercises. The teacher may wish to assign the reading of the summary and the completion of the review exercise as:

   a) homework to be reviewed in class the following day,

b) self-study with time allowed the following day for student questions, or

c) classwork.

7. <u>Tests</u>. At the end of each chapter commentary, the teacher will find a series of suggested test items. The teacher should again feel free to choose, add, or alter any of these problems in constructing a test for his own class. An additional source of test items, once altered, would be the review exercises appearing at the end of each chapter in the text.

   <u>Cumulative review</u> exercises appear at the end of Chapters 3, 6, 9, and 13 in this commentary. These items, when grouped and/or altered to constitute a test should point out areas of weakness that will, in turn, demand re-enforcement.

8. <u>Unified approach</u>. The teacher should be alert to related topics and concepts throughout the entire course. The students should be able to grasp key ideas that weave a continual thread throughout the main body of the text. Properties and relations must continually be placed in the foreground and mathematics should be viewed as a united subject rather than a series of disjoint branches of learning.

<u>Teachers Commentary of Unified Modern Mathematics Course I</u> is an expansion of the original commentary written by the authors of the text. It was revised by the following nine pilot teachers in the SSMCIS Project:

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It is hoped that the teaching experience of this team will be reflected in a practical list of suggestions and a reasonable estimation of time allotments for the whole of this commentary.

Chapter 1

Finite Number Systems

Commentary For Teachers

Time Estimate for Chapter 1: (13 to 16 days)

Chapter 1 introduces the possibility of having arithmetics other than the familiar whole number arithmetic. Primary concern centers on the following three objectives:

(1) To review in a new setting ideas that may have been introduced to the students in the Elementary School. Such ideas include those of identity element, commutativity, and arithmetic system. (Throughout the chapter primarily in exercises)

(2) To introduce to the student, in the context of clock arithmetics, the important concepts of finite set, open sentence, inverses, associativity, and distributivity. (Sections 1.4, 1.7, 1.20 and 1.22)

(3) To extend the student's understanding of operations on the set of whole numbers by defining and examining corresponding operations on sets of clock numbers. (Sections 1.1, 1.2, 1.4, 1.5, 1.13, 1.15, 1.17, 1.18).

Notations:

1. This chapter assumes that the student understands the definition and notation for set and empty set.
2. Since the primary goal of this chapter is an under-

standing of certain mathematical properties that will appear throughout the text, these properties should be emphasized and taught with care. The teacher should feel free to omit or limit exercises that stress computational skills; this will depend upon the nature of each class and/or individual students.

3. Certain sections have been marked as self-study units. In each instance, the student may read the exposition and answer some or all of the exercises in the following section. For the most part, these sections consist of simple applications of previously learned concepts.

4. A supplementary unit will be found at the end of the commentary for this chapter. The unit deals with Slide Rules for Finding Products in (Zm,.). Its origin can be found in section 1.12, problem 6.

5. A chapter examination and answer key will be found at the end of the commentary for this chapter. These questions reflect concepts within the chapter. Because the test is a suggested one, the teacher need not hold to it rigidly. In fact, he should be encouraged to be creative in constructing other test items relative to the work studied.

1.1 and 1.2 Jane Anderson's Arithmetic and Clock Arithmetic

(Both sections together should take one day).

After reading this brief story, students might be asked such questions as the following:

"What is eleven plus two?" "Was Jane right?" "Have you ever used Jane's method?" "What time is it six hours after ten o'clock?" "What is ten plus six?" "How do we assign a sum to ten and six?"

Students might be asked to find out how time is kept by the Army or the Navy, i.e., using a 24-hour clock. Perhaps a student knows of, or has access to, a special clock such as an "Egg-timer" clock. In any case, get students thinking about how numbers are <u>assigned</u> to pairs of numbers as <u>sums</u>.

The notion of order in reading a table is most important. In this chapter (and all preceding ones involving the use of tables), the operation a * b will be defined as:

1. Select the element, a, from the index column at the left.
2. Select the element, b, from the index row at the top.
3. The solution to a * b will be the cell in which the row and column intersect.

*	a	b	c
a		x	
b			
c			

Here a * b = x

Order will not seem important at first since the systems are commutative. However, the teacher should stress order continually and, once subtraction is introduced, many examples of non-commutativity can be seen.

Here, in $(Z_3, -)$, $(Z_3, -)$

we see that

$2 - 1 = 1$

$1 - 2 = 2$

-	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

1.3 Exercises (Answers)

1. (a) 1 (b) 4 (c) 3 (d) 11 (e) 9 (f) 9
   (g) 1 (h) 1 (i) 10 (j) 9 (k) 9 (l) 12

2. (a) 8 (b) 6 (c) 7 (d) 11 (e) 12 (f) 4 and 10

3. (a) 1+6 since we are using a row-column approach.
   (b) It represents adding 1 to each number at the head of a column.
   (c) The teacher should be aware of several patterns. Here are a few:
      1. Each round is a shift left of the preceding one, with the extra number moved way to the right.
      2. The numbers in the cells are symmetric around the main diagonal (upper left to lower right).
      3. Every element appears only once in each column and each row.
   (d) The first column and the first row contain the same elements in the same order. The last column and the last row maintain a similar order. In general, the elements in row n appear in the same order as the elements in column n.

(e)

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

(f) All possible sums in Z_{12} can be shown. Some Z_{12} sums are different than the corresponding W sums. In fact, every sum below the lower left to upper right diagonal is different.

1.4 <u>(Z_{12},+) and (W,+)</u> (1 day)

The chief purpose of this section is to differentiate between <u>finite</u> and <u>infinite</u> sets. The terms "finite" and "infinite" are used in a most naive way in this Section. After the students think of "atoms," "electrons," etc., then the answers given to Question 2 can become quite exciting. Possible answers for Question 3 might be the set of even whole numbers, the set of natural numbers (that is, the set of whole numbers with 0 deleted), etc.

Note that the <u>commutative</u> and <u>identity properties</u> in (W,+) are mentioned within the reading for the first time. After reviewing and explaining these properties within (W,+), the students are asked to look for similar properties within $(Z_{12},+)$

<u>For the student</u>: Commutative Property in (W,+)
If x and y are numbers in (W,+), then
$x.y = y.x$

<u>For the teacher</u>: Commutative Property in (S,*) or the set S under the operation *,
$\forall x,y \in S$: $x*y = y*x$

<u>For the student</u>: ADDITIVE IDENTITY in (W,+) = 0
since $0 + x = x$ and $x + 0 = x$ where x is any number from W.

<u>For the teacher</u>: IDENTITY in (S,*) = e where $\forall x \in S$:
$e*x = x*e = x$

<u>Special notation</u>: Some confusion has developed with regard to the symbol "+". The authors have decided to write "11 + 2 = 13" in (W,+) and "11 + 2 = 1" in $(Z_{12},+)$.

To avoid the above confusion one could specify that clock addition is to be performed on a set by introducing a symbol such as "$^+12$". Then we would have "$11 +_{12} 2 = 1$".

One could extend such a symbolism and eventually obtain such expressions as "$(Z_{12},+_{12})$", or "$3 \cdot_{12} 12 = 12$" or even

"$3 \cdot_{12} (5 +_{12} 4) = (3 \cdot_{12} 5) + (3 \cdot_{12} 4)$. Both pedagogical and printing considerations encouraged us to avoid introduction of any new operational symbols but instead we specify the context; e.g., $(Z_{12}, +)$, where in such operations are to be performed. Teachers, of course, can decide on the symbolism they wish to use.

Helpful device: A large cardboard clock listing the elements of Z_{12} (and consequently, later models listing the elements of other Z_m systems) could aid in student understanding. A hand on such a clock will prove to be most useful when other operations (subtraction, multiplication, division) are introduced in subsequent sections.

REFERENCES:

Many students are intrigued with the notion of "infinity". An excellent outside reading for students is the article entitled "New Names for Old" by E. Kasner and J. R. Newman in the World of Mathematics, Volume III, Simon and Schuster, New York, 1956. pp. 1996-2011. (Teachers and better students will enjoy "Infinity" by Hans Hahn, same volume, pp. 1593-1613. See also Lee Zippin's Uses of Infinity, L. W. Singer Co., 1962).

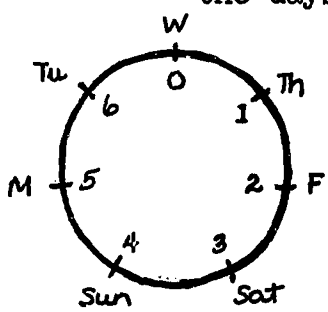
1.5 Calendar Arithmetic (1 day)

This section could be assigned to introduce $(Z_7, +)$ in a practical situation. Problems 5 and 6 in Section 1.6 are most important in developing the properties and concepts studied to this point.

1.6 Exercises

1. (a) Monday, Sunday
   (b) Monday
   (c) Wednesday
   (d) Thursday

(A useful device for problem 1 would be a clock showing the set Z_7 and the days of the week).



2. (a) 0
   (b) 1
   (c) 1
   (d) 6
   (e) 2
   (f) 2
   (g) 3
   (h) 4
   (i) 5
   (j) 1
   (k) 6
   (l) 0
   (m) 0
   (n) 0
   (o) 0

3. (a) 5 (b) 4 (c) 3 (d) 5

4. NOTE: Instruct students to save this $(Z_7,+)$ as well as all other $(Z_n,+)$ tables for later study.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) Table shows every possible sum.

(b) Yes, 0 since $0 + x = x$ and $x + 0 = x$.

(c) Yes.

(d) There is a shift of every element 1 to the left, with the first element moving to the position at the far right, whenever you compare row (n) with row (n + 1).

(e) There is a shift upward 1 element, with the uppermost moving to the position at the very bottom, whenever you compare column (n) with column (n + 1).

(f) There are a variety of explanations. Among them:

(1) (Upper) right to (lower) left diagonal = set of ordered pairs whose sum is 6. As one number of the pair increases by 1, its mate decreases by 1.

(2) (Upper) left to (lower) right diagonal = the set of ordered pairs (n,n).

5. (a) A variety of speculations and observations. Some include:

1. Where $x + y$ is less than 6, the tables match exactly.
2. Where $x + y \geq 6$ (thinking of whole numbers), the matching elements in the tables differ by 7.

   Ex: In $(Z_{12},+)$, $3 + 5 = 8$

   In $(Z_7,+)$, $3 + 5 = 1$
3. All properties relating row and column for one table (order, commutativity, etc.) will hold for both tables.

(b) Yes. The more important is the main diagonal (upper left to lower right). The symmetry about this diagonal implies commutativity. For the other diagonal, all elements in $(Z_n,+)$ add up to $(n - 1)$.

6. Possible similarities: both $(Z_{12},+)$ and $(W,+)$ have an additive identity element; both are commutative under addition.

   Possible differences: the number of entries in the addition table for $(Z_{12},+)$ is finite whereas the number of possible sums in $(W,+)$ is not finite; we can show all possible cases of commutativity of pairs of elements under addition by means of the table for $(Z_{12},+)$ whereas this cannot be done for $(W,+)$.

7. (a) Yes.

   (b) Yes, particularly in the first row and column and the last row and column.

   NOTE: It is suggested here that "12" be replaced by "0" in $(Z_{12},+)$ in order to conform to the other $(Z_n,+)$ systems.

1.7 Open Sentences (2 days)

The primary purpose of this section is an understanding of open sentences and solution sets. A great many new terms are introduced in this section: mathematical sentences (true, false, open), variable, domain of the variable, equation, inequality, solution set, empty set. This section should be studied and amplified extensively, most reasonably by proving a variety of review exercises while studying the oncoming sections. These concepts and terms will play a basic role in chapters to come.

Teachers might wish to consider every-day examples of open sentences such as:

□ is the President of the United States.

Δ is the Empire State.

□ is the boy whose first name begins with the letter R.

Δ is the girl whose first name begins with the letter X.

□ is older than 12 years.

Have the students identify the variable; the domain of the variable; true and false sentences as they result; the solution set. Alter domains and study the varions solution sets obtained.

Ex: Find the solution set of $x - 6 = 6$

Domain	W	Z_7	Z_{12}	Odd numbers
Solution	{12}	{5}	{0}	$\emptyset$

In the exercises, students should be asked to explain how they obtained solution sets for open sentences. Did they use a table? Note that no formal approaches are given for solving open sentences at this time. Intuition and simple substitution should be the typical approach of most students.

Set notation should be carefully explained and noted in all exercises. The solution of $x + 2 = 5$ is not $x = 3$. In truth, $x = 3$ is the simplest equivalent open sentence for $x + 2 = 5$. Since the braces, { }, are read as "the set whose members are", we enclose our solution with these braces.

Stress that the solution depends upon the domain. Here, the solution of $x = 3$ is $\{3\}$ if the domain is W and $\{\}$ or $\emptyset$ if the domain is the set of Even Numbers.

Teachers might wish to point out that the concept of <u>order</u> which applies to the set of whole numbers has no analogue for sets of clock numbers. However, there are no exercises in this text involving order in finite systems. To examine one: Solve $x < 10$ using Z_{12} as the domain of the variable. There can be <u>no solution</u>. By simply studying the clock, the students should see that 3 could come before 10 or after 10.

1.8 <u>Exercises</u>

1. (a) True (b) False (c) open (d) True (e) open
   (f) open (g) False (h) open
2. (a) False: $11 + 7 = 6$ in $(Z_{12},+)$
   (b) True: Check table for $(Z_7,+)$
   (c) True: Check table for $(Z_7,+)$
   (d) True: Check table for $(Z_{12},+)$
3. (a) $\{8\}$ (b) $\{3\}$ (c) $\{12\}$ (d) $\{8\}$ (e) $\emptyset$.
   (f) $\{3,4,5,\ldots\}$ (g) $\{1,2,3,4,\ldots\}$
   (h) $\{0,1,2,3\}$ (i) $\{0,1,2,\ldots 9\}$ (j) $\{5\}$
4. (a) $\{0\}$ (b) $\{0\}$ (c) $\{4\}$ (d) $\{2\}$
   (e) $\{5\}$ (f) $\{6\}$
5. (a) $\{1\}$ (b) $\{7\}$ (c) $\{6\}$ (d) $\{5\}$ (e) $\{0\}$
   (f) $\emptyset$.
6. Answers will vary. Some examples might be:

(a) $x + 4 = 10$ (b) $x + 4 = 3$ (c) $x + 8 = 2$

7. Answers will vary. Some examples might be:

(a) $x + x = 2x$ (b) $x + 5 = 2$ (c) $x < 1$.

8. (a) not equivalent (b) not equivalent (c) equivalent

(d) equivalent (e) not equivalent

1.9 <u>New Clocks</u> (1 day)

This section can be assigned for self-study. Since it is simply an extension into the Z_4, Z_5 and Z_6 systems, all students should be able to read this section and answer the exercises in section 1.10 with complete understanding.

1.10 <u>Exercises</u>

Students at this stage should see that the periodicity of the entries in clock addition tables makes them easy to construct. Also as soon as the entries for a row are determined, then commutativity allows us to complete a column.

1. (a)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(b) (1) 3 (3) 3 (5) 0 (7) 0

(2) 0 (4) 2 (6) 3 (8) 1

(c) LOW

(d) Addition in Z_4 is commutative since entries in table for $(Z_4,+)$ are symmetric with respect to the main diagonal.

2. Answers are similar to those in Section 1.6, numbers 5 and 6.

3. (a)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(b) (1) 1   (3) 0   (5) 1

(2) 0   (4) 3   (6) 2

(c) Answers are similar to those in Section 1.6, number 5.

(d) Answers vary. Here are some examples:

High + $\square$ = Off

Low + $\square$ = Low

Simmer + Medium = $\square$

4. (a) Answers may vary depending on the number of channels and positions such as "OFF", "UHF", etc.

(b) Answers vary. Some examples include: the hours in a day are given by $(Z_{24},+)$, the minutes on a clock by $(Z_{60},+)$, the seasons by $(Z_4,+)$.

5. (a) Answers vary. Here are some:

(1) Identity element of 0 does not alter any number.

(2) Row n and column n have the same elements in the same order.

(3) If written in order (n, n+1, n+2, ...), each row shifts one element to the left.

(4) In $(Z_n,+)$ no element appears more than once in any one row or in any one column.

(b)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

1.11 Rotations

This section requires only a simple explanation that would serve as introduction to a new type of system. It should not be treated as a full lesson requiring one day of class time. By constructing part or all of the (Hexagon, rotation) table in the class, the students should be able to answer Questions 1 through 5 in Section 1.12. Note that some students might have trouble in identifying $r_6 = r_0$.

Students, if given straight edge and compasses, might discover how easy it is to inscribe a regular hexagon in a circle.

1.12 Exercises

1. (a) r_3 (b) r_4 (c) r_1 (d) r_0 (e) r_3
2. (a) The resulting position after r_6 leaves the hexagon in the same position as r_0.
   (b) The rotation r_0 will not alter the position of the

hexagon after the last rotation.

(c) The identity rotation.

3. (a) (1) A rotation of 60° followed by a rotation of 60° is equivalent to a single rotation of 120°.

(2), (3), and (4) similar reasoning.

(b)

rot.	r_0	r_1	r_2	r_3	r_4	r_5
r_0	r_0	r_1	r_2	r_3	r_4	r_5
r_1	r_1	r_2	r_3	r_4	r_5	r_0
r_2	r_2	r_3	r_4	r_5	r_0	r_1
r_3	r_3	r_4	r_5	r_0	r_1	r_2
r_4	r_4	r_5	r_0	r_1	r_2	r_3
r_5	r_5	r_0	r_1	r_2	r_3	r_4

(c) The table for (Hexagon, rotation) has the "same form" as the table for $(Z_6,+)$.

4. (a) r = rotation of 90° clockwise; r_2 = rotation of 180°; r_3 = rotation of 270°.

(b)

rot.	r_0	r_1	r_2	r_3
r_0	r_0	r_1	r_2	r_3
r_1	r_1	r_2	r_3	r_0
r_2	r_2	r_3	r_0	r_1
r_3	r_3	r_0	r_1	r_2

(c) The table for (Square, rotation) has the "same form" as the table for $(Z_4,+)$.

5. The table for (Heptagon, rotation) has the "same form" as the table for $(Z_7,+)$. Use the addition table for $(Z_7,+,\cdot)$ using the elements of Z_7 as subscripts for the rotations.

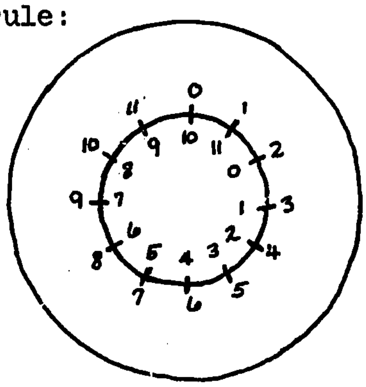
6. By matching 0 from ruler B with 2 from ruler A, every element on ruler A is 2 larger than its corresponding element on ruler B. Thus, 5 from B matches with 7 from A.

In general, (n) from B matches with (n+2) from A.

NOTE: A supplementary unit on Circular Slide Rules for Finding Products in $(Z_m, \cdot)$ can be found at the end of this chapter's commentary.

To answer the question on $(Z_{12}, +)$ slide rule:

By matching 0 from the "inner" ruler with 2 from the "outer" ruler, every element on the outer ruler is the sum of 2 and the element on the inner ruler.



(1 day)

1.13 Subtraction in Clock Arithmetic

The purposes of this section are to introduce a new operation called subtraction with Z_m systems and to examine the operation. Important observations should include:

(a) Whereas subtraction is restricted in (W,+), subtraction in $(Z_m, +)$ is unrestricted. 3 - 5 cannot be answered in W but it can be answered in Z_6.

(b) Subtraction is defined in terms of addition: stress that for all a, b and c in Z_m, a - b = c if and only if c + b = a. The students should be shown that such a "c" exists and also that "c" is unique (there is one and only one c).

(c) Subtraction in $(Z_m, +)$ is not commutative. The only exception to this rule is the Z_2 system.

Special Hints: Some students might find it difficult to subtract numbers by the use of an addition table. Setting up a subtraction table (section 1.14, exercise 2) might offer a greater obstacle. Here, a teacher or student constructed clock will prove most helpful. By moving the hand or dial in a counter clockwise movement, subtraction becomes a relatively easy operation.

Some teachers might wish to introduce the notion of inverse of 3 is written as -3 and $-3 = 2$. Thus, $1 - 3 = 1 + (-3) = 1 + (2) = 3$.

Note: The terms "restricted operation" and "always possible operation" are used in this chapter. In Chapter 2, when operations are studies, the term "operation" will be used to refer only to "always possible" operations.

1.14 Exercises

Exercise 2 shows that subtraction is an unrestricted operation in $(Z_6,+)$ and could be presented in class. Exercise 3 shows an interesting situation. In developing addition we went from a real world clock or dial to an abstract table. However, subtraction was first obtained with an abstract table and then an application to the real world was made in connection with moving "counter clockwise" on a dial. For exercise 7, note that subtraction is not commutative.

NOTE: The teacher should feel free to assign only a "selection" from exercises 1, 4, 5, 8 and 9 since much of this is repetitive.

1. (a) 2 (b) 1 (c) 4 (d) 2 (e) 0
   (f) 3 (g) 1 (h) 4 (i) 3 (j) 3

2. (a) (b) Yes

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

   (b) Yes. Every possible pair is assigned an element from $(Z_5, +)$.
   (c) Subtraction in $(Z_5, +)$ is unrestricted, but subtraction is restricted in W. In W, there is no solution for 2 - 3.

3. (a) "4"; 2 - 3 = 4
   (b) Move the pointer to "1" and then move the pointer counterclockwise through 2 intervals. The pointer will then be directed at "4". Thus, 1 - 2 = 4 in Z_5.
   (c) (1) 2 (2) 2 (3) 3 (4) 3

4. (a) 3 (b) 3 (c) 3 (d) 3 (e) 5
   (f) 2 (g) 5 (h) 2

5. (a) 2 (b) 6 (c) 5 (d) 3 (e) 4 (f) 6
   (g) 1 (h) 0 (i) 5 (j) 4 (k) 0 (l) 4

6. (a) no (b) no (c) no

For $m > 2$ there does not exist an identity element for subtraction in $(Z_m,+)$. (Note: for $m > 2$, 0 is a right identity element for subtraction, i.e. for all x, $x - 0 = x$. But for $m > 2$, 0 is not a left identity element for subtraction, i.e. for all x we do not have $0 - x = x$. Therefore, for $m > 2$, there is no identity element).

7. (a) No. Not all entries are symmetric with respect to the main diagonal.

(b) No. (c) No.

For $m > 2$ subtraction is not commutative in $(Z_m,+)$.

8. (a) {3} (b) {0} (c) {2} (d) {1}
(e) {2} (f) {4} (g) {0} (h) {4}
(i) {3} (j) {0} (k) {3} (l) {3}

9. (a) {4} (b) {3} (c) {5} (d) {4} (e) {3}
(f) {2} (g) {3} (h) {4} (i) {4} (j) {0}

1.15 Multiplication In Clock Arithmetic (1-2 days)

The purposes of this section are to introduce the operation of multiplication in Z_m systems, to examine relations between multiplication and other operations and to study properties in $(Z_m,\cdot)$.

Possible Motivation: Before defining multiplication, students might be asked to construct a multiplication table for a set of clock numbers. These tables could then be studied for various properties they might contain, even if the elements within the table are incorrect. As an example, a student could construct the $(Z_4, \cdot)$ table incorrectly as:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	2	0

Errors within the box.

The zero property of multiplication and the multiplicative identity are correctly stated. However, the errors would lead the student to a false conclusion of non-commutativity. Note the inconsistency in claiming $2.2 = 3$ while $2 + 2 = 0$ in the $(Z_4,+)$ table; the student should realize that $2 + 2 = 2.2$

Relations: Just as in $(W,+,\cdot)$, multiplication in $(Z_m,+,\cdot)$ can be expressed as repeated addition. The use of a physical clock would be helpful at this point. However, multiplication in $(Z_m,\cdot)$ relates directly with division in the whole numbers, taking its products from the set of remainders.

The properties that students might discover are fully stated in section 1.17 which follows the exercises. The teacher may merge both sections together or allow the students to explore this section in an intuitive manner before firmly establishing the properties that hold.

1.16 Exercises

1. (a)

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(b) There are many. Some include

(1) The 0 row and the 0 column have all 0 entries since $0.x = x.0 = 0$

(2) Row n and column n contain the same elements in the same order.

(3) There is a symmetry about the main diagonal.

(4) Every non-zero row and non-zero column contain each element from Z_5 only once.

2. (a) {2}
(b) {1}
(c) {0}
(d) {0,1,2,3,4}
(e) {4}
(f) {0}
(g) {3}
(h) {3}
(i) {1}
(j) {4}
(k) {2}
(l) ∅

3. (a)

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(b) Patterns 1, 2, 3 are the same as those in exercise 1, part (b), of this section. However, row 2 and column 2 do not yield every element from Z_4 only once; there is a series using products of 0 and 2 only. [Some students may conjecture that rows beyond the 0 row will repeat in a $(Z_m, \cdot)$ table whenever m is not a prime].

$(Z_6, \cdot)$

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$(Z_7, \cdot)$

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(a) Both are commutative finite operational systems with a multiplicative identity element 1, etc. Solution for $2.x = 0$ is $\{0,2\}$ in $(Z_4, \ )$ whereas the solution set for the corresponding equation in $(Z_6, \cdot)$ is $\{0,3\}$ etc.

(b) Similar: Both are finite operational systems; both are commutative over multiplication etc.

Different: In $(Z_7, \cdot)$ zero appears as an entry in only the first row and first column whereas this is not true in $(Z_6, \cdot)$; In $(Z_7, \cdot)$ entries do not repeat in a row or column except in the zero row and zero column whereas this is not true in $(Z_6, \cdot)$, etc.

5. (a)	(1)	{2}	(2)	{1.3}	(3)	{0,2}	(4)	{3}
	(5)	{0,1,2,3}	(6)	∅	(7)	{3}	(8)	∅
(b)	(1)	{4}	(2)	{4}	(3)	{3}	(4)	{0}
	(5)	{4}	(6)	∅	(7)	{2}	(8)	{0}
(c)	(1)	{3}	(2)	∅	(3)	∅	(4)	{0,3}
	(5)	{2,5}	(6)	∅	(7)	{1,3,5}	(8)	{0,2,4}
	(9)	∅	(10)	{1,4}	(11)	{0,3}	(12)	{2,5}

1.17 Comparison of $(W,\cdot)$ and Clock Multiplication (1 day)

This is a new section which can probably best be used as a basis for class discussion comparing the various multiplicative systems. It might be helpful to have several of the $(Z_n,\cdot)$ tables displayed in class to facilitate comparison.

Important here are the properties. The questions (a)-(e) are leading the student to a discovery that repetition of elements will occur in a non-zero row of $(Z_m,\cdot)$ whenever m is not prime. The rows that will contain these repeating series will be the prime factor rows of m and all multiples of those prime factors. In $(Z_{10},\cdot)$ repeating rows will be 2, 4, 5, 6 and 8.

The "secret code" research problem may be treated as enrichment. The answers to questions (a) - (h) follow:

(a) E B O H F S

(b) H E L P

(c) "A"

(d) There are 26 letters in the English alphabet.

(e) $x' = x + 3$ in $(Z_{26},+)$

(f) (1) $x' = x$ in $(Z_{29},+)$ GOOGOL IS THE SPY.

(2) $x' = x + 3$ in $(Z_{29},+)$

IN ENGLISH TEXT, THE LETTER E OCCURS MOST OFTEN.

(g) $2 \cdot 13 = 0$ and $2 \cdot 0 = 0$. No.

1.18 Division in Clock Arithmetic (1 day)

Here the purposes include an introduction to the operation of division and an examination of relations and properties with the operation.

Division, as with the whole numbers, is defined in terms of multiplication. Essentially, bring out that for all a, b and c in Z_m, $a \div b = c$ if and only if $c \cdot b = a$. Again we must show that c exists and is unique.

Division may be spoken of as repeated subtraction and here a physical clock will prove helpful. Some teachers may wish to use the concept of inverse at this point, equating $a \div b$ with $a \cdot \frac{1}{b}$ where $\frac{1}{b}$ is the multiplicative inverse of b.

Certain divisions must be noted as "not defined". We know that $6 \div 0 = x$ cannot be solved in $(W,\cdot)$ since there is no x to satisfy $x \cdot 0 = 6$. In the $(Z_m,\cdot)$ systems employing division, many more numbers are restricted as divisors.

Ex: In $(Z_{[illegible]},\cdot)$, $4 \div 3$ is not defined since $3 \cdot x = 4$ cannot be solved in the system.

Note that division is not commutative and that the systems possess no identity. Some difficulty might arise in looking at division problems in the form $\frac{a}{b}$ as fractions existing in a clock system.

1.19 Exercises

In their reports on division students might note that there is no x in Z_4 such that $1 \div 2 = x$ and also note that the solution set of $2 \div 2 = x$ is $\{1, 3\}$. Thus both existence and uniqueness fail for division in Z_4.

1. (a) 4 (b) 2 (c) 4 (d) 2 (e) 1
   (f) 4 (g) 3 (h) 4 (i) 0 (j) 0
   (k) 3 (l) 2 (m) 2 (n) 3 (o) not defined
2. (a) - (d) are not defined
3. No. For example, given 3 and 0, in this order, we cannot assign an element of Z_5 to this pair as quotient, i.e. $3 \div 0$ is not defined in Z_5.

   Division is restricted in Z_5 and in W; subtraction is restricted in W and unrestricted in Z_5. Note that division is restricted in Z_5 only when dividing by 0. In W, division may be restricted more often, such as $2 \div 3$.
4. (a) $\{3\}$ (b) $\{1\}$ (c) $\{4\}$ (d) $\{2\}$
5. There are no entries in the first column of the division table whereas all cells in the multiplication table contain entries.
6. Answers will vary. Some notations could include:
   (a) 2,3 and 4 have no multiplicative inverses

(b) while $\frac{4}{2} = 5$, it is also true that $\frac{4}{2} = 2$ since $2.2 = 4$.

1.20 Inverses in Clock Arithmetic (1 - 2 days)

Teachers might wish to point out how the idea of inverses can be applied to solving open sentences. The idea of inverse elements in an operational system will be encountered frequently later, so mastery is not crucial at this point.

1.21 Exercises

Exercises 7, 9, 10 and 12 call for generalizations. The teacher may wish to do these exercises in class or allow time for class discussion after assigning them as homework. The teacher may wish to reassign certain numerical problems in this section, simply changing the system for $(Z_5,+,\cdot)$ to $(Z_7,+,\cdot)$.

1. (a) 3 (b) 4 (c) 0 (d) 2 (e) 1 (f) 2
2. (a) 3 (b) 4 (c) 1 (d) 2 (e) does not have a multiplicative inverse. (f) 2
3. (a) (1) 4 (2) 1 (3) 0 (4) 3 (5) 2
   (b) (1) 3 (2) 1 (3) 4 (4) 2 (5) not defined
4. (a) 1 (b) 2 (c) 1 (d) 0 (e) 4 (f) 0
5. (a) 2 (b) 1 (c) 1 (d) 2
6. (a) (1) {0} (2) {2} (3) {1}
   (b) (1) {1} (2) {1} (3) {3}
7. (a) 2
   (b) (1) 4 (2) 1 (3) 3 (4) 0
   (c) 3 (d) $-(-x) = x$

8. (a) If $1 \div 2 = x$, then $1 = 2.x$. The solution set for x is $\emptyset$. Or simply, in Z_6, 2 does not have a multiplicative inverse.

   (b) No. (c) 0 (d) 1

9. (a) (1) 1 (2) 1 (3) 4 (4) 4 (5) 3 (6) 3

   (b) $-(2 + 3) = -(0) = 0$

   (c) $-2 + -3 = 3 + 2 = 0$

   (d) $-(x + y) = -x + -y$

10. In Z_6 less the 0 element under multiplication, or in the set $\{1,2,3,4\}$ or Z_6 numbers, $\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y}$

11. (a) True (b) True (c) False since not true for $x = 0$.

12. (a) True (b) False, $2.2 = 0$ does not imply that $2 = 0$.

1.21 The Associative and Distributive Properties (1 day)

John wrote "had". James wrote "had had". Thus "John, where James had had "had had," had had "had." The idea of associativity will again be discussed in Chapter 2. Stress that the purpose of a distributive property is to tell how two operations can be combined. This also will be seen again later so it need not be mastered here.

For the teacher: Associativity in $(S,\star)$:

$\forall a, b, c \in S$: $a \star (b \star c) = (a \star b) \star c$.

For the teacher: Distributivity in $(S,\star,\Delta)$

Right hand distributivity exists, or $\star$ distributes over Δ when $\forall a, b, c \in S$: $a \star (b \Delta c) = (a \star b) \Delta (a \star c)$.

Left hand distributivity of $\star$ over Δ is shown as:

$$\forall a, b, c \in S: \ (a \,\Delta\, b) \star c = (a \star c) \,\Delta\, (b \star c).$$

For the students: Stress that one operation distributes over a second operation. In the systems noted, it is multiplication that distributes over addition. The reverse will not be true. Some students will write $a \cdot (b + c) = a \cdot b + c$. The use of parentheses, $(a \cdot b) + (a \cdot c)$, on the right may overcome this error. Numerical substitutions, always performing operations within parentheses first, will help the student in discovering his error.

1.23 Exercises

If the teacher wishes to select a limited number from exercises 1 through 4, he should be aware that pairs of exercises are grouped to provide drill with the properties studied. Parts (a) and (b) are coupled; Parts (c) and (d) are coupled; etc. Exercises 6 and 7 are optional but they form a basis for an excellent classroom discussion.

1. (a) 4 (b) 4 (c) 1 (d) 1
   (e) 2 (f) 2 (g) 0 (h) 0
2. (a) 4 (b) 4 (c) 1 (d) 1
   (e) 0 (f) 0 (g) 1 (h) 1
3. (a) 4 (b) 4 (c) 0 (d) 0
   (e) 2 (f) 2 (g) 4 (h) 4
4. (a) 0 (b) 0 (c) 1 (d) 1

5. (a) Multiplication is distributive over subtraction in Z_6, i.e. $a \cdot (b - c) = a \cdot b - a \cdot c$

   (b) Division is not distributive over addition in Z_6 (Note $4 \div (1 + 0) = 4 \div 1 = 4$, however $(4 \div 1) + (4 \div 0)$ is not defined.)

6. There is a right hand distributive law $(b + c) \cdot a = b \cdot a + c \cdot a$ in $(Z_6, +, \cdot)$

7. (a) $a \cdot (b + c) = a.b + a \cdot c$ (Distributive Law of . over +)

   $= b \cdot a + c.a$ (Commutative Property of $\cdot$)

   $= c \cdot a + b \cdot a$ (Commutative Property of +)

   (b) Yes. $a \cdot (b + c) = c \cdot a + b \cdot a$ holds in all $(Z_m, +, \cdot)$.

1.24 Summary

Students should study this section at home, noting the key topics discussed in the chapter. A possible homework assignment would include reading the summary and answering the review questions in the following section. Any problems that arise could then be brought to the attention of the teacher the following day.

1.25 Review Questions

$(Z_8,+)$

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

$(Z_8,\cdot)$

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

1. (a) 5 (b) 0 (c) 4 (d) 4 (e) 5 (f) 5
2. (a) 2 (b) 0 (c) 4 (d) 4 (e) 1 (f) 1
3. (a) 4 (b) 4 (c) 4 (d) 4
4. It is false. $2 \cdot x = 0$ does not imply $x = 0$ because we could also have $x = 4$. Similarly try $4 \cdot x = 0$ or $6 \cdot x = 0$.
5.

x =	0	1	2	3	4	5	6	7
-x =	0	7	6	5	4	3	2	1

6.

x	0	1	2	3	4	5	6	7
$\frac{1}{x}$	not def.	1	not def.	3	not def.	5	not def.	7

7. (a) {2} (b) {4} (c) {7} (d) {0,4} (e) {4} (f) {0, 2, 4, 6} (g) {5} (h) {5} (i) ∅ (j) ∅
8. Saturday. Using $(Z_7,+)$, adding 1000 is equivalent to adding 6.
9. 7.

Chapter Examination

Items 1 through 10 would constitute a comprehensive full period test for this chapter. To show the teacher a variety of exercises, some of which could be truly challenging, additional items (11 through 13) are provided for the teacher's perusal. Again, this examination is a suggested one and the teacher should feel free to add or delete items as he chooses.

1. Construct an addition table for $(Z_4, +)$.
2. Construct a multiplication table for $(Z_4, \cdot)$.
3. (a) Why is $(Z_4, +)$ a restricted operational system?
   (b) Is multiplication commutative in $(Z_4, \cdot)$? Explain your answer.
4. Compute the following in $(Z_4, +, \cdot)$:
   (a) $3 + 3$
   (b) $2 + (3 + 2)$
   (c) $(2 + 3) + 2$
   (d) $2 \cdot (3 + 1)$
   (e) -1
   (f) $3 \cdot (1 - 3)$
   (g) $-(-3)$
   (h) $-2 \cdot [(-3) + (-1)]$
   (i) $\frac{1}{3} \cdot (1 + 3)$
   (j) $2 \div 0$
5. Solve the following open sentences in $(Z_4, +, \cdot)$:
   (a) $2 + 3 = x$
   (b) $3 + y = 1$
   (c) $2 \cdot z = 1$
   (d) $2 \cdot x = 0$
   (e) $\frac{2}{y} = 3$
   (f) $x \cdot x = 1$
6. (a) Give an example of a non-finite set.
   (b) Give an example of a large but finite set.
7. Give an example which shows that each of the following is False.

(a) Subtraction is commutative in $(Z_4, \cdot)$.

(b) The solution set of $2 \cdot x = 2$ is $\{1\}$ in $(Z_4, \cdot)$.

(c) Every element has a multiplicative inverse in $(Z_4, \cdot)$

(d) Subtraction is distributive over multiplication in $(Z_4, +, \cdot)$

8. State two similarities between $(Z_4, +)$ and $(W, +)$.

9. State two differences between $(Z_4, \cdot)$ and $(W, \cdot)$.

10. Fill - in by writing the missing word, phrase or symbol in an answer column

(a) Mathematical sentences that are either true or false are called ______________________.

(b) In an open sentence, the symbol which may be replaced by symbols for numbers is called the ______________.

(c) The set of all elements from the domain which give only true statements when replaced for the variable is called the __________ ______________________set.

(d) A set containing no elements is called the __________ ________________set.

(e) In $(Z_8, \cdot)$ for every element x, $x \cdot 0 = 0 \cdot x = 0$.
This illustrates what property?____________________

(f) In $(Z_9, +)$ for every element x, $x + 0 = 0 + x = x$.
This illustrates what property?____________________

(g) In $(Z_7, \cdot)$ for every element x, $x \cdot 1 = 1 \cdot x = x$
This illustrates what property?____________________

11. Circle the systems which are "unrestricted":

$(W, +)$ $(W, -)$ $(C, \cdot)$ $(C, \div)$

$(Z_{12},+)$ $(Z_8,-)$ $(Z_5,\cdot)$ $(Z_{893},+)$

$(E,\cdot)$ $(O,+)$

Here, let C = set of counting numbers = {1,2,3...}

E = set of even whole numbers = {0,2,4...}

O = set of odd whole numbers = {1,3,5,...}

12. Give all possible solutions for each open sentence using each of the four domains for the variable

	W	Z_5	Z_8	Z_{12}
$4 + 3 = x$				
$2 \cdot 4 = x$				
$x + x = 0$				
$x + 4 = 2$				
$2 - x = 3$				

13. Answer True or False

(a) The commutative property holds in $(Z_5,-)$.

(b) The commutative property fails in $(Z_2,-)$.

(c) $5 + 3 = 8$ is a true statement in $(Z_7,+)$.

(d) The identity element in subtraction is always 0.

(e) In $(W,+)$, $x + 2 < 3000$ yields an infinite solution set.

(f) In $(Z_{10},+)$, the solution set of $x > 7$ is {8,9}.

(g) In every $(Z_n,\cdot)$ there is always at least one row with repeating elements.

(h) In $(Z_7,+)$, $5 + 3 = 1$ is an example of a false statement.

(i) In any system, $5 + x = 12$ is always considered to be an open sentence.

Answers to Chapter Examinations.

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

3. (a) For every ordered pair of elements in Z_4 we can assign one and only one element of Z_4 to this pair as their sum.

   (b) Yes (1) There is a symmetry about the main diagonal or (2) Row n and column n both contain the same elements in the same order.

4. (a) 2 (b) 3 (c) 3 (d) 0
   (e) 3 (f) 2 (g) 3 (h) 0
   (i) 0 (j) not defined

5. (a) $\{1\}$ (b) $\{2\}$ (c) $\emptyset$ or $\{\ \}$
   (d) $\{0,2\}$ (e) $\{2\}$ (f) $\{1,3\}$

6. (a) e.g. W; or the set of even whole numbers; etc.
   (b) the set of electrons in the earth; etc.

7. (a) Any of the following: $1 - 0 \neq 0 - 1$; $3 - 0 \neq 0 - 3$; $2 - 1 \neq 1 - 2$; or $3 - 2 \neq 2 - 3$.

   (b) No. The solution set is $\{1, 3\}$

   (c) No. 0 does not have multiplicative inverse in $(Z_4, \cdot)$. Neither does 2.

   (d) No. For example

   $$3 - (2 \cdot 3) \neq (3 - 2) \cdot (3 - 3) \text{ because } 1 \neq 0.$$

8. Both $(Z_4,+)$ and $(W,+)$ have an additive identity element and both have commutative and associative properties for addition; etc.

9. $(Z_4,\cdot)$ has a finite number of products in its multiplication whereas $(W,\cdot)$ has a non-finite number of such products. Also $x\cdot y = 0$ implies $x = 0$ or $y = 0$ in $(W,\cdot)$ but not in $(Z_4,\cdot)$; etc.

10. (a) statements
(b) variable
(c) solution
(d) empty or null
(e) zero property of multiplication
(f) (additive) identity
(g) (multiplicative) identity

11. "Unrestricted" systems include:
$(W,+)$ $(C,\cdot)$ $(Z_{12},+)$ $(Z_8,-)$
$(Z_5,\cdot)$ $(Z_{893},+)$ $(E,\cdot)$

12.

W	Z_5	Z_8	Z_{12}
{7}	{2}	{7}	{7}
{8}	{3}	{0}	{8}
{0}	{0}	{0,4}	{0,6}
Ø or { }	{3}	{6}	{10}
Ø or { }	{4}	{7}	{11}

13. (a) False
(b) False
(c) False
(d) False
(e) False
(f) False
(g) True
(h) False
(i) True

Supplementary Unit: Circular Slide Rules for Finding Products in $(Z_m, \cdot)$

It is sometimes possible to construct circular slide rules in order to compute products for certain $(Z_m, \cdot)$ operational systems. For example, consider the set of numbers 1,2,3,4. This is the set Z_5 with 0 removed. Arrange these four numbers clockwise on a dial in the order 1,2,4 and 3. A larger dial with the same numbers in the same order is placed beneath the first dial. If the numerals are placed at equal intervals then we have a device similar to that in Figure 1a.

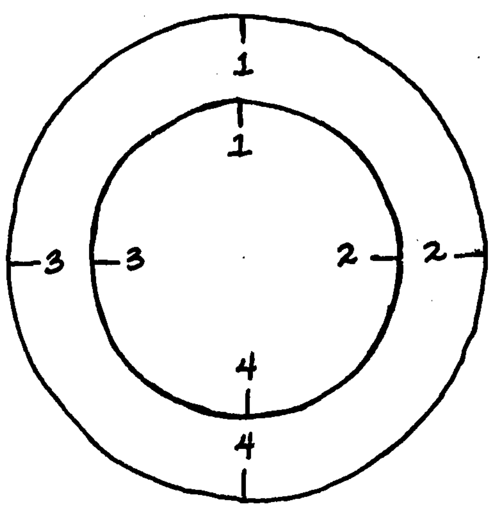


Fig. 1a

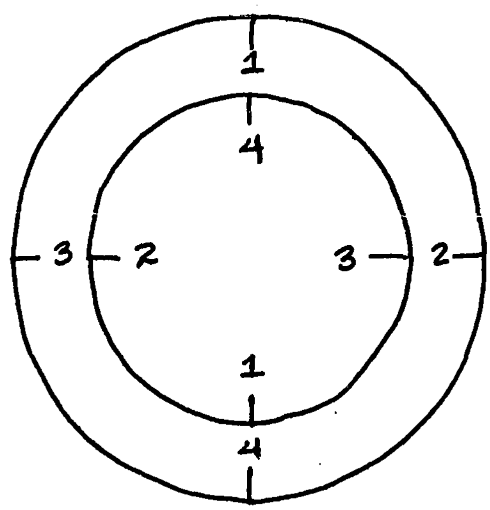


Fig. 1b

How can we compute such products as 4.2 or 4.4 in $(Z_5, \cdot)$? If we wish to find the product of a number in Z_5 (except 0) and 4 we proceed as follows: First, we rotate the smaller dial until the numeral 4 on the smaller dial is directly below 1 on the larger dial. Then to find the product of 4 and n we find n on the larger dial and the number directly next to n on the smaller dial is the product. For example, to find the product of 4 and 2 we rotate the smaller dial until 4 is directly below the 1 on the larger dial. (see figure 1b) We locate the 2 on the larger dial and note that the number directly next to 2 on the smaller dial is 3. Since the product of 4 and 2 is 3 in $(Z_5, \cdot)$ we see that this agrees with our earlier work with products.

Possible Lesson on Circular Slide Rules:

(a) Make a sketch to show how we can use a circular slide rule to compute $3 \cdot 1$, $3 \cdot 2$, $3 \cdot 3$, and $3 \cdot 4$. Do these results agree with our earlier work with products in $(Z_5, \cdot)$?

(b) What products are indicated in Figure 1a?

(c) Would you be able to obtain products if the same numbers were put on the dials in the order 1,2,4 and 3 rather than 1,2,3 and 4? (Answer: No)

(d) In $(Z_5, \cdot)$ compute the product of

(1) 2 and 2

(2) 2 and the product found in (1)

(3) 2 and the product found in (2)

(4) 2 and the product found in (3)

How are the above four results related to Figure 1a? If we had started with 3 and multiplied repeatedly by 2, what set of numbers result?

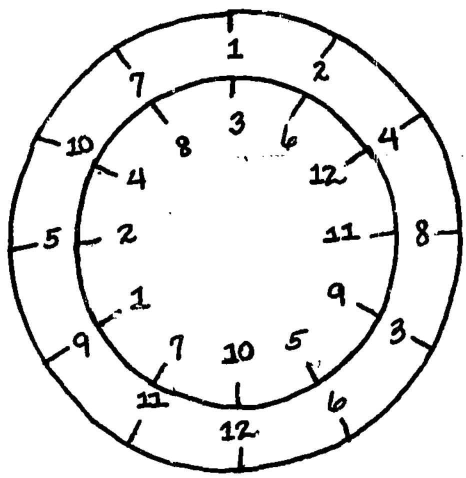
(e) If you arranged the numbers on the two dials in the order 1,3,2, and 4, would you be able to find the products in $(Z_5, \cdot)$? Take the number 3 and multiply it repeatedly by 3 in $(Z_5, \cdot)$ What set of numbers results? Do you see a pattern emerging? If you take any number of the set {1,2,3,4} and multiply it repeatedly by 4, do you obtain the same results as when you multiplied repeatedly by 3? What does this imply?

(f) Construct a slide rule for finding products in Z_7 with 0 omitted by placing on two dials the following numbers in the order indicated 1,3,2,6,4, and 5.

Show how this slide rule can be used to find the product of each of the above numbers and 4.

(g) In figure 2 a circular slide rule is shown which indicates the products of the numbers in Z_{13} (with 0 omitted) and the number 3.

Figure 2:
Circular Slide
Rule for $(Z_{13}, \cdot)$ less {0}



With the use of the above we can readily compute the products obtained when we multiply numbers in Z_{13} by 3.

(1) Explain how the entries of the partially filled table in Figure 13 can be obtained by use of Figure 12.

·	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3	3	6	9	12	2	5	8	11	1	4	7	10

Figure 13: Partial Table for $(Z_{13}, \cdot)$ less {0}

(2) Which of the following ways of arranging the number 1.2,3,... 12 on dials would yield a slide rule for determining products in $(Z_{13}, \cdot)$ less {0}?

(a) (1, 11, 4, 5, 3, 7, 12, 2, 9, 8, 10, 6)
(b) (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)
(c) (1, 7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2)
(d) (1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12)

CHAPTER 2

Time for Chapter 13 - 15 days

COMMENTARY FOR TEACHERS

Chapter 2 is designed principally to accomplish the following two objectives:

1) Extend the student's understanding of what is meant by a binary operation on a set.

2) Extend the student's understanding of certain key properties of operations and operational systems.

The first of these objectives is the concern of the following sections of the chapter (together with exercises which follow each of them):

Section 2.1, which reviews the notion of assigning one and only one element to an ordered pair of elements;

Section 2.3, which contains a formal definition of a binary operation on a set;

Section 2.5, which provides computational experience with several different operations on the whole numbers;

The second objective is accomplished principally in the following sections:

Section 2.9, which contains formal definitions of some operational properties the student has met earlier;

Section 2.11, which deals with cancellation laws in an operational system.

These five sections then constitute the heart of the chapter.

Section 2.7 provides further experience with solutions of open sentences, and at the same time helps to reinforce the concept of operation. Emphasis should be placed on solution set, and the use of set notation for root or roots of the open sentence.

Section 2.13 introduces no new ideas, but introduces instead two special systems which help to reinforce the ideas already introduced.

Section 2.15 simply introduces the word "group" for a particular kind of operational system, examples of which have been dealt with throughout the chapter.

Exercises which involve computation may (at the discretion of the teacher and dependent upon the proficiency of the class) be shortened.

2.1 Ordered Pairs of Numbers and Assignments (2 days including 2.2)

By means of such processes as division and raising to a power, the student's intuitive appreciation of the importance of ordered pairs of numbers should be increased. The principal objective of this section is to make the student aware of the fact that, confronted with a pair of numbers, he already knows many special ways of assigning a third number to that pair. All of this work is preparation for the formal definition of a binary operation, coming in Section 2.3.

One possible class activity, to accompany this introductory section, consists of having a student make assignments to certain ordered pairs, with other students guessing how he decided upon the assignments. Hopefully, students will come up with some unusual schemes which can be saved and used later, as examples (or counterexamples) of operations.

So called "Cayley Tables" or operational tables appear in this section and throughout the chapter. Presumably, students are familiar with them from Chapter 1, but be especially sure that they see how the notion of ordered pair depends upon reading the table correctly. Reading of the table should be stressed in terms of first element relating to the second element.

Special attention should be given to exercise 11 of 2.2. Do not assign as homework but develop in class. Exercise 12 may be developed later.

ANSWER TO EXERCISES

2.2 Exercises

1. (a) 5 (b) 5 (c) 12 (d) 583 (e) 583 (f) 1000
   (g) $\frac{13}{15}$ (h) $7\frac{1}{4}$ (i) 1.10 (j) 10,000,000.
2. (a) (0,5), (1,4), (2,3), (3,2), (4,1), (5,0).
   (b) (0,1), (1,0) (c) (0,0).
3. (a) (1,24), (2,12), (3,8), (4,6), (6,4), (8,3), (12,2), (24,1).
   (b) (1,13), (13,1). (c) (0,0), (0,1), (1,0), (0,2),...

(There are of course an endless number of such pairs.)

4. (a) 0 (b) 0 (c) 48 (d) 5406 (e) 5406 (f) $12\frac{3}{8}$
(g) $\frac{1}{2}$ (h) 1 (i) .3 (j) .2241

5. (a) 8 (b) 4 (c) 2 (d) 10 (e) 100 (f) 1000
(g) 10,000 (h) 1,000,000 (i) 25 (j) 32 (k) 64
(l) 81 (m) 27 (n) 1

6. (a) 81 (b) 64 (c) 16 (d) 16 (e) 243 (f) 125

7. (a) (2,4), (4,2), (16,1). b) (10,1)

8.

+	5	682	17	8	0	1	1720
5	10	687	22	13	5	6	1725
682	687	1364	699	690	682	683	2402
17	22	699	34	25	17	18	1734
8	13	690	25	16	8	9	1728
0	5	682	17	8	0	1	1720
1	6	683	18	9	1	2	1721
1720	1725	2402	1737	1728	1720	1721	3440

9.

	5	682	17	8	0	1	1720
5	25	3410	85	40	0	5	8600
682	3410	465,124	11,594	5456	0	682	1,173,040
17	85	11,594	289	136	0	17	29,240
8	40	5456	136	64	0	8	13,760
0	0	0	0	0	0	0	0
1	5	682	17	8	0	1	1720
1720	8600	1,173,040	29,240	13,760	0	1720	2,958,400

10. (a) In all cases

    (b) Whenever a = b. Also $4^2 = 2^4$.

11. (a) 0; 2; -2; undefined; 8; 5

    (b) 12; 14; 10; $\frac{1}{8}$; $8\frac{1}{8}$; 21

    (c) 75; 77; 73; $\frac{1}{50}$; $8\frac{1}{50}$; 255

    (d) 300; 302; 298; $\frac{1}{200}$; $8\frac{1}{200}$; 2005.

    (e) 30,000; 30,002; 29,998; $\frac{1}{20,000}$; $8\frac{1}{20,000}$; 2,000,005.

    (f) $\frac{3}{4}$; $2\frac{3}{4}$; $-1\frac{1}{4}$; 2; 10; $5\frac{1}{4}$

Note: for negative numbers, the response may be "no known number."

12. (a) 2 (b) 6 (c) 5 (d) 100 (e) 3 (f) 1 (g) 1 (h) 4 (i) 21 (j) 21

2.3 What is an Operation? (2-3 days - including 2.4)

The principal purpose of this section is to establish understanding of formal definition of a binary operation on a set. In order to assure understanding of the phrase "one and only one" which is used in that definition, it is important to use two kinds of counterexamples. In one kind, there are pairs to which no assignment can be made; see, for instance, problems 3 and 4 of section 2.4. In the other kind, there are pairs to which <u>more than one</u> assignment can be made; see, for instance, problem 5 of Section 2.4.

The introduction of an operational symbol such as "*" can be troublesome. But students should see that the operation of addition makes the assignment $(3,5) \rightarrow 8$ which can be written "3 + 5 = 8," if one understands that "+" identifies the operation in question. In the same way, if we are thinking of some operation that makes the assignment $(3,5) \rightarrow 45$, we simply agree to let "$*$" identify that operation for the time being, we can write "$3 * 5 = 45$."

The word "binary" is included in the definition of operation since that is the kind of operation with which we are concerned (as contrasted, for instance, to a unary operation which assigns an element to a single element, or a ternary operation which assigns an element to a triple of elements).

An example of a unary operation is: assign the successor to an element. Another example would be $x \longrightarrow x^2$. An example of a ternary operation is take an ordered triple and assign a 3 digit number in that order (or reverse order) that is $(6,3,7) \longrightarrow 6\ 3\ 7$ or $(6,3,7) \longrightarrow 7\ 3\ 6$ etc.

Another example of a ternary operation is: select 3 non-collinear points ordered either clockwise or counter-clockwise and assign a parallelogram. Another example would be a triangle and assign the center of gravity.

While the student should see the obvious connection between the word "binary" and the notion of pairs, the word is not emphasized in the text at this time.

Exercises 7-9 of 2.4 should be covered in class even though L.C.M. is not used again until chapter 11. Exercise 11 and 12 can be handled in an intuitive fashion with rigor in so far as proof is concerned, dependent upon the class.

2.4 Exercises

1. (a) 0 (b) 1 (c) 1 (d) 15 (e) 15 (f) 100 (g) 2010 (h) 1000 (i) a + 1 (j) a (k) a
2. (a) 6 (b) 8 (c) 12 (d) 4 (e) 3 (f) 680 (g) 136 (h) 8536 (i) 161 (j) 820,000
3. No. for instance, 2-5 is not a whole number. So subtraction does not assign a whole number to <u>every</u> pair of whole numbers.
4. No. for example, $2 \div 3$ is not a whole number.
5. No. Many pairs have <u>more than one number</u> assigned to them. For example, the pair (8,12) can be assigned 1, 2, and 4.
6. (a) No. One counterexample ruins it. There is no number to assign to the pair (0,0).

   (b) Yes. The pair (0,0) has been removed in this case.
7. (a) 6 (b) 10 (c) 10 (d) 77 (e) 77 (f) 5 (g) 5 (h) 100 (i) 630 (j) 42
8. All answers are 0.
9. (a) Yes.

   (b) Yes. Every ordered pair$\longrightarrow$0. $(a,b)\longrightarrow 0$. $0 \in W$, and 0 is unique.
10. (a) No. 1 + 1 = 2. So 2 is assigned to the pair (1,1), but 2 is not in S.

+	0	1
0	0	1
1	1	2

(b) Yes.

1	0	1
0	0	0
1	0	1

11. (a) Yes. An even whole number is assigned to every pair.
    (b) Yes. An even whole number is assigned to every pair.
    (c) No, not unless we give meaning to such expressions as $2^0 = 1$, $1 \notin E$. $\infty = 0^0$.
12. (a) No. An <u>even</u> number is assigned to each pair.
    (b) Yes, since an odd number is assigned to every pair.
    (c) Yes.
13. (a)

+	0	1
0	0	1
1	1	0

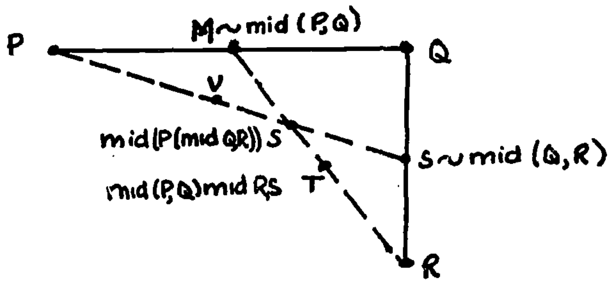
(b) Yes, and here are all the assignments:

$(0,0) \rightarrow 0$; $(0,1) \rightarrow 1$;
$(1,0) \rightarrow 1$; $(1,1) \rightarrow 0$.

14. Yes, the table does define an operation. To every pair is assigned a member of the set $\{a,b\}$.
    If we consider "a" and "b" as labels, remove "a" and replace it by "0", remove "b" and replace it by "1", we have the table from exercise 13(b). Thus, the two systems are "alike except for the labels used." (Note: examples such as these are intended to pave the way for the concept of isomorphism, coming in chapter 3 of the second course.)
15. (a) No, since c is not in the set.
    (b) No. There are no assignments for such pairs as (a,c), (c,a), (b,c), etc.
16. (a) Yes (b) Yes
    (c) Yes. Assign the point Q itself.

(d) Yes. To every pair of points is assigned a point. (This problem deserves some emphasis, since students need to understand that operations do not have to be concerned with "numbers.")

(e) No; "mid" is not associative.



17. (a) $\frac{3}{8}$ (b) $3\frac{15}{16}$ (c) 4.6 (d) $10\frac{1}{2}$ (e) .7495 (f) $9\frac{7}{8}$

2.5 Computations with Operations (and 2.6 - 2 days)

In this section, we wish to do the following:

1. Reinforce the definition of binary operation.
2. Introduce the student to the use of parentheses in mathematical expressions.
3. Provide instances of commutativity, associativity, identities, and inverses, prior to formal statement of properties in Section 2.7.

The ordinary operations -- addition and multiplication -- of arithmetic are so familiar to the students that the three objectives above are best accomplished by introduction of some unfamiliar operations (all on the whole numbers) so that the student must constantly think about what he is doing. That is the reason for the six operations introduced in this section. It might be a good idea to have these six definitions reproduced on separate sheets so that students may have them readily at hand, without having to refer to the text; they will be used not only in this section, but in some subsequent sections.

Every teacher probably has his own special way of helping students with parentheses. "Working from the inside out" is a phrase commonly used, for instance, when working with parentheses within parentheses. It is probably best to avoid, however, instructions such as "Do what is in parentheses first." With such instructions and confronted with the expression "$3 + (5 + 7)$," the student's thinking may proceed as follows:

First, $5 + 7$ is 12. Then 3 more is 15.

Unwittingly he has used commutativity, by taking $12 + 3$, rather than the intended $3 + 12$. Of course, since addition is commutative, the final result in this case is correct. But for a non-commutative operation, the procedure would lead to an incorrect result. For instance,

$$32 *_{10} (4 *_{10} 56) = 32 *_{10} 456 = 32{,}456 \text{ (not } 45{,}632).$$

In the exercises of Section 2.6, problems 2 and 6 are directed to associativity, problems 3 and 4 to commutativity, and problem 5 to identities. Also, problems 15 and 21 (a), (b) deal with distributivity of multiplication over addition, if you care to mention it at this time. Since this chapter culminates in the group properties, where only one operation is involved, the distributive property does not play a part in it.

2.6 Exercises

1. (a) 10 (b) 7 (c) 5 (d) 5 (e) 17 (f) 29
2. (a) 8 (b) 8 (c) 1160 (d) 194
3. (a) 111 (b) 111 (c) 109 (d) 111
4. (a) 58 (b) 32 (c) 17 (d) 17
5. (a) 42 (b) 42 (c) 42 (d) 615 (e) 615 (f) 615
6. (a) 12 (b) 12 (c) 298 (d) 116
7. (a) 15 (b) 13
8. (a) 61 (b) 87
9. (a) 1445 (b) 17
10. (a) 170
11. 897 12. 1479 13. 34250 14. 14 15. 33 16. 33
17. 36 18. 17 19. 17 20. 293
21. (a) $\frac{17}{24}$ (b) $\frac{17}{24}$ (c) 65 (d) $14\frac{1}{3}$ (e) 24 (f) $\frac{1}{12}$
22. Answers vary.
23. (a) $n = 0$. (b) $\{0, 1, 2, 3, 4, 5\}$

2.7 Open Sentences (and 2.8, 1 - 2 days)

In this section, the six operations introduced in Section 2.5 are used to extend the student's understanding of open sentences and his ability to solve them. Again, emphasis should be placed on solution <u>set</u>.

2.8 <u>Exercises</u>

1. 3 2. no solution 3. 2 4. no solution 5. 22
6. no solution 7. 11 8. no solution 9. {0, 1, 2, 3, 4, 5}
10. {0, 1, 2, 3, 4, 5, 6} 11. no solution 12. 42
13. any whole number 14. any whole number 15. no solution
16. 3 17. 3 18. 4 19. no solution 20. no solution
21. 60 22. no solution 23. no solution 24. no solution
25. 289 26. (a) 1 (b) 0 (c) {0, 1, 2,...,832} (d) any whole number (e) no solution (f) no solution
27. (a) any whole number (b) any whole number (c) any whole number (d) 3 (e) any whole number (f) any whole number
28. (a) 6 (b) 5 (c) no solution (d) 121 (e) no solution (f) no solution (g) no solution (h) 5 (i) 5 (j) no solution
29. (a) 2 (b) any whole number (c) 23 (d) omit if possible or do on an intuitive level $(a *_6 a) = 68$,
$a^2 + (a^2 + a^2)^2$, $a^2 + (2a^2)^2, a^2 + 4a^4 = 68$, $a^2(1 + 2a^2) = 68$.
$a = 2$. (e) 108 (f) 3

2.9 <u>Properties of Operations</u> (and 2.10, 2 - 3 days)

In this section, we formalize the notions of commutativity, associativity, identity, and inverse, all of which the students have encountered in a more informal way earlier. Be sure to stress the importance of saying "For every a, etc." in the statement of the properties. Thus, $8 - (5 - 0) = (8 - 5) - 0$, but subtraction is not associative, since such a statement is not true for every a, every b, and every c. In the same vein, stress the importance of finding a counterexample to show the falsity of a general statement. Thus, the single counterexample $(8 - 3) - 2 \neq 8 - (3 - 2)$ establishes that subtraction is not associative.

We deal here with a general identity element, rather than with the more sophisticated notion of right hand and left hand identities. Thus, for us, an identity element must "work on both sides;" i.e., it must commute. For example, 0 is not an identity for subtraction although $n - o = n$, for any n, it is false that $0 - n = n$ for any n. But 0 could be called a "right-hand" identity for subtraction.

Section 2.10 Exercises 1-13 may be covered in class. Exercises 14-19 may be assigned.

2.10 Exercises

1. (a) 599 (b) 599 (c) 435 (d) no whole number (e) 42,394 (f) 42,394 (g) 102 (h) no whole number
2. (a) and (c) are true for all whole numbers.
3. (a) and (c) are true.
4. (a) The statement is true whenever a = b.
   (b) The statement is true whenever a = b.
5. The following is not commutative: $^{*}4$ All others are.

6. (a) 20 (b) 20 (c) 8 (d) 4 (e) 144 (f) 144 (g) 1 (h) 4

7. (a) and (c) are true for all whole numbers.

8. (a) and (c) are true.

9. (a) Whenever $c = 0$, and a and b are any whole numbers, with $a > b$

   (b) Whenever $c = 1$, and a is a multiple of b

   (Emphasize that lack of associativity means that the operation is not associative in every case: there may be true instances of associativity even for a non-associative operation.)

10. The following is not associative: $*_6$ All others are.

11. (a) 15, 15, 312, 312

    (b) 0. It is the only identity element.

    (c) 15, 15, 312, 312

    (d) 1. It is the only identity element.

12. $1 *_3 a = 1$

13. There is no identity element for division. $n \div 1 = n$, for all n. However, $1 \div n \neq n$, in general.

14. (a) It is commutative. The table is symmetric about the diagonal.

    (b) It is associative.

    (c) Yes. 0.

    (d) 1 and 5 are inverses. 2 and 4 are inverses. 3 is its own inverse. 0 is its own inverse.

15. (a) It is commutative.

    (b) It is associative.

    (c) Yes. 1.

    (d) 1 is its own inverse. 5 is its own inverse. No other elements have inverses.

16. (a) Yes (b) No (Students should give counterexamples.)

17. (a) $P * P = P$ (b) There is exactly one assignment for each pair.

(c) No. For example, $P * Q = R$, but $Q * P = S$

(d) No

(e) No

S P Q R

P S Q R $P{\star}Q$

$S = (P{\star}Q){\star}R$

P Q R $Q{\star}R$ T

$T = P{\star}(Q{\star}R)$

18. (a) Yes (b) Yes (c) Yes. b (d) a and c are inverses of each other. b is its own inverse.

19. (a) 8 does not have an inverse. 0 is its own inverse.

(b) 8 does not have an inverse. 1 is its own inverse.

2.11 Cancellation Laws (and 2.12 - 2 days)

Cancellation laws are important, for instance, in solutions of equations. Later, when the student is working with real numbers, he will solve the equation "$2x + 7 = 15$" as follows:

$$2x + 7 = 15$$
$$2x = 8$$
$$x = 4.$$

The steps in this chain can be explained by the cancellation laws of addition and multiplication in the set of real numbers. Thus, we could write:

$$2x + 7 = 15$$
$$2x + 7 = 8 + 7$$
$$2x = 8 \quad \text{(by cancellation law of addition)}$$
$$2 \cdot x = 2 \cdot 4$$
$$x = 4 \quad \text{(by cancellation law of multiplication)}$$

It might be of advantage at this point to demonstrate this diagram for the students in preparation for Course II section on equivalent equation.

In this section, illustrations both of systems with a cancellation law and of systems without a cancellation law are given. The student should see why there can be no general cancellation law if any element appears <u>more than once</u> in any row and column of the multiplication table; it must be excluded from any cancellation law formulated for multiplication. Later, in algebra, the student cannot reason that if $x \cdot x = 2 \cdot x$, then $x = 2$ (since 0 is obviously a solution also).

<u>Section 2.12</u> - Exercises 1-4 could be covered in class. Exercises 5-12 may be assigned.

2.12 <u>Exercises</u>

1. $a = 19$ and $b = 19$
2. $x = 38$ and $y = 38$
3. (a) $a = b$ (b) $a = b$ (c) no conclusion
4. (a) No. Foe example, 4 max 3 = 4 max 2, but $3 \neq 2$.

   (b) No. (See counterexample in part (a).)
5. Yes
6. (a) yes (b) yes (c) no cancellation law (d) no cancellation law
7. You can make the conclusion from statements a, b, c, f, g, h, i, j, but not from statements d, e, k.
8. We run into trouble here, since * is not a commutative operation. We do not have a left cancellation law; that is, if P*Q = P*R, then Q = R. And we have a right cancellation law; if Q*P = R*P,

then Q = R. However from a statement such as P*R = Q*P, we can conclude nothing as the following counterexample shows:

$P*R = S$
s
$Q*P = S \quad Q \neq R$

This problem gives an opportunity to discuss with students the necessity of right hand and left hand cancellation laws in systems that are commutative.

9. No. This shows up in the table since b appears twice in a row and in a column. Thus, $b*a = b*c$, but $a \neq c$.
10. Answers vary.
11. (a) Yes (b) Yes (c) Yes, even (d) Yes (e) Yes
12. (a) Yes (b) Yes (c) Yes, odd (d) No (e) No

•	Even	Odd
Even	Even	Even
Odd	Even	Odd

2.13 <u>Two Operational Systems</u> (and 2.14 - These 2 sections may be omitted if pressed for time).

