DOCUMENT RESUME

ED 046 731 SE 010 493

AUTHOR Buchman, Aaron; Zimmerman, Robert
TITLE Eleventh Year Mathematics.
INSTITUTION New York State Education Dept., Albany. Bureau of Secondary Curriculum Development.; State Univ. of New York, Albany.
PUB DATE 70
NOTE 45p.; Reprint

EDRS PRICE EDRS Price MF-$0.65 HC-$3.29
DESCRIPTORS Algebra, *Curriculum, *Curriculum Guides, Grade 11, Instruction, Mathematics, Modern Mathematics, *Secondary School Mathematics, *Teaching Guides, Trigonometry

ABSTRACT

This outline for Eleventh Year Mathematics in New York adheres closely to the recommendations of The Commission of Mathematics of the College Entrance Examination Board and thus presents a unified development of certain aspects of algebra, trigonometry, and analytic geometry. Its aim is both as a terminal course in mathematics and as a solid foundation for college mathematics. The outline attempts to involve to a great degree the "modern" emphasis on the understanding of broad, basic, unifying mathematical concepts. (Author/CT)

Eleventh Year Mathematics

ED 046731

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY

THE UNIVERSITY OF THE STATE OF NEW YORK/THE STATE EDUCATION DEPARTMENT
BUREAU OF SECONDARY CURRICULUM DEVELOPMENT/ALBANY, NEW YORK 12224

SE 010 493

ED046731

ELEVENTH YEAR

MATHEMATICS

1970 Reprint

The University of the State of New York
New York State Education Department
Bureau of Secondary Curriculum Development
Albany 1968

THE UNIVERSITY OF THE STATE OF NEW YORK

Regents of the University (with years when terms expire)

1984 Joseph W. McGovern, A.B., LL.B., L.H.D., LL.D., D.C.L., Chancellor ---------- New York

1985 Everett J. Penny, B.C.S., D.C.S., Vice Chancellor ---------- White Plains

1978 Alexander J. Allan, Jr., LL.D., Litt.D. ---------- Troy

1973 Charles W. Millard, Jr., A.B., LL.D., L.H.D. ---------- Buffalo

1972 Carl H. Pforzheimer, Jr., A.B., M.B.A., D.C.S., H.H.D. ----- Purchase

1975 Edward M. M. Warburg, B.S., L.H.D. ---------- New York

1977 Joseph T. King, LL.B. ---------- Queens

1974 Joseph C. Indelicato, M.D. ---------- Brooklyn

1976 Mrs. Helen B. Power, A.B., Litt.D., L.H.D. ---------- Rochester

1979 Francis W. McGinley, B.S., LL.B., LL.D. ---------- Glens Falls

1980 Max J. Rubin, LL.B., L.H.D. ---------- New York

1971 Kenneth B. Clark, A.B., M.S., Ph.D., Litt.D. ---------- Hastings on Hudson

1982 Stephen K. Bailey, A.B., B.A., M.A., Ph.D., LL.D. ---------- Syracuse

1983 Harold E. Newcomb, B.A. ---------- Owego

1981 Theodore M. Black, A.B. ---------- Sands Point

President of the University and Commissioner of Education
Ewald B. Nyquist

Executive Deputy Commissioner of Education
Gordon M. Ambach

Deputy Commissioner of Education

Associate Commissioner for Instructional Services
Philip B. Langworthy

Assistant Commissioner for Instructional Services (General Education)
Bernard F. Haake

Director, Curriculum Development Center
William E. Young

Chief, Bureau of Secondary Curriculum Development
Gordon E. Van Hooft

Director, Division of General Education
Ted T. Grenda

Chief, Bureau of Mathematics Education
Frank S. Hawthorne

FOREWORD

The earlier outline for Eleventh Year Mathematics and this revision both adhere closely to the recommendations of the Commission on Mathematics of the College Entrance Examination Board that this course be a unified development of certain aspects of algebra, trigonometry, and analytic geometry. As the terminal course in the three-year sequence in mathematics, the Eleventh Year Mathematics course is designed to provide both a solid foundation for college level mathematics and the mathematical literacy needed to function effectively in an increasingly technological society.

Recent developments in the field of mathematics and the thinking of contemporary leaders in the mathematics community have challenged those responsible for curriculum direction and revision to make mathematics teaching still more effective by involving to a greater degree the "modern" emphasis on the understanding of broad, basic, unifying mathematical concepts. This revision is an attempt to help teachers meet this challenge.

In May 1967, an Ad Hoc Eleventh Year Mathematics Committee met at the Department to consider revisions which would reflect to a greater degree the "modern" concepts as well as best provide for continuity with the new syllabus for Ninth Year Mathematics. Members of this committee included: Benjamin Bold, Stuyvesant High School, New York City; Margaret Dickson, Scotia-Glenville High School; Leo Dustman, Spring Valley High School; Eva Hanneman, Waverly High School; Charles Kissam, Northport High School, and Sister Ann Xavier, St. Agnes School, Rochester.

The recommendations of this committee were incorporated in the first outline of the revision of the Eleventh Year Mathematics syllabus which was developed during the summer of 1967 by Eva Hanneman and Sister Ann Xavier, working under the direction of Frank Hawthorne, Chief of the Bureau of Mathematics Education. Copies of the first outline were sent to the members of the committee for suggestions and revisions; and additional pertinent comments and suggestions were made by Harry Schor, Abraham Lincoln High School, New York City.

In a series of meetings the suggestions for modifications were evaluated by Mr. Hawthorne and the staff of the Bureau of Mathematics Education: Bruno Baker, Aaron Buchman, Agnes Higgins, Melvin Mendelsohn, Fredric Paul, and John Sullivan. The final draft was prepared and edited by Mr. Buchman and Robert Zimmerman, Associate in Secondary Curriculum Development.

Gordon E. Van Hooft
Chief, Bureau of Secondary Curriculum Development

William E. Young
Director, Curriculum Development Center

4

ELEVENTH YEAR MATHEMATICS

Scope and Content

I. Number System
   A. Structure of the number system through complex numbers (1)
   B. Postulates of a number field (2)
   C. Operations on numbers in specified sets
      1. Operations on real numbers involving
         a. Factoring (3)
         b. Fractions (4)
         c. Exponents and radicals (5)
      2. Operations on complex numbers
         a. Algebraic (6)
         b. Graphic (optional) (7)

II. Functions and Relations
   A. Definitions - relation, domain, range, function, inverse of a relation (8)
   B. Polynomial functions
      1. Linear (9)
      2. Quadratic (10)
   C. Trigonometric functions
      1. Definitions and values of the six trigonometric functions (11)
      2. Graphs of the sine, cosine, and tangent functions (12)
   D. Inverse trigonometric functions (13)
   E. Exponential functions (14)
   F. Logarithmic functions
      1. Definition (15)
      2. The laws of logarithms (16)
      3. Computation (17)
   G. Direct and inverse variation (18)
   H. Quadratic relations (19)

III. Open Sentences (20)
   A. Conditional equations (21)
      1. Fractional (22)
      2. Irrational (23)
      3. Exponential (24)
      4. Quadratic (25)
      5. Trigonometric (26)
      6. Absolute value (27)
      7. Literal - include the transformation of formulas
   B. Identities
      1. Algebraic (28)
      2. Fundamental trigonometric (29)
      3. Reduction formulas (30)

4. Sine, cosine, and tangent of the sum and difference of two angles; of double angles and half angles (31)
5. Derivation and use of product expressions for the sum and the difference of two sines and two cosines (optional)
6. Proving identities involving two or more functions of a single angle or double angle (32)

C. Inequalities
   1. First and second degree in one variable (33)
   2. Two of the first degree in two variables, graphic solution
   3. One first and one second degree in two variables, graphic solution (optional) (34)
   4. Absolute value (35)

D. Verbal problems (36)

E. Conjunction of open sentences
   1. Linear - quadratic systems (37)
   2. Systems of three linear equations in three variables (optional)

IV. Solution of Triangles
   A. Law of Sines and Law of Cosines (38)
   B. Area of triangles in terms of two sides and included angle (39)

V. Sequences and Series (optional) (40)

ERIC

SUGGESTIONS FOR TEACHING

1. The structure of the real number system should be studied and extended to the set of complex numbers. The study of the reals should include not only the properties of the field of real numbers, but also the relationships among the various sets of numbers which make up the set of reals. (See appendix, section A).

- Real numbers
  - Rational numbers
    - Integers
      - Natural numbers
      - Zero
      - Negative integers
    - Nonintegral rationals
  - Irrational numbers

Complex numbers should be defined in the binomial form a + bi, in which a and b are real numbers. The real numbers may be thought of as complex numbers of the form a + 0i. Imaginary numbers are complex numbers of the form a + bi where $b \neq 0$.

- Complex numbers a + bi
  - Real numbers a + 0i
  - Imaginary numbers a + bi ($b \neq 0$) (Called pure imaginary if a = 0 and $b \neq 0$)

2. The postulates of a field should be reviewed and discussed in reference to each set of numbers under consideration. Students should be able to recognize which postulates are or are not satisfied by the various number sets.

The following properties should also be considered:

Properties of equality (for all real numbers a, b, c)

a. Addition property (If $a = b$, then $a + c = b + c$.)
b. Multiplication property (If $a = b$, then $ac = bc$.)
c. Reflexive property ($a = a$)
d. Symmetric property (If $a = b$, then $b = a$.)
e. Transitive property (If $a = b$ and $b = c$, than $a = c$.)

Properties of inequality (for all real numbers a, b, c)

a. Trichotomy property ($a > b$, $a = b$, or $a < b$)
b. Addition property (If $a > b$, $a + c > b + c$, for all c.)
c. Multiplication property (If $a > b$, $ac > bc$ for $c > 0$. If $a > b$, $ac < bc$, for $c < 0$.)
d. Transitive property (If $a > b$ and $b > c$, then $a > c$.)

It should be noted that the complex numbers are not ordered.

3. Students should know that factoring involves the use of the distributive postulate. A good balance should be maintained between the development of factoring skill and the knowledge of the use of postulates. Factoring should include the following areas:
   a. Common monomial or binomial factor:
      $2\cos^2 x + \cos x$
      $b^2 + ab + b + a = b(b + a) + 1(b + a) = (b + a)(b + 1)$
   b. Difference of two squares
      $4\sin^2 x - 9\cos^2 x$
      $4x^2 - \frac{1}{9}$
   c. Quadratic trinomial, including perfect square trinomials
      $4x^2 + 4x - 3$
      $2\sin^2 x - 9\sin x - 5$

   NOTE: The importance of the universe in factoring should be pointed out. Although $x^2 + 1$ is not factorable in the set of real numbers, $x^2 + 1 = (x + i)(x - i)$ in the set of complex numbers. Although $x^2 - 3$ is not factorable in the set of rational numbers, $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$ in the set of real numbers.

