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ABSTRACT

Three different approaches to the creation of popularity indices from sociometric data are described. One involves a factor analysis of the sociometric matrix and the other two are different approaches to the weighting of sociometric choices. All turn out to have the same mathematical solution when the relationships are symmetric: it is certain eigenvectors of the sociometric matrix. This technique has the additional benefit of giving the clique structure at a glance. The technique is compared to Hubbell's (1965) method for clique identification. The method is then illustratively applied to structural data on the pattern of overlap in membership among a set of high school activities. A measure of "centrality" in this structure, analogous to individual popularity in sociometric structures, is calculated for each of the activities and the results are compared to common sense expectations about high school activities. (Author)

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MATHEMATICAL ANALYSES OF HIGH SCHOOL SOCIAL STRUCTURES

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Los Angeles

December 1970

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INTRODUCTION

The aim of this paper is to propose a general technique for the analysis of structural data. It will be shown that three different approaches to analyzing structural data all have essentially the same solution, and also, as an added bonus, that the technique automatically give the "cliques" in the structure. The technique is then illustratively applied to data on the pattern of overlap in membership among voluntary activities in a high school.

In considering the three different approaches, we will let W be a real symmetric matrix of relationships or sociometric choices. All values in W are assumed to lie between zero and one inclusive. For example, we might let $W_{ij} = 1$ if i and j are friends and $W_{ij} = 0$ otherwise. The elements on the main diagonal are zero. We will now look at three approaches to devising measures of popularity or centrality in this structure.

THE FACTOR ANALYSIS APPROACH

The bonds that form in a group could be conceived of as the result of "interaction potentials" possessed by each individual. If W were a matrix of friendships, S_i would be individual i's propensity to form friendships. The actual bond between i and j, W_{ij}, should be close to S_iS_j. In matrix notation, we want to calculate a column vector S such that the sum of the squared differences between SS and W is minimized. W is like a peculiar correlation matrix with zero communalities and the criterion for S is identical to the criterion for the first principal components factor of a correlation matrix (minimizing squared differences). The vector S (and the first principal components factor) is the eigenvector of the largest eigenvalue of the matrix standardized so that its length is the eigenvalue.

An interesting byproduct of the factor analysis is that there is an exact parallel between the factor structure and the clique structure. Define a clique as a set of individuals such that no relationships extend outside the clique and all individuals within the clique are related directly or indirectly through chains of choices. Each clique will be represented in the factor structure by a factor all of whose elements are greater than or equal to zero (Appendix C). The members of the clique will have nonzero positive loadings on this factor and all those who are not members of that clique will have zero loadings (Appendix A). Thus, the clique structure can be read at a glance; each clique will have its own vector of popularity scores.

The individuals who load high on a factor all of whose elements are greater than or equal to zero will be especially popular in their clique. For each clique the magnitude of the eigenvalue of its eigenvector of popularity scores will be a measure of how good the eigenvector is in summarizing the relationships in the clique. In a factor analysis of a correlation matrix these eigenvalues tell how much of the variance each factor accounts for.

It is well known that W is factorable even when it is not symmetric. If W is asymmetric it is possible to factor it into a row and a column factor whose product is a least squared error estimate of W (Wright and Evitts, 1961). The row factor is an eigenvector of WW' and the column factor is an eigenvector of W'W. W is assumed to be symmetric in this paper because then there is the necessary convergence between the three approaches. If W were not symmetric the weighting techniques to be described next may have no real number solution at all.

THE CONVERGENCE OF AN INFINITE SEQUENCE

A simple measure of poularity is just the number of friends each person has. It might be desired to create second order indices of popularity by weighting each person's choice of others by the number of choices he receives. But there is no reason to stop at this point. If the second order indices are better than the first order indices (the number of friends), then third order indices in which each person's choice of others is weighted by his second order measure of popularity should be even better.

Let S_0 be a column vector of ones. Then S_1, the first order popularity measure, is just $S_1 = WS_0$. The second order measure is $S_2 = WS_1 = W^2S_0$. The mth order measure is $S_m = W^mS_0$.