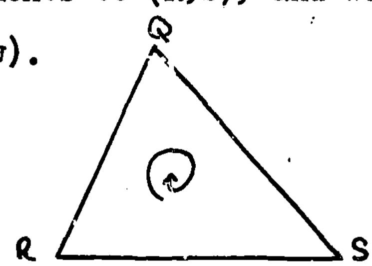
In this section, two additional operational systems, one numerical and one geometric, are introduced in order to have one more look at operational properties before summarizing the properties of a group.

Besides the digital multiplication system discussed in the text, other systems the student might enjoy investigating are:

{1, 3, 7, 9} and digital multiplication;

{1, 3, 5, 7, 9} and digital multiplication.

In the system (P, tri), rely on the student's intuitive notions of plane, equilateral triangle, and "clockwise," which should be strong enough to enable him to understand how the assignments are made. A good discussion question grows out of the stipulation that R-S-T be clockwise in orientation. Why is this necessary? Students should see that except for this restriction, there would be two assignments to (R,S), and we would not have a binary operation (see below).



Show that Q ≠ T

In other words, the restriction to clockwise preserves the notion of a binary operation.

2.14 Exercises

1. (a) yes (b) symmetric about diagonal
2. (b) yes; it can be concluded from the fact that ordinary multiplication is associative (the digits come from ordinary multiplication).
3. (a) yes, 6 (b) no
4. (a) 2 (b) 6 is its own inverse; 4 is its own inverse.
5. (a) a = b
   (b) Yes. No element appears more than once in a row or column.
6. (a) 8 (b) 6 (c) 6,4 (d) no solution (e) 8 (f) 8
7. no
8. no
9. no
10. There can be no inverses, since there is no identity.
11. There is a right hand cancellation law, and a left hand cancellation law.
12. (a) Yes; commutativity, associative, identity, inverses.
    (b) No.

2.15 What is a Group? (and 2.16 - 1 day)

This section should not be dwelled upon at length, for there is plenty of time for the group concept to develop spirally. Here we intend simply to introduce the word "group" as a name for a certain kind of operational system largely by citing systems already encountered. Examples of groups will occur throughout the remainder of the course.

2.17 and 2.18 - Summary and Review - 1 day

These two sections may be assigned for homework and then discussed in class.

Time should be allowed for quizzes and tests.

2.16 Exercises

1. commutative group
2. not a group, lacks inverses
3. not a group, lacks inverses
4. commutative group
5. not a group

2.18 Review Exercises

1. (a) 9 (b) 14 (c) 7 (d) 9 (e) 2
2. (a) (0, 4), (1,3), (2,2), (3,1), (4,0)
   (b) (1,4), (2,2), (4,1)
   (c) (0,4), (1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1), (4,0)
   (d) (0,4), (1,3), (2,2), (3,1), (4,0), (5,11), (6,10), (7,9), (8,8), (9,7), (10,6), (11,5)

(e) (1,4), (2,2), (4,1), (8,2), (2,8), (4,4), (4,7), (4,10), (5,8), (7,4), (8,5), (8,8), (8,11), (10,4), (10,10), (11,8)

3. (a) 1112 (b) 1112 (c) 622 (d) no whole number (e) 867 (f) 867 (g) 435 (h) 435 (i) no whole number (j) no whole number (k) 27 (l) 81 (m) 64
4. The following are operations, a, b, e
(f is an operation only if you define expressions with zero exponent)
5. The following are true: a, b, e
6. (a) 20 (b) 20 (c) 4 (d) 8 (e) 144 (f) 144 (g) 1 (h) 4 (i) 12 (j) 12
7. (a) 4096 (b) 65,536
8. The following are associative: a, d, e
9. (a) 12 (b) 15 (c) no
10. (a) 55 (b) 202 (c) 2500 (d) 2400 (e) 184
11. (a) 61 (b) 0 (c) no solution (d) 7
(e) no solution (f) 1 (g) 0 (h) 0, 1, 2, 3, 4
(i) 5 (j) no solution (k) 2 (l) no solution
(m) any whole number is a solution (n) no solution
(o) 50 (p) 10 (q) no solution (r) 3 (s) 100 (t) 0, 1, 2
12. (a) 10 (b) 16 (c) 64 (d) 49 (e) 29 (f) 49 (g) 27 (h) 21 (i) 26 (j) 25 (k) 25
13. The conclusion can be made from the following: a, b, c, f
14. (a) Assign Q itself (b) yes; one and only one point is assigned to every ordered pair of points (c) and (d) no (students should show counterexamples) (e) no (f) There

is a right hand cancellation law, and a left hand cancellation law.

15. There are sixteen binary operations on the set [0,1]. One way to analyze the situation is as follows: Take the pair (0,0). There are two possibilities for an assignment, 0 and 1. Having made a choice for this assignment, take the pair (0,1); there are also two choices here. Hence, there are four different ways to make assignments to the pairs (0,0) and (0,1). Continuing in this way, there are eight different ways to make assignments to the three pairs (0,0), (0,1), (1,0). And finally there are sixteen ways to make assignments to the four pairs (0,0), (0,1), (1,0) and (1,1).

All sixteen operational tables are printed below. It would be profitable for students to discuss properties of these systems, deciding which are groups.

	0	1
0	0	0
1	0	0

	0	1
0	0	1
1	0	0

	0	1
0	0	0
1	1	0

	0	1
0	0	1
1	1	0

	0	1
0	0	0
1	0	1

	0	1
0	0	1
1	0	1

	0	1
0	0	0
1	1	1

	0	1
0	0	1
1	1	1

	0	1
0	1	0
1	0	0

	0	1
0	1	1
1	0	0

	0	1
0	1	0
1	1	0

	0	1
0	1	1
1	1	0

	0	1
0	1	0
1	0	1

	0	1
0	1	1
1	0	1

	0	1
0	1	0
1	1	1

	0	1
0	1	1
1	1	1

SUGGESTED TEST ITEMS

(The test is too long for one class period. Selections may be made from these items.)

PART ONE. COMPLETION. In each of the sentences below, there is a blank. Decide what word or expression must go in the blank in order to make a true statement. Then write that word or expression in the blank preceding the statement. The example, number 0, has been completed correctly.

0. ____12____ The number ___ is the sum of 8 and 4.

1. ________ Addition is a binary operation on the set W of whole numbers which assigns the number ___ to the ordered pair (6,3).

2. ________ Multiplication is a binary operation on the set W of whole numbers which assigns the number 1 to the ordered pair ___.

3. ________ 3^2 = ___.

4. ________ 2^3 = ___.

5. ________________ If $*$ is a binary operation such that $a*b = b*a$ for every whole number a and every whole number b, then the operation $*$ has the ___ property.

6. ________________ $18-(10-2) =$ ___.

7. ________________ $(18-10)-2 =$ ___.

8. ________________ If $*$ is an associative operation, then for all numbers a, b, and c, $a*(b*c) =$ ___.

9. ________________ The number ___ is the identity element for addition of whole numbers.

10. ________________ For every whole number a, $a \cdot 1 =$ ___.

11. ________________ In $(Z_3, +)$ the inverse of 2 is ___.

12. ________________ In $(Z_3, \cdot)$ the element ___ has no inverse.

PART TWO. TRUE-FALSE Decide whether each of the following statements is true or false. Then write the complete word "true" or the complete word "false" in the blank preceding the statement. The example, number 0, has been answered correctly.

0. ____false____ The sum of 8 and 2 is 16.

1. ________________ Subtraction is a binary operation on the set of whole numbers.

2. ________________ If a and b are whole numbers, and $a + 2 = b + 2$, then $a = b$.

3. ________________ If a and b are whole numbers, and $0 \cdot a = 0 \cdot b$, then $a = b$.

4. ____________ The solution set of the equation "$0 \cdot X = 0$" is the entire set W of whole numbers.

5. ____________ The system (W,+) has a cancellation law.

6. ____________ The system (W,+) is a group.

7. ____________ In the system (W,+) every number has an inverse.

8. ____________ In the system (W,+), the equation "$5 + x = 2$" has no solution.

9. ____________ The third power of 4 is 12.

10. ____________ In any group, the identity element is its own inverse.

PART THREE "COMPUTATION WITH PARENTHESES" Find the simplest name for the number by each of the following expressions.

1. $2 + (17 + 5)$
2. $17 \cdot (5 + 2)$
3. $(17 \cdot 5) + 2$
4. 1^5
5. 5^1
6. $2 + (3^2)$
7. $(2 + 3)^2$
8. $[(8 \max 2)^2 + 3] \cdot 10$
9. $(17 \cdot 5) + (17 \cdot 2)$
10. $(14 \cdot 0)^3$

FOUR. In this section, we consider a set S with exactly three elements, called a, b, and c. That is,

$$S = \{a,b,c.\}$$

The operational tables below define two binary operations, $*_1$ and $*_2$, on set S.

$*_1$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$*_2$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

So, we have two operational systems, $(S,*_1)$ and $(S,*_2)$. All questions in this section refer to these systems.

1. $a *_1 c =$
2. In the system $(S,*_2)$, list <u>all ordered pairs</u> to which <u>c</u> is assigned.
3. $(a *_1 b) *_2 c =$
4. $a *_1 (b *_2 c) =$
5. $c *_2 (c *_1 c) =$
6. $(c *_2 c) *_1 c =$
7. Is $*_1$ a commutative operation?
8. Is $*_2$ a commutative operation?
9. (a) Is there an identity element in the system $(S,*_1)$? If so, what is it?

   (b) Is there an identity element in the system $(S,*_2)$? If so, what is it?
10. Give the inverse of each element in $(S,*_1)$. If the element has no inverse, say so.

    (a) The inverse of <u>a</u> is

    (b) The inverse of <u>b</u> is

(c) The inverse of <u>c</u> is

11. Give the inverse of each element in $(S,*_2)$. If the element has no inverse, say so.

(a) The inverse of <u>a</u> is

(b) The inverse of <u>b</u> is

(c) The inverse of <u>c</u> is

12. In which of the two systems is there a cancellation law?

13. Solve the following equations. That is, find what element <u>x</u> must be in order to make the statement true. If the equation has <u>no solution</u>, say so. If there is <u>more than one</u> solution, be sure to list them all.

a) $b *_1 x = a$

b) $c *_2 x = c$

c) $a *_2 x = a$

d) $a *_2 x = b$

e) $x *_1 x = b$

Answers to suggested test items

Part I.

0. 12
1. 9
2. (1,1)
3. 9
4. 8
5. Commutative
6. 10
7. 6
8. a*(b*c)
9. 0
10. a
11. 1
12. 0

Part II

0. false
1. false
2. true
3. false
4. false
5. true
6. false
7. false
8. true
9. false
10. true

Part III

1. 24
2. 119
3. 87
4. 1
5. 5
6. 11
7. 25
8. 670
9. 119
10. 0

Part IV.

1. c
2. (b,c),(c,b)
3. c
4. c
5. c
6. a
7. yes
8. yes

9.a) yes, a
 b) yes, b

10.a) a
 b) c
 c) b

11.a) none
 b) b
 c) c

12. *1

13.a) { c }
 b) { b }
 c) {a,b,c}
 d) ∅ - no solution
 e) { c }

TEACHERS' COMMENTARY

Chapter 3 Mathematical mappings

APPROXIMATE TIME FOR CHAPTER: 13-16 days

The main purposes of this chapter are:

(1) To develop the concept of a mapping as a special kind of assignment.

(2) To develop some elementary procedures for investigating the properties of mappings, particularly mappings of sets of numbers.

(3) To introduce the operation of composition of mappings.

(4) To develop the notion of inverse and identity mappings, and sufficient conditions for a mapping to have an inverse under composition.

(5) To study two classes of mappings of sets of whole numbers, specifically $n \rightarrow n+a$ and $n \rightarrow an$, and the properties they possess as a class.

It should be understood at the outset that a mathematical mapping is considered by the author to be synonymous with a function from a set A to a set B. The fundamental notion underlying the concept of a mapping is that of assignment. An assignment can be treated as a relation or a correspondence, of course, but it is our desire at this level to give the function concept a more dynamic connotation. Also the word "assignment" carries with it the element of directionality which is an inherent part of the mapping concept, e.g. a mapping from A to B.

It must be stressed that before we can begin to talk about whether an assignment is a mapping we must be clear as to the set of elements to which objects are assigned, and clear as to the set from which the assigned objects may be chosen. There must also be a clear method, process, rule, display, or diagram by which the individual objects are assigned.

However, the particular method, process, or rule by which objects are assigned is absolutely immaterial to whether or not an assignment is a mapping. The fundamental and unchanging criterion for us is that each element of the first set be assigned exactly one element of the second set. This is stated more formally in the following definition.

<u>Definition</u>: A mapping is an ordered triple (f,A,B) such that A and B are sets and such that to each $x \in A$ there is assigned by f exactly one element $y \in B$.

To indicate that $a \in A$ is assigned $b \in B$ we write $a \xrightarrow{f} b$. b is then called the image of a by f or the image of a under f. The "exactly one" condition may then be stated as: if $a \xrightarrow{f} b_1$

and $a \xrightarrow{f} b_2$ then $b_1 = b_2$.

Note that in this definition the phrase "there is assigned" and the symbol "f" are left undefined. Thus, the notion of an assignment is an additional undefined term in this development which is not found in the definition of a function as "a set of ordered pairs such that no two pairs have the same first element." "Assignment" could be given a precise definition in terms of set theory but we choose not to do so at this time. The purpose of the examples in section 3.1 is to make clear what is meant by an assignment, as well as to provide a basis for the definition of a mapping.

We shall name a mapping in the course in various ways. For defined sets A and B, "the mapping f of (from) A to B" may be given by:

(1) the rule $n \rightarrow$ ________.

(2) the table...

(3) the arrow diagram...

(4) ... (some verbal description of the way images are assigned).

The domain of a mapping is <u>always</u> the first set in the mapping. The range <u>may</u> or <u>may not be</u> all of the second set in the mapping. No special name is usually given to the second set. If you wish, you may refer to it as the codomain. Then the range is a subset (a proper subset in many instances) of the codomain.

3.1 <u>Assignments and Mappings</u> - (Approximate Time: 1 day)

In this section there are examples designed to make intuitively clear what we mean by an "assignment" and "assigns". There is an inherent difficulty with these words grammatically; however, that should be faced at the outset. Suppose we have in an arrow diagram

$$C3 \longrightarrow \text{Helen.}$$

Now we say that "to C_3 is assigned Helen" so that the direction of the mapping is clear. But we might more naturally say that Helen is assigned to C_3, in which case the assignment is read in the reverse direction to the actual sense of the mapping. Care should be exercised in this regard.

The essence of this chapter is a special type of assignment called a mapping.

The use of "arrow" criteria for an assignment to be a mapping is recommended. e.g.,

(1) each element of A is the origin of an arrow

(2) no element of A is the origin of more than one arrow, or, equivalently,

(3) each element of A is the origin of exactly one arrow.

The student should have a clear conception of what a mapping is and should be made to realize that every assignment is not a mapping. He should feel comfortable in his ability to recognize a mapping. The terms domain and range and their relation to a mapping should also be emphasized.

Note that in this set of exercises and in subsequent sets of exercises, it is not necessary to assign every exercise (is this assignment a mapping?). Exercises felt to be absolutely essential to the development will be noted here.

An additional activity here might be to have students think of natural assignments that are or are not mappings. Exercise 4 should be assigned. It is the forerunner of the step function in analysis. Suggested exercises: exercises 1 and 2 may be done in class. The rest may be assigned as homework. Exercise 4 is a good review of fractions.

3.2 Solutions to Exercises

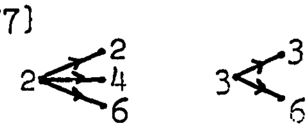
1. Example 1: (a) No. C1 (b) Yes. C2 assigned Judy and Mary, C4 assigned Louise and Sandra, C5 assigned Janice and Carol, [Note double arrow from C_2] and C_1 has no assignment.

   (c) No. (d) ____________.

   Example 2: (a) Yes. (b) No. (c) Yes.

   (d) Domain: {John, Al, Bert, Fred, Steve}

   Range: {68,70,73,77}

   Example 3: (a) Yes. (b) Yes. 

   (c) No. (d) __________.

   Example 4: (a) Yes. (b) No. (c) Yes.

   (d) Domain is set of states of the United States of America. Range is the set of their capital cities.

<u>Example</u> 5: (a) Yes. (b) No. (c) Yes.

(d) Domain is W. Range is {5,6,7....}

<u>Example</u> 5: (a) Yes. (b) No. (c) Yes.

(d) Domain: {Mary, Steve, Joe, Janet, Peter, Harry}.

Range: {Mr. Brown, Mr. Jones, Mr. Ross, Mr. White}.

2. (a) Yes. Each element of A is assigned exactly one element of B. [Or, each number in A is the origin of one and only one arrow.]

(b) No. $5 \genfrac{}{}{0pt}{}{\nearrow 4}{\searrow 8}$. That is, 5 is assigned more than one element of B. [Or, 5 is the origin of more than one arrow.]

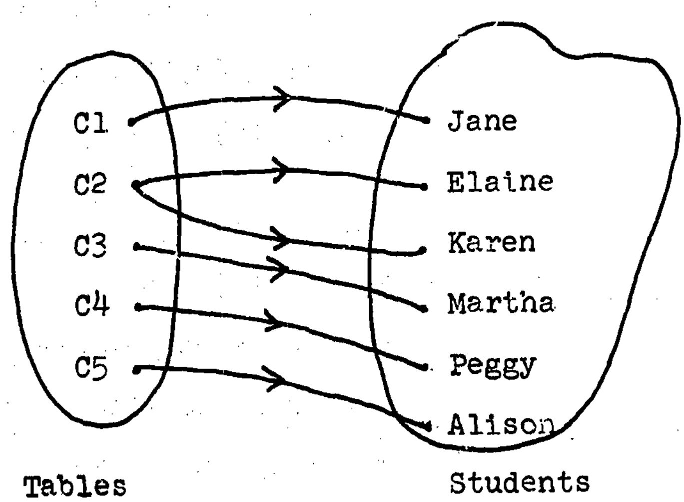
(c) Yes. Each element of A is assigned exactly one element of B. [Or, no number in A is the origin of more than one arrow.]

(d) Yes. Each number in A is the origin of exactly one arrow.

(e) No. 1, 3, 5, and 7 are each the origin for two arrows.

3. (1)

(a)

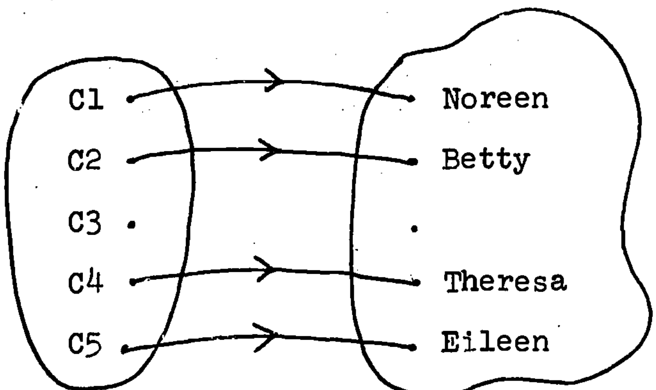


(b) Not a mapping

(c) C2 is the origin of two arrows.

(2)

(a)



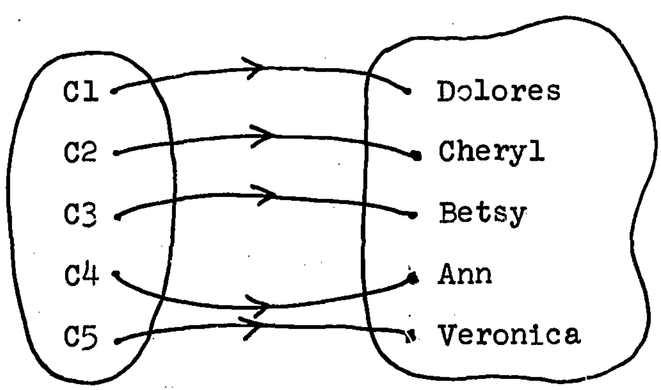
(b) Not a mapping.

(c) C3 is assigned no student.

(C3 is not the origin of an arrow)

(3)

(a)

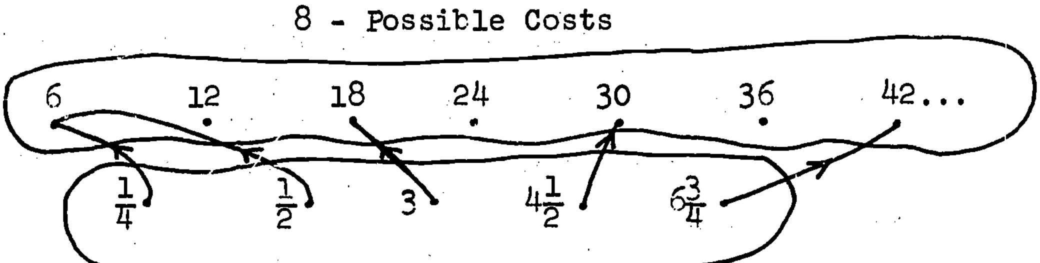


Tables Students

(b) A mapping.

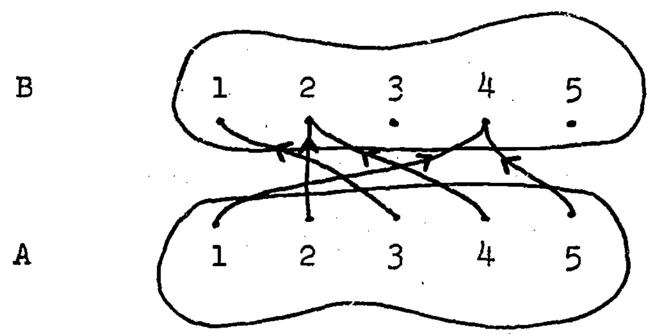
(c) Each table is the origin of exactly one arrow.

8 - Possible Costs

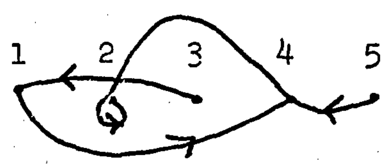


A - Weights

5. (a)



or



Or other non-linear arrangements of {1,2,3,4,5}

(b) 1.

(c) {1,2,4}

(d) No. 3 and 5 are contained in B, but are not contained in the range.

(e) Yes, 2 and 4.

3.3 <u>Mappings of Sets of Whole Numbers</u> - (Approximate Time: 1 - 2 days)

The purpose of this section is to further extend the notion of a mapping to the set of whole numbers and to introduce the notion of a mapping given by a rule n⟶ ______.

Examine the examples in the text and stress the technical definition of a mapping.

It should be pointed out in example 2 that one should be very cautious about such arrow diagrams. Without a rule, we really know nothing about the images for whole numbers not in the diagram:

The answers to the questions following example 4 are:

(1) each whole number has an image.

(2) each whole number has exactly one image.

(Please note that the use of the word "image" should be confined to the objects assigned by a mapping.)

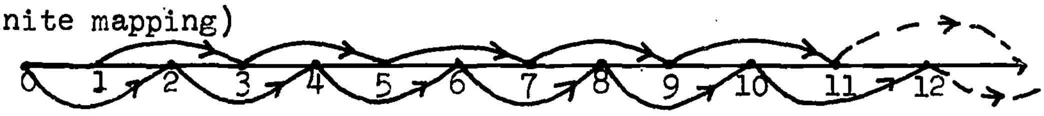
(3) $3 \longrightarrow 5$, $4 \longrightarrow 6$, $7 \longrightarrow 9$.

(4) Yes. 1, 2, 5.

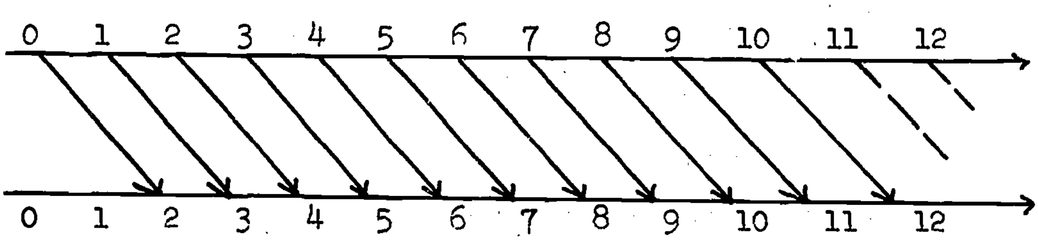
In example 6, $2 \longrightarrow 4$, $5 \longrightarrow 3$, and $6 \longrightarrow 11$. The range is {2,3,4,7,9,11}.

In this section, all mappings are represented on a single line. It should be noted here that the same mapping can be pictorially represented using two lines. This idea is first introduced in section 3.13 but can aid the students in the earlier chapters in distinguishing the domain from the range.

Consider Example 4 of this section. The pictorial representation is as follows: (Note that dotted lines indicate an infinite mapping)



Another suggested representation is as follows:

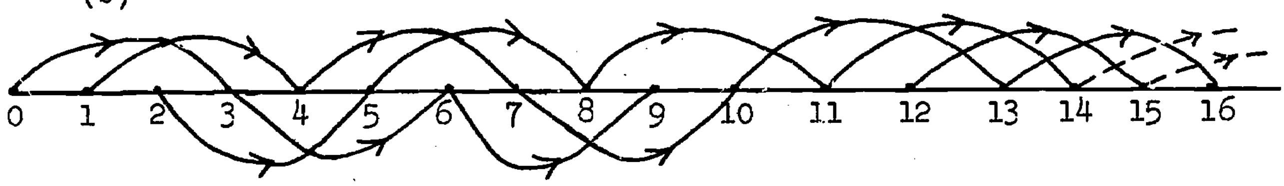


Suggested assignment: for 3.4: Exercise 1,2 and 6 may be done in class. Exercise 6 is a good discussion question. Exercises 3,4,5,7 are excellent homework problems.

3.4 Solutions to Exercises

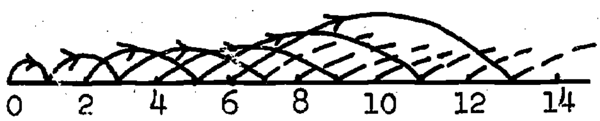
1. (a) 3, 41, 1362.

(b)



2. No. Let A = {3,4,5}. Any set of whole numbers which excludes 0, 1, and 2 is permissible, the simplest being {3} = A.

3.



etc.

4. (a) True: $3 \to 6$, $4 \to 8$, $5 \to 10$

(b) True: $3 \to 10$, $4 \to 13$, $5 \to 16$

(c) False: $3 \to 8$, $4 \to 11$, $5 \to 14$

(d) False: $3 \to 9$, $4 \to 16$, $5 \to 25$

(e) False: $3 \to 9$, $4 \to 8$, $5 \to 7$

5. (a) The set A may be any subset of W.

(b) A may be any subset of {0,2,4,6,...}

(c) A may be any subset of W.

(d) A may be any subset of {2,3,4,5,...}

(e) A may be any subset of W.

(f) A may be any subset of N.

The arrow diagrams will vary according to choice of A.

6. (a) (1) Yes. (2) Domain = {0,1,2,3}; Range = {1,2,3,4}

(3) $n \to n + 1$ (4) Yes.

(b) (1) Yes. (2) Domain = {0,1,2,3,4,5}

Range = {0.2,4,6,8,10}

(3) $n \to 2n$ (4) Yes.

(c) (1) Yes. (2) Domain = {1,2,3,4,5,6,7,8}

Range = {0,1,2,3,4,5,6,7}

(3) $n \to n-1$ (4) Yes.

(d) (1) Yes. (2) Domain = {10,11,12,...,18}

Range = {10,11,...,17}.

(3) Rule not easily expressed: $n \to 10$, for $n = 10$

$n \to n - 1$ for $n \neq 10$

(4) No.

(e) (1) Yes. (2) Domain = W

Range = {0,1,4,9,16,...}

(3) $n \to n^2$ (4) Yes.

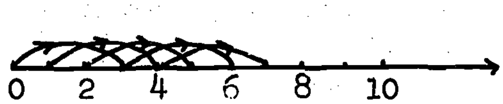
(f) (1) Yes. (2) Domain = W

Range = {1,3,5,7,9,...}

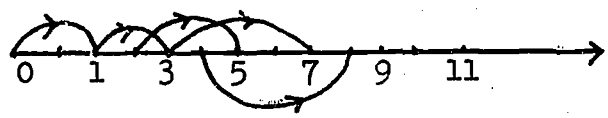
(3) $n \to 2n + 1$ (4) Yes.

(g) (1) No. $3 \to 6$ and $3 \to 0$; $6 \to 9$ and $6 \to 3$.

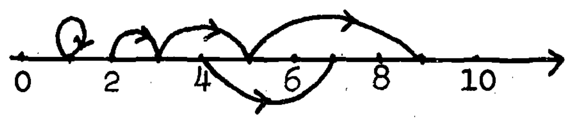
7. (a)



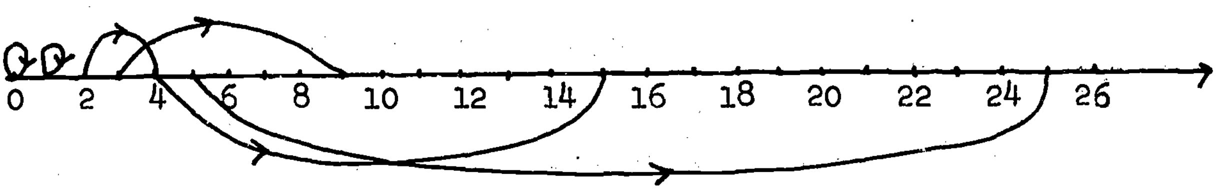
(b)



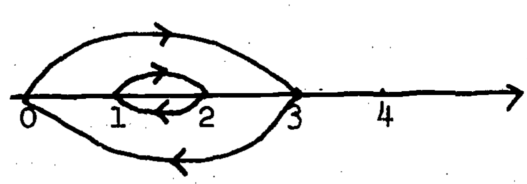
(c)



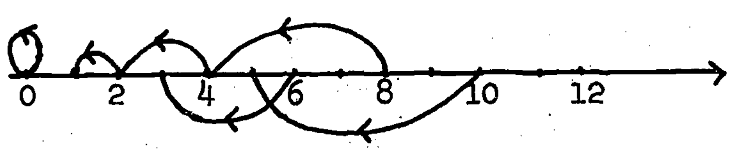
(d)



(e)



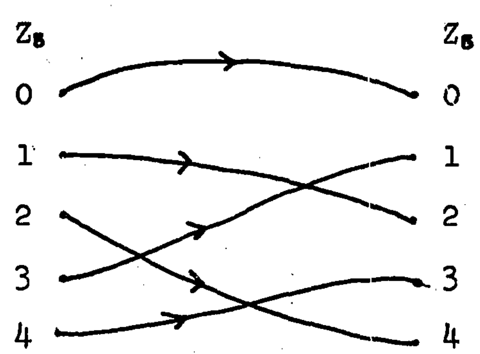
(f)



3.5 <u>MAPPINGS OF CLOCK NUMBERS</u> (Approximate Time: 1 day)

Such mappings provide an opportunity to show that two seemingly different rules make the same assignment of images. Stress, that, regardless of external details, two mappings are the same (equal) if and only if (1) they have the same first and second sets, and (2) for each element of the first set the images assigned are the same.

An excellent opportunity again arises here to study the information that can be obtained about a mapping by studying the "arrows." Also, alternative diagrams like the following might be used.



One might begin this section with a discussion and explanation of W and 2n. A useful technique might be to consider a particular rule $n \rightarrow$ _____ using first W and then 2n as your domain and range.

3.6 Solutions to Exercises

1. Because . in Z_5, $n-3 = n + (-3) = n+2$
2. (1) a. yes b. domain = range = $\{0,1,2\}$ c. $n \rightarrow n-1$ d. No
   (2) a. yes b. domain = range = $\{0,1,2,3\}$ c. $n \rightarrow n+2$ d. No
   (3) a. no 0 has two images.
   (4) a. no 1 has two images.
   (5) a. yes b. domain = range = $\{0,1,2,3,4,5\}$ c. $n \rightarrow n$ d. No

(6) a. yes b. domain = {0,1,2,3,4}, range = {0}

c. $n \rightarrow 0$ d. yes, 0.

(7) a. yes b. domain = {0,1,2,3,4}, range = {4}

c. $n \rightarrow 4$ d. yes, 4.

(8) a. yes b. domain = {2,3,4}, range {1,0,5}

c. no easy rule d. no

3.7 <u>SEQUENCES</u> - (Approximate Time: 1 - 2 days)

The purpose of this section is to examine a particular type of mapping with domain N.

Take care to point out that when we write down a sequence as, for example 3, 5, 7, 9, 11, 13, ..., we really have indicated a mapping because we have put 3 in the "first position," 5 in the "second position," etc. so that we have

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 7 & 9 & 11 & 13, \end{array}$$

where the order of "positions" is from left to right.

Suggested assignment: Exercise 1 may be a classroom assignment. Exercises 3 and 4 may be a homework assignment. Exercise 2 should be assigned and discussed. The point here is that a mapping can be generated by chance phenomena. Also exercise 5 should be assigned and discussed, but prior discussion is needed to prevent difficulty with notation.

3.8 Solution to Exercises

1. (a) (1) N
   (2) N
   (3) $\{1,2,3,4,5\}$
   (4) N
   (5) $\{1,2,3,4,5,6\}$
   (6) N

   (b) (3) and (5) are finite. The rest infinite.

   (c) (1), (2), and (4) are infinite because the domain is N.

2. Many answers. Discuss in class.

3. (a) 1, 2, 3, 4, 5, 6
   (b) 11, 10, 9, 8, 7, 6
   (c) 485, 658, 831, 1004, 1177, 1350
   (d) $37\frac{5}{17}$, $37\frac{10}{17}$, $37\frac{15}{17}$, $38\frac{3}{17}$, $38\frac{8}{17}$, $38\frac{13}{17}$
   (e) 157, 160, 165, 172, 181, 192
   (f) $79\frac{1}{2}$, 81, $83\frac{1}{2}$, 87, $91\frac{1}{2}$, 97

4. (a) 2, 1, 6, 2, 10, 3, 14, 4, 18, 5
   (b) 39

5. (a) 7, 16, 43, <u>124</u>
   (b) $a_7 = 3 \cdot a_6 - 5 = 3 \cdot 1096 - 5 = 3288 - 5 = \underline{3283}$
   $a_8 = 3 \cdot a_7 - 5 = 3 \cdot 3283 - 5 = 9849 - 5 = \underline{9844}$

3.9 Composition of Mappings - (Approximate Time: 2 days)

It should be emphasized that, where defined, the composition of two mappings is again a mapping. The reason for this order is that we wish $[g \circ f](x) = g(f(x))$ when we get the $f(x)$ notation. Be sure that in each example the conditions for the composite mapping to be defined are met.

When finding a rule for $g^\circ f$ in terms of rules for g and f, stress that "n" in "n ———>3n + 2" for instance, is a <u>place holder</u>. Thus, if k + 2 is in the domain of the mapping, we may write "k + 2 ———>3(k + 2) + 2." Use of "□" might be helpful also, e.g.,

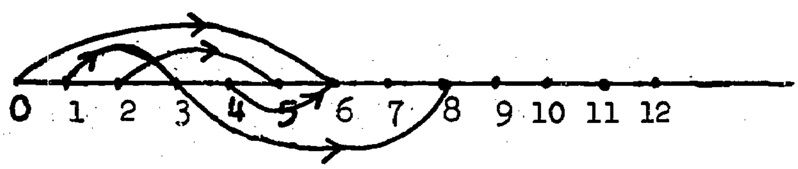
$$\square \longrightarrow 3 \cdot \square + 2.$$

It should be stressed that for the composition to exist the range of the first must be contained in the domain of the second. The use of colored chalk and colored pencils can be an essential aid in distinguishing the assignments in the composition of mappings. the notation <u>$f^\circ g$</u> implies that the mapping g must be done first and then f. In representing a composition of mappings pictorially, the best method may be to consider three parallel lines as in Figure 3.29.

3.10 Solutions to Exercises

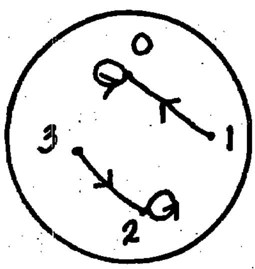
1. (a) 6 (b) 5

(c)

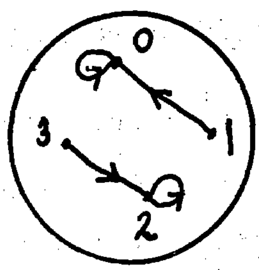


(d) Not possible. 5, 6, 8, 9, and 11 are not in domain of f.

2.



g ° f



f ° g

They are the same mapping since their arrow diagrams are the same, i.e. they each assign the same images to 0, 1, 2, and 3.

3. (a) 201 (d) 207 (g) 203
   (b) 69 (e) 405 (h) 403
   (c) 135 (f) 137 (i) 139

4. (a) 11 (b) 50 (c) 5 (d) 14
   (e) There is no whole number n such that $n \xrightarrow{h} 4$.

5. (a) 4 (b) 4
   (c) (1) [1, 4, 7, 10] (3) [2, 5, 8, 11]
   (2) ∅ (4) ∅

3.11 Inverse and Identity Mappings -(Approximate Time: 2-3 days)

The student is now introduced to the inverse and identity mapping. It should be noted that the identity mapping on a given set A, a mapping of A to A with the rule $n \longrightarrow n$, is designated by the symbol j_A. The symbol j_B denotes the identity mapping of B to B. The inverse mapping is introduced here by demonstrating that in certain instances a given mapping f of A to B has associated with it a mapping g of B to A such that $g^o f = j_A$ and $f^o g = j_B$. If A = B, of course, $g^o f = f^o g = j_A$. Then, the goal is to see what properties of A and B made this possible. They are, of course, that f and g are both one-to-one and onto mappings. We have not developed the true statement that if $g^o f = j_A$ and $f^o g = j_B$ then f and g are both one-to-one and onto. This will be considered in Course II. However, it is a legitimate question for exploration, but it is a more subtle question. The student should realize that if g is the inverse of f then f is the inverse of g.

3.12 Solutions to Exercises

1. (a) Example 2: Has no inverse. Every element of the second set is the image of some element in the first set but 73 is the image of both John and Fred.

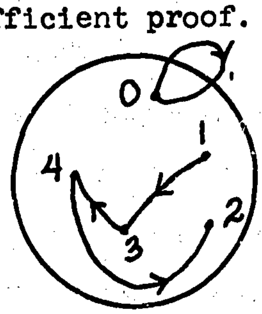
Example 4: Has an inverse. Every element of the second set is the image of exactly one element of the first set.

Example 5: Has no inverse. Some elements of the second set (0, 1, 2, 3, and 4) are the image of no element in the first set.

Example 6: Has no inverse. Every element of the second set is the image of some element of the first set but Mr. Jones and Mr. Ross are each the image of 2 elements of the first set.

(b) f has an inverse. Both conditions hold.

Diagram is a sufficient proof.



(c) g has an inverse. Arrow diagram is sufficient to show that conditions (1) and (2) hold.

(d) h has no inverse. 1,2,4, and 5 are each not the image of any element in Z_6; also 0 and 3 each have 3 pre-images.

(e) This mapping has no inverse. Condition (2) is not satisfied. Each country has many people assigned to it.

2. (a) Example 4: Assign to each capital city the state for which it is capital.

(b) An arrow diagram is a satisfactory answer or rule $n \longrightarrow 2n$.

(c) An arrow diagram is a satisfactory answer or rule $n \longrightarrow n + 3$.

3. (a) For every n in W,

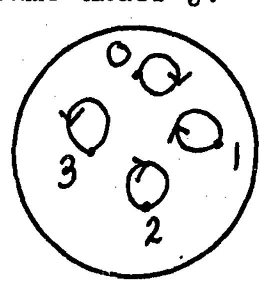
$n \xrightarrow{f} 2n \xrightarrow{j} 2n$ so that $n \xrightarrow{jof} 2n$.

and

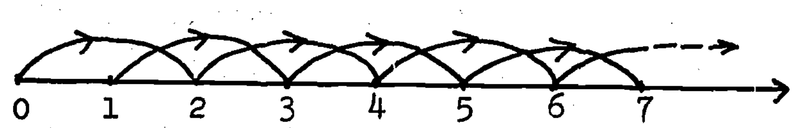
$n \xrightarrow{j} n \xrightarrow{f} 2n$ so that $n \xrightarrow{foj} 2n$.

(b) Yes. Because whatever happens under f, the image remains the same under j.

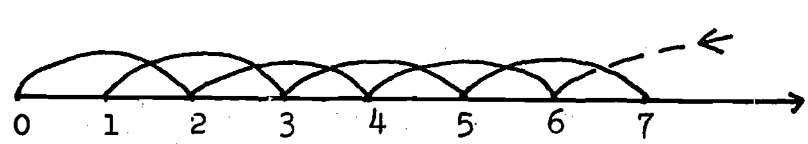
4.



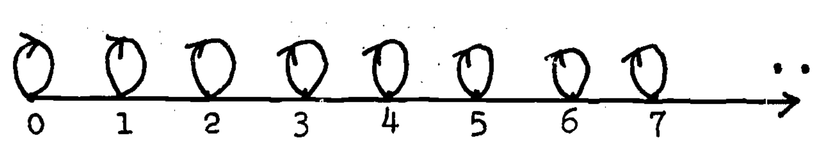
5. (a)



(b)

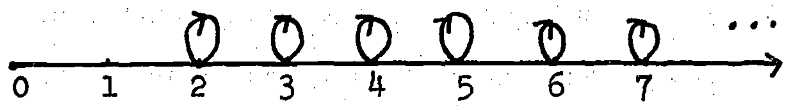


(c)



In (c) some argument should be given to justify the diagram.

(d)



Again, some argument should be given, e.g.

$2 \xrightarrow{f} 0 \xrightarrow{h} 2$ so that $2 \xrightarrow{h \circ k} 2$.

*Since 0 and 1 are not in the domain of K, the arrow diagram starts with 2.

6. (a) $D = W$, $R = \{1,3,5,7,9,\ldots\}$

The inverse mapping of R to D has the rule

$$n \longrightarrow \frac{n-1}{2}$$

(b) $D = N = \{1,2,3,4,\ldots\}$.

$R = \{1,4,7,10,13,16,\ldots\}$

The inverse mapping of R to D has rule

$$n \longrightarrow \frac{n+2}{3}$$

(c) $D = \{2, 3, 4, 5, 6, \ldots\}$

$R = \{0, 1, 2, 3, \ldots\} = W$.

The inverse mapping has the rule $n \longrightarrow n + 2$.

(d) $D = W$, $R = \{2, 3, 4, \ldots\}$

The inverse mapping of R to D has rule $n \longrightarrow n - 2$.

(e) $D = W$, $R = \{25, 73, 121, 169, \ldots\}$

The inverse mapping of R to D has rule

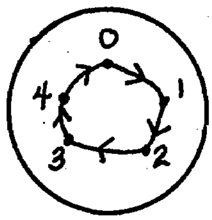
$$n \longrightarrow \frac{n-25}{48}$$

(f) $D = W$, $R = \{1800, 1808, 1816, 1824, \ldots\}$

The inverse mapping of R to D has rule

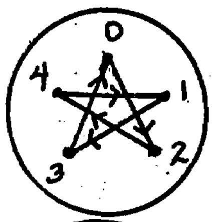
$$n \longrightarrow \frac{n-1800}{8}$$

7. (a)



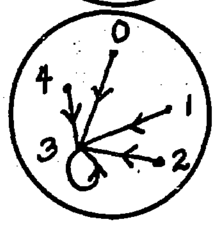
This mapping has an inverse, since it is one-to-one and onto. The rule is either $n \longrightarrow n - 1$ <u>or</u> $n \longrightarrow n + 4$.

(b)



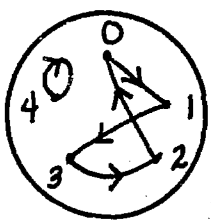
This mapping has an inverse, since it is a one-to-one and onto. A rule is $n \longrightarrow n + 3$ <u>or</u> $n \longrightarrow n - 2$.

(c)



The mapping has no inverse. It is neither one-to-one nor onto.

(d)



This mapping has an inverse, since it is one-to-one and onto. The rule is

$n \longrightarrow \frac{1}{2}(n - 1)$ <u>or</u>

$3(n - 1)$ or $3(n + 4)$ <u>or</u>

$\frac{1}{2}(n + 4)$

3.13 <u>Special Mappings of W to W</u> - (Approximate Time: 1 day)

The goals of this section are to:

(1) lay the groundwork for the study of linear functions of the real numbers to the real numbers as mappings with rules $n \longrightarrow n + b$, $n \longrightarrow an (a \neq 0)$, and their compositions.

(2) point out the lack of inverses for such mappings of W to W even though it seems natural to conceive of such an inverse in terms of a physical interpretation.

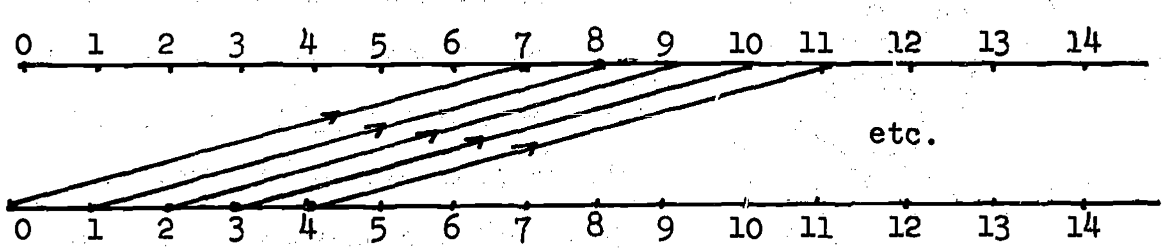
(3) point out incidentally some geometric properties related to these mappings. Here <u>only</u> observations or discoveries on the student's part are intended.

3.14 Solutions to Exercises

1. (1) (a) R = {7,8,9,10,...}

(b) Rule type: n ——>n + a, a = 7

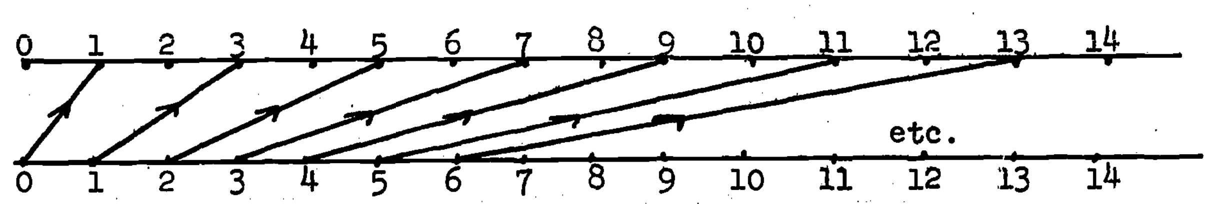
(c)



(2) (a) R = {1,3,5,7,9,...}

(b) Rule not of either type

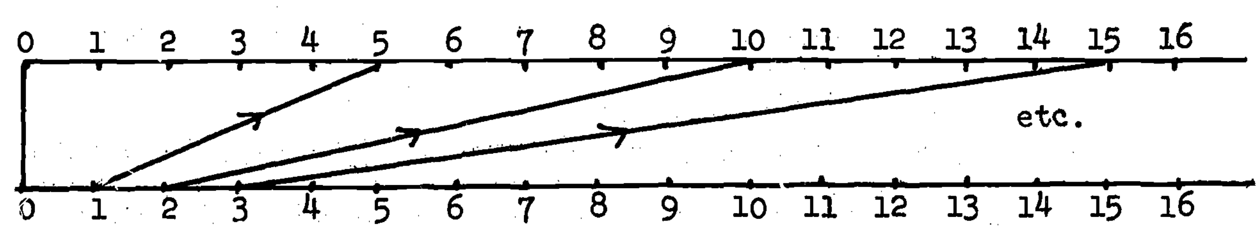
(c)



(3) (a) R = {0,5,10,15,20,...}

(b) Rule type: n——>an, a = 5.

(c)



Note: In each of these mappings a different scale may be used.

Some discussion should be made in class as to the relationship between the appearance of the diagram and the actual mapping.

(4) (a) R = {3,7,11,15,...}

(b) neither rule type

(c) ______ as before.

(5) (a) R = {25,26,27,...}

(b) $n \longrightarrow n + a$, $a = 25$

(c) ______

(6) (a) R = {0,60,120,180,...}

(b) $n \longrightarrow an$, $a = 50$.

(c) ______

2. (a) Yes. Yes.

(b) No. Yes.

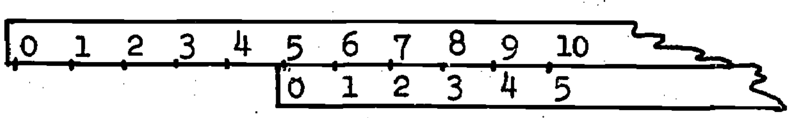
(c) Yes. This inverse only exists if h is one-to-one and onto.

(d) You move 7 hours in a clockwise direction.

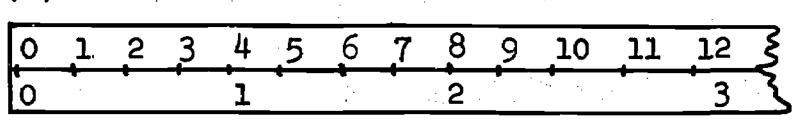
(e) Yes. A move of 7 hour positions in a counterclockwise direction. Rule is $n \longrightarrow h - 7$ or $n \longrightarrow n + 5$.

3. (a) Move it one unit to the right.

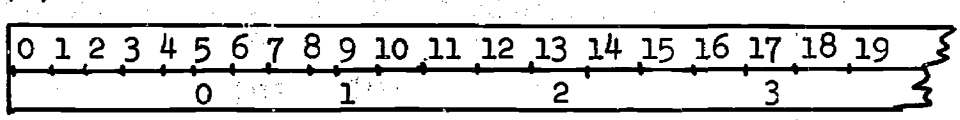
(b) (1)



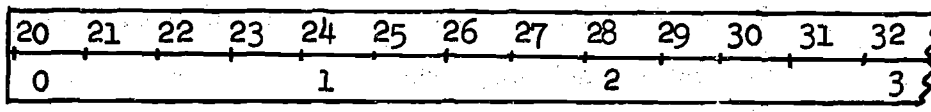
(2)



(3)

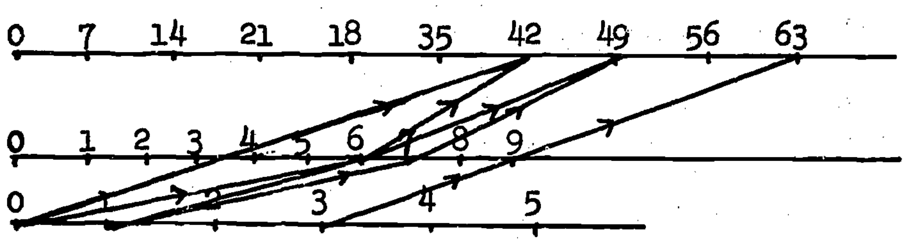


(4)

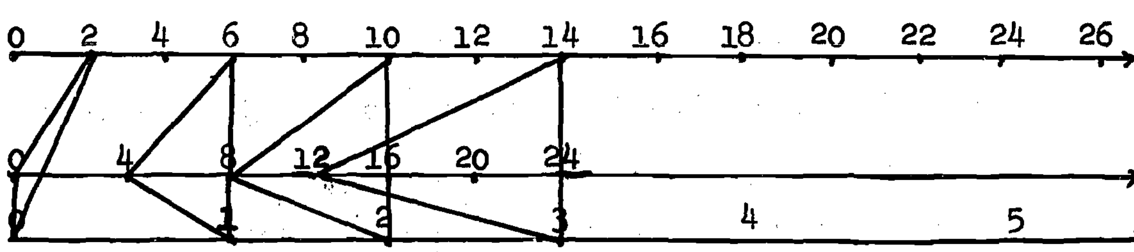


4. Only one or two arrows need be drawn. Choose scales to allow construction easily.

(a)



(b)



5. (a) -- (b) -- (c) yes

(d) f = koh where $n \xrightarrow{h} 77n$

and $n \xrightarrow{k} n + 1306$

(e) If $n \xrightarrow{f} an$

and $n \xrightarrow{g} n + b$

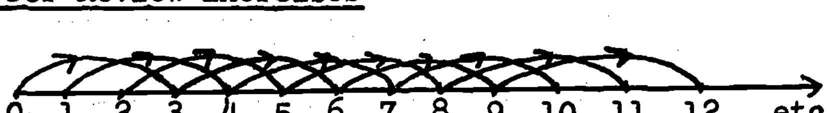
then $n \xrightarrow{f} an \xrightarrow{g} an + b$

$\therefore n \xrightarrow{g \circ f} an + b$ so that p is a and q is b.

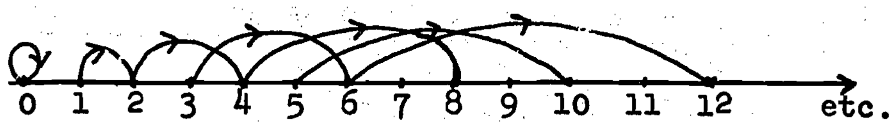
3.16 Solutions for Review Exercises

1. (a)

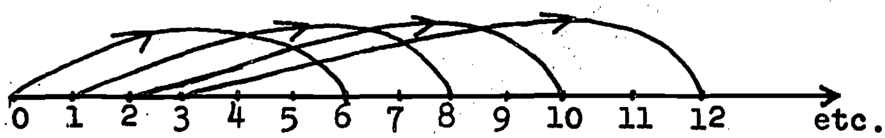
f:



g:



gof:



(b) $n \xrightarrow{gof} 2n + 6$, $n \xrightarrow{fog} 2n + 3$

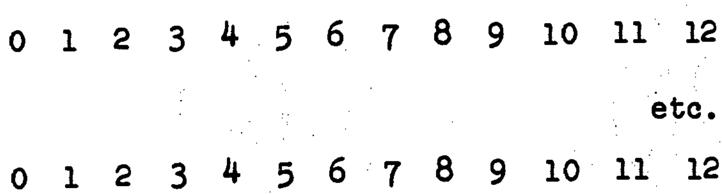
(c) $637 \xrightarrow{gof} 1280$, $637 \xrightarrow{fog} 1277$

No. No. Because 637 has a different image under gof than it does under fog.

(d) {9}

(e) { } or $\emptyset$

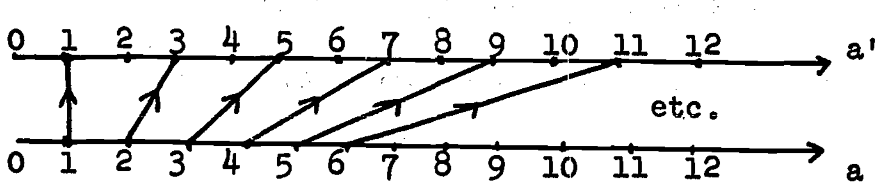
2. (a)



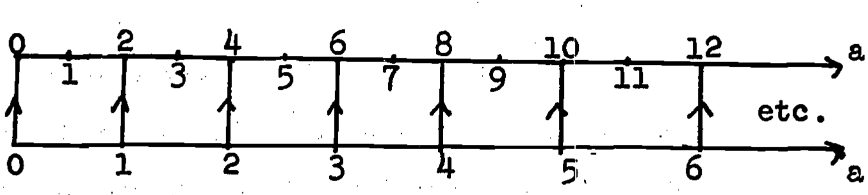
(b) No. 2 is the image of no whole number (also 0, 1 and 3).

(c) Yes. Each image is the image of exactly one whole number.

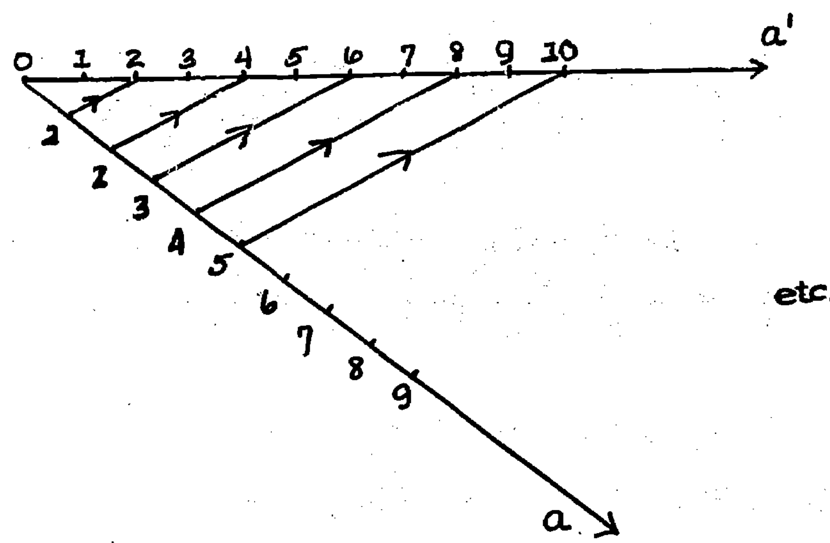
3. (a)



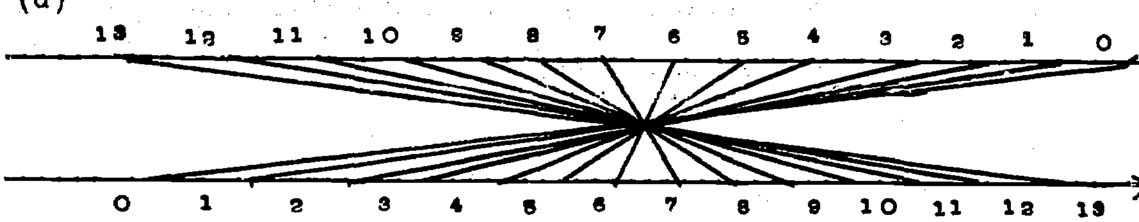
(b)



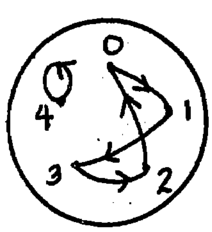
(c)



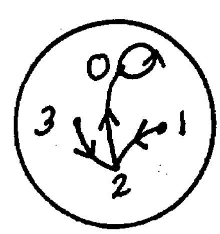
(d)



4. (a)



(b)



(c) (a) is one-to-one

(a) is onto

(a) has an inverse because of previous answers.

The rule is $n \longrightarrow \frac{1}{2}(n - 1)$ or

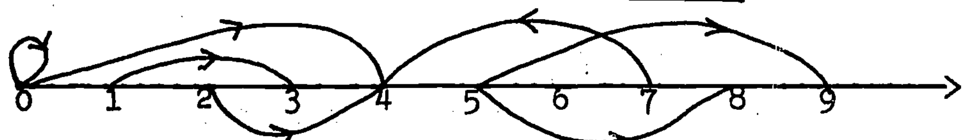
$3(n - 1)$ or $3(n + 4)$ or $\frac{1}{2}(n - 4)$

Suggested Items for a Chapter Test

1. Determine whether or not each of the following assignments is a mapping. In each case give a reason for your answer.

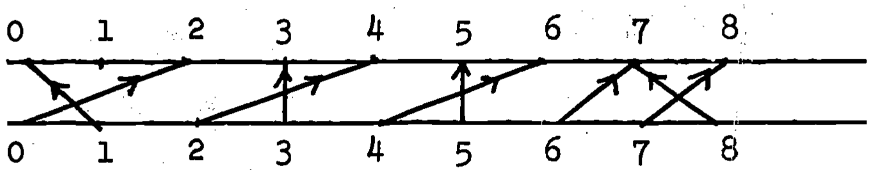
   (a) A gym teacher assigns the boys in a gym class activities according to the following chart:

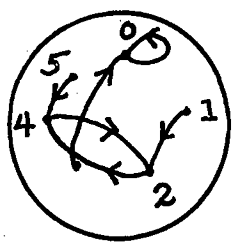
Name	Activity
John	Flying Rings
Steve	Rope Climbing
Henry	Tumbling
Allan	Flying Rings
George	Rope Climbing
Howard	Tumbling
Tom	Tumbling

(b) 

(Remember that in an arrow diagram the first set is the set of numbers that are at the origin of some arrow and the second set is the whole numbers.)

(c)



(d) 

(Here the first set and the second set are Z_6.)

2. (a) Choose a set A of whole numbers such that $n \longrightarrow \frac{1}{3}n - 7$ is a rule for a mapping of A to W.

   (b) List at least one whole number that cannot be in the domain A of any mapping of A to W having this rule.

3. Draw an arrow diagram on a line for the mapping of $\{2,4,6,8,10\}$ to W having the rule $n \longrightarrow \frac{1}{2}n - 1$.

4. In this problem the domain for each sequence is $\{1, 2, 3, 4, 5\}$ and the range is contained in the set of numbers of arithmetic.

   (a) Find the range of the sequence with rule

   $n \longrightarrow 2n - 1.$

   (b) Find the range of the sequence with rule

   $n \longrightarrow \frac{3}{4}n^2 + \frac{2}{3}.$

   (c) Find the range of the sequence given by the rule that

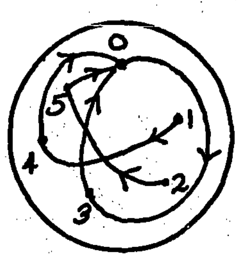
   (1) $1 \longrightarrow 23$ and

   (2) $a_{n+1} \longrightarrow 3a_n - 7.$

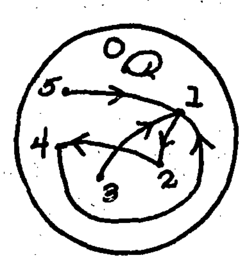
   (d) Find a rule like that of (c) for (a).

5. Two mappings f and g of Z_6 to Z_6 are given by the following arrow diagrams:

f:



g:



   (a) Find the image of 3 by f.

   (b) Find the image of 4 by g.

   (c) Find the image of 3 by gof.

   (d) Find the image of 3 by fog.

   (e) Draw an arrow diagram for fog.

   (f) Does fog = gof? Explain your answer.

6. f and g are mappings of W to W given by the rules:

   $n \xrightarrow{f} 12n + 7$ $\qquad\qquad$ $n \xrightarrow{g} 13n + 16.$

(a) Find the image of 4 by f.

(b) Find the image of 8 by g.

(c) Find the image of 5 by gof.

(d) Find the image of 5 by fog.

(e) Find a rule of the form $n \longrightarrow \square$ for gof.

(f) Find a rule of the form $n \longrightarrow \square$ for fog.

7. For the mappings in (6):

   (a) List the set of whole numbers whose image of f is 127, if possible.

   (b) List the set of whole numbers whose image by g is 355, if possible.

8. Determine whether or not the mapping f of A to B where $A = W$ and $B = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ and $n \xrightarrow{f} 2n - 1$ has an inverse. If so, give reasons for your answer and the rule of the inverse mapping g.

9. Given the rule $n \longrightarrow \frac{1}{2}n - 3$, find a domain D and a range R consisting of whole numbers so that the rule defines a one-to-one mapping of D onto R. Then find the rule of the inverse mapping of R onto D.

10. Given $A = \{0,1,2,3,4\}$. Draw an arrow diagram on a line for the identity mapping j_A of A to A.

11. If the mapping f of A to B and the mapping g of B to A are inverse mappings:

   (a) fog = ?

   (b) gof = ?

12. Given the rule $n \longrightarrow 2n$.

(a) If $n \longrightarrow 2n$ is the rule for a mapping f of W to W, is there an inverse mapping for F? Why?

(b) If $n \longrightarrow 2n$ is the rule for a mapping f of Z_8 to Z_8 is there an inverse mapping for F? Why?

(c) If $n \longrightarrow 2n$ is the rule for a mapping of Z_8 to Z_8 is there an inverse mapping for f? Why?

Answers to Suggested Test Items

1. (a) Yes. Each boy is assigned exactly one activity?

(b) No. $0 \longrightarrow 0$ and $0 \longrightarrow 4$; $5 \longrightarrow 8$ and $5 \longrightarrow 9$.

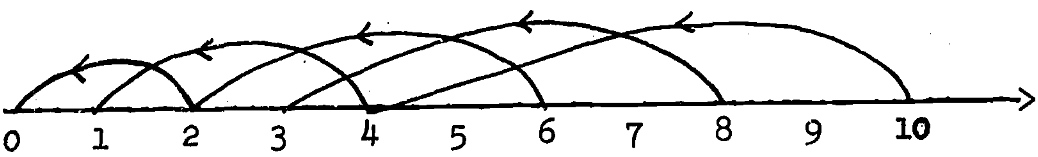
(c) Yes. Each whole number 0-8 is assigned exactly one whole number.

(d) Yes. Each element of Z_8 is assigned exactly one element of Z_8.

2. (a) Any non-empty subset of {21,24,27,...{

(b) Any number not in the set {21,24,27,...}

3.



4. (a) 1, 3, 5, 7, 9.

(b) $\frac{17}{12}$, $\frac{11}{3}$, $\frac{89}{12}$, $\frac{38}{3}$, $\frac{233}{12}$.

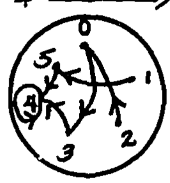
(c) 23, 62, 179, 530, 1583.

(d) $1 \longrightarrow 1$

$a_{n+1} \longrightarrow a_n + 2.$

5. (a) $3 \xrightarrow{f} 0$ (c) $3 \xrightarrow{gof} 0$

(b) $4 \xrightarrow{g} 1$ (d) $3 \xrightarrow{fog} 0$

(e) 

(f) No - e.g. $3 \xrightarrow{gof} 0$ and $3 \xrightarrow{fog} 4$

6. (a) $4 \xrightarrow{f} 55$ (d) $5 \xrightarrow{fog} 81$

(b) $8 \xrightarrow{g} 117$ (e) $n \xrightarrow{gof} 156n + 107$

(c) $5 \xrightarrow{gof} 67$ (f) $n \xrightarrow{fog} 156n + 199$

7. (a) $\{10\}$ (b) $\{\ \}$ or $\emptyset$ or not possible.

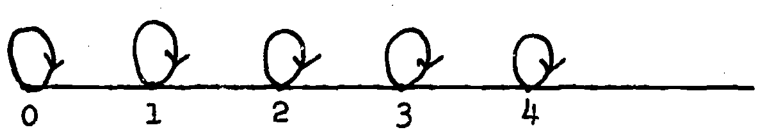
8. Has an inverse because it is one-to-one and onto.

The mapping g has the rule $n \longrightarrow \frac{1}{2}(n + 1)$.

9. D may be any non-empty subset of $\{6,8,10,12,\ldots\}$.

R must then be properly selected. Rule of inverse in any case is $n \longrightarrow 2(n + 3)$ or $n \longrightarrow 2n + 6$.

10.



11. (a) $fog = j_B$ (b) $gof = j_A$

12. (a) No. Because f does not map W <u>onto</u> W.

(b) Yes. f is then one-to-one and onto.

(c) No. f is neither one-to-one nor onto.

CHAPTER 4

TEACHERS' COMMENTARY

The Integers and Addition (13 days)

General Introduction

The purpose of this chapter is the introduction of a new set of numbers (the integers) in which every equation of the type $x + a = b$ has a solution. First a need is shown for a new type of numbers. Then addition is introduced by gain and loss in a game situation. The general properties of $(Z,+)$ are then formally discussed. Line translations are used to add a geometric interpretation to addition.

Subtraction is introduced informally and then developed formally in terms of addition. Equations are solved by use of translations and also by use of the cancellation law. Ordering the integers is discussed, followed by the idea of absolute value as a maximizing operation. The entire chapter could take 12 days plus 1 day for a test.

Section 4.1 (1 day)

The purpose of this section is to review the solution of equations of the kind studied in chapters 1 and 2, with considerable emphasis on the fact that not every equation of the type "$x + a = b$" has a solution in $(W,+)$. This emphasis serves to preview the purpose of the chapter -- the introduction of a new set of numbers, the integers.

Exercise 1 may be used as a class exercise.

4.2 Exercises

1. (a) {4} (b) {3} (c) {3} (d) {0} (e) {4} (f) {3}
   (g) {6} (h) {5} (i) {4}
2. (a) {2} (b) {2} (c) {1} (d) {1}
3. Yes -- in this case, it is not too laborious to write every possible equation.
4. Answers vary.
5. Yes. See question 3
6. (a) {11} (b) {19} (c) ∅ (d) {112} (e) ∅
   (f) {22} (g) {953} (h) {311} (i) ∅

4.3 Some New Numbers (1 day)

This section introduces some of the negative integers informally by means of a simple interpretation -- winning and losing points in a game. Emphasize the agreement to add partial scores in order to obtain the total score, for it is this agreement which forces us to create new numbers (rather than subtracting). It might be profitable to actually play a simple game in class, with students using the whole numbers and the new negative numbers to keep score.

In a later section (4.13), after the integers have been more formally developed, the point is made that all equations of the type "x + a = b" are solvable in the set of integers. Even in the present section, however, it might be pointed out that the new number $^{-}5$, for instance, is a solution of the equation "6 + x = 1"; and, as the student saw in section 4.1, this equation has no whole number solution. In this way, section 4.1 and present section may be related.

Exercises 5, 6, 7 may be done in class. Problem 6 and 9 could also be used to introduce addition of integers. Problem 11 could be optional.

