DOCUMENT RESUME

ED 045 178	24	PS 003 333

AUTHOR	Carey, Russell L.; Steffe, Leslie P.
TITLE	An Investigation in the Learning of Equivalence and Order Relations by Four- and Five-Year-Old Children.
INSTITUTION	Georgia Univ., Athens. Research and Development Center in Educational Stimulation.
SPONS AGENCY	Office of Education (DHEW), Washington, D.C. Cooperative Research Program.
REPORT NO	RP-17
BUREAU NO	BR-5-0250
PUB DATE	Dec 68
CONTRACT	OEC-6-10-061
NOTE	212p.
EDRS PRICE	EDRS Price MF-$1.00 HC-$10.70
DESCRIPTORS	Age Differences, Cognitive Development, *Conservation (Concept), Correlation, Curriculum Research, *Mathematical Concepts, Mathematical Vocabulary, *Mathematics Education, Measurement Instruments, *Preschool Children, Statistical Analysis, Student Characteristics, Tables (Data), Training Techniques
IDENTIFIERS	Piaget

ABSTRACT

This study is one of a series which attempts to arrive at generalizations about the learning of mathematics and the use of its terminology in the context of mathematical structure by young children. The first half of the document describes an experimental training program designed to integrate mathematical concepts of metric space, arc length, and transformations with Piagetan notions of conservation, transitivity, and symmetry. Subjects were 20 four-year-old and 34 five-year-old children who participated in three units of small group instruction. They were pretested and posttested individually. Seventy-four tables and nine diagrams are interspersed within the text to statistically support points as they are made. Results, conclusions, discussion, and implications are given and indicate that generally there are slight significant relationships between the psychological and mathematical constructs after training. The second half of the document contains the bibliography and three appendixes. Appendix I lists student characteristics in tabular form; Appendix II gives the details of the three instructional units; and Appendix III lists directions and materials for making the four measuring instruments used in the pre- and posttests. (WY)

ED0 45178

PA-24
BR-5-025

Research Paper No. 17

AN INVESTIGATION IN THE LEARNING OF EQUIVALENCE AND ORDER RELATIONS BY FOUR- AND FIVE-YEAR-OLD CHILDREN

by
Russell L. Carey and Leslie P. Steffe
University of Georgia
and
R & D Center in Educational Stimulation

December, 1968

Research and Development Center in Educational Stimulation
University of Georgia
Athens, Georgia

The research reported in this publication was performed as part of the activities of the Research and Development Center in Educational Stimulation, University of Georgia, pursuant to a contract with the United States Department of Health, Education and Welfare, Office of Education under Provisions of the Cooperative Research Program.

Center No. 5-0250 Contract No. OE 6-10-061

PS 003333

Foreword

The present study is one of a series of important studies which attempt to identify generalizations about the learning of mathematics. It is, however, somewhat unique in that it attempts to arrive at some generalizations about the learning of mathematics and the use of its terminology in the context of mathematical structure. As such, the opportunity to comment on the general effort now being made in this all-important field is a privilege.

The authors of this study recognize how difficult it is to arrive at "hard core" generalizations when studying four and five-year-olds. One is never sure whether the difficulty the child faces is one of semantics or one of not possessing a concept which the task is trying to establish. But this difficulty is not one faced exclusively by the child. Experimenters also have this difficulty. Witness the comment made by the authors as regards the work of Piaget and Smedslund. "When Smedslund and Piaget say that conservation of length is a necessary condition for transitivity, of what type of conservation of length do they write?" (p. 34). Experimenters have not been too careful to define the terms which they are studying in terms of an operational definition. A few examples will suffice to make the point.

In his classical studies Piaget speaks of the conservation of number. But is he really dealing with number or is he studying a relation? When studying two sets of objects A and B, the question, "Does A have as many objects as B?" or "Does A have more objects that B?" are questions about the relations "as many as" and "more than." They are not questions about number at all. If Piaget lines up two rows of objects which are perceptually, and obviously, in one-to-one correspondence and asks, "Are there as many a's as b's?", he is not asking about number at all. The question really means, "Are the elements of a in one-to-one correspondence with the elements of b?". The child can determine the answer without counting. Then, when one of the sets of objects is distorted so as to make the perceptual one-to-one correspondence less evident, the child may lose sight of the one-to-one correspondence and by shifting his perceptual focus on something other than the one-to-one correspondence, arrive at a "wrong" response.

From the point of view of mathematics, and possibly from the point of view of learning, one-to-one correspondence is prior to the idea of numerousness. From the point of view of mathematics, the relations "as many as," "more than" and "fewer than" are basic to the development of number. It is on the basis of these relations that the cardinal numbers can be ordered, thereby arriving at the counting skills. The fact that studies, such as this one, must be made in terms of simple mathematical structures has been made by Piaget himself.

> Learning is possible in the case of these logical-mathematical structures, but on one condition. . .that is, the structure you want to teach can be supported by simple, more elementary, logical-mathematical structures (19, p.16).

One-to-one correspondence is simple, more elementary, and basic to the mathematical concept called "number."

In the same sense, the phrases "as long as", "longer than" and "shorter than" refer to relations between two objects and do not require the higher logical-mathematical structure of measurement. For children the phrase "as long as" is operationally defined by selecting two sticks and having the child place them end-to-end so that the endpoints coincide. The phrase "the same length as" then means "we can make the endpoints coincide." The phrase "longer than" is defined by showing that the endpoints of sticks A and B coincide at the left (right) but that the endpoint of A extends beyond that of B on the right (left). We then say that A is longer than B. It is now possible to study these relations, but experimenters must be careful not to confuse the study of these relations with that of measurement. Measurement has something to do with number and these relations are simpler and prior to the study of the application of number to line segments.

Piaget is most certainly right when he says that one mathematical structure must be studied in terms of simpler mathematical structures. However, it has long been known that all words cannot be defined in terms of simpler words. There comes a time when one reaches the simplest word and there are no words left to construct an intelligent definition. The same situation exists in terms of logical-mathematical structures. When ordering mathematical structures in terms of "simpler than" (assuming for the moment that it can be done), there comes a time when you reach the simplest In terms of the present discussion, the simplest structure in terms of mathematics and learning are such relations as "as many as" and "longer than." These must be defined operationally for the child and, once defined, questions about the symetric and transitive properties of the relation can be studied.

The authors of the present study are to be congratulated for recognizing the need for embedding their study in basic mathematical structures. The emphasis in itself is a contribution to the study of learning mathematics.

H. Van Engen
The University of Wisconsin

CONTENTS

LIST OF TABLES

Table		Page

Table | Page

Table | Page

LIST OF DIAGRAMS

CHAPTER I

INTRODUCTION

Different types of curriculum research in mathematics are certainly possible and, moreover, are necessary. At least three types of research, two of which will be mentioned here, have been identified (21, pp. 102-104). Evaluative research may be divided into at least two categories: formative evaluative research and summative evaluative research. Formative evaluative research is to "identify those aspects of a course where revision is necessary . . . . Formative evaluation . . . makes it possible to observe and measure the effectiveness of aspects of the innovations as they are being developed" (21, p. 102)*. Summative evaluation takes place only after the innovation has been completely developed.

Theory-oriented research may be described as research that "attempts to identify generalizations about the learning of mathematics . . ." (21, p. 103). For example, research in which the hypotheses have been formulated as a result of a theory may be called theory-oriented research.

Models have been formulated which may be interpreted as a guide to methods of curriculum research in mathematics (15, p. 21). However, as Rosskopf has pointed out, "Can it be that one researcher's model becomes the straitjacket of another? Isn't it possible that creative

*See bibliography beginning page 136. These references give the serial number in the bibliography, followed by the page number in the source book.

efforts in research might go curtailed because they seem beyond the bounds of the model" (22, p. 116)? Van Engen expresses concern when he says: "It seems to me that for the present we need answers to some pressing questions. Most certainly, we need more answers to specific questions before we construct general theories. . .. Theories are built on facts, and we have too few facts to indulge in serious theory building for the present" (28, p. 114). He goes on to say:

> The variables involved in researching methods of teaching on a broad scale are so many and varied and so many value judgments must enter into method that, for the present, they are hopelessly complex. . . .within broad limits, I doubt it makes much difference whether the commutative law is taught by a team, a teacher in a self-contained classroom, a specialized teacher, discovery methods, good expository teaching, and so on ad infinitum (28, p. 114).

Rosenbloom states that:

> I have a hunch that to a certain extent, Piaget's results are culturally determined; and by giving the child different experiences, by changing his environment, one might change his course of development. The implications of this notion for head start programs and nursery schools are important as they develop formal educational programs.
>
> The implications of Piaget's theories for mathematics education have not yet been realized. Studies by competent researchers involving American children are badly needed. New curricular materials, based on sound psychological evidence should be written (20, p. 49).

The present exploratory study is concerned with the development of an equivalence relation "the same length as" and two order relations "shorter than" and "longer than" in children at the ages four and five years. It is necessary to outline the mathematical and psychological background which gave rise to the study, which is done in the next section.

MATHEMATICAL BACKGROUND

Metric Spaces

Real numbers have associated properties other than algebraic. In particular, real numbers along with a distance function, $\rho (x,y) = |x-y|$, form a metric space, which is a non-empty set of elements M together with a real-valued function ρ defined on M x M so that for all x, y, and z in M:

(1) $\rho (x,y) \geq 0$

(2) $\rho (x,y) = 0$ if and only if $x = y$

(3) $\rho (x,y) = \rho (y,x)$

(4) $\rho (x,y) \leq \rho (x,z) + \rho (z,y)$ (23, p. 109)

The function ρ is called a metric.

It is easy to see that the length of a line segment (a closed interval) is equal to the distance between the endpoints of the segment. However, the length of an arbitary continuous curve defined on an interval [a,b] is not necessarily equal to the distance between the endpoints.

In E^n (n-space), those segments for which the distance between the endpoints is 1 are called unit segments, or in short, units. A segment may be arbitrarily selected to represent a unit segment. Once a selection is made, the length of each segment may be represented as a multiple of the length of the unit segment.

If [p,q] is any arbitrary segment and $P = \{[p_1, p_2], [p_2, p_3], \ldots, [p_{n-1}, p_n]\}$ is a partition of [p, q], then $\sum_{i=1}^{n-1} \rho (p_i, p_{i+1}) = \rho (p,q)$.

Moreover, if p is a regular partition (the lengths of all the subintervals are the same), then ρ (p,q) = (n - 1) [ρ (p_i, p_{i+1})] for an arbitary i.

If [p_e, p_m] is a unit segment, then ρ (p,q) = [n - 1)$\cdot$ k] ρ (p_e, p_m). Selecting the greatest integer r in (n - 1). k, [p, q] can be partitioned into r unit segments and one other segment of length (n - 1). k - r. If r = (n - 1). k, then the original partition is made up of n - 1 unit segments each of which would then be congruent to [p_e, p_m].

In the next section on arc length, the notion of a polygonal path and its length need to be clarified. A polygonal path is formed by consecutive line segments which join p_0 to p_1; p_1 to p_2; p_2 to p_3; etc., where p_0, p_1, p_2, p_3, . . . are points in E^n. The length of the polygonal path is just $\sum_i \rho$ (p_i, p_{i+1}).

Arc Length

The term "curve" has more than one meaning in mathematics. The definition adopted in this study is given as follows:

> A curve γ in n-space is a mapping or transformation from E^1 into E^n (10, p. 251).

A curve, then, is a mapping. A point p in E^n is said to <u>lie on</u> a given curve if there is a t in E^1 for which p = γ (t). The set of all points which lie on γ is called the <u>trace</u> of the curve (10, p. 251).

If a curve is defined on an interval [a, b] of E^1, then γ (a) and γ (b) are called the <u>endpoints</u> of the curve. If γ (a) = γ (b), the curve is closed. If p lies on the curve γ, then p is called a multiple point if there is more than one t in [a, b] for which $\gamma(t) = p$. A simple curve has no multiple points. A simple closed curve is a closed curve in which the only multiple points are two endpoints.

The length (or measure) of a continuous curve defined on an interval [a, b] of E^1 is defined as the least upper bound of the lengths of all possible inscribed polygons (10, p. 258). If $\{[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n]\}$ forms a partition of [a, b] and if $p_j = \gamma(t_j)$, the consecutive line segments joining $p_0, p_1, p_2, \ldots, p_n$ form a polygonal path C which is said to be inscribed in γ.

If the lengths of two curves are finite, then the two lengths may be compared by using the trichotomy law of real numbers; that is, if a and b are the lengths of two curves A and B respectively, then exactly one of the following holds: $a < b$; $a = b$, or $a > b$. If $a < b$, then A is shorter than B; if $a = b$, then A is the same length as B; and if $a > b$, then A is longer than B. It is easy to see that "the same length as" is an equivalence relation and "longer than" and "shorter than" are order relations.

Measurement

Measurement is a process whereby a number is assigned to some mathematical object. Consider, for example, a segment, or more concretely, a physical representation of a segment. Once a unit segment is determined, a number (an integer) of unit segments which corresponds to the greatest integer r discussed in the section on Metric Spaces may be determined. The length of the segment is, of course, a member of $[r, r + 1]$ and depends on the completeness property of the real numbers.

The foci of this study are determined by the manner in which the integer r is selected. Certainly, a regular partition of a sub-segment of the original segment is characterized by the unit segment, where the intervals of the partition are congruent to the unit segment. Herein lies the crux of the matter. Given two segments, one to be considered as a unit segment and one to be considered as a segment to be measured, a unit iteration process may be performed (mentally or overtly) and terminated when the integer r is determined. The relations "the same length as," "longer than" and "shorter than" are implicitly involved in the process described. These relations may be defined in terms of "equals" or "greater than" for real numbers, as already has been done in this study. The relations are involved in the measuring process at least in the following manner. When the unit segment A is applied to the segment B, one of the following is determined; A is the same length as B, A is shorter than B, or A is longer than B. If A is shorter than B, then A is the same length as a subsegment of B (say B'). B' is the same length as A and B is longer than A. If r times the length of A is the same length as B, then r subsegments of B; B_1, B_2, . . . B_r, have been determined each of which is the same length as A. By virtue of the transitive property of "the same length as", then B_1 is the same length as B_2, etc.

Transformations

In this section, the notions of an isometry and isometric metric spaces will be discussed. Let $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ be metric spaces (ρ and σ are the metrics), and let F be a one-to-one continuous function from X onto Y such that F preserves distances, and such that F^{-1} (the inverse of F) is continuous. F is called an isometry between the two spaces, and the spaces are called isometric. An example of an isometry of the plane is the transformation described by $(x, y) \to (x + a, y + b)$. A transformation which does not leave distance invariant is given by $(x, y) \to (ax, by)$ where $a, b > 1$. A more general example of a transformation of the plane which preserves distance is given by $(x, y) \to (x \cos \theta - y \sin \theta + a, x \sin \theta + y \cos \theta + b)$. Such a transformation preserves not only the measure of geometrical figures, but also their shape, i.e., the figure and its transform are congruent. When this transformation is restricted to particular geometrical figures such as segments, the relations, "the same length as", "longer than" or "shorter than" (as the case may be) are conserved between given segments and their transforms, between one of two congruent segments and the transformation of the other, or between the transformations of both. If the transformation is restricted to finite point sets, relations such as "as many as," "more than" or "fewer than" are conserved in an analogous manner. However, there are transformations which are not necessarily distance preserving which also leave such relations invariant (such as $(x, y) \to (2x,y)$). There are also transformations which are not distance preserving, but which leave such relations as "same length as", etc., or "same area as," etc., invariant. The only transformations that will be considered in this study are transformations that conserve the relations "the same length as", "longer than" and "shorter than".

PSYCHOLOGICAL BACKGROUND

Conservation of Length

Piaget emphasizes the importance of conservation of length measurment. He states that: "Underlying all measurement is the notion that an object remains constant in size throughout any change in position" (18, p. 90). One of Piaget's classical experiments of conservation of length is described as follows: "We present the child with two sticks of the same length. He satisfies himself that they are the same length by comparing them. The smallest children will tell you that when you push one stick out beyond the other one, it becomes longer. It's longer because it goes out farther than the other one" (19, p. 27).