These popularity measures tend to become infinitely large. However, there is a small modification that does allow S_m to converge to a set of nonzero popularity scores as m approaches infinity. Let λ_1 be the highest eigenvalue of W. At each stage divide the result by the largest eigenvalue. $S_m = WS_{m-1}/\lambda_1 = W^mS_0/\lambda_1^m$. S_m converges to an eigenvector of λ_1 (Appendix B). Thus, the solution is almost identical to the factor analytic solution. The final popularity scores are described by an eigenvector of the largest eigenvalue. This is equivalent to the most powerful factor in a factor analysis, if the eigenvector is standardized so that its length is the eigenvalue. However, if there is more than one clique the limit of the S_m scores will only give the popularity scores for the clique with the largest eigenvalue, which will tend to be the largest clique; entries for all nonmembers of the clique, including both isolates and those who belong to smaller cliques, will be zero. Thus, this approach is a little less general than the other two.

SIMULTANEOUS LINEAR EQUATIONS

Suppose we look at the limit of the previously described process. We want to weight each person's contribution to the popularity of others by his own popularity. We can set up a system of homogeneous linear equations for the unknown popularity scores. For each individual i:

$$S_i = W_{i1}S_1 + W_{i2}S_2 + \ldots W_{in}S_n$$

Each popularity score should be positive or zero.

This system of equations is matrix form is $S = WS$, or $(W-I)S = 0$, where the vector S is the unknown popularity scores. These equations have a nonzero solution only under the unlikely condition that $\det(W-I) = Q$. We would modify the equations above by multiplying the left hand side by a constant λ . This modification does not violate the spirit of the model and it allows a solution to the equations.

$$\lambda S_i = W_{i1}S_1 + \ldots + W_{in}S_n$$

Then the equations in matrix form are $WS = \lambda S$, or $(W-\lambda I)S = 0$. This is the familiar problem of finding eigenvalues and eigenvectors. λ is an eigenvalue and S is an eigenvector.

Although every W will have eigenvalues and eigenvectors, the question remains as to whether there is a desirable solution. λ should be positive and each S_i should be positive. There should be just one solution; we should not be faced with an arbitrary choice of possible eigenvectors. In Appendix C it is demonstrated that each clique will have a positive λ such that all elements of its eigenvector are greater than or equal to zero. Moreover, Appendix A shows that these solutions do not contradict one another.

Thus, all three approaches have the same solution. The solution is always the eigenvector of the largest eigenvalue for each clique. There are minor differences. In the factor analysis approach the eigenvectors are standardized so that their lengths are their eigenvalues. The limiting process gives a solution only for the clique with the largest eigenvalue (which makes no difference if there is just one clique).

The chief difficulty will be with the assumption of symmetry. Some sociometric structures are naturally symmetric or can reasonably be made so, and the technique can be applied directly. For example, being friends might be defined so that if they do not choose each other they are not friends. Talking to one another, spending time together, dating, and other behavioral relations are naturally symmetric. An index of <u>centrality</u> in a communication structure might be desired, as a clue to how fast information will spread from an individual or to how much power an individual has because of his key position. This structure will be symmetric if the communication channels are two-way.

However, other relations will inalterably asymmetric, such as nominations for the most powerful or the most popular in a group. A plausible agrument can be given for making a matrix of nominations symmetric. $(W+W')/2$ is a symmetric matrix and it could be used instead of W. One reason for giving the nominations of the more powerful or popular more weight is that they may have better knowledge of what the status system is. If this is true then it is consistent to let status be determined in part by the statuses of those one nominates as well as the statuses of one's nominators, and the analysis of $(W+W')/2$ will do this.

HUBBELL

The closest approach to that presented here is Katz's (1953) method of calculating status indices and Hubbell's (1965) method of clique identification. Since Hubbell's technique includes Katz's, only Hubbell's will be discussed here.