4. a. The four fundamental operations on fractions should be limited to those cases in which the denominators are monomials or binomials. In all work with fractions, attention should be given to the fact that a zero denominator makes the fraction meaningless.
   b. Simplification of complex fractions should be limited to cases such as:

$$\frac{R + \frac{a}{b}}{S - \frac{c}{d}}\ ; \quad \frac{\frac{a}{b} + \frac{c}{d}}{\frac{m}{n} - \frac{p}{q}}\ ; \quad \frac{\frac{1}{\cos A} + \frac{1}{\sin A}}{\frac{1}{\cos A} - \frac{1}{\sin A}}$$

5. Exponents and Radicals
   a. The following laws should be developed in class (for positive integral exponents)
      (1) $x^a \times x^b = x^{a + b}$
      (2) $x^a \times y^a = (xy)^a$
      (3) $x^a \div x^b = x^{a - b}$ (Where $a > b$)($x \neq 0$)
      (4) $x^a \div y^a = \left(\frac{x}{y}\right)^a$ ($y \neq 0$)
      (5) $(x^a)^b = x^{ab}$
   b. Meaning and use of negative, zero, and fractional exponents
      Define $x^0 = 1$ where $x \neq 0$.
      $x^{-a} = \frac{1}{x^a}$ where $x \neq 0$.

$x^{a/b} = (\sqrt[b]{x})^a = \sqrt[b]{x^a}$ where a is an integer, b is a positive integer and x is a positive real number.

Show that these definitions are consistent with the laws stated in a on the previous page. Use should include scientific notation.

c. Fundamental operations with irrational numbers should be reviewed and extended to include the rationalization of the denominator of expressions such as:

$$\frac{3}{\sqrt{2}} \quad \frac{3}{4\sqrt{2} - 1} ; \quad \frac{\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} - \sqrt{3}}$$

Although the usual applications involve rationalizing the denominators, a few examples of rationalizing the numerator may be included.

6. The student should be able to apply the four fundamental operations to complex numbers and should be familiar with the additive and multiplicative identities and inverses of complex numbers.

7. The student should be able to represent complex numbers graphically and find sums and differences by graphic methods.

8. A relation is a pairing of the elements of one set with the elements of another set. The order of the pairing must be specified. Thus, a *relation* may be defined as a set of ordered pairs. The set consisting of all the first members of the ordered pairs is the *domain* of the relation. The set consisting of all the second members of the ordered pairs is the *range* of the relation.

If the elements of each ordered pair of a given relation are interchanged, a new relation is formed which is called the *inverse* of the relation. From this definition, it can be seen that the range of the original relation is the domain of its inverse and that the domain of the original relation is the range of its inverse. The graph of the inverse relation is symmetric to the graph of the original relation, with respect to the line $y = x$. (Appendix, Section I gives supplementary information on inverse functions.)

A function is a special type of relation in which each element of the domain is paired with a unique element of the range. Thus a *function* may be defined as a set of ordered pairs in which no two ordered pairs have the same first element.

Graphically, this means that a relation is a function if and only if no vertical line meets the graph of the relation at more than one point. This is sometimes called the vertical line test.

In elementary mathematics, a function is generally specified in one or more of the following ways: a table, a graph, or an equation. Thus, if f is the function defined by $y = f(x) = \sin x$, (where x is real), f may be expressed in terms of the familiar table, the graph, or the equation $y = \sin x$.

The symbol f(x) is often used as the second element of an ordered pair and is customarily called the value of the function. Also, if we write the ordered pair as (x,y) y is called the value of the function. Thus, y and f(x) are given the same meaning and we may write $y = f(x)$.

The work of the eleventh year concerns itself, primarily with the study of polynomial functions, trigonometric functions, logarithmic functions, and exponential functions. Future reference to any of these functions, unless otherwise stated, will imply real functions, that is, those whose domain and range are subsets of the real numbers.

9. The study of the linear function should include the following:
   a. In the development of this syllabus, the y-intercept of a line will be defined as the ordinate of the point at which the line crosses the y-axis. If the line intersects the y-axis at (0,b), we shall say that the y-intercept is b. Similarly, the x-intercept is the abscissa of the point at which the line crosses the x-axis. It must be pointed out that textbook writers are not uniform, nor even consistent, in their usage of the term intercept, and teachers may wish to familiarize their pupils with other usages current in the literature. Also, the difference between a line with slope 0 and a line with no slope should be clarified.
   b. Review of the slope-intercept form $y = mx + b$ and study of the point-slope form $y - y_1 = m(x - x_1)$.
   c. Parallel and perpendicular lines. Show that nonvertical parallel lines have the same slope, and conversely. Show that two oblique lines are perpendicular if the product of their slopes is -1, and conversely. (See Appendix, Section B).

      Problems such as the following should be considered:
      (1) Write an equation of the straight line which is parallel to the line $2x - 3y = 6$ and contains the point (3, -1).
      (2) Write an equation of the straight line which is perpendicular to $y = 2x + 3$ and contains the point (-2, 1).

10. The study of the function $\{(x,y) \mid y = ax^2 + bx + c,\ a \neq 0\}$ should include:
    a. The graph of the function and the effect upon the curve due to changes in the coefficients a, b, or c.

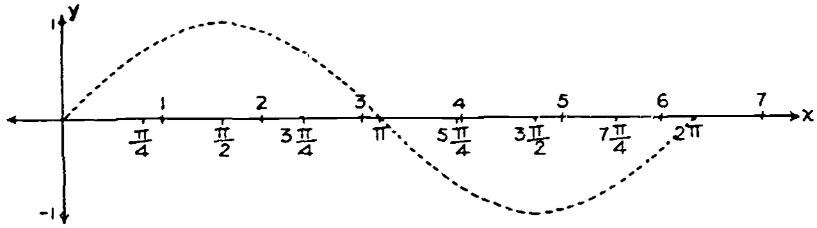
b. The equation of the axis of symmetry
c. Maximum or minimum values of $ax^2 + bx + c$
d. Use of the graph to obtain the solution set of the equation $ax^2 + bx + c = k$

11. The six trigonometric functions should be defined for all angles, both positive and negative. The values of the functions of the special angles, that is, the quadrantal angles and multiples of 30° and 45° should be studied. Radian measure should be used extensively throughout the study of trigonometric functions. The student should know the relationships of central angle, arc, and radius, and should be able to express the measure of an angle in either degrees or radians. (See Appendix, Section C).

The function F, defined by $F = \{(t,s) \mid s = \sin t \text{ and } t \text{ is a real number}\}$ has mathematical significance independent of the concept of angle and its association with the coordinate axis. Applications of the trigonometric functions of numbers may be found in the advanced study of such periodic phenomena as harmonic motion, vibration, light, sound, and electricity.

12. a. When graphing the trigonometric functions, care must be taken in the labeling of the axes. The graph of any function consists of a set of points in the coordinate plane located by using ordered pairs of real numbers. In the trigonometric functions, although the second element is a real number, the first element is usually thought of as being the measure of an angle. However, if the measure of the angle is expressed in radians, then both elements will be nondenominate real numbers and the same scale may be used for labeling both axes.

The graph of the sine function with domain $0 \leq x \leq 2\pi$



b. Graphs of the sine, cosine, and tangent functions for both positive and negative values of x.

In sketching trigonometric function curves, pupils should consider form, amplitude, and period.

(1) The two axes should be labeled accordingly.
(2) The scale of values used should be indicated.
(3) The values of the x-axis should be expressed in radians.
(4) The graphs are curved lines and should *not* be drawn as straight lines.
(5) Graphs should be sketched to pass through such key points as maximum and minimum points and x- and y-intercepts.
(6) Graph paper which is appropriate in size and scale for the type of question involved should be supplied to pupils. To safeguard against loss or errors in rating, the name of the pupil and the school should appear on each sheet of graph paper submitted.

c. Graphs of the functions defined by a sin bx and a cos bx for values of b equal to ½, 1 and 2 to include the meaning of the terms amplitude and period and a study of these graphs as a and b vary.

d. Graphic solution of systems of trigonometric equations based on c above.

13. a. In the case of the trigonometric functions, the first element of each ordered pair is the measure of an angle θ, expressed as a real number. The second element, for example, $\sin \theta$, is another number. The number θ uniquely determines $\sin \theta$. To obtain the inverse of the sine function, we interchange θ and $\sin \theta$. However, $\sin \theta$ does not determine θ uniquely. In fact, there are infinitely many values of θ corresponding to a given value of $\sin \theta$. Therefore the inverse of the sine function is a relation which is not a function and should not be referred to as an "inverse function." The same is true of each of the other trigonometric functions. If $\sin \theta = k$, an ordered pair of the inverse relation would be (k, θ). The notation commonly used for θ is $\theta = \text{arc} \sin k$ or $\theta = \sin^{-1} k$. Either of these is read "θ is an angle whose sine is k."

If a trigonometric function is made one-to-one by restricting its domain, then its inverse will be a function. If the domain of the sine function is restricted to $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the function is one-to-one. Its inverse is also a function since for each value of $\sin \theta$, there is a unique value of θ. In a similar manner, the domain of the other trigonometric functions may be restricted to produce inverse functions. (The graphs of the trigonometric functions or those of their inverse relations would be helpful in explaining this procedure to students.) To distinguish the inverse functions as just defined from the inverse relations, the initial letter of the name is capitalized when referring to the inverse function.

Thus the range of each of the inverse functions, and, by implication, its definition, is

$$\frac{-\pi}{2} \leq \text{Arc sin } k \leq \frac{\pi}{2} \quad \text{or} \quad \frac{-\pi}{2} \leq \text{Sin}^{-1} k \leq \frac{\pi}{2}$$

$$0 \leq \text{Arc cos } k \leq \pi \quad \text{or} \quad 0 \leq \text{Cos}^{-1} k \leq \pi$$

$$\frac{-\pi}{2} < \text{Arc tan } k < \frac{\pi}{2} \quad \text{or} \quad \frac{-\pi}{2} < \text{Tan}^{-1} k < \frac{\pi}{2}$$

In restricting the domain and range of the trigonometric functions we limit ourselves to a discussion of what are usually called the "principal values" of the inverse functions. The equation y = arc sin ½ has many solutions, some of which are 30°, 150°, -210°. The equation y = Arc sin ½ may be read "y is the principal value of the angle whose sine is ½." This gives y = 30° or $y = \frac{\pi}{6}$.

b. Applications
   (1) To express the roots of a trigonometric equation in inverse trigonometric form. When solving a trigonometric equation, values are often obtained which are not in the range of the function. All such values should be discarded. For example, if when solving a second degree equation $\sin\theta = \frac{5 + \sqrt{3}}{4}$ and $\sin\theta = \frac{5 - \sqrt{3}}{4}$ are obtained, the first should be discarded and the roots written $\theta = \text{arc sin } \left(\frac{5 - \sqrt{3}}{4}\right)$
   (2) To evaluate expressions such as:

$$\sin(\text{Arc tan } \tfrac{1}{2}) = \frac{\sqrt{5}}{5}$$

$$\cos(\text{Arc sin } (-\tfrac{3}{5})) = \frac{4}{5}$$

$$\text{Arc sin } (\cos \tfrac{5\pi}{6}) = -\frac{\pi}{3}$$

$$\sin[\sin^{-1}(-1) + \cos^{-1}(-1)] = 1$$

14. The exponential function may be used as an introduction to logarithms. When so used, it is of the form $\{(x,y) \mid y = a^x, a > 0, a \neq 1\}$ with domain the set of real numbers and range the set of positive real numbers. Although a may have values such that 0 < a < 1, values of a greater than one are most frequently used. Application should be limited to graphing functions such as $\{(x,y) \mid y = 2^x\}$ and $\{(x,y) \mid y = 10^x.\}$

15. The logarithmic function is of the form $\{(x,y) \mid y = \log_a x, a \neq 1, a > 0\}$ with domain the set of positive real numbers and range the set of all real numbers. As with the exponential function, a is usually greater than one. This function should be introduced as the inverse of the exponential function.