In such experiments, it is entirely possible that a semantic difficulty arises. That is, do "longer" and "farther out" mean the same thing to the child? If so, then "longer" may become associated with distance traveled or ordinal position. He may know, in fact, that the two sticks are indeed of equal length. Piaget gives evidence that the above response type is not a linguistic confusion but instead a real situation of non-conservation. He states:

> In other words, this study of anticipatory imagery enables us to confirm our interpretation that we have a real situation

> of non-conservation, that is, an inferential or deductive insufficiency, and not simply a semantic confusion (19, p. 29).

According to Piaget, children pass through stages with regard to the development of the ability to conserve length. The following five transitional steps from non-conservation to conservation of length are noted by Piaget.

> First: Regulations of a purely perceptual kind. In origin, these are independent of the judgment as such, but they influence judgment in the direction of equality. Thus, Pel is less convinced of the inequality of the test objects when their absolute size is greater which makes a stagger of 1 or 2 cm. relatively less: hence, Pel judges 5 cm. staggered sticks as unequal while recognizing that pairs of 7 cm. and 10 cm. sticks are equal.
>
> Second: The second step may be called intuitive regulation and relates rather to the decentering of attention than to that of perception. Thus, Per and Lep notice that when one of the strips is longer on the right, the other is longer on the left. Their response marks the beginning of a relationship between the two paired extremities (i.e., the four extremities taken in pairs), as against an intuitive centering on the leading extremity.
>
> Third: Somewhat more advanced is the intuitive regulation shown by Mil, who recognizes the conservation of length when both sticks are moved simultaneously in opposite directions, but fails to recognize it when the change of position is applied to only one.
>
> Fourth: A number of subjects, like Froh, come nearer still to operational reasoning. They note that the sticks are equal when arranged in exact alignment, and then, because they are not sure whether that equality is maintained when one of the sticks is staggered, they realign it to convince themselves of that conservation! Their action testifies to the genuineness of the uncertainty felt by children as to the conservation of length when objects undergo a change in position. However, their method of varification does not imply operational reversibility, and is no more than an empirical or intuitive return to the starting point. Reversibility is foreshadowed but not yet complete, as is proved by the responses of Froh who uses the method to convince himself of the equality of the sticks,

but immediately denies that equality when he sees one stick lying at an agle of 45^{o} from the mid-point of the other.

Fifth: Finally, conservation is discovered. After first believing that the length of objects really changes, a number of subjects are finally persuaded by mutually compensating contradictory intuitions (as elicited by variations in the size and relative position of the test objects) to dissociate the reconstruction of reality from perceptual or intuitive appearance. Thus, Lob ends by saying: "It looks longer, but it's the same thing after all". and Led: "They're the same, but you pulled it," suggests the logical and necessary character of conservation which belongs to stage III. For the rest, the acceptance of conservation remains somewhat tentative.

Why do these subjects not regard conservation as logically necessary? They are willing to compare differences in length resulting from a given set of positional changes with those produced by others. It is these comparisons which lead to increasing compensation of an intuitive nature. In the end they guess at conservation, without basing this notion on an exact composition of the spaces left empty by the change in position of the test objects and the corresponding spaces which are occupied: they do not realize that in every change of position these two factors are mutually compensating. Their thought does not yet embrace a system of fixed sites and deals only with the transformation of objects. That limitation precludes the operational conservation of length. It does, however, admit of an intuitive conservation of relations of equality, which anticipates operation and may even come near to it (18, pp. 100, 101).

It should be noted that an operation, for Piaget, "is an interiorized action . . . . it is a reversible action; . . . it can take place in both directions, . . . joining or separating . . . . it is a particular type of action which makes up logical structures" (19, p. 8). He also emphasizes that "concrete operations . . . operate on objects, and not yet on verbally expressed hypotheses" (19, p. 9).

Piaget concedes that experience is a basic factor (but not sufficient) in the development of cognitive structures. The two reasons he cites for this view are that (1) conservation of substance is a logical necessity and not a function of experiences and (2) physical experiences and logical-mathematical experiences are psychologically very different. A physical experience is merely knowledge gained about the objects by an abstraction from the objects. Logical-mathematical experiences are much more profound and are characterized by knowledge gained from action effected upon the objects, which is quite different from a physical experience (19, pp. 11-12).

Piaget goes on to say that:

> . . . coordination of actions before the stage of operations needs to be supported by concrete material. Later, this coordination of actions leads to the logical mathematical structures (19, p. 12).
>
> . . . learning is possible in the case of these logical-mathematical structures, but on one condition--that is, that the structure you want to teach to the subjects can be supported by simple, more elementary, logical-mathematical structures (19, p. 16).
>
> . . . learning of structures seems to obey the same laws as the natural development of these structures. In other words, learning is subordinated to development and not vice-versa . . . (19, p. 17).

The question arises then, that if a child is led to compare the lengths of two curves, is the experience logical-mathematical experience?

> . . . the child can receive valuable information via language or via education directed by an adult only if he is in a state where he can understand this information. That is, to receive the information, he must have a structure which enables him to assimilate this information. This is why you cannot teach higher mathematics to a five-year-old (19, p. 13).

Duckworth interprets Piaget's stand on teaching as follows:

> Good pedagogy must involve presenting the child with situations in which he himself experiments, in the broadest sense of that

> term . . . trying things out to see what happens, manipulating things, manipulating symbols, posing questions, and seeking his own answer reconciling what he finds one time with what he finds at another, comparing his findings with those of other children (19, p. 2).

Piaget further elaborates the role of experience in intellectual development.

> Experience is always necessary for intellectual development . . . . But I fear that we may fall into the illusion that being submitted to an experience (a demonstration) is sufficient for a subject to disengage the structure involved. But more than this is required. The subject must be active, must transform things, and find the structure of his own actions on the objects.
>
> When I say "active", I mean it in two senses. One is acting on material things. But the other means doing things in social collaboration, in a group effort. This leads to a critical frame of mind, where children must communicate with each other. This is an essential factor in intellectual development. Co-operation is indeed co-operation (19, p. 4).

The process of equilibration also sheds light on the above question, but in a brighter context. "In the act of knowing, the subject is active, and consequently, faced with external disturbance, he will react in order to compensate and consequently, he will tend toward equilibrium" (19, p. 4). An example of a process of equilibration is given in the development of the idea of conservation by a child in the case of a physical transformation of rolling a ball of plasticene into a sausage. The most probable focusing is on one dimension. Only if the child notices both dimensions and oscillates between the two, will he come to see that they are related. "You will . . . find a process of . . . equilibration . . . which seems to me the fundamental factor in the acquisition of logical mathematical knowledge" (19, p. 14).

Duckworth states:

Piaget sees the process of equilibration as a process of balance between assimilation and accommodation in a biological sense. An individual assimilates the world--which comes down to saying he sees it in his own way. But sometimes something presents itself in such a way that he cannot assimilate it into his view of things, so he must change his view--he must accomodate if he wants to incorporate this new item (18, p. 4).

Adler mentions that:

Piaget's critics have often complained that his emphasis on inward maturation and growth leaves no room for the effects of a stimulating environment. This view involves a partial misunderstanding of his theory, and the difficulty could be resolved easily by the realization that Piaget assumes continuous interaction between the child and his environment (1, p. 300).

The real cause of the failure of formal education must be sought primarily in the fact that it begins with language (accompanied by illustrations and fictitious or narrated action) instead of beginning with real practical action. The preparation for subsequent mathematical teaching should begin in the home by a series of manipulations involving logical and numerical relationships, the idea of length, area, etc.; and this kind of practical activity should be developed and amplified in a systematic fashion throughout the whole course of primary education . . . (1, p. 301).

Piaget also takes a stand on the role of language in education.

Words are probably not a short-cut to a better understanding . . . . The level of understanding seems to modify the language that is used, rather than vice-verse . . . . Mainly, language serves to translate what is already understood; or else language may even present a danger if it is used to introduce an idea which is not yet accessible (19, p. 5).

I believe that logic is not a derivative of language. The source of logic is much more profound. It is the total coordination of actions, actions of joining things together or ordering things, etc. This is what logical-mathematical experience is. It is an experience which is necessary before there can be operations. Once the operations have been attained this experience is no longer needed and the coordination of actions can take place by themselves in the form of deduction and construction for abstract structures (19, p. 13).

Relative to Piaget's views on the role of language in education, Lovell states:

> I know of no evidence to refute the view of Piaget that, although language aids the formation and stabilization of a system of communications constituted by concepts, it is not in itself sufficient to bring about the mental thought and which make possible the elaboration of mathematical concepts (16, p. 212).

Lovell has further stated:

> For Piaget, mathematical concepts cannot be brought about by using the symbols of mathematics, rote learning, or verbalizations. They are arrived at my manipulating things; not the things themselves but from an awareness of the significance of actions performed with them (16, p. 216).

Wohlwill (29) has shown that mastery of the verbal labels "one," "two," and so on, plays an important role in helping the child pass from a stage where number is responded to wholly on a perceptual basis to a stage where number is responded to conceptually in a sense that four green circles can be matched with four red triangles. Levell comments, "for the schools, this suggests the need for active concrete experience and the stimulation of discussion to go along together" (16, p. 214). Moreover, Bruner, et al. have commented, "Where does the language begin and the manipulation of materials stop? The interplay is continuous" (8, pp. 54-55).

For Piaget, concrete operations are reversible. In the fourth transitional step from non-conservation to conservation (see page 9), he emphasizes the role that reversibility plays in the child's discovering conservation of length. Bruner, et al. state "construction, unconstruction, and reconstruction provide reversibility in overt operations until the child . . . can internalize such operations . . . " (8, p. 52).

Conservation of Length and/or Transitivity of Length

Lovell, et al. have conducted a replication study of conservation of length (relations). In this study, two rods, each 5 cm. in length, were used. The rods were placed in front of the children with their endpoints coinciding. The children were to agree that the rods were of the same length after which one rod was transformed by (a) being pushed about half a centimeter ahead of the other, (b) being placed perpendicular to the other (in such a way that a T was formed), or (c) being placed at an angle to the other but touching it. In each case, the children were asked if the rods were still of the same length. This experiment was repeated using two rods, each 10 cm. in length.

Lovell then divided the children into three Piagetian stages: I and II A of children who judged the stick that was moved to be the longer; II B of children making mixed responses; and III of children who recognized that the sticks were by necessity of the same length. Of 10 five-year-old children, 9 were in the low stage (I and II A) and 1 in stage II B. Of 15 six-year-old children, 11 were in the low stages and 2 were in each of the upper two stages. Of 15 seven-year-old children, 10 were in the low stage, 1 in the middle stage and 4 in the upper stage. Of 15 eight-year-old children, 6 were in the low stage, 1 in the middle stage and 8 in the upper stage. Of 15 nine-year-old children, 3 were in the low stage, 2 in the middle stage and 10 in the upper stage (17).

Smedslund, in a study conducted to determine interrelations of specific acquisitions of ability for concrete reasoning, has observed that of 63 children between the ages of 4 years and 3 months and 7 years, (inclusive), 22 children passed a conservation of length item. Of 38 children between the ages of 4-3 and 6-0 (inclusive) only 7 passed (25).

The results of the above two studies show quite clearly that children below six years of age (and many seven-year-olds) are unable to conserve length (relations). Lovell's subjects were selected from a primary school in England and Smedslund's subjects were selected from an elementary school in Boulder, Colorado.

In the same study as above, Smedslund used Müller-Lyer figures to assess transitivity of length (longer than). In the beginning, the experimenter (E) placed the two sticks, the lengths of which were to be ultimately compared by the use of a third stick, with the ends nearest E coinciding, so that the children could see the remaining two ends. The procedure was as follows:

> (Black sticks placed close together, longer stick to the right, ends nearest experimenter coinciding.)
> (1) Which one of these two is longer? (Black sticks placed 20 in. from each other. M-L figures under the stick, longer stick to the right.)
> (2) Which one of the two looks longer? Don't count these (M-L figures), only the sticks! That's a very easy question! (Distance between sticks 20 in., M-L figures, longer stick to the left, blue stick compared with longer stick, ends toward experimenter coinciding.)
> (3) Which of these two is longer? (Blue stick compared in the same way with shorter stick.)
> (4) Which one of these two is longer?
> (5) Do you remember which was longer, this one (longer stick) or this one (blue stick)?

(6) Do you remember which was longer, this one (shorter stick) or this one (blue stick)? (If the answer to 6 was wrong, both 5 and 6 were repeated.)

Test questions:

(Blue stick removed from the table but held visible in the experimenter's hand)
(7) Which one is the longer of these two (longer and shorter black sticks)?
(8) Why do you think so? (25, pp. 14-15).

In 1-6, the child was corrected if wrong. The conservation of length (longer than) task included the same materials as above, with the exception of the blue stick. The procedure was as follows:

(Sticks placed on M-L figures, longer stick to the right.)
(1) First a very easy question. Which one of these two sticks looks longer? Don't count these (M-L figures), only the sticks! (In two or three cases the subject did not respond according to the M-L illusion. In these cases it was sufficient for the experimenter to ask, with doubt in his voice, "Do you really think that one looks longer?" in order to bring about a reversal of judgment. (Both sticks held upright and close together with lower ends on the table.)
(2) Which one is longer now?
(3) Do you remember which one was longer when they were upright? (If the answer was incorrect, both sticks were held upright again, and the procedure was repeated from Question 2.)

Test questions:

(4) Which one is longer now?
(5) Why do you think so? (25, pp. 15-16).

Two additional items were presented to the children, but using identical materials. The only difference in the items was a left-right orientation of the longer stick.

The first in each pair of test questions was listed as correct or incorrect. The explanations were categorized as adequate, inadequate, or ambiguous. A subject received a pass mark if he gave at least one correct response followed by an adequate explanation.

Of the 160 subjects involved in the study (ages 4-3 to 11-4), 26 passed conservation of length and failed transitivity; 1 failed conservation of length but passed transitivity; 56 failed both and 77 passed both (25, p. 22).

In this study, Smedslund differentiates among the following: (1) percepts (2) goal objects and (3) inference patterns. Percepts are the stimulus situations, as perceived by the subjects; goal objects are what the subject is instructed to attain (length); and the inference patterns are transitivity or conservation (25, p. 26). He goes on to amplify:

> The third aim of this study, namely to investigate the relations between inference patterns, is now seen as the central and theoretically most important one and is intimately connected with the analytic task of finding exact relations. Concrete types of inference patterns, and the unitary nature of the construct requires that these patterns should be exactly related. The search for such relations represents an attempt to determine what has come to be known as construct validity.
>
> In view of the narrow situational scope of the acquisitions of concrete reasoning, exact relations between inference patterns can only be discovered when goal objects and percepts are held rigidly constant. The present test contains only two comparisons approaching this methodological ideal (conservation and transitivity of length, and - + and + -), and a few others where the results were so clear that differences in goal objects and percepts obviously had had little effect (25, p. 27).

One must be extremely cautious, however, about concluding that a child is able to make a logical inference (e.g., conservation of "longer than") by his response to one or two identical items. In a study on the effects of selected experiences of the ability of children to conserve numerousness, Harper et al. observed a large range in correlations (-.06 to .80) between corresponding items of two tests designed to measure identical abilities (14). This observation stresses the importance of basing such inferences discussed above on a substantial number of items, each of which is designed to elicit the same inference pattern, but which varies from every other item on differing dimensions.

Braine has reported a study in which he examined the two relations "longer than" and "shorter than" by using non-verbal techniques. A series of three phases were utilized. In phase 1, the child was taught the relations "longer than" and "shorter than." In phases 2 and 3, Braine assumed his testing technique was sufficient to differentiate those children who were able to use the transitive property from those who could not. A procedural difficulty may be present, however. In the phase 1 training session, a child was trained to select, e.g., the shorter of two uprights by a process of reinforcement. A candy was always placed under the shorter stick. During phase 2 and 3, a child may have been cuing on the measuring stick and the upright which was shorter than the stick. There is no guarantee that this did not happen. Braine reported:

> At the end of the task several Ss were asked how the use of the measuring stick helped them find the candy. All except two said they did not know. Of these two, one answer was, "Because it (the stick) makes it (the upright) smaller." The other answer was, "If the thing (measuring stick) is bigger than there (upright) and the other one the stick's smaller, it's there" (7, p. 15).