4.4 Exercises

These exercises not only provide drill but constitute an important part of the development. The student has an opportunity to work with a number of physical situations (e.g., temperature change, profit and loss, etc.) which help to give meaning to negative numbers and to addition. It is not intended that any formal rule for addition be discussed this time. Students should find sums intuitively, using the physical interpretations as a guide. The verbalization of the addition process is probably too complicated to be considered at this time; at the end of the chapter, a formal definition of addition (in terms of absolute value) is presented.

1. (a) $^-14$ (b) 7 (c) $^-7$ (d) $^-22$ (e) 22 (f) $^-14$
2. What is x if you score x points on the first play, 7 points on the second play, and your total score after two plays is 4? (Emphasize again that we add scores to obtain the total score.)
3. You score 5 points on the first play, x points on the

second, and your total score after two plays is 0.

4. $^{-}2$

5. (a) 4 (b) 0 (c) $^{-}4$ (d) 1 (e) 0 (f) $^{-}1$ (g) $^{-}5$
(h) $^{-}15$ (i) $^{-}20$ (j) $^{-}25$ (k) $^{-}7$ (l) $^{-}189$

6. (a) 5;0 (b) 7;2 (c) $^{-}5$;$^{-}10$ (d) In part (a) the number must represent a five degree <u>rise</u>; in part (c) the number must represent a five degree <u>fall</u>. (e) 0;$^{-}5$.

7. (a) 1 (b) $^{-}5$ (c) $^{-}2$ (d) 10 (e) $^{-}10$ (f) 0 (g) $^{-}2$
(h) 2 (i) $^{-}2$ (j) 0 (k) $^{-}4$ (l) 9 (m) 4 (n) 15 (o) 30.

8. (a) 25;225 (b) 0;200 (c) $^{-}25$;175 (d) In part (a) the number must represent 25 <u>more</u>; in part (c) the number must represent 25 <u>less</u>.

9. (a) 5; 14 (b) $^{-}5$;4 (c) In part (a) the number must represent a <u>gain</u> of five yards; in part (b) the number must represent a <u>loss</u> of five yards (d) $^{-}9$;0 (e) $^{-}15$;$^{-}6$.

10. (a) 122 (b) $^{-}48$ (c) 48 (d) $^{-}122$ (e) $^{-}18$ (f) $^{-}102$
(g) 20 (h) $^{-}106$ (i) 106 (j) $^{-}60$ (k) 60 (l) 75
(m) 75 (n) $^{-}125$ (o) $^{-}500$ (p) $^{-}100$ (q) $^{-}1000$ (r) $^{-}1500$

11. Answers vary.

4.5 <u>The Integers and Opposites (1 day)</u>

In this section, the student learns the words "integer," "positive integer," and "negative integer." The existence of the entire set Z of integers is simply assumed; the work of the preceding sections should make the assumption a reasonable one.

Opposites are introduced as a pair of numbers (integers) whose sum is zero. The symbol "-a" should be introduced with care, with emphasis on the fact that -a is not necessarily a negative number. In order to maintain the distinction between the two concepts of <u>opposite of an element</u> and a <u>negative element</u>, we are using both the "upper dash" (as in $^-2$, a negative number) and the "lowered dash" (as in -a, the opposite of a). In section 4.11, after the concepts have had time to take root, we shall discontinue use of the upper dash.

Should the occasion arise in class to make a distinction between the concepts of opposite and negative, you might use a system such as $(Z_4,+)$ with which the students are already familiar. In this system, the element 3, for instance, has an inverse; it is 1, since 3 + 1 = 1 + 3 = 0. Therefore we may write "-3 = 1," as in fact we did in chapter 1. However, there are no <u>negative</u> elements in Z_4.

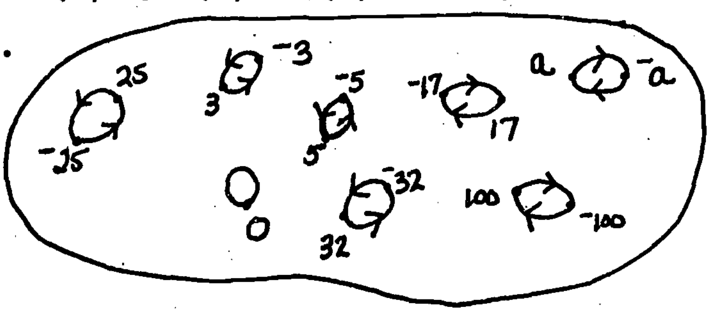
Be sure that students do not become confused with the use of parentheses in expressions such as "-($^-2$)" and "-(-a)." They simply separate the dashes in order to facilitate reading.

Exercises 3, 6, 7 may be done in class.

4.6 Exercises

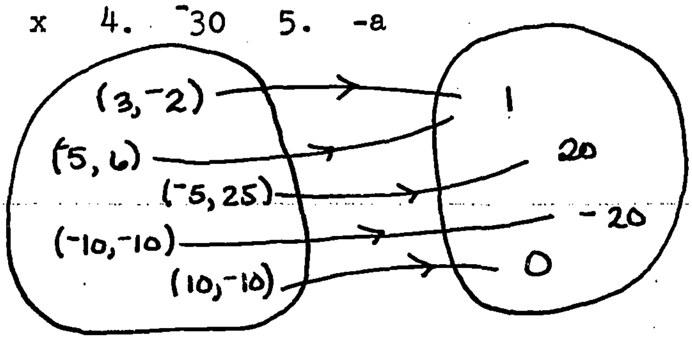
Exercise 5 might be used as a starting point for encouraging students to discover an interesting generalization. That is, -(-a) = a; -(-(-a)) = -a; -(-(-(-a))) = a; etc. Some students will probably discover that an even number of oppositing symbols produces <u>a</u> itself, while an odd number of oppositing symbols produces -a.

1. (a) $^-3$ (b) 2 (c) $^-5$ (d) 72 (e) 0 (f) $^-15$

2. 

Since the loop from "0 to 0" may be traversed in either direction, no arrow is needed.

3. x 4. $^-30$ 5. -a

6. 

7. Answers vary. 8. Answers vary.

9. (a) {0,1,$^-1$,2,$^-2$,3,$^-3$,...} Of course, there need be no particular order, and these particular integers need not be used.)

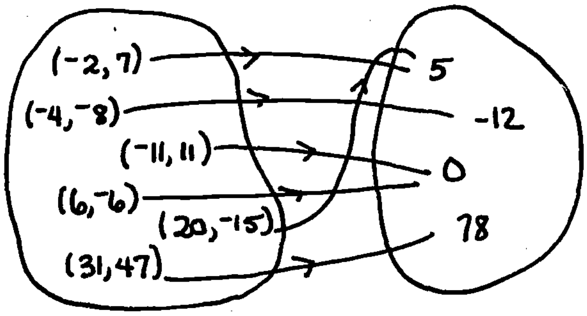
   (b) Answers vary. (c) Answers vary. (d) {0}

4.7 <u>Properties of (Z,+) (1 day)</u>

Students should accept by this time the fact that they can assign an integral sum to any ordered pair of integers. Thus, at this time, we make note of the fact that (Z,+) is an operational system (operational systems having been defined in chapter 2). Since much work has already been done with the properties of commutativity, associativity, identity element, and inverse elements (in both chapter 1 and chapter 2), they are introduced here with little fanfare as properties of the system (Z,+). In terms of the physical situations which initially motivated creation of the integers, all of the properties should seem reasonable to the student.

Of course, with these properties the system (Z,+) is seen to be a commutative group (defined in section 2.15). The student's attention is directed to this in exercise 4 of the following section.

4.8 Exercises

1. (a) $^-4$ (b) $^-3$ (c) $^-3$ (d) 11 (e) 11 (f) 90 (g) $^-36$ (h) $^-36$ (i) 44
2. (a) 2 (b) $^-3$ (c) 20 (d) $^-50$ (e) 16 (f) 28 (g) 165 (h) $^-195$ (i) 9 (j) $^-14$ (k) $^-24$ (l) 105
3. 

4. (a) $(Z_4,+)$ has all four properties (b) Not every element in (W,+) has an inverse; all the other properties are present (c) All four properties (d) All three systems are commutative groups.

4.9 The Integers and Translation on a Line (2 days)

In this section we return to the problem (first raised in chapter 3) of working with assignments of the kind $n \longrightarrow n + a$. In this section, however, a is an integer, and so we are able to speak of translations of the line. That is, rules of the kind $n \longrightarrow n + a,\ a \in Z$, may be used to describe translations. In a line translation, every point of the line has exactly one image, and every point is the image of exactly one point. Thus, the translation is a one-to-one onto mapping; the domain is the set of all points on the line, and the range is the set of all points on the line.

Translations of the kind described by $n \longrightarrow n + a$, $a \in Z$, where a is a negative integer, help us to associate the negative integers with points of the line. Furthermore, composition of line translations gives us still another interpretation of addition of integers.

Exercise 4 could be assigned as homework and then discussed in class. Problem 7 a - c may be done in class and the remainder done for homework, Problem 11, 12, may be used by the teacher as part of the classroom explanation of this section.

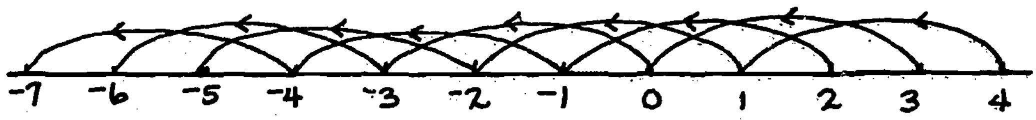
4.10 Exercises

In exercise 4, use known instances of identity elements to help make the answer "identity translation" reasonable. For instance, in addition of integers, 0 is the identity element, since addition of 0 (in either order) to any integer produces that integer. In the same way, the composition of $n \longrightarrow n + 0$ (in either order) with any translation produces that translation.

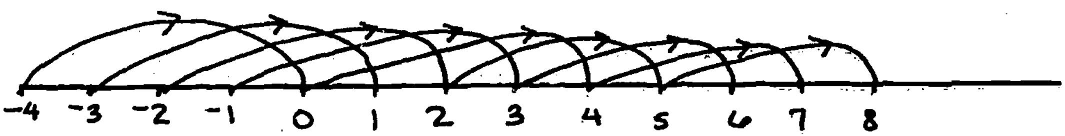
Exercise 12 is again concerned with group properties. Showing the translations on a number line can be helpful in convincing students that composition of translations is associative. (See exercise 11i and 11j.)

Exercise 11 is worth considerable attention, since it provides a good visual way of adding integers; it may be particularly helpful to any student who has been having trouble with addition.

1.

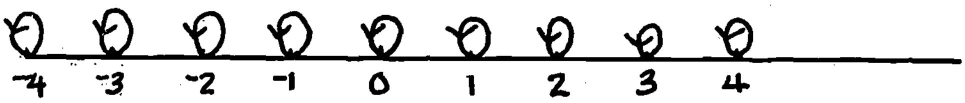


2.



3. $n \longrightarrow n + 4$

4.



5. $n \longrightarrow n + 0$ (6) $n \longrightarrow n + (-a)$

7. (a) $n \longrightarrow n + (^{-}12)$ (b) $n \longrightarrow n + (^{-}13)$

(c) $n \longrightarrow n + 45$ (d) $n \longrightarrow n + 66$

(e) $n \longrightarrow n + (a + b)$ (f) $n \longrightarrow n + 63$

(g) $n \longrightarrow n + 0$

8. Yes.

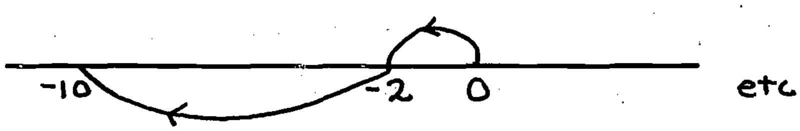
9. (a) $n \longrightarrow n + (^{-}14)$ (b) $n \longrightarrow n + (^{-}54)$

(c) $n \longrightarrow n + 0$ (d) $n \longrightarrow n + (^{-}127)$

(e) $n \longrightarrow n + (^{-}10)$ (f) $n \longrightarrow n + (-27 + (-a))$

10. (a) $n \longrightarrow n + 4$ (b) $n \longrightarrow n + 4$

11.



12. "Yes" for all parts

4.11 Subtraction in (Z,+) ($\frac{1}{2}$ to 1 day)

In this section we ask the student to use an understanding of subtraction he has had since the elementary grades: $a - b = c$ if and only if $c + b = a$. As pointed out in the text, the student has probably used this fact to "check" subtraction problems in arithmetic. The desire to preserve this relationship between subtraction and addition motivates the way in which we determine the difference of any two integers. Since the student is able to relate this to his earlier work in arithmetic, this introduction is probably more meaningful than one which simply defines $a - b$ as $a + (-b)$. The latter principle is considered in section 4.13.

It is at the beginning of this section that we agree to discontinue use of the "upper dash." The lower dash may now be used for both a negative number and to indicate an opposite. Thus, "-2" may be read as either "negative two" or "opposite of two," since the opposite of two _is_ negative two. However, in the case of an expression containing a variable, such as "-x," the preferred reading is "opposite of x." Reading this latter expression as "negative x" may cause some students to consider it as a negative number.

Sections 4.11 and 4.13 could be done in one day. The exercise in 4.12 could all be done in class.

4.12 Exercises

1. 3 2. 7 3. -7 4. -3 5. 3 6. -3 7. 8 8. -8
9. -1 10. -5 11. -7 12. 200 13. 0 14. -100
15. 300

4.13 Subtraction as Addition of Opposites ($\frac{1}{2}$ to 1 day)

Through the use of numerical instances, the student should come to accept the principle "$a - b = a + (-b)$" as a perfectly reasonable one. It is of course the definition of subtraction in any additive group.

Only two numerical instances are discussed in the text. In most classes, many more will probably be called for. For example, each of the exercises in section 4.12 might be used: $5 - 2 = 5 + (-2)$;$5 - (-2) = 5 + 2$; $-5 -2 = -5 + (-2)$; etc.

Exercise 1 - 30 could be done in class. Problem 21 could be used as part of class explanations and class discussion of $-(a + b) = (-a) + (-b)$, an important fact.

4.14 Exercises

1.	100	6.	-7	11.	-249	16.	a + 2
2.	-100	7.	-21	12.	110	17.	-2 + a
3.	50	8.	-182	13.	0	18.	-a + b
4.	7	9.	8	14.	4,366,269	19.	x + y
5.	21	10.	85	15.	5,233,568	20.	a + (b + c)

21. Yes. Subtraction assigns exactly one integer to every ordered pair of integers.

    Subtraction is neither commutative nor associative.

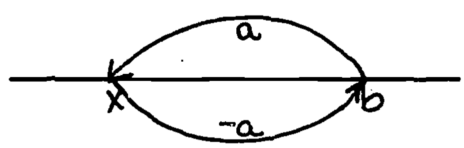
22.	1	29.	-40
23.	-6	30.	17
24.	37	31.	-27
25.	188	32.	-95
26.	-371	33.	0
27.	-261	34.	0
28.	-32	35.	0

36. (-x) + (-y), or -x -y
37. x + (-y), or x - y
38. x + y
39. -7 -a
40. -a + 4 (Note that this is -(a + -4) = -a + -(-4) = -a + 4)
41. -a + b
42. -a - b - c
43. -a - b + c   44. -a + b + c   45. x + y

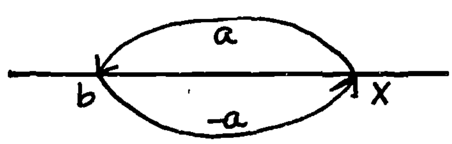
4.15 Equations in (Z,+) (1 day)

At the beginning of this chapter, we observed that not all equations of type "x + a = b" are solvable in (W,+). The purpose of the present section is to make clear to the student that every such equation is solvable in (Z,+).

Line translations and their inverses are used to lead to the general equation (x + a = b) which appears as exercise 30 in the section following. Note that in terms of line translations we have



if <u>a</u> is positive; and we have



if <u>a</u> is negative. In either case, however, x + a = b is seen to be equivalent to b + (-a) = x, or b - a = x. (Here again we see the importance of understanding -a as an opposite, not necessarily a negative number.) Exercise 1 - 10 may be done in class.

4.16 <u>Exercises</u>

1.	-2	6.	-10	11.	13
2.	0	7.	10	12.	13
3.	-2	8.	10	13.	60
4.	0	9.	-8	14.	23
5.	5	10.	-8	15.	13

16. 15	21. b - 3	26. b + (-15), or b - 15
17. 12	22. b - 5	27. b + 10
18. -18	23. b - 100	28. b + 10
19. -103	24. b + 6	29. b + 14
20. 103 (Here note that -a = -103; so a = 103.)	25. b + 6	30. b + (-a), or b - a

31. Yes; the solution is in fact b - a.

32. Answers vary.

4.17 Cancellation Law (1 day)

Looking back, the work of this section should be related to that in chapter 2 in which cancellation laws in operational systems were considered. Looking ahead, it is important that the cancellation law in (Z,+) be understood at this time, since it will play an important role in developing multiplication of integers in chapter 6.

Be sure that students understand that the equation solving in this section is to see if they understand the cancellation law; if the only purpose were to find solutions, they might well prefer the method of the preceding section. However, be certain that the students are able to solve equations by the cancellation law.

4.18 Exercises

1. Yes, since (Z,+) possesses the commutative property.

2. (a) (W,+) is one; see chapter 2 for others.
   (b) $(Z_4, \cdot)$ is one; see chapter 2 for others.
3. (a) -5 (b) -5 (c) -3 (d) -3 (e) -10 (f) -14
   (g) 19 (h) -19
4. (a) 12 (b) 16 (c) -3 (d) -3 (e) 55 (f) -32
   (g) 44 (h) 5 (i) 21 (j) 64 (k) 4 (l) b - a

4.19 Ordering the Integers (1 day)

The definition of ordering presented here is the usual one. That is, a > b is and only if a - b is a positive integer. Point out to students that this means all positive integers are greater than zero, and all negative integers are less than zero, a fact they have probably accepted intuitively long ago. Also point out that the ordering of the integers induces an ordering of the points on the number line. Problem no. 1 could be done in class and no. 6 is optional.

4.20 Exercises

1. (a) 2, since 2 - (-6) is a positive integer.
   (b) 6, since 6 - (-2) is a positive integer.
   (c) -2, since -2 - (-6) > 0. (In view of the remark in the preceding section, "> 0" means "is positive."
   (d) 0, since 0 - (-1) > 0.
   (e) 1, since 1 - 0 > 0.
   (f) -6, since -6 - (-7) > 0.

2. -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5

3. a; s; c = d

4. (a) $7 > -3$ (g) $5 < 14$
   (b) $-5 > -15$ (h) $-5 > -14$
   (c) $-8 < 0$ (i) $7 < 15$
   (d) $8 > 0$ (j) $-7 > -15$
   (e) $-100 < 2$ (k) $-3 < 5$
   (f) $1 > -2500$ (l) $3 > -5$

5. If $a < b$, then $-a > -b$.

6. (a) 7 (b) 5 (c) 13 (d) 52 (e) 33 (f) 97

7. (a) y (b) y (c) x

8. $a < c$ ("<" is a transitive relation in Z.)

4.21 Absolute Value (1 day)

Exercise 6 of the preceding section and the study of the "max" operation in Chapter 2 have prepared the student for the definition of absolute value which is presented in this section. For any integer a, the absolute value of a is the greater of the pair (a,-a); i.e., max(a,-a). Thus if a is positive, $|a| = a$; and if a is negative, $|a| = -a$. Of course, $|0| = 0$. Be sure that students understand that $|a| = a$ is not necessarily true; it is true only if a is positive or zero.

Also be sure that students see that $|a - b| = |b - a|$. This follows since $a - b$ and $b - a$ are opposites; and if two numbers are opposites, they have the same absolute value. The application of this to finding the distance between two points on a line is an important one.

A second approach to absolute value is to introduce the idea of distance between a and 0.

Exercise 1 may be done in class as well as 11 a - f.

Exercises 5, 7, 13, 16, 17 are optional problems.

4.22 Exercises

1. (a) 7 (b) 7 (c) 215 (d) 215 (e) 215 (f) 3 (g) 83 (h) 83 (i) 100 (j) 100 (k) 3 (l) 6

2. 0

3. (a) 4 (b) 4 (c) 82 (d) 82 (e) 7 (f) 7 (g) 21 (h) 7 (i) 25 (j) 25 (k) 28 (l) 28

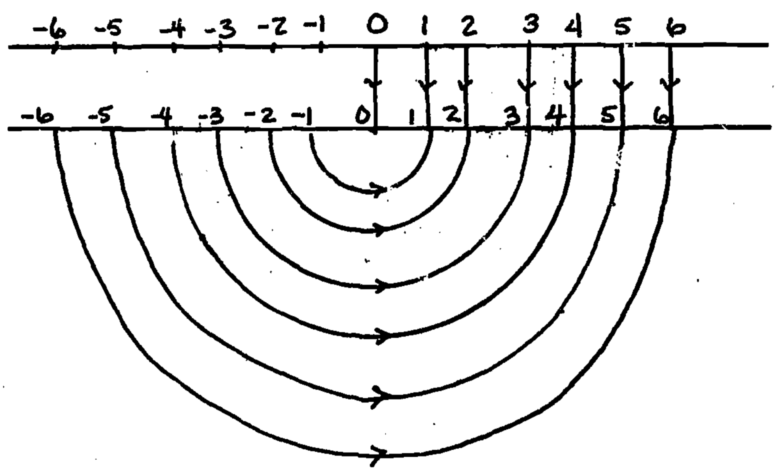
4. (a) 43 (b) 10 (c) 23 (d) 53 (e) 20

5. (a) 7 (b) 7 (c) 7 (d) 7 (e) 18 (f) 18 (g) 18 (h) 18 (i) 100 (j) 100 (k) 100 (l) 100

6. (a) 7 (b) 7 (c) 12 (d) 12 (e) 20 (f) 20 (g) 32

7. 31

8. $|p|$ (It is <u>not</u> p, unless p is positive or zero.)

9. 

10. $|a| < 0$ is never true

11. (a) {5,-5} (d) {100,-100} (g) {1,-5} (j) {6,-6}
    (b) {0} (e) {8,-8} (h) {1,-1} (k) ∅
    (c) ∅ (f) {8,-10} (i) {-4,6} (l) {-7}

12. (a) {0,1,2,-1,-2}
    (b) Union of two sets: integers greater than 2; integers less than -2.
    (c) {-4,-3,-2,-1,0,1,2,3,4}
    (d) Union of two sets: integers greater than 5; integers less than -5.
    (e) ∅
    (f) All integers except 0
    (g) All integers between -100 and 100 $(-100 < a < 100)$
    (h) Union of two sets: integers greater than 100; integers less than -100. $(a < -100) \cup (a > 100)$

*13. (a) -3, -2, -1
    (b) 1, 2, 3
    (c) -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

(d) -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3

14. True (On the number line, the points are the same distance from the origin.)

15. (a) True (b) True (c) True

16. (a) You may simply add the absolute values.

(b) Add the absolute values; then take the opposite.
$-(|a| + |b|)$

(c) Subtract the absolute value of b from the absolute value of a; then take the opposite. $-(|a| - |b|)$

*17. It is true in the following cases:

when $a = 0$ or $b = 0$

when a and b are both positive

when a and b are both negative

4.23 Many of the following problems can be done in class as a review. Some can be omitted according to the needs of the class.

4.24 <u>Review Exercises (1 day)</u>

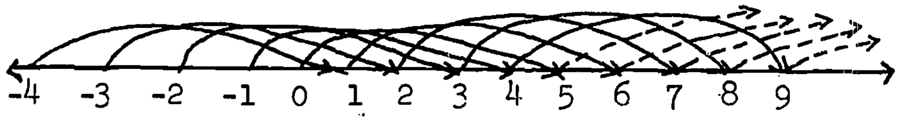
1. -33	7. 66	13. -63	19. -132	25. 63
2. -117	8. -12078	14. -21	20. 38	26. -1
3. 71	9. 280	15. 55	21. -15	27. -111
4. 180	10. -1616	16. -155	22. 15	28. -44
5. -447	11. 40	17. 71	23. 5	29. 4277
6. 7	12. -72	18. -570	24. -5	30. 2902

31. -1000, -109, -88, -42, -10, -3, 0, 3, 68, 72, 215

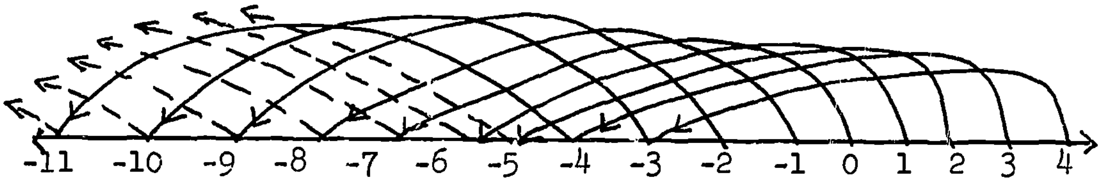
32. (a) > (b) = (c) > (d) = (e) <

(f) = (g) < (h) < (i) = (j) =

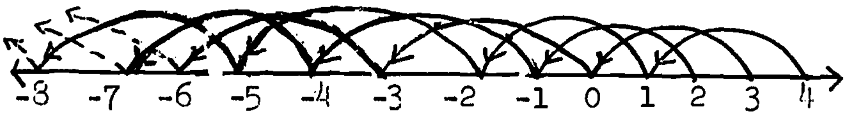
33. (a) n——>n + 5



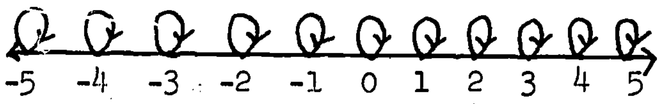
(b) n——>n - 7



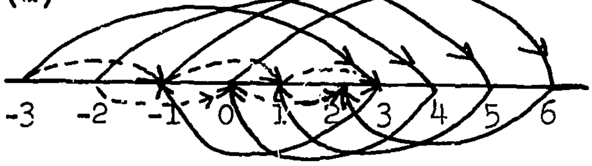
(c) n——>n + (-3)



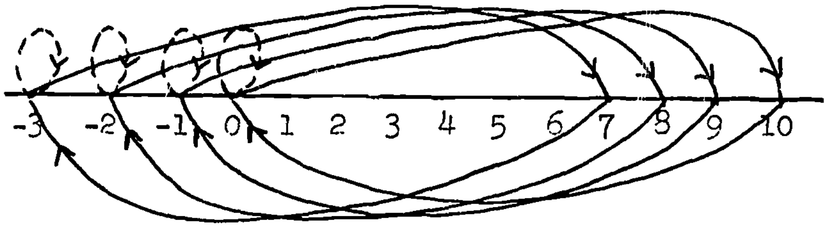
(d) n——>n + 0



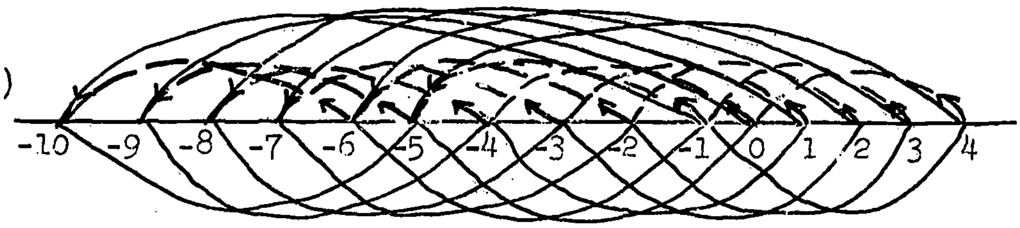
34. (a)



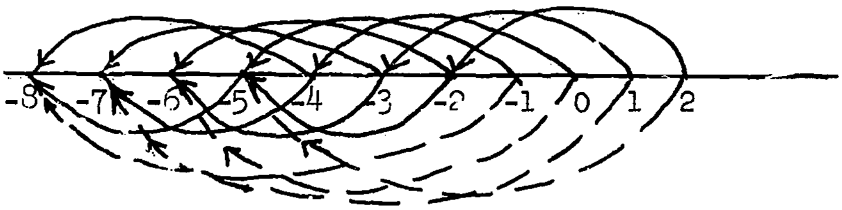
(b)



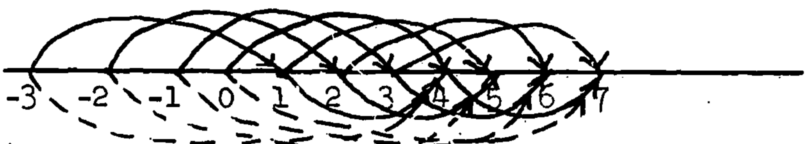
(c)



(d)



(e)



35. (a) The composition is n——>n + 2

(b) n——>n + 0 (c) n——>n - 4 (d) n——>n - 7

(e) n——>n + 7

36. (a) {4}

(b) {-4}

(c) {10}

(d) {10}

(e) {25}

(f) {-25}

(g) {187}

(h) {187}

(i) {b - 7}

(j) {13 - a}

(k) {b - a}

(l) {r - t}

37. (a) $\{5,-5\}$ (f) Zero and all positive integers or $\{x: x \geq 0\}$

(b) $\emptyset$

(c) $\{0\}$ (g) all negative integers or $\{x: x \leq 0\}$

(d) $\{5,-5\}$ (h) All integers

(e) $\{5, -9\}$

38. (a) All integers between -15 and 15 or $-15 < x < 15$

(b) Union of two sets: integers greater than 15; integers less than -15 or $-15 > x > 15$

(c) All integers

(d) $\emptyset$

(e) All integers between -3 and 7 or $7 > x > -3$

(f) All integers between -7 and -3 or $-7 < x < -3$

39. (a) $-a + (-b)$, or $-a - b$ (b) $-a + b$ (c) $a + (-b)$ or $a - b$

(d) $a + b$ (e) $-x + y - z$

40. (a) They all lie in a line, from upper left to lower right

(b) Symmetry about the principal diagonal

(c) Zero appears exactly once in each row and in each column

Suggested Test Items

Chapter 4

1. $32 + (-8) =$
2. $32 - (-8)$
3. $25 + (-3)$
4. $25 - 3$
5. $42 + (-2) + (-8)$
6. $25 - 5 - 10$
7. $13 + (-7) + (-13)$
8. $-7 -8 + 15$
9. $21 - 18 + 14 - 7 - 10$
10. $102 - 18 + 15 - 75$
11. $a + (-a)$, where $\underline{a}$ is an integer

12. $a - a$, where a is an integer

13. List the properties of a commutative group. Use integers to give an illustration of each property.

In problems 14 - 25, solve the given equation for x

14. $x + 2 = 5$ 16. $x + 75 = 60$ 18. $3 + x = 0$

15. $x + 5 = 2$ 17. $x - 75 = 60$ 19. $-5 + x + 7 = 1$

20. $-2 + x + (-3) = 5$

21. $14 + x = -6$

22. $x + 9 = 12$

23. $x + 4 = b$, where b is an integer.

24. $x + a = b$, where a and b are integers.

25. $x + n = t$, where n and t are integers.

26. $|-3| =$

27. $|3| =$

28. max. $(3,-3) =$

29. if a is positive, $|a| =$

30. if a is negative, $|a| =$

31. $|7 - 15| =$

32. $|15 - 7| =$

33. $|14 - 32 + 7| =$

34. $|a + (-a)| =$

35. $|7 + (-3)| =$

36. $|7| + |(-3)| =$

In problems 37 - 45, solve the given equation for x.

37. $|x| = 4$ 40. $|x| + 1 = 3$ 43. $|x| = |-7|$

38. $|x| = 0$ 41. $|x + 1| = 3$ 44. $|x + 1| > 3$

39. $|x| = -4$ 42. $|x| = |7|$ 45. $|x + 1| < 3$

In problems 46 - 55, answer "true" or "false."

46. The sum of two negative integers is a negative integer.

47. If $a < b$, then $|a| < |b|$.

48. $(Z,+)$ is a commutative group.

49. If $a < b$, then $-a < -b$

50. If r - s is a positive number, then $r > s$

51. $|a + b| = |a| + |b|$

52. If $a < b$ and $b < c$, then $a < c$.

53. Every equation of type "$x + a = b$" has a solution in the set of integers.

54. If $x + a = x + b$, then $a = b$.

55. If $a = b$, then $x + a = x + b$.

In problems 56 - 60, insert "=" or "<" or ">" whichever gives a true sentence.

56. $-2 \quad 7$

57. $2 \quad -7$

58. $|3 + (-2)| \qquad |3| + |-2|$

59. $a \quad -a$, where <u>a</u> is a positive integer.

60. $a \quad -a$, where <u>a</u> is a negative integer.

Answers to Suggested Test Items

Chapter 4

1. 24
2. 40
3. 22
4. 22
5. 32
6. 10
7. -6
8. 0
9. 0
10. 24
11. 0
12. 0
13. Operational system with
    1. Associativity
    2. Identity element
    3. Inverses
    4. Commutativity
14. {3}
15. {-3}
16. {-15}
17. {135}
18. {-3}
19. {-1}
20. {10}
21. {-20}
22. {-21}
23. {b - 4}
24. {b - a}
25. {t - n}
26. 3
27. 3
28. 3
29. a
30. -a
31. 8
32. 8
33. 11
34. 0
35. 4
36. 10
37. {4,-4}
38. {0}
39. ∅
40. {2,-2}
41. {2,-4}
42. {7,-7}
43. {7,-7}
44. $x > 2$ or $x < -4$
45. $-4 < x < 2$
46. True
47. False
48. True
49. False
50. True
51. False
52. True
53. True
54. True
55. True
56. $<$
57. $>$
58. $<$
59. $>$
60. $<$

Course I Chapter 5

Probability and Statistics

Commentary for Teachers

(Estimated time for chapter: 10-12 days)

A. General Comments

Although the chapter on probability and statistics is not completely in the path of a sequential development of algebraic and geometric ideas, it serves many useful purposes at this point:

1. It is sufficiently different from the preceding chapters in nature and in the type of activities performed by the student to provide variety.
2. It provides practice in computation with rational numbers and maintains skills developed in elementary school with "fractions". This is needed since the formal introduction to rational numbers comes late in the program.
3. It provides examples of uses of topics such as mappings, graphing conditions, function notation, measure and others, thereby serving to bring together many diverse topics in one setting.
4. Through its use of statistics, probability at this stage gives the students a glimpse of a branch of mathematics which is extremely fruitful both in a theoretical and applied sense.
5. Some idea of scientific method can be developed in connection with the experiments.

There are certain concepts that students hopefully should acquire from this chapter. A list of these may serve as a guide to teachers in connection with tests of student achievement:

1. An experiment as a set of trials and a corresponding set of outcomes.
2. The idea of an outcome set or outcome space and an event as a subset of an outcome set.
3. The relative frequency of an outcome or an event as a number.
4. The relative frequency of an event as a number greater than or equal to zero and less than or equal to one.
5. The relative frequency is 1 if the event is certain and 0 if it is impossible.
6. The relative frequency of an event is the sum of the relative frequencies of its simple outcomes.
7. The sum of the relative frequencies of the simple outcomes in an outcome set is 1.
8. The concept of the stability of relative frequencies as a tendency.
9. Probability as a prediction of relative frequency.
10. Equally likely outcomes.
11. The meaning of "and" and "or" as it is often used in mathematics.
12. Some thought about events that have common outcomes and $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
13. Methods of presenting graphical data.

B. Some suggestions on sections and answers to exercises.

(5.1 - 5.3 Estimated time - 1 to 2 days).

5.1 Introduction

The words "estimate", "confidence", "control", "assembly line", "sample", and "population" may be difficult for the student. The teacher should discuss the meaning of these words before the students read the text. The word "estimate" should be discussed as intelligent guessing.

5.2 - 5.3 Die-Tossing Experiment

Start with examples of events such as: the event that the outcome is less than 3 etc.; or the event that the outcome is even, and lead the students to the notion that an event is a subset of an outcome space. The definition was too abrupt.

It is important to point out that there are many possible events that can be picked out of the outcome set or outcome space for the same experiment. There are also many ways of describing any event within the outcome space. For example, in the die-tossing experiment:

$$\{1,2,3,4,5,6\}$$

$$\{\text{odd, even}\}$$

$$\{0_i < 3;\ 0_i \geq 3\}$$

$$\{0_i = 2;\ 0_i \neq 2\}$$

These are all descriptions of the outcome space itself. ("0_i" refers to any definite but unspecified simple event and i can have the values 1,2,3,4,5 or 6.)

The terms "two-headed coin", "frequency", "relative frequency", and "cumulative frequency" need to be discussed and explained.

Exercise 11, p. 251, should be stressed; perhaps expanded to a day's lesson. It might be desirable to add a column to show each student's observation. Note that relative frequencies should be expressed as a decimal, to show the convergence.

Note 1

(However, when relative frequency and probability are first introduced, it is important that the students be made aware that these can be expressed interchangeably as a decimal and as a fraction, and that they have practice in making the conversion.)

5.3 Exercises (after the die-tossing experiment)

1. (a) {1,2} (d) {2,3} (f) {1,2,3,4,5,6}
   (b) {6} (e) { } (g) {2,3,5}
   (c) {1,2,6}
2. (a) The outcome is even.
   (b) The outcome is odd.
   (c) The outcome is less than 2 greater than 5.
   (d) The outcome is a prime number. (There are other possible answers.)
3. Answers to (a), (b), and (c) will vary.
   (d) The sum should be 1 in each case.
   (e) The sums should be 1.
   (f) Because the sum of the frequencies of the outcomes is equal to the number of trials.
4. (a) 100 (b) 1 (c) Yes (d) 1
5. (a) 0 (b) 0 (c) Impossible
6. Zero

7. $\frac{5}{24} : \frac{2}{24} : \frac{5}{24}$

8. $\frac{12}{24} = \frac{1}{2}$

9. $\frac{1}{2}$

10. The relative frequency of an event is the sum of the relative frequencies of the simple outcomes in the event.

5.4 <u>The Thumbtack Experiment</u> (Estimated time - 1 day)

In the table the consecutive differences should be given in decimal form. The dots in the second graph to illustrate the stability of relative frequency should be connected by straight segments only to show the tendency from point to point.

5.5 <u>The Probability of an Event</u> (5.5 to 5.8 - 1 to 2 days)

At the end of the section, listing the properties of probabilities, express properties 1, 2, and 3 in words and relate them specifically to the summary at the end of Section 5.3. All 5 of these properties are extremely important to later work in probability and statistics. Make sure these 5 properties are clearly understood, illustrated, and tested experimentally. Interesting discussions can result from the variation between the expected and the experimental results of Section 5.4.

5.6 and 5.7 <u>A Game of Chance and Equally Probable Outcomes</u> (Which might best be done in class)

The game of chance in 5.6 should lead the students to realize that not all the events (i.e. sums) are equally likely. Section 5.7

develops the concept of equally probable outcomes. The words "fair" and "random" may not be understood and their meaning should be discussed at an intuitive level for future use. (Tetrahedron)

5.8 <u>Exercises</u>

Ex. 2 and Ex. 3 should both be stressed to show results when events are independent and when they are not independent. A comparison of these results should be made in a class discussion (as in Ex. 3 (g)).

Exercise 4 is optional: however, it is a good motivational exercise. It makes the point that each trial is independent. For example, in tossing a die, if the probability of getting a 6 on the first toss is $\frac{1}{6}$, and a 6 is obtained, the probability of getting a 6 on the second toss is <u>still</u> $\frac{1}{6}$.

<u>Answers to Exercises</u>:

1. (d) 12
   (e) 24
   (f) 36
   (g) $\frac{12}{36} = \frac{1}{3}$: $\frac{24}{36} = \frac{2}{3}$
   (h) $\frac{2}{3}$: $\frac{1}{3}$

2. (d) $\frac{15}{36}$: $\frac{15}{36}$: $\frac{30}{36}$

3. (c) Yes: (4, 4)
   (d) $\frac{6}{36}$: $\frac{5}{36}$: $\frac{1}{36}$
   (e) $\frac{10}{36}$: $\frac{11}{36}$: No
   (g) Yes

(5.9 - 5.11 -- Estimated time one day).

5.10 Counting with Trees

The point should be made that each path represents an ordered triple and each ordered triple respresents a simple outcome.

5.11 Preview

This section formalizes the results of examples 2 and 3 in Section 5.8. If necessary, recall these at this time. If 5.8 is completely understood, 5.11 can be omitted.

5.12 Exercises (Estimated time one day.)

Ex. 9, 10, 11 (together) should be stressed to show that $P(A \text{ and } B) = P(A) \cdot P(B)$ (when A and B are independent events).

Answers to Exercises:

1. (a) Greater for tetrahedron: $\frac{1}{6}$ for dies $\frac{1}{4}$ for tetrahedron.
   (b) Most students will pick the cubical die.
   (c) Most students will select tack on left.
2. (a) { (1,4), (2,4), (3,4), (4,4), (5,4), (6,4)
   (1,3), (2,3), (3,3), (4,3), (5,3), (6,3)
   (1,2), (2,2), (3,2), (4,2), (5,2), (6,2)
   (1,1), (2,1), (3,1), (4,1), (5,1), (6,1) }
   (b) { (H, 1, a) (H, 1, e) etc. for 40 triples.}
3. If $P(R) > P(L)$, you will tend to reach +3 first.
4. (a) $\frac{3}{8} : \frac{5}{8}$
   (b) $\frac{5}{7}$
5. 14

6. P (vowel) = $\frac{5}{26}$ NOTE: A, E, I, O, U are considered as vowels and Y is considered a consonant.

   P (consonant) = $\frac{21}{26}$

7. 385 NOTE: Letters must be random.

8. $\frac{7}{12}$

10. (a) Approximately 190

    (b) Approximately 384

    (c) 8

11. (a) The middle spinner

    (b) $\frac{1}{4} : \frac{1}{3} : \frac{1}{12}$ may be reasonable estimates.

13. If a 6-child family is selected at random, the probability that the 6 are all boys is $\frac{1}{64}$.

14. (a) $\frac{11}{36}$ (c) $\frac{1}{36}$

15. $\frac{1}{49}$

16. (a) (T, 3) (d) $\frac{2}{3}$ (f) $\frac{r}{n}$

    (c) $\frac{4}{12} = \frac{1}{3}$ (e) $\frac{8}{12} = \frac{2}{3}$ (g) $\frac{1}{4}$

17. (b) $\frac{1}{12}$

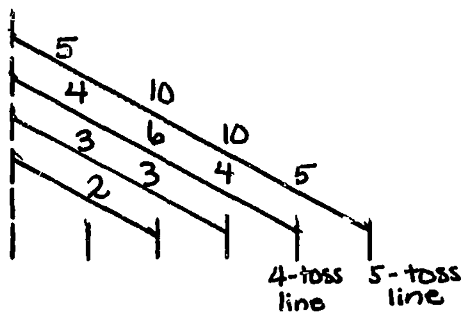
    (c) $\frac{1}{6}$

5.13 <u>Research Problems</u> (Estimated time one day.)

This section may be omitted if time is short. Challenging and worthwhile discussions can arise from these problems, particularly in an exceptionally good class. If the class has had little difficulty with this material and has enjoyed it, this section would be most fruitful.

5.13 Research Problems

1. The number of paths to each state follows the pattern of Pascal's Triangle:



Probabilities line-wise:

$\frac{1}{32}$				
$\frac{1}{16}$	$\frac{5}{32}$			
$\frac{1}{8}$	$\frac{4}{16}$	$\frac{10}{32}$		
$\frac{1}{4}$	$\frac{3}{8}$	$\frac{6}{16}$	$\frac{10}{32}$	
$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{4}{16}$	$\frac{5}{32}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

2. The probability that some two people in a group of 20 people (unselected with respect to birthdays) will have the same birthday anniversary is .41, for 30 people it is .71 and for 40 it is .89.

(Note: The full solution of this problem is beyond the tools used in this course. It might be best, therefore, to simply ask the students to guess what they think the probabilities will be. Then try the experiment in your class. Most of them will underestimate and will be greatly surprised by the above result. If they ask "why" be honest and simply state that this is an advanced problem in probability that they will encounter later.

It is sometimes desirable to indicate, by presenting problems like these, that a subject often goes far beyond the simple problems encountered at a first introduction.)

(Est. time for 5.14 - 5.16 -- 2 days)

5.14 Statistical Data and 5.15 Presenting Data in Tables

In these sections the student is introduced to statistics and arrangement of data. These sections should be done carefully as the ideas introduced here will be used again, both in this course and later ones. In particular, the meaning of the words "interval" and "frequency" should be completely understood.

Further work involving organization of data in tables, charts, and graphs will be done in Chapter 13.

The teacher might begin these two sections by presenting some statistical data that one might collect given in "raw" form. For example, the data might represent the number of cars passing a given street-corner during successive one-minute intervals, and might look like:

+, +, 1, 4, 3, 5, 3, 2, 5, 5, etc.

The point should be made that this "raw" data is not very useful. To improve the presentation of the data, we might use a frequency table, and then calculate relative frequencies and add them to the table:

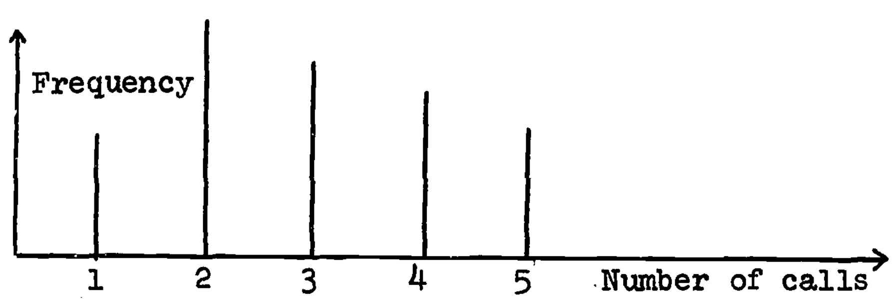
Number of Calls		Frequency	Relative Frequency
0	~~1111~~	5	.
1	~~1111~~ 1	6	.
2	.	.	.
3	.	.	.
4			
5			
6			
7			

Now have the students collect some data at home and organize it in a frequency table (including relative frequencies). Then give a set of data concerning a continuous variable (of course the words discrete and continuous are not used!) for example:

(a) A set of 50 observations of the time it takes a worker to perform the same task.

(b) The lengths of 50 leaves taken from a tree.

Next discuss grouping of the data (discrete data are usually not grouped but continuous data are). Follow this with exercises, then treat <u>bar diagrams</u>, the way discrete data are usually given.



(In this case it now has meaning to draw rectangles (see commentary for Section 5.17.))

5.16 <u>Exercises</u>

1. Answers will vary.

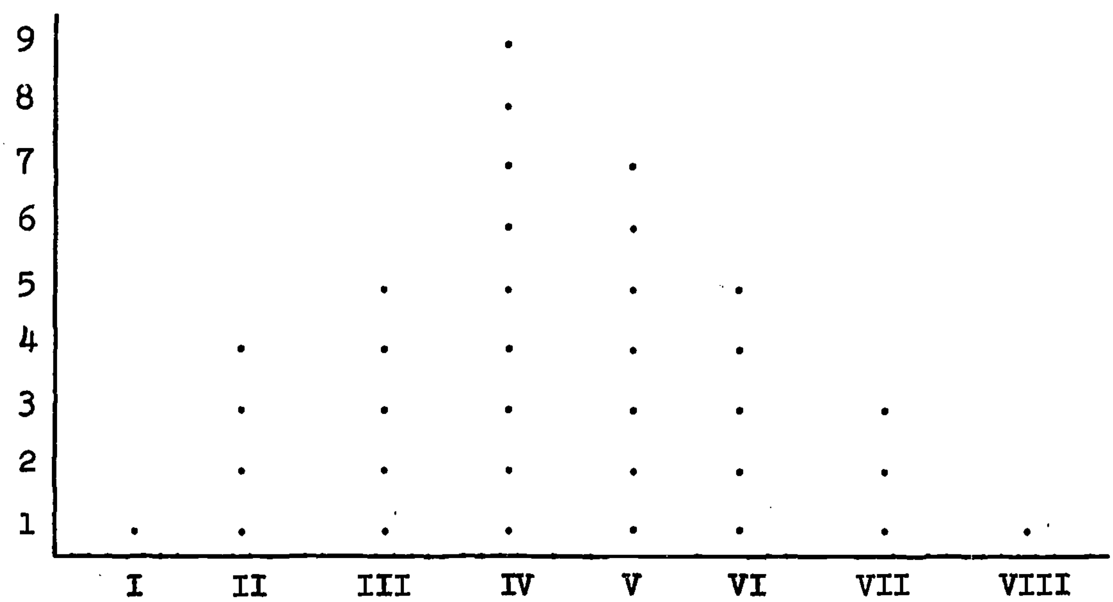
2. (a) girls - 3 boys - 3

   (b) girls - 7 boys - 6

   (c) girls - 8 boys - 8

	Interval	Tally	Frequency
3b and	I. 61 - 65	1	1
3c	II. 66 - 70	1111	4
	III. 71 - 75	~~1111~~	5
	IV. 76 - 80	~~1111~~ 1111	9
	V. 81 - 85	~~1111~~ 11	7
	VI. 86 - 90	~~1111~~	5
	VII. 91 - 95	111	3
	VIII. 96 - 100	1	1

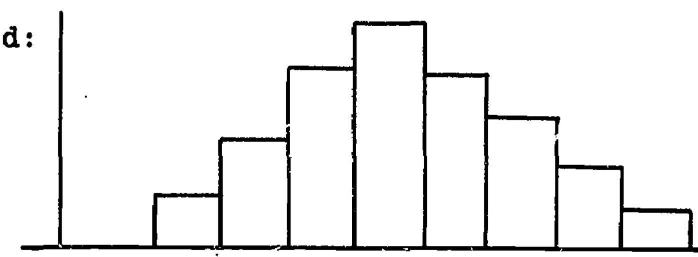
3d



(Est. time for 5.17 - 5.18 -- 2 days)

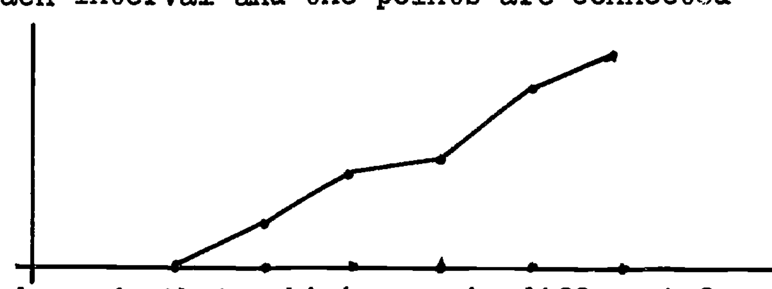
5.17 The Frequency Histogram and the Commulative Frequency Histogram

In this section the histogram and comulative frequency histogram are developed using the concepts of the preceding two sections. First, the histogram is introduced:



The area of a rectangle above a class is equal to the frequency or relative frequency of the class. A special kind of histogram is a population pyramid. For instance, the one for the U.S.A.

Next, the cumulative relative frequency is given for the right end point of each interval and the points are connected with straight lines:



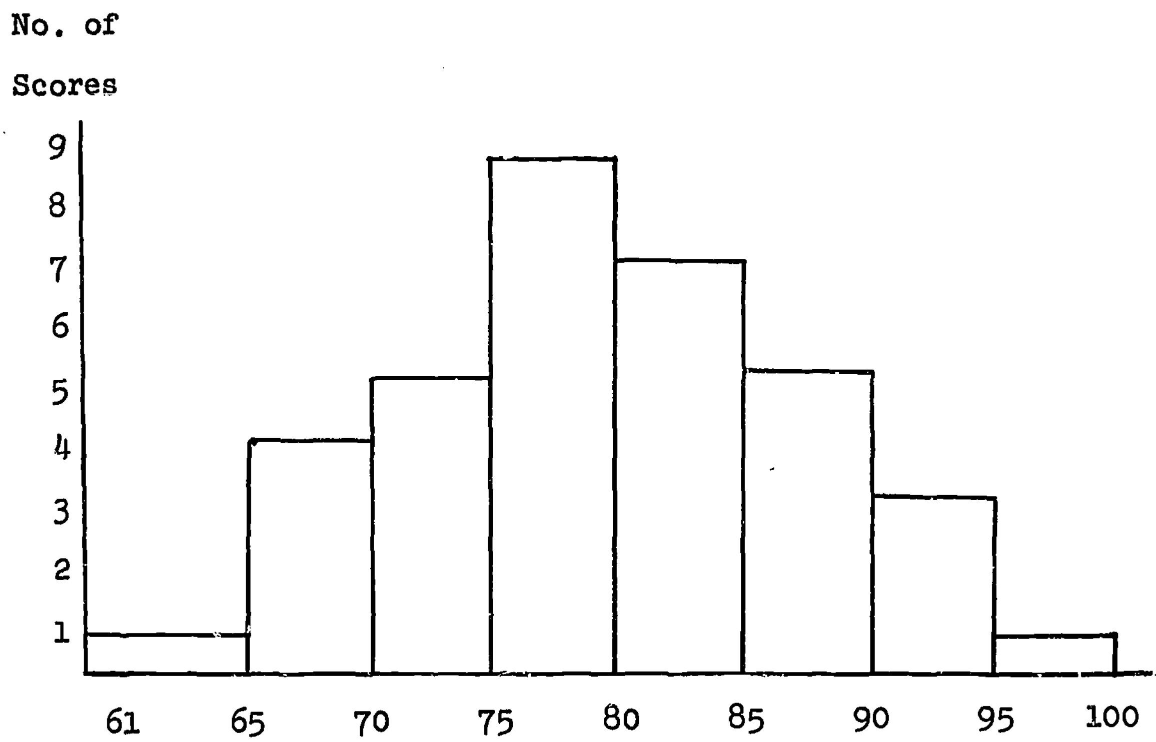
The point might be made that a histogram is different from a bar graph. In a bar graph the bars are separated by a space. Bar graphs are not used for statistical purposes (beyond simply displaying the data). For this reason bar graphs will not be mentioned in the text of this or later courses.

The extension from the cumulative frequency table (or graph) to the cumulative frequency polygon is immediate, and students will be asked to do these in the exercises.

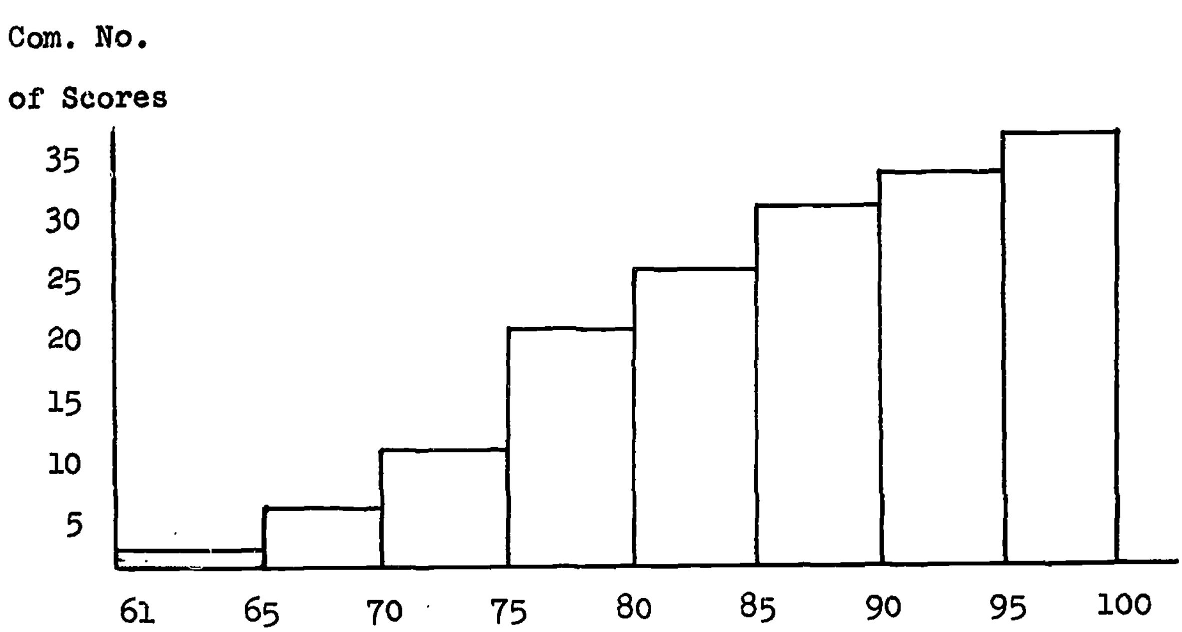
5.18 Exercises

All of these should be assigned as homework and gone over carefully, especially Example 3.

1. (a)


No. of
Scores
9
8
7
6
5
4
3
2
1
61
65
70
75
80
85
90
95
100

1. (b)


Com. No.
of Scores
35
30
25
20
15
10
5
61
65
70
75
80
85
90
95
100

3a.

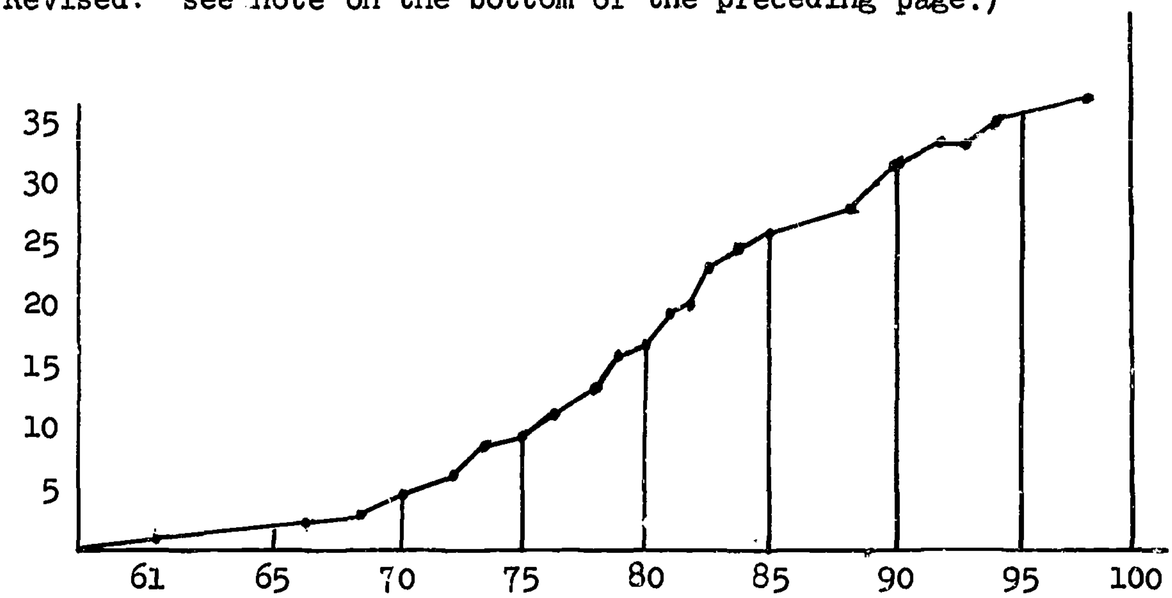
Score	Frequency	Cum. Frequency	Score	Frequency	Cum. Frequency
62	1	1	81	2	21
67	1	2	82	1	22
69	1	3	83	2	24
70	2	5	84	1	25
72	2	7	85	1	26
73	2	9	88	2	28
75	1	10	89	1	29
76	2	12	90	2	31
78	4	16	92	1	32
79	2	18	93	2	34
80	1	19	98	1	35

3b.

		*Frequency	Cum. Frequency
I.	61 - 65	1	1
II.	66 - 70	4	5
III.	71 - 75	5	10
IV.	76 - 80	9	19
V.	81 - 85	7	26
VI.	86 - 90	5	31
VII.	91 - 95	3	34
VIII.	96 - 100	1	35

*This column is repeated from Example 3, Section 5.16.

3c. (Revised: see note on the bottom of the preceding page.)



(5.19 and 5.20 - Est. time, 1 day)

5.20 Review Exercises

1. (a) {(bus, train), (bus, plane), (train, plane)} or, if the student considers order important, {(bus, train), (train, bus), (bus, plane), (plane, bus), (train,plane), (plane, train)}

   (b) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

   (c) {(a,a), (a,e), (a,i), (a,o), (a,u), (e,e), (e,i), (e,o), (e,u), (i,i), (i,o), (i,u), (o,o), (o,u), (u,u)} or you could use the set of 25 permutations. The outcome set is selected for the purpose at hand.

   (d) {(1,1), (1,2), (1,5), (1,10), (2,2), (2,5), (2,10), (5,5), (5,10), (10,10)} or the permutations or {2, 3, 4, 6, 7, 10, 11, 12, 15, 20} when outcome set is the set of possible scores.

(e) {(3 blue, 0 red) (2 blue 1 red) (1 blue, 2 red)}

(f) {(JJJ), (JJS), (JSJ), (SJJ), (JSS), (SJS), (SSJ), (SSS)} or {(J-2, S-0), (J-2, S-1), (J-1, S-2), (J-0, S-3)}

2. (b) 25 (c) $\frac{1}{25}$ (d) $\frac{4}{25}$ (e) $\frac{16}{25}$

(g) {2, 3, 4, 5, 6, 7, 8, 9, 10}

(h) $\frac{5}{25} : \frac{4}{25} : \frac{4}{24}$

(i) $\frac{4}{25} : \frac{8}{25} : \frac{5}{25}$

4. (a) {(1,H), (2,H), (3,H), (4,H), (5,H), (6,H), (1,T), (2,T), (3,T), (4,T), (5,T), (6,T)}

(c) $\frac{1}{6}$ (d) $\frac{5}{6}$ (e) 1 (f) 1 (g) $\frac{1}{6}$ (h) No

(i) $\frac{1}{12}$ (j) Yes

8. (a) $\frac{1}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

9. 12 ways

10. 6 ways, $\frac{2}{3}$

D. Chapter Test on Probability and Statistics

I. Answer the following questions based on the information in the table:

Record of a Baseball Player for 25 Times at Bat

Event	Frequency
Made a hit (event H)	7
Made an out (event O)	10
Walked (event W)	3
Made first on an error (event E)	5
Times at Bat	25

144

1) What is the relative frequency of event:

   (a) H (b) O (c) W (d) E?

2) What is the sum of the relative frequencies of H, O, W, E?

3) What is the relative frequency of

   (a) H or O (b) W or E (c) H or W (d) O or E?

4) What is the relative frequency of H and O?

5) What is the relative frequency of the event that the player did not get a hit?

6) What is the sum of the relative frequencies of H and (not H)?

7) Express the relative frequency of H as a decimal to the nearest thousandth.

8) What is the relative frequency of the event that every time the player went to bat he made a hit or made an out or walked or made first on an error?

9) (a) What events are included in not-W? Not-E?

   (b) What events are included in both not-E and not-W?

10) What is the probability of not-W or not-E?

II. True-False

(a) The probability of an event is never greater than 1.

(b) The probability of an event is sometimes greater than .99.

(c) The probability of an event cannot be an odd number.

(d) If a coin is "fair", the probability of heads is less than $\frac{1}{2}$.

(e) The relative frequency of an event is a prediction of the probability of the event.

(f) As the number of trials increases in an experiment, the relative frequency of an event tends to "stabilize".

(g) If you select a vowel at random, the probability of selecting "e" is $\frac{1}{4}$.

(h) The sum of the relative frequencies of the simple outcomes in an experiment is always 1.

(i) If P(E) = R, then P(not-E) = R - 1.

(j) Every experiment has exactly one outcome set.

III. Tabulate an outcome set for the following experiments:

(a) Toss a penny and a nickel and observe the pairs of outcomes (penny first).

(b) Toss a pair of dice and observe the sum of the outcomes.

(c) Toss 3 thumbtacks, each with a different color, and observe ordered triples as outcomes.

IV. Organize the data in the table below:

Heights of Thirty-Six Seventh Grade Students

(in inches)

56, 55, 58, 59, 57, 61, 54, 56, 59, 60
57, 59, 56, 59, 60, 55, 57, 56, 58, 50
60, 52, 57, 60, 59, 54, 64, 62, 54, 59
62, 61, 53, 58, 56, 62

Interval	Frequency	Relative Frequency	Cumulative Relative Frequency
49.5 - 53.5			
53.5 - 57.5			
57.5 - 61.5			
61.5 - 65.5			

Course I Chapter 6

Multiplication of Integers

Commentary for Teachers

(6 - 9 days)

This chapter has 3 main objectives:

1. To extend the definition of multiplication from W to Z;
2. To give computational practice with the new definition and develop some simple yet useful properties of (Z,+,·);
3. To show that (Z,·) can be interpreted as a set of dilation mappings of a line with the operation of composition.

Recalling Chapter 2, it is important to emphasize the fact that there are many possible ways to define "·" on Z. All that need be given is some rule which assigns to each ordered pair of integers (x,y) a unique integer which is called the product of x and y. For example:

$$x \cdot y = 0 \qquad \forall_{x,y}$$
$$x \cdot y = x \qquad \forall_{x,y}$$
$$x \cdot y = y \qquad \forall_{x,y}$$

are all bona fide operations on Z and could be chosen as the definition of multiplication. However, 1 it is only reasonable to expect that "·" for Z should be an extension of "·" for W; that is, if a, b $\geq$ 0, a · b is the whole number product. Furthermore, 2 we would hope that the useful properties of (W,+,·) are also properties of (Z,+,·) and 3 that our definition of "·" makes sense wherever multiplication of integers is the appropriate model of a physical situation.

6.1 Operational Systems (W,·) and (Z,·) (and 6.2: 1 day)

To accomplish 1 and 2 above, we review the definition, properties, and computation of $(W,\cdot)$. Particular emphasis shall be placed on the distributive property since it is central in later sections. A successful class procedure with this section has been a closed book review: "Name and give an illustration of one property of $(W,\cdot)$,..."

6.2 Exercises

1. (a) Identity or Commutative
   (b) Identity
   (c) Identity
   (d) Associative
   (e) Commutative
   (f) Associative
2. (a) $(43 \times 28) \times 76 = 76 \times (43 \times 28)$ C
   $= (76 \times 43) \times 28$ A
   (b) $87 \times (43 \times 76) = 87 \times (76 \times 43)$ C
   $= (87 \times 76) \times 43$ A
   (c) $8 \times (69 \times 25) = (69 \times 25) \times 8$ C
   $= 69 \times (25 \times 8)$ A

   (Other orders are possible.)
3. For all a and b in W; $a + b = b + a$.
4. For all a, b, and c in W; $a + (b + c) = (a + b) + c$.
5. 0
6. 0; 1; 0; 1; 0; 0; 1; none $0 - a \neq a'$.
7. 504; 504; 504; 504; 4700; 2162 (note use of properties is in aid here)
8. Variations on pattern of the example done in text
   Product is the same

10. 189; 189; 2300; 3800; 470; 430; 130; 65

11. No, $13 \neq 63$

12. No, see 11 for counterexample

6.3 Multiplication for Z (Sections 6.3, 6.4, 6.5, 6.6 -- 2-3 days)

Although to define · for all ordered pairs in Z we must technically consider an infinite number of cases, it seems highly reasonable to assume at once $|r| \cdot |s| = |r \cdot s|$, thus making signs the main object to be questioned. We will not consider "a negative and b positive" only because we want multiplication to be commutative. This condition need not be satisfied since there are non-commutative operations. (We will not be too happy with any definition for multiplication that somehow turns out non-commutative.) Since multiplication of positive integers is actually multiplication of whole numbers, the rule "positive times positive is positive" is forced on us.

6.4 Multiplication of a Positive Integer and a Negative Integer

You might want to suggest--before looking at the pattern--the following definition of · for Z:

$$a \circ b = \begin{cases} a \cdot b, & a,b \text{ in W} \\ 0, & a,b \text{ negative integers (either or both)} \end{cases}$$

is associative, commutative, and has the multiplication property of zero, as you can show with numerical examples:

$$3\ (-4) = 0 = (-4)\ 3$$

$$7\ (2\ (-5)) = 7\ 0 = 0 = 14\ (-5) = (7\ 2)\ (-5)$$

However, cancellation, multiplication property of 1, and distributivity fail--which you might challenge the students to prove by exhibiting appropriate counter-examples. For example, after defining and illustrating ask, "Will this be okay for a definition of multiplication in Z? Will the desired properties hold?"

The pattern shown in this section then <u>suggests</u> the definition "positive times negative is negative". It remains to be seen if this rule of assignment meets the expectations we have for it. Some simple physical situations of multiple loss are usually convincing. Section 6.7 will be more effective if you hold off on the distributivity argument for now.

6.5 <u>The Product of Two Negative Numbers</u>

Students will question the adequacy of this pattern as a basis for "negative times negative is positive". This weakness must be admitted, but it is certainly fair to say, "Let us give it a try and see if it meets the criteria set up in 6.3 for multiplication in Z." The next set of exercises does just this.

In Chapter 4, -a was defined as the opposite of a such that a + -a = 0, and -(-a) was defined as the opposite of the opposite of a, or a itself. It might be wise at this point to demonstrate (by reading aloud) this terminology. Numerical examples (of the following general case) should be done first, followed by this generalization:

If $a, b > 0$ then (1) $ab > 0$ and (2) $-(ab) = a(-b) < 0$.

Therefore

$$
\begin{aligned}
(-a)(-b) &= -(a(-b)) \\
&= -(-(ab)) \\
&= ab,
\end{aligned}
$$

and $(-a)(-b) = ab > 0$.