Their relationship to each other may be shown by graphing both functions on the same set of axes and showing that their graphs are symmetric with respect to the line $y = x$. Although graphing may be limited to bases such as 2 and 10, logarithms should be defined so that students should be able to find the value of x in expressions such as:

$\log_2 8 = x$; $\log_x 3 = \frac{1}{2}$; $\log_5 x = 2$

16. Proofs should be given for the following:
    a. $\log ab = \log a + \log b$
    b. $\log \frac{a}{b} = \log a - \log b$
    c. $\log a^n = n \log a$
    d. $\log \sqrt[n]{a} = \frac{1}{n}(\log a)$

17. This work should emphasize not only the evaluation of abstract expressions such as $\sqrt[3]{\frac{8.103 \times 0.574}{45.9 \sin 27° 12'}}$, but also evaluation in connection with actual formulas which come within the range of the pupil's understanding; for example, find the radius of a sphere whose volume is 89.6 cu. in. Interpolation of numbers to 4-figure accuracy and of values of functions of angles to the nearest minute should be included.

18. Both direct and inverse variation should be defined using a constant of variation and should involve higher powers of the variables.

19. Students should be able to graph a circle with center at the origin and a hyperbola of the form $xy = c$. Also, if a, b, and c are positive, $ax^2 + by^2 = c$ should be recognized as the equation of an ellipse or circle; and $ax^2 - by^2 = \pm c$ as the equation of a hyperbola.

20. Open sentences, whether equations or inequalities, are of three types. Identical equations (identities) and absolute inequalities are true for all elements of the replacement set for which the statement has meaning. Conditional equations and conditional inequalities are those which are true for some, but not all elements of the replacement set. These first two types are well known. However, there are also statements of equality and inequality which are never true. For example:

    $\frac{1}{x - 1} - \frac{1}{x + 1} = \frac{1}{x^2 - 1}$; $|x + 2| < -3$

21. Equations are solved by obtaining a series of new equations which we want to be equivalent to the original equation, that is, each has the same solution set as the original equation. Operations which may lead to nonequivalent equations are

multiplying or dividing by zero, multiplying or dividing by an expression containing the variable, or raising both members to the same power, possibly fractional.

Since the solution of irrational and fractional equations involves raising to a power or multiplying by an expression containing the variable, the solution set of any derived equation will contain all of the solution set of the original equation, but may contain other elements as well. Hence a check is essential.

If each member of an equation is divided by an expression containing the variable, the solution set of the derived equation may not contain all of the elements of the solution set of the original equation. Therefore, students should be advised not to use this operation.

22. The study of fractional equations should be extended to include those having binomial denominators.

23. Here the work should be limited to those cases which have only one radical involving the variable and which lead to equations of the first or second degree, for example,

$\sqrt{x^2 + 5} - 1 = x$; $\sqrt{x - 1} = x - 7$; $\sqrt[3]{x - 1} = 2$;

$\frac{3}{\sqrt{x + 2}} = 1$; $\sqrt{3 - 2\cos x} = 2$

NOTE: When solving the second equation above, the derived equation is not equivalent to the original one. Checking will reveal which value should not be included in the solution set. (Avoid calling this an "extraneous root".)

24. Students should be able to solve exponential equations of the type $4^x = 8$; $3^{x+1} = 27^x$.

25. The study of quadratic equations should include:
   a. Review of the solution by factoring.
   b. Derivation (see Appendix, Section D) and use of the formula.
   c. The discriminant and its use in determining the nature of the roots.
   d. Derivation and use of the relationship between roots and coefficients. Use should include checking the roots of a quadratic equation.

26. Trigonometric equations involving two or more functions of the single angle or the double angle should be solved. The use of the quadratic formula should be included.

27. The equation $| x + a | = b$, where b is non-negative, is true if $x + a = b$ and if $x + a = -b$. See Section H for the definition of absolute value. Thus, the equation $| x + a | = b$, where b is non-negative, is solved by finding the solution sets of the two equivalent equations, $x + a = b$ and $x + a = -b$, and taking the union of the two sets. The solution set of the equation $| x + 5 | = 7$ is $\{2, -12\}$.

The solution set of the equation $| x | = k$, where k is negative, is always empty since there is no number whose absolute value is negative. Care must be taken when solving equations of the form $| ax + b | = cx + d$ since values of x which make $cx + d$ negative cannot be members of the solution set of this equation. Thus, the equation $| 2x + 11 | = x + 4$ is not equivalent to either $2x + 11 = x + 4$ or $2x + 11 = -x - 4$. The equation $| 2x + 11 | = x + 4$ is internally inconsistent and its solution set is empty. Again, $| 2x + 1 | = 3x + 4$ and $2x + 1 = -(3x + 4)$ are equivalent equations and have the same solution set which is $\{-1\}$. Equations containing absolute values should first be inspected to see if there are any obvious restrictions on the values of the variable. Then any solutions of the two derived equations should be checked in the original equation to determine whether or not they are members of the solution set of the original equation.

28. A few simple algebraic identities should be presented so that students will be made aware of the fact that identities are not limited to trigonometry. Examples:

$$(a + b)^2 = a^2 + 2ab + b^2; \quad \frac{a^2 - b^2}{a - b} = a + b \text{ (when } a \neq b).$$

29. The student should know the following trigonometric identities and be able to use them in the proofs of other identities and in the solution of simple equations.
   a. Reciprocal: $\sin\theta \csc\theta = 1$, $\cos\theta \sec\theta = 1$, $\tan\theta \cot\theta = 1$
   b. Quotient: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$
   c. Pythagorean: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\cot^2\theta + 1 = \csc^2\theta$

Students also should be able to express any function of an angle in terms of any other function of that angle, for example:

$$\tan\theta = \pm \frac{\sin\theta}{\sqrt{(1 - \sin^2\theta)}}$$

30. This should include expressing the values of function of $-\theta$, $180° \pm \theta$, $360° \pm \theta$ in terms of a function of θ.

If these identities are developed using θ as an acute angle, students should be made to recognize that the identities hold for all values of θ. (Values of functions of $90° \pm \theta$ and $270° \pm \theta$ may be omitted here and considered as applications of the sum or difference of two angles.)

31. The derivations of these identities are required. The order in which the identities for the sine and cosine of the sum and the difference of two angles are developed may vary. While no one sequence is required, one suggested sequence may be found in the Appendix, Section E.

32. Students should be made aware of the fact that certain replacement values of the variable may make one or both members of an identity meaningless. These values, if any, should be determined for each identity.

33. Students should know that inequalities may be true for all, some, or no elements of the replacement set. The algebraic solution of linear inequalities in one variable should be reviewed. The study of quadratic inequalities should include those of the form:

$ax^2 + bx + c > 0$
$ax^2 + bx + c < 0$
$ax^2 + bx + c \geq 0$
$ax^2 + bx + c \leq 0$

when $a \neq 0$ and a, b, and c are real numbers. Application should be limited to those which do not require the use of the quadratic formula. Suggested method for solution may be found in the Appendix, Section F.

34. Study of quadratic inequalities in two variables should be limited to the graphic solution of cases involving the circle with center at origin, the parabola, and the hyperbola of the form $xy = c$. Conjunction of any of these with the linear inequality in two variables should be included. (See Appendix, Section G).

35. Inequalities involving absolute value should be limited to cases such as $| x | > 2$ and $| x + 2 | < 7$. (See Appendix, Section H).

36. A majority of the verbal problems customarily included at this level are artificial and impractical. Many of them are mere puzzles and as such have no relation to actual life situations. Even those which seem to bear some resemblance to actual affairs have been formulated so as to contain the exact amount of information necessary to solve the problem and in most cases evidently the answer has been predetermined. This, of course, is seldom true of a real problem. The practicality of problems, however, should not be the only criterion in deciding the case for or against problem solving as such.

ERIC

Most real problems from fields such as mechanics, science, economics, engineering, and the like presuppose a background which is beyond the grasp of pupils at this stage. Furthermore, to explain this background satisfactorily, even if this were possible, would require more time than is available. On the other hand, the techniques and mental processes involved in the solving of simple "puzzle" problems of the traditional types are the same as those which can be applied at a later time, and with profit, to those of a genuinely practical nature. Finally, it should be pointed out that whatever the educative value of problem solving may be, that value can seldom be realized if teachers and pupils persist in using *special* procedures for special types of problems. Instead, pupils should be encouraged to develop and to understand *generalized* procedures which can be used in attacking all problems regardless of their particular type. Emphasis should be placed on the analysis of the problem and the expression of the quantities involved as related variables.

If the preceding cautions are kept in mind and if the appropriate concepts are emphasized during problem solving activities, use can still be made of the familiar problem settings such as number (including digit problems), geometric, mixture, motion, work, and business. Additional verbal problem settings, consistent with pupil background and ability, may be provided.

37. These systems should be solved both algebraically and graphically. The graphic solution should be limited to those cases in which the quadratic equation is that of a circle of the form $x^2 + y^2 = r^2$, a parabola of the form $y = ax^2 + bx + c$, or a hyperbola of the form $xy = c$.

38. Suggested derivation of the law of sines and the law of cosines may be found in the Appendix, Section E. Use of the law of sines should include computation with logarithms. The solution of the oblique triangle by means of the law of sines or the law of cosines should include the following four cases:
    a. One side and any two angles
    b. Three sides
    c. Two sides and the angle opposite one of them (the ambiguous case)
    d. Two sides and the included angle

    Solution of trigonometric verbal problems should be limited to those cases in which the figure is easily obtained from the description or in which the figure is given. Problems of navigation involving bearing should be omitted and vector problems should be limited to those of a simple parallelogram of forces. Vectors may be treated in the traditional manner or as an ordered pair of numbers. For the latter, see Appendix, Section J.

39. The derivation and use of a formula for the area of a triangle in terms of two sides and the included angle are required. For a derivation, see Appendix, Section E.

40. A sequence $\{a_n\}$ may be defined as a function that associates with each positive integer, n, a real number, a_n. A series is an indicated sum of the terms of a sequence. The terms of the sequence should be indicated as $a_1, a_2, \ldots a_n$. The general term (sometimes called the last term) is a_n. In order to further illustrate the idea of a general term, it may be well to consider some sequences other than the arithmetic or geometric.

$1, 4, 9, 16, \ldots, n^2, \ldots \quad (a_n = n^2)$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \quad (a_n = \frac{1}{n})$

Study should include development and use of the general term, a_n, and the sum of n terms, S_n, for arithmetic and geometric sequences. The study of geometric sequences should include the infinite case.

Section A THE REAL NUMBER SYSTEM

The eleven postulates listed below form part of the foundations of the entire subjects of arithmetic and ordinary algebra. They are the axioms of an abstract system called a field. The letters a, b, and c stand for arbitrary elements of a set of numbers.