Hubbell's basic equation for status scores is (1965: 382):

$$S = E + WS$$

S is a column vector of status scores. W is called by Hubbell a "structure" matrix. All values of W are less than or equal to one in absolute value. E is a vector of "exogenous contributions" to status in the system, aspects of a person's status that are not reflected in his nominations by other group members. Each individual's status is the sum of his nominations by others weighted by <u>their</u> status plus contributions of other factors (represented by e_i).

There is no discussion in the article of how the e_i's are to be independently determined. It is suggested that in the absence of external evidence E might as well be a vector of 1's. Thus, it seems fair to say that with sociometric data, in which W is given but not E, E is a mathematical convenience that turns a set of unsolvable homogeneous equations ($(W-I)S=0$) in to a set of solvable nonhomogeneous equations ($(W-I)S=E$), just as λ was introduced as a convenience.

It is easy to see that Hubbell's technique is very similar to the technique proposed in this paper. What are their relative advantages and disadvantages?

1. Hubbell's approach requires the <u>arbitrary</u> assumption of some E vector. This appears to be true of the technique when it is applied to sociometric choice matrices rather than to the economic problems it was designed for. On the other hand, λ in the "factor analysis" approach is not arbitrarily supplied by the researcher; it comes from the data itself.

2. The solution vector S in the factor analysis model has a data reduction interpretation. SS' is a least squared error estimate of W. This is not an interpretation that fits Hubbell's model.

3. Cliques can easily be identified using the factor analysis approach. Hubbell produces a refined measure of the relationship between every pair of group members, the sum of all the direct and indirect paths between them, but cliques must still be grouped together "by hand."

4. Hubbell's technique has a nice interpretation that the factor analysis approach lacks. The elements of S are the sums of all the direct and indirect paths between a given individual and all other individuals.

5. Hubbell's approach permits negative values in the matrix W and, most importantly, it does not require that W be symmetric."

This summary was not meant to favor Hubbell's approach or the factor analysis approach but only to show that they had different advantages and disadvantages.

OVERLAPPING GROUPS AND CENTRALITY

Instead of applying the technique to matrices describing friendships or communication patterns between individuals, a sociologically relevant but not often examined kind of data will be analyzed. This is the pattern of overlap in membership between groups, a relationship between groups rather than between individuals. Data about overlap are often easily available and they can be important.

In a structure of overlapping groups "centrality" is analogous to popularity. The centrality of a group is related to the extent of its overlap with other groups, but it is also affected by the centrality of the groups it overlaps with; a group is more central if it overlaps with central groups than if it overlaps with noncentral groups.

There are many instances in which centrality is important. Consider the following:

1. In studying the spread of rumors and other communications in a system of groups, the centrality of a group in the pattern of overlapping memberships could be a clue about how rapidly information in a given group will be transmitted to the social system as a whole. Information should spread especially fast from groups that overlap with other central groups.

2. In examining the pattern of interlocking directorates among the largest corporations, centrality might be a clue about influence. The directors of central corporations would have more widespread and extensive contacts than the members of less central corporations not only because they belong to more boards but also because these other boards would also contain active directors with many contacts. Centrality might also

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bear a relation to influence in the pattern of voluntary organizations in a small community.

3. In a high school central activities might tend to be those with the highest status in the high school. Central clubs and activities would be those that active students belonged to.

In calculating centrality indices, each overlap of a group with another group is weighted by the centrality of the other group. Thus, the calculation of centrality indices is identical to the calculation of popularity indices. For each isolated set of groups (analogous to a clique) the eigenvector of the largest eigenvalue of W is the desired vector of centrality scores.

A MEASURE OF OVERLAP BETWEEN GROUPS

We need a measure of overlap between groups that is standardized so that the sheer sizes of the two groups does not by itself affect the measure, just as the product moment correlation coefficient is the covariance standardized so that it is unaffected by the variance of the two variables. The ideal measure of overlap would take the value .00 if there were no overlap and 1.00 if there were the maximum possible overlap between the two groups. It would take some standard value, say .05, if membership in the two groups were statistically independent. Standard measures of association are inadequate because they are zero if there is statistical independence and negative if there is no overlap. We want the measure to be positive in the former case (there is some overlap) and zero in the latter.