Another approach for mature students, since the proof of multiplication of the integers is developed in Course II in the chapter on Groups is the following: Demonstrate the following:

$-(ab) + ab = 0$	Definition of Inverses
$a(-b) + ab = a(-b + b)$	Distributive Property of $\cdot$ over +.
$= a \cdot 0$	Definition of inverses
$= 0$	Multiplication of 0

hence since $-(ab) + ab = 0$ and $a(-b) + ab = 0$

then	$-(ab) + ab = a(-b) + ab$	Replacement
	$-(ab) = a(-b)$	Right Cancellation

$ab + -(ab) = 0$ to show $(-a)(-b) = ab$ can be shown in the same way.

6.6 Exercises

1. -540: -1221: -540: 112: 112: -1221. Note that this seems to indicate that multiplication as defined in 6.5 is commutative. So far, so good!
2. 470: -4300: -300: -300: 470: -4300. This suggests associativity.

3. 170: 170: 370: -6700: -6700: 370: 4600. This suggests distributivity. (Watch out for (g), though. It is not distributivity since it is + over x.)
4. -: +: -: +: -. Note -- You might want to ask the class for a generalization on the sign of a product of n factors.

6.7 Multiplication of Integers through Distributivity (and 6.8 1 - 2 days)

It might be wise to review the distributive property and the cancellation laws of (W,+,·) before discussing this section. Also, more examples of the Replacement Assumption including examples such as (-a + a) = 0 or (7 + -7 = 0), (4 + -6) = -2. Have the youngsters give examples. This will be helpful in handling proofs in Course II in the Chapter on Groups.

Here is where you deliver the coup de grace to doubting Thomases. Negative times negative must be positive or the distributive property won't hold. As a further proof you might show what happens if we define negative times negative equals negative:

1) Assume (-5)(-5) = -25
2) We already have defined (-5)(5) = -25
3) Therefore, (-5)(-5) = (-5)(5)
4) But this implies that -5 = 5 by cancellation.

Thus either distributivity or cancellation can be used to infer

(-)(-) = (+).

6.8 Exercises

1. -70: -70: 48: 48: 156: 156: -418: -418. Note that this exercise suggests $(-r)(s) = (r)(-s)$, $r \cdot s = (-r)(-s)$, and $-(r \cdot s) = (-r)(s)$. Draw these generalizations from the class. Point out that in the generalizations, $-r$ need not represent a negative number. It is the opposite of r, whatever r is. Thus $-r \cdot s = r \cdot (-s)$ also implies that $(-(-3))(-5) = (-3)(5)$.
2. -90: -90: 210: 210: 210: -90: -90: 210: 210: Note that $a(b - c) = ab - ac = a(b + (-c))$ is suggested here. You may or may not want to prove it using 1 as follows:

$$\begin{aligned} a(b - c) &= a(b + (-c)) \\ &= ab + a(-c) \\ &= ab + (-(ac)) \\ &= ab - ac \end{aligned}$$

3. -11: 3: 3: -8: 64: 64: 252: 252: 252: 16: -64: 112: -13: 13: 13: -19: 65: -33. Note that again many generalizations are illustrated here.
4. $\{\pm 2\}$: $\{\pm 2\}$: $\emptyset$: $\{0, \pm 1\}$: $\{0, \pm 1\}$ $\{1, -5\}$: $\{0, 6\}$: $\{-2, 3\}$: $\{0, -1, -2, -3\}$: $\{0, 1, 2, 3\}$: $\{0, -3\}$.
6. (a) $3(0 + 0) = 3 \cdot 0$

$$3 \cdot 0 + 3 \cdot 0 = 3 \cdot 0 + 0$$
$$3 \cdot 0 = 0$$

(b) $3(2 + (-2)) = 3 \cdot 0$

$$3 \cdot 2 + 3(-2) = 0$$
$$6 + 3(-2) = 0$$
$$3(-2) = -6$$

(c) $-3(2 + (-2)) = -3 \cdot 0$

$(-3)(2) + (-3)(-2) = 0$

$-6 + (-3)(-2) = 0$

$(-3)(-2) = -(-6)$

$(-3)(-2) = 6$

7. If $a \cdot b = 0$ and neither a nor b are zero then we would have either $(+)(+) = (+)$, $(-)(-) = (+)$, or $(+)(-) = (-)$ neither of which involves a zero product.

8. T: F: T: T: T: T.

9. (a) $2x = -12$: $\{-6\}$

(b) $2x = x - 3$: $\{-3\}$

(c) $4 < x^2 < 20$: $\{\pm 3, \pm 4\}$

(d) $x + (x + 1) = -7$: $\{-4\}$

(e) $x(x + 1) = 42$: $\{6, -7\}$

10. (a) $2x + 3x = (2 + 3)x$
$= 5x$

(b) $2x + x = (2x + 1x)$
$= (2 + 1)x$
$= 3x$

(c) $5x - 2x = 5x + (-2)x$
$= (5 + (-2))x$
$= 3x$

(d) $2x - 5x = 2x + (-5)x$
$= (2 + (-5))x$
$= -3x$

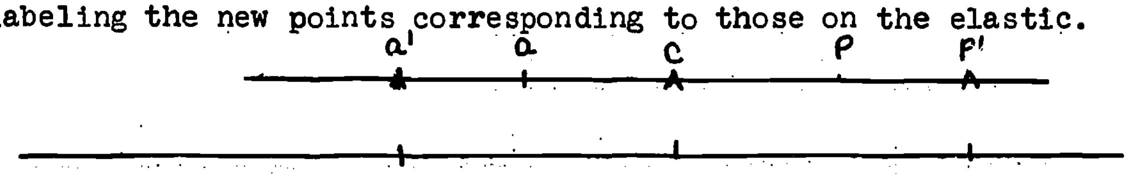
(e) $5x - 4x = 5x + (-4)x$
$= (5 + (-4))x$
$= x$

(f) $x - 5x = 1x + (-5)x$
$= (1 + (-5))x$
$= -4x$

11. $\{4\}$: $\{-4\}$: $\{20\}$: $\{4\}$: $\{-20\}$: $\{-20\}$: $\{-20\}$:
$\{0, \pm 1, \pm 2\}$: $\{0, \pm 1, \pm 2, -3\}$.

6.9 Dilations and Multiplication of Integers (and 6.10: 1 - 2 days)

This section develops a mathematical system (D',x) which is isomorphic to (Z,·). The set D' is a set of mappings--the elements of D' are themselves mappings. The operation "x" is composition of mappings. Intuitively, the mapping 2' takes an elastic line and stretches it to twice its length. The midpoint of the line is the only fixed point. This could be demonstrated by labeling various points on a piece of elastic, fixing their original locations with respect to points on the chalkboard, and then stretching the elastic and labeling the new points corresponding to those on the elastic.



Care should be taken in this section with the terms reflection in C, and symmetry in Point C as these terms are not defined here (they will be in chapters 9 and 10.) Students should understand these on an intuitive level.

In section 6.10, exercise 5 should be developed from an intuitive approach. The "more rigorous" proof may be demonstrated on the board after the students have volunteered their own solutions.

6.10 Exercises

1. 42: -42: 7: 0
2. -42: 42: -7: 0

3. -6: 6: 0: -60: 60

4. 42': -42': -42': -42': 225': 625': 1225': 2025': -24': -24': -1700': -1700'.

*5. (a) $|r' \times s'| = |r'|\cdot|s'|$ $= |s'|\cdot|r'|$ $= |s' \times r'|$ Also the number of reflections in r' x s' is the same as the number in s' x r'. Therefore they are equal mappings.

(b) $|(r' \times s') \times t'| = |(r' \times s')|\cdot|t'|$ $= (|r'|\cdot|s'|)\cdot|t'|$ $= |r'|\cdot(|s'|\cdot|t'|)$ $= |r'|\cdot|s' \times t'|$ $= |r' \times (s' \times t')|$ Again, same number of reflections in both.

In (a) and (b) the manipulations are possible because magnitudes of dilations are whole numbers and obey those properties.

(c) The comment about (a) and (b) applies here too.

$$|1' \times r'| = |1'|\cdot|r'| = 1\cdot|r'| = |r'|$$

6. 0': A ⟶ C for all A, where C is the center of the dilation. 0' is a mapping.

7. Yes.

8. Yes. But only for 1'. Every point is mapped to itself. -1 only has 1 fixed point. The length is equal to the original segment but it reflects through the fixed point.

6.11 and 6.12 Summary and Review (1 day)

These two sections may be assigned as homework and discussed the next day in class.

Allow 1 day for a chapter test. Quizzes may be given during class.

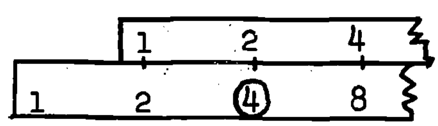
6.12 Review Exercises

1. 2: 16: -2: 63: -63: 144: 207: 207: 376: -390: -580: -95.

2. {$\pm$3}: {$\pm$1}: {-4} {0,$\pm$1,$\pm$2}: Ø: {0,-2}: { } {1,-3}: {0,+1,+2}: {0,-1}.

4. T: T: F: F: T: F: T: F.

5. (a) 1



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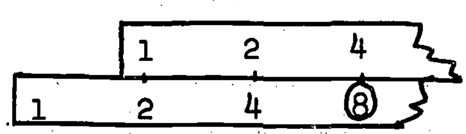

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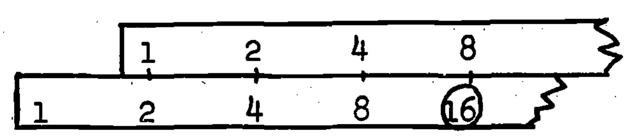

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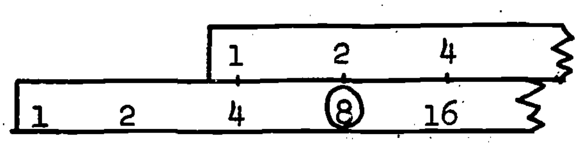

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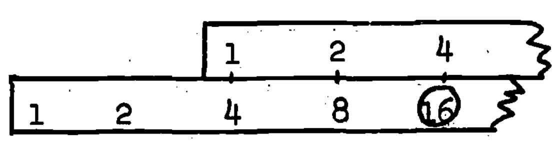
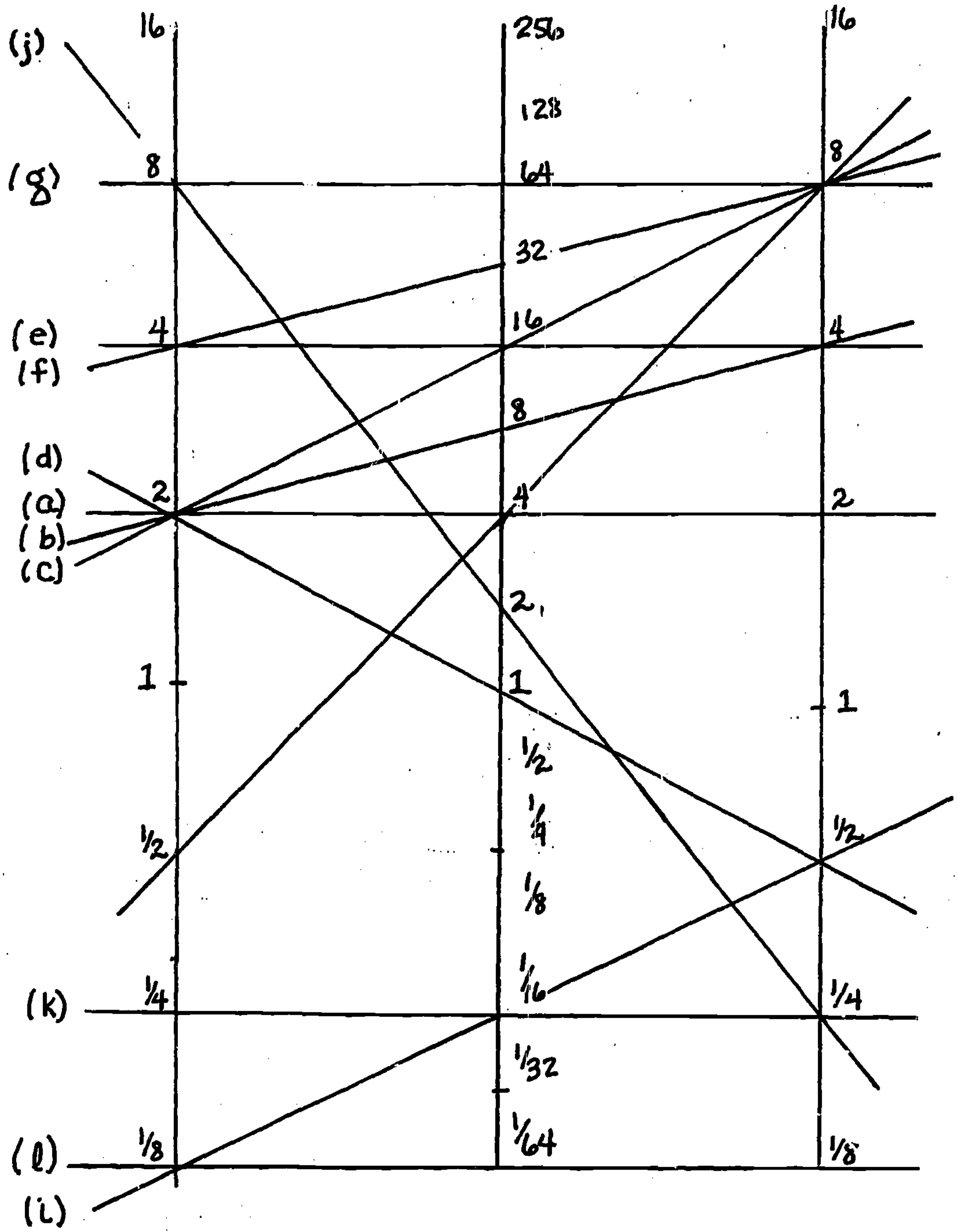


(b) No. The midpoint should be less. In fact, the square root of the product. Notice that the midpoint for points 2 and 8 is 4 and not 5.


(j)
(g)
(e)
(f)
(d)
(a)
(b)
(c)
(k)
(l)
(i)
16
8
4
2
1
1/2
1/4
1/8
256
128
64
32
16
8
4
2
1
1/2
1/4
1/8
1/16
1/32
1/64
16
8
4
2
1
1/2
1/4
1/8

7. (a) The hot rod is moving to the left at a speed of 4 fps.
   (b) 12 feet to the right of 0. -3 seconds may be interpreted as 3 seconds ago.
   (c) 4×-2 locates the hot rod where it was 2 seconds ago, at -8. -4×2 locates the hot rod where it will be in 2 seconds, but it has been moving to the left, so that it will be -8. -4×-2 locates the hot rod where it was 2 seconds ago, but it has been moving to the left, so that it was at 8.

Quiz for Chapter 6

I. Compute:

(a) $-16 \cdot -14$

(b) $26 \cdot -24$

(c) $-7\ (8 - (-2))$

(d) $(-7 \cdot 23) + (-3 \cdot 23)$

(e) $(17 \cdot (-29)) - (7 \cdot (-29))$

II. Find the solution set from the set of integers.

(a) $2x + 7 = 3$

(b) $2y + 7 = -3$

(c) $-2x - 7 = -3$

(d) $7 - s = 10$

(e) $8 - s = -1$

(f) $|3r| = 6$

(g) $s^2 = 9$

(h) $(x + 1)\ (x - 1) = 0$

(i) $|3s - 1| = 2$

(j) $|2t + 1| < 3$

III. Answer Sometimes (S), Always (A), or Never (N), whichever fits. Use Sometimes only when the sentence may be true on some occasions and false on others. r, s, and t are integers.

(a) $rs = sr$

(b) $-(rs) = (-r \cdot s)$

(c) $|rs| = |-r| \cdot |-s|$

(d) $|-2r| = 2 \cdot |r|$

(e) $(rs)t < r(st)$

(f) $r(s + 1) = rs + r$

(g) $2r > r$

(h) $(-1)r = -r$

(i) $r^2 < 0$

(j) $r^2 < r$

Key to Quiz

I. (a) 224 (b) -624 (c) -70 (d) -230 (e) -290

II. (a) $\{-2\}$

(b) $\{-5\}$

(c) $\{-2\}$

(d) $\{-3\}$

(e) $\{9\}$

(f) $\{2,-2\}$

(g) $\{3,-3\}$

(h) $\{1,-1\}$

(i) $\{1\}$

(j) $\{0,1,-1\}$

III. (a) A

(b) A

(c) A

(d) A

(e) N

(f) A

(g) S

(h) A

(i) N

(j) N

TEACHERS' COMMENTARY

Chapter 7 Lattice Points and the Plane

Approximate Time for Chapter: 13 - 16 days

Chapter 7 has three major objectives:

(1) To develop the idea of assigning pairs of integers to lattice points in a plane in such a way that the result is a two-dimensional coordinate system.

(2) To apply the idea of a coordinate system to the study of open sentences in two variables.

(3) To apply the idea of a coordinate system to the study of such mappings as translations, dilations and others.

The relationship between the lattice points and the set of ordered integers will form the basis eventually for the plane coordinate system. But the material of this chapter, even though limited, has an interest of its own.

Emphasize the fact that when the expression "two lines" is used, it means two different lines. However we often meet situations when it is necessary to use two different names for the same line until we are sure that they are, in fact, the same line. For example you may wish to discuss line r and line s and still leave open the possibility that they are the same line as in an indirect proof where you assume the line r is not equal to line s and show that this leads to a concentration.

You will notice that our coordinate system uses oblique axes. This is done to avoid the impression left with many students that axes in a plane coordinate system are necessarily perpendicular to each other. The perpendicularity relation between lines becomes important in rectangular coordinate systems, the systems appropriate to comparison of distance on different lines. Our interest here is in the affine plane where such comparisons are not made.

You may have to remind students that we are working with a lattice, a set of points that are separated from each other. There are empty spaces in a lattice just as there are numbers missing in the set of integers. If this confuses students they may be assured that the empty spaces and missing numbers will be filled in subsequent developments. For this chapter we are interested in mappings over the points in a lattice and the set of ordered integers.

Students should be supplied with graph paper, rulers, and colored pencils to facilitate the work of this chapter. Teachers may find an overhead projector valuable in teaching this material.

7.1 Lattice Points and Ordered Pairs of Integers (1 to 2 days)

The purpose of this section is to introduce lattice points and ordered pairs of integers. The motivation used to introduce lattice points is the geodesic dome. However, the dome does not exist in a plane and has to be flattened first. Unfortunately a certain amount of stretching takes place in this flattening process which destroys the size and shape of some triangles. Nevertheless the interest that many have in the dome can be used as a starting point. It can suggest the lattice in figure 7, where all triangles are equilateral and where points are evenly spaced.

Ample practice should be given in (a) finding the coordinates of a given point in the lattice and (b) finding the point for a given pair of coordinates. It might be worthwhile to mention that the lattice system is divided by the axes into four regions each called a quadrant.

Exercises 1 and 2 may be done in class. Exercises 4 and 6 are good reasoning questions.

A good exercise for reinforcing the students' skill in locating coordinate points is the following:

<u>A Fun Exercise in The Use of Lattice Points</u>

Directions: Locate each of the following points, starting with A, and join each successive point with a straight line segment. For example: (1) locate points A and B and draw a line segment between them, (2) locate point C and draw a line segment from B to C, (3) locate point D and draw a line segment from C to D, etc.

Note: For best results, use a rectangular coordinate system.

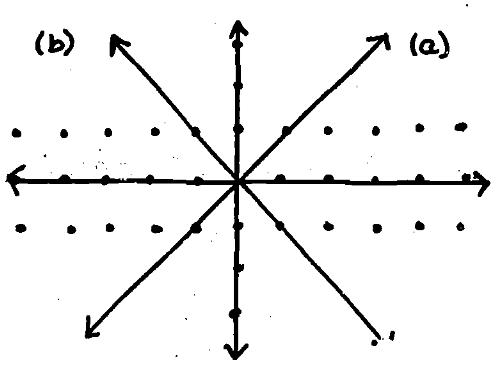
A:	(2,13)	E:	(30,6)	I:	(0,-10)
B:	(17,8)	F:	(20,3)	J:	(-11,-17)
C:	(26,8)	G:	(12,4)	K:	(-14,-17)
D:	(36,14)	H:	(1,-4)	L:	(-13,-15)

M: (-11,-15)	R: (-4,-7)	W: (-21,4)
N: (-2,-9)	S: (-6,-5)	X: (-22,5)
O: (-2,0)	T: (-7,0)	Y: (-20,8)
P: (-5,0)	U: (-10,0)	Z: (-14,14)
Q: (-5,-4)	V: (-18,6)	

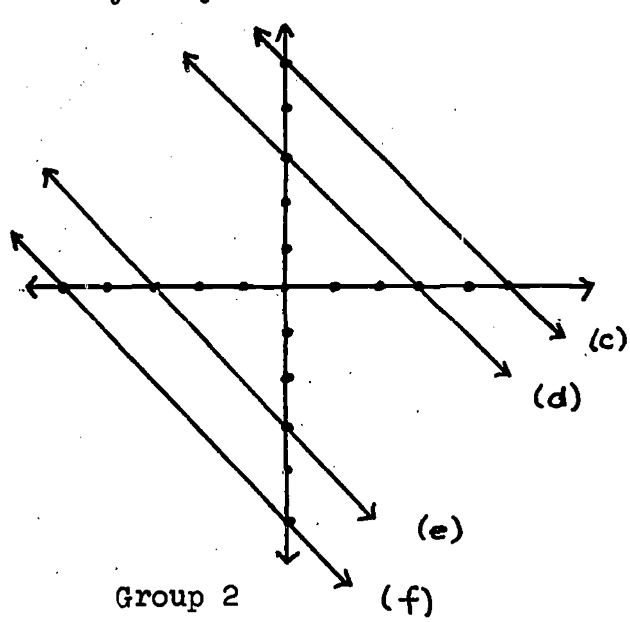
AA: (-15,11)   BB: (-17,10)   CC: (-11,8)   DD: (2,13)

7.2 Solutions to Exercises

1. a. A(3,1), B(-1,1), C(-2,-1), D(1,-2), E(2,0).
   b. A(1,1), B(1,-2), C(-1,-1), D(-2,2), E(0,2).
   c. A(-1,2), B(-2,-2), C(2,0), D(0,-1), E(2,2).
   d. Note axes; A(2,4), B(-1,4), C(0,-2), D(1,-3), E(-3,0).
2. yes; yes; no, $2\frac{1}{2} \notin Z$.
3. to be shown on lattice paper of student
4. (-1,0), (0,0), (1,0), (3,0), (4,0), (5,0).
5. (0,-4), (0,-3), (0,-1), (0,0).
6. In each the angle between axes may vary.



Group 1



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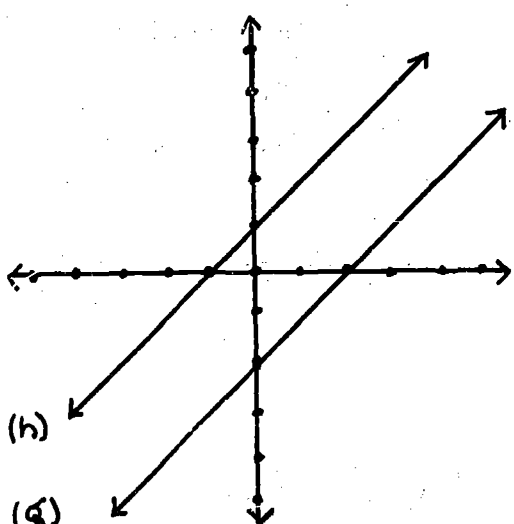
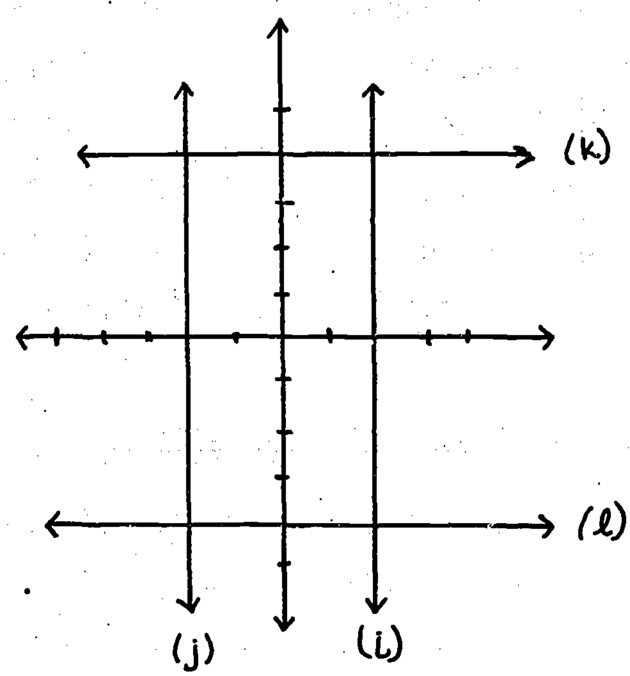


Group 3



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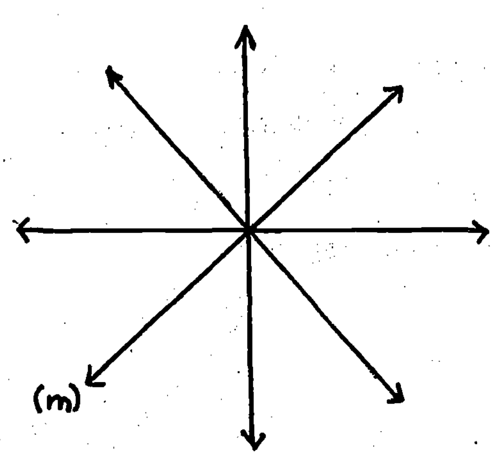
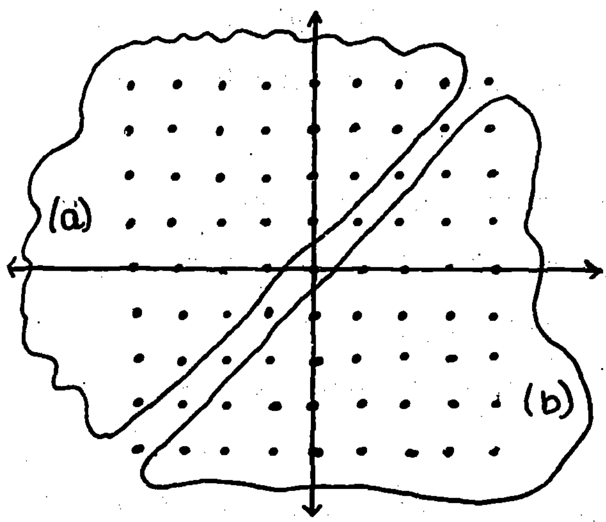


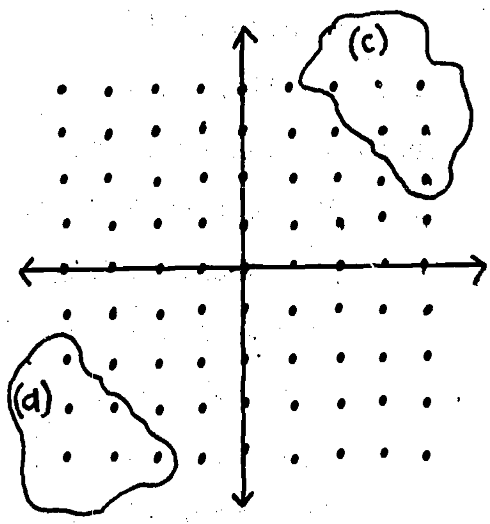
Group 5

Note: (m) includes 2 lines.

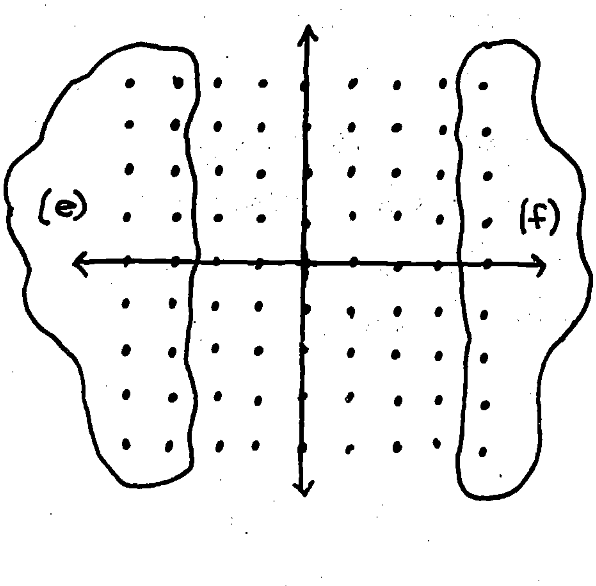
7.(a),(b)



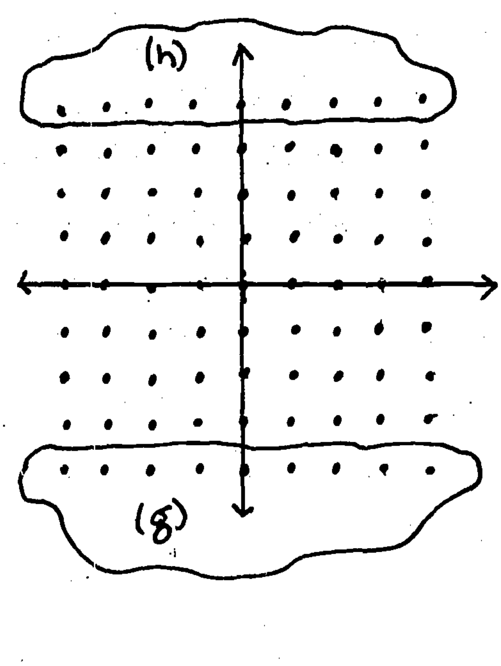
(c),(d)



(e),(f)



(g),(h)



7.3 Conditions on Z x Z and their Graphs (2 - 3 days)

The purpose of this section is to relate the Cartesian set Z x Z to conditions in two variables. Many important ideas are included in this section:

(1) The set of lattice points associated with the ordered pairs satisfying a given condition is called the graph of the solution set of the condition.

(2) The translation of verbal sentences into mathematical symbols.

(3) The translation of a mathematical sentence into a verbal statement.

No formal method of finding the ordered pairs satisfying a condition is given; the approach is primarily intuitive.

If a graph consists of a set of collinear points, its structure can be indicated by drawing a "line" through the points. However, it is more accurate when working with conditions on Z x Z, to show the graph by enclosing the points as a set or individually as the case may require.

The teacher should be selective in assigning exercises in this section. Exercises 1, 3, 6, 7 and 11 may be classwork problems. Some of the remaining exercises may be assigned as homework; problems 12 and 13 are especially good.

7.4 Solutions to Exercises

1.	(a)	$x = y$	(e)	$x + y = -5$	(i)	$x = -2$
	(b)	$x = -y$	(f)	$x - y = 2$	(j)	$y = 4$
	(c)	$x + y = 3$	(g)	$x = y = -1$	(k)	$\|x\| = \|y\|$
	(d)	$x + y = -3$	(h)	$x = 2$		

2. These will be the same as the graphs in 7.2 Exercise 6. The difference is that the students are now graphing the solution sets of open sentences rather than just finding points that satisfy certain conditions.

3. (a) Six more than the first coordinate is equal to the second coordinate.
   (b) The difference of the second coordinate and first coordinate is three.
   (c) The second coordinate is equal to the absolute value of the first coordinate.
   (d) The second coordinate is two less than the first coordinate.
   (e) The second coordinate is the absolute value of three less than the first coordinate.
   (f) The first coordinate is seven.
   (g) The second coordinate is one.

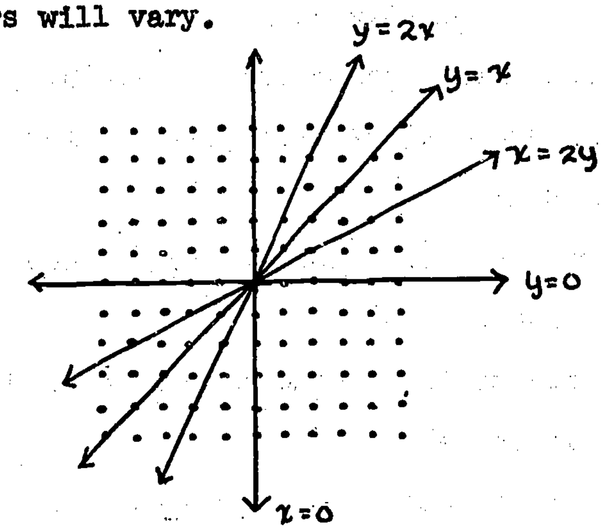
4. (a) $x < y$ (b) $x > y$ (c) $x + y > 5$ (d) $x + y < -5$
   (e) $x < -2$ (f) $x > 3$ (g) $y < -4$ (h) $y > 3$

5. (a) $y = 2x$ (b) $x = 2y$ (c) $y = 3x$ (d) $x = 3y$

6. (a) The second coordinate is five times the first coordinate.
   (b) The first coordinate is five times the second coordinate.
   (c) The second coordinate is the square of the first coordinate.
   (d) The second coordinate is zero.
   (e) The second coordinate is less than zero.
   (f) The first coordinate is greater than zero.
   (g) The product of the coordinates is six.
   (h) Two times the first coordinate is three times the second coordinate.

7. Answers will vary.

8.



9. The origin; y = 0; x = 0; y = x, y = 2x, and x = 2y.

10. (a) $y = 2x + 1$ (b) $x = 3y - 5$

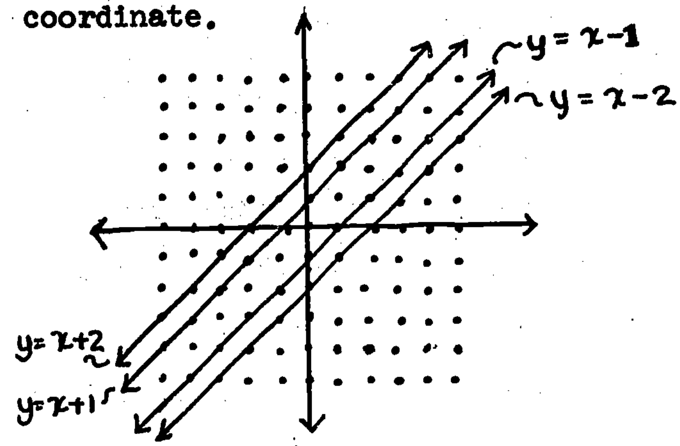
11. (a) The second coordinate is one more than the first coordinate.

(b) The second coordinate is one less than the first coordinate.

(c) The second coordinate is two more than the first coordinate.

(d) The second coordinate is two less than the first coordinate.

12.



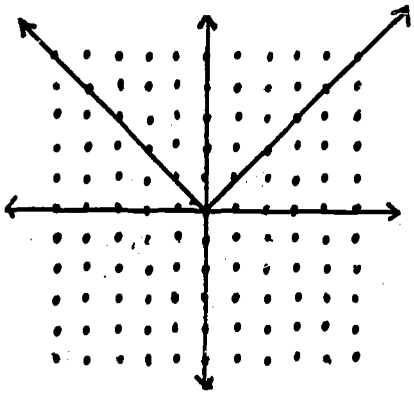
13. They were parallel: (0, 1), (0, -1), (0, 2), (0, -2).

7.5 Intersection and Unions of Solution Sets (1 - 2 days)

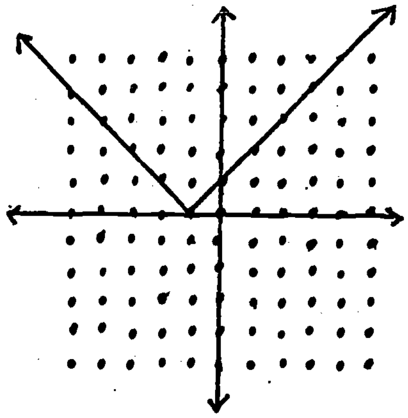
The purpose of this section is to introduce compound conditions in Z x Z. For the first time the concept and symbolism for union and interesection are used. Previous knowledge of these ideas is assumed; however, a review may be necessary. To differentiate among graphs of different conditions one can use different Geometric Figures to enclose the points. This makes it easy to visualize intersections and unions of graphs and the related solution sets. The exercises on inequalities are important in providing a background for graphing absolute value conditions.

7.6 Solutions to Exercises

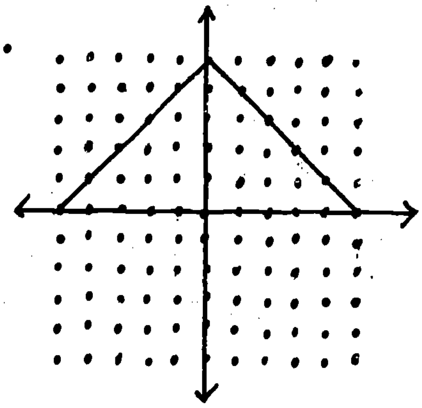
1.



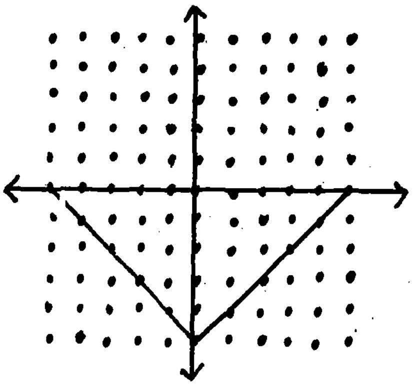
2.



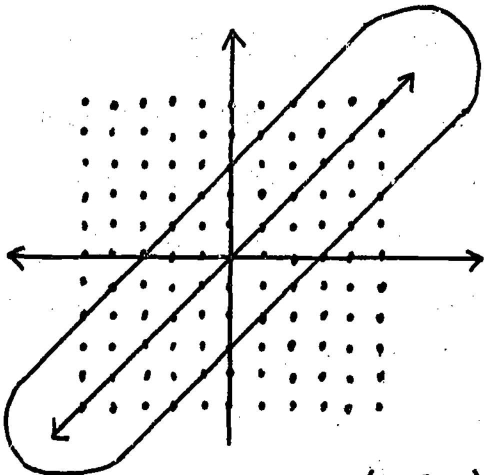
3.



4.



5.



7.7 Absolute Value Conditions (1 day)

The purpose of this section is to investigate absolute value conditions. Recall that we defined the absolute value of an integer a as max (-a, a). Now the definition of absolute value is broken down into cases. These cases correspond to certain regions of a plane. For example, consider the condition $y = |x|$:

Case I: $x > 0$
Case II: $x < 0$

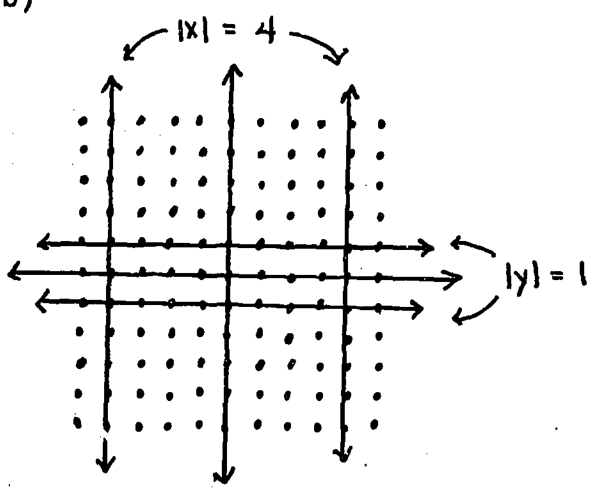
If you examine the case where $x > 0$, the corresponding region is the half-plane on the side of the y-axis where x is positive; in the same way, if you examine the case where $x < 0$, the corresponding region is the half-plane on the side of the y-axis where x is negative. Therefore, we consider the graph of an absolute value condition in parts, according to whether the expression within the absolute value sign is positive or negative.

Exercise 1 may be a class assignment, a short review exercise. Exercises 6 and 7 may be omitted; however, many teachers have found success with problem 7. The remaining exercises may be assigned as homework.

7.8 Solutions to Exercises

1. (a) 7 (b) 15 (c) 0 (d) 1 (e) 999

2. (a), (b)



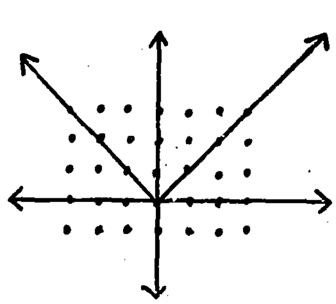
(c) {(4,1), (4,-1), (-4,1), (-4,-1)}

(d) The 4 lines

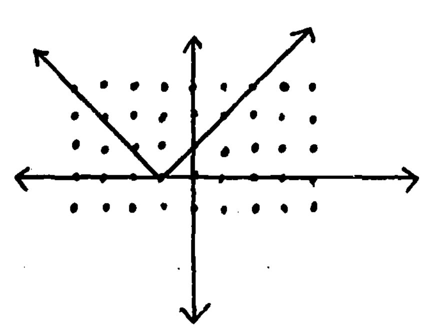
(e) (c) is determined by the intersection of (a) and (b).

(c) is determined by the union of (a) and (b).

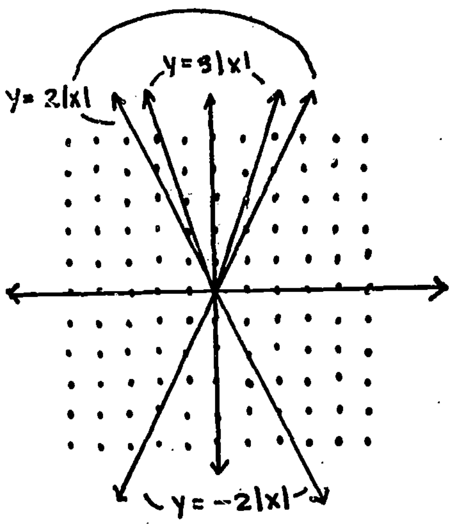
3. $y = |x|$



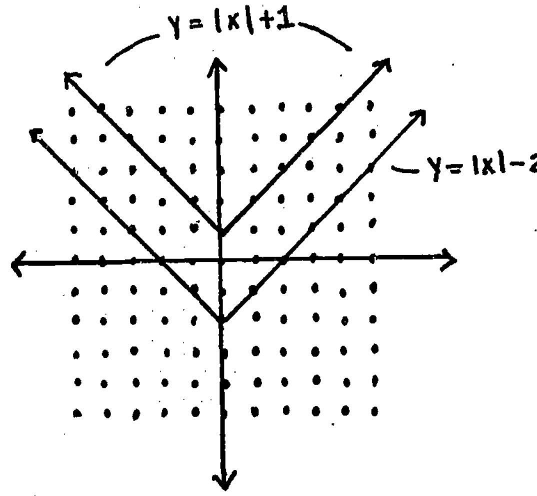
4. $y = |x + 1|$



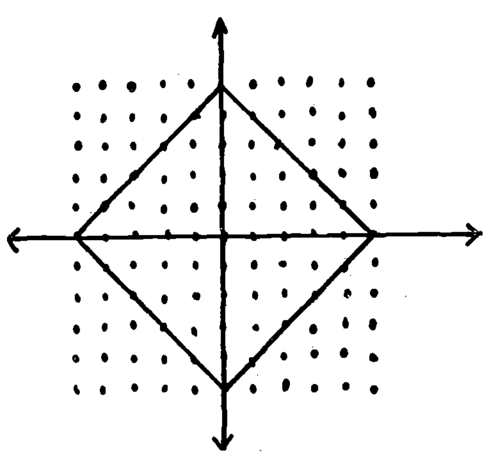
5.



6.



7.



7.9 Lattice Point Games (1 day)

A lattice point game first appears as a suggested exercise in Section 7.1 of the teachers' commentary. Here, further such games are presented. The teacher may use his own discretion in omitting or using this as an optional exercise. It has been suggested that this section be saved for a day before vacation as review work.

7.10 Sets of Lattice Points and Mappings of Z into Z (1 day)

Every lattice point serves to relate its coordinates. That is, the integer, x, obtained from the x-axis is related to (or mapped onto) an integer, y, obtained from the y-axis, by the point that has (x,y) as coordinates. A set of lattice points, then, (where no two have the same first coordinate) represents a mapping of a subset of Z into Z. The domain is the set of first coordinates and may be related to the points on the x-axis with coordinates (x,0). The range is the set of second coordinates and may be related to the points on the y-axis with coordinates (0,y).

Students should have practice in finding the image, y, for an integer, x, by substituting x in the rule for the mapping and computing to find y. The table method of relating domain and range is used here. This is a great help in graphing a mapping when given a rule relating x and y.

Note: for exercise 1, the teacher should state a domain.

7.11 Solutions to Exercises

1. (a) $y = x^2$

Domain	Range
0	0
1	1
-1	1
2	4
-2	4
3	9
-3	9

(b) $y = 2x + 1$

Domain	Range
0	1
1	3
-1	-1
2	5
-2	-3
3	7
-3	-5

(c) $y = x^2 + 1$

Domain	Range
0	1
1	2
-1	2
2	5
-2	5

(d) $y = 2x - 1$

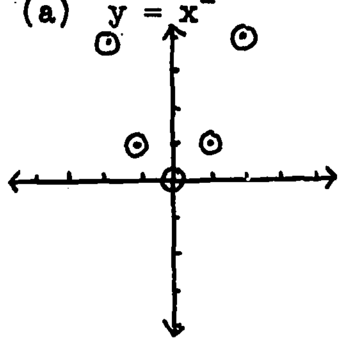
Domain	Range
0	-1
1	1
-1	-3
2	3
-2	-5

(e) If x is even $y = 9$.

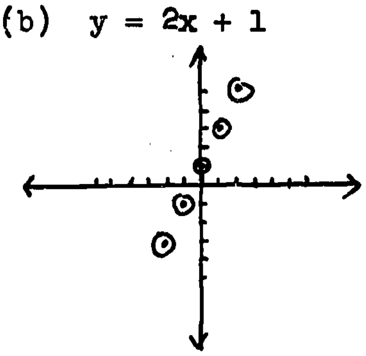
If x is odd $y = 1$.

Domain	Range
0	9
1	1
2	9
3	1
4	9

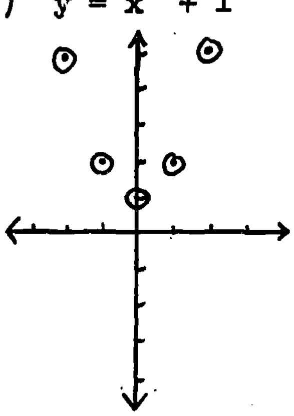
2. (a) $y = x^2$



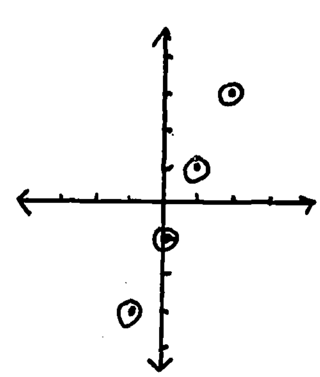
(b) $y = 2x + 1$



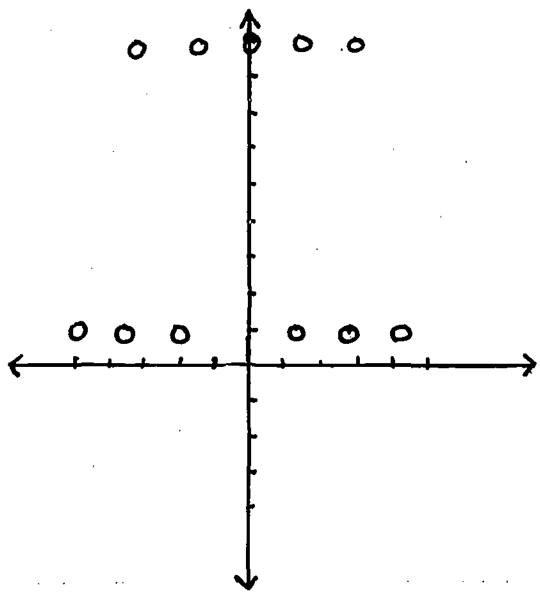
(c) $y = x^2 + 1$



(d) $y = 2x - 1$



(e) $y = \begin{cases} 9 \text{ if } x \text{ is even} \\ 1 \text{ if } x \text{ is odd} \end{cases}$



7.12 Lattice Points in Space (1 day)

The purpose of this section is to give the student some idea of the need for three coordinates in space and to indicate that there is an extension of the ideas developed for Z x Z and lattice points to three dimensions. Students may have difficulty in visualizing three dimensions. In this case the teacher may refer to physical anologies such as the corners of the room or models. For students faced with this problem, Exercise 2 of Section 7.13 may be assigned. Exercise 1 may be done in class.

7.13 Solutions to Exercises

1. (a) EDGF, BCDG, AOCB, EFAO, FGBA, EDCO
   (b) 3: AOCB, EFAO, EDCO
   (c) Diagram with vertices (0,0,0), (2,0,0), (0,3,0), (0,0,4), (2,3,0), (0,3,4), (2,0,4), (2,3,4).
2. (a) Figure should resemble a "corner".

7.14 Translations in Z x Z (1 day)

If students have covered the first 6 chapters, they will have a background of ideas on translations and compositions of mappings. In this case, Section 7.14 will be an extensions of these ideas to lattice points and coordinates.

The notation, $T_{a,b}$, designates the translation which maps (0,0) onto (0 + a, 0 + b) or (a,b), or in general (x,y) onto (x + a, y + b). The main activity in this section is to find the images of points in a geometric figure under translation and see what properties are preserved.

Note: Exercise 1 introduces the concept of an inverse translation. It might be advisable to treat this as a classwork exercise. Exercise 2 may be difficult for some students and the teacher may choose to do this problem with the class and assign exercise 4, which is an analogous problem.

7.15 Solutions to Exercises

1. (a) $T_{0,0}$ (b) $T_{0,0}$ (c) $T_{0,0}$

2. $T_{a,b} \circ T_{c,d} = T_{c+a,\ d+b}$ definition of composition

$= T_{a+c,\ b+d}$ commutativity of (Z,+)

$= T_{c,d} \circ T_{a,b}$ definition of composition

3. The commutative property for composition of translations.

4. $T_{a,b} \circ (T_{c,d} \circ T_{e,f}) = T_{a,b} \circ T_{e+c,\ f+d}$ def. of composition

$= T_{(e+c)+a,\ (f+d)+b}$ def. of composition

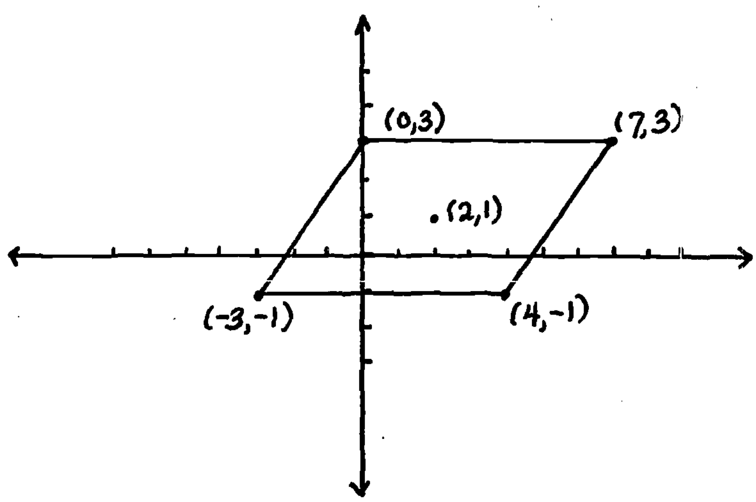
$= T_{e+(c+a),\ f+(d+b)}$ assoc. of $(Z,+)$

$= T_{c+a,\ d+b} \circ T_{e,f}$ def. of composition

$= (T_{a,b} \circ T_{c,d}) \circ T_{e,f}$ def. of composition

5. The associative property for composition of translations

6.



7. Yes.

8. (-5,-2); (-2,2); (5,2); (2,-2)

9. Yes.

10. Yes.

7.16 Dilations and Z x Z (1 day)

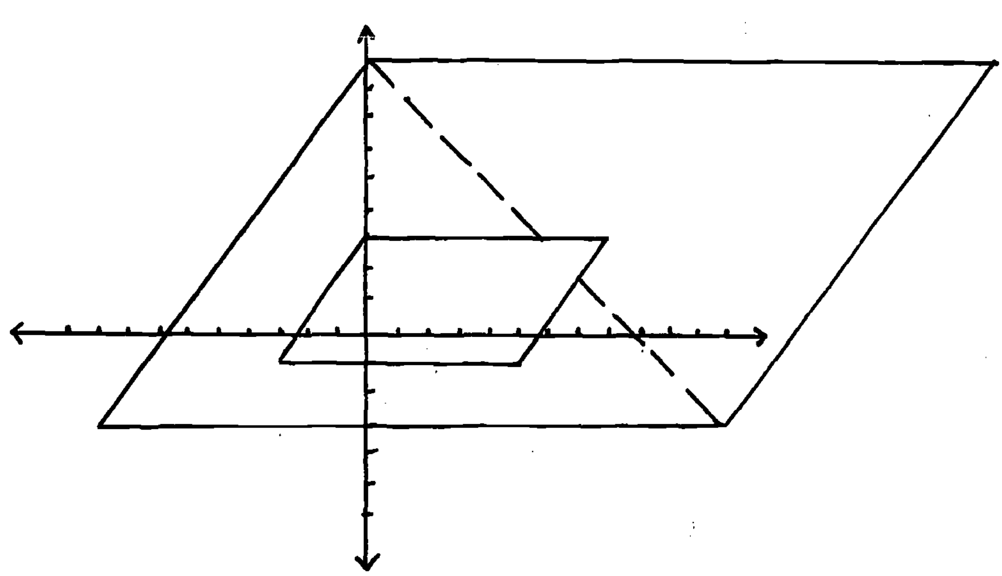
If students have covered the first 6 chapters, they will have a background of ideas on dilations and compositions of mappings. In this case, Section 7.16 will be an extension of these ideas to lattice points and coordinates.

A dilation can be expressed by $(x,y) \longrightarrow (ax, ay)$, $a \neq 0$. If 1, this is sometimes called a "stretching"; if $a = 1$, every point maps onto itself and we have the identity dilation; if a = -1, then each point is "reflected" in the origin which means that a point and its image are symmetric with respect to the origin (this is sometimes called a half-turn); if a < -1, then the dilation is a composition of the point reflection in the origin and a "stretching". We don't allow a = 0, but if we did, the rule for a dilation would map each lattice point onto the origin where they would stay because no other dilation would be able to map the origin anywhere but onto the origin.

Note: Exercise 3 discusses some of the improtant properties of dilations.

7.17 Solutions to Exercises

1. (-3,-1) ⟶ (-9,-3)
   (0,3) ⟶ (0,9)
   (7,3) ⟶ (21,9)
   (4,-1) ⟶ (12,-3)



2. (a) a parallelogram

   (b) No. Yes.

   Note: A comparison of this problem and problem 6 of Section 7.15 can lead to a comparison of dilations and translations.

3. (a) D_1 or $(x,y) \longrightarrow (1x,1y)$

   (b) Yes. Yes.

   Note: These properties are dependent on the same properties for Z.

   (c) D_1 and D_{-1}.

7.18 Some Additional Mappings and Z x Z (1 day)

This section provides an opportunity to investigate the properties of several other kinds of mappings. The concepts of the previous sections are reinforced. If a student still wishes (or needs) other examples of mappings, try:

(a) $(x,y) \longrightarrow (ax,y)$

(b) $(x,y) \longrightarrow (x,ay)$, where a is an integer.

This section may be omitted at the discretion of the teacher.

7.20 Solutions to Review Exercises

1. Answers will vary.
2. (a) $2x - 3y = 7$ (b) $x = 2|y| - 3$ (c) $x > 0$ and $y < 2$.
3. (a) The second coordinate is two less than the square of the first coordinate.

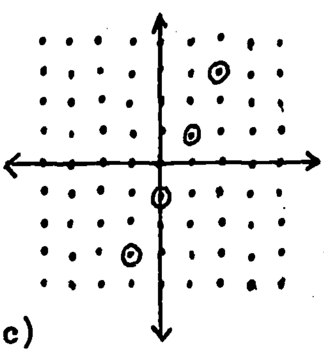
   (b) The absolute value of the sum of the coordinates is five.

   (c) The second coordinate is greater than two or the first coordinate is less than three.

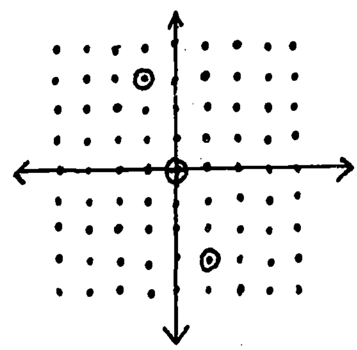
4. (a) $\{(4,1)\}$ (b) $\{(-1,1)\}$

5. (a) (b)

$y = 2x - 1$

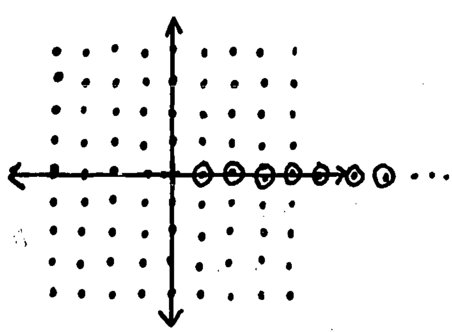


$y = -3x$



(c)

$x > 0$ <u>and</u> $y = 0$



6. (a) First Quadrant (b) 1st, 2nd, 3rd, or 4th.

   (c) Third Quadrant (d) Second Quadrant

7. These points should lie on a circle in the first quadrant if the axes are drawn perpendicular to each other and have the same unit distances on both axes. Otherwise, they lie on an ellipse.

8. Same as 7.

9. (a) (0,0), (0,10), (4,0)

   (b) (0,0), (0,10), (-4,0)

   (c) (0,0), (0,-10), (-4,0)

   (d) (0,0), (0,-10), (4,0)

   It might prove valuable to discuss the results of these mappings.

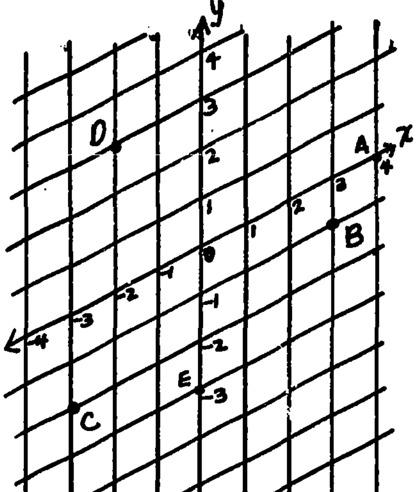
10. (a) (3,4), (3,7), (7,7), (7,4)
(b) (0,0), (6,3), (10,3) (4,0)
(c) (5,0), (5,-3), (9,-3), (9,0)
(d) (0,0), (0,0), (4,0), (4,0)

It might prove valuable to discuss the results of these mappings.

<u>Suggested Test Items</u>

Part I:

1. Find the coordinates of the points named.



A ________

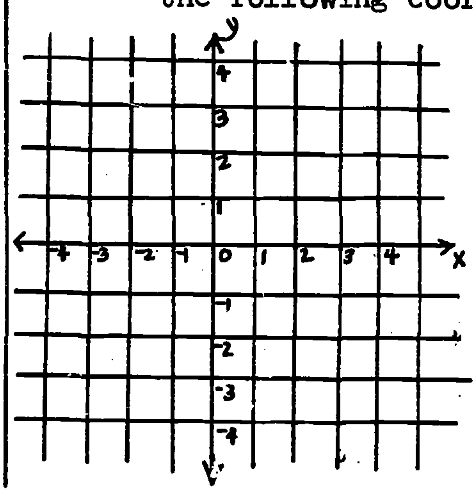
B ________

C ________

D ________

E ________

2. Locate the points that have the following coordinates.



A. (-3,0)

B. (-4,4)

C. (0,2)

D. (-1,-2)

E. (0,0)

Part II: Complete the following sentences:

1. The coordinates of the origin are ________________.

2. If two points are on a line parallel to the x-axis, they have the same __________ __________.

3. If two points are on a line parallel to the y-axis, they have the same ______________ __________.

4. If a point is on the x-axis, its second coordinate is ________; if the first coordinate of a point is zero, it is on the ____________________.

5. If $x > 0$ and $y > 0$, then (x,y) is in the ______ ______.

6. If $x > 0$ and $y < 0$, then (x,y) is in the _______ _______.
7. $(-5,-6)$ is in the ______________ quadrant.
8. If $x = -3$ and $y = |x|$, then $y =$ ______________.
9. If $x = 3$ and $y = 5x - 16$, then $y =$ ______________.
10. If $y = |x|$ and $y \neq x$, then $y =$ ______________.
11. If $x < 0$, then $|x| =$ ______________.
12. If $x + y = 7$ and $x - y = 1$, then $(x,y) =$ ______________.
13. If the points of a circle are mapped onto points by the rule, $(x,y) \longrightarrow (5x,5y)$, the image will be a __________ and will be _______ _______ than the original.
14. If a parallelogram is in the first quadrant and is translated by $(x,y) \longrightarrow (x + 5, y + 3)$, the image will be a _______ in the ______________ quadrant.

Part III: Graph the following conditions:

1. $Y = x + 1$
2. $y = 2x - 4$
3. $y < -3x$
4. $x = |4|$
5. $y = |x + 3|$
6. $2y - 1 = 2x - 1$
7. $x < 0$ <u>and</u> $y = -2$
8. $x > 0$ <u>and</u> $x < 5$ <u>and</u> $y < 5$ <u>and</u> $y > 0$.

Part IV: For the following mappings find the image of triangle $(0,0)$, $(2,3)$, $(5,0)$ and tell what quadrant or quadrants it is in:

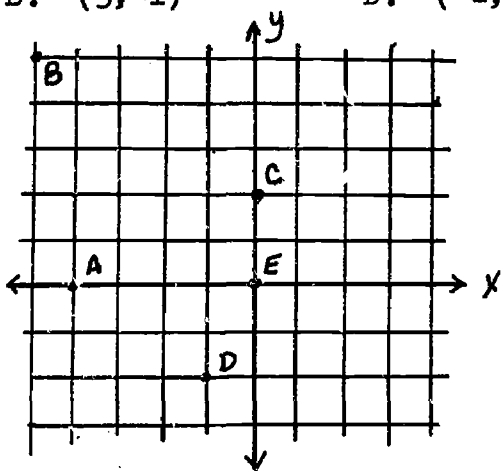
1. $(x,y) \longrightarrow (x + 1, x + 3)$
2. $(x,y) \longrightarrow (-3x, -3y)$
3. $(x,y) \longrightarrow (-x, y)$
4. $(x,y) \longrightarrow (x, -y)$
5. $(x,y) \longrightarrow (x-3y, y)$

Answers to Suggested Test Items

Part I:

1. A. (4,0)   C. (-3,-2)   E. (0,-3)

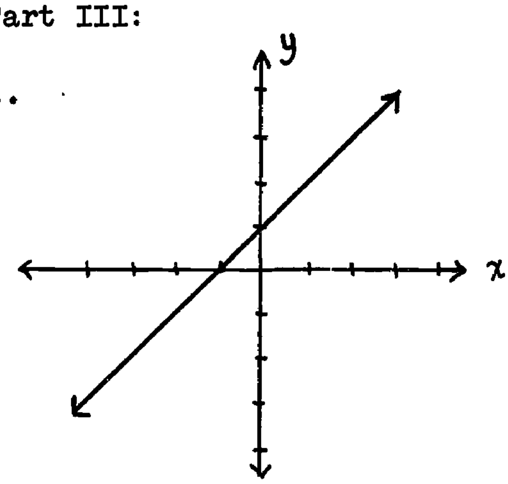
   B. (3,-1)   D. (-2,3)

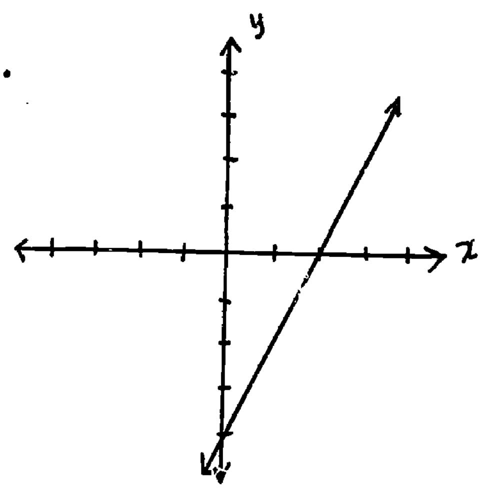
2. 

Part II:

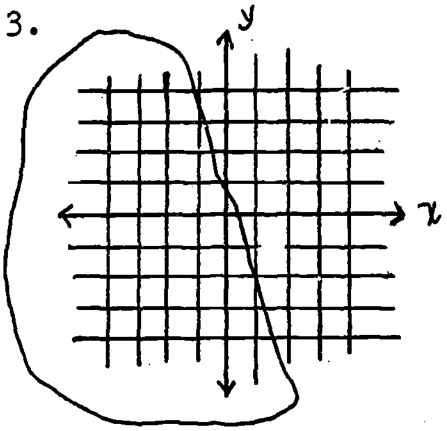
1. (0,0)
2. y-coordinate
3. x-coordinate
4. zero; y-axis
5. 1st quadrant
6. 4th quadrant
7. 3rd quadrant
8. 3
9. -1
10. -x
11. -x
12. (4,3)
13. circle; larger
14. parallelogram; first

Part III:

1. 

2. 

3.



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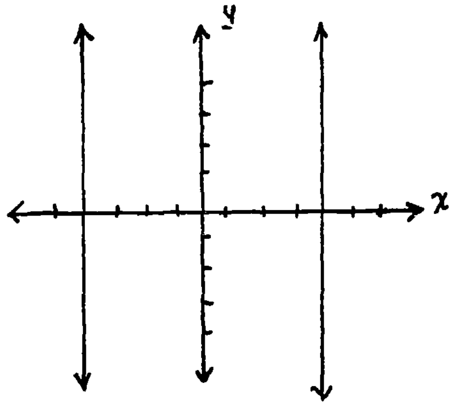

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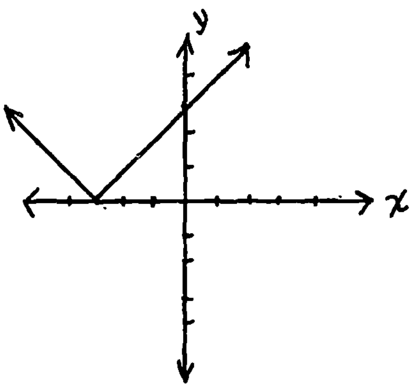

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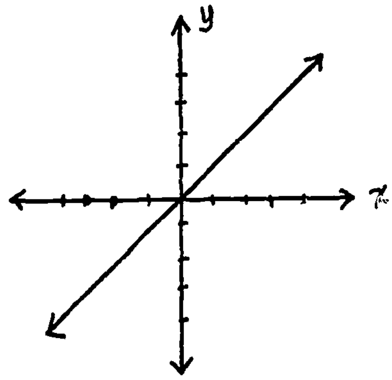

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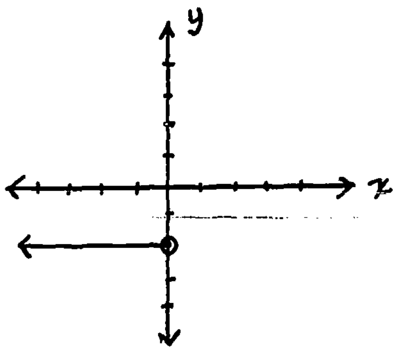

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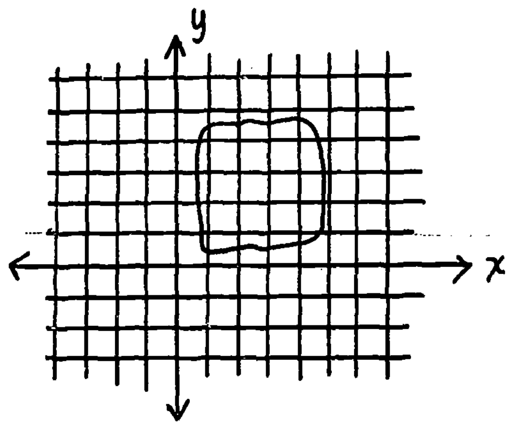


Part IV:

1. $(0,0) \longrightarrow (1,3)$
   $(2,3) \longrightarrow (3,6)$ 1st quadrant
   $(5,0) \longrightarrow (6,3)$

2. $(0,0) \longrightarrow (0,0)$
   $(2,3) \longrightarrow (-6,-9)$ 3rd quadrant
   $(5,0) \longrightarrow (-15,0)$

3. $(0,0) \longrightarrow (0,0)$
   $(2,3) \longrightarrow (-2,3)$ 2nd quadrant
   $(5,0) \longrightarrow (-5,0)$

4. $(0,0) \longrightarrow (0,0)$
   $(2,3) \longrightarrow (2,-3)$ 4th quadrant
   $(5,0) \longrightarrow (5,0)$

5. $(0,0) \longrightarrow (0,0)$
   $(2,3) \longrightarrow (-7,3)$ 1st and 2nd quadrants
   $(5,0) \longrightarrow (5,0)$

TEACHER'S COMMENTARY

Chapter 8 Sets and Relations

The main objectives of this chapter are:

1. To provide for the student precise meanings of basic set-theoretic terms. The terms considered (and the section in which they are discussed) include the following:

set (8.1), equality of sets (8.3), subset (8.3), proper subset (8.3), null set (8.3), universal set (8.5), union of two sets (8.7), intersection of two sets (8.7), complement of a set (8.7), disjoint sets (8.7), cartesian product set (8.9).