In conclusion, Braine stated:

The inference A > B.B > C. $\supset$.A > C . . . , is generally available to children at least two years before the age at which Piaget located its developemnt . . . . The difference between Piaget's experimental procedures and those used here suggests that these factors are the development of skill and interest in the technique of length measurement, and perhaps vocabulary development . . . (7, p. 39).

Of Braine's results, Smedslund said:

Four categories of data support the hypothesis that Braine's Ss did not have genuine transitivity.

First, the results reported in this article indicate that the development of transitivity of length, as measured by the percentage of Ss with transitivity in each age group, starts at zero somewhere between 5 and 6 years and does not reach the 50 per cent level until around 8 years. This clearly contradicts Braine's finding that the 50 per cent level is reached somewhere between 4-2 and 5-5.

Secondly, as mentioned above, Vinh-Bang's data on conservation of length, obtained by means of an objective and standardized procedure, support the present findings against Braine's. Since conservation is a necessary condition for transitivity and since practically no children between 4 and 5 years appear to have conservation, it seems highly unlikely that Braine's subjects had genuine transitivity.

Thirdly, the absence of explanations in Braine's data (with one exception) is consistent with the assumption that there was no genuine transitivity. However, this argument is not in itself compelling, since transitivity need not necessarily be accompanied by a verbalization.

Finally, the reported data on the use of a simple nontransitive hypotheses "A > B means candy by A" definitely indicate that this hypotheses is readily adopted by nearly all subjects, under conditions similar to Braine's. Although no data are available on other nontransitive hypotheses such as "A > B and C < B means candy by A" and "C < B means candy by A", the reported findings on one nontransitive hypothesis are enough to strengthen further the suspicion that the occurrence of such hypotheses can account for Braine's findings.

Although none of the preceding considerations are absolutely compelling they justify the following conclusions:

(1) Braine has failed to eliminate important variables (non-transitive hypothesis) which are not involved in the definition of the processes he sets out to investigate. Consequestly, his findings are equivocal.

(2) Theoretical considerations and available data strongly suggest that what Braine observed was not transitivity of length (26, pp. 404-405).

In reply to Smedslund's criticism, Braine stated:

In Smedslund's experiment, S repeatedly had to judge which was the longer of two sticks. The sticks were laid on top of V-shaped figures of black cardboard, so as to induce a fairly strong Muller-Lyer illusion. In the critical trials, the difference in length between the sticks was quite small, and the Muller-Lyer arms were set so that the shorter stic'. appeared longer. A measuring piece of intermediate length was carefully placed next to one of the sticks and then moved and placed next to the other stick. During this measuring, S and E reached a consensus for each stick as to whether it was longer or shorter than the measuring piece. After the measurement S was asked, "Which one of these two sticks is the longer one?" He could only arrive at a correct judgment through an inference of the form A > B.B > C, therefore, A > C (where A and C are the sticks and B is the measuring piece). It was found that children did not respond correctly on critical trials until about 8 years of age.

In the above experiment no steps are taken to ensure that the questions "Which is longer?" is construed not as "Which one looks longer?" but as "Which one is really longer?" Since the procedure denies the child any information as to the correctness of his interpretation of the question-all responses are approved with a smiling "Mmm" . . . the child is given no cue as to whether he should judge the apparent or real size of the sticks when these conflict. Actually, the child's or[illegible] guide to interpreting the question comes from the so[illegible] ditions of the experiment: a social astute 8-year-[illegible] well reason that E would not bother to measure the [illegible]n-less he wanted a judgment of their real length. It [illegible] that the conclusion that a majority of children und[illegible] of age do not understand the transitivity of length crucially dependent on the empirical truth of one assumption: that all children who understand the difference between real and phenomenal size will spontaneously construe the question "Which is longer", as a question about real rather than apparent size (6, pp. 800-801).

Braine, in the same study stated further:

The conclusion from the original experiment is therefore reaffirmed: that the relation, "longer than" becomes transitive for children at least two years earlier than the age found with classical Piagetian experimental techniques. The evidence from this and the other experiments cited indicates that order discrimination (1), conservation of size (2), and a grasp of transitivity of length can all be elicited in a majority of children by about 5 years of age (6, p. 807).

Effects of Training on Conservation Tasks

Several researchers have studied the effects of selected experiences on the ability of young children to perform certain Piagetian tasks. A noteworthy study is one performed by Beilen et al. (3). Two basic problems were of interest in the study. (1) Does a particular logical development result in the simultaneous ability to solve related problems in area and length measurement? (2) Are there limits to the acquisition of measurement operations associated with age when a deliberate effort is made to foster the acquisition through instruction (3, p. 608)?

The general intent of the instruction was to demonstrate and explain measurement by superposition and unit iteration methods as well as conservation of length and area. The aim was to generate, on the basis of specific examples, generalizations . . . necessary to measurement (i.e., conservation, transitivity, etc.).

The general technique was to ask questions of the children which, as much as possible, would elicit spontaneously the appropriate answers leading to measurement concepts which could then be generalized to a variety of specific physical problems. E drew the relevant generalization from the children's answers and from his own demonstrations (3, p. 612).

For length measurement, first graders improve considerably by virtue of instruction but the gains are also great for the noninstructed group. The extent of the noninstructed group's improvement leads to the conclusion that the testing itself facilitates learning. In area measurement, the gains are mostly to the "transitional" level; length measurement improvement is mostly to the "operational" level. Third graders, as noted, if not instructed show some loss; the instructed, however, show considerable improvement.

From these data we would conclude that first graders profit primarily from the task training and a little, though not much, from group instruction. Third graders, on the other hand, profit appreciably more from measurement concept instruction in that they are able to put to use the generalizations taught them. We reiterate that, in spite of training in the task as well as measurement instruction, no child is able to achieve operational area measurement in the first grade (3, p. 615).

This lends support to the view that the child's level of development places a limit on what he may acquire by virtue of experience or training at a particular time . . . . We believe, however, the present data warrant the view that from training some gains may be made in the direction of measurement acquisition at an early age (to "transitional measurement") even though full measurement is not possible. Whether a child will acquire operational measurement more adequately later having been exposed to early training is an open question. The notion that children with higher IQs may profit from such training to a greater extent than children with lower IQs is barely suggested by the data (3, p. 618).

Beilen has conducted a later study concerned with perceptual cognitive conflict of judgment in children in kindergarten through fourth grade about equal and unequal areas (5). His visual pattern board allowed the child to be confronted simultaneously with two-dimensional square regions. The child was asked to make judgments relative to the area of the two regions. Three types of comparison were presented:

(1) equality (congruent regions), (2) inequality and (3) quasi-conservation (equal areas but non-congruent regions). In the discussion of the results Beilen stated:

> We have assumed that the bases for these equality and inequality judgments were to be largely perceptual. It is evident from the data that the ability to make these perceptual discriminations and judgments is substantially available at the kindergarten age . . . (5, p. 222).
>
> . . . many children seem to have difficulty with the question "Which picture covers 'less' (or 'littler') space?" in contrast to the question "Which picture covers 'more' space?" This, on first examination may appear as a labeling difficulty . . . . What was evident, however, from an examination of the Ss was that many children who failed in the labeling failed in the understanding of the inverse logical relationship between "more" and "less."
>
> The quasi-conservation trials represent an even more difficult task . . . .
>
> The younger child . . . will think that when patterns are non-congruent they are necessarily unequal in area, and when they are congruent they are necessarily equal. In those instances where pattern arrangements are not congruent (but the areas are equal) the younger child will err . . . (5, p. 223).

In the same study, Beilen also found that in the quasi-conservation tasks, most subjects respond on all-or-nothing basis.

Beilen's two studies quoted above support the following contentions: (a) Experience either from group instruction or task training contributes to first graders' acquisition of concepts in length measurement. Task training, however, contributes more than group instruction. (b) The child's level of development may place a limit on concepts he/she is able to acquire. (c) First-grade children have little experience with the terms "less" or "littler," or they do not understand the logical relation between the terms "more" and "less" where the terms refer to regions in two-dimensional space.

Beilen (4) in a study on operational convergence in logical thought development had, as one of his major goals, the possible relations between learning and a unitary character of logical development. He included pretraining tests on number, length, and area conservation, training with four experimental procedures on number and length conservation and post-training transfer tests of number, length, and area conservation.

The correlation between pretests was as follows: number-length .44; length-area .23; number-area .05 (4, p. 333). Fifteen subjects "passed" exactly one of the tests, and four "passed" two or more. The subjects were 170 kindergarten children. The correlation between post-tests was: number-length .72; for number-area and length-area little or no difference in the correlations was reported. Thirty-two children passed one test only and 48 passed two or more. The above study points out the importance of the particular relations involved. From the way Beilen constructed his tests, however, one cannot definitely conclude that children who passed were responding on the basis of a mathematical relation. Twelve trials were given on which a child had to be successful on five of the last six in order to "pass." However, on each trial, a child was rewarded for a correct response. An exemplary item was given by Beilen as follows:

Trial 2A	Trial B
0 0 0	0 0 0
	0
0 0 0	0 0 0
	0
0 0 0	0 0 0
0 0	0
0 0	0

> The S was instructed to choose the row which was "like" the middle one and to respond by pressing a button at the base of either of the response columns. If correct, he heard a buzzer and was given a token. After S responded, E expanded or contracted the stimulus column so that the first and last corks were aligned with the first and last corks of either the shorter or longer (irrelevant cue) response column. No corks were removed or added. All contractions and expansions were made in sight of S. After each change, S was again asked to choose the column that was "like" the middle one, and his correct responses were reinforced in the same manner. It was presumed that the reinforcement would serve to provide S with information as to which of the concepts (length covered vs. number) represented in the array was the one sought in the test.
>
> On half the trials the incorrect (i.e., irrelevant cue) column was shorter than the middle one and on half it was longer. The number combinations changed in each trial (4, pp. 321-322).

It must be pointed out that in the above procedure the child may, by virtue of the reinforcement, learn to select the same column each time, i.e., not to change his response. Whether this actually happened, however, is an open question. If it did happen, then those children who met criterion on the last six items may know very little about the mathematical relation involved.

As noted, Beilen employed four experimental methods of training while trying to induce number and length conservation. The methods were: (1) Non-Verbal Reinforcement, (2) Verbal Orientation Reinforcement,

(3) Verbal Rule Instruction, and (4) Equilibration. The Non-Verbal Reinforcement procedure was a repeat of the non-verbal pretest (with 36 trials) which has already been discussed (see page 25). The VOR procedure included verbalization of the concepts in the instruction (the child was told what he had to figure out). The VRI procedure was an extension of the VOR procedure in that the children were asked to verbalize why they chose a particular column. In the event of an incorrect or inadequate (as determined by E) answer, the principle was explained to the child. The EQ procedure included successive rearrangements of one of two columns of sticks. After each rearrangement, the child was asked, for example, whether the two rows of sticks were of the same length. No reinforcement was given. The VRI group was the only group which showed any improvement over a control group. It has been already pointed out, however, that even the control group received training similar to the NVR group since the training received by the NVR group was merely an increase in trials over the pretest. It does suggest, however, that verbalization in the form of explanation may facilitate the acquisition of conservation of length (as Beilen views it).

DISCUSSION OF MATHEMATICAL AND PSYCHOLOGICAL BACKGROUND

Elkind (11) has categorized Piaget's conservation problem into two categories; conservation of identity and conservation of equivalence.

With regard to conservation problems, he stated that:

> Regardless of the content of the problems, they routinely involved presenting the subjects with a variable (V) and a standard (S) stimulus that are initially equivalant in both the perceptual and quantitative sense. The subject is then asked to make a judgment regarding their quantitative equivalence. Once the judgment is made, the variable stiumlus is subjected to a transformation, $V \rightarrow V'$, which alters the perceptual but not the quantitative equivalence between the variable and standard. After completion of the transformation, the subject is asked to judge the quantitative equivalence between the standard and the transformed variable (11, p. 16).

In the above conceptualization, a judgment of conservation may be relative to conservation of a quantitative relation or relative to the identity of V and V' (11, p. 16).

> It is probably true, none the less, that from the point of view of the subject, the conservation of identity is a necessary condition for the conservation of equivalence (11, p. 17).

From the point of view of conservation expressed above, one is led into many pitfalls when assessing children's judgments about conservation. For example, consider the relation "as many as." This relation is defined as follows: If the elements of set A are in one-to-one correspondence with the elements of set B, then A has as many elements as B, and vice versa, or $A \sim B$, where "$\sim$", means the elements of A and B are in one-to-one correspondence. In a conservation problem involving "$\sim$", if the child is asked to make a "quantitative judgment," one must be assured that the child associates at least a one-to-one correspondence with the phrase "as many as."

Smedslund has given three possible factors which may contribute to a possible misinterpretation of the absence of transitivity. These are: (1) no understanding of instructions, which is viewed as a lack of understanding of the words, (2) failure to perceive the two initial comparisons, and (3) forgetting (26, pp. 391-392). In a test on transitivity, Smedslund attempted to control these three factors. One may take the point of view, however, that even though a child may be able to point to the longer of two sticks, he may be basing his judgment on two endpoints only without regard to the relative position of the remining endpoints. In this case, should one be willing to accept that he "understands" the term "longer," or that he perceives the initial relation? Clearly a comprehension of relational terms, such as "longer than" or "as many as," is a prerequisite to problems in conservation of the relation as well as transitivity. As noted earlier, Piaget takes the point of view that in order to receive information via language, the child must have a structure which allows him to assimilate this information. However, if the child has had no opportunity to experience an equivalence relation, how can he assimilate it? The phrase "the same length as" has a quite different referent than does "as many as." While both are "Equivalence Relations," they still are different relations. Thus, there seems to be no reason to believe that the ability to conserve one of the two relations implies the ability to conserve the other. The role of experience in conservation problems is certainly an open one. Almy has stated:

> It seems clear from current experimentation that the question of just what is involved in the transition from nonconservation to conservation, or from thought that is predominately perceptual and intuitive to thought that is more conceptual and logical, or in Piaget's terms, "operational" is far from settled. Most of the evidence seems to weigh against the possibility that the transition can be accelerated by any short term manipulation. What might be accomplished by more pervasive intervention also remains an open question. But it may be well to bear in mind the fact that the ability to conserve represents only one dimension of the child's developing intellectual power (2, p. 47).

Smedslund in his study on concrete reasoning, observed that 31 of the children failed one of the two conservation problems involving "same as" and "longer than" while 32 failed both and 97 passed both (23, p. 22), which supports the contention that the ability to conserve a particular relation does not imply an ability to conserve another. Moreover, in a conservation problem, the initial relation need not be an equivalence relation. It may be, in fact, an order relation (e.g., "fewer than"). It is well known that an order relation is also transitive.

In his discussion of conservation problems, Elkind noted that it is a quantitative equivalence which is being conserved. If A and B are curves of finite length, then A is the same length as B if and only if $L(A)=L(B)$, where $L(A)$ is a number denoting the length of A, and $L(B)$ is the number denoting the length of B. If $T(B)$ is a transformation of B which is length preserving, then $L(B)=L(T(B))$ implies that A is the same length as $T(B)$. If children cannot associate a number with A and B, then there is no reason to believe that "the same length as" has any quantitative meaning for them. Therefore, under these conditions, there is no reason to expect children to conserve a quantitative equivalence between A and B, i.e., deduce that A and $T(B)$ are of the same length. Piaget et al allude to "qualitative" measuring and transitivity.

> The qualitative measuring of level III A which consists in transitive congruence differs from a true metrical system only in that the latter involves changes of position among the subdivisions of a middle term in a metrical system (one such subdivision is chosen as a unit of measurement and applied to the others), whereas in qualitative measuring, one object in its entirety is applied to another (18, p. 60).

While conservation, and hence qualitative transitivity, are achieved at a mean age of 7 1/2, measurement in its operational form (i.e., with immediate insight and not by trial-anderror) is only achieved at about 8 or 8 1/2. The time lag is particularly interesting because here the techniques of measurement are demonstrated to the subject and not left entirely to his initiative as in Ch. II. It confirms the hypothesis of a difference in kind as between qualitative operations and those which are truly metrical (18, p. 126).

The study of children's sponvanteous behavior in a measuring situation (Ch. II) revealed that the notion of a metric unit is evolved only at level III B and depends on the previous mastery of qualitative operational transitivity (level III A) and on the coordination of changes of position at the level of representation, itself a function of a system of reference (18, p. 128).