Addition

1.	$a + b$ is a unique element of the set	[Closure]
2.	$a + b = b + a$	(Commutative Postulate)
3.	$(a + b) + c = a + (b + c)$	(Associative Postulate)
4.	$a + 0 = 0 + a = a$	[Identity]
5.	$a + (-a) = (-a) + a = 0$	[Inverse]

Multiplication

6.	$a \cdot b$ is a unique element of the set	[Closure]
7.	$a \cdot b = b \cdot a$	[Commutative Postulate]
8.	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	[Associative Postulate]
9.	$a \cdot 1 = 1 \cdot a = a$	[Identity]
10.	$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ for $a \neq 0$	[Inverse]

Distributive Postulate

11. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

If by the natural numbers we mean the positive integers, the union of the sets of natural numbers, the negative integers and zero is the set of integers. The set of integers satisfies all of the field postulates except the inverse for multiplication. If a and b represent any integers, a number of the form $\frac{a}{b}$ where $b \neq 0$ is a rational number. The set of rational numbers satisfies all of the field postulates.

The numbers, $\frac{3}{8}$, 0.75, $\frac{22}{7}$, $\frac{-5}{3}$, 0.666..., $5 - \sqrt{9}$, sin 30°, tan 45°, and 1.7321 are rational numbers as each is the quotient of two integers. Each is written as, or may be written as, a numeral in fractional form. Any integer, a, is a rational number since it is equal to $\frac{a}{1}$. Any rational number may be expressed in fractional form. Any rational number may also be expressed in decimal form (terminating or repeating). For each fraction there is a repeating or terminating decimal representing the same number, and conversely.

<u>Examples:</u> $\frac{3}{4} = 0.75$, $\frac{27}{16}$, $= 1.6875$, $\frac{29}{12}$, $= 2.4166...$

$0.25, = \frac{1}{4}$, $3.73 = \frac{373}{100}$, $0.5454... = \frac{6}{11}$

The following method is suggested for changing a repeating or periodic decimal to the form $\frac{a}{b}$ where a and b are integers.

Example 1

Change 0.5454... to the form $\frac{a}{b}$ where a and b are integers.

Let A = 0.5454...
then 100A = 54.5454...
and 99A = 54

$A = \frac{54}{99}$ or $\frac{5}{11}$

Example 2

Change 0.02727... to the form $\frac{a}{b}$ where a and b are integers.

Let N = 0.02727...
1000 N = 27.2727...
10 N = 0.2727...
990 N = 27
N = 27/990 or 3/110

These are, of course, special cases of the general method for summing an infinite geometric series with $| r | < 1$.

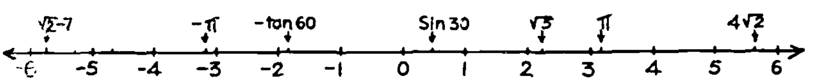
Some numbers when expressed as decimals do not repeat or terminate. These numbers we call irrational numbers. Examples: $\sqrt{3}$, $\sqrt{5} - 5$, $\sqrt[3]{15}$, π, - sin 10°, tan 30°.

There are five important theorems that should be known regarding irrational numbers:

1. If the rational root of an integer lies between two consecutive integers, then this root is an irrational number. In particular, if the square root of an integer is between two consecutive integers, then the root is irrational. It might be better to assume this theorem although a proof is not too hard if one uses the unique prime factorization theorem for integers.
2. The sum of a rational number and an irrational number is an irrational number. The proof of this theorem is appreciated by students for its simplicity and power. *Proof:* Suppose that there were a rational number b, and irrational number x such that their sum were the rational number c. Then we would have b + x = c or equivalently x = c - b. But c - b is a rational number, as the difference between any two rational numbers is a rational number. This contradicts the assumption that x is irrational. Hence, no such b and x exist.
3. The product of a rational number different from zero and an irrational number is an irrational number. The proof is similar to that above.
4. The sum of two irrational numbers may be either rational or irrational. $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ irrational; $(5 + \sqrt{2}) + (5 - \sqrt{2}) = 10$ rational.
5. The product of two irrational numbers may be either rational or irrational. $(\sqrt{2})(\sqrt{2}) = 2$ rational; $(\sqrt{2})(\sqrt{3}) = \sqrt{6}$ irrational.

The union of the set of rational numbers and the set of irrational numbers is the set of real numbers. The set of real numbers also satisfies all of the field postulates.

If a scale is established then the set of real numbers may be placed in one-to-one correspondence with the set of points on a straight line.



With every real number there is associated one and only one point on the number line.

With every point on the number line there is associated one and only one real number.

The associated point is called the *graph* of the number.

Students should know that the various number systems, such as the real numbers may be developed as a deductive system. The amount of work done in this area should depend on the background and maturity of the student.

Simple proofs of theorems such as those which follow might be presented. The following definitions will be considered:

1. $a + b + c$ means $(a + b) + c$
2. $a + b + c + d$ means $([a + b] + c) + d$ etc.
3. $a \times b \times c$ means $(a \times b) \times c$
4. $a \times b \times c \times d$ means $([a \times b] \times c) \times d$ etc.

Simple exercises in the use of field postulates and definitions may be demonstrated. Example: Show that $(b + c + d) \cdot a = ab + ac + ad$.

$(b + c + d) \cdot a = a \cdot (b + c + d)$	Commutative law for multiplication
$= a \cdot [(b + c) + d]$	Definition 1
$= a \cdot (b + c) + a \cdot d$	Distributive law
$= [ab + ac] + ad$	Distributive law
$= ab + ac + ad$	Definition 1

Example: Show that $(u + v)(x + y) = ux + uy + vx + vy$

After reviewing the operations on signed numbers and positive integral exponents, examples such as the following may be considered:

1. *Theorem:* If a and b represent real numbers then $(a + b)(a + b) = a^2 + 2ab + b^2$

Proof:

$(a + b)(a + b) = (a + b)a + (a + b)b$	Distributive law
$= a(a + b) + b(a + b)$	Commutative law for multiplication
$= (aa + ab) + (ba + bb)$	Distributive law

$= (a^2 + ab) + (ab + b^2)$ Commutative law for multiplication

Definition of exponent 2; a^2 means $a \cdot a$

$= [(a^2 + ab) + ab] + b^2$ Associative law for addition

$= [a^2 + (ab + ab)] + b^2$ Associative law for addition

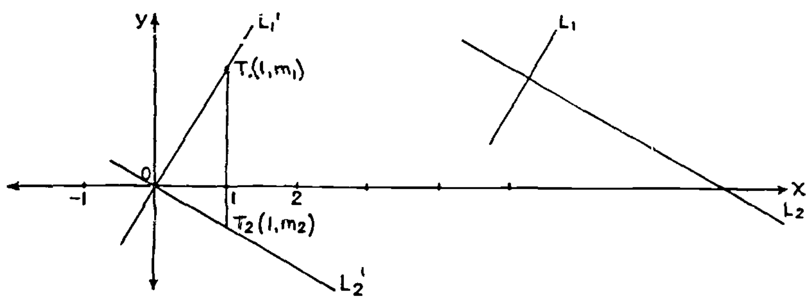
$= (a^2 + 2ab) + b^2$ Substitution

$(a + b)(a + b) = a^2 + 2ab + b^2$ Definition 1.

2. *Theorem:* If a and b are real numbers then $(a + b)(a - b) = a^2 - b^2$
3. Prove: $(3 \sin x - 2)(\sin x + 2) = 3 \sin^2 x + 4 \sin x - 4$
4. Find the product of $3 \sin^2 x \cos y$ and $5 \sin x \cos^2 y$
5. Divide $2 \sin^2 \theta \cos \theta + 5 \sin \theta$ by $\sin \theta$
6. Divide $3 \tan^2 A + \tan A - 2$ by $\tan A + 1$
7. Add $\frac{4}{3 - \sin A} + \frac{2}{\sin A - 3}$

Section B SLOPES OF PERPENDICULAR LINES

Theorem: Two nonvertical lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other. $(m_1 \cdot m_2 = -1)$



Proof: Given two nonvertical lines L_1 and L_2 with slopes m_1 and m_2 respectively. If these two lines are parallel, their slopes are equal and $m_1 \cdot m_2 \neq -1$. Thus we need only consider the case in which L_1 and L_2 intersect. Draw the lines L_1' and L_2' through the origin and parallel to L_1 and L_2 respectively. Lines L_1' and L_2' have slopes m_1 and m_2 respectively. Draw the line whose equation is

$x = 1$, intersecting L_1' at T_1: $(1,m_1)$, and L_2' at T_2: $(1,m_2)$. Triangle $T_1\ 0\ T_2$ is a right triangle if and only if:

$$(T_1T_2)^2 = (T_10)^2 + (T_20)^2$$

$$(m_2 - m_1)^2 = (1^2 + m_1^2) + (1^2 + m_2^2)$$

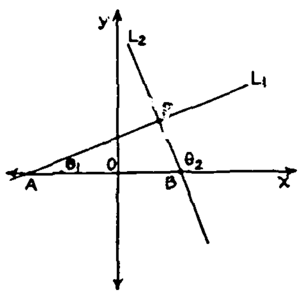
$$m_2^2 - 2m_1 m_2 + m_1^2 = 2 + m_1^2 + m_2^2$$

$$-2m_1 m_2 = 2$$

$$m_1 m_2 = -1$$

Alternate Proof

Proof: Let L_1 and L_2 be two nonvertical lines with slopes m_1 and m_2 respectively. If these two lines are parallel, their slopes are equal and $m_1 \cdot m_2 \neq -1$. Thus we may assume that L_1 and L_2 intersect at Point P. Triangle APB is a right triangle if and only if:



$$\theta_2 = 90 + \theta_1$$

$$\tan \theta_2 = \tan (90 + \theta_1)$$

$$= -\cot \theta_1$$

$$= -\frac{1}{\tan \theta_1}$$

Since $\tan \theta_2 = m_2$ and $\tan \theta_1 = m_1$

$$m_2 = -\frac{1}{m_1} \text{ or } m_1 \cdot m_2 = -1$$

Section C ANGLES AND THEIR MEASURE

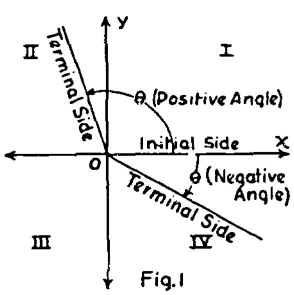


Fig.1

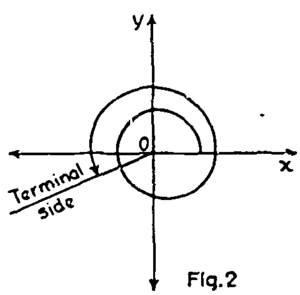


Fig.2

24

In our earlier work in mathematics we defined an angle as the figure formed when two half-lines or rays are drawn from the same point. At this time we prefer to think of an angle as being generated by the rotation of a half-line or ray about a fixed point (the vertex of the angle) from some initial position to a terminal position, the measure of the angle being the amount of rotation. Thus a straight angle is ½ of one complete rotation, a right angle is ¼ of one complete rotation, and a degree is 1/360 of one complete rotation. Since there are two possible directions of rotation we say that if an angle is generated by a counterclockwise rotation, the rotation and the angle are positive. If the angle is generated by a clockwise rotation, we say that the rotation and the angle are negative. (Fig. 1)

In a rectangular coordinate system an angle is said to be in *standard position* when its vertex is at the origin and its initial side coincides with the positive side of the x-axis. The terminal side may fall in any position through the origin. (Fig. 2)

If the terminal side of an angle falls within the first quadrant, the angle is called a first quadrant angle. If the terminal side falls within the second quadrant the angle is called a second quadrant angle. The same terminology is used for the third and fourth quadrants angle. If the terminal side of an angle falls on one of the axes, the angle is called a *quadrantal angle*. Thus angles of 0°, 90°, 180°, 270°, 360°, 450°, -90°, -180° are examples of quadrantal angles.