The following measure r is one among many possible standardized measures of overlap that have the desired properties. Let c be the value r should have when the amount of overlap between groups A and B is the amount expected if membership in the two groups were independent. Let u_{AB} be the expected overlap (if membership in the two groups were independent) divided by the maximum possible overlap, and let t_{AB} be the actual overlap between groups A and B divided by the maximum possible overlap. Define r_{AB} as follows:

$$r_{AB} = \left(t_{AB} \right)^{\frac{\log c}{\log u_{AB}}} \quad \text{when } A \neq B$$

$$r_{AA} = 0$$

It can easily be checked that this measure has the required properties.

DATA

The pattern of overlap among a set of voluntary student activities in a high school was examined. A high school year-book was used as a source of data. The year-book that was used was not selected in any systematic way; it happened to be easily available. The high school was Menlo-Atherton High School in Atherton, California and the year was 1956. This makes it more comparable with Gordon's (1957) study, with which it will be compared, than if a more recent year-book had been used.

No additional data outside the year-books were collected. There is no data on student attitudes toward the groups or about the informal social system in the high school. Without other sources of data as a check the technique can not be validated; it has only face validity. However, it will be shown that the results of the analysis correspond to common sense expectations about high schools.

The eigenvectors and eigenvalues of the matrix of standardized overlaps were computed. All the entries in the eigenvector with the largest eigenvalue were strictly positive. Therefore, all activities were connected directly or indirectly and this eigenvector gives all the centrality scores. The list of activities from the most central to the least central is as follows.

(Table 1 about here)

This list is consistent with common sense expectations, for the most part. The sole entrance criterion of the most central activity was that the student participate in many school activities. Activities with junior and seniors are more central than equivalent activities with freshmen and sophomores. For example, the Junior and Senior Boards are more central than the Freshman and Sophomore Boards, and the varsity sports are all higher than their frosh-soph counterparts. Among the varsity sports basketball, football, and baseball are the highest and they are followed by track, swimming, waterpolo, tennis, and wrestling; the major sports score higher than the minor sports. The student newspaper is more central than the yearbook staff. Clubs that do not represent the school nor perform a function for the school as a whole but merely satisfy members' private interests (model airplane club, astronomy club, archery club, Future Teachers of America, etc.) tend to score low.

One interesting feature is one of the two science-related clubs, the astronomy club, scores very low, while the other, the radio club, scores in the middle. It might be interesting to see if the status of science clubs has changed since 1956.

Centrality seems to be related to status or prestige. In Gordon's (1957) book, The Social Structure of the High School, a summary of

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student evaluations of the prestige of 50 student activities is given. If it is assumed that prestige is related to centrality and that the prestige ranking of activities in Gordon's high school and the high school under study are similar, then the centrality measures and the prestige ratings ought to be positively associated for those activities that were in both high schools. Not only should it be positively correlated. The centrality indices, which are weighted sums of overlaps, should be more highly correlated with prestige than are the unweighted sums, the row sums of the matrix of overlaps, if anything has been gained by weighting.

There were 21 activities that existed in both high schools. The Spearman rank order correlation between prestige (from Gordon's data) and the unweighted amount of overlap (the row totals) is .41. The correlation of the prestige scores with the centrality measures is .49. This is a small difference, but it is in the right direction.[1]

In comparing the two lists, the centrality scores and Gordon's status scores, there seems to be one systematic difference. Sports are higher in Gordon's list, especially the major sports. This could reflect a difference in the high schools. Gordon's high school was in a small Mid-western town and Menlo-Atherton is located in a wealthy San Francisco suburb.

In the systems we are examining it is assumed that there are two different types of groups: membership groups which are real, which meet, etc., and categories, which never function as groups and which the members may not even be aware of. The centrality of a category is the centrality of the groups its members belong to. The centrality scores for categories will be a weighted combination of their overlap with the groups in the system, where each overlap is weighted by the centrality of that group. Students were categorized from the yearbook according to their year in school, sex, and race. It was found that centrality increased from the freshman to the senior years, that females were more central than males, a tendency that increased with the years, that Negroes were very low in centrality, but that their centrality also increased with the years. These common sense findings support the technique.