It seems necessary, then, to define the relations "longer than" "shorter than" and "the same length as" on a basis that does not assume number. Let A, B, and C be segments. A is the same length as B if and only if, when segments (or their transforms) lie on a line in such a way that two endpoints coincide (left or right), the two remaining endpoints coincide. A is longer than B if and only if, the remaining endpoint of B coincides with a point between the endpoints of A. Also, in this case, B is shorter than A.

The above definitions are acceptable from a mathematical point of view. As noted earlier, the length of a curve is the least upper bound of the lengths of all inscribed polygons. Intuitively then, one could think of the length of a curve as the length of a line segment where, of course, the lengths are identical. It is essential to note that, in the definitions given above, children do not have to possess a concept of length as a number in order to learn the relation. The definitions are given entirely in terms of a line, the endpoints of curves, betweenness for points and coincident points on a line. Piaget, as already noted, has stated:

> Learning is possible in the case of these logical-mathematical structures, but on one condition . . . that is, the structure you want to teach can be supported by simple, more elementary, logical-mathematical structures (19, p. 16).

The relations "same length as," "longer than," and "shorter than," and their properties are more elementary and logically precede measurement. The definitions of the relations given above are the results of an attempt, on the part of the investigators, to define the relations in as simple a manner as possible but in such a way that they are still mathematically acceptable.

The relation should not be presented to the preschool child by the use of words alone. It needs to be operationally defined for the child, i.e., defined by physical operations with concrete objects. The physical operations eventually should be performed by the child himself. Central to Piaget's theory is the fact that the child is active; he gains knowledge through his own actions. Note part of the extract from Piaget on page 12:

> But I fear that we may fall into the illusion that being submitted to an experience (a demonstration) is sufficient for a subject to disengage the structure involved. But more than this is required. The subject must be active, must transform things, and find the structure of his own actions on the objects.

Operationally, then, for a child to find a relation between two "rods," say rod A and rod B, he will place the rods side by side and align two of the endpoints. The relative extension of the two remaining endpoints then determines the relations(s) for him. If rod A is in fact shorter than rod B, the child upon placing A by B determines that A is shorter than B. Though an equivalent action, imagined or overt, hild determines that B is longer than A. It is through the coordin n of these actions that logical-mathematical structures evolve for t ild. Piaget has related, as noted earlier, that:

> Coordination of actions before the stage of operations needs to be supported by concrete material. Later, this coordination of actions leads to the logical-mathematical structures (19, p. 12).

The following statements are all logical consequences of the manner in which the relations are defined, both mathematically and operationally.

(a) A shorter (longer) than B is equivalent to B longer (shorter) than A.

(b) A the same length as B implies A is not shorter (longer) than B.

(c) A shorter (longer) than B implies A not longer (shorter) than B.

If A, B, and C are open curves, then the relation "the same length as" possesses the following properties:

(1) Reflexive: $A \sim A$

(2) Symmetric: If $A \sim B$, then $B \sim A$

(3) Transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$

The relations "shorter than" and "longer than" possess the following properties:

(1) Non-Reflexive: $A \alpha A$

(2) Asymmetric: If $A \alpha B$, then $B \not\alpha A$

(3) Transitive: If $A \alpha B$ and $B \alpha C$, then $A \alpha C$

In the case of "the same length as," the reflexive property can be deduced from the symmetric and transitive properties (note: for each curve A, it is assumed there is a curve B (or which $A \sim B$). If "$\sim$" denotes "the same length as"; $A \sim B$ and $B \sim A$ implies $A \sim A$. One must note that $A \sim A$ is a logical deduction. If "α" denotes "shorter than" or "longer than," the asymmetrical and transitive properties may be verified, i.e., $A \alpha B$ implies $B \not\alpha A$ and $A \alpha B$ and $B \alpha C$ implies $A \alpha C$. If it were true that $A \alpha A$, then by the asymmetric property $A \not\alpha A$, which is a logical contradiction. Hence, "α" is nonreflexive. The nonreflexitivity of "α" is obtained as a logical consequence of the asymmetric property.

When Smedslund and Piaget say that conservation of length is a necessary condition for transitivity, of what type of conservation of length do they write? Elkind seems to think (at least in the case of Piaget) it is conservation of identity. He states:

> It is thus clear that Piaget's discussion of conservation is primarily aimed at explaining the conservation of identity and not the conservation of equivalence (11, p. 19).

It is feasible to view conservation of identity as a test of the reflexive or nonreflexive properties given above. On a logical basis, there is no reason to expect that this type of conservation is a necessary condition for the transitivity property since in the case of "the same length as," the reflexive property is a consequence of transitivity, and in the case of "longer than" and "shorter than," nonreflexivity does not imply transitivity. In fact, nonreflexivity is a logical consequence of the asymmetric property.

If a child establishes a relation between two curves in accordance with the definitions given earlier, then the conservation of the relation may simply be a realization, by the child, that the relation obtains regardless of any length preserving transformation on one or both of the curves. In other terms, the child realizes that if the curves are moved back, side by side as in the original state, the ends will still be in the same relative positions. Viewed in this manner, the conservation of the relation is essential for the transitivity property. Take the example of a child who is presented with two fixed-line segments, say, of the same length but not obviously so, and a third segment the same length as the first two and is questioned about the relative lengths of the two fixed segments (which he must not overtly compare). The child will have to realize that, once he has established a relation between the lengths of two segments, that relation obtains regardless of the proximity of the segments. None of the studies reviewed have been concerned explicitly with the symmetric or asymmetric properties, or logical consequences of the relations (such as consequences a, b, and c above). At this point, it is appropriate to define the pupil abilities that were of concern in the study.

Definitions of Pupil Abilities

In the following definitions, A, B, and C represent open curves of finite length. Again, a curve and a physical representation of its trace will not be distinguished. The possibility of "straightening" a curve will be assumed.

(1) Length comparison between two curves:

Given two curves A and B, a child is said to be able to establish a length relation "*" ("longer than", "shorter than", or "same length as") between A and B if and only if he

(a) places each curve on a line in such a way that two endpoints (left or right) coincide,

(b) compares the relative positions of the two remaining endpoints, and then,

(c) on the basis of (a) and (b), deduces that A*B, if in fact it is true that A*B.

(2) Conservation of length of a curve (Reflexive and Non-reflexive Properties):

Given a curve A and a length-preserving transformation T, a child is said to be able to conserve the length of A if and only if he deduces that A and T(A) are of the same length and that A and T(A) are not of different lengths.

(3) Conservation of a length relation between two curves:

A length relation between two curves A and B is conserved by a child if and only if the relation is (a) established by the child and then (b) retained, regardless of any length preserving transformation on one or both of the curves.

(4) Transitive property of length relations:

A child is said to be able to use the transitive property of the relation "*" (where "*" may be replaced by "longer than," "shorter than," or "the same length as") if and only if from establishing that A*B and B*C, he is able to deduce that A*C and no other relation holds between A and C.

(5) Symmetric (asymmetric) property of length relations:

A child is said to be able to use the symmetric (asymmetric) property of "*" (where "*" may be replaced by "the same length as," "longer than,"or "shorter than") if and only if from establishing that A*B, he is able to deduce that B*A, or that B$\not*$A, depending on the replacement for *.

(6) Consequences*

*See page 33 for the statements of the consequences.

(a) A child is said to be able to use consequence (a) if and only if from establishing $A\alpha B$, he is able to deduce that $B*A$ ("α" may be replaced by "longer than and "*" by "shorter than" or vice versa).

(b) A child is said to be able to use consequence (b) if and only if from establishing A the same length as B, he is able to deduce that A is not longer (shorter) than B.

(c) A child is said to be able to use consequence (c) if and only if from establishing A shorter (longer) than B, he is able to deduce that A is not longer (shorter) than B.

In the above abilities, it is important to recognize that no quantitative judgments are necessarily present. Once a child has the ability to conserve a length relation, there seems no reason to believe (other than logical insufficiency) that he/she would not have the ability to use the transitive property, the symmetric or asymmetric properties, or consequences of the relations. It also seems necessary that a child be able to conserve length relations before he/she is able to conserve length (use of the reflexive or nonreflexive properties).

Basic Questions of the Study

The following questions are of basic concern. The children who were the subjects of this study are four and five years of age.

(1) What is the level of performance of children when establishing a length relation between two curves A and B without formal experiences?

(2) What is the effect of selected experiences on the ability of children to establish a length relation between two curves?

(3) (a) If children are able to establish a length relation between two curves, are they able to conserve that relation without formal experiences?

(b) If children are able to establish a length relation between two curves, are they able to conserve the relation involving properties or logical consequences of that relation without formal experiences?

(4) (a) What is the effect of selected experiences on the ability of children to conserve length relations?

(b) What is the effect of selected experiences on the ability of children to conserve length relations involving properties or logical consequences of length relations?

(5) Are children able to conserve length without formal experiences?

(6) What is the effect of selected experiences on the ability of children to conserve length?

(7) If children are able to establish a length relation between two curves, can they use the transitive property with formal experience only in establishing length relations?

(8) If children are able to establish a length relation between two curves, can they use the transitive property with formal experience only in establishing length relations, conserving length relations and conserving length?

(9) Is the ability to conserve length necessary (sufficient) to enable children to use the transitive property with or without having had formal experiences in conserving length?

(10) Is the ability to conserve length relations necessary (sufficient) to enable children to conserve length with or without formal experiences in each?

(11) Is the ability to conserve length relations necessary (sufficient) for children to be able to use the transitive property with or without formal experiences in conserving length relations?

(12) Is the ability to conserve length necessary (sufficient) for children to be able to conserve length relations with or without formal experiences in each?

(13) What is the relationship between certain student characteristics and scores earned on Comparison, Conservation, and Transitivity Test?

CHAPTER II

PROCEDURE

Subjects

The subjects for the present study were 20 four-year-old and 34 five-year-old children in the Lillie E. Suder Elementary School, Jonesboro, Georgia. At the initiation of the study, the range of ages was 47-57 months for the group considered as four-year-olds and 59-69 months for the group considered as five-year-olds (Appendix I). The children were in three self-contained classrooms with some of both age groups in each room. The verbal maturity and intelligence of the subjects, as measured by the Peabody Picture Vocabulary Test and Stanford Binet Intelligence Scale, Form L-M, respectively, are given in Table 1.

TABLE 1

VERBAL MATURITY AND INTELLIGENCE

	Verbal Maturity		Intelligence	
Age Group	Range	Mean	Range	Mean
4	83-119	102.6	98-145	119.6
5	55-120	97.7	81-130	109.1

According to the Hollingshead Two Factor Index of Social Position, the social classes of the subjects range from I (high) to V. The number of subjects in each social class according to age level is given in Table 2.

TABLE 2

SOCIAL CLASSES BY AGE GROUP

Age Group \ Social Class	I	II	III	IV	V
4	3	4	9	4	0
5	3	8	13	6	4

Instructional Units and Measuring Instruments

Three instructional units were designed (Appendix II). Instructional Unit I was designed to develop the ability of children to establish a length relation between two curves. Instructional Unit II was designed to develop the ability of children to conserve length, and Instructional Unit III was designed to develop the ability of children to conserve length relations. From the discussion of psychological background in Chapter I, the following principles were extracted and employed in the design of the units.

(1) Mathematical concepts are not implicit in a set of physical materials. A child gains mathematical knowledge from a set of physical materials by the actions he performs on or with the materials.

(2) Mathematical concepts cannot be acquired by young children who use only the symbols of mathematics or verbalizations. Explanations which accompany the child's actions, however, may facilitate his acquisition of mathematical concepts.

(3) There should be a continuous interplay between the spoken words which symbolize a mathematical concept and a set of actions a child performs while constructing something that makes the concept tangible.

(4) In order to teach a concept, it is necessary to use different assortments of physical materials and different types of activities all of which are related to the development, by the child, of the same concept(s).

(5) The principle of reversibility should be employed (i.e., returning a transformed set of conditions to an original set of conditions).

(6) Situations must be contrived in which the children are led to multiple focusing (i.e., if A is the same length as B, then B is also the same length as A).

(7) Situations must be contrived which involve more than one child, so that the children may interact.

(8) The principle of equilibration should be employed.

Instruments were constructed to measure the pupil abilities outlined in Chapter I (see Appendix III). The first instrument, called the Length Comparison Test, was designed to measure the ability of children to establish a length relation between two curves. Six different material sets were used. Three items, one relative to "longer than," one to "shorter than," and one to "the same length as" were presented to the child in the case of each material set.

The second instrument, called the Conservation of Length Relation Test, consisted of two parts. In the first part of each item, the children were asked to compare the lengths of two curves. Since the materials used in the items differed from those used in either Instructional Unit I or the Length Comparison Test, these eighteen first parts were considered as an Application Test for Instructional Unit I (hereafter called the Length Comparison Application Test). The second part of each item involved the ability of the child to conserve the length relation he/she had just established. Nine of the items also involved the ability of the child to use the asymmetric property of "longer than" and "shorter than" or logical consequences given earlier in Chapter I. The remaining nine items did not demand that the child use any properties or consequences of the relations. These nine items comprised an instrument which will be designated as Conservation of Length Relations: Level I. The remaining nine items comprise an instrument which will be designated as Conservation of Length Relations: Level II.

As noted in Chapter I, Lovell, when assessing whether children were able to "conserve length," asked the children if the rods were still of the same length. Following Lovell, a similar question was used in the case of Conservation of Length Relations: Level I. In the case of Level II, an adaptation of the questioning procedure was employed.

"Yes" was the correct response in the case of nine items of Level I. The response, however, was given in the presence of a perceptual conflict so that, if a child based his response on visual perception, he would give an incorrect response. Item I may be an exception due to the way the red stick is positioned and the relation involved.

"No" was the correct response in the case of the nine items of Level II. Again, the children were required to respond in the presence of a perceptual conflict, so that if a child based his response on visual perception, he would give an incorrect response.

With regard to any relation at each level, the material underwent three distinct transformations, each of which was length preserving. Moreover, different material sets were used.

Smedslund has given twelve methodological rules which may be employed to maximize diagnostic reliability of sets of items such as are given here (25, pp. 4-5). The rules are set out below and several are followed by some comment.

(1) The tasks should not be solvable on the basis of perceptual processes. This can be insured if the initial events are absent at the moment of solution.

A transformation was effected on each material set so that initial conditions could not be physically present at the time of solution.

(2) The tasks should not be solvable on the basis of a readily available hypothesis with a nonlogical structure.

Guessing is always a factor. However, the way in which a performance criterion is obtained guards against solution by guessing. This is discussed further in the section on Experimental Design.

(3) Tasks which can be solved on the basis of specific previous information, which may have been available to some children, should be avoided.

In the treatments, all the children receive the same instructional units.

(4) Items involving practical skills that are likely to be taught in some environments should be avoided.

(5) The possibility of being correct by guessing should be minimized.

All the perceptual cues were biased against the correct answers.

(6) All information available to the subject should be in the form of perceived events. Verbally communicated hypothetical premises should be avoided.

All items involve physical materials.

(7) It must be insured that the subject actually perceives the relevant events. He must be asked to label them as they are presented.

In this study, the child had to establish the relation before he was asked to conserve it.

(8) It must be insured that the subject actually remembers the relevant information. He must be asked to recall this information immediately prior to the moment of solution.

As noted in (7) above, the child had to establish the relation before he was asked to conserve it. The items proceeded as fart as possible. However, after a child established a relation, it was felt that the sequence of events were not conducive to a recall of the relation. The answer would have been given to the child if recall was involved.

(9) Comprehension of the instructions should be ascertained. The subject's usage of terms suggested as difficult should be checked.

Same comment as in (7) above.

(10) The test responses should be so simple that effects of variation in general motor development, verbal fluency, etc., are excluded.

(11) There should be no differential reinforcement during the test. This is important in order to maintain the spontaneity and confidence in all subjects and in order to avoid differential learning effects and highly reliable guessing behavior.

The experimenters were, insofar as possible, instructed to be constant within each child and across childten.

(12) The same type of materials should be used throughout the items, as far as possible, in order to keep constant any

effects of the types of materials.

Just the opposite was considered to be a rule in this study. An inference pattern must at least be operative across materials and transformations to be operative. Stringent performance criteria were established, as will be discussed in the section on Experimental Design.