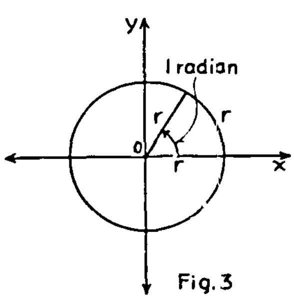


Fig. 3

COMPLETE ROTATIONS	DEGREES	RADIANS
0	0°	0
¼	90°	$\frac{\pi}{2}$
½	180°	π
¾	270°	$\frac{3\pi}{2}$
1	360°	2π
2	720°	4π

Fig. 4

Consider a circle of radius r with its center at the origin in the rectangular coordinate system. If the central angle θ intercepts an arc equal in length to the radius r then the measure of the angle is said to be one radian. (Fig. 3)

If the angle intercepts an arc equal in length to nr, the measure of the angle is n radians. Since the circumference (length) of a circle is equal to 2 π r, an angle whose measure is one complete rotation has a measure of 2 π radians. An angle whose measure is ½

ERIC Full Text Provided by ERIC

of one complete rotation has a measure of π radians. (Fig. 4) Note that the measure of an angle is presumed to be in radians unless otherwise specified. If the angle is measured in degrees the symbol for degrees is used. No special symbol is used for radians. Example:

1. Represent the following angles in standard position in a rectangular coordinate system using a curved arrow to show the direction in which the angle is generated. Express each angle in another equivalent form.

   45°, 240°, -135°, -300°, 480°, -72°, $\pi/6$, $-3\pi/4$, $5\pi/6$, $-4\pi/9$, $11\pi/3$, 2.
2. Find the length of an arc of a circle of radius 6 inches which subtends a central angle of 1½ radians.
3. If a point on a wheel turns through an angle of 1.4 radians in 0.1 second, find the RPM of the wheel.

Section D POLYNOMIALS

The general polynomial of degree n in a single variable x may be written in the form $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} \ldots + a_{n-1} x + a_n$ where n is a non-negative integer and $a_0, a_1, a_2, \ldots a_n$ are real numbers, $a_0 \neq 0$. The function defined by the polynomial is called a polynomial function.

If n = 1, the polynomial is called first degree or linear. If n = 2, the polynomial is called second degree or quadratic.

Derivation of the Quadratic Formula. The formula may be derived by completing the square as it is done in most textbooks. A less familiar but simple method called the Hindu Method is given in the following:

	$ax^2 + bx + c = 0, a \neq 0$
	$ax^2 + bx = -c$
Mult. by 4a	$4a^2x^2 + 4abx = -4ac$
Add b^2	$4a^2x + 4abx + b^2 = b^2 - 4ac$
	$(2ax + b)^2 = b^2 - 4ac$
	$2ax + b = \pm \sqrt{b^2 - 4ac}$
	$2ax = -b \pm \sqrt{b^2 - 4ac}$
	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Section E TRIGONOMETRIC DERIVATIONS

Rectangular Coordinates in Trigonometric Form

Definition: A circle is a set of all points in a plane which are a given distance, called the radius, from a fixed point called the center.

Let a circle of radius R (in which R is, of course, non-negative) be drawn with its center at the origin of a rectangular coordinate system. Let a point P with coordinates (x,y) be taken on the circle. Using the distance formula we get $\sqrt{(x-0)^2 + (y-0)^2} = R$, or $x^2 + y^2 = R^2$.

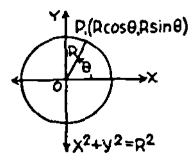
Hence, any point on the given circle has coordinates which satisfy the equation $x^2 + y^2 = R^2$.

Conversely, any point P whose coordinates satisfy the equation $x^2 + y^2 = R^2$ must also satisfy (1) $\sqrt{x^2 + y^2} = R$. If the coordinates satisfy (1) they will satisfy the equation $\sqrt{(x - 0)^2 + (y-0)^2} = R$ which implies that the distance from P to the origin is equal to R.

Hence, any point whose coordinates satisfy the equation $x^2 + y^2 = R^2$ lies on the circle.

Therefore, the equation of a circle with center at the origin of a rectangular coordinate system and radius equal to R is $x^2 + y^2 = R^2$. If R = 0 the graph is called a point circle.

Let OP be drawn forming an angle θ with the positive side of the x-axis, P being any point on the circle whose coordinates are (x,y). By definition $x/R = \cos\theta$ and $y/R = \sin\theta$. Then $x = R\cos\theta$ and $y = R\sin\theta$. Thus the rectangular coordinates of P may be written $(R\cos\theta, R\sin\theta)$.

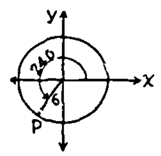


Example: Find the rectangular coordinates of P when R = 6 and θ = 240°. Locate point P on a sketch.

$x = 6\cos 240°$ $\quad y = 6\sin 240°$

$x = 6(-\frac{1}{2})$ $\quad y = 6(-\sqrt{3}/2)$

$x = -3$ $\quad y = -3\sqrt{3}$



Suggested Exercises:

1. Draw a figure and write the rectangular coordinates of point P when
   a. $R = 10,\ \theta = 30°$
   b. $R = 8,\ \theta = 60°$
   c. $R = 6,\ \theta = \pi/2$
   d. $R = 12,\ \theta = -225°$
   e. $R = 15,\ \theta = -2\pi/3$
   f. $R = 16,\ \theta = 19\pi/6$
   g. $R = a,\ \theta = \pi$
2. Express the coordinates of a point P in the real plane in terms of R when the angle θ is in standard position and equal to:
   a. 45° b. 72° c. 150° d. $\pi/2$ e. $4\pi/3$ f. 0° g. 270° h. 180° i. $(A - B)°$ j. $(A + B)°$
3. Verify that $x = R\cos\theta$ and $y = R\sin\theta$ satisfy the equation $x^2 + y^2 = R^2$.
4. If P and P' are two points in the real plane with coordinates indicated below, find the length of PP'.
   a. P(3,5) ; P'(7,2)
   b. P(0,8) ; P'(8 cos θ, 8 sin θ)
   c. P(0,a) ; P'(b cos θ, b sin θ)

The Law of Cosines

Let P be a point on the initial side of an angle θ in standard position, at a distance a from the origin. Let P' be a point on the terminal side of θ at a distance b from the origin. See. Fig. 1 . The coordinates of P are (a,0). The coordinates of P' are (b cos θ, b sin θ). Let the length of PP' be denoted by c. Using the distance formula to find c we have

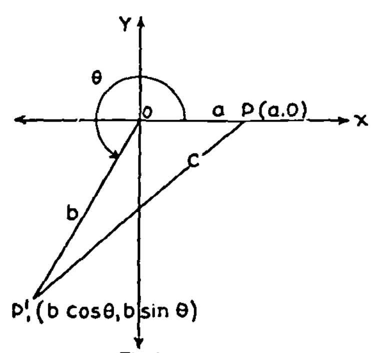


Fig.1

$$c = \sqrt{(b\cos\theta - a)^2 + (b\sin\theta - 0)^2}$$
$$c^2 = b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta$$
$$c^2 = a^2 + b^2(\sin^2\theta + \cos^2\theta) - 2ab\cos\theta$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

As derived, this formula holds for all values of θ. Why?

However, if θ is greater than 0° and less than 180° $(0° < \theta < 180°)$ then the angle θ may be thought of as an angle of a triangle included between two sides of length a and b respectively with c representing the length of the side opposite the angle θ.

The formula $c^2 = a^2 + b^2 - 2ab\cos C$ is known as the "law of cosines" in symbolic form.

This formula may now be used to find the third side of a triangle when two sides and the included angle of the triangle are known. It may be employed also to find the angles of a triangle when the three sides are known.

Exercises:

1. Solve the formula $a^2 = b^2 + c^2 - 2bc \cos A$, for $\cos A$ in terms of a, b, c.
2. If the three sides of a triangle are 5, 7, and 8 units in length, find the number of degrees in the angle opposite the side whose length is 7.

The Trigonometric Functions of the Sum and Difference of Two Angles

1. Prove: $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$

   Let the circle whose equation is $x^2 + y^2 = 1$ be drawn as shown in Fig. 2. Since $OP = OP' = 1$, the coordinates of points P and P' are respectively $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$. The circle whose equation is $x^2 + y^2 = 1$ is often referred to as *the unit circle*.

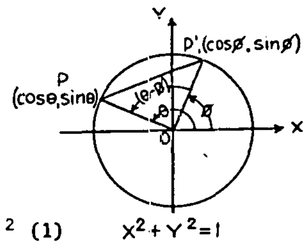


Fig. 2

   Using the distance formula,
   $(PP')^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2$ (1)
   Using the cosine law, $(PP')^2 =$
   $1 + 1 - 2(1)(1)\cos(\theta - \phi)$. (2)
   Equating (1) and (2), simplifying and regrouping terms, we get $2 - 2\cos(\theta - \phi) = (\sin^2\theta + \cos^2\theta) + (\sin^2\phi + \cos^2\phi) - 2(\cos\theta \cos\phi + \sin\theta \sin\phi)$.