The finding that females were more central than males, especially in the senior year, is interesting. It could be that high school was more of a high point in the life of females than males. From high school many of the boys would rise in status through attendance at college and through their occupation. For many of the girls their independent life would end soon after high school in marriage,and so females may have directed more energy to their "last chance" for individual accomplishment.

EDUCATIONAL APPLICATIONS

In this paper the technique was used to devise centrality measures for voluntary activities in one high school. Centrality appeared to be related to status. This measure of the status of activities could be used in conjunction with other measures or it has the advantage of being usable even if one has only year-books.

Some interesting possible applications of the model would involve comparing high schools. For example, how has the centrality of the various activities changed with time? A study of which groups have risen and which groups have fallen in centrality-status would be interesting. Have sports risen or fallen in centrality? Have science clubs risen or fallen in centrality in the past decade?

Another approach is to study the relationship between how much an individual participates in "central" organizations and the probability that the student goes to college. The correlation between the centrality of the organizations an individual participates in and whether or not he goes to college could be used as a way of characterizing high schools; some high schools might be more effective "launching pads" than others.

CONCLUSIONS

There are situations in which one wants to count the number of relationships that an individual has: in order to measure popularity, or power, or centrality in a communication system as examples. There are ways of modifying and perhaps improving this simple operation. One approach is to calculate a (column) vector of "interaction potentials" such that SS' is a close approximation to the matrix of relationships; S_iS_j is close to the relationship between individuals i and j. A second approach appears to be quite different. Instead of simply adding up the number of relationships each person has one might wish to weigh these relationships, because a relationship with a more central or popular individual contributes more to ones own popularity or centrality. In the paper two somewhat different approaches toward calculating these weights are described.

In asymmetric matrices "factoring" and "weighting" have different solutions (illustrated by Wright and Evitts, 1961, and Hubbell, 1965, respectively). However, when W is symmetric certain eigenvectors of W are both "factor" solutions and "weighting" solutions. A solution vector S can be interpreted <u>both</u> as a least squared error reduction of the matrix of relationships <u>and</u> as a set of weights for relationships. Moreover, cliques are easily identified.

The pattern of overlaps among a set of groups would seem to be a distinctively sociological object of study. When applied to data on the pattern of overlapping memberships among a set of groups, this technique gives a solution vector that is the best (least squared error) set of measures of the tendencies of the groups to overlap with each other and is also a set of weights so that each group's centrality is affected by the centrality of the groups it overlaps with.

The pattern of overlapping relations among a set of voluntary activities in a high school was examined. It was suggested that centrality would be a convenient measure of the status of these activities in a high school was examined. It was suggested that centrality would be a convenient measure of the status of these activities that could be derived solely from year-books.

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TABLE 1

Centrality Measures

1.	Golden Key Society (honorary club for active students)	.291
2.	California Scholarship Federation (honorary club for students with high GPA	.253
3.	Senior Board (legislature of senior class, elected	.235
4.	Girls Block Society (honorary athletic society	.228
5.	Junior-senior student council (student legislature, elected	.227
6.	"A" student court (tries violators of student conduct rules)	.226
7.	French Club	.213
8.	Cultural Board (planned school entertainment)	.212
9.	Student newspaper staff	.205
10.	Spanish Club	.200
11.	Girls Association (planned events relevant to girls)	.197
12.	Girls Athletic Board (promoted and planned intrmural girls' sports)	.171
13.	Publications Board (supervised student publications)	.171
14.	A Capella (choir)	.169
15.	Board of Welfare (cited violators of conduct rules)	.166
16.	Yearbook staff	.165
17.	Social Board (planned school dances)	.162
18.	Literary magazine staff	.147
19.	Boys Block Society (honorary athletic society)	.143
20.	Choraliers	.139
21.	School Spirit Board (staged rallies	.133
22.	Pom Pom girls	.128
23.	Cub Staff (assistants to newspaper staff)	.115
24.	Junior Board	.114
25.	Forensics (debating club)	.114