The third instrument, called the Conservation of Length Test, involved six items of a diversified nature. Three of the items involved the reflexive property of "the same length as" and three items involved the nonreflexive property of "longer than" or "shorter than." Five different material sets were employed. Whenever applicable, Smedslund's twelve methodological rules were employed, with the exception of rule 12.

The fourth instrument, called the Transitivity Test, involved six items of a diversified nature. For three of the items, "yes" was the correct response. For these three items, each of the relations "longer than," "shorter than," and "the same length as" was included.

For three of the items, "no" was the correct response. Each of these items involved transitivity of "the same length as." If "$\sim$" means "the same length as," then from establishing that $A \sim B$ and $B \sim C$, the child must conclude that it is not true that $A \alpha C$. To conclude this, he must know that $A \sim C$.

Again, whenever applicable, of the twelve methodological rules given by Smedslund all except no. 12 were employed. Moreover, it was not possible for the child to use a nontransitive hypothesis to arrive at a correct response, since all of the perceptual cues were biased against a correct response, and a child was not allowed to compare directly the two curves under consideration.

Instructional and Evaluational Sequence

Small group instructional procedures were utilized in each room. An instructional group generally consisted of six children. Teacher Aids were present to guide the remaining children. The pupils were treated as clinical cases on a one-to-one basis for evaluation. All pre- and post-tests for any one unit were identical. Due to absences on evaluation days, every test instrument was not administered to all the subjects. The tests were administered by specially trained evaluators. The instructional and evaluational sequence was:

1. Length Comparison Pretest
2. Length Comparison Instructional Unit (Unit I)
3. Length Comparison Posttest
4. Length Comparison Application Test (First Administration)
5. Conservation of Length Relations Pretest
6. Conservation of Length Pretest
7. Transitivity of Length Relations Test (First Administration)
8. Conservation of Length Instructional Unit (Unit II)
9. Conservation of Length Relations Instructional Unit (Unit III)
10. Length Comparison Application Test (Second Administration)
11. Conservation of Length Relations Posttest
12. Conservation of Length Posttest
13. Transitivity of Length Relations Test (Second Administration)

The Length Comparison Pretest was administered to all the children in the sample in early November, 1967. Then instruction with the material in the Length Comparison Instructional Unit proceeded for a sequence of seven sessions of 20-30 minutes. Due to small group instructional procedures, the total instructional time was more than seven days for any one class. However, any one child received only seven instructional sessions. The Length Comparison Posttest, Length Comparison Application Test, Conservation of Length Relations Pretest, Conservation of Length Pretest, and Transitivity of Length Relations Test were administered during the days immediately following the last instructional session. Since one class earned a high mean score on the Length Comparison Pretest, it was not posttested. These pupils were taught the material in the Length Comparison Instructional Unit to support the interpretation of the Conservation and Transitivity tests.

Instruction with the materials in Unit II: Conservation of Length, and Unit III: Conservation of Length Relations began immediately after the testing period following Unit I. Unit II was taught for a sequence of three sessions of 20-30 minutes. The instructional time for Unit III was five sessions of 20-30 minutes. The administration of the Length Comparison Application Test, Conservation of Length Posttest, Conservation of Length Relations Posttest, and Transitivity of Length Relations Test began one day after the last instructional session.

Testing Procedures

For any one child, each test was individually administered in one sitting. The items were assigned at random to each child so that each had a different sequence of the same items. The Conservation of Length Relations Test Level I and Level II were administered simultaneously so that a child would be forced to respond "yes" or "no" in a random sequence.

The Length Comparison Test was scored on a basis of the number of correct comparisons a child was able to perform. In the pretest some latitude was allowed in the scoring procedure. For example, if a child, when asked to find a pipe cleaner longer than a particular stick, selected the correct pipe cleaner and attempted to justify his selection by an approximate comparison (i.e., by not necessarily aligning two endpoints as precisely as possible), he/she was given credit for scoring the item correctly. On the posttest, however, no such latitude in the scoring procedure was permitted.

On the Conservation of Length Relations Test, if a child established a relation, regardless of whether he established a "correct" or "incorrect" relation, he was tested on his/her ability to conserve that relation. In the case of the transitivity test, unless a child established two correct comparisons, no measure was obtained on his/her ability to use the transitivity property of that relation.

Experimental Design

Question 1: What is the level of performance of children when establishing a length relation between two curves A and B without formal experiences?

Question 2: What is the effect of selected experiences on the ability of children to establish a length relation between two curves?

Figure 1 is a diagram of the design used to study the profiles of four- and five-year-old children with regard to the Length Comparison Pre- and Posttests (13). In particular, the design allows for testing of the following hypotheses, which provide information relative to the above questions.

1. The mean score of the four-year-old children does not differ from the mean score of the five-year-old children on the pre- and posttests.

2. The mean score on the posttest does not differ from the mean score on the pretest.

3. The profile of the four-year-olds does not differ from the profile of the five-year-olds.

Table 3 is the ANOVA Table for the design given in Diagram I (13). In addition to this design, an item analysis along with internal consistency reliability coefficients on the pre- and posttests will be reported. Similar data will be reported on the first and second administrations of the Application Test. Correlation coefficients between the scores on the Length Comparisons Posttest and Application Test will also be reported.

Question 3a: If children are able to establish a length relation between two curves, are they able to conserve that relation without formal experiences?

Question 3b: If children are able to establish a length relation between two curves, are they able to conserve the relation involving properties or logical consequences of that relation without formal experiences?

The nine items of the Conservation of Length Relations Test for which a response of "yes" was correct involves only conservation of a particular relation. These nine items are considered to exemplify a Level I. The nine items for which a response of "no" was correct

Diagram I

Outline of Design: Length Comparison Pre- and Posttest Analysis

Age	Individual	Pretest	Posttest	
4	1	X_{111}	X_{121}	
	.	.	.	
	.	.	.	
	.	.	.	
	N_4	$X_{N_4 11}$	$X_{N_4 21}$	
	Means:	$X_{.11}$	$X_{.22}$	$X_{..1}$
5	1	X_{112}	$X_{.22}$	
	.	.	.	
	.	.	.	
	.	.	.	
	N_5	$X_{N_5 12}$	$X_{N_5 22}$	
	Means:	$X_{.12}$	$X_{.22}$	$X_{..2}$
		$X_{.1.}$	$X_{.2.}$	$X_{...}$

involved not only conservation of a particular relation, but also the asymmetric property or logical consequences. These nine items are considered to exemplify a Level II.

One may think of each student's response set as being an ordered 18-tuple where each element is either "yes" or "no." If each response set is considered to be a random sample from 2^{18} such response sets, it has probability of 2^{-18} of occurring (12, p. 29). If a child is guessing

TABLE 3

ANOVA TABLE

Source of Variation	DF	SS	MS
<u>Between</u>			
A (Age)	1	Q_2	$F_2 = 31 \times Q_2/Q_3$
Subjects within groups	31	Q_3	
<u>Within</u>			
B (Tests)	1	Q_1	$F_1 = 31 \times Q_1/Q_5$
AB	1	Q_4	$F_3 = 31 \times Q_4/Q_5$
B x Subjects within groups	31	Q_5	

during the test, then one may consider his responses as being nothing more than an 18-tuple of "yes's" or "no's" for elements, where "yes" or "no" for any one entry each has probability of 1/2 of occurring. In this case, his/her response set may be considered as a random sample from 2^{18} such 18-tuples. The probability of that child obtaining at least six correct "yes" responses and six correct "no" responses is no greater than .06.

For a child to be classified at Level I and Level II he/she then must have at least six of the nine items which were written to exemplify Level I and six of the nine items which were written to exemplify Level II correct.

If one considers the nine items written at Level I or Level II regardless of the nine items written at the other level, a probability of only approximately .02 exists that a child can respond correctly to eight or nine items if he is guessing. Thus, if a child is not categorized at Level I and Level II on the basis of the performance criterion set, one may consider his responses to one of the two item sets written at Level I or Level II. Clearly, a high probability exists that those children who have at least eight or nine correct items for a particular set may be responding on a basis other than guessing to those nine items. These children, then, may be candidates for being classified at just Level I or Level II. One cannot, however, with any degree of confidence, assert that in fact these children do not possess a response bias unless the remaining nine items are considered. For example, if a child is able to score eight or nine on Level I items, and responds on a basis of guessing on Level II items, then a probability of only .02 occurs that the child will have at most one correct "no" response. If this unlikely event occurs, whether a response bias exists or whether the child is responding on the basis of the perceptual cue is an open question. For a child to be classified as just Level I or Level II, he must respond correctly to eight or nine items of the level in question and no less than two items of the other level.

The proportion of children who meet criterion on Level I and Level II will be investigated. It must be pointed out that the criterion is a conservative one since it is known that children do respond on the basis of perceptual cues (27).

A principle component factor analysis will be conducted on the 18 items to investigate possible factors which will be used in the interpretation of the above criterion. Item difficulties will also be reported as well as internal consistency reliabilities, both of which support interpretation of the criterion established.

In order to check the hypothesis that the distribution of total scores on Level I and Level II tests does not differ from a theoretical distribution based on random responses, a "goodness of fit" test will be employed (24, pp. 42-46).

Question 4a. What is the effect of selected experiences on the ability of children to conserve length relations?

Question 4b. What is the effect of selected experiences on the ability of children to conserve length relations involving properties or logical consequences of length relations?

The McNemar test for the significance of changes will be utilized to provide answers to question 4a and 4b (24, pp. 63-67). Those children who meet criterion for Level I and Level II will be given a "1" and those children who do not meet criterion will be given a "0." Thus, only a nominal scale of measurement is employed. Seigel states, "The McNemar test for the significance of changes is particularly applicable to those 'before and after' designs in which each person is used as his own control in which measurement is in the strength of either a nominal or ordinal scale" (24, p. 63).

Explicitly, the null hypothesis is:

H_0: For those children who change, the probability P_1 that any child will change from C (criterion) to `C (non-criterion) is equal to the probability $P_2$ that he will change from `C to C.

The alternative hypothesis is:

H_1: $P_1 < P_2$

Question 5: Are children able to conserve length without formal experiences?

The set {"yes", "no"} represents the possible responses for the six items of the Conservation of Length Test. Other responses are possible, but they occurred with zero probability in the testing sessions. There are 2^6 different six-tuples with "yes" or "no" as elements. If a child is guessing, then the probability that any one of the 2^6 six-tuples will occur is 2^{-6}. Under these conditions, the probability of receiving "at least five or six correct responses" is approximately .11. It must be pointed out, however, that children do respond on the basis of perceptual cues, so that the actual probability that a child who does not possess the ability to conserve length could obtain five or six may be much lower than .11.

If a child responds on the basis of a bias (always says "yes" or "no"), then he/she would not obtain a five or six. Moreover, if a child possesses only the ability to use either the reflexive or nonreflexive property, he/she also will not achieve a five or six. Hence, for a child to possess the ability to conserve length, the performance criterion of a total score of five or six is established. The proportion of children who reached this criterion will be used to answer Question 5. Moreover, a "goodness of fit" test will be employed to test the hypothesis that the distribution of total scores does not differ from a distribution based on random responses.

Question 6: What is the effect of selected experiences on the ability of children to conserve length?

The McNemar test for significance of changes will be utilized to provide an answer to Question 6. Explicitly, the null hypothesis is:

> H_0: For those children who change, the probability P_1 that any child will change from C to -C is equal to the probability P_2 that he will change from -C to C.
>
> The alternative hypothesis is:
>
> H_1: $P_1 < P_2$
>
> Question 7: If children are able to est[illegible]lish a relation between two curves, can they use the transitive [illegible]roperty with formal experiences only in establishing lengt[illegible] [illegible]ations?

Based on the average item difficulty o[illegible] [illegible] Application Test, a parameter will be established which may be re[illegible] as an efficiency level of the child's ability to establish len[illegible] [illegible]lations between curves. Using this parameter, r, the probab[illegible] [illegible]at a child could establish a correct relation in each o[illegible] [illegible]o necessary overt comparisons on any item in the transitivity [illegible] r^2. This assumes that the comparisons are performed independe[illegible]

If a child responds on a random basis [illegible]e probability p of a correct response on any item is $\frac{r^2}{2}$. Usin[illegible] [illegible]lue of p, a performance criterion will be established and a "go[illegible] [illegible]f fit" test will be performed on the [illegible]stribution of total scores to the theoretical distribution of total scores based on guessing. The same procedure will be followed to provide information relative to Question 8.

Questions 9-12 will be answered by observation. For example, to establish whether the ability to conserve length is necessary (sufficient) to enable children to use the transitive property, an inspection will be made of those children who met criterion on each test instrument. If the ability to conserve length is necessary for the ability to use the transitive property, then each child who attains criterion on the Transitivity Test must also meet criterion on the Conservation of Length Test. If the ability to conserve length is sufficient for the ability to use the transitive property, then each child who meets criterion on the Conservation of Length Test must also meet criterion on the Transitivity Test.

> Question 13: What is the relationship between certain student characteristics and scores earned on the Comparison, Conservation, and Transitivity Tests?

To answer this question, correlation studies will be conducted as well as observational studies.

CHAPTER III

RESULTS OF THE STUDY

The results of the study are partitioned into sections as follows: Section 1 contains the results of the Length Comparison Test; Section 2, the Length Comparison Application Test; Section 3, the Conservation of Length Relations Test; Section 4, the Conservation of Length Test; Section 5, the Transitivity Test; Section 6, Conservation and Transitivity Relationships.

Section 1

Length Comparison Test

An extensive internal-consistency reliability study was conducted on the pre- and posttests and subtests thereof. Results of this study are contained in Table 4.

The reliabilities associated with the total test scores are quite substantial and support analyses of the data. In the case of the pretest, the reliabilities for the subtests are also substantial. For the posttest, however, the reliability for the six items which were designed to measure the ability of children to establish the relation "shorter than" is low. Various reasons may be given, the most apparent of which is the high mean and relatively small standard deviation, as reported in Table 5. It is well known that easy tests may be unreliable.

TABLE 4

RELIABILITIES OF LENGTH COMPARISON PRE- AND POSTTESTS
(KUDER-RICHARDSON # 20)

Test	Reliability
<u>Pretest</u>	
Total	.91
Longer Than	.82
Shorter Than	.87
Same Length As	.77
<u>Posttest</u>	
Total	.83
Longer Than	.71
Shorter Than	.43
Same Length As	.73

Table 6 contains the item difficulties (proportion answering item correctly) for the eighteen items. Table 7 contains item difficulties arranged by material sets. The only discernable differences depend on the fact that the item difficulties for Material Sets 2 and 4 are consistently higher than for the other four material sets. These two material sets involve sticks only, which apparently are easier for children to manipulate than strings.

Table 8 contains the analysis of variance for the pre- and posttest total scores. No differences are statistically discernable for the variable of Age which is at two levels: four- versus five-year-olds. The mean score on the posttest is significantly greater than the pretest. No interaction of Age and Tests occurs which indicates that the difference between the means of the posttest scores for each group is not significantly different than the differences between the means of the pretest scores. Table 9 contains the mean scores of the pre- and posttests by age groups.

TABLE 5

MEAN AND STANDARD DEVIATIONS OF LENGTH COMPARISON PRE- AND POSTTESTS

Test	Mean	Standard Deviation
<u>Pretest</u>		
Total	10.68	5.35
Longer Than	4.38	1.91
Shorter Than	3.29	2.32
Same Length As	3.00	2.02
<u>Posttest</u>		
Total	14.55	3.53
Longer Than	4.94	1.43
Shorter Than	5.12	1.07
Same Length As	4.49	1.70

TABLE 6

ITEM DIFFICULTY OF LENGTH COMPARISON: PRETEST AND POSTTEST

Item*	Difficulty	
	Pretest	Posttest
1	.68	.79
2	.71	.91
3	.74	.82
4	.85	.88
5	.68	.76
6	.74	.76
7	.50	.79
8	.56	.94
9	.53	.82
10	.62	.85
11	.59	.91
12	.50	.79
13	.38	.65
14	.53	.68
15	.41	.56
16	.44	.65
17	.65	.94
18	.59	.82

*Items 1-6: Longer Than
7-12: Shorter Than
13-18: Same Length As

Tables 10 and 11; 12 and 13; 14 and 15 contain the results of analyses on the subtests "longer than," "shorter than," and "the same length as," respectively. Each analysis is parallel to that in Tables 8 and 9. On the subtest of "longer than," the children started with relatively high mean scores (68 and 75 percent for the four- and five-year olds, respectively) and ended with mean scores 78 and 86 percent, a non-significant gain, statistically. In the case of the subtest, "shorter than," a large gain was noted for both the four- and five-year-olds (from 43 to 76 percent for age four and 62 to 91 percent for age five). Again, age was not significant. In the case of the subtest, "same length as," a substantial increase was again present (48 to 57 percent for age four and 46 to 80 percent for age five). Age was again non-significant as was the interaction of Age and pre- posttest scores. On the basis of the pre- posttest scores alone, one may hypothesize that an interaction occurred. The nonsignificant interaction may be due to the power of statistical test involved.