   $2 - 2\cos(\theta - \phi) = 2 - 2(\cos\theta \cos\phi + \sin\theta \sin\phi)$

   Finally, $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$ (3)

   Again this formula holds for all values of θ and ϕ. It states "the cosine of the difference of two angles is equal to the product of the cosines of the two angles plus the product of their sines."
2. Using the formula for $\cos(\theta - \phi)$
   a. Let $\theta = 90°$ and show that $\cos(90° - \phi) = \sin\phi$. Let $\phi = 90° - \theta$ to obtain that $\cos\theta = \sin(90° - \theta)$
   b. Let $\theta = 0°$ and show that $\cos(-\phi) = \cos\phi$
   Using the formula $\sin\phi = \cos(90 - \phi)$, let $\phi = -\theta$ and show that $\sin(-\theta) = -\sin\theta$

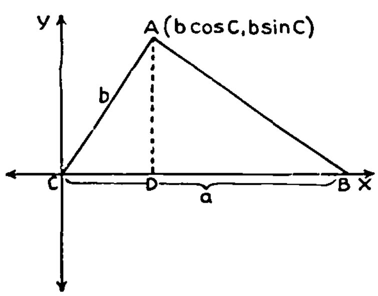
3. Prove: $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
Using the formula $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
Replace ϕ by $(-\phi)$
$\cos(\theta - (-\phi)) = \cos\theta\cos(-\phi) + \sin\theta\sin(-\phi)$
But $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$
Therefore $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

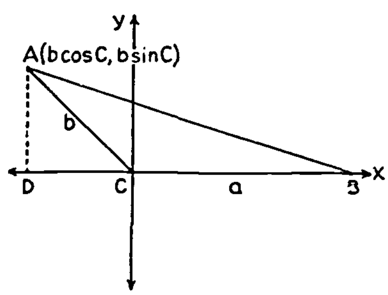
4. Prove: $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
$\sin\alpha = \cos(90° - \alpha)$
Replace α by $(\theta - \phi)$
$\sin(\theta - \phi) = \cos[90° - (\theta - \phi)]$
$\sin(\theta - \phi) = \cos[(90° - \theta) + \phi]$
$\sin(\theta - \phi) = \cos(90° - \theta)\cos\phi - \sin(90° - \theta)\sin\phi$
$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

5. Prove: $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
Replace ϕ by $(-\phi)$
$\sin[\theta - (-\phi)] = \sin\theta\cos(-\phi) - \cos\theta\sin(-\phi)$
$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

Area of a Triangle

Theorem: The area of a triangle is equal to one half the product of two sides and the sine of the included angle. ($K = \frac{1}{2} ab \sin C$).





Place triangle ABC with the vertex C at the origin and side CB coinciding with the positive direction of the x-axis. The coordinates of A are ($b\cos C$, $b\sin C$). Draw AD, the altitude to side BC. The length of the altitude AD is the y-coordinate of A, $b\sin C$.

$K = \frac{1}{2}(CB)(AD) = \frac{1}{2}(a)(b\sin C)$
$= \frac{1}{2} ab \sin C$

By choosing the coordinate system appropriately, we can obtain

$K = \frac{1}{2} bc \sin A$
$K = \frac{1}{2} ac \sin B$

The Law of Sines

The sines of the angles of a triangle are proportional to the lengths of the opposite sides.

A very simple proof of the law of sines which is perfectly general, makes use of the formula for the area of a triangle which we have just derived.

$K = \frac{1}{2} cb \sin A$, or $K = \frac{1}{2} ac \sin B$ or $K = \frac{1}{2} ab \sin C$

From these we obtain the inequalities

$\frac{1}{2} cb \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$

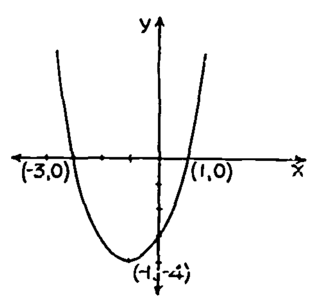
Dividing by $\frac{1}{2}$ abc we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Section F QUADRATIC INEQUALITIES IN ONE VARIABLE

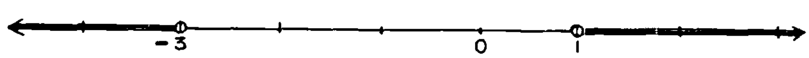
Methods of finding solution sets of inequalities will be illustrated by examples.

Example 1: Find the solution set of the inequality $x^2 + 2x > 3$. Transform the inequality to $x^2 + 2x - 3 > 0$



Solution 1:

The graph of $y = x^2 + 2x - 3$ indicates that when $x = -3$ and $x = 1$, y (or $x^2 + 2x - 3$) is zero. If $-3 < x < 1$, then y (or $x^2 + 3x - 3$) is negative. If $x < -3$ or $x > 1$, then y (or $x^2 + 2x - 3$) is positive. We are to determine the values of x for which y (or $x^2 + 2x - 3$) is greater than zero. Therefore, the solution set of $x^2 + 2x - 3 > 0$ is $\{x \mid x < -3 \text{ or } x > 1\}$. This solution set may be shown on a number line.



Solution 2:

The graph of $y = x^2 + 2x - 3$ indicates that the points (-3,0) and (1,0) may be used to determine the intervals for which $x^2 + 2x - 3$ is positive or negative. That is, the required interval is either $\{x \mid -3 < x < 1\}$ or $\{x \mid x < -3 \text{ or } x > 1\}$. Which set is the solution set may be determined by testing a value of x from each interval.

If $x = -1$	If $x = 2$
$x^2 + 2x - 3 = -4$	$x^2 + 2x - 3 = 5$
Therefore:	Therefore:
$x^2 + 2x - 3 < 0$ for $-3 < x < 1$	$x^2 + 2x - 3 > 0$ for $x < -3$ or $x > 1$

Thus, the solution set of $x^2 + 2x - 3 > 0$ is $\{x \mid x < -3 \text{ or } x > 1\}$

NOTE: Observe that the abscissas of the points (-3,0) and (1,0) are the zeros of the function, $\{(x,y) \mid y = x^2 + 2x - 3\}$. These zeros may be determined by setting $x^2 + 2x - 3 = 0$ and solving the equation. Also, if a quadratic function $\{(x,y) \mid y = ax^2 + bx + c, a \neq 0$ has a maximum value, we know that for values of x between the zeros of the function, the values of the function are positive. In the same way if the function has a minimum value, the value of the function is negative for all values of x between the zeros. If the function has no zeros, its value is positive for all x or it is negative for all x.

Solution 3:

$$x^2 + 2x > 3$$
$$x^2 + 2x - 3 > 0$$
$$(x + 3)(x - 1) > 0$$

This last inequality will be true if and only if the factors of the left member are either both positive or both negative.

Both Positive	or	Both Negative
$x + 3 > 0$ and $x - 1 > 0$		$x + 3 < 0$ and $x - 1 < 0$
$x > -3$ and $x > 1$		$x < -3$ and $x < 1$
The intersection of the solution sets of these inequalities is		The intersection of the solution sets of these inequalities is
$\{x \mid x > 1\}$		$\{x \mid x < -3\}$

Therefore the solution set of $x^2 + 2x - 3 > 0$ is the union of these sets which is $\{x \mid x < -3 \text{ or } x > 1\}$.

Example 2: $2x^2 - 3x \leq 5$
This inequality will be solved by the last method only

$$2x^2 - 3x - 5 \leq 0$$
$$(2x - 5)(x + 1) \leq 0$$

This inequality will be true if and only if one factor of the left member is positive or zero and the other negative or zero.

	or	
$2x - 5 \geq 0$ and $x + 1 \leq 0$		$2x - 5 \leq 0$ and $x + 1 \geq 0$
$x \geq 5/2$ and $x \leq -1$		$x \leq 5/2$ and $x \geq -1$
The intersection of the solution sets of these inequalities is ϕ		The intersection of the solution sets of these inequalities is
		$\{x \mid -1 \leq x \leq 5/2\}$

ERIC Full Text Provided by ERIC

The solution set of $2x^2 - 3x \leq 5$ is the union of these sets which is $\{x \mid -1 \leq x \leq 5/2\}$.

Section G GRAPHS OF INEQUALITIES IN TWO VARIABLES

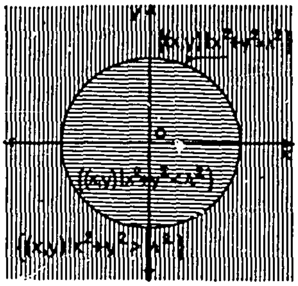


Fig. 1

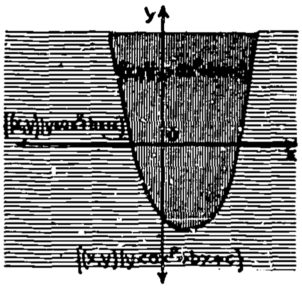


Fig. 2

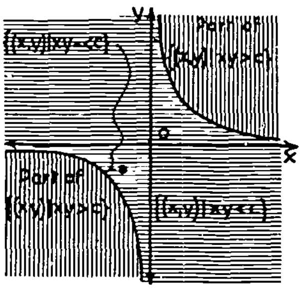


Fig. 3

When graphing an inequality, one should first graph the corresponding equality which divides the plane into three sets of points, as shown in figures 1,2,3. For example; if the graph of $\{(x,y) \mid y > ax^2 + bx + c\}$ is required, first draw the graph of $\{(x,y) \mid y = ax^2 + bx + c\}$. Substituting the coordinates of a point which is "inside" or one which is "outside" the parabola, will show that the region "inside" the parabola is the required set of points (see figure 2). To show that the parabola itself is not a

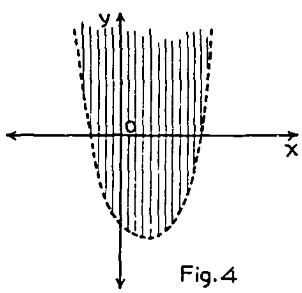


Fig. 4

part of the solution set, a dashed line may be used. Then the solution set $\{(x,y) \mid y > ax^2 + bx + c\}$ may be graphed as in figure 4.

Conjunction of Inequalities

Illustration:

Graph $\{(x,y) \mid xy < -6 \text{ and } y \geq -3x + 3\}$

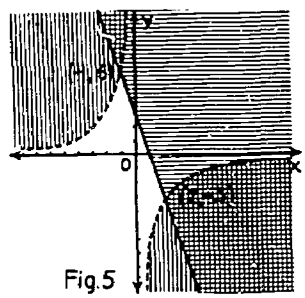


Fig.5

The solution set of the system is the intersection of the solution sets of the two inequalities. It is the doubly shaded region shown in figure 5. The solution set includes the portion of $y = -3x + 3$ which is a border of the doubly shaded region, but includes no part of $xy = -6$.

Section H ABSOLUTE VALUE

The absolute value of a real number a, symbolized by $|a|$, is never negative.

$|a| = a$ when $a \geq 0$

$|a| = -a$ when $a < 0$

On the number line, the absolute value of a number may be thought of as the nondirected distance between the graph of the number and the origin.

Example 1: Find the solution set of $|x| < 2$
The distance from 0 to x must be less than 2. The graph is shown in figure 1.

Fig.1

To solve the inequality algebraically, use the definition of absolute value. Case 1: If $x \geq 0$, then $x < 2$. (Replace $|x|$ by x in $|x| < 2$). This gives $\{x \mid x \geq 0 \text{ and } x < 2\}$. Case 2: If $x < 0$, then $-x < 2$ or $x > -2$. This gives $\{x \mid x < 0 \text{ and } x > -2\}$.

The solution set of the inequality is the union of the two above sets $\{x \mid x \geq 0 \text{ and } x < 2\} \cup \{x \mid x < 0 \text{ and } x > -2\}$ or $\{x \mid -2 < x < 2\}$.

Example 2: Find the solution set of $| x - 2 | \leq 5$.
The distance between 2 and x is not greater than 5. The graph is shown in figure 2.

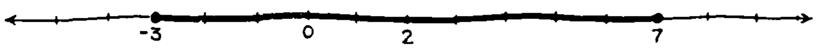


Fig.2

To solve the inequality algebraically, use the definition of absolute value. Case 1: If $x - 2 \geq 0$, then $x - 2 \leq 5$ or $x \geq 2$ and $x \leq 7$. This gives $\{x \mid x \geq 2 \text{ and } x \leq 7\}$. Case 2: If $x - 2 < 0$, then $-(x - 2) \leq 5$, or if $x < 2$, then $x \geq -3$. This gives $\{x \mid x < 2 \text{ and } x \geq -3\}$.