26.	Boys Athletic Board (exercised student control over intramural and extramural athletics)	.109
27.	Finance Board (raised money for student activities)	.104
28.	Varsity basketball	.100
29.	Varsity football	.099
30.	Band	.095
31.	Audio visual crew	.084
32.	Radio Club	.079
33.	Varsity baseball	.077
34.	Varsity track	.076
35.	"B" student court	.075
36.	Players Club (planned and acted school plays)	.074
37.	Golf	.073
38.	Varsity swimming	.069
39.	Cheerleaders	.067
40.	Dance band	.067
41.	Varsity water polo	.066
42.	Orchestra	.053
43.	Publicity Board (publicized school events)	.046
44.	Junior Statesmen (attended annual state convention patterned after state government)	.043
45.	"B" basketball	.036
46.	Tennis	.033
47.	Future Teachers	.030
48.	Freshman Board	.030
49.	Frosh-soph water polo	.028
50.	Frosh-soph student council	.027
51.	Archery Club	.026
52.	"B" swimming	.024
53.	Majorettes	.023
54.	Frosh-soph football	.023
55.	Wrestling	.022
56.	Red Cross	.021
57.	Sophomore Board	.020
58.	"C" swimming	.019
59.	"C" track	.018
60.	"B" track	.016
61.	"D" basketball	.015
62.	Frosh-soph basketball	.013
63.	"C" basketball	.010
64.	Astronomy Club	.008
65.	Model Airplane Club	.001

Appendix A

In Appendix C it is shown that there is associated with each clique an eigenvector all of whose elements are greater than or equal to zero. Here we wish to show that if an eigenvector of the matrix W has all positive elements the relationship between any of the individuals with strictly positive entries and any of the individuals with zero entries will be zero, and thus the eigenvectors with all positive or zero elements separate the system into sets of individuals who are not related to each other.

1. We have the following system of equations: $(W-\lambda I)S = 0$, where $W_{ii} = 0$, $W_{ij} \geq 0$ for every i and j.

2. Suppose that a solution exists for which $S_1 > 0$, $S_2 = S_3 = \ldots = S_n = 0$. An examination of the above equations shows that this implies that $W_{21} = W_{31} = \ldots = W_{n1} = 0$. The first group does not overlap with any other group.

3. Suppose that a solution exists for which $S_1 > 0$, $S_2 > 0$, ... $S_k > 0$, $S_{k+1} = S_{k+2} = \ldots = S_n = 0$, where $k \geq 2$.

   The last n-k equations now express n-k relationships among the first k (strictly positive) status scores. All the relationships sum to zero and the k(n-k) coefficients are all the relationships between the first k and the last n-k individuals. If any of the relationships were greater than zero others would have to be negative because every sum is zero and the first k status scores are strictly positive. But no relationships are negative. Therefore all of these relationships W_{ij}, $i \leq k$, $j > k$, are zero.

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Appendix B

We wish to show that a sequence of status scores converges to an eigenvector of the largest eigenvalue of W.

Let λ_1 be the eigenvalue of W that is greatest in absolute magnitude, assume it is of multiplicity one, and that it is strictly greater in absolute value than any other eigenvalue. The qualifications will almost certainly be true for any actual matrix W. Let $\{u_1, u_2, \ldots, u_n\}$ be a set of orthonormal eigenvectors for W, where u_1 is the eigenvector of λ_1. The eigenvalues of W are real and the eigenvectors are orthogonal because W is symmetric. It is shown in Appendix C that λ_1 is positive and that every element of u_1 is greater than or equal to zero (or else every element is less than or equal to zero - it makes no difference).