Table 16 contains the correlations of the pre- and posttest and subtests thereof with the variables (1) Verbal Maturity, (2) IQ, (3) Age, and (4) Social Class. All the correlations are low but some differ significantly from a zero correlation. Age correlates significantly (but low) with the posttest scores except for the subtest "longer than." One other correlation coefficient is statistically significant (.41) between Social Class and the subtest "same length as" of the pretest.

To assist in the interpretation of the data, Tables 17, 18, and 19 were constructed.

TABLE 7

ITEM DIFFICULTIES: MATERIAL SETS BY RELATIONS

Material Set	1	2	3	4	5	6
<u>Pretest</u>						
Longer Than	.68	.71	.74	.85	.68	.74
Shorter Than	.50	.59	.50	.62	.53	.56
Same Length As	.53	.59	.38	.65	.44	.41
Mean	.57	.63	.54	.71	.55	.57
<u>Posttest</u>						
Longer Than	.79	.91	.82	.88	.7[illegible]	.76
Shorter Than	.79	.91	.79	.85	.82	.94
Same Length As	.68	.82	.65	.94	.65	.56
Mean	.75	.88	.75	.89	.74	.75

TABLE 8

ANOVA

TOTAL SCORES: PRE- AND POST- LENGTH COMPARISON TESTS

Source of Variation	DF	SS	MS	F
<u>Between</u>	<u>32</u>			
A (Age)	1	82.31	82.31	2.65
Subjects within groups	31	961.45	31.01	
<u>Within</u>	<u>33</u>			
B (Tests)	1	221.84	221.84	22.45**
AB	1	2.35	2.35	<1
B x Subjects within groups	31	306.31	9.88	

$^{**}p < .01$

TABLE 9

INTERACTION TABLE

TOTAL SCORES: PRE- AND POST- LENGTH COMPARISON TESTS

Age	Pretest	Posttest
4	9.50	12.67
5	11.43	15.38

TABLE 10

ANOVA

PRE- AND POST- LENGTH COMPARISON TESTS: "LONGER THAN"

Source of Variation	DF	SS	MS	F
Between	32			
A (Age)	1	3.21	3.21	<1
Subjects within groups	31	117.46	3.79	
Within	33			
B (Tests)	1	6.06	6.06	2.85
AB	1	0.0	0.0	<1
B x Subjects within groups	31	65.93	2.13	

TABLE 11

INTERACTION TABLE

PRE- AND POST- LENGTH COMPARISON TESTS: "LONGER THAN"

Age	Pretest	Posttest
4	4.08	4.67
5	4.52	5.14

TABLE 12

ANOVA

PRE- AND POST- LENGTH COMPARISON TESTS: "SHORTER THAN"

Source of Variation	DF	SS	MS	F
Between	32			
A (Age)	1	14.91	14.91	3.42
Subjects within groups	31	135.12	4.36	
Within	33			
B (Tests)	1	54.55	54.55	26.35**
AB	1	.31	.31	<1
B x Subjects within groups	31	64.14	2.07	

**$p < .01$

TABLE 13

INTERACTION TABLE

PRE- AND POST- LENGTH COMPARISON TESTS: "SHORTER THAN"

Age	Pretest	Posttest
4	2.58	4.58
5	3.71	5.42

It is revealed that even though the means of the pre- and post-test of the relation "longer than" do not differ statistically, some children made large gains. Only a small number of children made negative gains on any subtest. In general, each negative gain was small.

The students were further divided into three categories: (A) those who did not earn a score of five or six on the posttest, (B) those who earned a five or six on the posttest but not on the pretest, and (C) those who earned five or six on both tests. Mean IQ's, Verbal Maturity scores, and mean Age are included in Tables 20-25. These tables suggest that the mean Verbal Maturity and IQ scores are greater for category C than category A for the five-year-olds. The differences are not marked, however. In the case of the four-year-olds, the Verbal Maturity means for categories A and B are similar and less than the mean for category C. Generally, very little difference exists between characteristics among categories of any one age group.

TABLE 14

ANOVA

PRE- AND POST- LENGTH COMPARISON TESTS: "SAME LENGTH AS"

Source of Variation	DF	SS	MS	F
Between	32			
A (Age)	1	10.61	10.61	2.04
Subjects within groups	31	161.33	5.20	
Within	33			
B (Tests)	1	24.24	24.24	14.18**
AB	1	4.78	4.78	2.80
B x Subjects within groups	31	52.98	1.71	

$^{**}p < .01$

TABLE 15

INTERACTION TABLE

PRE- AND POST- LENGTH COMPARISON TESTS: "SAME LENGTH AS"

Age	Pretest	Posttest
4	2.91	3.42
5	2.71	4.81

TABLE 16

CORRELATION MATRIX

STUDENT CHARACTERISTICS WITH PRE- AND POST- LENGTH COMPARISON TESTS

Test	Characteristics			
	Verbal Maturity	IQ	Age	Social Class
Pretest				
Total	.19	.02	.10	-.33
Longer Than	.10	.00	.07	-.20
Shorter Than	.29	.05	.21	-.24
Same Length As	.08	.01	-.05	-.41*
Posttest				
Total	.32	.06	.42*	.00
Longer Than	.26	.19	.22	-.03
Shorter Than	.33	.02	.41*	.05
Same Length As	.23	-.04	.42*	.00

*Significantly different from a zero correlation: $p < .02$

TABLE 17
FREQUENCY OF GAIN SCORES FOR FOUR- AND FIVE-YEAR-OLDS
ON THE PRE- POST- LENGTH COMPARISON "LONGER THAN"
TESTS ACCORDING TO POSTTEST SCORES

Gain Score by Age Group	Number of Children by "Longer Than" Posttest Score					
	1	2	3	4	5	6
5-year-olds:						
4				1		
3						2
2						4
1	1				1	1
0					1	6
-1					1	
-2			1	1		
-3			1			
4-year-olds:						
5					1	1
4						
3						1
2						
1					1	
0		1				3
-1					1	
-2		1	1	1		

TABLE 18
FREQUENCY OF GAIN SCORES FOR FOUR- AND FIVE-YEAR-OLDS
ON THE PRE- POST- LENGTH COMPARISON "SHORTER THAN"
TESTS ACCORDING TO POSTTEST SCORES

Gain Score by Age Group	Number of Children by "Shorter Than" Posttest Score					
	1	2	3	4	5	6
5-year-olds:						
6						1
5					1	2
4					1	
3			1			
2						1
1				2	1	1
0					1	7
-1					1	
4-year-olds:						
4				1	3	
3				1		
2				1		1
1		1	1			
0						2
-1					1	

TABLE 19

GAIN SCORES FOR FOUR- AND FIVE-YEAR OLDS ON
THE PRE- POST LENGTH COMPARISON "SAME LENGTH AS" TESTS
ACCORDING TO POSTTEST SCORES

Gain Score by Age Group	Number of Children by "Same Length As" Posttest Score					
	1	2	3	4	5	6
5-year-olds:						
5						1
4				2		1
3					1	2
2		1		1		4
1						1
0		1	1		1	2
-1				1		
-2						
-3						
-4		1				
4-year-olds:						
3						1
2		1				
1	1	1	3			1
0						
-1	1			1	2	

TABLE 20

CHARACTERISTICS OF FOUR-YEAR-OLDS WHO DID NOT EARN SCORES OF FIVE OR SIX (CATEGORY A) ON THE LENGTH COMPARISON POSTTEST

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	4/12	5/12	8/12
Verbal Maturity Range	82-105	82-105	82-114
Mean Verbal Maturity	95.0	94.4	97.9
IQ Range	103-136	103-134	103-145
Mean IQ	119.5	117.8	123.0
Age Range (months)	49-57	49-57	49-57
Mean Age (months)	52.3	52.2	53.1
Social Class Range	I-IV	I-IV	I-IV

TABLE 21

CHARACTERISTICS OF FOUR-YEAR-OLDS WHO EARNED SCORES OF FIVE OR SIX (CATEGORY B) ON THE LENGTH COMPARISON POSTTESTS BUT NOT THE PRETESTS

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	4/12	4/12	1/12
Verbal Maturity Range	92-102	95-114	97
Mean Verbal Maturity	96.5	102.0	97.0
IQ Range	113-145	114-145	114
Mean IQ	122.0	126.7	114.0
Age Range (months)	52-57	52-57	57
Mean Age (months)	55.2	55.2	57.0
Social Class Range	I-IV	I-IV	II

TABLE 22

CHARACTERISTICS OF FOUR-YEAR-OLDS WHO EARNED SCORES OF FIVE OR SIX (CATEGORY C) ON THE LENGTH COMPARISON PRETESTS AND POSTTESTS

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	4/12	3/12	3/12
Verbal Maturity Range	105-114	105-107	105-107
Mean Verbal Maturity	107.8	105.6	105.6
IQ Range	107-134	107-134	107-134
Mean IQ	122.2	119.0	119.0
Age Range (months)	48-57	48-57	48-57
Mean Age (months)	52.8	53.0	53.0
Social Class Range	II-III	II-III	II-III

TABLE 23

CHARACTERISTICS OF FIVE-YEAR-OLDS WHO DID NOT EARN SCORES OF FIVE OR SIX (CATEGORY A) ON THE LENGTH COMPARISON POSTTEST

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	5/21	3/21	8/21
Verbal Maturity Range	79-116	79-108	70-116
Mean Verbal Maturity	97.2	92.0	99.6
IQ Range	85-130	101-120	85-130
Mean IQ	105.0	109.3	105.2
Age Range (months)	59-69	60-67	60-67
Mean Age (months)	64.6	63.7	64.5
Social Class Range	I-IV	II	I-IV

TABLE 24

CHARACTERISTICS OF FIVE-YEAR-OLDS WHO EARNED SCORES OF FIVE OR SIX (CATEGORY B) ON THE LENGTH COMPARISON POSTTESTS BUT NOT THE PRETESTS

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	7/21	8/21	9/21
Verbal Maturity Range	90-117	79-116	79-117
Mean Verbal Maturity	100.0	98.8	99.2
IQ Range	96-120	85-130	87-122
Mean IQ	108.7	103.6	111.6
Age Range (months)	62-69	59-69	62-69
Mean Age (months)	65.9	65.5	66.2
Social Class Range	II-IV	I-V	II-V

TABLE 25

CHARACTERISTICS OF FIVE-YEAR-OLDS WHO EARNED SCORES OF FIVE OR SIX (CATEGORY C) ON THE LENGTH COMPARISON PRETESTS AND POSTTESTS

Variable	Test		
	Longer Than	Shorter Than	Same Length As
Ratio of Group	9/21	10/21	4/21
Verbal Maturity Range	70-114	91-117	86-114
Mean Verbal Maturity	102.6	104.3	104.8
IQ Range	97-126	97-126	103-126
Mean IQ	112.1	113.8	112.2
Age Range (months)	60-68	60-68	59-67
Mean Age (months)	64.0	64.5	62.0
Social Class Range	I-IV	I-IV	I-III

Section 2

Length Comparison Application Test

In order to ascertain that the ability of children to compare lengths of curves is not restricted to six specific material sets, the Length Comparison Application Test was administered. Table 26 contains the pair-wise correlations of the six tests comprised of the total test and subtests thereof. The correlation of .81 between total scores along with the significant pair-wise correlations of the respective subtests of the two tests indicate a high degree of relationship.

TABLE 26

CORRELATION MATRIX

LENGTH COMPARISON POSTTEST AND APPLICATION TEST (FIRST ADMINISTRATION)

Test	1	2	3	4	5	6	7	8
Posttest								
1. Total	1.00	.82**	.79**	.88**	.81**	.71**	.78**	.65**
2. Longer		1.00	.52**	.53**	.77**	.71**	.77**	.58**
3. Shorter			1.00	.57**	.69**	.60**	.65**	.59**
4. Same As				1.00	.58**	.50**	.56**	.48**
Application								
5. Total					1.00	.83**	.93**	.89**
6. Longer						1.00	.79**	.60**
7. Shorter							1.00	.72**
8. Same As								1.00

**Significantly different from zero correlations; $p < .01$

As noted in the chapter on procedure, the Length Comparison Application Test was administered twice, once before and once after the completion of Units II and III. The pupil performances on this second administration are of interest since Unit III contained additional exercises on Length Comparisons.

Table 27 contains the results of the reliability study on each test and subtests thereof. All the reliabilities on the first administration are substantial. In the second administration, however, the reliability of the subtest "shorter than" is very low. High mean scores (Table 28) and small standard deviation may be contributing to these modest reliabilities. No apparent changes in the mean scores are observable across administrations.

TABLE 27

RELIABILITIES OF LENGTH COMPARISON APPLICATION TESTS

Test	Reliability
First Administration	
Total	.85
Longer Than	.71
Shorter Than	.77
Same Length As	.68
Second Administration	
Total	.76
Longer Than	.63
Shorter Than	.18
Same Length As	.65

TABLE 28

MEANS AND STANDARD DEVIATIONS OF LENGTH COMPARISON

APPLICATION TESTS

Test	Mean	Standard Deviation
First Administration		
Total	14.10	3.89
Longer Than	5.02	1.39
Shorter Than	4.34	1.81
Same Length As	4.74	1.50
Second Administration		
Total	14.44	3.14
Longer Than	5.16	1.22
Shorter Than	4.86	1.03
Same Length As	4.42	1.58

Tables 29 and 30 contain the item difficulties at each administration of the test and all possible correlations between the total score and score on each subtest of each administration, respectively. Some modest correlations do exist between the total scores of the first and second tests of "longer than," "shorter than," and "same length as," even though the respective means are quite comparable. These correlations reflect a considerable fluctuation in scores at the "top" of the test scales.

Table 31 contains the correlation of the variables (1) Verbal Maturity, (2) IQ, (3) Age, and (4) Social Class with the total test scores and subtests thereof on the first and second administration. All but one

TABLE 29

ITEM DIFFICULTY OF LENGTH COMPARISON APPLICATION TEST

Item	Difficulty	
	First Administration	Second Administration
1	.92	.86
2	.78	.90
3	.92	.82
4	.80	.78
5	.76	.84
6	.84	.96
7	.72	.82
8	.68	.70
9	.68	.84
10	.68	.86
11	.78	.86
12	.80	.78
13	.88	.80
14	.72	.66
15	.84	.70
16	.76	.82
17	.78	.74
18	.76	.70

TABLE 30

CORRELATION MATRIX

LENGTH COMPARISON APPLICATION TEST

Test	1	2	3	4	5	6	7	8
First Administration								
1. Total	1.00	.81**	.89**	.77**	.50**	.49**	.35**	.39**
2. Longer Than		1.00	.64**	.41**	.29*	.27*	.32*	.15
3. Shorter Than			1.00	.51**	.42**	.47**	.22*	.32*
4. Same Length As				1.00	.53**	.44**	.36**	.48**
Second Administration								
5. Total					1.00	.82**	.76**	.86**
6. Longer Than						1.00	.52**	.53**
7. Shorter Than							1.00	.45**
8. Same Length As								1.00

Significantly different from a zero correlation: ** $p < .01$
* $p < .05$

of the significant correlations involves age, and this is consistent with the correlations reported earlier from the Length Comparison Pre- and Posttests.