2. To introduce to the student the following ideas and terms which deal with relations and properties of relations on sets:

relation (8.9),
reflexive property (8.11), symmetric property (8.11), transitive property (8.11), equivalence relation (8.11), equivalence class and partition of a set (8.13).

3. To expose the student to certain tools that may enable him to work more effectively with sets and relations.

These include:

set notation (8.1), Venn diagrams (8.5 and 8.7), arrow diagrams for relations (8.9).

Teachers should note that the odd numbered sections 8.1 -8.13 are content sections. All of the even numbered sections 8.2-8.16 are exercise sections.

Abundant motivating material for a discussion of sets and relations can be found in Chapters 1-7. With regard to sets, the following were introduced: "set" and "set of whole numbers", "set of clock numbers" and "empty set" in Chapter 1, "subset" in Chapter 2, "set of integers" in Chapter 4, "outcome set", "union", and "intersection" in Chapter 5, etc. Such notions as "operation", "mapping", "an integer is a set of ordered pairs of whole numbers", "the set of lattice points" all involve ideas from relation theory. Thus this chapter provides an opportunity to bring together for a close scrutiny many ideas concerning sets and relations which have been introduced earlier.

8.1 Sets (Estimated Time = 1 day.)

Students will enjoy thinking up examples of sets with interesting and/or familiar collective names. Student's examples can be used to bring out the idea that we distinguish between a set and the elements that make up a set. e.g. The set of all boys in the class is not a boy; or the set of desks in the room is not a desk. Students may be familiar with the ideas of number and numeral where we distinguish between a set and the name of a set. This point can also emerge by asking students if {2,2} is the same set as {2}.

Teachers may wish to point out early that when the elements of a set are themselves sets, we often speak of this as a family of sets or a class of sets rather than say "a set of sets".

Observe that the ten examples given verbally in 8.1 are repeated again in 8.1 using set notation. In the second presentation of these examples, use is made of the roster method in the odd cases and the rule or set builder method in the even cases. Note that Example 8 introduces the null set.

Be aware that in this chapter primary concern should be directed to developing a working vocabulary in connection with sets rather than developing an abstract theory of sets. If certain usages of language appeal to a given class (or teacher) they can be adopted temporarily. Later on it can be pointed out that a certain term, or synonym of this term, is used most often in mathematics. Teachers may wish to point out that mathematics is truly a universal language of science.

8.2 Exercises

All the exercises in this section should be done as homework and gone over in class. Exercise 8 affords an excellent opportunity for the students to display their ingenuity and imagination: although there is only one null set, there is an endless variety of defining properties which describe a null set. Exercise 10 provides another opportunity to discuss finite versus non-finite sets (recall Chapter 1). Exercise 9(c) should make the students aware of the difference between {0} and { }. This difference should be understood thoroughly.

Answers to Exercises

1. Maine, Missouri, Mississippi, Montana, Maryland, Minnesota, Michigan, and Massachusetts.
2. Indianapolis, Indiana; Oklahoma City, Oklahoma; Dover, Delaware; Honolulu, Hawaii.
3. They contain the same elements, i.e. $A_1 = A_{10}$.
4. Every element of A_9 is an element of A_1, i.e. A_9 is a subset of A_1.
5. (a) $A_2 = \{7\}$

   (b) $A_8 = \emptyset$ (or $\{\ \}$)
6. (a) A possible answer is $A_6 = \{x : x$ is a divisor of $24\}$

   (b) $A_7 = \{x : x$ is an integer, $x < 3$, $x > -3\}$

   (c) $A_9 = \{x : x$ is a whole number, $x > 0$, $x < 7$, x is odd$\}$
8. The set of whole numbers x, such that

   (1) $x \neq x$

   (2) $x = x + 1$

   (3) x is divisible by 6 but not by 3

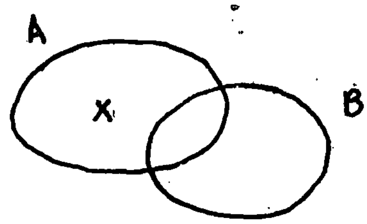
   (Many other properties possible)
9. (a) no (b) yes (c) no (d) yes.
10. They are all finite sets.

8.3 Set Equality, Subsets (Estimated Time - 1 day)

Ask the class why the following is not an adequate definition.

Set A is the same as set B if every element in set A is contained in set B.

Venn diagrams can be introduced to illustrate Remark 1: If set A is not a subset of set B, then set A contains at least one element x that is not contained in set B.



However, a detailed discussion of Venn diagrams is reserved for Sections 8.5 and 8.7.

Remark 2 can be proved as a theorem. It is sometimes called the "working definition" of equality since we make use of it often to establish that sets specified in different ways may actually be the same set.

For remark 3 note that set A is a subset of itself, but A is not a proper subset of itself. Remark 4 provides an opportunity to introduce the notion of indirect proof. Also note that $\emptyset$ is a proper subset of every set except itself.

8.4 Exercises

Exercise 1 highlights the fact that the order in which we display the elements of a set is immaterial. Exercises 5, 6 and 7 provide an excellent experience in formulating generalizations. Exercises 9 and 10 make good classroom exercises and actual students can be used. Many exercises should be done in class.

1. Set G is equal to set H, since both sets contain exactly the same elements.
2. (a) $G \subset L$ because every element in G is also an element of L.

   (b) $G \neq L$ because L is not a subset of G.
3. (a) $B \nsubseteq R$ since Dick $\in$ B but Dick $\notin$ R.

   (b) $G \subseteq R$ since every element of G is an element of R.
4. (a) yes (b) no (c) yes (d) yes (e) no (f) no (g) yes.
5. (a) $\emptyset$, $\{5\}$ (b) $\emptyset$.
6. (a) $\emptyset$, $\{5\}$, $\{7\}$, $\{9\}$, $\{5.7\}$, $\{5.9\}$, $\{7.9\}$, $\{5,7,9\}$

   (b) $\emptyset$, $\{5\}$, $\{7\}$, $\{9\}$, $\{5.7\}$, $\{5.9\}$, $\{7.9\}$
7. (a) 16 (b) 15 (c) 32 (d) 31 (e) 2^n.
8. There is at least one element in set B that is not an element in set A: $B \nsubseteq A$.
9. (a) $X \subset Z$ (b) $T \subset S$ (c) $M \subset Q$ (d) $T \subset G$

   (e) Since $A \subseteq Q$ and $Q \subseteq R$ we have that $A \subseteq R$. Also since we know that $R \subseteq A$ we conclude that $A=R$.

   (f) Nothing concerning sets P and R.
10. (a) yes (b) no (c) yes (d) yes (e) no (f) yes.
11. They are all the same.
12. $\emptyset$ is a set containing no elements whereas $\{\emptyset\}$ is a set containing an element, namely, the empty set.
13. (a) True since $X \subset Z$ (b) Not necessarily true since y could be an element of Y and still not be an element of X. (c) Not necessarily true since, for example, p could be an element of Z and not an element of Y. (e) True

since q $\notin$ Y and X $\subset$ Y. (f) True since r $\notin$ Z and X $\subset$ Z.

8.5 Universal Set, Subsets and Venn Diagrams (Estimated time 2-3 days.)

Encourage students to use a variety of shapes, besides the circular, to represent sets. Be very careful to have them discriminate between the geometric figures of the diagram and the actual sets portrayed by these figures. For example a region in the Euclidian plane will have infinitely many points, yet may be used in a Venn Diagram to represent a finite set or even an empty set! Note that an "x" is used to indicate that a set has at least one element but does not show how many. The "$\emptyset$" indicates the absence of all elements in a set or its subsets.

8.6 Exercises

Exercises 1 and 3 can be answered using only the notions of equality, subset, proper subset, and empty set. These have already been discussed. A discussion of unions, intersections, and complements is reserved for the next section (8.7). However, there is no harm if the students use some of that language here in responding to Exercises 1 and 3.

In Exercises 2, 4 and 5, encourage the students to use the more general diagram with intersecting regions, although alternate solutions submitted by students should be accepted, if they are correct.

All exercises should be done carefully; however, many problems can be done in class with student discussions.

8.6 Exercises (Solutions)

1. a. $B \subset A$

   b. $B \subset A$, $A \neq \emptyset$.

   c. $A = B$ (i.e. $A \subseteq B$ and $B \subseteq A$)

   d. $A \subset B$, $A \neq \emptyset$, $B \neq \emptyset$

   e. $A = B = \emptyset$

   f. $A = B$, $A \neq \emptyset$, $B \neq \emptyset$

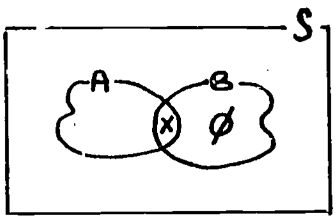
   g. $A \subset B$, $A,B \neq \emptyset$

   h. $A = 0$ and $B \neq \emptyset$

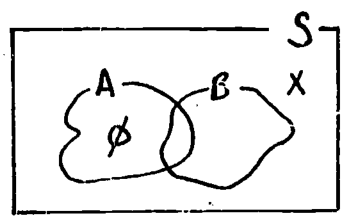
2. Note 2(c) should also state "$B \neq \emptyset$".

   and 2(e) should show "$A \neq B \neq S \neq \emptyset$".

a.



d.



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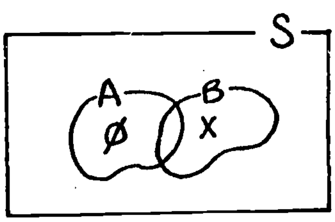

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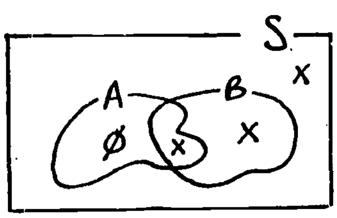

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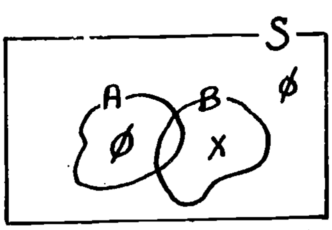

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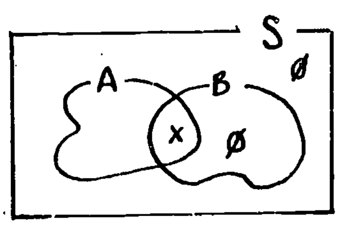


3. a. $A \subset C,\ B \subset C$

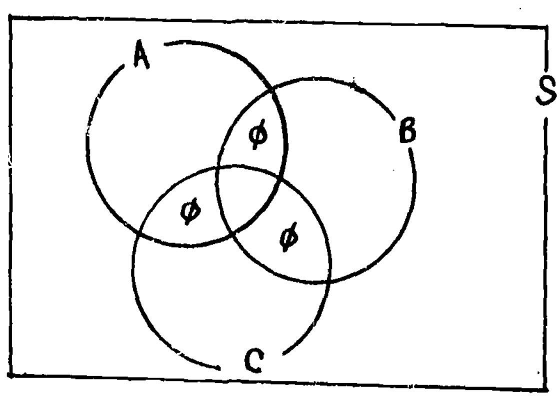
   b. $A \subset C,\ B \subset C,\ A \subset B$

   c. $A \subset B,\ A \subset C,\ B \subset C,\ A \neq C,\ B \neq C$

   d. $A \subset C,\ A \subset B,\ B \subset C,\ A \neq C,\ A \neq B$

   e. $A = B = C$ f. $C \subset B \subset A \subset S$ and $C \neq B \neq A \neq S$

4 a.



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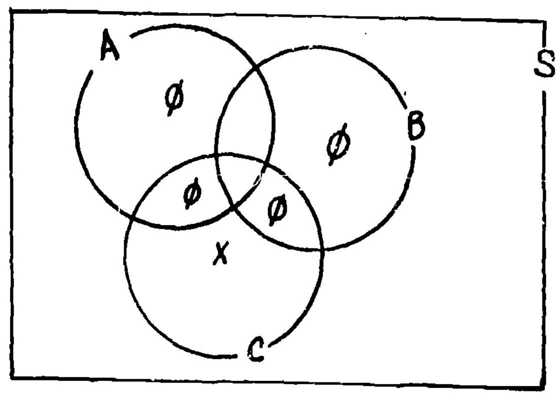

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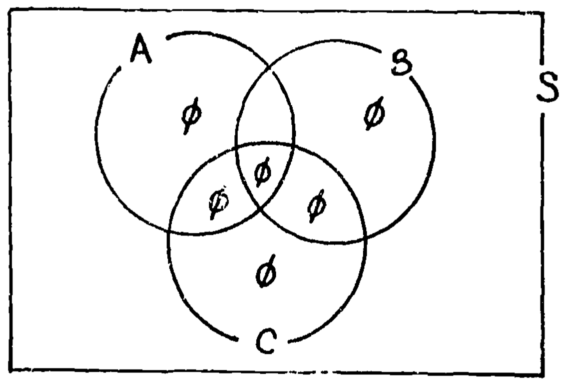
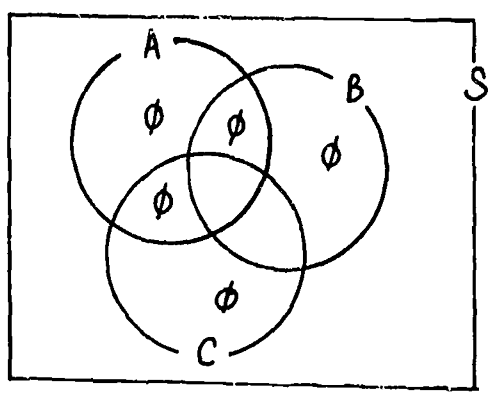
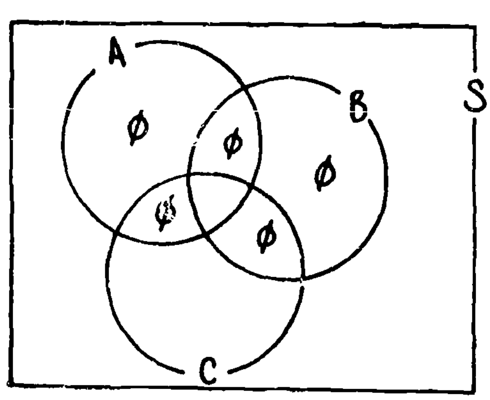


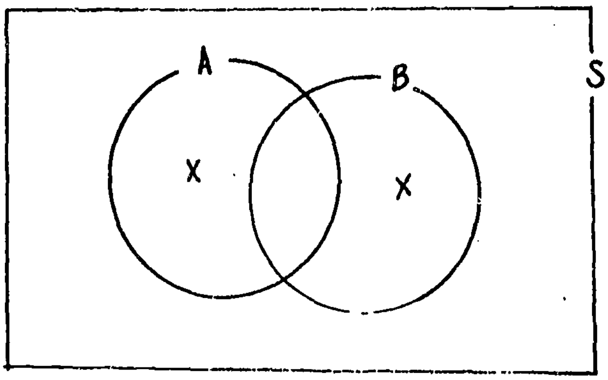
4 d.



e.



5 a.



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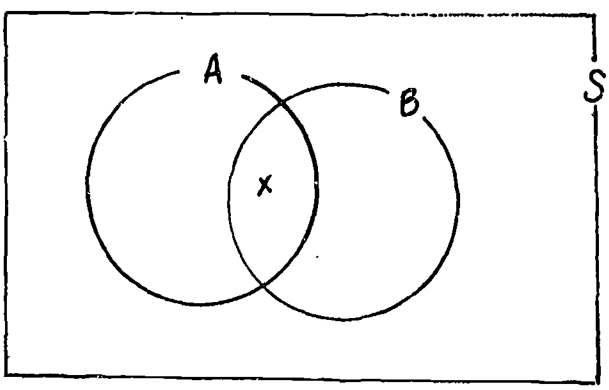
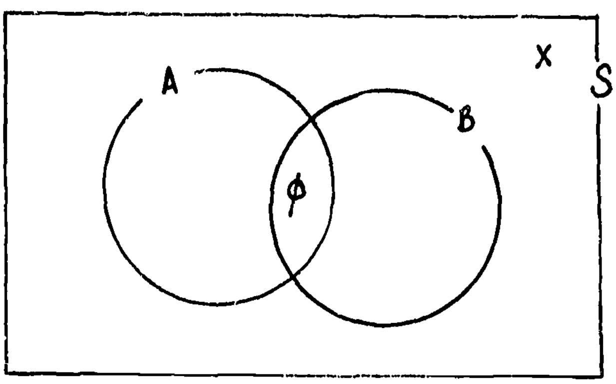
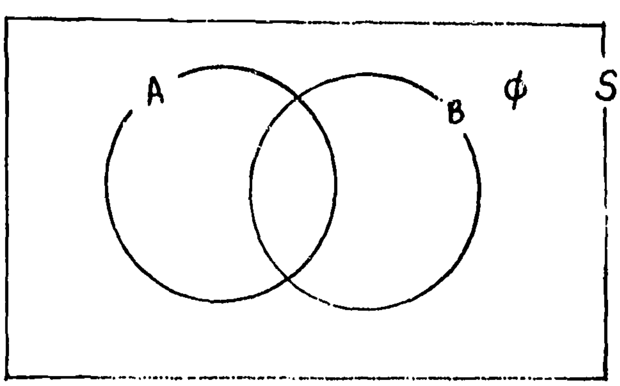


c.



d.



6. a. Yes
   b. Yes
   c. No
   d. Yes
   e. Maybe
   f. No
   g. Yes
   h. Maybe
   i. Yes
   j. No
   k. Yes
   l. No
   m. Maybe
   n. Yes
   o. Yes

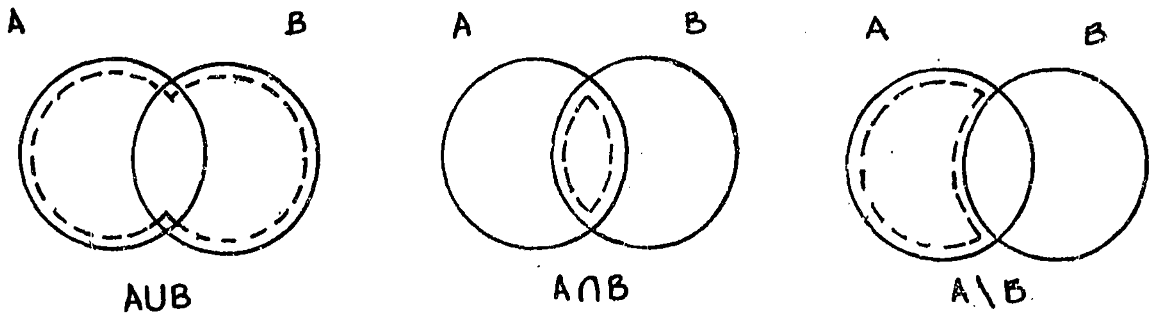
8.7 Unions, Intersections, Complements (Estimated time 1-2 days)

The Boolean properties of sets should be developed informally and semi-formally by

(a) examples using specific sets

(b) reasoning from definitions and previous remarks made in text

(c) shading, or otherwise marking, Venn Diagrams. Instead of shading we can use dashed lines to indicate such sets as $A \cup B$, $A \cap B$, $A \setminus B$, etc. Colored chalk is often very helpful for this purpose.



Besides using Venn diagrams, teachers might wish to illustrate set relations as Papy does in his Mathematique Moderne.* He makes effective use of color to display such operations as union, intersection, difference, etc. The overhead projector, if available, is a great aid in this section.

8.8 Exercises

Note that Exercises 1 through 10 provide an opportunity to develop some of the general Boolean properties of sets. Observe that Exercises 2,3, and 4 go together as do Exercises 5 and 6. Exercise 14 depends on Exercise 12. These two exercises can be considered optional, as can exercises 7 and 15. Class use of alternating problems in an exercise (say a, c, e, g, etc.) with the remaining problems (b,d,f,etc) as homework can save time on homework and yet yield better results. Exercises 12 and 14 should be done in class.

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*Modern Mathematics, Vol. 1. G. Papy, Macmillan Co., New York, N.Y. 1968.

8.8 Exercises

1. The set of all students in the school who are

   a. not seventh graders.

   b. girls

   c. (both) seventh graders and boys.

   d. (both) boys and bus to school.

   e. seventh graders or boys (or both).

   f. boys or who bus to school (or both).

   g. (both) seventh graders and girls.

   h. not seventh graders or are boys.

   i. girls, and do not bus to school.

   j. girls, or students who do not bus to school.

   k. seventh graders, or boys, or who bus to school.

   l. seventh graders, and are boys who bus to school.

   m. seventh graders who are girls or are students who bus to school.

   n. not seventh graders, or are boys, or do not bus to school.

   o. not seventh graders, and not both boys and students who bus to school.

   p. not seventh graders, and not (either) girls or students who bus to school.

2. (a) S (b) ∅ (c) [0,2,3,4,5,6,7,8] (d) [2]

   (e) [1,2,3,5,7,9] (f) [3,5,7] (g) B (h) A

   (i) [0,1,4,6,8,9] (j) [0,1,3,4,5,6,7,8,9]

   (k) [0,1,3,4,5,6,7,8,9] (l) ∅

3. (a) $S \cup C = S$ (b) $[0, 2, 4, 6, 8] \cup [1, 2, 3, 5, 7, 9] = S$

   (c) $\{0, 2, 4, 6, 8\} \cap \{3, 5, 7\} = \emptyset$ (d) $\emptyset \cap \{3,5,7\} = \emptyset$

4. (a) $\{0, 2, 4,6,8\{ \cup \{3,5,7\} = \{0,2,3,4,5,6,7,8\}$

   (b) $S\cap\{0,2,3,4,5,6,7,8\} = \{0,2,3,4,5,6,7,8\}$

   (c) $\{0,2,4,6,8\} \cap \{1,2,3,5,7,9\} = \{2\}$

   (d) $\emptyset \cup \{2\} = \{2\}$

5. a. $A \cup B = \{-4, -3, 0, 3, 4, 8, 16\}$

   b. $\overline{A \cup B} = \{7\}$

   c. $\bar{A} \cap \bar{B}$

   d. $A \cap B = \{8\}$

   e. $\overline{A \cap B} = \{-4, -3, 0, 3, 4, 16\}$

   f. $\bar{A} \cup \bar{B} = \{-4, -3, 0, 3, 4, 16\}$

   g. $A \cap (B \cup C) = \{0,8\}$

   h. $(A \cap B) \cup (A \cap C) = \{0,8\}$

   i. $A \cup (B \cap C) = \{-4, 0, 8, 16\}$

   j. $(A \cup B) \cap (A \cup C) = \{-4, 0, 8, 16\}$

   k. $A \cup (A \cap B) = \{-4, 0, 8, 16\}$

   l. $A \cap (A \cup B) = \{-4, 0, 8, 16\}$

   m. $\overline{A \cup (B \cup C)} = \emptyset$

   n. $\bar{A} \cap (\bar{B} \cap \bar{C}) = \emptyset$

6. Answers may vary.

   Some correct conjectures which students might make from above exercises are as follows:

   $\overline{A \cup B} = \bar{A} \cap \bar{B}$

   $\overline{A \cap B} = \bar{A} \cup \bar{B}$

   $\bar{S} = \emptyset$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A \cap (A \cup B)$$

$$\overline{A \cup (B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$$

However there are a number of <u>incorrect</u> conjectures which the students might make. These should be discussed carefully to see where they break down, eg. from 5 i: it may appear that $A \cup (B \cap C) = A$ or from 5m: $\overline{A} \cap \overline{B} \cap \overline{C} = \emptyset$
Venn diagrams can be helpful in explaining why these conjectures are false.

7. (a) <u>True</u>: because $N \subset W$, i.e. every element of N is already contained in W

   (b) <u>False</u>: actually $N \cap W = N$

   (c) <u>True</u>: because 0 is the only element of W which is not an element of N.

   (d) <u>True</u>: the complement of the universal set is always $\emptyset$

   (e) <u>False</u>: because $\overline{N}$ contains an element, 0

   (f) <u>True</u>: because $W = \emptyset$ and the intersection of any set with $\emptyset$ is $\emptyset$.

   (g) <u>True</u>: because $W \cup N = W$ and hence its complement is $\emptyset$.

   (h) <u>False</u>: actually $\overline{W \cap N} = \overline{N} = \{0\}$.

8. (a) By definition of intersection we have that

   $A \cap B = [x : x \in A \text{ and } x \in B]$

   Since every element of $A \cap B$ is an element of A, we conclude from our definition of subset that $(A \cap B) \subseteq A$

(b) By the definition of union, we have that

$A \cup B = [x : x \in A$ or $x \in B$ or x is an element of both A and $B]$

From exercises 6 (a) above we know that $(A \cap B) \subseteq A$. Thus we conclude that every element of $A \cap B$ is an element of $A \cup B$. In short, $(A \cup B) \subseteq (A \cup B)$.

9. a) By definition $A \cup A$ is the set that contains those and only those elements (of S) that belong either to A or to A. This condition is satisfied by every element of A and only the elements of A. Hence $A \cup A = A$.

   b) By definition $A \cap A$ is the set that contains those and only those elements (of S) that belong to both A and A. This condition is satisfied by every element of A and only the elements of A. Hence $A \cap A = A$.

   c) By definition $A \cup \bar{A}$ is the set that contains those and only those elements (of S) that belong either to A or to $\bar{A}$. This condition is satisfied by every element of S (and only by elements of S). Hence $A \cup \bar{A} = S$.

   d) By definition $A \cap \bar{A}$ is the set that contains those and only those elements (of S) that belong both to A and $\bar{A}$. But no element (of S) can belong both to A and $\bar{A}$. Hence $A \cap \bar{A} = \emptyset$

e) By definition $\overline{S}$ is the set of all those elements of S which do not belong to S. Since there can not be any such elements, $\overline{S} = \emptyset$.

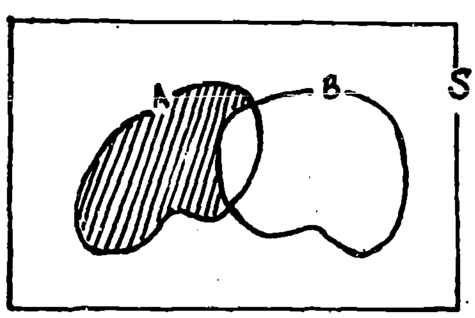
f) By definition $\overline{\emptyset}$ is the set of all those elements of S, which are not in $\emptyset$. Since this condition is satisfied by every element of S, it follows that $\overline{\emptyset} = S$.

g) By definition $A \cup S$ is the set that contains those and only those elements (of S) that belong either to A or S, or both. This condition is satisfied by every element of S (and only the elements of S).
Hence $A \cup S = S$.

h) By definition $A \cap \emptyset$ is the set that contains those and only those elements (of S) that belong both to A and to $\emptyset$. Since $\emptyset$ is empty, there are no elements (in S) that satisfy this condition. Hence $A \cap \emptyset = \emptyset$

10. $\overline{\overline{A}} = A$, because by definition $\overline{\overline{A}}$ is the set of all elements of S, which do not belong to $\overline{A}$. But by definition of $\overline{A}$ the elements of S that do not belong to $\overline{A}$ are the elements of A, itself.

11.



$A \cap \overline{B}$ is shaded.

12. (a) Yes: $A \;\; B = A \cap \overline{B}$

(b) No: $A \;\; B \neq B \;\; A$

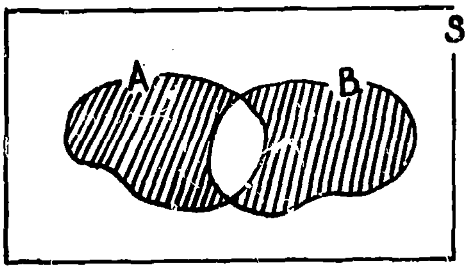
(c) Yes: $A \;\; B \subset A$

(d) $(A \;\; B) \cup (A \cap B) \cup (B \;\; A) = A \cup B$

(e) $(A \;\; B) \cap (B \;\; A) = \emptyset$

These results are easily verified by means of Venn diagrams.

13.



$(A \cap \overline{B}) \cup (\overline{A} \cap B)$ is shaded.

14. a) Yes b) Yes c) Yes d) $A \cap B$

15. a) $\emptyset$

b) $\{x: x \in Z \text{ and } x \neq 0\}$

c) B

d) C

e) Z

f) $\{x: x \in Z \text{ and } -5 < x < 10\}$ or by listing:

$\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

g) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

h) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

i) A

j) D

8.9 Cartesian Product Sets, Relations (Estimated time 2-3 days)

This section could be motivated by reviewing some topics considered earlier such as the outcome set for the tossing of a pair of dice or the idea of a set of lattice points, etc. Some student might wish to make a report to the class on the life of Rene Descartes.

Many seventh graders enjoy playing tic-tac-toe on a finite lattice where you lose your turn if you give incorrect coordinates (e.g. if the point has already been taken or if the point is not part of the lattice set being considered). Robert Davis has used this device with great success with very young children.

Note that both a tree and an arrow diagram are special instances of directed graphs or "diagraphs". This is an interesting topic and could be the subject of a report by some interested student. A good reference is Oystein Ore's Graphs and Their Uses (Singer). Note that we have assigned no direction to loops. When a member maps to itself, there is no need for the arrow.

Note also that the most general definition of a relation as subset of A x B (any set of ordered pairs) is not stressed here. Emphasis is placed almost entirely on relations which are subsets of A x A, i.e. relations on a set A.

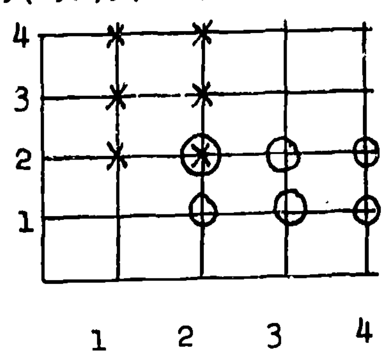
8.10 Exercises

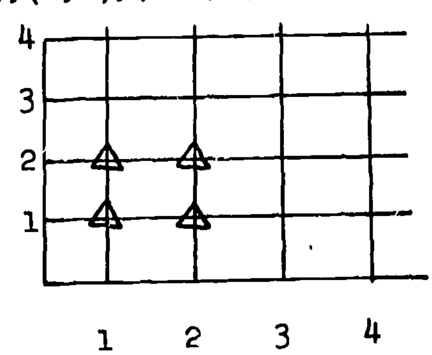
Exercise 8 provides an opportunity to review the idea of operational system. Exercise 10 can be related back to

Chapter 3 but is optional.

8.10 Exercises (Answers)

1. a. {(T,H), (T,B), (T,M), (T,J), (H,E), (B,G), (B,A)}
   b. {(H,B) (H,M),(H,J),(B,H),(B,M),(B,J),(G,A),(,F),(F,P)}
   c. ∅
   d. {(B,E),(B,P),(B,F),(H,G),(H,A),(H,P),(H,F)}
   e. {(M,H),(M,B),(M,J),(J,H),(J,B),(J,M),(A,G)}
2. a. {(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)}
   b. {(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)}
   c. {(1,1),(1,2),(2,1),(2,2)}
   d. {(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)}

3. a and b: 

c: 

   d. (1) {(2,2)}
      (2) {(2,1),(2,2)}
      (3) {(1,2),(2,2)}
      (4) P x {2} = {(1,2),(2,2)}
      (5) {(1,1),(1,2),(2,1),(2,2)} ∪ {(1,2),(1,3),(1,4),
          (2,2),(2,3),(2,4)}
      (6) P x {1,2,3,4} = {(1,1),(1,2),(1,3),(1,4),
          (2,1),(2,2),(2,3),(2,4)}

e. Various answers will be given and should be discussed. Students will learn from wrong answers as well as right answers. Same results are:

(1) $(P \times Q) \cap (Q \times P) = (P \cap Q) \times (P \cap Q)$

(2) $P \times (P \cap Q) = (P \times P) \cap (P \times Q)$

(3) $P \times (P \cup Q) = (P \times P) \cup (P \times Q)$

4. a. (1) $\{(1,2),(1,3),(2,2),(2,3)\} \cup \{(1,4),(1,5),(2,4),(2,5)\}$
$= \{(1,2),(1,3),(1,4),(1,5)(2,2),(2,3),(2,4),(2,5)\}$

(2) $\{1,2\} \times \{2,3,4,5\} = \{(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5)\}$

b. $M \times (N \cup P) = (M \times N) \cup (M \times P)$, so for this example at least, "x" is distributive over $\cup$.

5. a. $\{(2,0),(2,1),(4,0),(4,1),(4,2)\}$

b. A X B

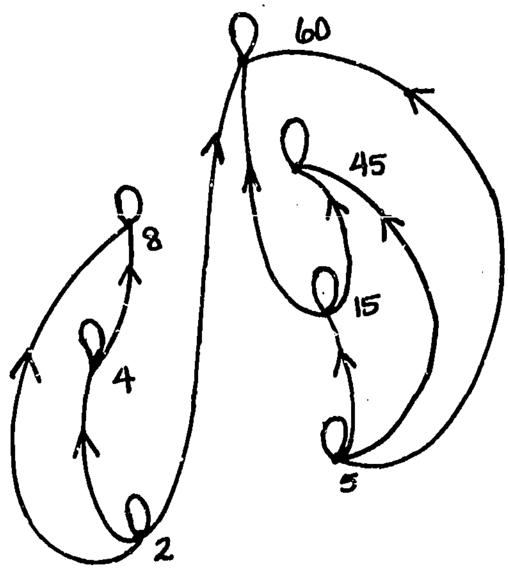
c. Not true, because 0 is not greater than 2, (also (0,2) is not in the set of pairs listed in part (a).)

d. Not true, because while $4>3$ is true, $3 \notin B$.

6. a. $\{(2,2),(2,4),(2,8),(2,60),(4,4),(4,8),$
$(4,60),(5,5),(5,15),(5,45),(5,60),$
$(8,8),(15,15),(15,45),(15,60),(45,45),(60,60)\}$

b. A x B

c.

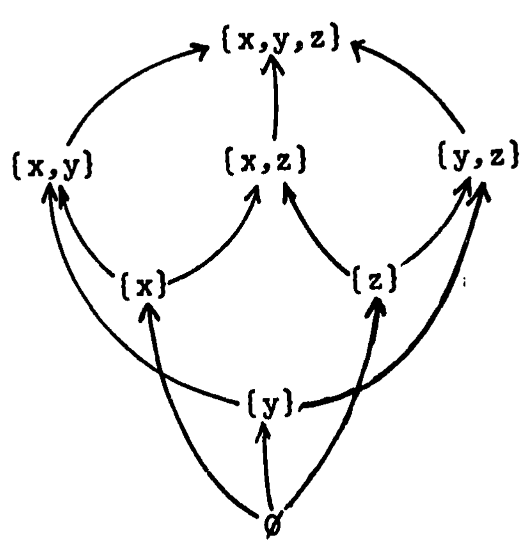


d. From the above diagram, a b will be true if and only if there is an arrow <u>from</u> a <u>leading</u> <u>to</u> b.

7. a. {x}

∅

b.



8. a. (1)F (2)T (3)F (4)T (5)T (6)F (7)T (8)F

b. {(1,2),(2,2),(4,2)}

c. {(4,1),(4,2)}

9. a. No, because to be a mapping you must have exactly one 2nd element for every possible 1st element in a pair. For example:

   $\{(1,2),(1,3),(2,3)\}$ is not a mapping because 1 as a first element has both 2 and 3 as possible 2nd element.

   $\{(1,2),(2,3)(4,2)\}$ is not a mapping of the set $\{1,2,3,4\}$ because 3 is not mapped; i.e. there is no pair with 3 as a first element. This set of pairs does describe a mapping of the set $\{1,2,4\}$, however.

   b. Yes, because a mapping of set A into set B can always be written as a set of ordered pairs (a,b) where $a \in A$ and $b \in B$, and this will be a subset of A x B. This subset defines a relation.

   c. If the pairs are indicated on the graph by x's, we look to see if exactly one x appears in every column. R is a mapping if and only if this holds.

10. There are mn possible pairs in the set A x B. There are 2^{mn} possible subsets of A x B, so there are 2^{mn} possible relations that could be defined.

8.11 Properties of Relations (Estimated time 2 days)

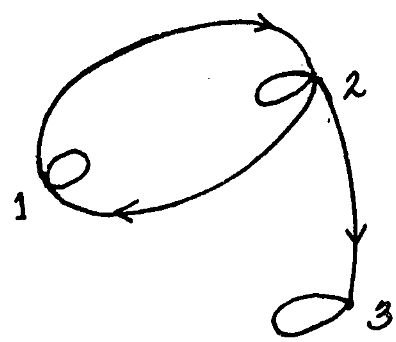
This section contains many important ideas. Have the students examine the examples carefully. Encourage them to construct further examples as well as counter-examples, i.e. reflexive relations vs. non-reflexive ones, symmetric vs. non-symmetric, transitive vs. non-transitive. The goal here is the notion of an equivalence relation, a notion which is very important in future work.

Terms such as "irreflexive", "anti-symmetric", etc. are not introduced here, but they are included in a few optional exercises in sections 8.12 and 8.14.

8.12 Exercises

Teachers may have students suggest relations besides those found in these exercises, and have these relations discussed. Family relationships are interesting and instructive.

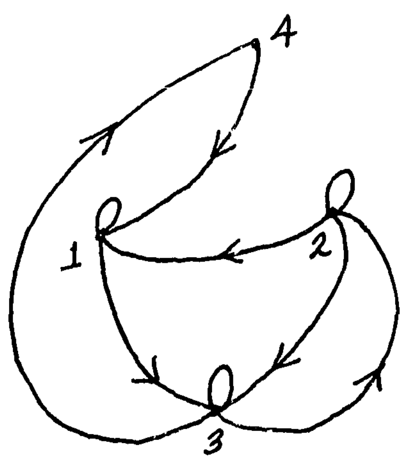
1. a. R is a subset of E x E.



c. It is reflexive, because every element has an arrow running to itself. It is not symmetric, because while there is an arrow running from 2 to 3, there is none from 3 to 2.

8.12 Exercises Continued

2. a.



b. Not reflexive, since $(4,4) \notin S$.

Not symmetric, since $(1,3) \in S$ but $(3,1) \notin S$.

Not transitive, since $(2,3) \in S$ and $(3,4) \in S$ but $(2,4) \notin S$.

3. a. Yes b. No c. Yes d. No e. Yes f. No g. Yes
4. a. No b. Yes c. No d. No e. Yes f. No g. Yes
5. a. Yes b. No c. Yes d. Yes e. Yes f. No g. Yes
6. (e) and (g)
7. a) When there is a number n in A such that (n,n) is not in the relation.

   b) When there is an ordered pair (a,b) in the relation but (b,a) is not in the relation.

   c) When (a,b) and (b,c) are in the relation but (a,c) is not. OR when aRb and bRc are in true but aRc is not.
8. (a) R_5 is reflexive

   (b) R_4 and R_5 are symmetric

   (c) R_3, R_4 and R_5 are transitive

9. (a) "is a brother of" is not reflexive

   (b) is not symmetric

   (c) is transitive

10. (a) (1) R_3 is reflexive

   (2) R_2, R_3 and R_4 are symmetric

   (3) R_1, R_3, R_4 and R_5 are transitive

   (b) R_3 is an equivalence relation on A

11. (a) The relation is reflexive

   (b) The relation is symmetric

   (c) It is transitive

   (d) It's an equivalence relation on the set of lines.

12. (a) (i) is reflexive.

   (b) (iii) is symmetric.

   (c) (i) and (ii) are transitive.

13. (a): (b) and (d) are irreflexive.

   (b) R is irreflexive.

   (c) R_5 is irreflexive

   (d) (ii) and (iii) are irreflexive.

14. (a) (b) (c) (d) and (f) are anti-symmetric.

   (b) R_3 and R_4 are anti-symmetric.

   (c) R_1, R_4 and R_5 are anti-symmetric.

   (d) (ii) is anti-symmetric.

8.13 Equivalence Classes and Partitions (Estimated time 2 days)

Some teachers may wish to introduce modular arithmetic at this time since this provides a clear example of how an equivalence relation partitions a set into equivalence classes.

(For example, for Z_3, the equivalence classes produced are $\{0, \pm 3, \pm 6, \pm 9, \ldots\}$

By now the students should know the meaning of an equivalence relation on a set. Now the students will learn that the relation effects a separation of the elements of a set into disjoint subsets. The collection of subsets produced by the equivalence relation R on G is called a partition of Y. Each of these subsets is called an equivalence class. Each must also be non-empty.

8.14 Exercises

Exercise 6 can be amplified by asking the students to supply other relations between lattice points which yield interesting equivalence classes. For example, the relation R_3 on lattice points defined by $(a,b)\ R_3\ (c, d)$ if and only if $a-b = c-d$ partitions z x z into equivalence classes. These also consist of "parallel lines". These exercises 8 and 9 are honor problems and only a few students, with help, will make progress with them. Exercise 10 is also optional. It introduces the notion of a partial order and pre-suppose Exercise 14 in Section 8.12.

8.14 Exercises (Answers)

1. (a) No. 4 is not in the union of the sets.

   (b) No. The sets are not all disjoint

$2 \in \{1, 2\}$ and $2 \in \{6, 2\}$

(c) Yes. $\{1, 3, 5\} \cup \{2, 4, 6\} = A$ and

$\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$

(d) Yes. The union of the sets = A

Intersections of pairs of sets = $\emptyset$

(e) Yes. For the reasons in (d)

(f) No. $4 \in \{1,2,3,5\}$ and $4 \in \{4,5,6\}$

(g) Yes. The two conditions are satisfied.

(h) No. $\{1, 2\} \cup \{3, 4\} \neq A$

2. $\{\{1, 2\}\}$; $\{\{1\}, \{2\}\}$

3. "<" does not partition W because "less than" is not an equivalence relation on W.

4. cRa means aRc by symmetric property.

aRc and cRb means aRb

by the transitive property.

5. $\{\{1\}, \{2\}, \{3\}, \{4\}\}$

$\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\}, \{2,3\}\}$

$\{\{1,2\},\{3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$, $\{\{1,4\},\{2,\},\{3\}\}$

$\{\{2,3\},\{1\},\{4\}\}$, $\{\{2,4\},\{1\},\{3\}\}$, $\{\{3,4\},\{1\},\{2\}\}$

$\{\{1,2,3\},\{4\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1,3,4\}, \{2\}\}$

$\{\{2,3,4\},\{1\}\}$, $\{\{1,2,3,4\}\}$

6. (a) The relations are reflexive, symmetric and transitive.

(b) R_1 divides the set of lattice points into subsets where each subset contains the lattice points whose first coordinate is the same. Each lattice point is in one and only one subset and the union of the subsets is the set of lattice points.

R_2 similarly divides partitions the set of lattice points by the second coordinate being the same for each point in the subset of S.

7. (a) R_1, R_4 and R_5 are equivalence relations on P.

(b) R_1 forms equivalence classes containing people living in a state.

R_4 forms equivalence classes of people who belong to the same political party. (Students will question the validity of this since some people do not belong to any political party. Do not try to resolve the question.)

R_5 forms equivalence classes of people whose IQ scores are the same.

8. $B_a = \{x:xRa\}$

Suppose $y \in B_a$. Then yRa is true. Then y is in the same equivalence class as a. Hence, Ba is a subset of the equivalence class Ca containing a.

Now suppose Z∈Ca; that is, Z is in the equivalence class containing a. Then by definition of equivalence class, ZRa is true. But then, by our definition of Ba, Z∈Ba.

Hence Ca < Ba.

But BaCCa and CaCBa ⟶ Ba = Ca. Hence for each a, Ba <u>is</u> the equivalence class containing a, so the sets Ba are exactly the equivalence classes in the partition of A effected by R.

9. (a) $S = \{\{1,2\}, \{1\}, \{2\}, \emptyset\}$. R is the set

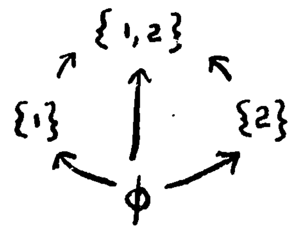
$\{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1,2\}), (\emptyset,\emptyset), (\{1\}, \{1\})$
$(\{1\}, \{1,2\}), (\{2\}, \{2\}), (\{2\}, \{1,2\}), (\{1,2), \{1,2\})\}$

This is reflexive, since $(\emptyset,\emptyset),(\{1\},\{1\}),(\{2\},\{2\}),$

$(\{1,2\},\{1,2\})$ are all in R.

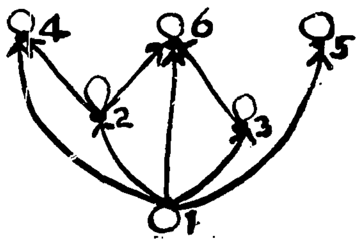
R is anti-symmetric, since whenever (a,b) and (b,a), then a = b. R is transitive by inspection.

(b)



Each arrow points in just one direction; i.e. we cannot have a<——>b (if a and b are different). Also, if one "follows" the arrows, it is not possible to "get back to" an element once having left it.

10. (a)



This relation is a partial ordering on E.

(b) "$a \angle a$" is never true, so the relation "$\angle$" is not reflexive, and hence is not a partial ordering.

8.16 Review exercises (Answers)

1. (a) (1) $\{-3,-2,-1,0,1,2,3\}$ or S
(2) $\{1\}$
(3) $\{-3,-2,-1,0,1,3\}$
(4) $\{-3,-1\}$
(5) $\{-3,-1,1,2,3\}$
(6) $\{1,3\}$
(7) A or $\{-3,-2,-1,0\}$
(8) $\{0\}$
(9) D or $\{0\}$
(10) B or $\{1,2,3\}$
(11) D or $\{0\}$
(12) S or $\{-3,-2,-1,0,1,2,3\}$

(b) (1) $\{1,2,3\}$ or B
(2) $\{-3,-2,-1,0\}$ or A
(3) $\{-2,0,2\}$
(5) $\{-2,0,1,2,3\}$
(6) $\emptyset$

(c) (1) D is a subset of A

(2) D is a proper subset of A

(3) A and B are disjoint

2. Some possible responses are

$\emptyset$ is a subset of A

A is a subset of the universal set

$A \cap A = A$

$A \cup A = A$

A is an improper subset of itself

$A \cap \bar{A} = \emptyset$

$A \cup \bar{A} =$ the universal set

$A \cup \emptyset = A$

$A \cap \emptyset = \emptyset$

$A \cap S = A$ ($S =$ universal set)

$A \cup S = S$

3. (a) {0,2}

(b) {0},{2},{0,2},$\emptyset$

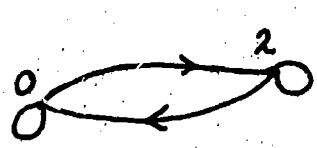
(c) {0},{2}

(d) {(0,0),(0,2),(2,0)(2,2)}

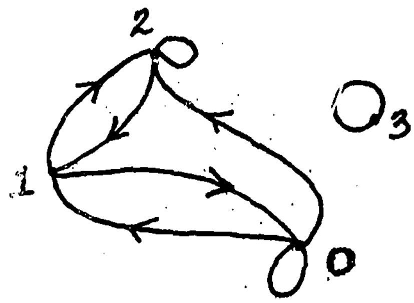
(e) No, since (0,1) is not an element of B x B.

(f) Yes.

(g)



4. (a)



(b) No. $(0,2) \in R$ but $(2,0) \notin R$

also $(1,1) \notin R$

(c) S would be an equivalence relation on U:

For, aRa, for each $a \in V$;

for each $(a, b) \in R$, $(b, a) \in R$;

and for (a,b) and (b,c) in R, (a,c) is in R.

5. (1) $(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B})$ by distributivity

(2) $= A \cap (S)$ by definition of $\overline{B}$

(3) $= A$ since $A \subset S$

6. a) "less than or equal to"

b) "is the same age as"

c) "is the father of"

d) "is greater than"

e) "is a classmate of"

7. a) False Example: $S = \{1,2,3,4,5\}$

b) True $A = \{2,3\}$ $\overline{A} = \{1,4,5\}$

$B = \{2,3,4\}$ $\overline{B} = \{1,5\}$

$A \subset B$ but $\overline{A} \not\subset \overline{B}$

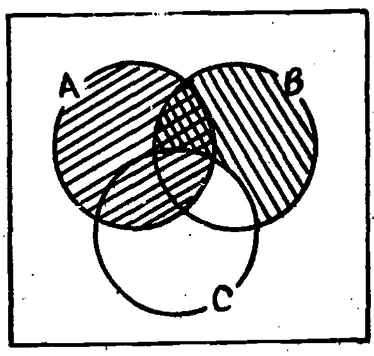
and $\overline{B} \subset \overline{A}$

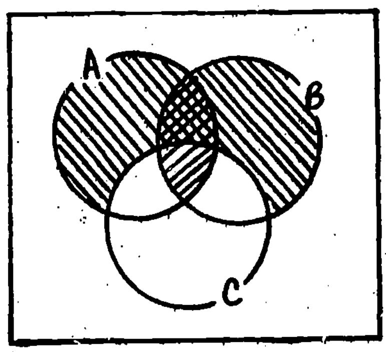
8. a) Not a partition of D since the sets are not all disjoint sets.

   b) It is a partition since the interaction of pairs of sets is the null set and the union of the three sets is D.

   c) It is not the partition since the union of the three sets is not D.

9.





$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Course I Chapter 9

Transformation of the Plane

Commentary for Teachers

Introduction (11 days)

The main objective of this chapter is to provide an experiential background for the following basic transformations in a plane:

(a) Reflection in a line.

(b) Reflection in a point.

(c) Translation.

(d) Rotation.

On the basis of this experience, the children should conjecture that all four transformations are one-to-one mappings of the plane onto itself preserving:

distance	midpoint
collinearity	angle measure
betweeness	parallelism

Under reflection in a point and translation, a line and its image are parallel.

If a figure and its image are identical under:

(a) some line reflection, then the figure is symmetric in a line.

(b) some point reflection, then the figure is symmetric in a point.

(c) some rotation which is not a multiple of a complete rotation, the figure has rotational symmetry.

Devices, other than those shown in text, may be suggested by your students for finding images under a mapping. A pin may be used to pierce holes in a creased paper. A soft lead pencil serves well in place of ink spots--just fold and use your pencil or finger nail to press over penciled sections.

Encourage your students to think of properties that are not mentioned in the text. For example: intersecting lines map into intersection lines, perpendicular lines map into perpendicular lines, circles map into circles, a composition of an even number of reflections in parallel lines corresponds to a translation, a composition of an even number of reflections in concurrent lines corresponds to a rotation.

You might challenge your students to find mappings that do not preserve distance, or collinearity, or parallelism. You might suggest dilation in a point (or line) of a plane.

If the notion of a group has been developed you can ask for sets of mappings that form a group under composition. For example, the following set of mappings are groups under composition:

(a) translations

(b) point reflection together with translations

(c) rotations in a fixed point

(d) reflections in a fixed line together with the identity map

(e) reflection in a fixed point together with the identity map

(f) reflections in two perpendicular lines together with point reflection in their intersection and the identity map.

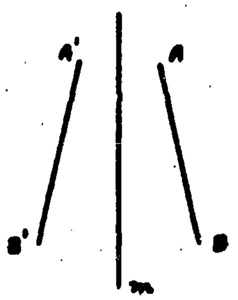
9.1 and 9.2 (2 days)

The purpose of these sections is to show experimentally the properties of a line reflection. The basic property demonstrated by the activities is to show that this mapping is an isometry. Having seen that it is an isometry, the preservation of betweeness and collinearity is also shown.

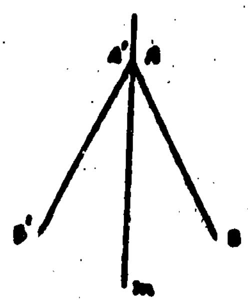
Many of these problems could be done in class. Parts of problems 5-8 could be done in class and parts at home. Problems 9-13 could be done at home as well as problem 15. Problem 11 is an important problem for future use and problem 14 can be considered optional.

9.3 Answers to Exercises

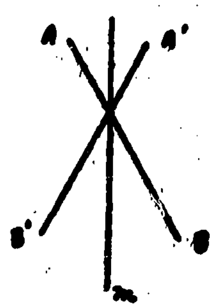
1. All points on the line of reflection.
2. The left hand.
3. Spinning counter-clockwise.
4. A,B,C.
5. (a)



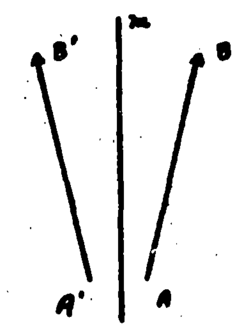
(b)



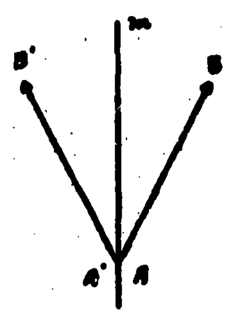
(c)



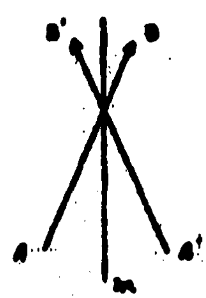
6. (a)



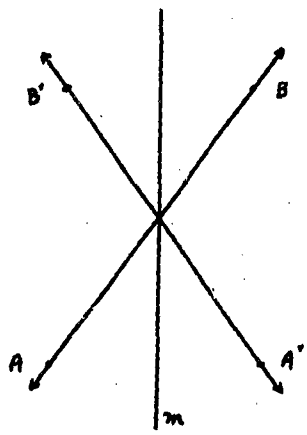
(b)



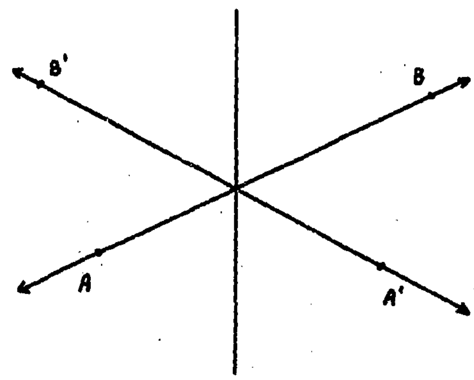
(c)



7. (a)



(b)



8. (a) A line containing A and perpendicular to m.

   (b) m and a line perpendicular to m containing A.

9. (a) Crease paper so that m folds on itself and the crease contains A.

   (b) Same as (a), but m is identical to its reflection in m.

10. The cutout figure is symmetric in the line at the crease.

11. A B C D E H I K M O T U V W X Y

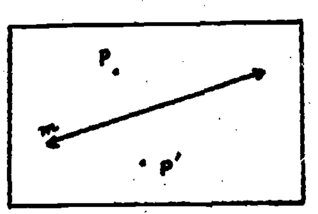
    No.

12. An activity.

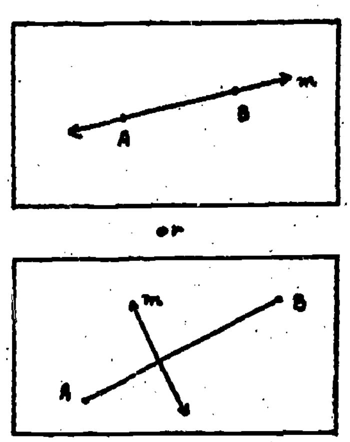
13. The name reads correctly.

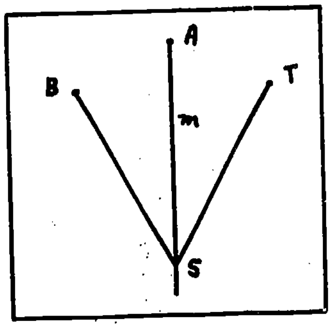
14. An activity.

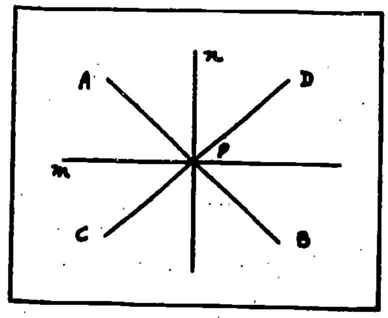
15. (a)



(b)



(c) 

(d) 

m
or
n

(e) In (a), m $\perp$ $\overline{PP'}$ and bisects PP'.

In (b), m $\perp$ $\overline{AB}$ and bisects AB or $\overline{AB} \subset$ m.

In (c), the angle at ST and SA is the same as the angle at SR and SA.

In (d), m $\perp$ n. Also many angles as in (c) are the same.

9.4 (1 day)

The object of this section is to introduce the idea of lines, rays, and segments. Stress should be placed upon the way that each of these is named and the essential difference between them. Note that the idea of an open halfline does not appear until Section 10.2 of Chapter 10. A ray is considered the same as a closed halfline or just a halfline. When naming a ray the procedure here is to place the arrow over the two letters so that it moves from left to right. Thus, "A ——— B"> is named AB and not BA.

9.5 <u>Exercises</u>

Problems 1 and 2 may be done in class and exercise 3 is a good one to be done for homework.

9.5 Answers to Exercises

1. (a) 3

   (b) AB, AC, BC

   (c) BA, CA, CB

2. (a) 6

   (b) 10

   (c)

No. of Points	2	3	4	5	6
No. of Lines	1	3	6	10	15

Children may recognize that the differences are 2, 3, 4, 5, increasing by one. Do not expect the generalization that the number of lines is $\frac{n(n-1)}{2}$ where n is the number of points.

   (d) Any one of n points may be selected and then any one of the remaining n-1 points. There are n(n-1) selections counting order. There will be two selections for each line.

3. (a) AB, AC, BC, BD, CD

   BA, CA, DA, CB, DB, DC

   (b) AB, BC, CD, BA, CB, DC

   accept also the ray opposite to AB and the ray opposite to DC.

   (c) 6 (or (8))

   (d)

No. of points	1	2	3	4	5
No. of rays	2	4	6	8	10

   (e) The number of rays is twice the number of points.

   (f) With the addition of each point two new rays are obtained.

(g) AB, AC, AD, BC, BD, CD

(h) 6

(i)

No. of points	2	3	4	5	6
No. of segments	1	2	6	10	15

(j) and (k) see exercise 2.

9.6 ($\frac{1}{2}$ day)

The purpose of this section is to define perpendicular lines as two lines such that either is its own reflection in the other.

9.7 ($\frac{1}{2}$ day)

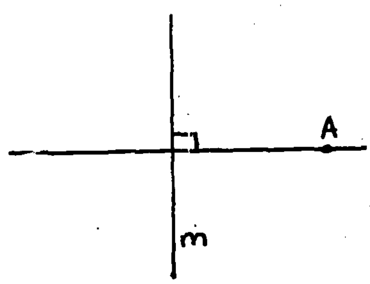
The purpose of this section is to introduce the student to a method of comparing the measure of angles by the measure of their span using a compass. The purpose of activities 5-7 is to demonstrate that angle measure is preserved by line reflections. Note that the idea of measuring an angle using a protractor will be discussed in section 10.19 of Chapter 10.

Activity 5 could be done in class while the others could be done as part of a homework assignment.

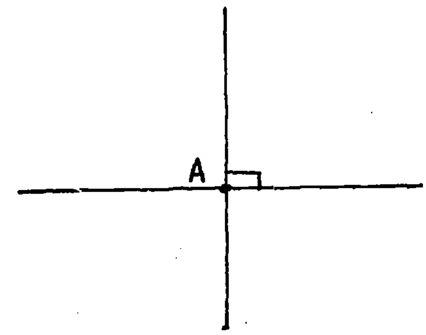
9.8 Exercises

Problem 4b previews the differences between similar and congruent triangles. A very informal discussion might be profitable in class.

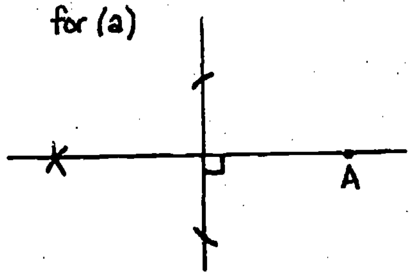
1. (a)



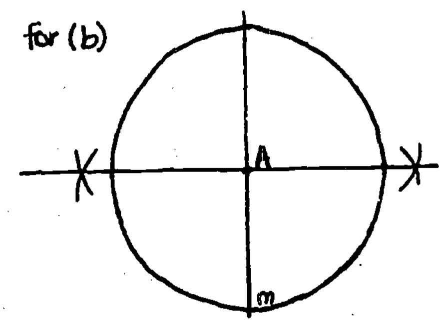
(b)



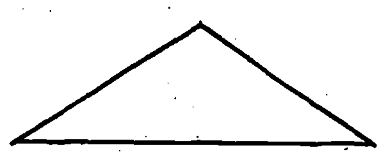
(c) for (a)



for (b)



2. (a) It has two sides of the same length.

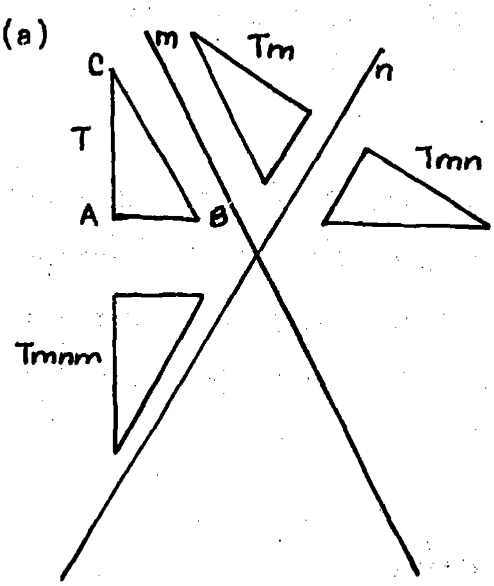


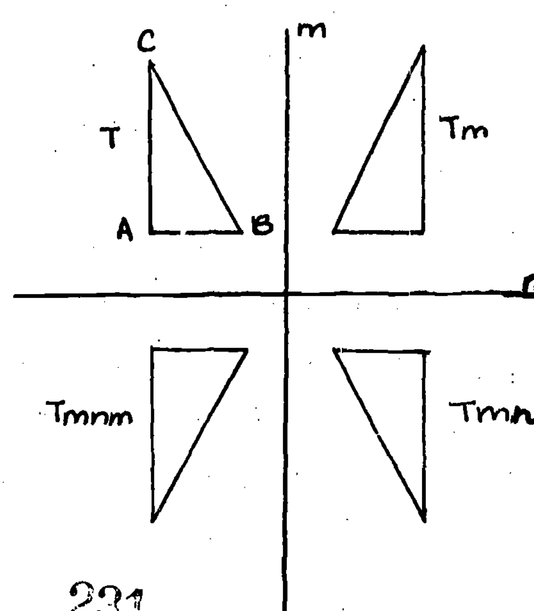
(b) There is no such triangle.

(c) All 3 sides have the same length.

(d) No.

3. (a)





(b) A variety of generalizations are possible. Among them are the fact that T_m and T_n commute under the operation "o". Also, all the triangles are the same size and shape and the same distance from the line of reflection.

4. (a) The radii $\overline{PA}$ and $\overline{QC}$ have different lengths and so cannot be used to compare angle measures.

   (b) Not necessarily. Consider 2 equilateral triangles with sides of different lengths.

5. For such rays a relatively large increase in the opening produces but a slight increase in the angle measure.

6. (a) No.

   (b) No.

9.9 (2 days)

The purpose of this section is to introduce the concept of reflection in a point and its properties. Comparison is made between line reflections and point reflections. Students should be encouraged to see the properties preserved by themselves rather than told what these properties are.

9.10 <u>Exercises</u>

Problem 3 helps clarify the meaning of symmetry in a point and symmetry in a line. Note that the names of the mappings are "reflection in a line" and "reflection in a point", while the properties of the figures are called "symmetry in a line" and "symmetry in a point". Problem 11 is optional.

1. P

2. (a) Yes. Its midpoint.

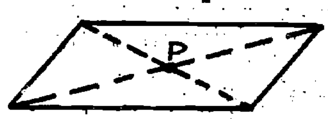
   (b) No.

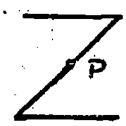
   (c) Yes. Any point in the line.

   (d) Yes. Any point in a midway parallel line.

   (e) Yes. The point at which the diagonals cross.

   (f) Yes. The midpoint of the diagonal bar.





3.

Letter	Symmetry in a point	Symmetry in a line
A	NO	YES
B	NO	YES
C	NO	YES
D	NO	YES
E	NO	YES
F	NO	NO
G	NO	NO
H	YES	YES
I	YES	YES
J	NO	NO
K	NO	YES
L	NO	NO
M	NO	YES
N	YES	NO
O	YES	YES
P	NO	NO
Q	NO	NO
R	NO	NO
S	YES	NO
T	NO	YES
U	NO	YES
V	NO	YES
W	NO	YES
X	YES	YES
Y	NO	YES
Z	YES	NO

Some may say that L is symmetric in a line. Accept it after their explanation.

4. (a) Yes. The $\perp$ bisector of the segment or the line that the segment lies along.

(b) Yes, the carrier of the ray.

(c) Yes. Any m $\perp$ l or the line itself.

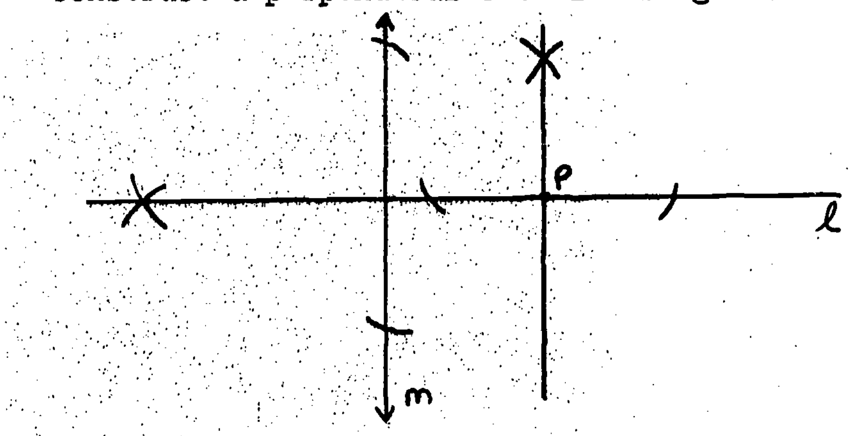
(d) No.

5. Activity.

6. Activity.

(a) Fold paper so that m falls on itself and the crease runs through P. Open up and fold so that the crease falls on itself and the new crease runs through P. The new crease should be parallel to m.

(b) First construct a perpendicular from P to m. Now construct a perpendicular to l through P.



7. (a) symmetry in a line / none

(b) symmetry in a line, in a point, and rotational symmetry

(c) symmetry in a line, a point, and rotational symmetry

(d) symmetry in a line, in a point, and rotational

(e) symmetry in a line

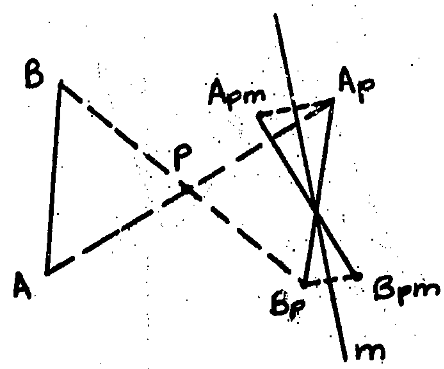
(f) symmetry in a line and rotational symmetry

(g) symmetry in a line, symmetry in a point, and rotational symmetry

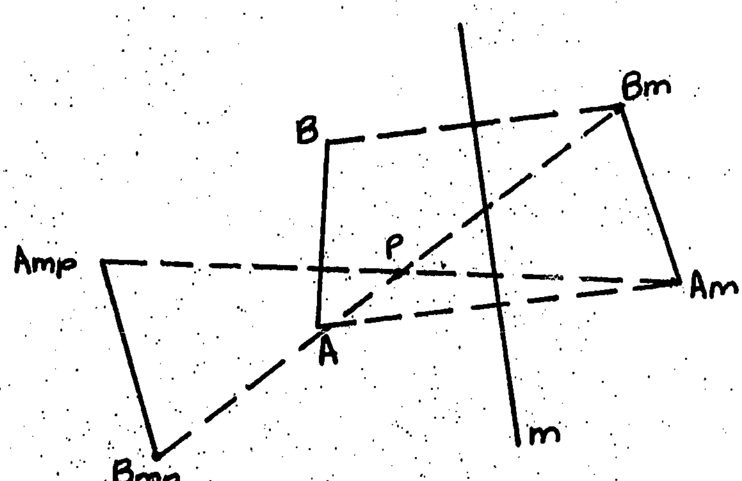
(h) symmetry in a point and rotational symmetry

(i) symmetry in a line

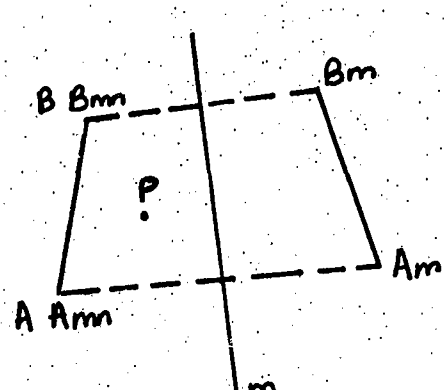
8. (a)



(b)

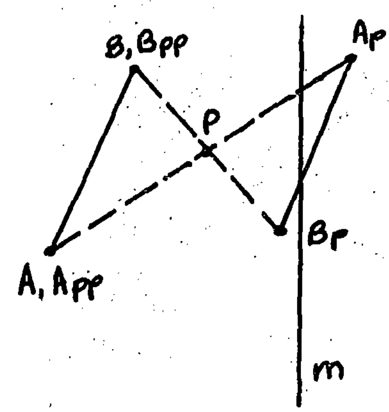


(c)

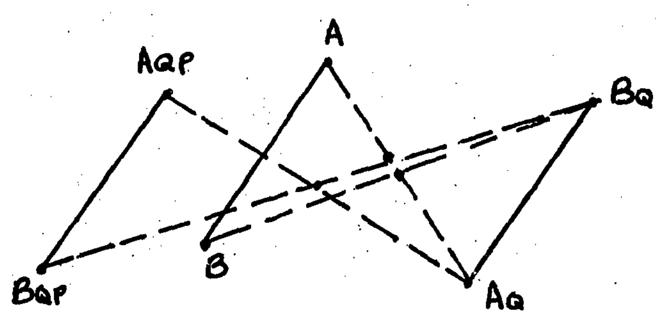


$S_P \circ S_P$ is the identity mapping.