Let S_0 be a column vector of ones. Because the orthonormal eigenvectors of W are a basis for the vector space, $S_0 = \Sigma C_i u_i$. Multiplying through by u_1', $C_1 = u_1' S_0 > 0$, because every element of u_1 is greater than or equal to zero but at least one element is nonzero. Define $S_m = WS_{m-1}/\lambda_1$.

$$S_m = W^m S_0 / \lambda_1^m = W^m \Sigma C_i u_i / \lambda_1^m = \Sigma C_i (\lambda_i / \lambda_1)^m u_i$$

$$= C_1 u_1 + \sum_{i \geq 2} C_i (\lambda_i / \lambda_1)^m u_i$$

The last term approaches zero as m increases. Therefore, S_m approaches $C_1 u_1$, which is also an eigenvector of λ_1 because $C_1 \neq 0$.

If the division were by any number smaller than λ_1 in absolute value S_m would diverge. If division were by any greater number S_m would converge to a zero vector. If division were by a number of the same magnitude but of opposite sign, S_m would oscillate.

Appendix C

We want to show that for each clique there exists a positive eigenvalue with an eigenvector all of whose elements are greater than or equal to zero. However, the eigenvalues and the eigenvectors of the matrix W are just the eigenvalues and the eigenvectors of the submatrices accociated with each clique. Therefore, all that will be shown is that the eigenvalue of W that is largestin absolute magnitude is also positive and that it has an eigenvector all of whose elements are greater than or equal to zero. Then it will follow that an eigenvector of the largest (positive) eigenvalue of the submatrix associated with each clique has elements all of which are greater than or equal to zero.

1. Let W be a symmetric matrix all of whose elements are greater than or equal to zero. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be its eigenvalues and let $\{u_1, u_2, \ldots, u_n\}$ be a corresponding set of orthonormal eigenvectors. The eigenvalues of W are real and the eigenvectors are orthogonal because W is symmetric. Assume, as in Appendix B, that λ_1 is greater in absolute value than any other eigenvalue.

2. $$u_1 = \begin{pmatrix} u_{11} \\ u_{21} \\ \cdots \\ \cdots \\ u_{n1} \end{pmatrix}$$ Assume without loss of generality that $u_{11} \neq 0$.

3. $$\text{Let } x = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

   Because the eigenvectors of the symmetric matrix W span the vector space, $\Sigma c_i u_i = x$. Multiplying through by u_1', $c_1 = u_1'x = u_{11} \neq 0$

4. Define x_m as the first column of the mth power of the matrix W. Because every element of W is positive or zero, it is clear that every element of W^m is also positive or zero, and hence every element of x_m is positive or zero.

5. $$x_m = W^m x = W^m \Sigma c_i u_i = \Sigma c_i W^m u_i = \Sigma c_i \lambda_i^m u_i$$

$$= c_1 \lambda_1^m u_1 + \sum_{i\geq 2} c_i \lambda_i^m u_i = u_{11}\lambda_1^m u_1 + \sum_{i\geq 2} c_i \lambda_i^m u_i$$

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6. We know that $\lambda_1 \neq 0$, $u_{11} \neq 0$. Therefore,

$$x_m / (u_{11} \lambda_1^m) = u_1 + (1/u_{11}) \sum_{i \geq 2} c_i (\lambda_i / \lambda_1)^m u_i$$

As m approaches infinity, the last term approaches zero because λ_1 is greater in absolute value than any other eigenvector Therefore, $x_m/(u_{11} \lambda_1^m)$ approaches u_1. All elements of x_m are positive or zero. Therefore, all elements of u_1 are greater than or equal to zero or they are less than or equal to zero, depending on the sign of $u_{11} \cdot \lambda_1^m$. If u_1 is an eigenvector, so is $-u_1$. Thus λ_1 has an eigenvector with all elements greater than or equal to zero.

7. Moreover, $\lambda_1 > 0$, because if λ_1 were negative, $x_m/(u_{11}\lambda_1^m)$ would oscillate from positive to negative values, instead of approaching the nonzero vector u_1.

Appendix D

Characteristics of students in Menlo-Atherton High School in 1956

Year in school

Freshmen	565
Sophomores	454
Juniors	421
Seniors	379

Sex

Male	929
Female	890

Race (identified from yearbook pictures)

Caucasian	1748
Negro	28
Oriental	43

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