The solution set of the inequality is the union of these two sets: $\{x \mid x \geq 2 \text{ and } x \leq 7\} \cup \{x \mid x < 2 \text{ and } \geq -3\}$ or $\{x \mid -3 \leq x \leq 7\}$.

An alternate algebraic analysis of inequalities, which high school pupils may find easier to use, may be based on the introduction of the following principle which can be verified by exercises similar to example 1 above. Inequalities involving absolute value may be solved by using this principle and the axioms of inequalities.

Principle: For any positive number a, the solution set of the inequality $| x | < a$ is $\{x \mid -a < x < a\}$. The solution set of $| x | > a$ is the union of $\{x \mid x < -a\}$ and $\{x \mid x > a\}$.

Example 3: Find the solution set of $| 2x - 7 | < 13$

Using the above principle,	$-13 < 2x - 7 < 13$
Adding 7,	$-6 < 2x < 20$
Dividing by 2,	$-3 < x < 10$

Therefore, the required solution set is $\{x \mid -3 < x < 10\}$.

Example 4: Find the solution set of $| 4x - 7 | \geq 13$

From the above principle, we must solve the two inequalities:

$4x - 7 \leq -13$	and	$4x - 7 \geq 13$
$4x \leq -6$		$4x \geq 20$
$x \leq -\frac{3}{2}$		$x \geq 5$

Therefore, the solution set is $\{x \mid x \leq -\frac{3}{2}\} \cup \{x \mid x \geq 5\}$ or $\{x \mid x \leq -\frac{3}{2} \text{ or } x \geq 5\}$.

Section I INVERSE FUNCTIONS

If two sets of elements $X = \{x_1, x_2, x_3, \ldots\}$, and $Y = \{y_1, y_2, y_3, \ldots\}$ are related by some rule which establishes a one-to-one correspondence between the elements of X and Y, then y is a function of x and x is a function of y. These two functions are called inverses of each other. The domain of one becomes the range of the other and vice-versa. That is, the domain and the range of these two functions are interchanged.

Let the linear function defined by $y = f(x) = 2x-5$ have as domain and range the set of real numbers. If we interchange x and y and solve for y in terms of x, we get $x = 2y-5$ and finally $y = \frac{x+5}{2}$ which also has a domain and range which consist of the set of real numbers. These two functions are inverses of each other. The inverse of function f is often indicated by f^{-1}. Thus in the example above: if $f(x) = 2x-5$ then $f^{-1}(x) = \frac{x+5}{2}$. It should be stressed that the -1 used in this notation is not an exponent.

In many cases a given function has no inverse function. However, by restricting the domain of definition of a function it is often possible to find an inverse function. The function defined by $y = f(x) = x^2$ has a domain which is the set of real numbers and a range consisting of the set of non-negative real numbers. Upon interchanging x and y we get $x = y^2$. Solving for y in terms of x, this gives us $y = \pm\sqrt{x}$ which does not define a function since for each positive real value of x there are associated *two* values of y. However, limiting the domain of x to the set of non-negative real numbers, we may confine the range of the inverse relation to the same set. Thus, when $f(x) = x^2$ is defined over the set of non-negative reals, $f^{-1}(x)$ may be defined by $f^{-1}(x) = \sqrt{x}$ having a domain and range consisting of the set of non-negative real numbers. How could we restrict the domain of x^2 so that $f^{-1}(x)$ is defined by $f^{-1}(x) = -\sqrt{x}$?

Exercises: (Assume the domain and range of each function are intended to be the largest possible subsets of the real numbers.)

Indicate the domain and range of each of the following functions and find an inverse function in each case by placing restrictions on the domain and range of f(x) if necessary.

1. $y = f(x) = 2x + 3$
2. $y = f(x) = x^2 - 2x$
3. $F = \{(x,y) \mid y = \sqrt{x-3}\}$
4. $F = \{(x,y) \mid y = -6/x\}$
5. $y = f(x) = \frac{2x+1}{x-2}$

Section J VECTOR ADDITION

The concept of vector which is a part of the trigonometric content of this course is usually introduced in the traditional manner by considering its geometrical applications to problems involving force or velocity.

Let us disassociate the concept of vector from all of its physical applications and define a vector as an ordered pair of real numbers (a,b).

We shall define the "addition" of two vectors by the equation (a,b) + (c,d) = (a+c,b+d). Let the vector (0,0) be called the zero vector. Using these new concepts and definitions it is easy to show that:

1. the sum of two vectors is always a vector
2. every vector has an additive inverse
3. "addition" is associative
4. "addition" is commutative

This abstraction of the notion of a vector and its properties under the operation of addition is easily linked with the more common geometric description of a vector as a line segment having the properties of magnitude and direction. As previously stated, the set of ordered pairs of real numbers may be placed in one-to-one correspondence with the set of points in the real plane.

The magnitude (length) of the vector (a,b) may be taken to be the distance from the origin (0,0) to the point whose coordinates are (a,b), that is, $\sqrt{a^2 + b^2}$. The direction of the vector (a,b) may be taken as the direction of the line through the points (0,0) and (a,b) having a sense in the direction from the point (0,0) to the point (a,b). With a vector (a,b) so oriented in the real plane, it is obvious that the x-component is the vector (a,0) and the y-component is the vector (0,b). Furthermore, (a,b) = (a,0) + (0,b).

It is readily seen that "addition" as performed by the use of the parallelogram law or triangle law is equivalent to "addition" as defined by the equation (a,b) + (c,d) = (a+c,b+d).

Thus, two vectors may be added by forming the vector whose x and y components are the respective sums of the x and y components of the given vectors.

Exercises such as the following may be used.

Problem: Two forces of 12 lbs. and 5 lbs. act upon a body at an angle of 70°. Find the magnitude and the direction of the resultant force.

Solution: Let the 12 lb. force act along the positive side of the x-axis and let the 5 lb. force make a 70° angle with the positive side of the x-axis.

Then the first force may be represented by the vector (12,0) and the second force by the vector (5 cos 70°, 5 sin 70°). Then the "sum" or resultant is represented by (12 + 5 cos 70°, 5 sin 70°).

The angle of the resultant with the 12 lb. force is given by arc tan $\left(\frac{5 \sin 70°}{12 + 5 \cos 70°}\right)$. The length of the vector or magnitude of the resultant is given by length = $\sqrt{(5 \sin 70°)^2 + (12 + 5 \cos 70°)^2}$. To complete the exercise evaluate these two expressions.

Solutions such as the above may be alternated with the solutions using the triangle laws to provide drill in both methods. The component method of operating on vectors is excellent preparation for later work in complex numbers and for more advanced engineering work.

Section K SUPPLEMENTARY EXERCISES

The material in this section is not intended to add to the required content of the syllabus. It does, however, contain supplementary exercises which the teacher may find useful.

1. Functions and Relations
   a. Which of the following relations are functions?
      (1) {(2,1), (3,2), (4,3), (5,5)}
      (2) {(1,1), (2,4), (1,9), (3,1)}
      (3) {(-2,1), (-4,2), (-3,2), (1,1)}
      (4) {(-2,4), (-3,1), (-8,2), (-3,5)}
   b. Which of the relations in example 1 have inverses which are functions?
   c. If {(3,4), (1,6), (x,3), (7,9)} is to be a function, list the integers which x may not represent.
   d. If the replacement set (universe) is {1, 2, 3, 4} determine which of the following relations are functions
      (1) $\{(x,y) \mid y = x\}$ (3) $\{(x,y) \mid y = |x|\}$
      (2) $\{(x,y) \mid y \neq x\}$ (4) $\{(x,y) \mid y = 1\}$
   e. The inverses of which of the relations in exercise 4 are functions?
   f. Does the following define a function? Explain.

$$f(x) = \begin{cases} x \text{ if } x < 0 \\ 1 \text{ if } x = 0 \\ -1 \text{ if } x > 0 \end{cases}$$

   g. If a function is defined by the equation y = 2x - 3, write in the form y = mx + b the equation which defines the inverse of this function.

h. When a real function is defined by an equation, unless specified otherwise, the domain is the set of all real numbers for which the equation has meaning. Determine the domain of:

(1) $f(x) = \frac{1}{x - 1}$

(2) $f(x) = \sqrt{\frac{1}{x}}$

2. Inequalities

a. On the same set of axes, sketch the graph of $y = \sin x$ and $y = \cos x$. From the graphs, find the values of $0 \leq x \leq 2\pi$ for which:

(1) $\sin x > \cos x$ (2) $\sin x \leq \cos x$

b. Draw the graph of the equation $y = x^2 - x - 6$. From the graph, indicate the values of x for which

(1) $y = 0$ (4) $y \leq -4$

(2) $y \leq 0$ (5) $y < -7$

(3) $y > 0$

c. (1) Graph the set of points represented by $\{(x,y) \mid x^2 + y^2 \leq 4\}$.

(2) Graph the set of points represented by $\{(x,y) \mid x^2 + y^2 > 16\}$.

d. Graph the set of points represented by $\{(x,y) \mid y > x^2 - 2x - 5 \text{ and } y \leq x - 5\}$.

e. Graph the set of points represented by $\{(x,y) \mid x^2 + y^2 < 25 \text{ and } xy \geq 12\}$.

f. Find the solution set of the following inequalities:

(1) $|x - 7| > 2$ (4) $|3x - 1| < 5$

(2) $|x + 3| - 1 \leq 4$ (5) $|x - 2/5| \geq 0$

(3) $|2x| > 7$ (6) $|2x - 5| + 3 < 1$

g. Find the solution set of:

(1) $x^2 + x - 2 \leq 0$ (4) $3x^2 - x - 4 \geq 0$

(2) $x^2 - 6x + 10 > 2$ (5) $4x^2 - 5x + 1 < 0$

(3) $2x^2 + x < 6$

3. Inverse Trigonometric Relations and Functions

a. Write the solution set for each of the following equations for $-2\pi < \theta < 2\pi$.