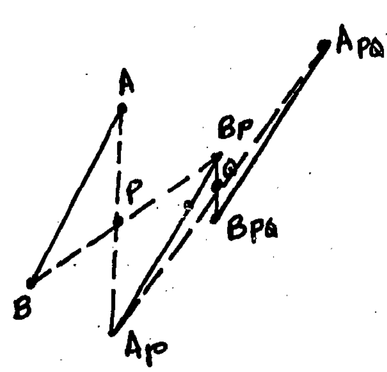
(d)



(e)



(f)



(g) b, c, d, e, f

9. There are many possible answers. We shall give but one.

(a) A translation or reflection through a point or line will do.

(b) A reflection in the line that bisects $\overline{AC}$ and $\overline{BD}$. Also, see (c)

(c) Use the midpoint of $\overline{AC}$ as the point for a point reflection. Then reflect in the line that bisects the angle formed by $\overline{CD}$ and the image of $\overline{AB}$.

10. Activity. [Note: $l_r \circ l_s$ means compose l_r with l_s.]

    (c) (1) No (2) Yes (3) Yes

11. (a) Reflect T in m.

    (b) Find the image of T under the symmetry in P.

9.11 (1 day)

The purpose of this section is to examine the properties of a translation as well as the properties of composition of translations.

9.12 <u>Exercises</u>

1. None (except for the identity)
2. (a) None

   (b) None

   (c) Any translation that is parallel to the given line.

   (d) Any translation

   (e) Any translation that is parallel to the edge of the halfplane.
3. An activity.

   The faces are a union of the original face and its image under 2, 4, 6, 8 taken one at a time.
4. Activity.

9.13 (2 days)

The purpose of this section is to describe rotations as geometric transformations. The term rotational symmetry applies to a particular figure when there is a point, a rotation which is less than a full rotation but not a zero rotation that maps the figure onto itself. Note that one must specify either a clockwise or counterclockwise direction when considering rotation. The problem answers have considered only a counter-clockwise direction.

9.14 Exercises

Problems 1-3, and 7-10 could be used in class. Problem 5 is a good homework problem. The table for 5d in the answer sheet is a row-column table in this sense: a $^\circ$ b means put a in the row and b in the column. The rotations are considered counter-clockwise.

1. H I N O S X Z
2. Preserved: Distance, collinearity, betweeness, midpoint, angle measure, parallelism, perpendicularity.

   Not preserved: Direction
3. (a), (b), (e)
4. (a) line symmetry

   (b) line symmetry

   (c) and (d) line symmetry, point symmetry, and rotational symmetry

5.

	e	l_m
e	e	l_m
l_m	l_m	e

	e	l_m	l_n	S_P
e	e	l_m	l_n	S_P
l_m	l_m	e	S_P	l_n
l_n	l_n	S_P	e	l_m
S_P	S_P	l_n	l_m	e

	e	P	P	P
e	e	P	P	P
P	P	P	P	e
P	P	P	e	P
P	P	e	P	P

	e	P	P	l_r	l_s	l_t
e	e	P	P	l_r	l_s	l_t
P	P	P	e	l_t	l_r	l_s
P	P	e	P	l_s	l_t	l_r
l_r	l_r	l_s	l_t	e	P	P
l_s	l_s	l_t	l_r	P	e	P
l_t	l_t	l_r	l_s	P	P	e

6. (a) l_m (b) S_P (c) $P_{3/4}$ (d) $P_{2/3}$ (e) $P_{1/2}$

7. All four preserve (a), (b), (c), (d), (e), (f)

8. None that we have studied--except perhaps dilations (a)

9. (a) The image of a point A, A', is such that $\overline{AA'}$ is $\perp$ to the plane and is bisected by the plane.

   (b) The image of a point A, A', is such that $\overline{AA'}$ is $\perp$ to the line and is bisected by the line.

   (c) and (d) natural extensions into space.

10. (a) The line.

    (b) The point.

    (c) The direction and magnitude.

    (d) The point and magnitude and direction.

Review Exercises (1 day)

1. All answers are "Yes".

2. (a) and (b) Reflection in a line.

   (c) Symmetry in a point.

   (d) Reflection in a line and symmetry in a point.

   (e) All except translation.

   (f) Symmetry in a point and rotational symmetry.

   (g) Translation.

   (h) Translation, symmetry in a point.

3. (a) The line.

   (b) The point.

   (c) None.

   (d) The point of rotation.

4. (a) If the line of a reflection is perpendicular to the given line or it is the given line.

   If the point of symmetry is in the given line.

   If the direction of a translation is parallel to the given line.

   If the rotation is a half turn about a point in the line.

   (b) If the line of reflection contains the ray.

   (c) If the line of reflection is the $\perp$ bisector of the segment or the line of the segment.

   If the point of symmetry is the midpoint of the segment.

   If the rotation is a half turn about the midpoint of the segment.

   (d) If the line of reflection bisects the angle determined by the rays.

(e) If the line of reflection contains a diagonal or the midpoints of two opposite sides.

If the point of symmetry is the intersection of the diagonals.

If the rotation is a multiple of the quarter turn about the center.

(f) If the line of reflection contains the midpoints of two opposite sides.

If the point of symmetry is the intersection of the diagonals.

If the rotation is a multiple of the half turn about the center.

(g) If the point of symmetry is the intersection of the diagonals.

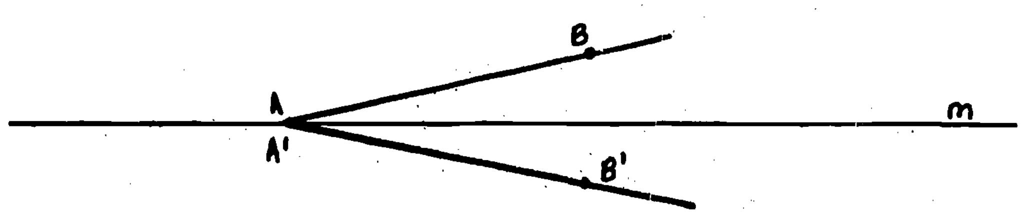
5. When either is its own image under a reflection in the other.

6. $m \perp n$.

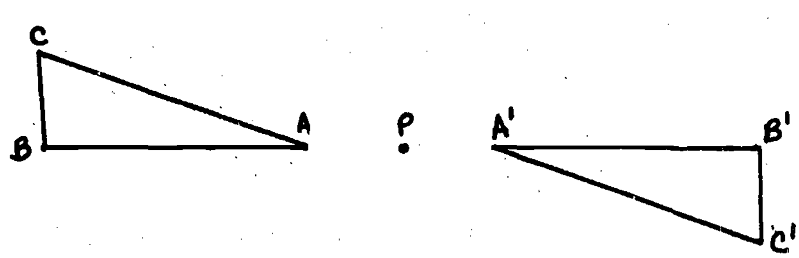
7. The point where m and n intersect.

8. (a) 1 (b) 2 (c) 1 (d) 2 (e) 3

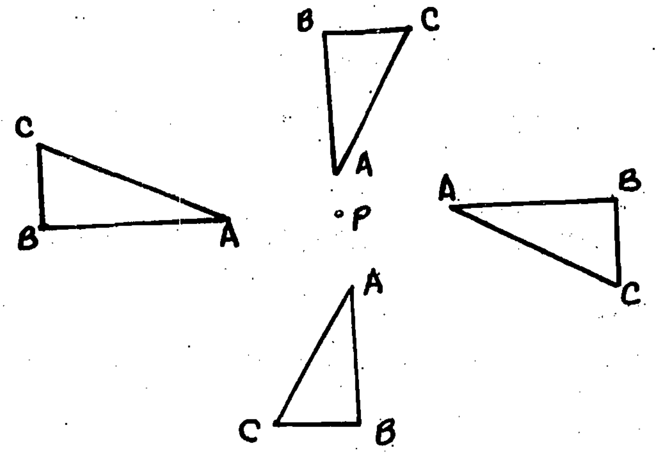
9.



10.



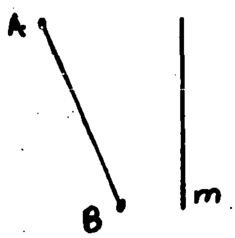
11.

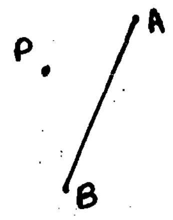


Test on Chapter 9

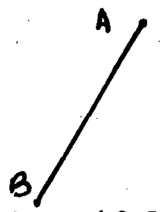
1. Without folding, find $\overline{A'B'}$ the image of $\overline{AB}$ under the

   (a) reflection in m   (b) reflection in P

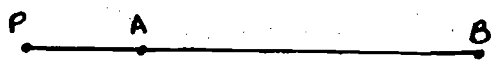




   (c) translation $\overrightarrow{2}$ (unit is the inch)



   (d) rotation $P_{1/4}$ (full credit is given for a good guess, but try to figure out a construction method)

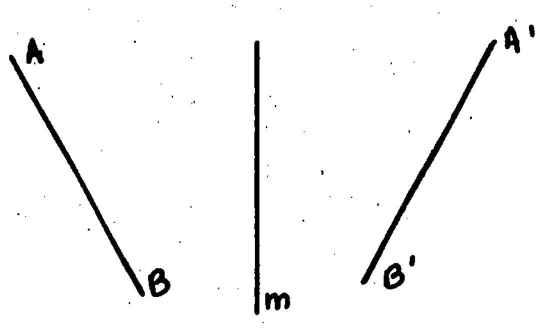


2. List five properties common to all the mappings listed in 1.
3. For which two of the four mappings studied are a line and its image always parallel?

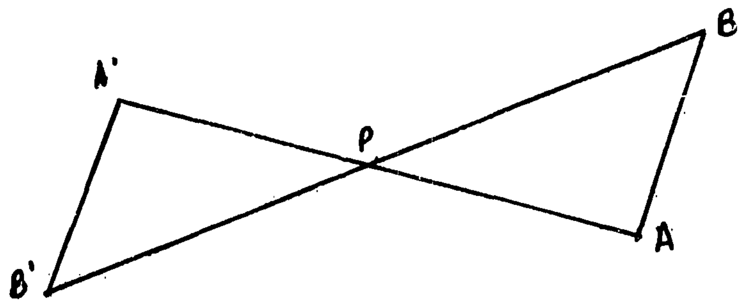
4. Under what circumstances will a line be identical with its image under a
   (a) reflection in a line
   (b) symmetry in a point
   (c) translation
   (d) rotation
5. If a point and its image are one and the same point, we call the point a fixed point under the mapping. What are the fixed points for
   (a) l_m (b) S_P (c) $\overrightarrow{2}$ (d) $P_{1/4}$
6. Which of the mappings listed in Exercise 5, when composed with itself, gives the identity mapping?
7. What kinds of symmetry does each of the following have? (Describe each specifically--give the point, or line, or turn, etc.)
   (a) a parallelogram
   (b) a square
   (c) an equilateral triangle
   (d) a circle
8. Define perpendicular lines in terms of reflection.
9. What is an isometry?

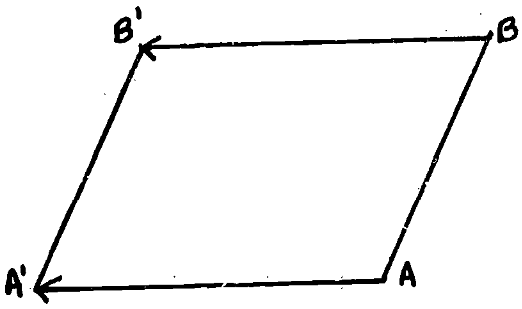
<u>Answers to Test Questions</u>

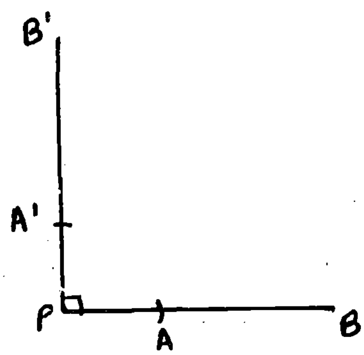
1. (a)



(b)







2. Preserve: distance, collinearity, betweenness, midpoint, angle measure.

3. Symmetry in a point and translation.

4. (a) When the line is $\perp$ to the line of the reflection or when it is the line of reflection.

   (b) When the line contains the point of the symmetry.

   (c) When the line is parallel to the direction of the translation.

   (d) When the rotation is a multiple of a half turn and the line contains the point of rotation.

5. (a) All the points of m.

   (b) P

   (c) None.

   (d) P

6. l_m and S_P.

7. (a) point symmetry.

   (b) line symmetry, point symmetry, and rotational symmetry.

   (c) line symmetry, point symmetry, and rotational symmetry.

   (d) line symmetry, point symmetry, and rotational symmetry.

8. Lines are perpendicular when either line is its own reflection in the other line.

9. An isometry is a mapping that preserves distance.

Course I Chapter 10

Segments, Angles and Isometries

(16-21 days)

Commentary for Teachers

The major purpose of this chapter is to sift out of students' experiences and intuition some mathematical concepts and principles concerning segments, angles, and isometries which may be use quickly and easily to derive some theorems. There is no persistent attempt to use formal procedures. The methods of deduction and experimentation are used indiscriminately leaving it to teachers and students to "do what is natural".

One may see this chapter in four parts:

Part 1. Segments. Sections 10.2-10.9.

Part 2. Coordinates and Isometries. Sections 10.10-10.16.

Part 3. Angles. Sections 10.17-10.23.

Part 4. Angles and Isometries. Sections 10.24-10.31.

One may also regard as the purpose of this chapter to create an organization in the set of points in a plane, for out of an organization (a set of basic relations) one can deduce theorems. The first attempt in this effort to organize is to consider, for points on a line, the Line Separation Principle. From this principle, that a point of a line separates it into two disjoint sets, we are led to a recognition of open halflines, halflines or rays, and segments. (Our definition of a halfline, which is synonymous with ray, is not the usual one. We prefer a halfline to resemble a halfplane, both of which are defined as including their boundaries. We could have defined them both as open sets, 10.2 <u>Lines, Rays, Segments</u> 10.3 - 1 day

The concept in this section to be emphasized is the halfline. The student has had experience with lines and rays in Chapter 9. The first Separation Principle is defined in this section.

The exercises may be done in class.

10.3 <u>Exercises</u>

1. $\overrightarrow{CD}$, $\overrightarrow{CB}$; $\overrightarrow{CE}$, $\overrightarrow{CB}$.
2. a. $\overline{AC}$ d. $\overrightarrow{AC}$ g. $\overline{AD}$ j. $\emptyset$
   b. $\overrightarrow{AC}$ e. $\overline{BC}$ h. $\overline{BD}$ k. $\emptyset$
3. a. $\overrightarrow{BE}$ b. $\overrightarrow{BA}$ c. $\overline{AC}$ d. $\overrightarrow{BD}$; yes $\overrightarrow{AC}$, $\overrightarrow{AD}$...
4. a. T b. F c. T d. F e. F f. T g. T
5. a. $-2 \leq x \leq 3$ c. $x \leq 3$ e. all numbers g. $x \geq 0$
   b. $x \geq -2$ d. $x > -2$ f. $0 \leq x \leq 3$ h. all numbers

10.4 <u>Planes and Halfplanes</u> 10.5 - 1 day

The second separation principle is defined in this section: that of the plane. The Exercises 1-7 may be done in class as they continue the development and reinforcement of this section. Exercise 8 may be assigned.

10.5 <u>Exercises</u>

1. T 2. F 3. T 4. F 5. F 6. T 7. F
8. $QII = H_{-x} \cap H_{+y}$; $QIII = H_{-x} \cap H_{-y}$; $QIV = H_{+x} \cap H_{-y}$

but because there are more occasions to refer to "closed halfplanes" than to "open halfplanes", we allow ourselves the convenience of defining halfplane to be a "closed halfplane".)

We use coordinates extensively, first line coordinates and then plane coordinates. It is hoped that students will learn to use them easily. Thus distances and midpoints are treated in terms of line coordinates in Part 1.

The second major step in organizing or structuring the set of points in a plane is taken by the Plane Separations Principle (also in Part 1) and this leads to halfplanes, which in Part 3, paves the way for angles.

In Part 2 we continue the use of coordinates to treat isometries. The important item in each case is the coordinate formula that serves as the rule of the isometry. In Section 10.10 we introduce coordinates by showing how useful they are in extending isometries from a pair of points to the points of the line containing the pair, or from a triplet of noncollinear points to the plane containing the points. This notion may prove to be too sophisticated for 7th year students, and if this is indeed the case, one may omit this section entirely. It has no accompanying set of exercises.

We list the coordinate formulas for isometries. It should be noted, however, that the formulas for the two line reflections are valid only for rectangular coordinate systems, while the other two are valid in any coordinate system.

Translation from (0,0) to (p,q): $(x,y) \longrightarrow (x + p, y + q)$

Reflection in the x-axis: $(x,y) \longrightarrow (x,-y)$

A study of the properties that are invariant (preserved) under these isometries use these coordinate formulas and when appropriate, equations and slopes of lines. Both equations and slopes serve in studies of collinearity and parallelism.

It is noteworthy that perpendiculars are introduced through line reflections and that rectangular coordinate systems follow naturally therefrom. (Perpendicularity is continued here after having been introduced in Chapter 9.)

In Part 3 we develop a concept of angle which differs from the one generally developed in the United States. It has been suggested by Professor G. Choquet as the result of his experiences in teaching children and the experiences of other professors in Europe. He believes (and we concur) that the notion of the angular region is closer to one's intuitive notion of an angle than the union of two noncollinear rays having the same endpoint. Our analysis of the analogy between segments and the region notion of an angle leads us to agree with Professor Choquet. In fact, we suggest that you exploit this analogy to the hilt in teaching the concept of angle and the measurement of angle. For some suggestions of this analogy, see Exercise 5 in Section 10.18.

Section 10.21, on Boxing the Compass, will probably be of interest to Boy Scouts as well as others. However, our major purpose in presenting it is to exhibit the bisection process. This process is also used in graduating an inch-ruler as well as other linear systems. Halving is a natural folk concept, as found for instance, in the measure "half of a quarter" instead of an "eighth", and as such has a firm place in one's intuition. The Process can also be used to construct a sequence of rational numbers whose limit is an irrational number. <u>However, this section may be omitted without loss of continuity</u>. (For this reason we have not written an accompanying set of exercises. If pressed for time--this may be omitted.)

In Part 4 we study the relationship between angles and these types of isometries. Quite early we come to the important property that ismetries, which by definition preserve distance, turn out to preserve angle measures also. This furnishes the underlying program for Sections 10.24-10.29. It leads quickly to a demonstration that the measures of the base angles of an isosceles triangle are the same, that similar properties are found for kites and parallelograms, and to the "Z angles" (alternate interior angles of parallel lines) and the "F angles" (corresponding angles of parallel lines). Part 4 ends with a lengthy study in which isometries are used to show the triangle-angle-measure-sum property. This basic property then provides the student with many opportunities to make simple deductions (corollaries). Thus this chapter ends with an intensive experience in deductions.

Students will use frequently the property of isometries that preserves angle measures. This may suggest the possibility that all mappings that preserve angle measure are isometries. They should be quickly disabused of this error. We included an exercise among the Review Exercises, Exercise 13, which dispels the notion. If students show interest in this matter they should be encouraged to discover for themselves that while distances are not preserved it is true that ratios of distances are preserved. We have here the beginnings of the mapping of the plane into itself which is called a similitude.

10.6 Measurement of Segments 10.7 - 1 day

Measurements of segments are discussed as mappings which assign the endpoints of a segment to corresponding numbers on a ruler. The section also points out that a translation preserves distance. It might be helpful to have the students experiment in class with rulers to emphasize the ideas of mappings and translations. The teacher should emphasize the idea of a line coordinate system and the coordinate of a point, as this will be discussed again in Course II--Coordinate Geometry.

10.7 Exercises

1. a. $|0 - 1\frac{1}{2}| = 1\frac{1}{2}$ e. $|\frac{1}{2} - 1\frac{1}{2}| = 1$ i. $|1\frac{1}{2} - 2| = \frac{1}{2}$

   b. $|0 - 3\frac{1}{2}| = 3\frac{1}{2}$ f. $|\frac{1}{2} - 2| = 1\frac{1}{2}$ j. $|4\frac{1}{4} - 1\frac{1}{2}| = 2\frac{3}{4}$

   c. $|0 - 5\frac{9}{16}| = 5\frac{9}{16}$ g. $|4\frac{1}{4} - \frac{1}{2}| = 3\frac{3}{4}$ k. $|3\frac{1}{2} - 4\frac{1}{4}| = \frac{3}{4}$

   d. $|4\frac{1}{4} - 0| = 4\frac{1}{4}$ h. $|5\frac{9}{16} - \frac{1}{2}| = 5\frac{1}{16}$ l. $|4\frac{1}{4} - 5\frac{9}{16}| = 1\frac{5}{16}$

2. 3 or -3
3. $|x - 8| = 2 \longrightarrow (x - 8 = 2 \text{ or } x - 8 = -2) \longrightarrow (x = 10 \text{ or } 6)$
4. $|x - 83| = 6\frac{1}{2} \longrightarrow (x = 89\frac{1}{2} \text{ or } 76\frac{1}{2})$

10.8 Midpoints and Other Points of Division 10.9 - 2 days

These two sections should be developed carefully. The formulas for midpoints and for other points of a segment are developed and then reinforced in the Exercises 1-4 of 10.9.

It would be wise to do these in class since they are needed later. Exercise 5 may be assigned for homework. (Exercises 2, 3, 4 develop the Triangle Inequality Property.)

10.9 Exercises

1. a. If x is assigned to B then $5 < x < 12$ guarantees that B is between A and C.

   b. $|5 - 8| + |8 - 12| = |5 - 12|$ or $3 + 4 = 7$

   $|5 - 11\frac{1}{2}| + |11\frac{1}{2} - 12| = |5 - 12|$ or $6\frac{1}{2} + \frac{1}{2} = 7$

   c. $AB = |5 - x| = x - 5$, $BC = |x - 12| = 12 - x$, $AC = |7 - 12|$. Therefore $AB + BC = AC$.

2. a. The additive property of betweenness for points; same reason

   b. $AB = AC + CB = r_1 + r_2$

   c. The additive property of betweenness for points; the substitution principle

3. The perimeter p_1 of $\Delta DAC = DA + AC + CD$

   The perimeter p_2 of $\Delta DBC = DB + BC + CD$

   To prove $p_1 > p_2$ it is sufficient to prove $DA + AC > DB + BC$ or $DA + AB + BC > DB + BC$ or $DA + AB > DB$

4. By the Triangle Inequality Property $AB + BC > CA$ or $AB > CA - BC$, or $CA - BC < AB$. The same proof can be given for $BC - AB < CA$ and $CA - AB < BC$.

5. (a), (e), (f)

10.10 Using Coordinates to Extend Isometries

10.11 Coordinates and Translations 10.12 Exercises - 2 days

Section 10.10 develops isometries further and builds an affine plane coordinate system. Emphasis should be placed on drawing lines parallel to the axis. Section 10.11 continues translations in the plane, develops the parallelogram, the concept of the diagonals bisecting each other and reinforcement of point symmetry.

10.12--Exercises--may be done partly in class and completed as an assignment.

10.12 Exercises (Ex. 1, 4-6 may be done in class.)

1. a. Midpoint of $\overline{AB'}$ has coordinates $(\frac{1}{2}(a + c + p),$ $\frac{1}{2}(b + d + q)$ and the midpoint of $\overline{A'B}$ has coordinates $(\frac{1}{2}(c + a + p),$ $(\frac{1}{2}(d + b + q))$. Hence $\overline{AB'}$ and $\overline{A'B}$ bisect each other.

   b. The sum of the x-coordinates of A and B' = a + c + p. The sum of the x-coordinates of A' and B = c + a + p. a + c + p = c + a + p by the associative and commutative properties of (Q,+). (This assumes a principle from preceding times.)

   c. b + d + q = d + b + q by associative and commutative properties of (Q,+).

2. a. C(3,2) c. D(-3,-5) e. A(-4,4) g. D(a+e-c,b+f-d)

   b. C(5,5) d. B(2,-1) f. C(a,b)

3. In the suggested coordinate system C has coordinates (1,1), E has coordinates $(\frac{1}{2},0)$, and F has coordinates $(\frac{1}{2},1)$. Then midpoint of $\overline{AC}$ and midpoint of $\overline{EF}$ have same coordinates, that is, $(\frac{1}{2},\frac{1}{2})$. Hence AECF is a parallelogram.

4. a. $\overline{PR}$ and $\overline{SQ}$ have the same midpoint (with coordinates (2,2)) and hence PQRS is a parallelogram.

   b. Using the coordinate system in a, the coordinates of B, A, C, respectively are (2,2), (1,3), (3,1) from which it follows that B is the midpoint of $\overline{AC}$. Since it it is also the midpoint of $\overline{PR}$, it follows that PCRA is a parallelogram.

5. To be suitable the sum of the x-coordinates of P and R should equal the sum of the x-coordinates of S and Q, and the sum of the y-coordinates of S and Q. The significance of taking any coordinate system is that the truth of the statement to be proved in 4b is independent of coordinate systems; or to put this in the jargon of mathematicians, the property is "coordinate-free". This suggests that the property is a geometric (rather than algebraic) one.

6. (This exercise should be done in class.)
   Let A and B have coordinates (a,b), (c,d) in some coordinate system. Then M, the midpoint of $\overline{AB}$ has coordinates $(\frac{a+c}{2}, \frac{b+d}{2})$. Let a translation of points of the plane (in which we are working) have rule $(x,y) \longrightarrow (x+p,y+q)$. Then $A(a,b) \longrightarrow A'(a+p,b+q)$, $M(\frac{a+c}{2}, \frac{b+d}{2}) \longrightarrow M'(\frac{a+c}{2}+p, \frac{b+d}{2}+q)$ and $B(c,d) \longrightarrow B'(c+p,d+q)$. It is an easy matter to show that the midpoint of $\overline{A'B'}$ is M' for

$\frac{1}{2}(a+p+c+p) = \frac{a+c}{2} + p$ and $\frac{1}{2}(b+d) + q$.

10.13 Perpendicular Lines 10.14 - 1 day

This section and the exercises may be combined to develop and reinforce perpendicularity. Exercise 4 defines the mid-perpendicular or perpendicular bisection of a line. This exercise and Exercise would be to the students advantage if they were done in class.

10.14 Exercises

1. a. $c \perp a$ because $c \neq a$ and a is its own image in the reflection in c.

   b. Yes; $b \perp c$.

   c. The results support the statement, since we started with $a \parallel b$, made $a \perp c$ and found $b \perp c$.

2. No. Through A there can be exactly one line perpendicular to BC and since we were told that $AC \perp BC$ then AB cannot be a second perpendicular.

3. No. If ℓ_1 and ℓ_2 intersect at some point, say P, then there would be two perpendiculars to line a containing P.

4. (in class)

   The image of P is P itself. Since $A \longrightarrow A'$ and $P \longrightarrow P$ $AP = A'P$. Let W be any point in ℓ. Then $Q \longrightarrow Q$ and $AQ = A'Q$.

5. (in class)

   a. (1) ℓ is the boundary of two halfplanes. Since $\overline{AB}$

intersects ℓ, A and B are on opposite sides of ℓ.

(2) E is on the same side of ℓ as B. Hence E and A are on opposite sides of ℓ.

(3) E and A are on opposite sides of ℓ.

(4) The additive property of betweenness for points.

(5) The Triangle Inequality Property.

(6) Every point in the midperpendicular of a line segment is as far from one endpoint of the segment as the other.

(7) The substitution principle (CA for CB).

(8) The additive property of betweenness for points.

b. The argument in a can be modified by replacing B with A and E with F.

c. Every point in one of the halfplanes determined by the midperpendicular of a line segment is nearer to the endpoint of the segment in that halfplane than to the other endpoint.

10.15 Using Coordinates for Line Reflection and Point Symmetrics

10.16 2 days

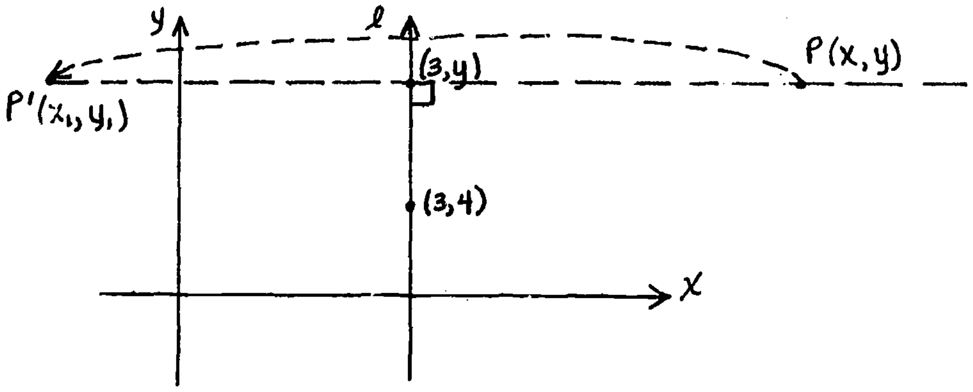
Rectangular coordinate systems are introduced here as a special coordinate system. Reflections in the axes yield the rules for reflections and the composition of these reflections. The composition of $\ell_x \circ \ell_y$ is a point symmetry in the origin. Students may volunteer other names for this composition.

The exercises 10.16 may be begun in class and continued at home, but they should be discussed in class.

10.16 Exercises

1. a. (3,-5); (-3,5); (-3,-5)
   b. (-3,-5); (3,5); (3,-5)
   c. (5,3); (-5,-3); (-5,3)
   d. (-3,5); (3,-5); (3,5)
   e. (2,0); (-2,0); (-2,0)
   f. (0,-5); (0,5); (0,-5)
   g. (-3,1); (3,-1); (3,1)
   h. (82,643); (-82,-643); (-82,643)

2. Consider the problem in this light:



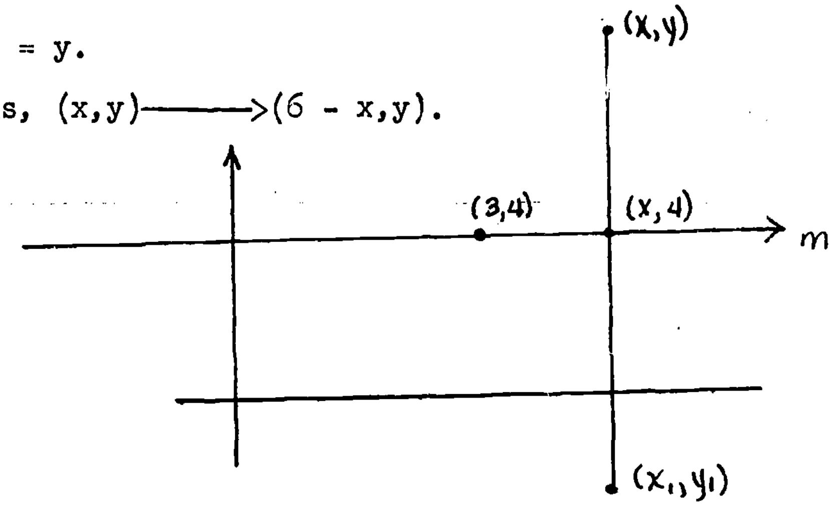
P'(x_1,y_1) is the image of P(x,y).

Then $x - 3 = 3 - x_1$

so $x_1 = 6 - x$.

Also, $y_1 = y$.

Thus, $(x,y) \longrightarrow (6 - x, y)$.



As in the above:

$x_1 = x$ and $y - 4 = 4 - y_1$, so

$y_1 = 8 - y$.

Hence, $(x,y) \qquad (x,8-y)$.

a.	(5,4)	c.	(3,2)	e.	(6,0)	g.	(-2,-3)
b.	(6,3)	d.	(9,-1)	f.	(-4,0)	h.	(6-x,y)

3.

a.	(1,4)	c.	(3,6)	e.	(0,8)	g.	(8,11)
b.	(0,5)	d.	(-3,9)	f.	(10,8)	h.	(x,8-y)

4.

a.	(-1,-4)	c.	(-3,-2)	e.	(0,0)	g.	(-8,3)
b.	(0,-3)	d.	(3,1)	f.	(-10,0)	h.	(-x,-y)

5. a. A'(1,5); B'(3,-1)

   b. M(2,3); M'(2,-3)

   c. $(2,-3) = (\frac{1+3}{2}, \frac{-5-1}{2})$

   (A line reflection preserves midpoints.)

6. $A(2a,2b) \longrightarrow A'(2a,-2b)$, $B(2c,2d) \longrightarrow B'(2a,-2d)$

   M, midpoint of $\overline{AB}$ $(a+c,b+d) \longrightarrow M'$ $(a+c,-b-d)$ and midpoint of $\overline{A'B'}$ also has coordinates $(a+c,-b-d)$. Hence ℓ_x preserves midpoints.

7. $A(2a,2b) \longrightarrow A'(-2a,-2b)$, $B(2c,2d) \longrightarrow B'(-2c,-2d)$.

   M of $\overline{AB}$ $(0,0) \longrightarrow M'(0,0)$ = midpoint of $\overline{A'B'}$. (Special case of Exercise 6)

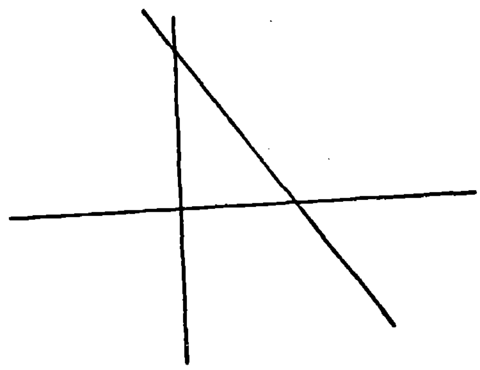
8. a. This can be done by observation or by use of slope formula, i.e. $\frac{3-1}{1-4} = \frac{-3-1}{10-4} = -\frac{2}{3}$.

   b. The images have coordinates (1,-3), (4,-1), 10,3). The new slopes are $\frac{-1+3}{4-1} = \frac{4}{6} = \frac{2}{3}$.

   c. The reflections preserve collinearity.

9. Images under point symmetry in O have coordinates (-1,-3), (-4,-1), (-10,3). Equality of slopes is preserved. (The line and its image are parallel.)

Problem 8



It might be advantageous to indicate that all lines have an equation in the form of $y = ax + b$. Then by using simultaneous equations you can demonstrate slope and also indicate that all 3 points satisfy the equation. (This will depend on the maturity of the class.)

(This may be optional) 9 is similar to 8.

Both problems may be done from an intuitive viewpoint.

10.17 <u>What is an Angle</u>? 10.18 - 1 day

Care should be used here to define an angle. This section should include a definition of angles of 0° and 180°. The teacher should be aware of the fact that these 2 "degenerate" angles do not satisfy the definition in the book. However, if they are defined as they are in the text, the student should be able to accept these as "special cases". It is important that they accept the 0° and 180° angles as angles, since they are necessary ones. The students should also be aware of the following:

Figure a.

a) $\angle AOB$ is the half-plane defined by the shading in Figure a.

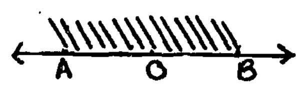


Figure b.

b) $\angle BOA$ is the half-plane defined by the shading in Figure b.

10.18 Exercises

1. $\angle BOD$ ... red-black pencil
   $\angle AOC$ ... blue-black ink
   $\angle BOC$ ... red-black ink
2. a. $\angle XOY$; $\angle ZOW$ and $\angle YOX$; $\angle WOZ$
   b. $\angle YOY$; $\angle ZOZ$; $\angle XOX$; $\angle WOW$
   c. $\angle YOZ$; $\angle ZOX$; $\angle XOW$; $\angle WOY$
3. a. $\angle AOC$; b. $\angle COB$; c. $\angle AOD$; d. $\angle COC$
4. $\angle AOB$, $\angle AOC$, $\angle AOD$, $\angle BOC$, $\angle BOD$, $\angle COD$
5. a. An angle has two endrays.
   b. An angle is a set of rays.
   c. The interior of an angle contains rays of the angle other than its endrays.
   d. If $\overrightarrow{VC}$ and $\overrightarrow{VD}$ are interior rays of $\angle AVD$, then every ray in $\angle VCD$ is in $\angle AVB$.
6. From Funk and Wagnalls: Standard College Dictionary.
   (1) Critical angle (optics) the least angle of incidence at which a ray is totally reflected
   (2) Angle of attack (aeronautics)

(3) Gliding angle (aeronautics)

(4) To angle a moving object in order to avoid a hazard

(5) Angle iron

(6) Angle meter (clinometer)

(7) Angle of attack (aeronautics)

(8) Angle of incidence (physics)

(9) Angle of view (optics)

(10) Angle of yaw (aeronautics)

(11) Angle plate (mechanics) (an angle worm has reference to an angler, one who fishes with a hook (an angle)).

10.19 Measuring an Angle 10.20 - 1-2 days

The students should each have protractors to use and each student should be checked to see that he is using a protractor correctly. The exercises can be done partly in class, perhaps Exercises 4, 8, 11 in class and the rest as homework.

10.20 Exercises

1.	a. 60	d. 150	g. 120	j. 180
	b. 45	e. 105	h. 0	k. 90
	c. 45	f. 135	i. 180	l. 90
2.	a. 70	c. 50	e. 30	
	b. 80	d. 10	f. 130	
3.	a. 70	b. 110	c. 260	

4. (b) $\angle ABC$ and $\angle CBD$

5. $\angle AVB$ and $\angle BVC$; $\angle BVC$ and $\angle CVD$; $\angle CVD$ and $\angle DVA$; $\angle DVA$ and $\angle AVB$; $\angle AVC$ and $\angle CVD$; $\angle AVC$ and $\angle AVD$; $\angle CVD$ and $\angle DVA$; $\angle CVD$ and $\angle CVB$.

6. In the figure for Exercise 5 $m\angle DOB + m\angle BOC > 180$, which is not a possible angle measure.

7. a. $m\angle AVB = 92$ d. $m\angle EVC = 120$ g. $m\angle BVF = 122$
   b. $m\angle DVC = 28$ e. $m\angle AVF = 28$ h. $m\angle AVD = 180$
   c. $m\angle AVC = 152$ f. $m\angle FVD = 152$

8. $\overrightarrow{OC}$; $\overrightarrow{OY}$; $\angle DOC$

9. a. $\angle AOB$; b. $\angle DOC$

10. $m\angle AOC = 130^\circ$, $m\angle BOD = 130^\circ$, $m\angle AOB = 50^\circ$, $m\angle COD = 50^\circ$.

11. $m\angle A = 71$, $m\angle B = 65$, $m\angle C = 44$
    (These are approximations.) Sum of measures = 180.

12. a. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ do not intersect the semicircular edge of the protractor and hence are not assigned numbers.
    b. No. The difference between any two numbers on the protractor cannot exceed 180.

13. $m\angle BVC = m\angle AVD$

14. 90

15. a. $m\angle AVB$ is approximately 30; then $m\angle BCV = 180 - 30$ or 150
    b. $180 - 70 = 110$

10.21 Boxing the Compass

This section may be omitted if pressed for time or students may read it on their own time.

10.22 More About Angles 10.23 - 1 day

Here the uniqueness of the midray is defined. The names of angles such as right, acute, and obtuse are also defined. Exercises 7,8 may be included in the classwork. The rest of the exercises can be assigned as homework.

10.23 Exercises

Answers for Exercises 1 and 2a should be the same except for position.

Answers for Exercises 2b - 6 vary with each student. Included in Answer 6 should be the idea that the sum of 2 right angles is 180°, and the sum of the three angles of a triangle is 180°, hence there is a contradiction.

7. a. no b. yes c. no; no; no d. yes
   e. A ray is between two other rays if it is an interior ray of the angles having the other rays as sides.
8. If VC is between VA and VB, then $m\angle AVC + m\angle CVB = m\angle AVB$.
   If C is between A and B, then AC + CB = AB.

10.24 Angles and Line Reflections 10.25 - 1-2 days

Isometries preserve distance. It is demonstrated here that they also preserve angle measure. This concept is helpful in demonstrating that base angles of an isosceles triangle are equal and leads to other deductions found in the exercises.

10.25 Exercises

These exercises should be completed as they develop the concepts presented in 10.24.

1. Since AB = AC, ΔABC is isosceles and $m\angle ABM = m\angle ACM$
$m\angle ABD = 180 - m\angle ABM$, $m\angle ACE = 180 - m\angle ACM$. Hence
$m\angle ABD = m\angle ACE$.

2. a. is true because BD = CE and BM = CM. It follows that DM = EM. Also AM $\perp$ BC. Hence AM is midperpendicular of $\overline{DE}$.

   b. is true because the endpoints of a segment are images under the line reflection in the midperpendicular of the segment.

   c. is true because ℓ: a ———>A, D ———>E. Hence AD = AE.

   d. is true because ℓ preserves collinearity.

   e. is true because ℓ: $\angle DAB$ ———>ℓEAC and isometries preserve angle measure.

3. Suppose that P is not in ℓ. Then it is one of the open halfplanes of ℓ and then would be nearer to Q than to P, and nearer to R than to Q. Both of these are false since PQ = QR. Therefore P is in ℓ.

4. a. isosceles; isosceles

   b. The midray of $\angle A$ is in the midperpendicular of $\overline{BD}$.
   The midray of $\angle B$ is in the midperpendicular of $\overline{BD}$.

   c. One. This leads oto: the midrays are in AC.

   d. $\angle ADB$ and $\angle ABD$; $\angle ADC$ and $\angle ABC$; $\angle DCA$ and $\angle BCA$; $\angle BDC$ and $\angle DCB$; $\angle DAC$ and $\angle BAC$; $\angle ACD$ and $\angle ACB$.

5. The line reflection in AC: D ———>B. Hence AC is the midperpendicular of $\overline{DB}$. The line reflection in DB: A ———>C.

Hence $\overleftrightarrow{DB}$ is the midperpendicular of $\overline{AC}$.

Another approach is to use midrays $\overrightarrow{AC}$ and $\overrightarrow{DB}$ and point that then we have 2 isosceles triangles and the midrays are midperpendiculars as well, hence the diagonals bisect each other and are $\perp$ to each other.

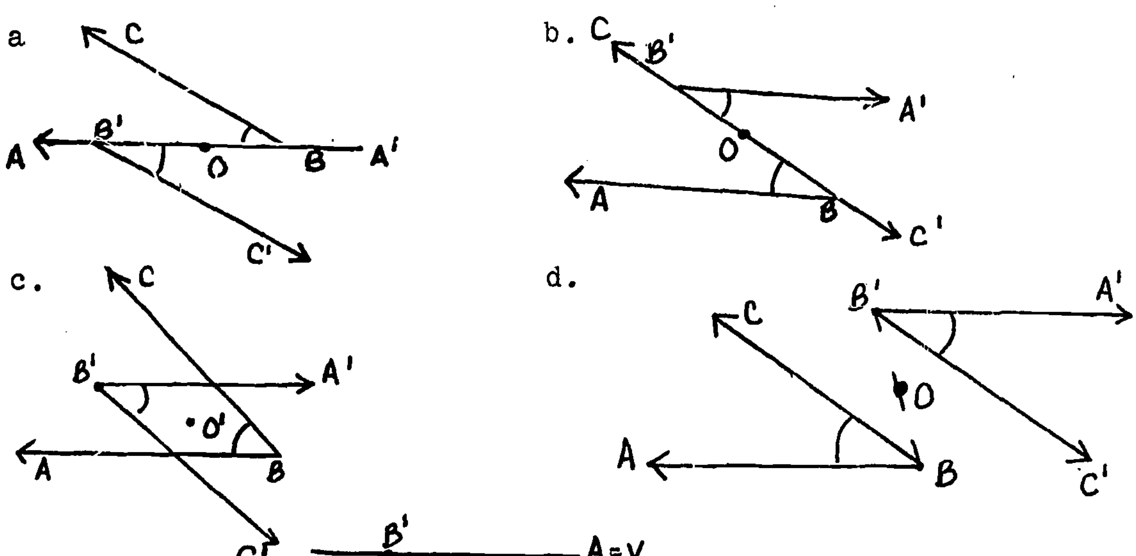
10.26 <u>Angles and Point Symmetries</u> 10.27--1 day

This section continues the development of isometries and angle measure and relates point symmetries to the idea of isometries. The exercises may be done in class.

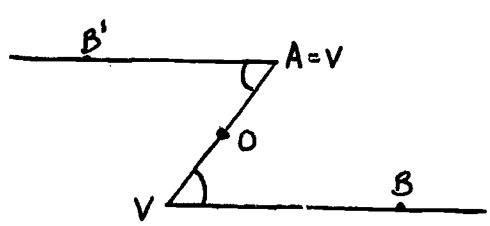
10.27 <u>Exercises</u>

1. If the measure of one angle is a, then the measure of the other three are 180 - a, a, 180 - a.

2\.



3\.



Show the parallelogram at the board.

V————>A, ∠BVA————>∠B'AV.

Since O is the midpoint of $\overline{VV'}$ and $\overline{BB'}$, VBV'B' is a parallelogram and it follows that AB' || BV (opposite sides of a parallelogram lie in parallel lines). ...lies in a line that is parallel to the line of the second side.

4. The quadrilateral is a parallelogram. In addition to all the properties of a parallelogram this quadrilateral (a rhombus) also has the following:

   The diagonals bisect each other a right angles.

   Each diagonal here is the midray of two angles.

   The sides have the same length.

   Please demonstrate this at the board so that the students can see the rhombus and that the diagonals are ⊥.

5. This quadrilateral is also a parallelogram but not a rhombus. Again, demonstrate at the board.

6. Yes. The intersection of $\overline{AC}$ and $\overline{BD}$. Under this symmetry A<————>C, D<————>B, $\overrightarrow{AB}$————>$\overrightarrow{CD}$, $\overrightarrow{AD}$————>$\overrightarrow{CB}$, ∠DAB————>∠BCD hence m∠A = m∠C, etc.

10.28 <u>Angles and Translations</u> 10.29--1 day

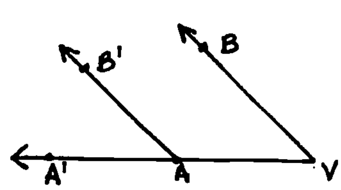
These two sections should be done as a unit.

10.29 <u>Exercises</u>

1. $\overrightarrow{VA}$————>$\overrightarrow{AA'}$

   VB————>AB'

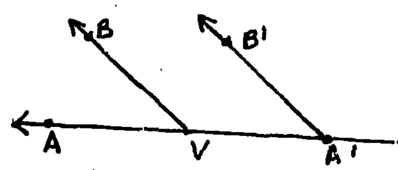
   ∠AVB————>∠A'AB'



2\. a. VA———>A'V'

VB———>A'B'

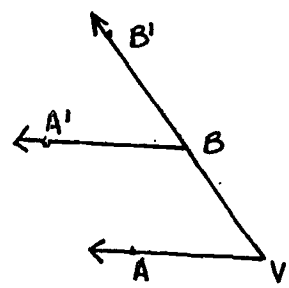
∠AVB———>∠VA'B'



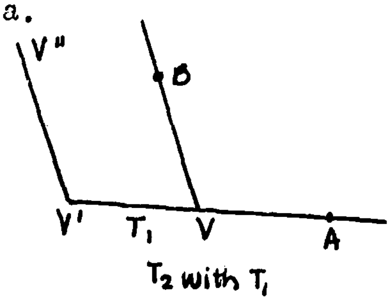
b\. VA———>BA'

VB———>BB'

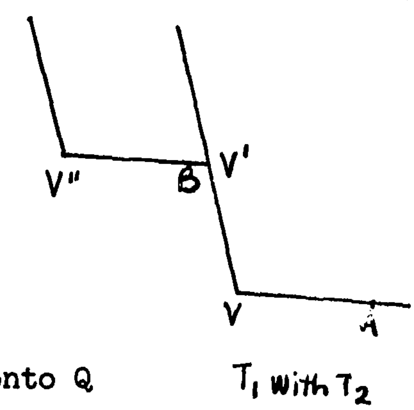
∠AVB———>∠A'BB'



3\. a.



b.



c. yes, yes

4\. a. The translation that maps V onto Q

b. The point symmetry in M

c. The point symmetry in Q

d. Same as c

e. c with a

f. They are the same.

10.30 <u>Sum of Measures of the Angles of a Triangle</u> 10.31--1-2 days

The sum of the measures of the angles of a triangle is 180°. In this section the "proof" is developed using the concepts of translations, and isometries. A kind of paragraph proof for the right triangle and for sum of angles of a quadrilateral.

Exercises 1, 5, 8, 10, 11 may be assigned.

Exercises 6, 7, 9, 12, 13 may be done in class.

10.31 Exercises

1. a. 70 b. 69 c. 11

2. a. 50 b. 80 c. 56 d. 69.5

3. 60

4. a. 30, 50 b. 42, 70 c. $56\frac{1}{4}$, $93\frac{3}{4}$

5. a. 98 b. 136 c. 72

6. The sum of the measures of three angles in 90.3 or 270. Hence the measure of the fourth angle is 360-270 = 90.

7. $m\angle A = m\angle C$, and $m\angle B = m\angle D$. But $m\angle A + m\angle B + m\angle C + m\angle D = 360$. Hence $2m\angle A + 2m\angle B = 360$ or $m\angle A + m\angle B = 180$. By a similar argument $m\angle C + m\angle D = 180$.

8. a. The measure of an obtuse angle is greater than 90. Therefore the sum of measures of two obtuse angles is greater than 180. The sum of the measures of angles of any triangle is 180. Therefore there is no triangle the sum of whose measures is greater than 180.

   b. The sum of the measures of the base angles is less than 180. Since these measures are the same, then each measure is less than 90. Hence the base angles of an isosceles triangle are acute angles.

9. a. By the isosceles triangle property $m\angle A = m\angle B = m\angle C$ since $m\angle A + m\angle B + m\angle C = 180$, then $m\angle A = 60$.

   b. In $\triangle ADE$, $m\angle E + m\angle EDA + m\angle DAE = 180$.
   In $\triangle ADC$, $m\angle ADC + m\angle DCA + m\angle DAC = 180$.

In ΔACB, $m\angle ACB + m\angle CBA + m\angle BAC = 180$.

The sum of the measures of the nine angles is 540.

Among these nine angles $m\angle EDA + m\angle ADC = m\angle EDC$, $m\angle DCA + m\angle ACB = m\angle DCB$ and $m\angle EAD + m\angle DAC + m\angle CAB = m\angle EAB$, because of the Betweeness-Addition Property of Angles.

Hence, the sum of the measures of the angles of ABCDE is 540.

c. $\frac{1}{5} \cdot 540 = 108$

10. a. $m\angle BCD = 110$

b. $m\angle BCD = 117$

c. Yes, $m\angle BCD = m\angle A + m\angle B$ (An exterior angle of a triangle is equal to the sum of the opposite interior angles.)

d. $m\angle BCD = 180 - m\angle BCA = m\angle A + m\angle B$

$180 - m\angle A + m\angle B = m\angle BCA$

$180 - m\angle BCA = m\angle A + m\angle B$. Then by transitive property it is proved.

11. a. $m\angle ADC = 98^{\circ}$

b. $m\angle PAD = 100$, $m\angle QDC = 82$, $m\angle ROB = 70$, $m\angle SBA = 108$.

c. 360

d. 360

e. 360

If A, B, C, D are the angles of ABCD, with measures a, b, c, d then the "arc" angles have measures $180 - a$, $180 - b$, $180 - c$, $180 - d$. The sum of the measures of these latter angles is $720 - (a+b+c+d) = 720 - 360 = 360$.

12. a. 720

b. 360

c. 120; 60

13. for 8 sides; 1040, 360, 45

for 10 sides; 1440, 360, 36

10.32 Summary 10.33 Review Exercises--1 day

These two sections should be assigned and reviewed in class. Time should be all for quizzes and 1 day allowed for a Chapter Test.

ANSWERS - REVIEW EXERCISES

1. a. $AB = |-2 -4| = 6$

b. $\frac{1}{2}(-2+4) = 1$

c. If x is assigned to C, then $-2 < x < 4$.

d. If x is assigned to D, then $|-2-x| = 2|x-6|$. Since $-2 < x < -6$, $-2-x = 2(x-6)$ and $x = 3\frac{1}{3}$

f. $|x-(-2)| = 6$ and $(x+2 = 6$ or $x+2 = -6)$ and $x = 4$ or $+8$ but since $\overline{AB} - 2 > A$, $4 \longrightarrow B$, then $E = B$.

2. a. $AB = |-12 - (-6)| = 6$

b. $\frac{1}{2}(-12-6) = -9$

c. $-12 < x < -6$

d. $|-12 - x| = 2\ |x + 6|$ and $x = -8$

e. $-12 - x = -2\ (x+6)$ and $x = 0$

f. $|x-(-12)| = 6$ and $x = -18$ or -6

3. a. $m\angle AVB = |10 - 110| = 100$

b. $\frac{1}{2}(10 + 110) = 60$

c. No

d. $10 < x < 110$

e. 100

f. $|y - 10| = 2|110-y|$ or $y = 76\frac{2}{3}$

4. a. $m\angle AVB = |122 - 38| = 84$

b. $\frac{1}{2}(122 + 38| = 80$

c. yes

d. $38 < x < 122$

e. 84

f. $122 - y = 2(y-38)$ or $y = 69\frac{1}{3}$

5. It cannot be done since the sum of the measures of two angles of a triangle cannot exceed 180. In the attempt to draw such a triangle two sides would diverge and thus the triangle would not "close".

6. a. A' (-4, -2), B' (1, 3), C' (6, -2)

b. No, since slope of AB $\neq$ slope of BC. No--same reason for A', B', C'.

c. AB = A'B' since line reflections are isometries.

d. $m\angle ABC = m\angle A'B'C'$ because line reflections, like all isometries preserve angle measures.

7. a. A'(4,2), B'(1,3), C'(-6,2)

b,c,d. same answers as in Exercise 6.--b, c, d.

8. a. A' (4, -2), B'(-1, 3), C' (-6, -2)

b, c, d. same answers as in Ex. 6.--b, c, d, modified to read "point symmetry in the origin" for "line reflection on the x-axis".

9. a. A' (6,2), B' (1,7), C' (-4,2)

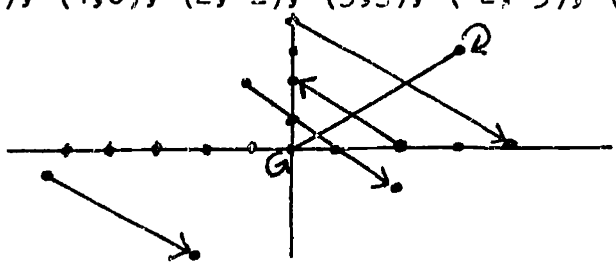
b,c,d, same answers of Exercise 6.--b, c, d, modified to read "point symmetry in P(1,2)" for "line reflection in the x-axis".

10. a. A(6,2), B(1,-3), C(-4,2).

b, c, d. same as Exercise 6.--b, c, d, modified to read "line reflection in the given perpendicular" for "line reflection in x-axis".

11. a. (0,2), (4,0), (2,-1), (3,3), (-2,-5), (0,0)

b.

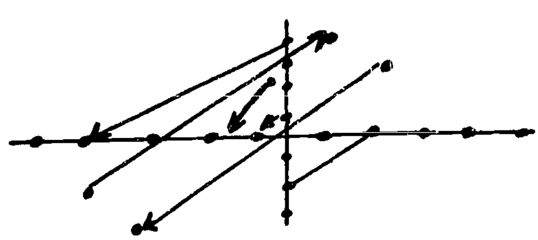


c. Line translations in the line with equation y = x. Domain = Range = set of points in plane. The rule of its inverse is $(x,y) \longrightarrow (y,x)$

d. the identity mapping

12. a. (0,-2), (-4,0), (-2,1), (-3,-3), (2,5), (0,0)

b.



c. It is a line reflection in the line with equation $x+y = 0$. Domain = Range = Set of points in plane.

d. yes

13. No. Distances are doubled.

14. The point symmetry in M maps B onto C, C onto B, and A onto A', where M is the midpoint of AA'. Also the translation that maps A onto C, maps B onto A'. Hence $m\angle A + m\angle B + m\angle C = m\angle A'CD + m\angle BCA' + m\angle C = 180$.

15. a. 120 b. 60 c. 135 d. 150 e. 162

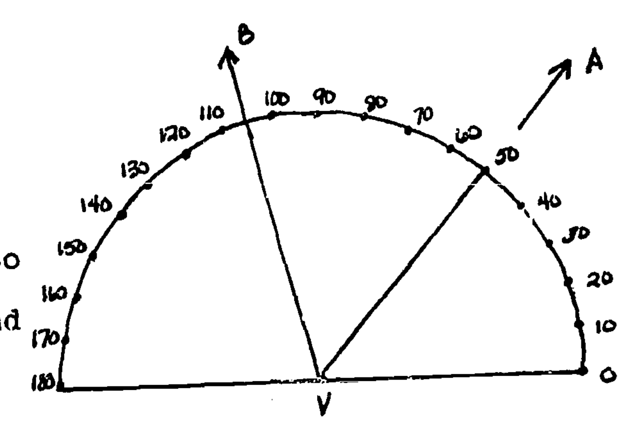
16. a. 60 b. 120 c. 45 d. 30 e. 18

17. The midperpendicular of $\overline{BC}$ contains both D and A. Hence, under the line reflection in this midperpendicular $D \longrightarrow D$, $A \longrightarrow A$, $B \longrightarrow C$ and $\angle DAB \longrightarrow \angle DCB$, or $m\angle DAB = m\angle DCB$.

Suggested Items for a Test - Chapter 10

1. Let a ruler assign -4 to A and 8 to B.
   a. Find AB.
   b. Find the number assigned to the midpoint of $\overline{AB}$.
   c. For what numbers assigned to C will C be between A and B?
   d. What numbers may be assigned to D if AD = 5?
   e. What numbers may be assigned to E if AE = 3 · EB?
   f. What numbers may be assigned to points in AB?

2. The numbers assigned to VA and VB by a protractor are 50 and 105.



   a. Find $m\angle AVB$.
   b. Find the number assigned to the midray of $\angle AVB$.
   c. For what numbers assigned to VC will VC be between VA and VB?
   d. What numbers may be assigned to VD if $m\angle AVD = 20$.

e. For what numbers assigned to VE will $\angle AVE$ be obtuse?

f. For what numbers assigned to VF will $m\angle BVF = 3m\angle AVF$?

3. In a certain rectangular coordinate system the coordinates of A, B, C are (-3,3), (-1,1), and (4,-4) respectively.

   a. Find the coordinates of their images under a line reflection in the x-axis.

   b. A, B, C collinear. Show that their images are also collinear by calculating slopes.

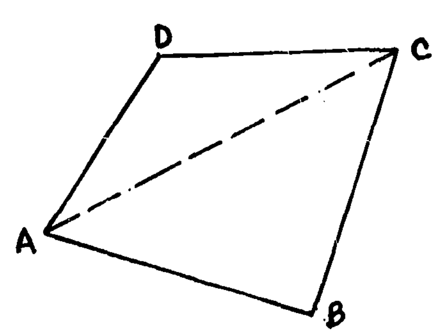
   c. Explain why the length of AB is the same as the length of its image.

4. In a certain coordinate system (not necessarily rectangular) the coordinates of P, Q, R are (-2,3), (0,0), (5,1) respectively.

   a. What are the coordinates of P', R', Q', the images of a translation under which the image of Q is Q'(1,2)?

   b. Compare the measures of $\angle PQR$ and $\angle P'Q'R'$. Justify your answer.

5. Explain why the sum of the measures of the angles of quadrilateral ABCD is 360.



6. Find the measure of each angle of a 10-sided figure if their measures are the same. Also find the measure of an

exterior angle.

Answers for Suggested Items for a Test

1. a. $AB = 12$ d. $|x-(-4)| = 5$ $x = 1$ or -9

   b. $\frac{1}{2}(-4+8) = 2$ e. $|x-(-4)| = 3|x-8|$ $x = 14$ or 5

   c. $-4<x<8$ f. $x \geq -4$

2. a. 55 d. 30 or 70

   b. $77\frac{1}{2}$ e. $140<x<180$

   c. $50<x<105$ f. $|105-x| = 3|x-50|$ $x = 63\frac{3}{4}$ or $22\frac{1}{2}$

3. a. $A'(-3,-3)$, $B'(-1,-1)$, $C'(4,4)$

   b. Slope $A'B'$ = slope of $B'C' = 1$

   c. A line reflection is an isometry--hence it preserves distance.

4. a. $P'(-1,5)$, $Q'(1,2)$, $R'(6,3)$

   b. They are the same. Isometries preserve angle measures and the translation maps $\angle PQR$ onto $\angle P'Q'R'$.

5. Consider ΔADC and ΔABC.

   $m\angle D + m\angle DAC + m\angle ACD = 180 = m\angle B + m\angle BAC + m\angle ACB$.

   Since $m\angle DAC + m\angle CAB = m\angle DAB$ or $m\angle A$, and $m\angle DCA + m\angle ACB = m\angle DCB$ or $\angle C$, we add and obtain $m\angle A + m\angle B + m\angle C + m\angle D = 360$.

6. $\frac{8}{10}(180) = 144$ $180 - 144 = \underline{36}$

TEACHERS COMMENTARY

Course I

Chapter 11

Elementary Number Theory

(Approximate Time 9-12 days)

It has been noted in the introduction of the commentary that Chapter 11 is not a key Chapter in the development of this program. Although this Chapter is not essential, the teacher should make some attempt to introduce its basic ideas. The teacher should be aware that this commentary has been written in the spirit that this Chapter will be covered in full, but the teacher may not have the time to do so. As a result, the teacher may approach this Chapter in different ways once the key Chapters have been completed. He may, of course, cover this Chapter in full. Another alternative that has been suggested is that this may be assigned as a self-study Chapter. In this case the teacher should be selective in assigning reading and exercises. Also, he may choose some major ideas from the section and introduce them in class. Once again, the time element is key note in determining the completeness of Chapter 11.

Purposes of this Chapter:

1. To examine $(N,+)$ and $(N,\cdot)$ and some of their properties through an axiomatic approach.
2. To reinforce the concepts of multiple, factor and divides.

3. To develop an elementary way of proof dependent on certain axioms.
4. To introduce the division algorithm and its applications.
5. To introduce prime and composite numbers.
6. To develop the concept of unique factorization of the natural numbers through a discussion of complete factorization.
7. To examine the sieve of Eratosthenes as a technique of discovering primes.
8. To study some important proofs of number theory such as Euclid's proof that there are an infinite number of primes.
9. To introduce Euclid's algorithm, a technique for finding the greatest common divisor of two natural numbers.

11.1 <u>(N, +) and (N, ·)</u>

approximate time (1 day)

The purpose of this section is to review the concept of an operational system, in particular, (N, +) and (N, ·) and to consider the definitions of (1) factor (2) multiple (3) divides. Here, for the first time, the student encounters an approach to mathematics. It should be emphasized at theis point thet math itself is built from axiomatic systems. The distinction between an axiom and a theorem should be clear in the minds of the students.

Recall that the relation "divides" has been previously discussed; it may be important to note that this relation is reflexsive, anti-symmetric, and transitive. Exercises 1 and 3 may be done as a classwork assignment.

11.2 Solutions to Exercises

1. (a) True. $2 + 3$ or 5 is a natural number.

   (b) True. $2 \cdot 3$ or 6 is a natural number.

   (c) False. $0 \in W$ but $0 + 0 \notin N$.

   (d) True. $(N, +)$ is an operational system.

   (e) False. $1 \in N$, $0 \in W$ but $(0 \cdot 1) \notin N$.

   (f) True. $(N, \cdot)$ is an operational system.

2. (a) Multiple.

   (b) x (and y) is a factor of z

   (c) r is a multiple of p (and r)

   (d) divisor

   (e) product of 7 and 8

   (f) product expression.

3.

(a) T	(e) T	(i) F
(b) F	(f) T	(j) F
(c) T	(g) T	(k) T
(d) T	(h) T	(l) F

4. (a) T $3 \cdot 13 = 39$

   (b) F $91 = 5 \cdot 17 + 6$

   (c) F $8 \times \frac{1}{2} = 4$

   (d) T $4 = 1 \cdot 4$

   (e) T $13 \cdot 5 = 65$

   (f) T $3 \cdot 2 = 6$, $3 \cdot 4 = 12$
   $3 \cdot 6 = 18$

   (g) T $2 \cdot a = n$, $a \in N$

   (h) T $1 \cdot n = n$

   (i) T $n(n + 3) = N^2 + 3n$
   by the distributive property.

5. (a) $6 \cdot 1$, $3 \cdot 2$, $2 \cdot 3$, $1 \cdot 6$

(b) $7 \cdot 1$, $1 \cdot 7$

(c) $1 \cdot 1$

(d) $12 \cdot 1$, $6 \cdot 2$, $4 \cdot 3$, $3 \cdot 4$, $2 \cdot 6$, $1 \cdot 12$

(e) $13 \cdot 1$, $1 \cdot 13$

(f) $2 \cdot 1$, $1 \cdot 2$

(g) $3 \cdot 1$, $1 \cdot 3$

(h) $35 \cdot 1$, $7 \cdot 5$, $5 \cdot 7$, $1 \cdot 35$

(i) $36 \cdot 1$, $18 \cdot 2$, $12 \cdot 3$, $9 \cdot 4$, $6 \cdot 6$, $4 \cdot 9$, $3 \cdot 12$, $2 \cdot 18$, $1 \cdot 36$

(j) $37 \cdot 1$, $1 \cdot 37$

11.3 Divisibility

(approximate time 3 days)

The purpose of this section is to develop an elementary proof dependent on 7 basic axioms. It should be stressed that the axioms given and the theorems discussed are, for the purposes of this Chapter, true only in N. In a proof, each statement should be justified by one of the following:

(a) Assumption (of given)

(b) Definitions

(c) Axioms

(d) Replacement Assumption

(e) Previously proven Theorems.

Depending on the type of class, the teacher's expectations in terms of rigor for a proof may vary;the format presented in the text, although a good introduction, need not be followed. It will be necessary to discuss each of the theorems in this section.

Axiom 7, the Division Algorithm, is not a new concept for the student. The basic difficulty with this notion will probably be the notation, which can be explained with a few simple examples.

Also in this section, the even and the odd numbers are formally defined. A teacher might ask the students to formulate their own definitions and then to compare them to those in the text.

Once again the teacher should use his own discretion in selecting exercises. It should be noted that if a proof of a theorem is not assigned the statement of the theorem should be discussed. Exercise 9 introduces a test for divisibility which can be helpful to students in future experiences. However, the notation and many of the proofs are difficult and need explanation, but are certainly worthwhile.

11.4 Solutions to Exercises

1. (a) Division Algorithm
   (b) Distributivity of multiplication over addition
   (c) Property of 1
   (d) Theorem A, 1
   (e) $7 \in N$ and 7 is not even
   (f) $2b + 1$ where b is same whole number

(g) "p implies q". Should a student interpret the sentence as "'if q is false implies p is false", a correct answer is "neither p nor q".

(h) (N, •) is an operational system.

2. (0, 13),(1, 10),(2, 7),(3, 4),(4, 1). All. (4, 1).

3. (a) Given: $3 \mid a$ and $3 \mid b$; $a, b \in N$

Prove: $3 \mid (a + b)$

Proof

$3 \mid a$ and $3 \mid b$; $a, b, \in N$	Assumption
For some $x, y \in N$, $a = 3x$, $b = 3y$	Def. of $r \mid s$.
$a + b = 3x + 3y$	Th. A, 1)
$= 3(x + y)$	Distributivity (A5)
$(x + y) \in N$	(N, +) is an operational system
$3 \mid (a + b)$	Def. of $r \mid s$

(b) Given: $c \mid a$, $c \| b$; $a,b,e \in N$

Prove: $c \mid (a + b)$

Proof

$c \mid a$, $c \mid b$; $a,b,c \in N$	Assumption
For some $x, y \in N$, $a = cy$, $b = cy$	Def. of $r \mid s$
$a + b = cx + cy$	Th. A, 1)
$= c\ (x + y)$	Distributivity (A5)
$(x + y) \in N$	(N, +) is an operational system
$c \mid (a + b($	Def. of $r \mid s$

4. Given: $a \mid b$, $b \mid c$; $a,b,c \in N$

Prove: $a \mid c$

Proof

$a\|b$, $b\|c$; a, b, $c \in N$	Assumption
For some x, $y \in N$ $b = ax$, $c = by$	Def. of $r\|s$
$\therefore c = (ax)y$	Replacement: $b = ax$
$c = a(xy)$	Associativity (A4)
$(xy) \in N$	$(N, \cdot)$ is an operational system
$a\|c$	Def. of $r\|s$

5. Given: $a|b$; a, b, $c \in N$

Prove: $a|bc$

Proof

$a\|b$; a, b, $c \in N$	Assumption
For some $x \in N$, $b = ax$	Def. of $r\|s$
$bc = bc$	Equality is reflexive
$bc = (ax)c$	Replacement: $b = ax$
$bc = a(xc)$	Mult. is associative (A4)
$a\|bc$	Def. of $r\|s$

6. (a) Given: $a \in E,\ b \in O$
Prove: $a + b \in O$

<u>Proof</u>

$a \in E,\ b \in O$	Assumption
For some $x, y \in W$	Def. of even and odd
$a = 2x,\ b = 2y + 1$	
$a + b = a + b$	Equality is reflexive
$a + b = 2x + (2y + 1)$	Replacement
$a + b = (2x + 2y) + 1$	Additiin is associative (A4)
$a + b = 2(x + y) + 1$	Distributivity (A5)
$x + y \in W$	(W, +) is an operational system
$a + b \in O$	Def. of odd number

(b) Given: $a, b \in O$
Prove: $a + b \in E$

<u>Proof</u>

$a, b \in O$	Assumption
For some $x, y \in W$	Def. of even number
$a = 2x + 1, \quad b = 2y + 1$	Equality is reflexive
$a + b = a + b$	Replacement
$a + b = (2x + 1) + (2y + 1)$	Addition of whole numbers
$a + b = (2x + 2y) + (1 + 1)$	is commutative and associative
$a + b = 2(x + y + 1)$	Distributivity: $1+1=2\cdot 1$
$x + y + 1 \in W$	(W,+) is an operational system
$a + b \in E$	Def. of even number

(c) Given: $a \in E$, $b \in O$

Prove: $ab \in E$

Proof

$a \in E$, $b \in O$	Assumption
For some $x \in W$, $a = 2x$	Def. of even number
$ab = ab$	Equality is reflexive
$ab = (2x)b$	Replacement: $a = 2x$
$ab = 2(xb)$	Mult. is associative (A4)
$xb \in W$	$(W, \cdot)$ is an op. system
$ab \in E$	Def. of even

(d) Given: $a, b \in O$

Prove: $ab \in O$

Proof

$a, b \in O$	Assumption
For some $x \in W$, $a = 2x + 1$	Def. of odd number
$ab = a\ b$	Equality if reflexive
$ab = (2x + 1)b$	Replacement
$ab = (2x)\ b + b$	Distributivity
$(2x)b \in E$	Previous exercise (c)
$ab \in O$	Previous exercise (a)

7. There are no three odd number totaling 30.

Proof: The sum of any pair of odd numbers is an even number from exercise 6(b). If this even sum be increased by any odd number the total is an odd number by exercise 6(a). Hence, the toal of any triple of odd numbers is an odd number. But as 30 = 2(15), 30 is an even number, so there are no three odd numbers totaling 30.

8. (a) False $2|6$ but $2\nmid(6 + 1)$

(b) True For assumptions needed, see exercise 5.

(c) True

Proof

$a\|b + c,\ a\|b\ a,b,c \in N$	Assumption
$(b + c - b) \in N$	(N, +) is an operational system
$a\|(b + c - b)$	Exercise 3b
$b + c - b = c$	(N,+) is an operational system
$\therefore a\|c$	Replacement

9. (a) As 10 is even, it follows that every natural number N may be expressed in the form

$N = 2x + y$ where $x, y \in W$

$2x + y$ is even iff y is even from exercise 3(b) and exercise 8(c).

$2x + y$ is odd iff y is odd from exercises 6(a) and 8(c).

Note: $2x = a_n10^n + \ldots + a_210^2 + a_110^1$ and $y = a_o$

(b) The proof will be sketched for a four-digit number. The same idea can be used for any number.

Let $N = a_3 10^3 + a_2 10^2 + a_1 10 + a$

But $a_3 10^3 = a_3(10^3-1) + a_3 = a_3(1000-1) + a_3 = 999a_3 + a_3$

$a_2 10^2 = a_2(10^2-1) + a_2 = a_2(100-1) + a_2 = 99a_2 + a_2$

$a_1 10 = 9a_1 + a_1$

Hence $N = (999a_3 + a_3) + 99a_2 + a_2) + (9a_1 + a_1) + a_o$

$N = (999a_3 + 99a_2 + 9a_1) + (a_3 + a_2 + a_1)$

$N = 3(333a_3 + 33a_2 + 3a_1) + (a_3 + a_2 + a_1)$

$N = 3x + y$

where $x = 333a_3 + 33a_2 + 3a_1 \in W$

$y = a_3 + a_2 + a_1 \in W$

Hence, N is a multiple of 3 iff y is a multiple of 3 from exercises 6(c) and 8(c)

(c) (1) iff $4 | (10a_1 + a_o)$

(2) iff $a_o = 0$ or 5

(3) iff $2|N$ and $3|N$

(4) iff $8| (100a_2 + 10a_1 + a_o)$

(5) iff $9| (a_n + a_{n-1} + \ldots + a_o)$

(6) iff $a_o = 0$.

(d) (1) For some $x \in W$, $N = 4x + 10a_1 + a_o$

(2) For some $x \in W$, $N = 5x + a_o$

(3) Let $N = 6x + y$ where $x, y \in W$ and $y < 6$. If $2|N$ and $3|N$ then $2|y$ and $3|y$ (Exercise 8(c)). As $y < 6$ y must be 0. Hence $N = 6x$ and $6|N$

(4) For some $x \in W$, $N = 8x + (100a_2 + 10a_1 + a_0)$

(5) A proof exactly similar to 9(b) may be given.

(6) For some $x \in W$, $N = 10x + a_0$

If $10|N$ then $10|a_0$ (Exercise 8(c)). But $a_0 < 10$.

Hence, $a_0 = 0$.

11.5 Primes and Composites

(approximate time 1 day)

This section discusses prime and composite numbers. The notion of a factor set is used to exemplify the factors of a number. Note that {1}, the set of prime numbers and the set of composite numbers form a partition of N.

11.6 Solutions to Exercises

1. (a) 1 and the number itself

   (b) a natural number divisible by a natural number other than 1 and itself.

   (c) 2

   (d) 1 or more than 2

2. (a) {1, 2, 5, 10}

   (b) {1, 13}

   (c) {1, 2, 3, 4, 6, 12}

   (d) {1, 2, 3, 4, 6, 8, 12, 24}

   (e) {1, 2, 17, 34}

   (f) {1, 5, 7, 35}

(g) {1, 2, 3, 4, 6, 9, 12, 18, 36}

(h) {1, 37}

3. (a) 13, 37

   (b) 10, 12, 24, 34, 35, 36

   (c) None

4. It must be composite.

5. (a) 47 (b) 4

6. It must be a composite number.

7. (a) {2} (b) {3,5,7,11,13,17,19}

8. The number of elements in the factor set of a composite number is not two.

9. (a) 4, 9, 25-that is, p^2

   (b) 6, 8, 10-that is, p_1p_2 or p^3

11.7 Complete Factorization

(approximate time 1 day)

The purpose of this section is to introduce the concept of the unique factorization of the natural numbers. This goal is reached by a discussion of complete factorization through the use of factor trees. The students have encountered the notion of the greatest common divisor in their previous study of operational systems; here a formal definition is given. To find the greatest common divisor, factor sets and complete factorization are both used.

Exercise 5 can be a fun problem which reinforces the notion of the uniqueness of a complete factorization. Prior discussion in reviewing a binary operation and its properties may be necessary for exercise 7. The construction of the table in exercise 8 may be a good homework problem followed by a class discussion on parts a - h of the same exercise. Exercises 9 and 10 introduce the concept of the least common multiple through the use of factor sets and complete factorization. It would be advisable for the teacher to discuss the least common multiple before assigning these exercises. A comparison and distinction between the least common multiple and the greatest common divisor should be attempted.

11.8 Solutions to Exercises

1. (a) $1 \cdot 9$, $3 \cdot 3$
   (b) $1 \cdot 10$, $2 \cdot 5$
   (c) $1 \cdot 15$, $3 \cdot 5$
   (d) $1 \cdot 100$, $2 \cdot 50$, $4 \cdot 25$, $10 \cdot 10$
   (e) $1 \cdot 24$, $2 \cdot 12$, $3 \cdot 8$, $4 \cdot 6$
   (f) $1 \cdot 16$, $2 \cdot 8$, $4 \cdot 4$
   (g) $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$, $6 \cdot 12$, $8 \cdot 9$
   (h) $1 \cdot 81$, $3 \cdot 27$, $9 \cdot 9$

2. (a) 3^3 (d) $2^2 \cdot 5^2$ (g) 3^4 (j) $2^2 \cdot 5^3$
   (b) $2 \cdot 5$ (e) $2^3 \cdot 3$ (h) $2 \cdot 3 \cdot 5 \cdot 7$
   (c) $3 \cdot 5$ (f) 2^4 (i) $2^3 \cdot 5^2$

3. 1, 4, 6, 8, 9, 12, 18, 72

4. (a) 2 will be a factor

(b) 2 will not be a factor

5. (a)

24
4 x 6
2x2 x 2x3

24
3 x 8
3 x 2x4
3 x 2x2x2

24
2 x 12
2 x 2x6
2 x 2x2x3

and others

(b)

96
8 x 12
2x4 x 3x4
2 x 2x2 x 3x2x2

96
16 x 6
4x4 x 2x3
2x2 x 2x2 x 2x3

(c)

625
25 x 25
5x5 x 5x5

625
5 x 125
5 x 5 x 25
5 x 5 x 5x5

(d)

1000
10 x 100
2x5 x 10x10
2 x 5 x 2x5 x 2x5

1000
25 x 40
5x5 x 4x10
5x5 x 2x2 x 2x5

and others

6. (a) 2 x 5 or 10

   (b) 1

   (c) 2 x 3 or 6

   (d) 13 x 17 or 221

7. g.c.d. is a binary operation on N. It has the following properties: commutativity, associativity, $a^{\circ}a = a$, $a^{\circ}1 = 1$, $p_1{}^{\circ}p_2 = 1$ if $p_1 \neq p_2$, $p^{\circ}n = p$ or 1.

8.

n	Factors of n	Number of factors	Sum of factors
9	1,3,9	3	13
10	1,2,5,10	4	18
11	1,11	2	12
12	1,2,3,4,6,12	6	28
13	1,13	2	14
14	1,2,7,14	4	24
15	1,3,5,15	4	24
16	1,2,4,8,16	5	31
17	1,17	2	18
18	1,2,3,6,9,18	6	39
19	1,19	2	20
20	1,2,4,5,10,20	6	42
21	1,3,7,21	4	32
22	1,2,11,22	4	36
23	1,23	2	24
24	1,2,3,4,6,8,12,24	8	60
25	1,5,25	3	31
26	1,2,13,26	4	42
27	1,3,9,27	4	40
28	1,2,4,7,14,28	6	56
29	1,29	2	30
30	1,2,3,5,6,10,15,30	8	72

   (a) 2,3,5,7,11,13,17,19,23,29

   (b) 4,9,27

   (c) 3

   (d) 4; 1 + p + q + pq or(1+p) (1+q)

   (e) K + 1

   (f) K + 1

(g) K + 1

(h) 6,28

9. (a) $30 = 2x3x5$, $45 = 3^2x5$, l.c.m. $= 2x3^2x5 = 90$

If p is a factor of either number then select p or the greater power of p. The product of all such p's will be the l.c.m. of the numbers.

(b) (1) $30= 2x3x5$, $108 = 2^2x3^3$ l.c.m. $= 2^2x3^3x5 = 540$

(2) $45 = 3^2x5$, $108 = 2^2x3^3$ l.c.m. $= 2^2x3^3x5 = 540$

(3) $15 = 3x5$, $36 = 2^2x3^2$ l.c.m. $= 2^2x3^2x5 = 180$

(4) $81 = 3^4$, $210 = 2x3x5x7$ l.c.m. $= 2x3^4x5x7 = 5670$

(5) $16 = 2^4$, $24 = 2^3x3$ l.c.m. $= 2^4x3 = 48$

(6) $200 = 2^3x5^2$, $500 = 2^2x5^3$ l.c.m. $= 2^3x5^3 = 1000$

(c) The product of the g.c.f. and l.c.m. of a and b is ab

10. l.c.m. is a binary operation on N. It has the following properties: commutativity, associativity,

$a^\circ a = a$, $a^\circ 1 = a$, if $p_1 \neq p_2$, $p_1 \circ p_2 = p_1 p_2$

11.9 The Sieve of Eratosthenes

(approximate time 1 day)

In this section the Sieve of Eratosthenes is encountered; a simple technique of discovering prime numbers. The historical implications of this section should be enjoyable and rewarding; perhaps some students will be motivated to undertake further study along these lines. The concept of twin primes is introduced in the reading and that of prime triplets is presented in exercise 5.

If this last idea is rather difficult, exercise 5 may be assigned as an extra credit problem. This section may be assigned as a self-study unit.

11.10 Solutions to Exercises

1. (a) (1) 4 (2) 9 (3) 25 (4) 49
   (b) p^2
   (c) 11 x 11 = 121 and 51 < 121
   (d) (1) They are not primes.
   (2) They are primes.
   (e) No. Multiples of 4 are multiples of 2.

2. (a) Primes are: 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47, 53,59,61,67,71,73,79,83,97,101,103,107, 109,113,127.
   (b) 25 (c) 31 (d) 127 (e) 11 because 130 < 13 x 13

3. (a) (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73)
   (b) Twin primes.
   (c) 8

4. (a) Primes are: 281, 283, 287
   (b) 281, 283, 287
   (c) 2,3,5,7,11,13,17

(d) If a prime larger than 17 appeared among the numbers less than 290 then the other factor would have to be less than 17 and therefore could have been eliminated.

5. (a) 3, 5, 7

(b) The triple would then have a composite number.

(c) If $k > 1$, $3k$ is a composite number.

(d) 1 or 2

(e) As $p \neq 3k$, $p = 3k + 1$ or $3k + 2$ for some $K > 1$.

If $p = 3k + 1$ then $3|(p+2)$, since $p+2 = (3k+1) +2 =$
$= 3k + 3 = 3(k+1)$.

If $p = 3k+2$ then $3|(p+4)$, since $p+4 = (3k + 2) +4 =$
$= 3k+6 = 3(k+2)$. In either case p, $p+2$, $p+3$ for $p > 3$ can not all be prime numbers.

(f) The only prime triple is (3, 5, 7).

11.11 On the Number of Primes

(Approximate time: 1 day)

As in the previous section, this section presents some important historical background. The proof of Euclid that there are an infinite number of prime numbers is developed. Unsolved problems such as, "Is the number of twin primes infinite or finite?" are brought to the students' attention. It is important that they realize that more than half of the mathematics we assume has yet to be proven.

This section may be either assigned as a self-study unit or considered as an optional topic.

11.12 Solutions to Exercises

1. (a) $10 = 3 + 7$ (f) $20 = 3 \div 17$

   (b) $12 = 5 + 7$ (g) $36 = 5 + 31$

   (c) $14 = 3 + 11$ (h) $48 = 5 + 43$

   (d) $16 = 3 + 13$ (i) $100 = 3 + 97$

   (e) $18 = 5 + 13$ (j) $240 = 113 + 127$

2. (a) Each number N is one more than a product of consecutive primes. P is the largest of the consecutive primes.

   Last N here should be 2.• 3 : 5 : 7 : 11 : 13 : 17 + 1

   (b) $2 + 1 = 3$ a prime

   $2 \cdot 3 + 1 = 7$ a prime

   $2 \cdot 3 \cdot 5 + 1 = 31$ a prime

   $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$ a prime

   $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$ a prime

   (c) checks

   (d) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1 = 510,511$

   $= 19 \times 26,869$

   (e) If 17 were the largest prime number then the number in (d) is larger than 17 and so must be composite. But none of the primes 2,3,5,7,11,13,17 divide the number in (d). Hence the number in (d) cannot have any prime factors, contradicting the fact that N is composite. Hence, 17 cannot be the largest prime.

   (f) There are an infinite number of consecutive primes to check. Computers can handle, at best, a finite number of numbers.

11.13 Euclid's Algorithm

(approximate time 2 days)

This section introduces Euclid's Algorithm, another technique for finding the greatest common divisor, of two natural numbers. The process involved is an intricate one involving successive applications of the division algorithm. The teacher should not belabor the method, but should try to convey the idea through the use of a few examples. The idea of relatively prime numbers should be mastered for it is important in the following Chapters.

Axiom 8 and the theorems following may be omitted at the discretion of the teacher. Perhaps, the more capable students will be interested in studying these topics on their own. Exercises 4 - 7 which involve proofs may be omitted depending on the depth of the development of this section.

Exercise 9 is extremely difficult. Not only is the concept of Fermat's Little Theorem involved, but the proof of this Theorem is beyond the capabilities of most junior high school students.

11.14 Solutions to Exercises

1. (a) $1122 = 10 \times 105 + 72$

$105 = 1 \times 72 + 33$

$72 = 2 \times 33 + 6$

$33 = 5 \times 6 + 3$

$6 = 2 \times 3 + 0$ g.c.d. $(1122, 105) = 3$

(b) $2244 = 5 \times 418 + 154$

$418 = 2 \times 154 + 110$

$154 = 1 \times 110 + 44$

$110 = 2 \times 44 + 22$

$44 = 2 \times 22 + 0$ g.c.d. $(2244, 418) = 22$

(c) $315 = 1 \times 220 + 95$

$220 = 2 \times 95 + 40$

$95 = 2 \times 40 + 15$

$40 = 2 \times 15 + 10$

$15 = 1 \times 10 + 5$

$10 = 2 \times 5 + 0$ g.c.d. $(315, 220) = 5$

(d) $19,656 = 21 \times 912 + 504$

$912 = 1 \times 504 + 408$

$504 = 1 \times 408 + 96$

$408 = 4 \times 96 + 24$

$96 = 4 \times 24 + 0$ g.c.d. $(19,650,912) = 24$

2. (1) $144 = 1 \times 104 + 40$

$104 = 2 \times 40 + 24$

$40 = 1 \times 24 + 16$

$24 = 1 \times 16 + 8$

$16 = 2 \times 8 + 0$ g.c.d. $(144, 104) = 8$

(2) $144 = 2^4 \times 3^2$, $104 = 2^3 \times 13$ a.g.c. $= 2^3 = 8$

3. (a) 1

(b) a

4. As $p \nmid b$, p and b are relatively prime. Hence these are integers x and y such that $px + by = 1$
Multiplying by c yields $cpx + cby = c \cdot 1$ or $cpx+cby = c$
But $p|(cpx)$ and $p|(cby)$ so that $p|(cpx + cby)$ hence, $p|c$.

5. From exercise 4 if $p \nmid a$ then $p \mid b$. If $p \nmid b$ then $p \mid a$.

6. As a and b are relatively prime there are integers x and y such that $ax + by = 1$
As $a|c$ and $b|c$ there are integers r and s such that

$$ar = c \quad \text{and} \quad bs = c$$

Multiplying by c yields

$$cax + cby = c \cdot 1$$
$$cax + cby = c$$
$$acx + bcy = c$$
$$a(bs)\,x + b(ar)\,y = c$$
$$ab(sx) + ab(ry) = c$$
$$ab(sx + ry) = c$$

Hence $(ab) \mid c$.

7. As d = g.c.d. (a,b) there are integers x and y such that

$$d = xa + yb$$

But $a = rd$ and $b = sd$. Hence

$$d = x(rd) + y(sd) \text{ or } d = (xr)\,d + (ys)d \text{ or}$$
$$d = (xr + ys)d$$

Dividing by d gives

$$1 = xr + ys$$

If r and s were not relatively prime then their common divisor would have to divide 1. But the only divisors of 1 are 1 and - 1. Hence, r and s are relatively prime.

8. Seven consecutive composites are

90,91,92,93,94,95,96

Eight consecutive composites are

114,115,116,117,118,119,120,121; also 119,120,121,122, 123,124,125,126.

Another approach that suggests a general method is to consider

$9! + 2$	$9! + 6$
$9! + 3$	$9! + 7$
$9! + 4$	$9! + 8$
$9! + 5$	$9! + 9$

which are eight consecutive composites, the first is divisible by 2, the second by 3, ... , the eighth by 9. In general, to find n consecutive composites consider the n numbers

$(n + 1)! + 2$

$(n + 1)! + 3$

....

$(n + 1)! + (n + 1)$

The k^{th} number listed is divisible by $k + 1$.

9. (a) $2^{5-1} - 1 = 2^4-1 = 16-1 = 15$ and $5|15 \therefore 5|(2^{5-1}-1)$

$4^{3-1} -1 = 4^2-1 = 16-1 = 15$ and $3|15 \therefore 3|(4^{3-1}-1)$

(b) If $p|a$ then $p \nmid (a^{p-1}-1)$ Example: $3|6$ but $3 \nmid (6^{3-1}-1) =$

$= 6^2 - 1 = 36-1 = 35$

(c) The conclusion need not follow. Thus $2^{4-1}-1 = 2^3-1 =$
$= 2^3-1 = 8-1 = 7$, $4 \nmid 7$. $\therefore 4 \nmid (2^{4-1}-1)$

(d) Note that p is a prime and $p \nmid a$. If p is divided into the following multiples of a:

$1 \cdot a, 2 \cdot a, 3 \cdot a, \ldots (p-1)a$

The smallest whole number remainders will be less than p and, as we shall show, be all different. That the remainders are different may be argued as follows. Let:

$$1 \cdot a = q_1 p + r_1 \qquad 0 < r_1 < p$$
$$2 \cdot a = q_2 p + r_2 \qquad 0 < r_2 < p$$
$$3 \quad a = q_3 p + r_3 \qquad 0 < r_3 < p$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(p-1) \cdot a = q_{p-1}p + r_{p-1} 0 < r_{p-1} < p$$

Where $q_1 q_2, \ldots, q_{p-1}$ are the respective quotients when p is divided into $1 \cdot 2$, $2 \cdot a$, $3 \cdot a$, ..., $(p-1)$. a. The respective remainders $r_1, r_2 r_3, \ldots . r_{p-1}$ are all less than p. As $p \nmid a$ and $p \nmid s$ for all natural numbers 1,2,3, ..., p-1 it follows that $p \nmid (sa)$ for $s = 1, 2, 3, \ldots, p-1$. Hence, for $s = 1, 2, 3, \ldots, p-1$ there will always be a non-zero remainder when dividing sa by p. Thus $r_1, r_2, r_3, \ldots, r_{p-1}$ are all greater than zero.

We shall now show that two remainders can not be the same. Suppose two remainders were the same say the u^{th} and the v^{th} remainders with $u < v$, so that $r_u = r_v$. It follows that

$$ua = d_u p + r_u$$

$$va = d_v p + r_v$$

and $u < v$

Then $va - ua = d_v p - d_u p$

and $(v - u)\, a = (d_v - d_u)\, p$

and $p \mid (v - u)\, a$.

But $0 < v - u < p$, so that p is relatively prime to $v - u$. Also, p is relatively prime to a. It then follows that p is relatively prime to $(v-u)a$ and $p \nmid (v-u)a$. This contradicts the previous result that $p \mid (v-u)a$. This contradiction arises from the assumption that remainders r_u and r_v were the same. Hence, no two remainders can be the same. But the remainders

$$r_1, r_2, r_3, \ldots, r_{p-1}$$

are $p - 1$ different natural numbers between 0 and p and so must be, in some order, the $p - 1$ natural numbers $1, 2, 3, \ldots, p - 1$. We shall need the following preliminary theorem.

Given: $ua = q_u p + r_u$

$va = q_v p + r_v$

Conclusion: $uva^2 = r_u r_v$ + an integer multiple of p

Proof

$ua = q_u p + r_u \qquad va = q_v p + r_v$ Assumption

$(ua)\ (va) = (ua)(va)$ Equality is reflexive

$(ua)\ (va) = (q_u p + r_u)\ (q_v p + r_v)$ replacement

$(ua)\ (va) = r_u r_v + r_u q_v p$
$\qquad + r_v q_u p + q_u q_v pp$ Distributivity (W, +) and (W, •) are operational systems.

$\qquad = r_u r_v + (v_u q_v + r_v q_u + q_u q_v p)p$ Distributivity

$(ua)\ (va) = r_u r_v + xp,\ x \in W$ (W,+),(W,•) are oper. systems.

We now multiply the left members and right members of the following equalities and make use of the above theorem repeated by

$1 \cdot a = q_1 p + r_1$

$2 \cdot a = q_2 p + r_2$

$3 \cdot a = q_3 p + r_3$

....

$(p-1) \cdot a = q_{p-1} p + v_{p-1}$

$1 \cdot 2 \cdot 3 \ldots (p - 1) \cdot a^{p - 1} = r_1 r_2 r_3 \ldots r_{p-1} +$

a multiple of p

But $r_1 r_2 r_3 \ldots r_{p-1} = 1 \cdot 2 \cdot 3 \ldots (p-1)$ where the remainders need not be in increasing order. Hence, as

$$1 \cdot 2 \cdot 3 \ldots (p-1) \cdot a^{p-1} = (p-1)! + \text{a multiple}$$

of p

We have $(p-1)!\ a^{p-1} = (p-1)! +$ a mult. of p

or $(p-1)!\ a^{p-1} - 1 =$ multiple of p

Hence $p \mid \{(p-1)!\ a^{p-1} - 1\}$. But p is relatively prime to $(p-1)!$

Hence $p \mid (a^{p-1} - 1)$ which is exactly what Fermat's Little Theorem says must be the case. This completes the argument.

11.16. Solutions to Review Exercises

1. (a) $50 = 10 \cdot 5$
   (b) 6 is a factor of 30 or $30 = 5 \cdot 6$
   (c) $6 \cdot 5 = 30$
   (d) $6 \cdot 1 = 6$
   (e) There is no integer x for which $7x = 30$
   (f) The natural number divisors of 7 are 1 and 7
   (g) $6 = 2 \cdot 3$
   (h) $91 = 7 \cdot 13$

2. (a), (b) If $ab = c$ then a and b are factors of c while c is multiple of a and of b.
   (c) If the factor set of a number consists of exactly 2 numbers, then the original number is a prime.

(d) If the factor set of a number has a cardinality other than two, then the number is composite provided it is greater than one.

3. (a) $38 = 2 \cdot 17$

   (b) $72 = 2^3 \cdot 3^2$

   (c) $96 = 2^5 \cdot 3$

   (d) $97 = 97$ a prime

4. (a) 2

   (b) 2

   (c) $2^3 \cdot 3 = 24$

   (d) 1

5. (a) $2^3 \cdot 3^2 \cdot 17 = 1224$

   (b) $2^5 \cdot 3 \cdot 17 = 1632$

   (c) $2^5 \cdot 3^2 = 288$

   (d) $2^3 \cdot 3^2 \cdot 97 = 6984$

6. Primes are: 131, 137, 139, 149

   (a) 4

   (b) 1

   (c) 11 because $13 \cdot 13 = 169 > 150$

7. $96 = 1 \times 72 + 24$

   $72 = 3 \times 24 + 0$ g.c.d. $(96, 72) = 24$

8. Given: $a|b$, $b|c$ $a, b, c \in N$

   Prove: $a|c$

Proof

$a\|b,\ b\|c \quad a, b, c \in N$	Assumption
For some $x, y \in N$	Def. of $r\|s$
$b = ax \quad c = by$	
$c = by$	
$= (ax)\ y$	Replacement
$= a(xy)$	Mult. is associative (A4)
$(xy) \in N$	$(N,\cdot)$ is an operational system
$\therefore a\|c$	Def. of $r\|s$

9. Yes. 9 and 10 are relatively prime. See Exercise 6 of set 11.14.

10. If $a \nmid b$ and a is prime then a and b are relatively prime as the only common factor is 1. Hence, g.c.d. (a, b) = 1.

SUGGESTED TEST ITEMS

1. Determine if the following are true or false. Explain your answers.
   a) 8 is a factor of 32.
   b) 47 is a composite number.
   c) 39 and 65 are relatively prime .
   d) 28 is a multiple of 7.
   e) 19 is a prime number.
   f) $12|12$

2. Give a complete factorization of each of the following:

   a) 48

   b) 351

3. Determine the following:

   a) g.c.d. (90, 126)

   b) l.c.m. (90, 126)

   c) g.c.d. (140,175)

   d) l.c.m. (324, 60)

4. If $7 \cdot 6 | c$ does it following that $7 | c$ and $6 | c$? Explain your answer.

5. In the following proof, supply the reasons for each step.

   If $c | a$ and $c | b$, then $c | a \cdot b$ where a, b, c are natural numbers.

   <u>Proof</u>:

   1. $c | a$ and $c | b$ where $a, b, c \in N$
   2. For some $x, y \in N$, $a = cx$ and $b = cy$.
   3. $a \cdot b = (cx) \cdot (cy)$
   4. $(cx) \cdot (cy) = c[x(cy)]$
   5. $a \cdot b = c[x(cy)]$
   6. $[x(cy)] \in N$
   7. $c | a \cdot b$

6. Prove: The sum of two even numbers is an even number.

ANSWERS TO SUGGESTED TEST ITEMS

1. a) True: $8 \cdot 4 = 32$
   b) False: The factor set of 47 = {1, 47}. Therefore, 47 is a prime number.
   c) False: g.c.d.(39, 65) = 13
   d) True: 7 is a factor of 28 or $7 \cdot 4 = 28$
   e) True: The factor set of 19= {1, 19}
   f) True: 12 is a factor of 12 or $12 \cdot 1 = 12$
2. a) $48 = 3 \cdot 2^4$
   b) $351 = 13 \cdot 3^3$
3. a) g.c.d. (90, 126) = 18
   b) l.c.m. (90, 126) = 630
   c) g.c.d. (140, 175) = 35
   d) l.c.m. (324, 60) = 1620
4. Yes. As 7 and 6 are relatively prime numbers, both 7 and 6 must belong to the factor set of c. Therefore, $7|c$ and $6|c$.
5. Step 1: Assumption
   Step 2: Definition of $r|s$
   Step 3: Theorem A_a : $a = b$ and $c = d \Rightarrow a \cdot c = b \cdot d$
   Step 4: Associativity of multiplication (A_4)
   Step 5: Replacement (Statements 3 and 4)
   Step 6: $(N, \cdot)$ is an operational system.
   Step 7: Definition of $r|s$

6. Given: $a \in E$, $b \in E$

Prove: $a + b \in E$

Proof:

1.	$a \in E$, $b \in E$	Assumption
2.	For some $x, y \in W$ $a = 2x$, $b = 2y$	Definition of even
3.	$a + b = 2x + 2y$	Theorem A_1
4.	$2x + 2y = 2(x + y)$	Distributivity
5.	$a + b = 2(x + y)$	Replacement (Statements 3 and 4)
6.	$(x + y) \in W$	$(W, +)$ is an operational system
7.	$a + b \in E$	Definition of even.

Course I

Chapter 12 - THE RATIONAL NUMBERS

Time Estimate for the chapter: (15-18 days)

In addition to the overall purpose of studying the rational number system, the approach taken in this chapter outlines several primary goals:

a) To develop the rational number system $(Q,+,\cdot)$ by extending the largest number system we know - the set of integers $(Z,+,\cdot)$. (Sections 12.1, 12.2, 12.4)

b) To define a rational number by studying equations of the form "$ax = b$" where $a \in Z$, $b \in Z$, $a \neq 0$. The objects which are solutions of such equations, denoted by $\frac{b}{a}$, are called rational numbers. (Section 12.6)

c) To study the properties of $(Q,+,\cdot)$ (Sections 12.8, 12.12)

d) To delineate $(Q,+,\cdot)$ as a field and to define an order relation on Q so that $(Q,+,\cdot)$ is presented (without using formal language) as an ordered field. (Section 12.16)

e) To study addition, subtraction, multiplication and division as operations with Q, using elements of $\frac{b}{a}$. (Sections 12.6, 12.10, 12.12, 12.14)

f) To study decimal fractions and infinite repeating decimals. (Sections 12.18, 12.20, 12.22)

(Sections 12.1 and 12.2 combined with exercises in 12.3 1-2 days)

12.1 W, Z and Z_7

By studying the similarities and differences of these three systems, the student is introduced to <u>a quest for a more encompassing number system</u>. In Chapter 4, W was extended to Z so that equations of the form "$a + x = b$" would always have solutions. Through this extension, subtraction on Z became an unrestricted operation. It should be evident that "$a + x = b$" will always contain solutions in Z_7.

By introducing equations of the form "<u>ax = b</u>", the students should see that solutions can not always be found in $(W,+,\cdot)$ and in $(Z,+,\cdot)$. Obviously, division is not an operation on W and on Z. Using $(Z_7,+,\cdot)$, students will see that solutions to "$ax = b$" can be obtained provided that $a \neq 0$. With this information, that $a \neq 0$, the students should see that we will try <u>to extend Z</u> to form a new number system where solutions of "$ax = b$" will always be possible.

Calling this new system Q (the set of rational numbers), the students should realize that its elements can be written in the form $\frac{b}{a}$ where a and b are integers and $a \neq 0$. It should follow that division is an operation on $Q \setminus \{0\}$.

As a bonus, it can eventually be seen that equations of the form "$ax = b$" will have solutions in Q when $a \in Q$, $b \in Q$ and $a \neq 0$. Please remember that this section is intended to give an overview; the actual extension will take place in preceding sections.

<u>Special note</u>: An excellent classroom discussion could arise in that $(Z_7,+,\cdot)$ is a field that cannot be ordered. In any ordered field with additive and multiplicative identities of 1 and 0, it is a theorem that $1 > 0$.

Then $1+1 > 0$, $1+1+1 > 0$, etc. until, in $(Z_7,+,\cdot)$ $1+1+1+1+1+1+1 > 0$ becomes $0 > 0$.

(Combined with Section 12.1 in time estimate)

12.2

This section describes a set Z' consisting of <u>reciprocals</u> of Z. The question is then asked: Will <u>$Z \cup Z'$</u> form the set that will contain all the solutions for equations of the form "$ax = b$"?

Although many properties will hold within $Z \cup Z'$, an <u>operational system cannot be attained.</u> It is inherent in what is done here and later that the new numbers obtained will obey the usual laws of commutativity, associativity, etc.

12.3 <u>Exercises</u>

Note that the exercises are extensive and time-consuming. The teacher should feel free to be selective. Please note that exercises 1 through 7 refer to $(Z_7,\cdot)$

There is a deliberate relation for exercises 1, 5, and 7 and a second deliberate relation for exercises 2, 3, 4 and 6.

1. (a) 4 (d) 3
   (b) 5 (e) 6
   (c) 2 (f) undefined

2. (a) 5 (e) 4
   (b) 4 (f) 2
   (c) 2 (g) 3
   (d) 5 (h) 6

3. (a) 5 (e) 4
   (b) 4 (f) 2
   (c) 2 (g) 3
   (d) 5 (h) 6

4. (a) {5} (e) {4}
   (b) {4} (f) {2}
   (c) {2} (g) {3}
   (d) {5} (h) {6}

5. (a) 4 (d) 3
   (b) 5 (e) 6
   (c) 2 (f) no solution

6. $b \cdot \frac{1}{a} = \frac{b}{a}$ and the solution to $ax = b$ is $\frac{b}{a}$. $(a \neq 0)$ in $(Z_7, +, \cdot)$ The answers for exercises 2, 3 and 4 must be the same.

7. $\frac{1}{a}$ is the solution in $(Z_7, +, \cdot)$ for $ax = 1$ if $a \neq 0$. The answers for exercises 1 and 5 must be the same.

8. (a) $\frac{1}{-7}$ (b) no multiplicative inverse
   (b) $\frac{1}{13}$ (f) 1
   (c) 17 (g) -18
   (d) -11 (h) -1

9. (a) -104 (f) $\frac{1}{35}$ (k) $\frac{1}{150}$
   (b) -104 (g) $\frac{1}{35}$ (l) $\frac{1}{150}$
   (c) -104 (h) $\frac{1}{-36}$ (m) $\frac{1}{150}$
   (d) -104 (i) $\frac{1}{-36}$ (n) $\frac{1}{-192}$
   (e) -104 (j) $\frac{1}{-36}$ (o) $\frac{1}{-192}$

10. (a) $9 = (3)(3) = (-3)(-3) = (9)(1) = (-9)(-1) = (1)(9)$
    $= (-1)(-9)$.

(b) $75 = (1)(75) = (-1)(-75) = (-3)(-25) = (5)(15)$
$= (-5)(-15)$, etc.

(c) $-15 = (-1)(15) = (1)(-15) = (3)(-5) = (-3)(5)$, etc.

11. (a) $\frac{1}{12} = \frac{1}{(2)(6)} = \frac{1}{(-2)(-6)} = \frac{1}{(3)(4)} = \frac{1}{(-3)(-4)}$
$= \frac{1}{(-3)} \cdot \frac{1}{(-4)} = \frac{1}{(12)} : \frac{1}{1}$, etc.

(b) $\frac{1}{75} = \frac{1}{(3)(25)} = \frac{1}{(-3\ (-25)} = \frac{1}{(5)(15)} = \frac{1}{(-5)(-15)}$, etc.

(c) $\frac{1}{-15} = \frac{1}{(-1)(15)} = \frac{1}{(15)(-1)} = \frac{1}{(3)(-5)} = \frac{1}{(-3)(5)}$, etc.

12. (a) $\frac{1}{5}$ (e) $\frac{1}{-5}$
(b) $\frac{1}{-12}$ (f) $\frac{1}{12}$
(c) $\frac{1}{-17}$ (g) $\frac{1}{17}$
(d) $\frac{1}{4}$ (h) $\frac{1}{-4}$

13. (a) {2} (g) {10}
(b) {-3} (h) {-40}
(c) {14} (i) {375}
(d) {-506} (j) {285}
(e) {-6} (k) {-30}
(f) {-2000} (l) {-15,340}

14. (a) {7}
(b) {-7}
(c) { } or ∅
(d) {9}
(e) { } or ∅
(f) {0}
(g) { } or ∅
(h) {100}
(i) {-100}
(j) {-3}
(k) {-12}
(l) { } or ∅

15. (a) 7 (e) 0 (i) none
(b) -7 (f) none (j) -701
(c) none (g) -3 (k) -2117
(d) -10 (h) 257

16. Every integer is a solution of "$0 \cdot x = 0$". Therefore "$\frac{0}{0}$" is indeterminate, it cannot be the name of a unique integer. The solution set of "$0 \cdot x = 0$" can be written as Z.

17. There is no solution in Z'.

12.4 $Z \cup Z'$ to Q (2 days)

Five important concepts are found within this section which finally brings us to the set of rational numbers, Q.

1) Q is created as the set which contains Z, Z' and the set of "missing" products $(b \cdot \frac{1}{a})$ where $b \in Z$ and $\frac{1}{a} \in Z'$. The students should review the definition of an operation on a set (from chapter 2) at this point. Then they should see that to this point they have $(Z,\cdot)$ and $(Z',\cdot)$ as operational systems but not $(Z \cup Z',\cdot)$.

   It should be made clear that in all of this we are assuming that the desirable commutative and associative laws hold for multiplication and that 1 is the identity element throughout. (Thus we do not worry about $(\frac{1}{a},b)$ because it should be true that $\frac{1}{a} \cdot b = b \cdot \frac{1}{a}$). Also, the question as to whether $(Q,\cdot)$ is in fact an operational system is left open in this section. It may be advisable to call this to the students' attention.

2) A rational number is a single element from Q, say $\frac{a}{b}$. However there are an infinite number of names for this single element, found by $\frac{na}{nb}$ when $n \in Z$. Just as one person has many names (Henry Allen Jones, H.A. Jones, Henry, Hank, Stinky, etc.) and just as one integer has many names ($9 = 8 + 1 = 9 + 0 = 10 + -1$, etc.), the concept of one rational number with many names should not be treated in a complicated manner.

The major problem considered here is what to do about $6 \cdot \frac{1}{8} = \frac{6}{8}$ and $3 \cdot \frac{1}{4} = \frac{3}{4}$.

First, we show that $\frac{b}{a}$ is a solution for the equation "$ax = b$". Then we show that $\frac{nb}{na}$, for $n \in Z$, is also a solution of "$ax = b$". We argue that "$ax = b$" must have a unique solution in Q. If not, the desirable cancellation property of Z does not carry over to Q. Finally, we show that $\frac{a}{b} = \frac{na}{nb}$ for every n in Z.

$$\begin{aligned} \frac{na}{nb} &= na \cdot \frac{1}{n \cdot b} \\ &= (n \cdot \frac{1}{n})(a \cdot \frac{1}{b}) \\ &= 1 \cdot (\frac{a}{b}) \\ &= \frac{a}{b}. \end{aligned}$$

That is, $\frac{a}{b}$ and $\frac{na}{nb}$ must be the same rational number.

3) By definition, a distinction is made in that a rational number is an element of the set Q while a fraction is a name for that rational number.

4) Two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, are said to be equivalent if and only if $a \cdot d = b \cdot c$ (the approach noted in part 2 above shows why this is true).

All equivalent fractions will form an equivalence relation (symmetry, transitive, reflexive properties hold). The set of fractions which are equivalent name one rational number and these fractions are said to belong to an equivalence class.

A second approach to this topic could include ordered pairs. It would then be stated that a rational number is defined as an equivalence class of ordered pairs where $(a,b) \sim (c,d)$ if and only if $a \cdot d = b \cdot c$.

5) The irreducible fraction is introduced as one in which the numerator and denominator contain no common factors. A second definition, relying upon concepts learned in Chapter 11, simply states that when the numerator and denominator are relatively prime, the fraction is called irreducible.

Ex: $\frac{9}{20}$ is an irreducible fraction.

Note that neither 9 nor 20 are prime but since they contain no common factors, they are relatively prime.

Ex: $\frac{5}{20}$ is not an irreducible fraction.

Both numerator and denominator contain a common factor of 5; even though 5 is prime, 5 and 20 are not relatively prime.

$$\frac{5}{20} = \frac{5 \cdot 1}{5 \cdot 4} = 1 \cdot \frac{1}{4} = \frac{1}{4}.$$

$\frac{5}{20}$ is equivalent to the irreducible fraction $\frac{1}{4}$.

12.5 Solutions for Exercises

Note: Exercise 5 is essential to future work. Exercises 4 and 5 could be started in class.

1. (a) $\frac{2}{3} = \frac{-2}{-3} = \frac{4}{6} = \frac{-4}{-6} = \frac{6}{9} = \frac{-6}{-9}, \ldots$

   (b) $\frac{-6}{-8} = \frac{6}{8} = \frac{3}{4} = \frac{-3}{-4} = \frac{9}{12} = \frac{-9}{-12}, \ldots$

(c) $\frac{-3}{5} = \frac{3}{-5} = \frac{-6}{10} = \frac{-9}{15} = \frac{9}{-15}, \ldots$

(d) $\frac{4}{10} = \frac{-4}{-10} = \frac{2}{5} = \frac{-2}{-5} = \frac{6}{15} = \frac{-6}{-15}, \ldots$

(e) $\frac{3}{1} = \frac{-3}{-1} = \frac{6}{2} = \frac{-6}{-2} = \frac{9}{3} = \frac{-9}{-3}, \ldots$

2. (a) $\{\frac{4}{5}\}$ (e) $\{\frac{2}{5}\}$

(b) $\{\frac{-1}{7}\}$ (f) $\{\frac{2}{5}\}$

(c) $\{\frac{-1}{2}\}$ (g) $\{\frac{2}{5}\}$

(d) $\{\frac{-1}{6}\}$ (h) $\{\frac{2}{5}\}$

3. (a) $9x = 7$ (d) $-3x = 1$

(b) $13x = -12$ (e) $3x = -1$

(c) $2x = 1$ (f) $-8x = -5$

other answers are possible.

4. (a)

$$\frac{6}{9} = \frac{2\cdot 3}{3\cdot 3}$$
$$= (2\cdot 3)\cdot\frac{1}{(3\cdot 3)}$$
$$= (2)(3)(\tfrac{1}{3})(\tfrac{1}{3})$$
$$= (2)(\tfrac{1}{3})(3)(\tfrac{1}{3})$$
$$= (2)(\tfrac{1}{3})(1)$$
$$= (2)(\tfrac{1}{3})$$
$$\frac{6}{9} = \frac{2}{3}$$

Follow the procedures as outlined at the left. Cross-multiplication ($ad = bc$) should be used only as a quick mental check.

a) True

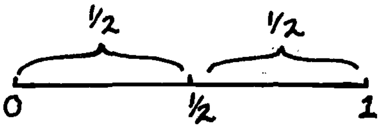
b) True

c) False

d) True

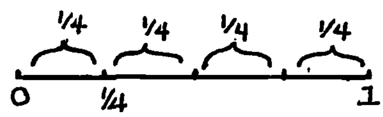
e) False

5. (a) (1)



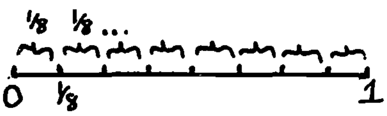
because $2 \cdot \frac{1}{2} = 1$

(2)



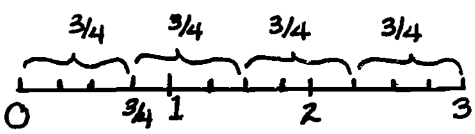
because $4 \cdot \frac{1}{4} = 1$

(3)



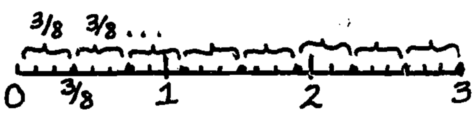
because $8 \cdot \frac{1}{8} = 1$

(4)



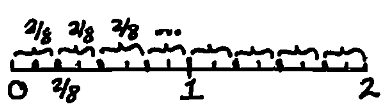
because $4 \cdot \frac{3}{4} = 3$

(5)



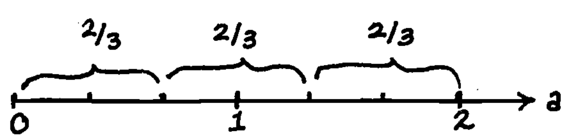
because $8 \cdot \frac{3}{8} = 3$

(6)

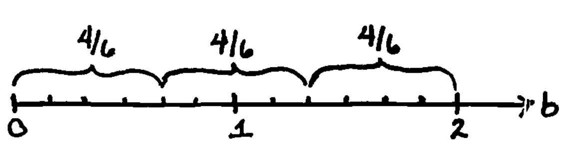


because $8 \cdot \frac{2}{8} = 2$

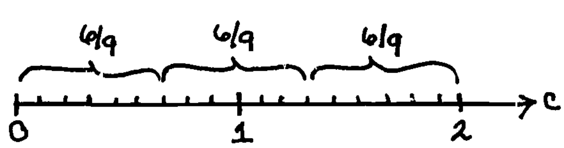
(b)



because $3 \cdot \frac{2}{3} = 2$



because $3 \cdot \frac{4}{6} = 2$



because $3 \cdot \frac{6}{9} = 2$

Each fraction $(\frac{2}{3}, \frac{4}{6}, \frac{6}{9})$ is a correct replacement for x in "3 · x = 2." Since there is only one solution to this equation, the three fractions must be equivalent.

12.6 <u>(Q, ·)</u> (1 day)

Here it should be pointed out again that we are assuming that · is commutative and associative. We now give the rational number $\frac{b}{a}$ a description as the <u>unique solution</u> of the equation "ax = b" where a and b are integers and $a \neq 0$.

For the sake of <u>standardization</u>, students should be told to place all fractional answers in the irreducible form equivalent to the answer obtained.

Some condusion might arise with the negative integers. The teacher may wish to form another convention:

a) fractions containing one negative integer should be written in the form $\frac{-a}{b}$ where a, b ∈ N.

b) fraction containing two negative integers should be written in the form $\frac{a}{b}$ where a, b ∈ N.

Thus $\frac{3}{-5} = \frac{-3}{5}$ since $\frac{3}{-5} = \frac{1 \cdot 3}{-1} = (\frac{1}{-1})(\frac{3}{5}) = (-1)\ (\frac{3}{5})$

$$= (\frac{-1}{1})(\frac{3}{5}) = \frac{-1 \cdot 3}{1 \cdot 5} = \frac{-3}{5}.$$

Similarly, $\frac{-2}{-7}$ becames $\frac{2}{7}$ since $\frac{-2}{-7} = \frac{-1 \cdot 2}{-1 \cdot 7} = (\frac{-1}{-1})(\frac{2}{7}) = (1)(\frac{2}{7})$

$$= (\frac{1}{1})(\frac{2}{7}) = \frac{1 \cdot 2}{1 \cdot 7} = \frac{2}{7}.$$

12.7 Solutions for Exercises

Note: Exercises 2 and 8 are essential to future work. If possible, these exercises should be introduced in class. The teacher should be attentive to properties illustrated by exercises 3,5 and 6.

1. (a) $\frac{1}{3}$

(b) $\frac{-7}{15}$

(c) $\frac{5}{2}$

(d) $\frac{65}{34}$

(e) $\frac{-15}{64}$

(f) $\frac{1}{1}$

(g) $\frac{0}{1}$ or $\frac{0}{-1}$

(h) $\frac{5}{32}$

2. (a) $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$, $\frac{9}{12} \cdot \frac{2}{3} = \frac{1}{2}$

(b) yes

(c) (1) $\frac{5}{24}$, $\frac{5}{24}$

(2) $\frac{-2}{21}$, $\frac{-2}{21}$

(3) $\frac{38}{7}$, $\frac{38}{7}$,

(4) $\frac{5}{16}$, $\frac{5}{16}$

(5) $\frac{1}{6}$, $\frac{1}{6}$.

(d) Confirm.

3. (a) $\frac{2}{15}$

(b) $\frac{15}{56}$

(c) $\frac{8}{11}$

(d) $\frac{8}{11}$

(e) $\frac{10}{1}$

(f) $\frac{21}{10}$

(g) $\frac{21}{10}$

(h) $\frac{8}{63}$

(i) $\frac{8}{63}$

(j) $\frac{0}{1}$

(k) $\frac{28}{45}$

(l) $\frac{28}{45}$

(m) $\frac{0}{1}$

(n) $\frac{35}{12}$

(o) $\frac{35}{12}$

4. (a) 10 (b) $\frac{10}{1}$ (c) 18 (d) $\frac{18}{1}$ (e) 56 (f) $\frac{56}{1}$ (g) 75 (h) $\frac{75}{1}$ (i) -6 (j) $\frac{-6}{1}$ (k) 24 (l) $\frac{24}{1}$

5. (a) $\frac{2}{5}$ (b) $\frac{4}{3}$ (c) $\frac{5}{6}$ (d) $\frac{10}{7}$ (e) $\frac{9}{5}$ (f) $\frac{3}{4}$

6. In all parts, the product is $\frac{1}{1}$.

7. (a) $\{\frac{5}{7}\} = \frac{-5}{-7} = \frac{10}{14} = \frac{-10}{-14} = \frac{15}{21} = \frac{-15}{-21} \ldots$

(b) $\{\frac{2}{3}\} = \frac{-2}{-3} = \frac{4}{6} = \frac{-4}{-6} = \frac{10}{15} = \frac{-10}{-15} = \ldots$

(c) $\{\frac{1}{4}\} = \frac{-1}{-4} = \frac{2}{8} = \frac{-2}{-8} = \frac{3}{12} = \frac{-3}{-12} = \ldots$

(d) $\{\frac{1}{10}\} = \frac{-1}{-10} = \frac{2}{20} = \frac{3}{30} = \frac{-3}{-30} = \ldots$

(e) $\{\frac{2}{5}\} = \frac{-2}{-5} = \frac{4}{10} = \frac{6}{15} = \ldots$

(g) $\{\frac{2}{-3}\} = \frac{-2}{3} = \frac{4}{-6} = \frac{-4}{6} = \ldots$

(h) $\{\frac{-2}{3}\} = \frac{2}{-3} = \ldots$

(i) $\{\frac{a}{b}\} = \frac{-a}{-b} = \frac{2a}{2b} = \frac{-2a}{-2b} = \frac{3a}{3b} = \ldots = \frac{na}{nb}$ where $M \in Z$.

8. (a) (1) $\frac{2}{3} \stackrel{?}{=} \frac{4}{6}$

$2 \cdot 6 \stackrel{?}{=} 3 \cdot 4$

$12 = 12$

Equivalent

(2) $\frac{-3}{10} \stackrel{?}{=} \frac{6}{-20}$

$(-3)(-20) \stackrel{?}{=} 10 \cdot 6$

$+60 = 60$

Equivalent

(3) $\frac{7}{8} \stackrel{?}{=} \frac{-14}{-16}$

$7(-16) \stackrel{?}{=} 8(-14)$

$-112 = -112$

Equivalent

(b) (1) true (6) true

(2) true (7) true

(3) true (8) false

(4) true (9) true

(5) false (10) true

12.8 Properties of (Q, $\cdot$) (1 day)

Here the properties suggested by the preceding discussions and exercises are formally stated. The students should clearly see that within the operational system (Q, $\cdot$), we have:

(a) commutativity

(b) associativity

(c) an identity equivalent to $\frac{1}{1}$

(d) multiplicative inverses or reciprocals.

These properties should be linked with earlier discussions of operational systems (e.g., Chapter 1, Chapter 2 Chapter 4.)

Note: Not every rational number has a multiplicative inverse. If the inverse of $\frac{a}{b}$ is $\frac{b}{a}$ and if $\frac{b}{a}$ is to be a rational number, then $a \neq o$. In short, every rational number

$\frac{a}{b}$ has a multiplicative inverse provided that $a \neq 0$ and $b \neq 0$.

12.9 Solutions for Exercises.

Note: In Exercises 2, the student should come to realize that the product of 0 and any rational number is 0. Note that in Exercise 5(c), $(Q, \cdot)$ fails to be a group for the one reason that 0 has no inverse. If this element is "removed," a group structure results. (5(d)). Be sure to consider Exercise 7.

1. (a) $\{\frac{1}{1}\}$ (b) $\{\frac{3}{4}\}$ (c) $\{\frac{1}{1}\}$ (d) $\{\frac{5}{6}\}$ (e) $\{\frac{5}{4}\}$ (f) $\{\frac{7}{10}\}$ (g) $\{\frac{1}{1}\}$ (h) $\{\frac{1}{1}, \frac{-1}{1}\}$

Note: In (h) $\frac{1}{1}$ is a solution because $x \cdot x = \frac{1}{1} \cdot \frac{1}{1} = 1$

and $\frac{-1}{1}$ is a solution because $x \cdot x = \frac{-1}{1} \cdot \frac{-1}{1} = 1$

<u>but</u> $\frac{1}{1} \cdot \frac{-1}{1} \neq 1$.

2. In all parts, the product is 0.

3. The product of 0 and any rational number is 0.

4. (a) $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \dots, \frac{k}{k}, \dots$ $(k \neq 0)$

   (b) $\frac{5}{2}, \frac{10}{4}, \frac{15}{6}, \dots \frac{5k}{2k}, \dots$ $(k \neq 0)$

   (c) 1, (d) $\frac{1}{1}$ and $\frac{-1}{1}$ (e) $\frac{0}{1}$.

5. (a) associativity, identity element, inverse elements

   (b) No; not every element has an inverse.

   (c) No; 0 has no inverse

   (d) Yes; it is a commutative group.

6. (a) -8; 14; -234; 55; 0; 0

   (b) $\frac{-8}{1}$; $\frac{14}{1}$; $\frac{-234}{1}$; $\frac{55}{1}$; $\frac{0}{1}$; $\frac{-14}{1}$

7. (a) $\frac{5}{6}$ (b) $\frac{7}{8}$ (d) 1 (e) $\frac{2}{9}$ (f) $\frac{5}{6}$ (g) $\frac{1}{2}$ (h) $\frac{6}{7}$

   (i) $\frac{2}{7}$ (j) $\frac{a}{f}$

12.12 Division of Rational Numbers. (1 to $1\frac{1}{2}$ days)

Division (as usual) is defined in terms of multiplication, and this section is designed to help the student see why it is reasonable when dividing by a rational number to multiply by the reciprocal instead. While eventually he will use this generalization almost automatically, it would be wise to work a number of examples such as those in the text. Incidentally, the phrase "multiply by the reciprocal" is preferable to "invert and multiply."

We also show now that the rational number $\frac{b}{a}$ may be interpreted as the quotient of b and a for $b \in Z$, $a \in Z$, $a \in Z$ and $a \neq 0$. This justifies the use of Q as a symbol for the rational numbers.

Also, we show that in constructing a system to solve equations "$px = q$" with $p \in Z$, $q \in Z$, and $p \neq 0$ we have also constructed a system $(Q, +, \cdot)$ in which "$px = q$" has a solution for $p \in Q$, $q \in Q$, and $p \neq 0$.

12.11 Solutions for Exercises.

In Exercise 4, if the student has trouble with equations such as $\frac{2}{3} \cdot \frac{x}{y} = \frac{3}{4}$, use analogies such as $5 \cdot n = 10$ to help him see that he may rewrite the equation as $\frac{x}{y} = \frac{3}{4} \div \frac{2}{3}$.

In Exercise 5(d), there is some ambiguity. Technically, division is not an operation since we cannot divide by zero; and this is the answer called for here. However, it is generally accepted that the rational numbers are closed under division, with the understanding that division by zero is not allowed.

1. (a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{15}{8}$ (d) $\frac{8}{15}$ (e) $\frac{-42}{5}$ (f) $\frac{-5}{42}$

2. (a) $\frac{98}{15}$ (b) $\frac{64}{81}$ (c) $\frac{1}{1}$ (d) $\frac{0}{1}$ (e) $\frac{16}{121}$

   (f) $\frac{8}{9}$ (g) $\frac{4}{3}$ (h) $\frac{16}{9}$ (i) $\frac{15}{4}$ (j) $\frac{5}{3}$ (k) $\frac{14}{1}$

   (l) $\frac{7}{2}$

3. (a) 3 (b) $\frac{3}{1}$ (c) $\frac{3}{1}$ (d) 4 (e) $\frac{4}{1}$ (f) 1 (g) $\frac{1}{1}$

   (h) In $(Z, \cdot)$ there is no solution. However, the quotient is $\frac{1}{2}$.

   (i) $\frac{1}{2}$

4. (a) $\{\frac{9}{8}\}$ (b) $\{\frac{8}{9}\}$ (c) $\{\frac{15}{14}\}$ (d) $\{\frac{10}{21}\}$ (e) $\{\frac{24}{25}\}$

(f) $\{\frac{6}{5}\}$ (g) $\{\frac{3}{2}\}$ (h) $\{\frac{7}{8}\}$ (i) $\{\frac{0}{1}\}$ (j) $\{\frac{2}{3}\}$.

5. (a) No. There is no number which when multiplied by $\frac{0}{1}$ give $\frac{2}{3}$. Or, $\frac{0}{1}$ has no multiplicative inverse.

(b) $\frac{0}{1}$

(c) $\frac{0}{1}$ or 0 because $\frac{0}{1}$ has no reciprocal. Or, $\frac{d}{c} \notin Q$.

(d) No, since one cannot divide by 0.

(e) Yes. (See commentary notes above).

(f) No.

12.12 <u>Addition of Rational Numbers</u> (2 to 3 days).

This section contains five major areas of study or purpose, the last of which is optional for classroom discussion:

1. <u>Computational</u> <u>understanding</u> regarding the addition of rational numbers in fractional form, $\frac{a}{b}$.

In general, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ where $\frac{a}{b}, \frac{c}{d} \in Q$.

The students must understand that $(Q, +)$ is an operational system.

2. An inherent understanding of the <u>least common denominator</u>. A good device for student comprehension of this concept can be found in listing two sets of fractions equivalent to the fractions to be added.

Ex: $\frac{3}{8} + \frac{1}{6}$

$$\frac{3}{8} = \{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \frac{15}{40}, \frac{18}{48}, \frac{21}{56}, \ldots\}$$

$$\frac{1}{6} = \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \frac{5}{30}, \frac{6}{36}, \frac{7}{42}, \frac{8}{48}, \frac{9}{54}, \ldots\}$$

Here, $\frac{3}{8}$ and $\frac{1}{6}$ same many common denominators: 24, 48, 72, 96, 120, etc. but <u>only one is least</u>, that is 24.

3. By studying the properties that hold in (Q, +), the students should realize that <u>(Q, +) is a commutative group</u>. In addition to being an operation system, we see that (Q, +) maintains:

   (a) commutativity

   (b) associativity

   (c) an identity, $\frac{0}{1}$

   (d) an inverse for every element. The additive inverse of $\frac{a}{b}$ is $\frac{-a}{b}$ where both are Rational Numbers.

4. An additional property relating (Q, +) with (Q, $\cdot$) is that of <u>distributivity</u>. Here, multiplication will

distribute over addition in $(Q, +, \cdot)$:

$$\forall \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q: \quad \frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$$

and

$$\left(\frac{a}{b} + \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \frac{e}{f} + \frac{c}{d} \cdot \frac{e}{f}$$

5. (optional discussion) <u>$(Q, +, \cdot)$ forms a structure called a field</u>. We see that the properties of the field are maintained in that:

(a) $(Q, +)$ is a commutative group.

(b) $(Q \setminus \{0\}, \cdot)$ is a commutative group.

(c) Distributivity of multiplication over addition holds.

12.13 1. (a) $\frac{5}{6}$ (b) $\frac{13}{6}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{14}$

(e) $\frac{29}{9}$ (f) $\frac{95}{36}$ (g) $\frac{65}{36}$ (h) $\frac{11}{48}$

(i) $\frac{-1}{12}$ (j) $\frac{xz + yw}{yz}$

2. (a) $\frac{7}{10}$ (b) $\frac{7}{10}$ (c) $\frac{29}{40}$ (d) $\frac{29}{40}$

(e) $\frac{89}{96}$ (f) $\frac{89}{96}$ (g) $\frac{-4}{221}$ (h) $\frac{-4}{221}$

3. Commutative property

4. (a) $\frac{67}{30}$ (b) $\frac{67}{30}$ (c) $\frac{11}{24}$ (d) $\frac{11}{24}$

5. Associative property.

6. $\frac{0}{1}, \frac{0}{2}$, etc.

7. (a) $\frac{0}{1}$ (b) $\frac{0}{1}$ (c) $\frac{9}{11}$ (d) $\frac{9}{11}$

(e) $\frac{0}{1}$ (f) $\frac{0}{1}$ (g) $\frac{0}{1}$ (h) $\frac{-81}{7}$

8. (a) 10 (b) $\frac{10}{1}$ (c) 7 (d) $\frac{7}{1}$ (e) -8

(f) $\frac{-22}{1}$ (g) -12 (i) $\frac{-12}{1}$

9. (a) Yes, it is a commutative group.

(b) Yes, it is a commutative group.

10. (a) $\frac{-2}{3}$ (b) $\frac{5}{3}$ (c) $\frac{0}{1}$ (d) $\frac{-15}{7}$

(e) $\frac{15}{7}$ (f) $\frac{a}{b}$

11. (a) $\frac{-3}{4}$ (b) $\frac{5}{2}$ (c) $\frac{-10}{3}$ (d) $\frac{75}{7}$

(e) $\frac{-2}{5}$ (f) $\frac{2}{5}$ (g) $\frac{7}{8}$ (h) $\frac{-7}{8}$ (i) $\frac{a}{b}$

12. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{2}{3}$ (d) $\frac{2}{3}$ (e) $\frac{3}{5}$

(f) $\frac{3}{5}$ (g) $\frac{7}{15}$ (h) $\frac{7}{15}$

12.14 <u>Subtraction of Rational Numbers</u> (1 day)

In general, subtraction is defined in terms of addition as follows:

$$A - B = A + (-B).$$

And this is the sort of definition we present here for subtraction of rational numbers. Notice the analogy between the cases for division and subtraction:

Instead of dividing by $\frac{a}{b}$, you may multiply by the (multiplicative) inverse.

Instead of subtracting $\frac{a}{b}$, you may add the (additive) invers. This analogy is probably worth mentioning to the students. Thus we speak of the system $(Q, +, \cdot)$ since theoretically division and subtraction are not needed as distinct operations.

Note that subtraction is neither commutative non associative. For lack of the latter property, it can be stated that $(Q, -)$ is not a group. Similarly lacking would be a left hand identity element.

12.5 Solutions for Exercises.

1. (a) $\frac{2}{5}$ (b) $\frac{5}{13}$ (c) $\frac{-5}{13}$ (d) $\frac{-3}{1}$

   (e) $\frac{4}{5}$ (f) $\frac{-4}{5}$ (g) $\frac{-2}{5}$ (h) $\frac{1}{8}$

   (i) $\frac{0}{1}$ (j) $\frac{-43}{24}$ (k) $\frac{83}{24}$ (l) $\frac{-83}{24}$

   (m) $\frac{43}{24}$ (n) $\frac{-9}{195}$

2. (a) $\frac{1}{15}$ (b) $\frac{1}{15}$ (c) $\frac{-3}{5}$ (d) $\frac{-1}{5}$

3. (a) $\frac{1}{2}$ (b) $\frac{5}{4}$ (c) $\frac{1}{2}$ (d) $\frac{-19}{48}$ (e) $\frac{26}{21}$

   (f) $\frac{26}{21}$ (g) $\frac{0}{1}$ (h) $\frac{0}{1}$

4. Yes

5. (a) No. (b) No. (c) No. (since $\frac{0}{1}$ does not commute)

6. No (it is not associative, and it lacks a left hand identity element)

12.16 Ordering the Rational Numbers (1 day)

The fact that (Z, +, ·) is embedded in (Q, +, ·) and the desire to reserve the properties of "<" with respect to Z is used to motivate the definition of the "<" relation in Q x Q. The purpose is to develop a sense on the line by which order is maintained. In terms of convention, we usually say that a < b if a is to the left of b. An interesting experiment is seen in indicating parallel lines with opposing senses and checking to see that order within each is maintained. In looking at the rationals, $\frac{a}{b}$ and $\frac{c}{d}$, the students must understand that b and d are positive integers. Since there are no stipulations regarding the numerators, a could be a positive integer, a negative integer or zero. The same holds for c.

Thus $\frac{a}{b} < \frac{c}{d}$ $\quad \frac{ad}{bd} < \frac{bc}{bd}$

$\frac{a}{b} < \frac{c}{d}$ if and only if ad < bc.

The number line (as in Examples 1 and 2 is often an excellent visual aid to help students get the "feel" of the "<" relation. It might also be wise to point out why it is correct to speak of "<" as a relation, in accordance with the concept developed in Chapter 8. One could, after all, list ordered pairs belonging to "<", e.g., (2, 5), $(\frac{1}{2}, \frac{3}{4})$, $(\frac{1}{3}, \frac{5}{9})$.

12.17 Solutions for Exercises.

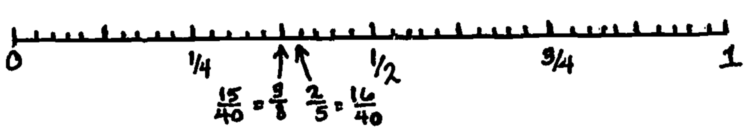
In Exercise 5, emphasize that we can partition Q into three classes called POSITIVE, ZERO, AND NEGATIVE.

Exercise 6 suggests the usual definition of a positive element in Q. That is, $\frac{a}{b}$ is a positive retional number if the integral product ab is a positive integer.

Exercise 9 deals briefly with the notion of density. In Q, it is always possible to find a third rational number which is between two given rational numbers. Another interesting way to show this is to show that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a+c}{b+d}$ is between them.

1. (a) $\frac{3}{8} < \frac{2}{5}$ since $\frac{15}{40} < \frac{16}{40}$ (c) $\frac{5}{4} < \frac{7}{5}$ since $\frac{25}{20} < \frac{28}{20}$

   (b) $\frac{5}{8} < \frac{3}{4}$ since $\frac{5}{8} < \frac{6}{8}$ (d) $\frac{9}{4} < \frac{8}{3}$ since $\frac{27}{12} < \frac{32}{12}$

2. (a) 

   (b) (c) (d) similar.

3. (a) true (b) false (c) false (d) false

   (e) false (f) true (g) false (h) false

   (i) false

4. (a) $\frac{1}{2} < \frac{5}{8}$ (e) $\frac{-14}{5} < \frac{-8}{3}$ (i) $\frac{-7}{4} < \frac{-5}{3}$

   (b) $\frac{-5}{8} < \frac{-1}{2}$ (f) they are equal

   (c) $\frac{7}{15} < \frac{11}{23}$ (g) $\frac{13}{7} < \frac{100}{51}$

   (d) $\frac{-11}{23} < \frac{-7}{15}$ (h) they are equal

5. (a) positive
   (b) negative
   (c) negative
   (d) positive
   (e) negative
   (f) positive
   (g) negative
   (h) positive
   (i) zero
   (j) negative
   (k) positive
   (l) positive

6. Yes, it is positive, since either numerator and denominator are both positive or both negative. We know this because the product is positive.

7. (a) No, since it is not true that $ab < ab$.
   (b) No, for if $\frac{a}{b} < \frac{c}{d}$, then it is not true that $\frac{c}{d} < \frac{a}{b}$.
   (c) Yes.