(1) $\theta = \text{arc sin } \frac{1}{2}$ (4) $\theta = \text{arc tan } (-1)$

(2) $\theta = \text{arc cos } \frac{1}{2}$ (5) $\theta = \text{arc sec } 2$

(3) $\theta = \text{arc sin } (-\frac{\sqrt{2}}{2})$ (6) $\theta = \text{arc cot } 0.7$

b. Evaluate:

(1) Arc sin 1 (2) Arc $\cos(-\sqrt{2}/2)$ (3) Arc tan(-1)

c. Find the positive value of each of the following:

(1) cos(arc sin 3/5) (4) tan(2 arc sin $\frac{1}{2}$)

(2) tan(arc cos 5/13) (5) csc(2 arc tan $\sqrt{2}/2$)

(3) sin(arc tan 1) (6) sin(arc tan 1/2 + arc tan 1/3)

d. Show that Arc sin a + Arc cos a = $\pi/2$.

e. Show that Arc tan 1/7 + 2 Arc tan 1/3 = $\pi/4$.

f. Find:
   (1) Arc sin(sin $5\pi/6$)
   (2) Arc sin[cos($-\pi/5$)]
   (3) Arc tan(sin $7\pi/3$)

4. Formulas, Variation and the Solution of Equations
   a. The area A of a regular pentagon whose side is s is given by the formula $A = \frac{5s^2 \tan 54^\circ}{4}$. Solve for s in terms of A.
   b. The radius R of a circle circumscribed about a regular decagon whose area is A is given by the formula $R = \sqrt{\frac{A}{5 \sin 36^\circ}}$. Solve for A in terms of R.
   c. Solve the following set of equations for all values of x and y greater than 0° and less than 360°:
      $4 \cos x + 3 \sin y = 5$
      $6 \cos x - \sin y = 2$
   d. Solve the following set of equations for $\tan \theta$ in terms of a and b:
      $\tan \theta + \cot \theta = a$
      $\tan \theta - \cot \theta = b$
      NOTE: These equations may be consistent or inconsistent, consider $a = b = 1$, for instance.
   e. Express y in terms of a, tan A, and tan B, given that $\tan A = y/x$ and $\tan B = \frac{y}{x+a}$ and $x \neq 0, -a$.
   f. Is the following statement true or is it false? Since $\sec \theta = 1/\cos \theta$, the secant of an angle varies inversely as the cosine of the angle.
   g. Find to the nearest tenth the value of sin x which satisfies the equation $\cos^2 x + 3 \sin x - 2 = 0$.
   h. Solve the following equation for x. Express the solutions in inverse form.
      $3 \cos^2 x + 2 = 5 \cos x$.
   i. Given the equation $m = \frac{\cos x + 1}{\cos x - 1}$, express x in terms of m.
   j. Solve the equation $\cos 2x = \frac{1}{2}$ for $0^\circ < x < 360^\circ$.

In the solution of equations such as $\cos 2x = \frac{1}{2}$ for all values of x from 0° to 360° care should be exercised, since in the solution one obtains $2x = 60^\circ$ or 300°, period of 360° and then $x = 30^\circ$ or 150°, period of 180°. Since the statement of the question asks for all x in a double period of 360°, it is necessary to add 180° to each of the results, 30° and 150°, obtaining 30°, 150°, 210°, 330° as the required solution set.

5. Identities

As an introduction to the usual work in the proof of trigonometric identities, it is suggested that exercises such as the following be considered.

a. Express $\sec \theta - \cos \theta$ as the product of two trigonometric functions.

b. Transform $\frac{1 - \sin \theta}{\cos \theta}$ to the form $\frac{\cos \theta}{1 + \sin \theta}$.

c. Express $\tan^2 A$ in terms of $\sin A$.

d. Express $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A}$ as a single trigonometric function.

e. Express $\frac{2 \tan \theta}{1 + \tan^2 \theta}$ as a single trigonometric function of 2θ.

f. Simplify: $\frac{\sin \theta}{\csc \theta - \cot \theta}$

g. Simplify: $\csc 2\theta - \cot 2\theta$.

h. Express $\tan 50°$ in terms of $\tan 20°$.

i. Show that $\frac{\sin 2A}{2} = \sin A \cos A$ is true for all values of angle A.

j. Show that $\tan A/2 = \frac{\sin A}{1 + \cos A}$ is true for all values of angle $A \neq (2n + 1)\pi$, n integral.

k. Is the statement $\frac{1}{2} \sin x = \sin x/2 \cos x/2$ an identity?

l. Is the statement $\cos^2 3x/2 - \sin^2 3x/2 = \cos 3x$ true for all values of angle x, only some values of x, or is it false for all values of angle x?

6. Simple Exponential and Logarithmic Functions

Exercises: (Base 10 is implied unless otherwise indicated)

a. Find $\log 10^x$

b. Find $\log 10^{a-b}$

c. Find $\log 10^x + \log 10^y$

d. Change the following to exponential form and find the value of x:

(1) $\log_2 2 = x$ (3) $\log_2 x = 4$

(2) $\log_x 4 = 2$ (4) $\log_2 x = -3$

e. Write each of the following in logarithmic form:

(1) $8 = 2^3$ (4) $x = 10^{-2}$

(2) $a = b^x$ (5) $10^3 = 1000$

(3) $25 = x^2$

7. Graphical Representations

a. Solve graphically and estimate nonintegral solutions to the nearest tenth:

(1) $y = 2^x$ $2x - y = 0$ $-2 \le x \le 3$	(2) $y = 2^x$ $y = x + 1$ $-2 \le x \le 3$	(3) $y = \sin x$ $y = \pm \frac{1}{2}$ $-\pi \le x \le +\pi$
(4) $y = \sin 2x$ $y = \cos x$ $0 \le x \le 2\pi$	(5) $y = \tan x$ $y = x$ $-\pi/2 \le x \le \pi/2$	(6) $y = \log_3 x$ $2y = x - 1$

b. Construct the graph of:
   (1) $y = x + \sin x$, $0 \le x \le 2\pi$ (2) $y = \sin(\pi/2 - x)$, $0 \le x \le 2\pi$

c. Solve graphically: $2 \sin x + 3 \cos x = 0,\ 0 \le x \le 2\pi$

8. Further Multiconcept exercises

a. Prove the identity $(a \ne 0)$, $\frac{1}{a} + \frac{a - 1}{a} = 1$

b. Prove the identity $(r \ne \pm 1)$, $\frac{r^2 - r}{(r - 1)^2} + \frac{r + 1}{r^2 - 1} = \frac{r + 1}{r - 1}$

After the trigonometric functions have been defined in terms of variables associated with the general angle in standard position on the coordinate axes, variety in the drill work may also be obtained on occasion, by requiring that a trigonometric identity be proved by showing the equivalence of the two algebraic expressions obtained by replacing the trigonometric functions by equivalent ratios.

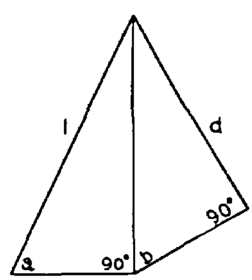
c. Prove the identity after replacing the functions by ratios.

$$\frac{1}{(1 + \cot^2 A)} = 1 - \cos^2 A$$

When the topic, inverse variation, is introduced, the teacher may return to the simple right triangle for additional practice material involving the trigonometric functions. Examples such as the following are useful since they review several concepts.

d. In the right triangle ABC with right angle C, side a is *constant*. If side b = 4 when tan A = 3, find tan A when side b = 3.

Other variations involving other formulas obtained from the right triangle will immediately suggest themselves to the teacher.



For example, one of the formulas which can be obtained from the diagram at the right is $d = \sin a \sin b$. By setting one of the three variables, d, sin a or sin b, equal to a *constant*, either direct or inverse variation may be illustrated and used in practice exercises.

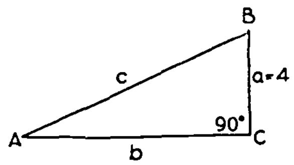
The law of sines, $a/\sin A = b/\sin B$, may be used as practice in variation if *two* of the variables are set equal to *constants*. Problems such as the following are useful and interesting, especially if the pupils are asked to draw several "movie frames" to illustrate how the figure may vary under the restrictions.

e. In triangle ABC, side a and angle B are given as *constants*. Use the law of sines in a discussion of the variation of side b with respect to sin A under given conditions. Draw a set of consecutive figures to illustrate how triangle ABC may be altered under the above restrictions.

f. A certain table of trigonometric functions lists the cosine of an acute angle as a two digit decimal. The tenth's digit of the cosine of a required angle is one more than twice the hundredth's digit. The sum of the digits of the cosine is 10. Find the *angle* to the nearest degree.

It is quite easy for the teacher to construct other problems concerning numbers and including some trigonometric content, which may be used to alternate with problems of purely algebraic content.

g. In a certain right triangle ABC with right angle C, side $a = 4$. If the hypotenuse of the triangle were decreased by 10 and angle A varied so that the sine of A were increased by 0.3, then side a would be increased by 1. Find the *sine of A* in the original triangle.



Solution: The hypotenuse is c.
The formula is $c \sin A = a$.
From the original conditions,
$c \sin A = 4$ so that $c = 4/\sin A$.
From the revised conditions,

$$\left(\frac{4}{\sin A} - 10\right)(\sin A + 0.3) = 5$$

etc.

Other examples of this nature in which either the angle or a function of the angle may be required, will suggest themselves to the teacher.

h. Find to the nearest ten minutes the smallest positive value of x that satisfies the equation $\sin^2 x + 2 \sin x = 1 = 0$.

The related topic, the formation of an equation from its roots, also permits the use, now and then, of trigonometric content. Such exercises may involve roots which are integral, decimal, or irrational real numbers, but it would be best to limit the range of the trigonometric functions used so as to correspond to a real domain of the angle involved.

ERIC Full Text Provided by ERIC

i. Write the quadratic equation with the roots cos $x = +\frac{1}{2}, +\frac{1}{4}$
j. Write the quadratic equation with the roots sin $x = 0.44, -0.25$
k. Write the quadratic equation with the roots tan $x = \pm\sqrt{3}$

Further correlation of the trigonometric content and the algebraic content may be carried out in the drill work accompanying the study of arithmetic series. Laboratory measurements involving cyclic phenomena such as light, radio, sound, etc. may be such that the intensities measured and therefore the amplitudes of the trigonometric functions used to describe the phenomena, may form an arithmetic series. Problems such as the following then do have meaning.

l. Find the common difference of the arithmetic series, $3 \sin a + 2 \sin b$, $5 \sin a + 3 \sin b$, $7 \sin a + 4 \sin b, \ldots$
m. Insert two arithmetic means between $11 \sin a - 5 \cos b$ and $19 \sin a + 4 \cos b$.

ELEVENTH YEAR MATHEMATICS

Unit	Topics	Time Allotment (Days)
I	Number system (1); Postulates of a field (2); Operations on the real numbers (3)(4)(5); Algebraic identities (28)	15
II	Fractional (22), irrational (23) and literal equations; Verbal problems (36); Inequalities of first degree (33); and Absolute value (27)(35)	15
III	Functions and relations, definitions (8); Direct and inverse variation (18); Definition of trigonometric function for positive and negative angles (11); Reduction formulas (30); Simple trigonometric equations (26); and Identities (29)	20
IV	Exponential functions (14); Logarithmic function (15); Laws of Logarithms (16); Computation (17); Exponential equations (24)	17
V	Complex numbers (6)	5
VI	Polynomial functions, linear (9) and quadratic (10); Quadratic relations (19); Linear - quadratic systems (37); Verbal problems (36)	17
VII	Quadratic equation (25); Verbal problems (36); Trigonometric equations (26); Second-degree inequalities (33)	17
VIII	Trigonometric graphs (12); Inverse trigonometric functions (13)	8
IX	Law of Cosines and simple applications (38); Sine, cosine and tangent of the sum and difference of two angles, of double angles, and half angles (31); Identities (32) and equations (26)	20
X	Area of triangle (39); Law of Sines (38); Verbal problems involving the law of sines and law of cosines (38)	10
		144

ERIC