8. (a) $<$ (b) $=$ (c) $>$

9. (a) No
   (b) Yes, for instance $\frac{5}{2}$, which is their average.
   (c) Yes. Since the average of two rational numbers is a rational number between them. We know $2\frac{1}{2}$ is between 2 and 3.

10. $\frac{1}{6}$	13. $\frac{49}{2}$	16. $\frac{-3}{16}$	19. 12
11. $\frac{31}{8}$	14. $\frac{-13}{8}$	17. $\frac{-16}{3}$	20. $\frac{4}{3}$
12. 2	15. $\frac{-19}{8}$	18. $\frac{81}{14}$	21. -6

22. -6

23. $\frac{81}{4}$

24. $\frac{-81}{4}$

25. -2

26. -2

27. 5

28. 10

29. $\frac{37}{6}$

30. 7

31. $-3 < \frac{-7}{3}$

32. $\frac{41}{3} < 14$

33. $4 < \frac{21}{5}$

34. $\frac{-21}{5} < -4$

35. $\frac{47}{8} < 6$

36. $\frac{999}{1000} < 1$

12.18 Decimal Fractions (1 day)

The purpose here is to view the set of rational numbers when using decimal notation in place of that of common fractions. Students will understand that every fraction (a rational $\frac{a}{b}$ where $a, b \in Z$ and $b \neq 0$) can be written in decimal notation. Again, changing the name of the rational number will not change the way in which it behaves, whether in operations or in the solution of equations.

Note that not all decimals are rational numbers. Only terminating decimals and infinite repeating decimals can be transformed to the form $\frac{a}{b}$ where $a, b \in Z$ and $b \neq 0$.

The student probably has a firm understanding of the place value concept which he brings with him from the elementary school. Hence, this section is in the nature of a review, with the possible exception that extension to the right of the decimal point may not be so familiar.

12.19 Solution for Exercises

One of the purposes of Exercise 2 is to emphasize that .5, for instance, is not a new kind of number; it is the rational number which is the solution of "$10 \cdot x = 5$" or "$2 \cdot x = 1$."

1. (a) $\frac{3}{10}$ (f) $\frac{3}{100}$ (k) $\frac{61}{20}$

   (b) $\frac{8}{25}$ (g) $\frac{3}{1000}$ (l) $\frac{251}{10}$

   (c) $\frac{8}{25}$ (h) $\frac{3}{1000000}$ (m) $\frac{5}{8}$

   (d) $\frac{13}{40}$ (i) $\frac{1}{2}$ (n) $\frac{85}{8}$

   (e) $\frac{73}{10}$ (j) $\frac{1}{200}$ (o) $\frac{33}{100}$

2. (a) $2 \cdot x = 1$
   (b) $10 \cdot x = 7$
   (c) $25 \cdot x = 2$
   (d) $100 \cdot x = 7$
   (e) $100 \cdot x = 7$
   (f) $1000 \cdot x = 33$
   (g) $10 \cdot x = 27$
   (h) $8 \cdot x = 3$
   (i) $5 \cdot x = 3$
   (j) $5 \cdot x = 3$
   (k) $1000000 \cdot x = 123456$
   (l) $1000000 \cdot x = 333333$
   (m) $-2 \cdot x = 1$
   (n) $-20 \cdot x = 1$
   (o) $-10 \cdot x = 27$
   (p) $-8 \cdot x = 3$

3. (a) .5
   (b) .25
   (c) .75
   (d) .2
   (e) .4
   (f) .6
   (g) .8
   (h) .125
   (i) .375
   (j) .625
   (k) .875

4. (a) . 5 = .50 = . 500 = . 5000 = . 500000

   (b) to (i) Similar.

5. (a) the numerator may be thought of as 5, and the denominator (understood) as 10.

   (b) 7 and 100000.

   (c) 82 and 10

   (d) Yes. The place value system determined the denominator.

6. (a) . 15 (b) .35 (c) .32 (d) .84

   (e) .390625 (f) .315

12.20 Infinite Repeating Decimals. (1 - 1 $\frac{1}{2}$ days)

Certain rational numbers, such as $\frac{1}{3}$, do not have a terminating decimal representation, and this section deals with them. Actually this is a very subtle topic, since the limit concept is involved in a statement such as $\frac{1}{3} = .3333 \ldots = .\overline{3}$ therefore such a statement is avoided at this time, and we merely show that we can approximate $\frac{1}{3}$ to as many decimal places as desired. Every rational number may be represented as an infinite repeating decimal (for instance, $\frac{1}{2} = .49999999 \ldots = .4\overline{9}$) ; but again we do not discuss the matter here. The student should see however that every rational number is represented by either a terminating decimal or a decimal that develops (sooner of later) a "repeating pattern."

12.21. Exercises.

In Exercise 5(e), the limit concept is not far beneath the surface. If the class seems prepared, then other questions of this nature might be asked.

Exercise 7 is a good one for class discussion. The student should be convinced that in perfoming the division $a \div b$, either he will get a zero remainder at some stage, or a repeating pattern will develop. (This results from the fact that only a finite number of remainders are available.)

1. (a) $\frac{1}{3000}$ (b) $\frac{1}{30000}$ (c) .3333

2. (a) $3 \cdot x = 1$ (b) $10 \cdot x = 3$ (c) $100 \cdot x = 33$
   (d) No, for they are different numbers.

3. (a) $\frac{4}{600}$ (c) .17 (e) $\frac{2}{6000}$ (g) .167
   (b) $\frac{2}{600}$ (d) $\frac{4}{6000}$ (f) .167 (h) .1667

4. (a) .8333 (d) .1818
   (b) .6667 (e) .0833
   (c) .0909 (f) .4167

5. (a) $\frac{1}{90}$ (b) $\frac{1}{900}$ (c) $\frac{1}{9000}$ (d) $\frac{1}{90000}$
   (e) .111111

6. (a) 2.333 (b) Yes

7. (a) Seven numbers--0,1,2,3,4,5,6 (b) .142857142857

8. 0

9. If you get 0 as a remainder, the decimal terminates. If you never get 0, some remainder must repeat, since there is a limited number of possible remainders.

10. 0;1 .09;.10 .0909; .0910 .090909; .090910 .09090909; .09090910

11. It is not the case that each of the intervals is contained in the interval before it.

12.22 Decimal Fractions and Order of the Rational Numbers.

(1 day)

Comparison of rational numbers represented by decimal fractions is very easy; and so, if place value is properly understood, this section should pose no difficulties.

12.23 Exercises.

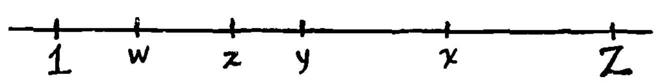
Exercise 3, (together with Exercise 9 of 12.18) should be used to strengthen the understanding of Q as a dense set. One cannot list consecutive elements of Q by the natural ordering, for between any two elements there is a third.

1. (a) $12.5 > 12.4$ (f) $826,33 > 826.30$
   (b) $8.33 < 8.34$ (g) $5.4793293 > 5.4789999$
   (c) $.1257 > .1250$ (h) $548 < 551$
   (d) $.1257 > .125$ (i) $1.9999 < 2$
   (e) $.6666 < .6667$ (j) $.9874 < .9875$

2. (a) $-3.567 > -3.582$
   (b) $-.12345 > -.12453$
   (c) $-.99 > -1$
   (d) $-100.555 > -100.565$
   (e) $-42\ 80 > -42.85$
   (f) $-42.8 > -42.85$
   (g) $-12.9999 < 12.9998$
   (h) $-4.378 < -4.3779$

3. Answers will vary. Here, the exact midpoints are shown:
   (a) .7
   (b) 2.37
   (c) 45.9615
   (d) 105
   (e) 5.425
   (f) 5.425
   (g) 3.85
   (h) 2.995

4.



5. No, it is not possible, for instance, to find an ineger between 1 and 2.

12.25 Solutions for Review Exercises.

1. (a) $\frac{3}{4}$
   (b) $\frac{4}{3}$
   (c) $\frac{3}{-4}$ or $\frac{-3}{4}$ or $-\frac{3}{4}$
   (d) $\frac{-3}{-4}$
   (e) $\frac{3}{4}$
   (f) $\frac{5}{12}$
   (g) $\frac{20}{3}$
   (h) 7
   (i) $\frac{5}{7}$
   (j) $\frac{8}{-3}$ or $\frac{-8}{3}$
   (k) $\frac{511}{102}$
   (l) $\frac{6}{-11}$ or $-\frac{6}{11}$
   (m) 1
   (n) 0
   (o) $\frac{a}{4}$

2. (a) $\frac{17}{12}$ (e) $\frac{64}{25}$ (i) $\frac{37}{12}$

(b) $\frac{8}{9}$ (f) 1 (j) $\frac{2}{3}$

(c) $\frac{27}{14}$ (g) $\frac{4}{3}$ (k) $6\frac{3}{4}$ or $\frac{27}{4}$

(d) $-\frac{27}{14}$ (h) $\frac{3}{4}$ (l) $\frac{32}{5}$

(m) $\frac{-27}{4}$ (o) $\frac{3}{7}$

(n) $\frac{5}{32}$ (p) $\frac{7}{3}$.

3. (a) $\frac{17}{8}$ (f) $\frac{145}{48}$

(b) $\frac{11}{15}$ (g) $\frac{59}{24}$

(c) $\frac{21}{10}$ (h) $23\frac{9}{10}$ or $\frac{239}{10}$

(d) $\frac{20}{9}$ (i) 1

(e) $\frac{80}{9}$ (j) $\frac{1}{2}$

4. (a) $\frac{6}{7}$ (c) $\frac{10}{3}$ (e) $\frac{ad}{bc}$

(b) $\frac{81}{4}$ (d) $\frac{32}{5}$

5. (a) $6 \cdot \frac{1}{10}$

(b) $6 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100}$

(c) $0 \cdot \frac{1}{10} + 6 \cdot \frac{1}{100} + 3 \cdot \frac{1}{1000}$

(d) $0 \cdot \frac{1}{10} + 0 \cdot \frac{1}{100} + 6 \cdot \frac{1}{1000} + 0 \cdot \frac{1}{10000} + 3 \cdot \frac{1}{100000}$

(e) $2.10 + 5 + 0.\frac{1}{10} + 8.\frac{1}{100}$

(f) $3 + 1 \cdot \frac{1}{10} + 7 \cdot \frac{1}{100} + 5 \cdot \frac{1}{1000}$

(g) $2 + 0 \cdot \frac{1}{10} + 0 \cdot \frac{1}{100} + 0. \frac{1}{1000} + 0 \cdot \frac{1}{10,000} + 0. \frac{1}{100,000} +$

$\frac{5}{1,000,000}$

(h) $3 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100} + 3 \cdot \frac{1}{1000} + 3 \cdot \frac{1}{10,000}$

6. (a) .5
   (b) .5
   (c) .75
   (d) .4
   (e) 3.4
   (f) .3333
   (g) .7
   (h) .70
   (i) .625
   (j) .1429

7. (a) $\frac{1}{2} < \frac{2}{3}$
   (b) $\frac{4}{7} > \frac{5}{9}$
   (c) $\frac{23}{5} > \frac{25}{7}$
   (d) $.3475 > .3429$
   (e) $\frac{1}{3} > .333333$
   (f) $.375 = \frac{3}{8}$
   (g) $.00001 > .000009$
   (h) $\frac{20}{7} > \frac{25}{12}$
   (i) $-\frac{3}{5} > \frac{-2}{3}$

8. Answers will vary. Midpoints are shown here.
   (a) $\frac{3}{4}$
   (b) $\frac{5}{8}$
   (c) $\frac{9}{16}$
   (d) $\frac{33}{64}$
   (e) $\frac{7}{18}$
   (f) .3455
   (g) $\frac{37}{15}$
   (h) $\frac{1}{200}$

9. (a) $\frac{9}{10}$ (c) $\frac{3}{8}$

(b) $-\frac{1}{15}$ (d) $\frac{-43}{10}$

Suggested Test Items -- Chapter 12

PART ONE. "RATIONAL NUMBER OPERATIONS." Perform the following rational number computations, in each case giving the result as an irreducible fraction.

1. $\frac{2}{3} - \frac{3}{5}$
2. $\frac{5}{6} \div \frac{3}{8}$
3. $\frac{3}{8} \div \frac{5}{6}$
4. $\frac{1}{2} + (\frac{2}{3} + \frac{3}{4})$
5. $(2 - \frac{1}{3}) \div \frac{1}{2}$
6. $-1 + (\frac{5}{8} \cdot \frac{8}{5})$
7. $(\frac{-2}{3} + \frac{2}{3}) \cdot \frac{5}{12}$
8. $(5 \cdot \frac{7}{5}) + (3 \cdot \frac{-2}{3})$
9. $\frac{36}{25} \cdot \frac{15}{24}$
10. $\dfrac{\frac{2}{5} + \frac{1}{3}}{\frac{7}{3} - \frac{1}{5}}$

PART TWO. EQUATIONS. Give the solution of each of the equations.

1. $2 \cdot x = 3$
2. $-7 \cdot x = 15$
3. $\frac{1}{2} \cdot x = 15$
4. $\frac{3}{5} \cdot x = \frac{1}{2}$
5. $x + x = \frac{3}{4}$
6. $\frac{2}{3} \cdot x = 0$
7. $\frac{2}{3} \cdot x = 1$
8. $\frac{2}{3} \cdot x = \frac{2}{3}$
9. $-\frac{7}{8} + x = 0$
10. $-\frac{7}{8} + x = -\frac{7}{8}$
11. $-\frac{7}{8} + x = \frac{7}{8}$
12. $x \div \frac{7}{9} = \frac{3}{21}$

6. $\frac{2}{3} \cdot x = 0$

PART THREE. "DECIMAL FRACTIONS."

1. Write a decimal fraction for each of the following rational numbers.

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{3}{4}$ (d) $\frac{7}{8}$ (e) $\frac{5}{2}$

2. Write an irreducible fraction $\frac{a}{b}$ which represents the same rational number as each of the following.

(a) .6 (b) .125 (c) .250 (d) 1.8

(e) .300

3. Write a decimal fraction which approximates $\frac{4}{9}$ correct to three decimal places.

PART FOUR. ORDERING AND DENSITY

1. In each of the following, place a "<", "=", or ">" so that a true statement results.

(a) $\frac{9}{14}$ $\frac{17}{24}$ (d) .3215 .3209

(b) $-\frac{9}{14}$ $-\frac{17}{24}$ (e) .6666 $\frac{2}{3}$

(c) $-\frac{1}{3}$ 0

2. Prove that $.33 < \frac{1}{3} < .34$.

3. (a) Find a rational number $\underline{x}$ such that $\frac{5}{7} < x < \frac{6}{7}$

(b) Find a rational number $\underline{x}$ such that $\frac{a}{b} < x < \frac{c}{d}$

PART FIVE. PROPERTIES AND RELATIONS

1. Complete each sentence with the word, phrase or term that will make the sentence a true statement for $(Q, +, \cdot)$

   (a) The additive inverse of $\frac{3}{5}$ is _____ .

   (b) The multiplicative inverse of $\frac{-1}{7}$ is _______.

   (c) $(\frac{35}{87} + \frac{9}{16}) + \frac{8}{97} = \frac{35}{87} + (\frac{9}{16} + \frac{8}{97})$ by the ____________ property.

   (d) Operational systems in Q include $(Q,+)$, $(Q\cdot)$ and _____.

   (e) $\frac{a}{b} = \frac{c}{d}$ if and only if $bc =$ _______.

2. Using $(Q,+,\cdot)$, answer TRUE or FALSE.

   (a) $(Q,+)$ is a group.

   (b) Every rational number has a reciprocal.

   (c) $(Q, \cdot)$ is a group.

   (d) If $\frac{a}{b} < \frac{c}{d}$, then a new rational number can be found so that $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$.

   (e) The identity in $(Q, +)$ is 1.

   (f) The identity in $(Q, -)$ is 0.

   (g) In $(Q, \cdot)$, $x = \frac{1}{x}$ only when $x = 1$.

   (h) Every rational has an additive inverse.

   (i) Addition distributes over multiplication in $(Q,+,\cdot)$.

3. Rewrite any three of the false statements from problem 2, correcting the errors to make them true statements.

Answer Key for Suggested Test Items.

Part One

1. $\frac{1}{15}$
2. $\frac{20}{9}$
3. $-\frac{9}{20}$
4. $\frac{23}{12}$
5. $\frac{10}{3}$
6. $\frac{0}{1}$
7. $\frac{0}{1}$
8. $\frac{5}{1}$
9. $\frac{9}{10}$
10. $\frac{11}{32}$

Part Two

1. $\{\frac{3}{2}\}$
2. $\{\frac{-15}{7}\}$
3. $\{\frac{6}{1}\}$
4. $\{\frac{5}{6}\}$
5. $\{\frac{3}{8}\}$
6. $\{\frac{0}{1}\}$
7. $\{\frac{3}{2}\}$
8. $\{\frac{1}{1}\}$
9. $\{\frac{7}{8}\}$
10. $\{\frac{0}{1}\}$
11. $\{\frac{7}{4}\}$
12. $\{\frac{1}{9}\}$

Part Three

1. (a) .5 (b) .2 (c) .75 (d) .875 (e) 2.5
2. (a) $\frac{3}{5}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{9}{5}$ (e) $\frac{3}{10}$
3. .444

Part Four

1. (a) $<$ (b) $>$ (c) $<$ (d) $>$ (e) $<$
2. $.33 < \frac{1}{3} < .34 \Longrightarrow \frac{33}{100} < \frac{1}{3} < \frac{34}{100} \Longrightarrow$

$$\frac{33}{100}\left(\frac{3}{3}\right) < \frac{1}{3}\left(\frac{100}{100}\right) < \frac{34}{100}\left(\frac{3}{3}\right) \Longrightarrow \frac{99}{300} < \frac{100}{300} < \frac{102}{300}.$$

3. Answers will vary

   (a) $\frac{11}{14}$ (midpoint), etc.

   (b) $\frac{ad + bc}{2bd}$ (midpoint); $\frac{a + c}{b + d}$; etc.

Part Five

1. (a) $-\frac{3}{5}$

   (b) $-\frac{7}{1}$

   (c) associative

   (d) (Q, -)

   (e) ad

2. (a) True

   (b) False

   (c) False

   (d) True

   (e) False

   (f) False

   (g) False

   (h) True

   (i) False

3. (b) Every rational number except zero has a reciprocal.

   (c) $(Q \setminus \{0\}, \cdot)$ is a group.

   (e) The identity in (Q, +) is 0 .

   (f) There is no identity in (Q -).

   (OR) The right hand identity in (Q, -) is 0.

   (g) In $(Q, \cdot)$ $x = \frac{1}{x}$ only when $x = 1$ or $x = -1$.

   (i) Multiplication distributes over addition in $(Q, +, \cdot)$.

Course I Chapter 13

Some Applications of the Rational Numbers

(total time 15 days)

Commentary for Teachers

The purpose of this chapter is to show some important applications of rational numbers. Dilations and translation are extended to a domain with rational numbers. Computation with decimals and percents are shown here. Ratio and proportion are also an important application of rational numbers. Presenting data in rectangular, circle and bar graphs is done in this chapter. An introduction to the idea of vectors is important for future work.

13.1 Rational Numbers and Dilations (3 days)

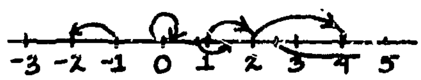
In view of the student's earlier work with dilations D_a where a is an integer (Chapter 7), the extension of the concept to include dilations D_x where x is a rational number should not be difficult for the idea is the same. The "distance" of any point from the origin is multiplied by the factor x, together with a reflection in the origin if x is a negative number.

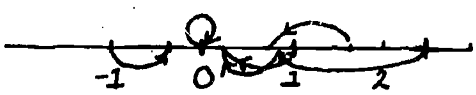
The text introduces rational number dilations in terms of a composition. Thus, $D_{3/2}$ is interpreted as a dilation D_3 (a "stretcher") followed by a dilation $D_{1/2}$ (a "shrinker"). Such an interpretation has some advantage in that students see that if a segment is stretched by 3, then shrunk by only 2, the final segment will be longer than the original. Hence, in a problem of the type, "find $\frac{a}{b}$ of x" (a number of which are included in this section) he can easily tell whether the result should be greater than x or less.

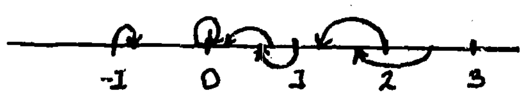
13.2 Exercises

In Exercise 2, the student should see another reason why we interpret $\frac{a}{b}$ and $\frac{c}{d}$ as representing the same number if ad = bc; for in such a case, they are associated with the same dilation.

In Exercise 4, the student should see that composition of dilations provides another reason for defining multiplication of rational numbers as we do. In fact, for the students not already familiar with multiplication of fractions this might well be the initial motivation.

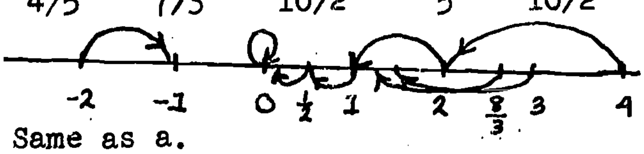
1. a. 

   b. 

   c. 

   d. $D_{2/3}$

   e. $D_{4/5}$, $D_{7/3}$, $D_{10/2}$ or D_5, $D_{10/2}$ or D_5

2. a. 

   b. Same as a.

   c. Yes; under these two dilations, every point has the same image.

   d. When ad = bc.

3. a. $\frac{10}{3}$ b. $\frac{20}{9}$ c. $\frac{20}{9}$ d. $D_{10/9}$

4. a. $D_{35/6}$ b. $\frac{36}{6}$

5. 65

6. 2100

7. $60.00

8. 30,000

9. $1955.00

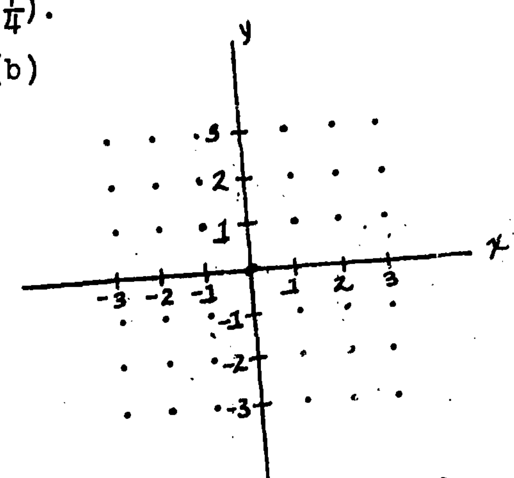
10. $2.25

11. a. Jim b. Sue c. They are the same height.

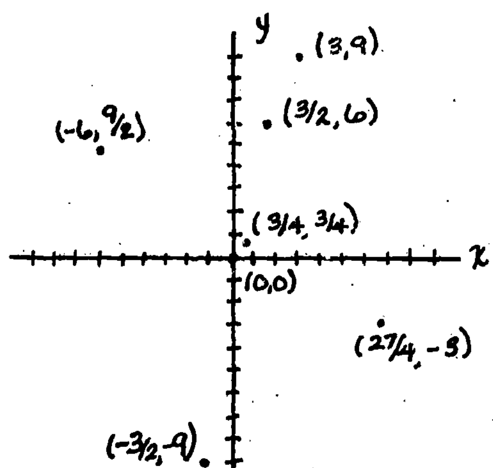
12. $\frac{7}{10}$

13. (c) and (c) In both cases, it is the segment joining (0,0) and $(\frac{3}{2},\frac{7}{4})$.

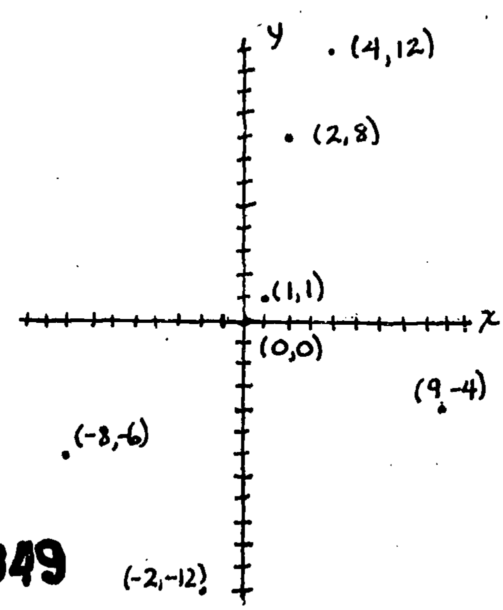
(a) + (b)



14. a.



b.



c. They are the inverse of each other.

15. a, b.

$$(X,Y) \xrightarrow{D_{\frac{b}{a}}} (\tfrac{b}{a}X, \tfrac{b}{a}Y) \xrightarrow{D_{\frac{a}{b}}} (\tfrac{a}{b} \cdot \tfrac{b}{a}X, \tfrac{a}{b} \cdot \tfrac{b}{a}Y)$$

$$(X,Y)$$

$$D_{\frac{a}{b}} \cdot D_{\frac{b}{a}} = D$$

c. $(X,Y) \xrightarrow{D_{\frac{b}{a}}} (X,Y) \xrightarrow{D_{\frac{a}{b}}} (X,Y)$

16. a. $D_{\frac{3}{2}}$ b. D_5 c. $D_{\frac{3}{4}}$ d. $D_{\frac{y}{x}}$, $x \neq 0$

17. a. The points are all mapped to (0,0).

b. The points are their own images.

c. The images are reflections about the origin.

13.3 Computations with Decimal Fractions (2 days)

Here we deal with the "decimal fraction computations" which we expect all students to be able to do. Some classes may already know this material very well; in others, it may be advisable to discuss in class a number of examples similar to those explained in the text. Although a full explanation is seldom fascinating to students who already know "how to do it", they should at least see how the basic properties (e.g. distributive property) are involved in the computational algorithms.

In Example 2, students should understand why we compute to "three decimal places" when our final answer is expressed correct to only "two decimal places". The '8' in the thousandths place tells us to use 29.13 instead of 29.12 (since .128 is closer to .130 than to .120).

13.4 Exercises

In Exercise 2, be sure that students see why all of the results are the same. Such examples can do much to end the confusion over "moving the decimal point" in a division problem.

1. a. 11.50 b. 46.220 c. -344.73 d. 11.25 e. 5.0625
   f. -1.40 g. -11.00 h. 29.1 i. 15.2 j. -15.2
   k. 2.025 l. -15.80
2. a. 1.7 b. 1.7 c. 1.7 d. 1.7
3. Any one of the quotients can be obtained from the other by multiplying by the multiplicative identity.
4. a. 15.69 b. 6.46 c. 181.25 d. .02 e. .02 f. 5149.53
5. $256.67
6. $285.18
7. a. $\frac{6}{5}$ b. 1.2
8. 6.75 inches
9. 12 yards

13.5 Ratio and Proportion (1 day)

We discuss ratio first as a comparison of sets; specifically, it is defined as the quotient of the measures of the sets. When the measures are those made on finite sets they are whole numbers and the ratio is a fraction when the measures are of continuous physical magnitudes. They are usually rational numbers and again their quotient is a rational number. This is the kind of definition we want in mathematics, although in the vernacular people often speak of the ratio of things.

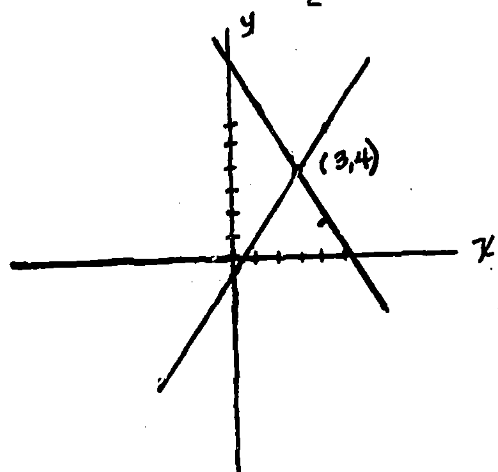
13.6 Exercises

In Exercise 6, it should be stressed that $\frac{a}{b} = \frac{3}{4}$ means the following:

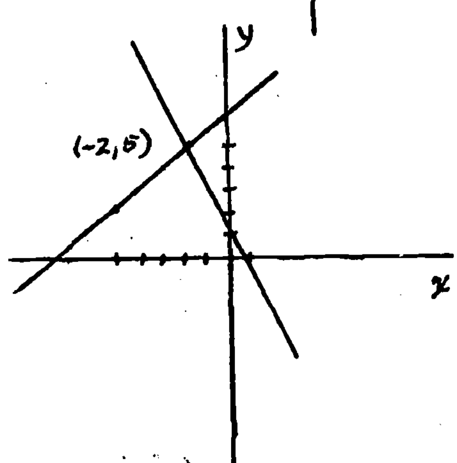
$$4 \cdot a = 3 \cdot b.$$

Therefore, $a = \frac{3}{4} \cdot b$, and $b = \frac{4}{3} \cdot a$.

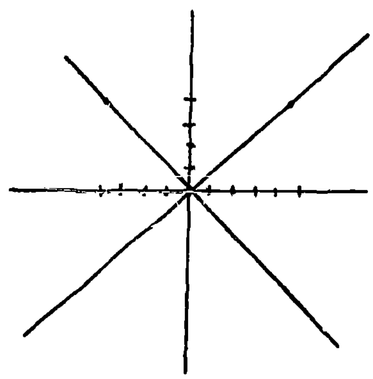
1. a. 2:5 b. $d_{2/5}$ c. 5.2 d. $D_{5/2}$ e. 1
2. a. 2:1 b. 1:2 c. 2:1 d. 66:25 e. 13:7 f. 7:13 g. 2:1
3. a. 17:20 b. 3:20 c. 1:1 d. 17:3 e. 3:17
4. (1,3), (2,6), (3,9),---.
5. a. d b. c c. c d. equal
6. a. $\frac{4}{3}$ b. $\frac{3}{4}$
7. a. 325 miles b. $4\frac{1}{2}$ miles
8. a, b.



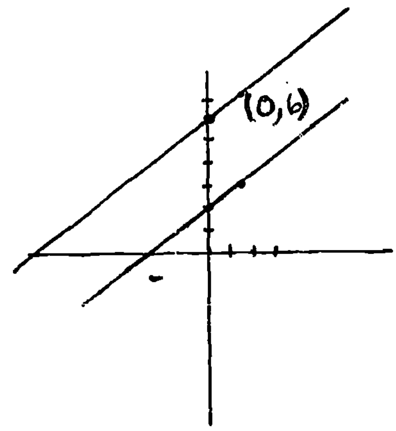
c, d.



e.



f.



The lines seem to be parallel.

13.7 <u>Using Proportions</u> (1 day)

This section presents a physical example to motivate the applications of proportions. The second example is strictly an algebraic approach to show the student how to solve a proportion.

13.8 <u>Exercises</u>

1. a. $\frac{5}{6} = \frac{10}{12}$, $\frac{15}{18} = \frac{20}{24}$ b. $\frac{1}{4} = \frac{2}{8}$, $\frac{3}{12} = \frac{4}{16}$ c. $\frac{1}{2} = \frac{2}{4}$, $\frac{3}{6} = \frac{4}{8}$
2. a. 27 b. 28 c. $16\frac{2}{3}$
3. a. 6 b. $4\frac{4}{5}$ c. 7 d. 1.47 e. $4\frac{1}{2}$ f. 18
   g. 2 h. 25 i. 5
4. 35
5. $35\frac{5}{7}$
6. $x = 12\frac{1}{2}$, $y = 15$

13.9 <u>Meaning of Percent</u> (1 day)

Percent is defined in the usual way. Specifically, we want an equivalent fraction with denominator equal to 100. The numerator of this fraction will be the percent.

13.10 Exercises

The percent equivalents in Exercises 4 and 5 are ones commonly used, and perhaps are worth commiting to memory.

1. a. $\frac{2}{4}, \frac{3}{6}$ etc. b. $\frac{1}{4}, \frac{2}{8}$ etc. c. $\frac{3}{2}, \frac{6}{4}$ etc.
   d. $\frac{1}{1}, \frac{2}{2}$ etc. e. $\frac{2}{1}, \frac{4}{2}$ etc.
2. a. 20% b. 10% c. 25% d. 45% e. 100%
3. a. 50% b. 50% c. 25% d. 250% e. 150%
   f. 125% g. 17% h. 117%
4.

$\frac{1}{4}$	.25	25%
$\frac{3}{4}$	.75	75%
$\frac{1}{5}$	.20	20%
$\frac{3}{5}$	.60	60%
$\frac{1}{5}$	.20	20%
$\frac{1}{8}$	.125	12.5%
$\frac{7}{8}$	.875	87.5%
$\frac{4}{5}$	.80	80%
$\frac{3}{8}$	.375	37.5%
$\frac{2}{5}$	.40	40%
$\frac{1}{10}$	.10	10%
$\frac{9}{10}$	.90	90%
$\frac{1}{1}$	1.00	100%

$\frac{7}{10}$	.70	70%
$\frac{1}{20}$	.05	5%
$\frac{3}{10}$	.30	30%
$\frac{1}{100}$	.01	1%

5. a. $66\frac{2}{3}\%$ b. $16\frac{2}{3}\%$

   c. $33\frac{1}{3}\%$ d. $8\frac{1}{3}\%$

13.11 Solving Problems with Percents (1 day)

Here we discuss standard problem types involving percents. No distinction is made as to "types", however; all are treated as proportion problems. Students should not be forced to use proportions however, as the text points out following Example 5. Also, confronted with a problem such as "Find 18% of 90", the student might well prefer to solve it as he did in Section 14.1:

$$18\% \text{ of } 90 = \frac{18}{100} \text{ of } 90 = \frac{18}{100} \cdot 90.$$

13.12 Exercises

Exercise 1 is intended to suggest to the student that if he first finds 1% of a number, it is relatively simple to find a% of a number by multiplying by a.

1. a. 5; 25; 2.5; 7.5; 250; .5; 50; 500

   b. 1.50; 15; 50; 2

   c. .24; 6.72; .18; .42; 18

   d. 80; 40; 120; 360; 4000

   e. .5; 50; 100; 120 f. .92; 92; 276; 322

2. 910

3. 15%

4. a. $\frac{2}{3}$ b. $66\frac{2}{3}\%$ c. $\frac{3}{2}$ d. 150%

5. $\frac{40}{20}$, 200%

$\frac{20}{25}$, 80%

$\frac{25}{20}$, 125%

$\frac{500}{400}$, 125%

$\frac{400}{500}$, 80%

$\frac{8}{80}$, 10%

$\frac{80}{8}$, 1000%

$\frac{16}{80}$, 20%

$\frac{80}{16}$, 500%

$\frac{4.2}{42}$, 10%

$\frac{42}{4.2}$, 1000%

$\frac{1.8}{180}$, 1%

$\frac{180}{1.8}$, 10000%

6. a. $22\frac{1}{2}\%$ b. 22 points

7. 55 points

8. a. 55 b. 160 c. $34\frac{2}{7}$ d. 40 c. 700 f. 8400

9. a. $1.60 b. $.60 c. $.50 d. $.13 e. $.04

f. $.40 g. $140 h. $139.96 i. $.40

10. a. $90 b. $180

11. a. $22.50 b. $56.25 c. $22.50

12. a. $20 b. $10 c. $5 d. $51 e. $25.50

f. \$12.75 g. \$165 h. \$123.75 i. \$31.50 j. \$15.75

13. $2\frac{1}{2}\%$

14. a. 1300 b. $66\frac{2}{3}\%$ c. 150% d. $111\frac{1}{9}$ e. 50
f. 270 g. 4.4 h. 4.4 i. 27 j. 11 k. $83\frac{1}{3}\%$

13.13 Presenting Data in Graphs (2 days)

The fundamental outcome of this section is to have students recognize that for approximate judgments, comparison of geometrical figures as seen by the eye are usually quicker and easier to grasp than similar comparisons of the numerical measures of these figures. Students must also develop judgments of the size of a unit to be used to make a graph of acceptable size and yet readily analyzed. In bar graphs we compare the lengths of the bars, in a rectangular graph, the lengths of the separate sections, and in circle graphs either the lengths of the area or the areas of the sectors. A good intuitive exercise is to show that ratios of arcs and the corresponding ratios of the areas of the sectors are equal.

13.14 Exercises

1. a. 2 b. 4 c. bicycle; walk d. circle or bar

4. percents 36; 44; 5; 15; Degrees; 129.6; 158.4; 18.0; 54.0

13.15 Translations and Groups (1 day)

There is no new concept to be developed here, merely extending a translation of a singleton onto a singleton to that of a translation of an ordered pair onto an order pair in the plane. The analogy should cause no trouble in making a composition of translations in the plane. The teacher should propose that if the translations $x \longrightarrow x + a$, $y \longrightarrow y + b$ and $(x,y) \longrightarrow (x + a, y + b)$ are understood, what meaning could be given to $(x,y,z) \longrightarrow (x + a, y + b, z + c)$.

13.16 Exercises

In Exercise 7 and 8 note that t represents only one particular translation for each (x,y). Then t^{-1} also represents only one particular translation.

1. Yes; the composition of any two translations is a translation.
2. $(t_1 \circ t_2) \circ t_3 = t_1 \circ (t_2 \circ t_3)$ (Associativity)
   There is an identity translation I such that $I \circ t = t \circ I = t$
   Every transformation t has an inverse t^{-1} such that
   $t \circ t^{-1} = t^{-1} \circ t = I$.
3. If the rule for t is $(x,y) \longrightarrow (x + a, y + b)$, then the rule for t^{-1} is $(x,y) \longrightarrow (x - a, y - b)$.
4. 0 is a rational number, and I is the translation with the following rule:

$$(x,y) \longrightarrow (x + 0, y + 0)$$

5. If t_1' $(x,y) \longrightarrow (x + a, y + b)$
   t_2' $(x,y) \longrightarrow (x + c, y + d)$
   t_3' $(x,y) \longrightarrow (x + e, y + f)$
   Then $(t_1 \circ t_2) \circ t_3 = t_1 \circ (t_2 \circ t_3)$ since $(x + e) + (c + a) = ((x + e) + c) + a$, and similarly for y.
6. t_1' $(x,y) \longrightarrow (x + a, y + b)$
   t_2' $(x,y) \longrightarrow (x + c, y + d)$

7. (a) $(x,y) \longrightarrow (x + \frac{2}{3}, y - 4\frac{1}{2})$

(b) $(x,y) \longrightarrow (x + 1, y - 6\frac{3}{4})$

(c) $(x,y) \longrightarrow (x + 1\frac{1}{3}, y - 9)$

(d) No. There is no identity, and there are no inverses.

8. (a) $(x,y) \longrightarrow (x - \frac{1}{3}, y + 2\frac{1}{4})$

(b) $(x,y) \longrightarrow (x - \frac{2}{3}, y + 4\frac{1}{2})$

(c) $(x,y) \longrightarrow (x - 1, y + 6\frac{3}{4})$

(d) No. There is no identity, and there are no inverses.

9. Yes; all necessary properties are satisfied.

The identity I is present. The inverse of t^k is t^{-k}.

10. It must be shown that the composition of two such translations is a translation of the same kind.

Let t_1 have the rule: $(x,y) \longrightarrow (x + p_1 a, y + q_1 b)$.

Let t_2 have the rule: $(x,y) \longrightarrow (x + p_2 a, y + q_2 b)$.

Then $t_2 \circ t_1$ has the rule: $(x,y) \longrightarrow (x + p_1 a + p_2 a, y + q_1 a + q_2 a)$.

or

$(x,y) \longrightarrow (x + (p_1 + p_2)a, y + (q_1 + q_2)b)$.

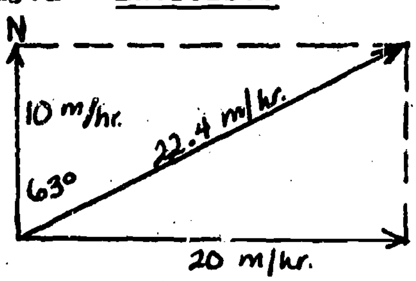
which is of the required form, since $p_1 + p_2$ is an integer, and $q_1 + q_2$ is an integer. The identity is obtained by letting $p = q = 0$. And the inverse of $(x,y) \longrightarrow (x + pa, y + qb)$ is $(x,y) \longrightarrow (x - pa, y - qb)$. (With students, numerical examples will be helpful.)

13.17 <u>Applications of Translations</u> (1 day)

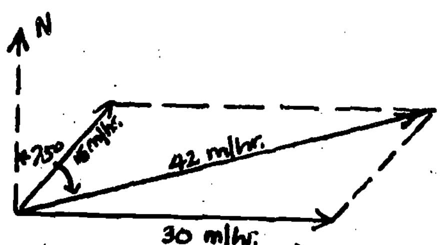
This section is important. It is the first notion of a vector in the physical sense of the word, and the section develops the additive group of translations in a plane as a composition group of free vectors in a plane. For further work on this section see Elementary Vector Geometry by Seymour Shuster, Wiley and Company.

Section 13.18 Exercises

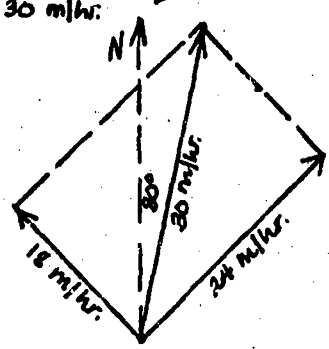
1.



2. (a)



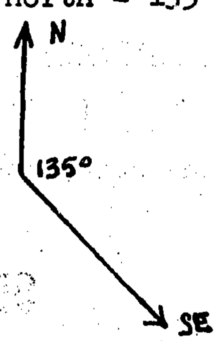
(b)

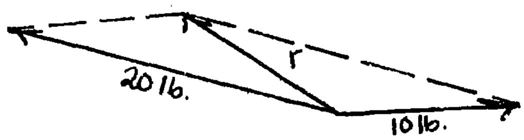


(c) Diagram is not necessary. The two forces act in the same direction.

Actual speed = 20 miles per hour.

Angle with the north = 135°.



3. 

resultant = r = 11 lb.

(1.1 inches)

4. a. 39 b. 29 c. 24
5. $22\frac{1}{2}$
6. 44
7. 39
8. Same magnitude, opposite directions.

13.20 Review Exercises

1. a. 4 b. 30 c. 227.5
2. $420.00
3. $176.00
4. $15\frac{5}{9}\%$
5. 200
6. $95.00
7. a. 53.202 b. 15.15 c. -17.5 d. 74.175 e. 49
   f. 1.32 g. 4.65
8. $\frac{7}{6}$, 6.7
9. 18
10. a. 21 b. 31.5 c. $2\frac{4}{7}$ d. 6 or -6
11. a. (-1, -1) b. (1,1) c. $(^-\frac{10}{3}, \frac{^-20}{3})$ d. (0, -15)
    e. (-15, 0) f. $(\frac{5}{3}, \frac{^-5}{3})$
12. a. $(x + \frac{10}{3}, y - \frac{4}{5})$ b. $(x + 5, y - \frac{6}{5})$ c. $(x - \frac{5}{3}, y + \frac{2}{5})$
    d. $(x - \frac{10}{3}, y + \frac{4}{5})$

Suggested Test Items (2 days)

1. Below are listed certain points on a line, together with

their coordinates. Write the coordinate of the image of each of these points under the dilation $D_{\frac{7}{3}}$:

P: 3 Q: -3 R: $\frac{7}{3}$ S: 6 T: $\frac{5}{2}$

2. John has visited 66% of the states. How many states has he visited?

3. Jane has visited 66% of the states. How many states has she visited?

4. Last year a man earned \$8200. This year he earned $\frac{5}{4}$ of that amount. How much has he earned this year?

5. Compute the following:

   a. $54.83 + 17.75$

   b. $54.83 - 17.75$

   c. $17.75 - 54.83$

   d. 18.4×7.6

   e. $\frac{45.9}{4.25}$

6. A picture measuring 10 inches (length) by 8 inches (width) is enlarged so that the new length is 14 inches.

   a. What is the ratio of old length to new length? (Express as irreducible fraction.)

   b. What is the ratio of new length to old length? (Express as irreducible fraction.)

   c. What is the new width?

7. Solve the following proportions:

   a. $\frac{2}{3} = \frac{x}{90}$ b. $\frac{2}{3} = \frac{x}{70}$ c. $\frac{5}{4} = \frac{28}{x}$

8. Express the ratio of $\frac{1}{2}$% to $\frac{1}{2}$ as an irreducible fraction.

9. A bank pays 4% interest. If Mr. Jones receives \$36.00 interest, how much money does he have in the bank? (Assume

that the interest rate is an annual one.)

10. In a school with 560 students, 70 of them are in the honor society. What percent of the students are in the honor society?

11. In the same school (560 students), 35% of the students are taking a foreign language. How many students are taking a foreign language?

12. The expenditures of a certain town for the past year amounted to $50,000. It is estimated that the expenditures for the present year will be 120% of that amount. What is the estimated amount (in dollars) of this year's expenditures?

13. a. 80 is ____% of 50.

    b. 50 is ____% of 80.

    c. 50 is 80% of ____.

    d. 80 is 50% of ____.

Test Answers

1. P: $3 \longrightarrow 7$, Q: $-3 \longrightarrow -21$, R: $\frac{49}{9}$, S: $\frac{42}{3} = 14$, T: $\frac{35}{6}$
2. 33
3. 33
4. $10250
5. a. 72.58 b. 37.08 c. -37.08 d. 139.84 e. 10.8
6. a. $\frac{5}{7}$ b. $\frac{7}{5}$ c. 11.2
7. a. 60 b. $46\frac{2}{3}$ c. $22\frac{2}{3}$
8. $\frac{1}{100}$
9. $900.00
10. $12\frac{1}{2}\%$

11. 196

12. $60,000

13. a. 160%

    b. $62\frac{1}{2}\%$

    c. $62\frac{1}{2}$

    d. 160

Comments for Teachers

Course I, Chapter 14

Algorithms and Their Graphs

This chapter introduces the basic language and techniques of flow charting with two goals in mind. First, computer capabilities have had a profound influence on the problem solving techniques of mathematics and the sciences. The first step in solving a problem is often analysis of processes into a sequence of simple steps each of which can be simulated on a computer. Thus it is helpful to have systematic standardized procedures for diagramming or graphing processes.

Second, successful flow chart analysis of any process requires, as a prerequisite, thorough understanding of the process. Therefore, the exercise of flow charting familiar algorithms has a pedagogical feature independent of the flow charting itself; familiar algorithms can be reviewed and analyzed from a fresh point of view.

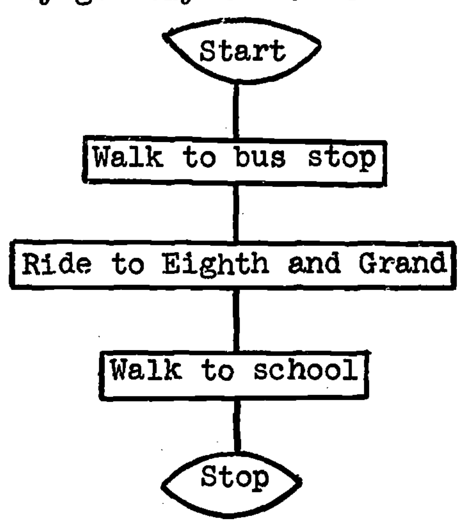
The sign painter flow chart is used as an organizing thread for the chapter, but the important processes are the mathematical algorithms. The chapter will probably take from 7 to 10 days of class time.

14.1 Each type of flow chart box has a characteristic kind of direction and, to indicate immediately the purpose, a characteristic shape. However, the only really crucial thing is to have directions within the boxes clearly describe the action to be taken. Flow charts composed by different programmers (students) will commonly vary in detail and organization. This is particularly true of the non-mathematical processes such as sign painting or finding one's way to school. While the absence of a single "right answer" might be initially upsetting to students, it is a fact of life even in many parts of mathematics and must be met.

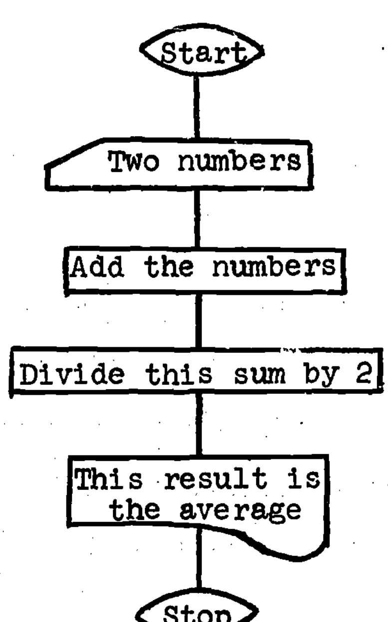
14.2 Solutions

1. Answers will vary greatly here. Two examples are given.

a.



d.



Quite often the order of the boxes is essential; occasionally some rearrangement will still produce the same result.

2. With the change suggested in (b) it is possible to use the same flow chart for a wide variety of signs.

   (d) (1) 21, (2) 12

   (e) (1) 8/7 inches, (2) 2 inches

3. (a) operation (b) input (c) input (d) input

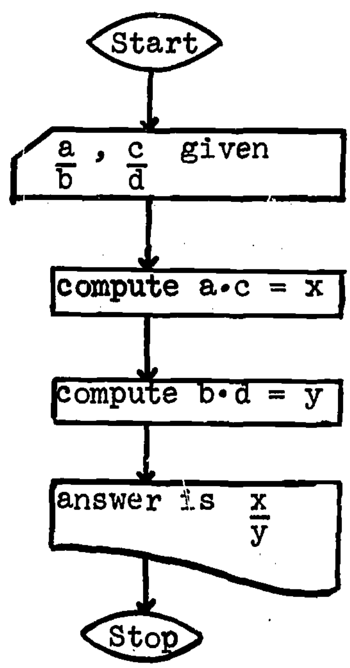
   (e) operation (f) input (g) output

4. (a) ball, basket, aim, shoot, score

   (b) needle, thread, cloth, cut, sew, dress

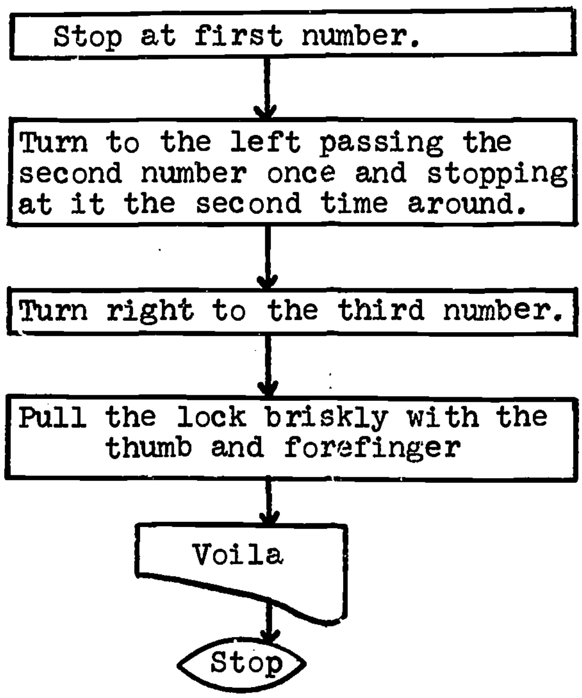
   (c) ball, glove, bat, throw, hit, catch, out

5. Answers will vary, but here is one possiblility.



6. See 1(d)

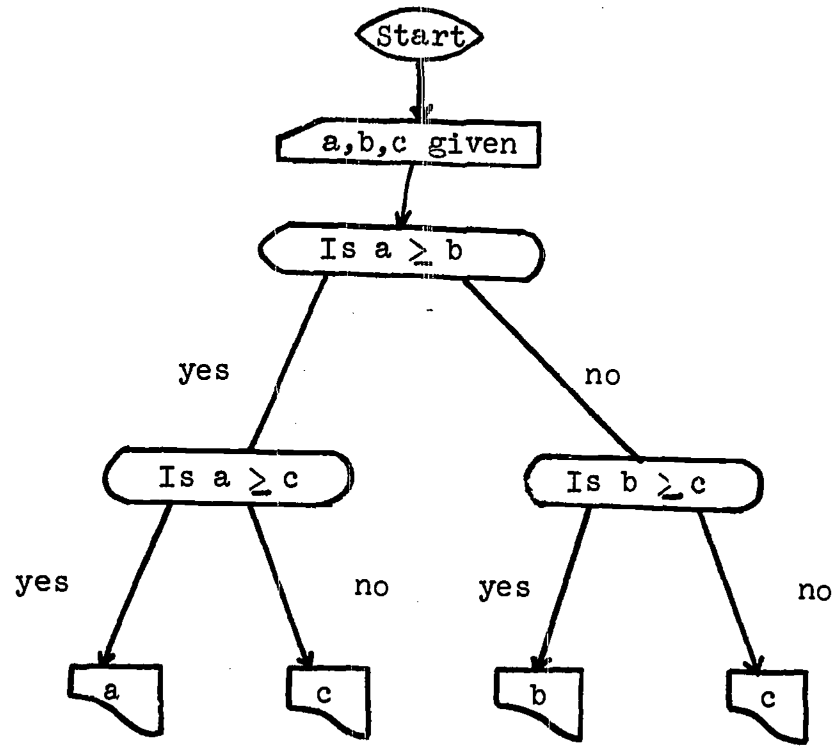
7. Remaining boxes might be:



14.3 Branching processes are very common in mathematics as the examples of this section illustrate. This is a good time to review the definitions and computational algorithms of many number operations.

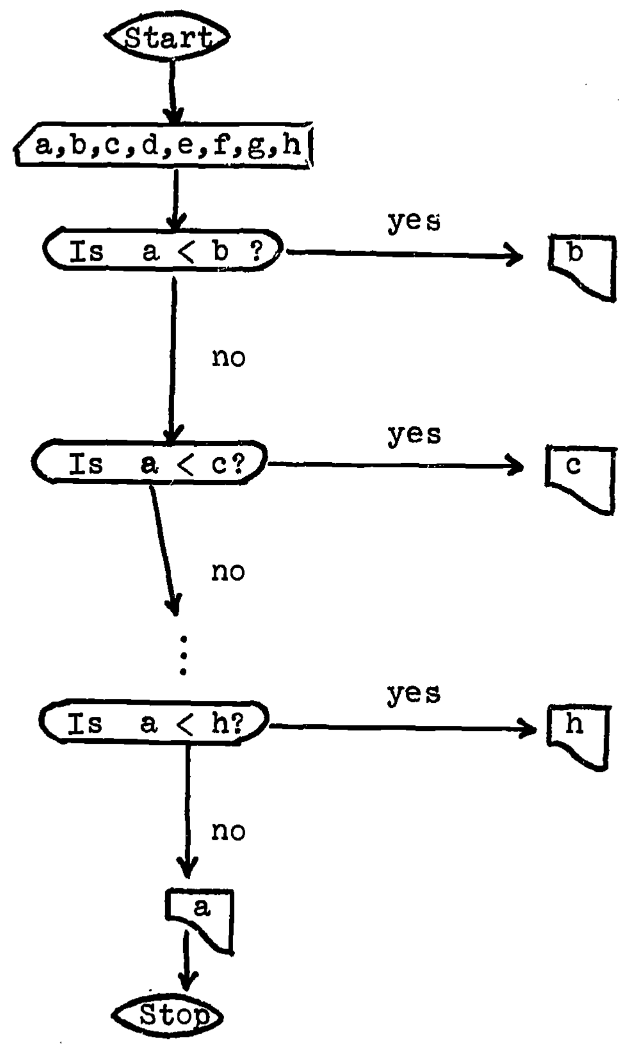
14.4 Solutions

1. (a) yes (b) no (c) yes (d) no (e) yes (f) no
2. This flow chart could be derived by minor modifications of that in Figure 14.8
3. One possibility:



Stop

4. This can be obtained from the chart in 3 by asking one more question along each branch. For instance in the far left or "yes" branch one must also ask "Is b $\geq$ c". If the answer is yes, then one can conclude that a $\geq$ b $\geq$ c. If the answer is no, then one can conclude a $\geq$ c $\geq$ b.
5. One way to remove the duplicates would be to interject several boxes of the type "Is a = b" at the beginning and then using appropriate short circuit arrows to the order decision box that is useful next.
6. One possibility:

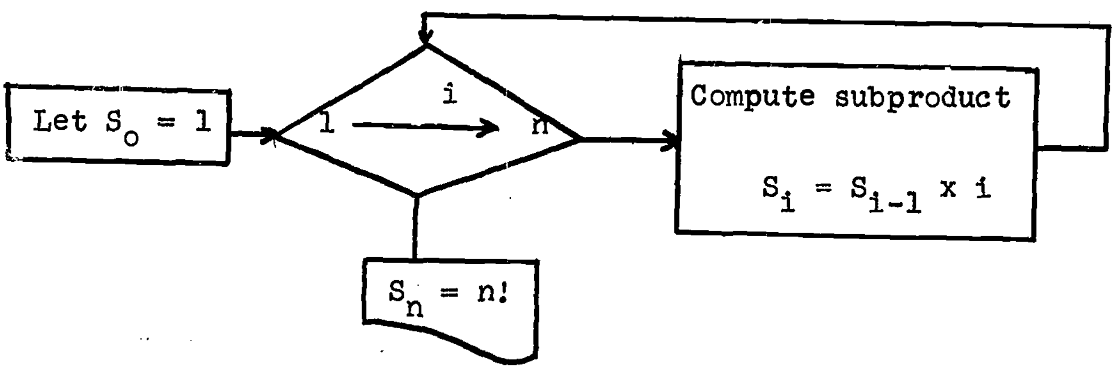


14.5 Writing a flow chart that makes use of a counter (diamond) for directing iterations of a process requires directions that are given in formula terms as well as the pesky business of subscript indices. This will probably need special teaching attention, but it is valuable for much future work in mathematics.

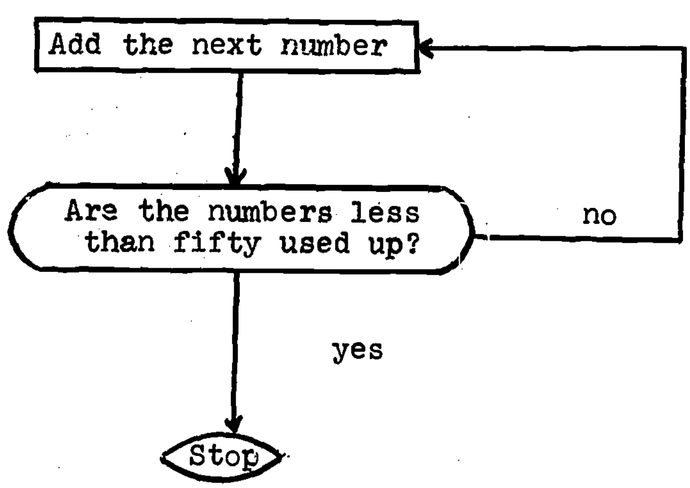
Note that there are several different ways to make use of a diamond in the flow chart for adding 100 numbers. The method given was chosen because it seemed to simplify things the best. However, encourage students to try their own formulation of the directions. It is certainly unimportant for them to memorize the particular flow chart given in Figure 14.3.

14.6 Solutions

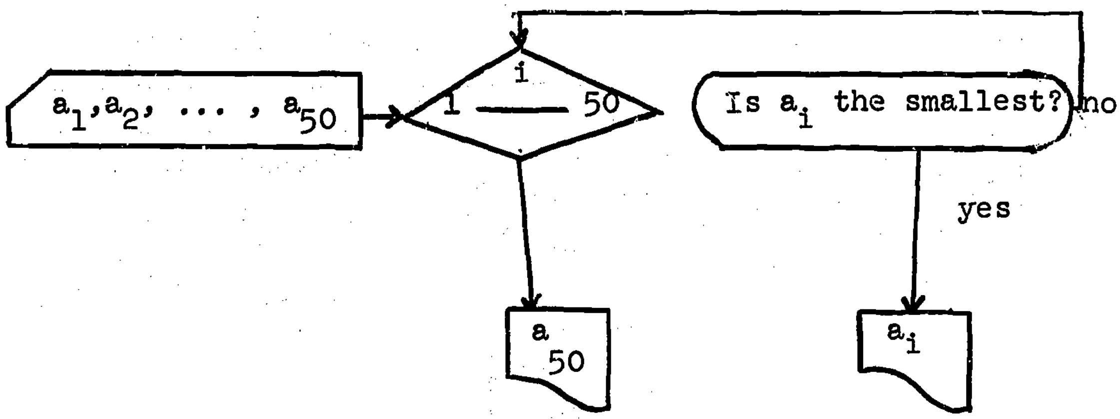
1. This was to be n! One possible chart is the following.



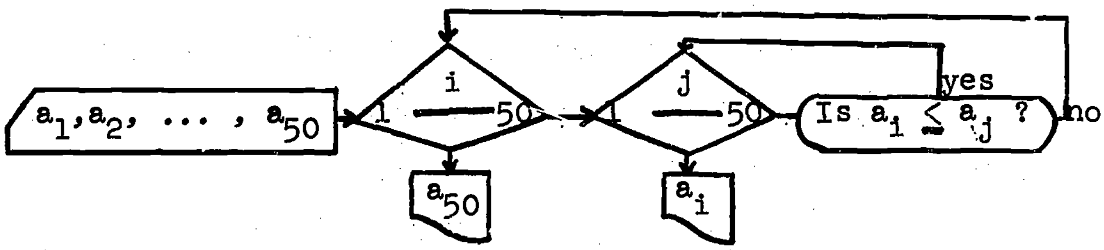
2. Of course one way to eliminate diamonds is to put each step in as an individual operation box. A more concise way would be to do something like the following.



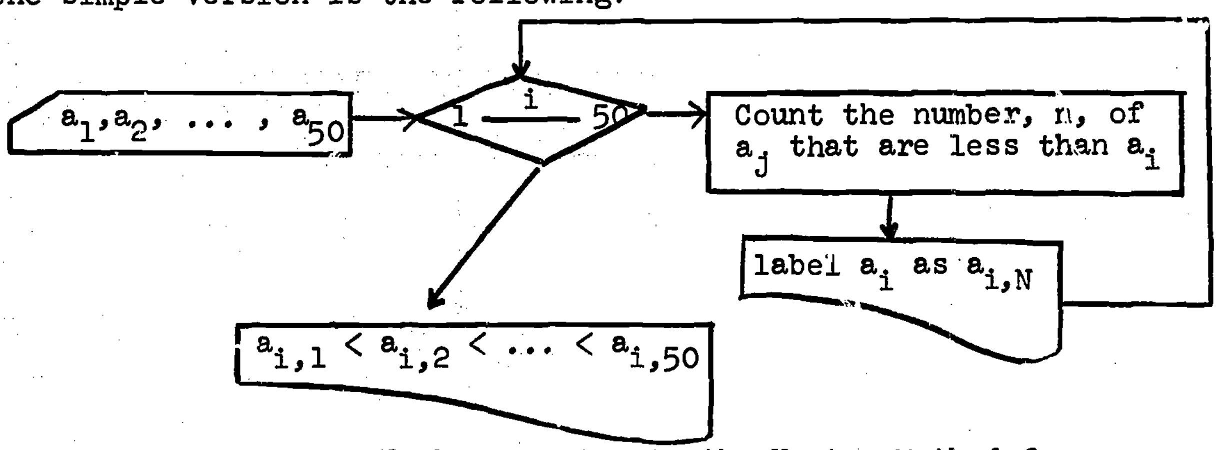
3. One simple version is the following:



Another more useful chart is the following:



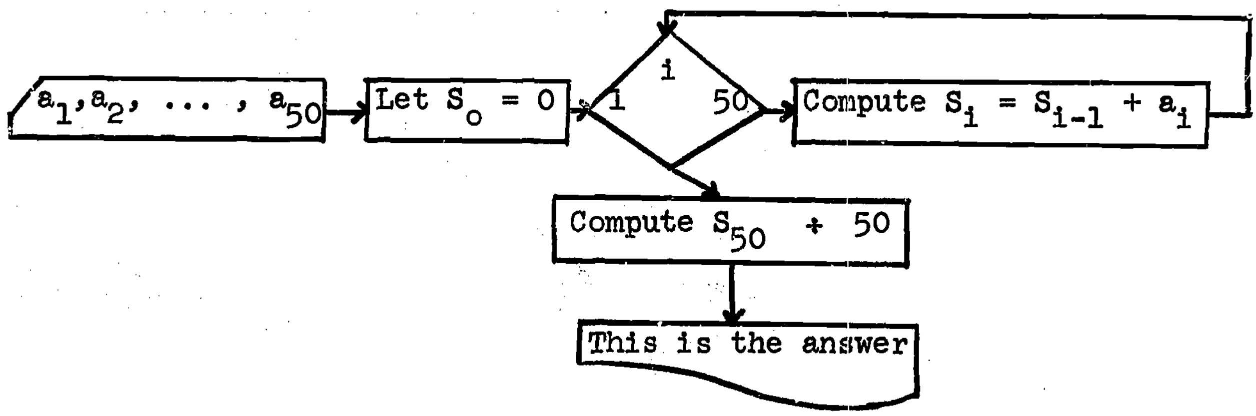
4. One simple version is the following:



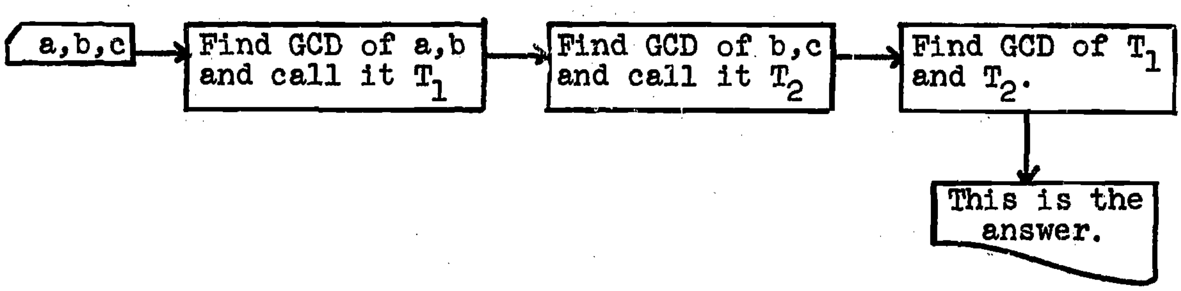
14.7 The important part of this section is the Newton Method for calculating approximations to square roots. It will be used heavily in later work--particularly statistics in Course II--and thus must be mastered.

14.8 Solutions

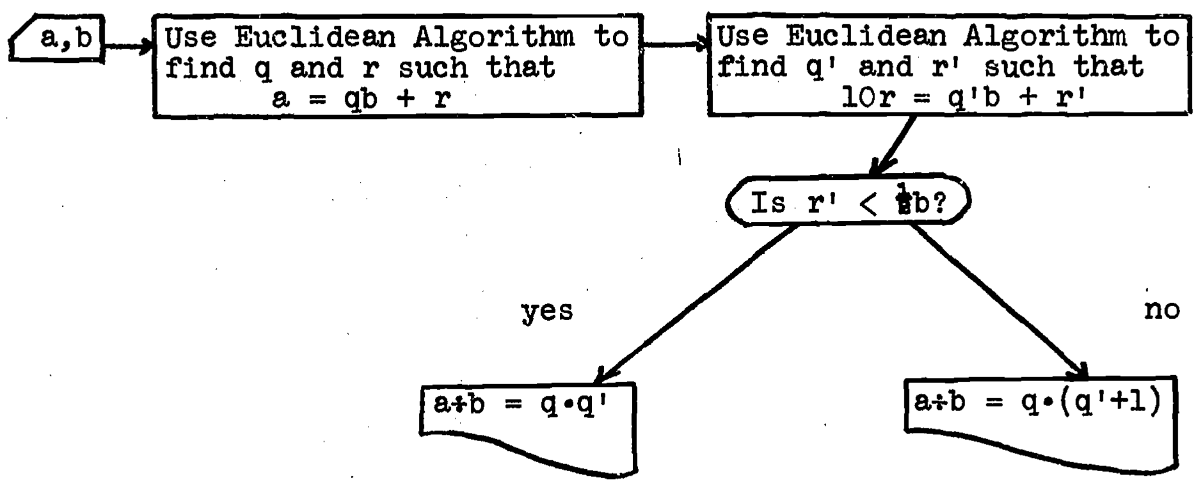
1. Various charts are possible. One example is the following:



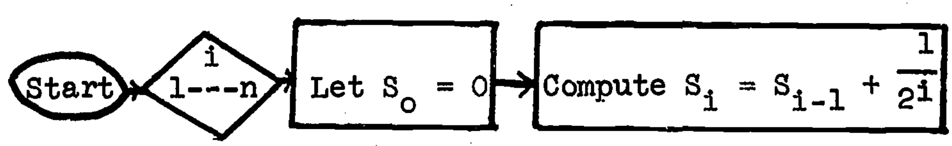
2. One possibility is the following:



3. One possibility is the following: To divide a by b



4. One possibility is the following:



The diamond is designed so that any chosen n can be inserted.

5. It calculates the LCM of two given integers.

6. If $\left| \frac{a}{e} - e \right| < \frac{1}{10}$ then $\left| a - e^2 \right| < \frac{e}{10}$.

$$\text{or } \left| \sqrt{a} - e \right| \cdot \left| \sqrt{a} + e \right| < \frac{e}{10}.$$

But since $e < \left| \sqrt{a} + e \right|$,

$$|e| \cdot \left| \sqrt{a} - e \right| < \frac{e}{10}.$$

Therefore, $\left| \sqrt{a} - e \right| < \frac{1}{10}$.