Section 3

Conservation of Length Relations Test

This section contains the results of the pre- and post-administration of the Conservation of Length Relations test: Level I and Level II. An internal consistency reliability study was conducted at

TABLE 31

CORRELATION MATRIX

STUDENT CHARACTERISTICS WITH LENGTH COMPARISON APPLICATION TESTS

Test	Characteristics			
	Verbal Maturity	IQ	Age	Social Class
First Administration				
Total	.10	.08	.36**	.02
Longer Than	.10	.14	.38**	-.04
Shorter Than	.14	.06	.23	-.02
Same Length As	-.01	.00	.30*	.13
Second Administration				
Total	.22	.03	.37**	.13
Longer Than	.26	.07	.39**	.18
Shorter Than	-.05	-.12	.29*	-.06
Same Length As	.28*	.08	.24	.15

Significantly different from a zero correlation: ** $p < .01$
* $p < .05$

each of the pre- and post-administrations. Table 32 contains the results of the data. The reliabilities are substantial and support interpretation of the data.

Table 33 contains the principal component factor analysis of the pre- and posttests. The two factors reported for the pretest were the only factors that had eigen values greater than one for that test. In the case of the posttest, however, two other factors had eigen values greater than one; one of 1.33 and one of 1.11. Only one loading, however, was greater than .5 on each of these remaining two factors.

TABLE 32

RELIABILITIES OF CONSERVATION OF LENGTH RELATIONS TEST:
LEVEL I AND LEVEL II
(KUDER-RICHARDSON # 20)

Test	Reliability
Pretest	
Level I	.88
Level II	.81
Posttest	
Level I	.88
Level II	.83

The first factor of the pretest is a bipolar factor with the items at Level I loading negatively and the items at Level II loading positively. Four of the five positive loadings which exceeded .5 were items involving the asymmetrical property of "longer than" or "shorter than." The remaining positive loading involved the statement, "If A is the same length as B, then A is not longer than B."

The six items which have loadings greater than .5 on the second factor include four involving logical consequences of the relations, one involving the asymmetrical property of "shorter than" and one which tests conservation of "the same length as."

TABLE 33

FACTOR ANALYSIS OF CONSERVATION OF LENGTH RELATIONS TESTS

Item		Pretest Factors		Posttest Factors	
		1	2	1	2
Level I					
Longer Than	1	-.5179	.4928	-.1938	.6206
	2	-.8139	.3029	.0778	.7065
	3	-.7534	.1310	.3150	.4925
Shorter Than	4	-.6982	.3444	-.2794	.5122
	5	-.6776	.4312	-.2651	.4185
	6	-.7831	-.0674	-.1996	.7985
Same Length As	7	-.6903	.5853	-.3807	.8069
	8	-.7261	.2393	-.3466	.3631
	9	-.8440	.4050	-.4118	.7599
Level II					
Longer Than	1	.5748	.4854	.5134	.3771
	2	.3730	.5885	.7704	.2654
	3	.6523	.0404	.8291	.2600
Shorter Than	4	.8592	.2577	.5999	.1304
	5	.7244	.5162	.7887	.2206
	6	.4528	.5671	.5580	.0838
Same Length As	7	.5512	.6195	.7462	.2527
	8	.3729	.2845	.6475	.2374
	9	.3865	.7055	.7429	.1681
Percent Communality		45.57	21.61	41.67	33.00

Of the two identifiable factors on the posttest, the items which have loadings greater than .5 are all at Level II. Moreover, each item at Level II has a loading greater than .5 on Factor I. The second factor clearly involves those items written at Level I. Factor I may be named "Conservation of Length Relations: Level II" and Factor II may be named "Conservation of Length Relations: Level I."

Table 34 contains the item difficulties and Table 35 contains the means and standard deviations for the pre- and posttests. From these two tables, it is apparent that the gain from pre- to posttest for Level II was not great.

Table 36 contains the McNemar Test for the significance of changes for those children who met performance criterion for Level I and Level II on the pre- and posttests. χ^2 is significant for the five-year-old children, and this indicates that the probability of change from ~C to C for any five-year-old is greater than the probability that he changed from C to ~C. χ^2 was not significant for the four-year-olds.

On the posttest, 17 children scored an eight or nine on the nine Level I items, but they scored at most five on the nine Level II items. These 17 children are candidates for Level I. Of these 17 children, seven had scores of either a 0 or 1 on the Level II items, which occurs with probability no greater than .02 if they were responding randomly on the nine Level II items. Whether these seven children were basing their responses to the nine Level II items on the perceptual cue or just said "yes" without reflection is an open question. The remaining ten children are much better qualified for classification at Level I. These children

TABLE 34

ITEM DIFFICULTY OF CONSERVATION OF LENGTH RELATIONS TESTS

	Item	Difficulty	
		Pretest	Posttest
	1	.59	.83
	2	.39	.69
	3	.55	.85
	4	.49	.87
Level I	5	.51	.83
	6	.51	.77
	7	.37	.73
	8	.49	.85
	9	.39	.73
	1	.47	.46
	2	.43	.58
	3	.49	.44
	4	.59	.58
Level II	5	.43	.44
	6	.43	.52
	7	.57	.56
	8	.43	.46
	9	.49	.58

TABLE 35

MEANS AND STANDARD DEVIATIONS OF CONSERVATION OF LENGTH RELATIONS TEST: LEVEL I AND LEVEL II

Test	Mean	Standard Deviation
Pretest		
Level I	4.29	3.17
Level II	4.33	2.80
Posttest		
Level I	7.13	2.56
Level II	4.62	2.92

did respond "no" at least twice, which indicates that guessing may have been the dominant factor in response to Level II items. Therefore, their responses to Level I items may not have been biased responses. As a total then, there may have been 29 children who meet performance criterion on the Level I items.

On the pretest, eight children scored eight or nine on Level I questions, but at most five on Level II questions. Of these eight children, four scored only a zero or one on the Level II questions. This leaves only four candidates to be placed at Level I. Of these four candidates, only one met criterion on the posttest for Level I and Level II. One other met the criterion for Level I on the posttest. The remaining two did not meet any criterion on the posttest. None of the second group of four met any criterion on the posttest.

TABLE 36

MCNEMAR TEST FOR THE SIGNIFICANCE OF CHANGE PRE- AND POST-CONSERVATION OF LENGTH RELATIONS LEVEL I AND LEVEL II

All Children

Pre \ Post	~C	C
C	2	4
~C	30	15

$\chi^2 = 8.47$: $p < .005$: C-Met criteria: ~C-Did not meet criteria

Four-year-olds

Pre \ Post	~C	C
C	0	3
~C	13	2

$\chi^2 = .5$; Not significant at .05

Five-year-olds

Pre \ Post	~C	C
C	2	1
~C	17	13

$\chi^2 = 6.67$: $p < .005$

Seven children scored eight or nine on Level II questions but at most five on Level I questions. Of these seven, five had only a zero or one on Level I questions. One of the remaining two candidates met criterion for Level I and Level II on the posttest. The other child did not meet any posttest criterion. No child met criterion for Level II only on the posttest.

Tables 37 and 38 contain comparisons between Mean Age, IQ and Verbal Maturity scores for those four- and five-year-old children who met criterion and those who did not meet criterion on the pre- and posttests for Level I and Level II. It appears that the mean IQ for the five-year-old "criterion" group is greater than the "not criterion" group.

TABLE 37

COMPARISON OF FOUR-YEAR-OLDS WHO MET CRITERION (LEVEL I AND LEVEL II) WITH THOSE WHO DID NOT MEET CRITERION ON THE CONSERVATION OF LENGTH RELATIONS TEST: PRE- AND POSTTEST

Achievement Level by Test	Ratio of Students	Mean Verbal Maturity	Mean IQ	Mean Age
Pretest				
Criterion	3/18*	104.3	118.3	56.7
Not Criterion	15/18	102.2	120.2	52.7
Posttest				
Criterion	5/18	104.8	120.2	55.8
Not Criterion	13/18	101.7	119.8	52.7

*These three students met the criterion on both the pre- and posttests

TABLE 38

COMPARISON OF FIVE-YEAR-OLDS WHO MET CRITERION (LEVEL I AND LEVEL II) WITH THOSE WHO DID NOT MEET CRITERION ON THE CONSERVATION OF LENGTH RELATIONS TEST: PRE- AND POSTTEST

Achievement Level by Test	Ratio of Students	Mean Verbal Maturity	Mean IQ	Mean Age
Pretest				
Criterion	3/33	105.3	122.0	64.7
Not Criterion	30/33	98.4	108.5	64.7
Posttest				
Criterion	14/33	98.6	116.1	65.4
Not Criterion	19/33	99.4	105.1	64.2

Table 39 contains correlations between the pretest Level I and Level II total scores and Verbal Maturity scores, IQ scores, Age, and Social Class. The only significant correlation is between IQ and the posttest Level II total scores. This correlation is, however, low. A phi-correlation coefficient of .14, calculated on Table 40, supports the correlation involving age given in Table 39. These nonsignificant correlations do not contradict the data obtained in Table 36, since that table involves children who changed from C to -C or vice versa.

If children do, in fact, respond "yes" or "no" in a random fashion to items presented, then the distribution of total scores should not depart from a theoretical frequency distribution based on random responses, except for chance fluctuation. Tables 41 and 42 contain, by age, such a theoretical frequency distribution and four actual frequency distributions. In the case of each actual frequency distribution, the test of whether that distribution departs from the theoretical distribution results in a significant χ^2. The shapes (see Diagrams II, III, IV, and V) of the distribution are not then explainable on the basis of random responses to the items.

TABLE 39

CORRELATION MATRIX

STUDENT CHARACTERISTICS WITH CONSERVATION OF LENGTH RELATIONS: PRE- AND POSTTEST LEVEL I AND LEVEL II SCORES

Test	Characteristics			
	Verbal Maturity	IQ	Age	Social Class
<u>Pretest</u>				
Level I	.08	-.01	-.14	.26
Level II	.16	.10	-.00	-.25
<u>Posttest</u>				
Level I	.26	-.10	.20	.20
Level II	.00	.34*	.08	.04

*Significantly different from a zero correlation: $p < .02$

TABLE 40

FREQUENCY TABLE: AGE BY CRITERION (LEVEL I AND LEVEL II POSTTEST)

Criteria \ Age	4	5
C	5	14
~C	13	19

TABLE 41

COMPARISON OF THEORETICAL AND ACTUAL FREQUENCY DISTRIBUTIONS: FOUR-YEAR-OLDS

N = 18

Total Score	0	1	2	3	4	5	6	7	8	9
Theoretical	.04	.32	1.27	2.95	4.43	4.43	2.95	1.27	.32	.04
Level I Pre-	2	3	1	1	0	3	0	2	1	5
Level I Post-	2	0	0	2	2	1	0	0	3	8
Level II Pre-	2	2	1	2	0	1	3	4	3	0
Level II Post-	0	3	1	2	1	2	3	2	4	0

Level I Pretest	$\chi^2 = 744.534$	$p < .005$
Level I Posttest	$\chi^2 = 1712.793$	$p < .005$
Level II Pretest	$\chi^2 = 140.769$	$p < .005$
Level II Posttest	$\chi^2 = 69.569$	$p < .005$

TABLE 42

COMPARISON OF THEORETICAL AND ACTUAL FREQUENCY DISTRIBUTION:
FIVE-YEAR-OLDS

N = 33

Total Score	0	1	2	3	4	5	6	7	8	9
Theoretical	.06	.58	2.32	5.41	8.12	8.12	5.41	2.32	.58	.06
Level I Pre-	3	6	5	4	2	2	3	2	3	3
Level I Post-	0	0	1	3	0	5	2	6	4	12
Level II Pre-	2	6	3	4	3	5	3	2	2	3
Level II Post-	4	5	3	2	1	2	6	3	3	4

Level I Pretest	$\chi^2 = 351.273$	$p < .005$
Level I Post	$\chi^2 = 2071.527$	$p < .005$
Level II Pretest	$\chi^2 = 267.042$	$p < .005$
Level II Posttest	$\chi^2 = 495.883$	$p < .005$

DIAGRAM II

THEORETICAL AND ACTUAL FREQUENCY OF SCORES

CONSERVATION OF LENGTH RELATIONS LEVEL I PRE- AND POSTTEST:

FOUR-YEAR-OLDS

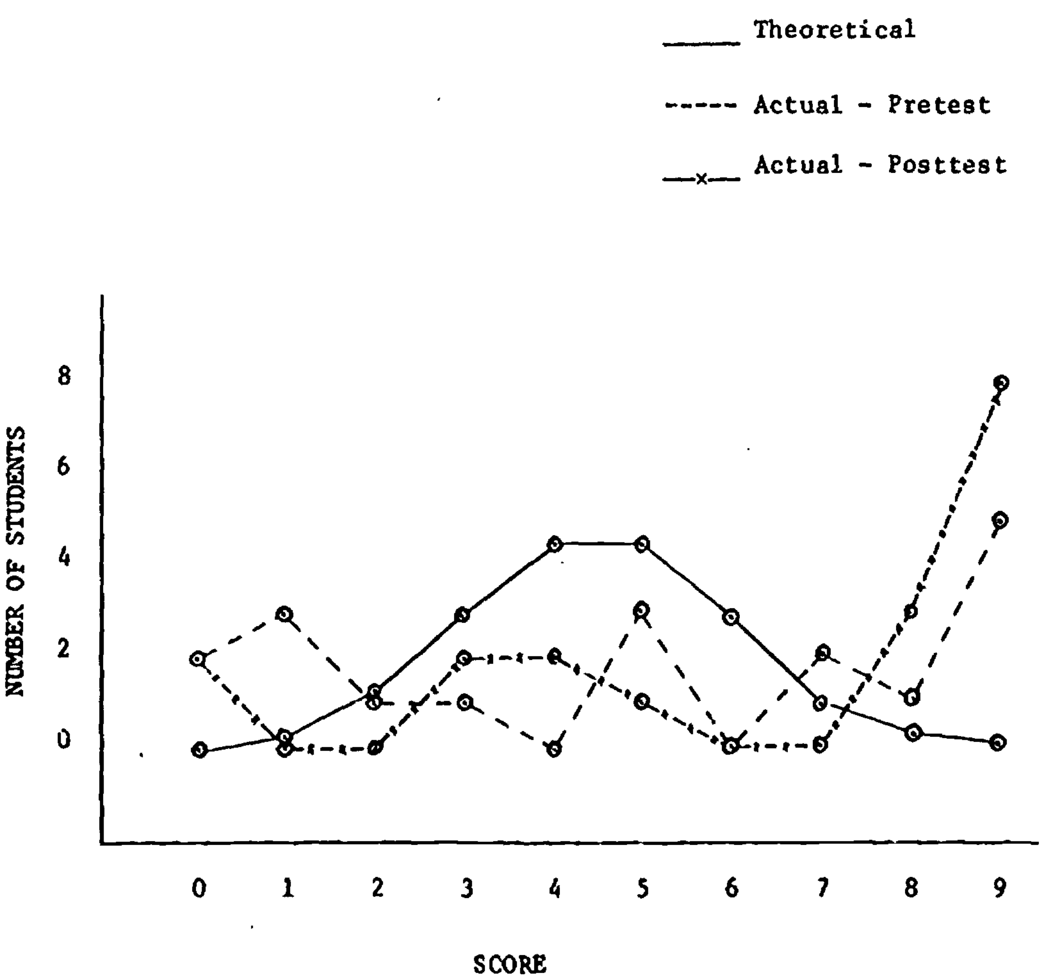


DIAGRAM III

THEORETICAL AND ACTUAL FREQUENCY OF SCORES

CONSERVATION OF LENGTH RELATIONS LEVEL II PRE- AND POSTTEST:

FOUR-YEAR-OLDS

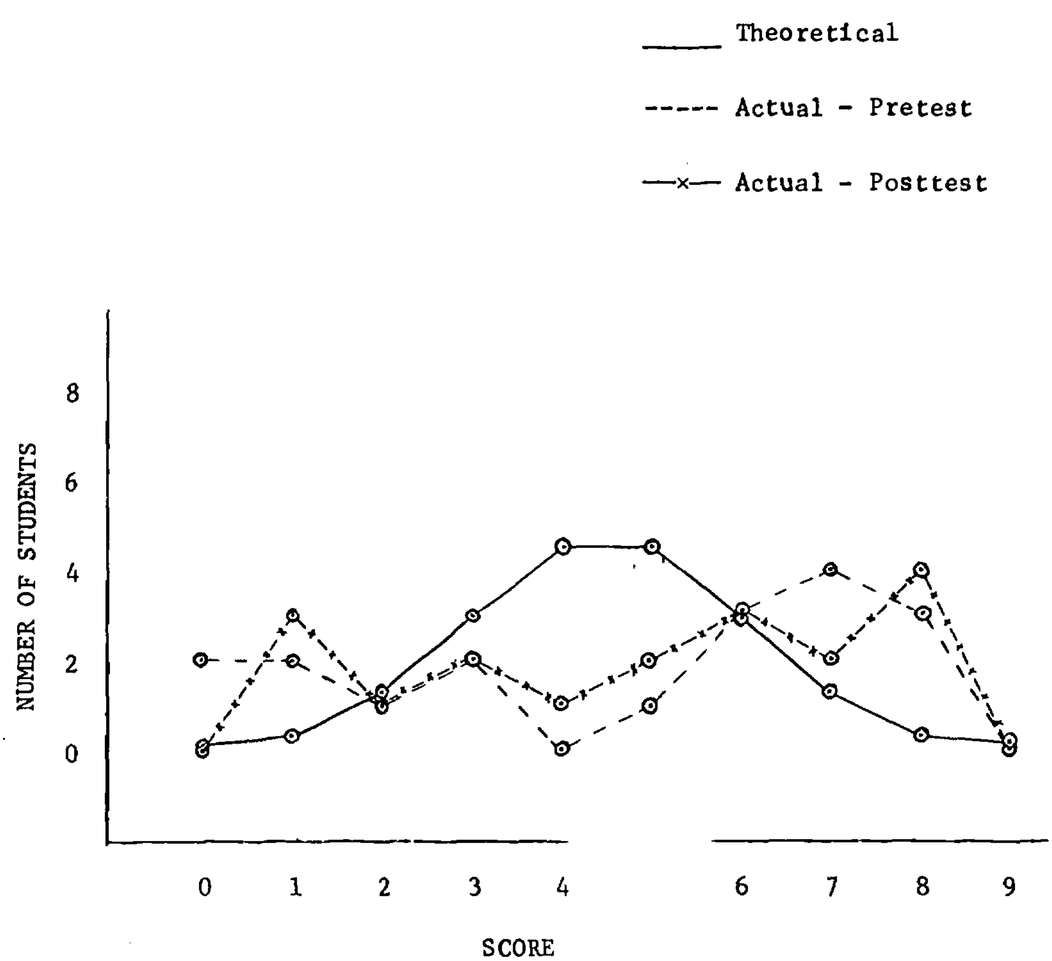


DIAGRAM IV

THEORETICAL AND ACTUAL FREQUENCY OF SCORES

CONSERVATION OF LENGTH RELATIONS LEVEL I PRE- AND POSTTEST:

FIVE-YEAR-OLDS

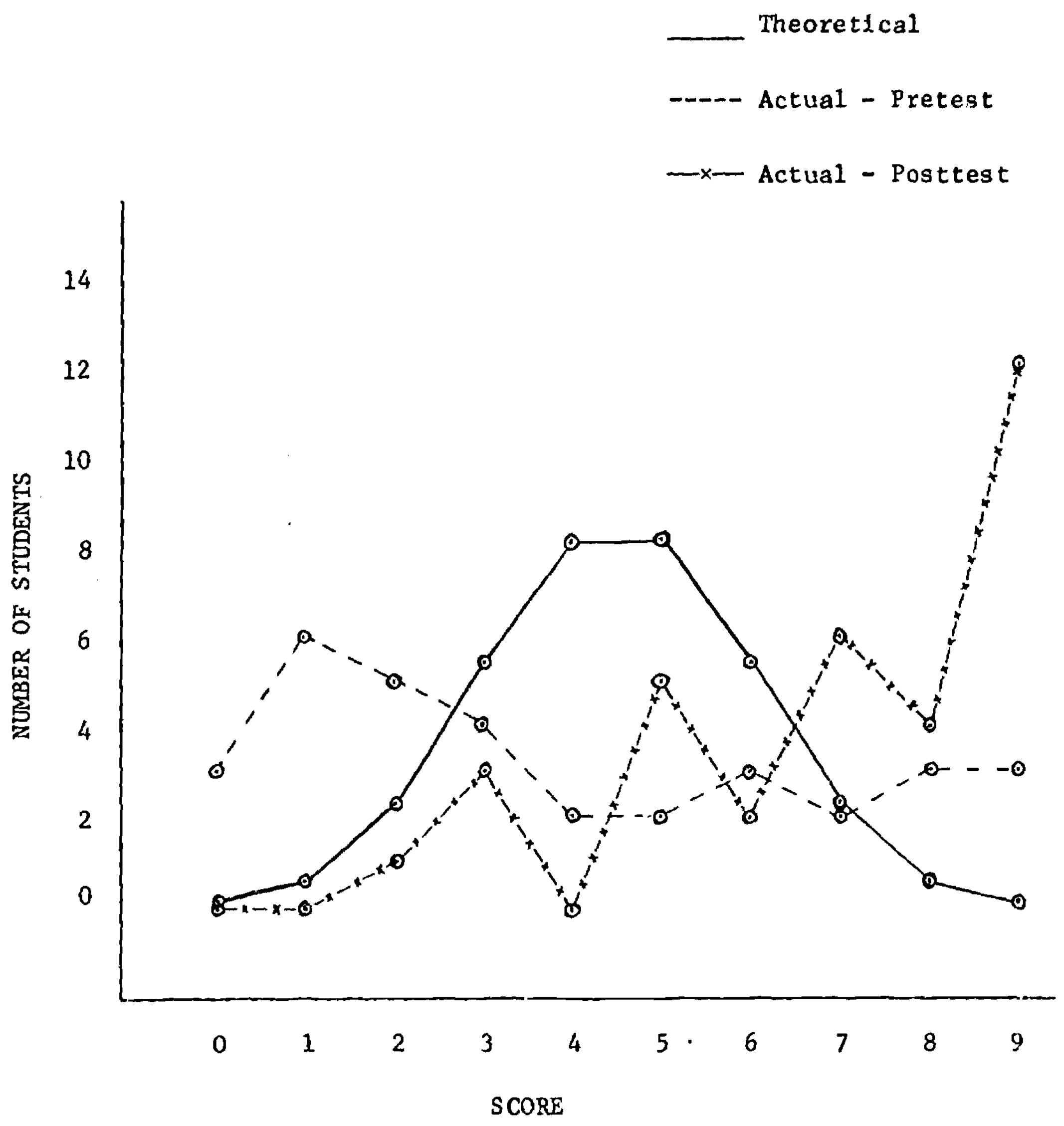
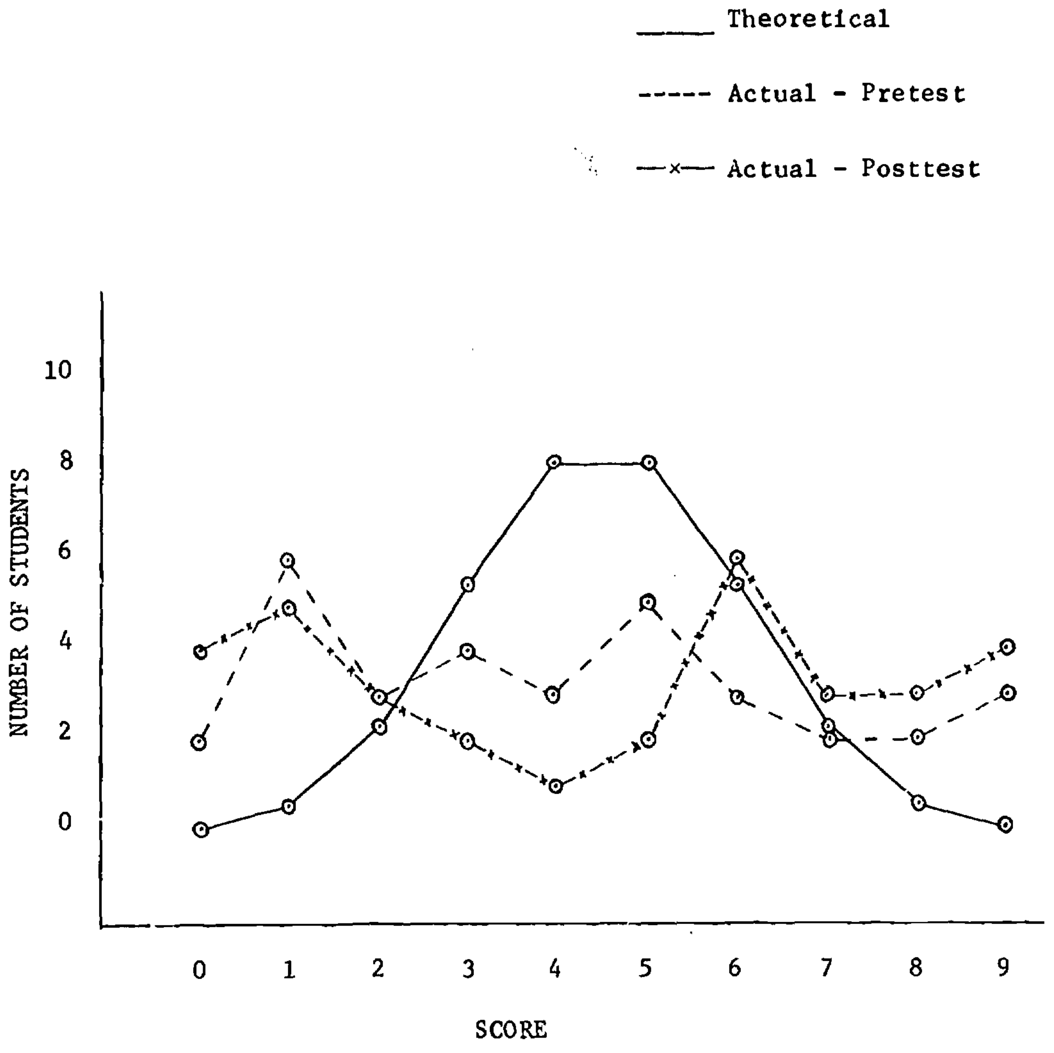


DIAGRAM V

THEORETICAL AND ACTUAL FREQUENCY OF SCORES

CONSERVATION OF LENGTH RELATIONS LEVEL II PRE- AND POSTTEST:

FIVE-YEAR-OLDS



The children's responses are analyzed further in Tables 43, 44, 45, and 46. Due to the way in which the test was scored, a child could conceivably make an "incorrect" comparison but still provide a measure of his ability to make a conservation response. Tables 43 and 44 contain, by age, two-by-two contingency Tables of Length Comparison by Conservation of Relations Posttest. These tables are given for each relation.

TABLE 43

CONTINGENCY TABLES

LENGTH COMPARISON BY CONSERVATION OF LENGTH RELATIONS POSTTEST: FOUR-YEAR-OLDS WHO MET CRITERION ON LEVEL I AND LEVEL II

Conservation Test		Length Comparison			
		Level I		Level II	
		C	I	C	I
Longer Than	C	13	2	11	1
	I	0	0	3	0
Shorter Than	C	14	1	8	4
	I	0	0	3	0
Same Length As	C	12	3	11	1
	I	0	0	1	2
Total	C	39	6	30	6
	I	0	0	7	2

C: Correct

I: Incorrect

TABLE 44

CONTINGENCY TABLES

LENGTH COMPARISON BY CONSERVATION OF LENGTH RELATIONS POSTTEST: FIVE-YEAR-OLDS WHO MET CRITERION ON LEVEL I AND LEVEL II

Conservation Test		Length Comparison			
		Level I		Level II	
		C	I	C	I
Longer Than	C	38	1	33	3
	I	3	0	6	0
Shorter Than	C	36	3	29	4
	I	2	1	8	1
Same Length As	C	32	4	30	6
	I	3	3	5	1
Total	C	106	8	92	13
	I	8	4	19	2

C: Correct I: Incorrect

Of the 342 comparisons made by the 19 children who met criterion for Level I and Level II, only 41 were "incorrect." Of these 41 "incorrect" comparisons, there were 32 corresponding correct conservation responses. There were 267 correct conservation responses among the remaining 301 comparisons. It is important to note that the correct Level II conservation responses compare quite favorably with the correct Level I conservation responses. It must oe noted that the frequencies in the contingency tables are fairly comparable across relations.

Tables 45 and 46 are analogous to Tables 43 and 44, except that they contain data for children who did not meet criterion on Level I and Level II.

TABLE 45

CONTINGENCY TABLES

LENGTH COMPARISON BY CONSERVATION OF LENGTH RELATIONS POSTTEST: FOUR-YEAR-OLDS WHO DID NOT MEET CRITERION ON LEVEL I AND LEVEL II

Conservation Test		Length Comparison			
		Level I		Level II	
		C	I	C	I
Longer Than	C	17	5	12	4
	I	8	9	19	4
Shorter Than	C	20	5	10	7
	I	8	6	15	7
Same Length As	C	14	7	10	9
	I	8	10	14	6
Total	C	51	17	32	20
	I	24	25	48	17

C: Correct I: Incorrect

The 32 children represented in Tables 45 and 46 made 143 "incorrect" comparisons out of 576 comparisons. The incorrect Level II conservation responses far exceeded the correct responses in the case of the five-year-olds. The opposite was true, however, for the Level I responses. In the case of the four-year-olds, the incorrect Level II conservation responses again exceeded the correct responses, but not in the same ratio as for the five-year-olds. Again, the frequencies are fairly comparable across relations.

Table 47 reveals that the classroom of which the children are members statistically appears to be related to the number of children in the classroom who meet criterion on the pretest. After instruction,

TABLE 46

CONTINGENCY TABLES

LENGTH COMPARISON BY CONSERVATION OF LENGTH RELATIONS POSTTEST: FIVE-YEAR-OLDS WHO DID NOT MEET CRITERION ON LEVEL I AND LEVEL II

Conservation Test		Length Comparison			
		Level I		Level II	
		C	I	C	I
Longer Than	C	43	3	12	2
	I	8	3	39	4
Shorter Than	C	42	6	14	4
	I	5	4	31	8
Same Length As	C	35	10	9	5
	I	9	3	31	12
Total	C	120	19	35	11
	I	22	10	101	24

C: Correct   I: Incorrect

TABLE 47

CONTINGENCY TABLE FOR THE LEVEL OF PERFORMANCE OF STUDENTS BY CLASS ON THE CONSERVATION OF LENGTH RELATIONS PRETEST

	Met Criteria	Did Not Meet Criteria	Total
Class # 1	5	13	18
Class # 2	1	15	16
Class # 3	0	17	17
Total	6	45	51

$\chi^2 = 7.164;\ p < .05$

TABLE 48

CONTINGENCY TABLE FOR THE LEVEL OF PERFORMANCE OF STUDENTS BY CLASS ON THE CONSERVATION OF LENGTH RELATIONS POSTTEST

	Met Criteria	Did Not Meet Criteria	Total
Class # 1	6	12	18
Class # 2	6	10	16
Class # 3	7	10	17
Total	19	32	51

χ^2 = .232; Not Significant at .05 level

this relationship was not statistically significant, as is shown in Table 48. The Chi Square of .232 is not significant at the .05 level.

Section 4

Conservation of Length

Table 49 shows that the reliabilities for the Conservation of Length test are low. A contribution to the low-test reliabilities is the existence of more than one factor in the test as indicated by Table 50. The items of both the pre- and posttest loaded on two factors. Factor 1 in the pretest was a combination of conservation involving the reflexive property and the type of transformation. Pretest Factor 2 was a combination of conservation involving the nonreflexive property and the type of transformation. The order of the factors reversed from pre- to posttest. It is noted that for both factors, two of the items involving the same property loaded with a higher value than the third. The third item always involved a different transformation.

TABLE 49

RELIABILITIES OF CONSERVATION OF LENGTH TESTS
(KUDER-RICHARDSON # 20)

Test	Reliability
Pretest	.43
Posttest	.53

TABLE 50

FACTOR ANALYSIS OF THE CONSERVATION OF LENGTH TEST

Item	Pretest		Posttest	
	1	2	1	2
1	.1971	.6889	.7808	.0553
2	-.1639	.6452	.8748	.0444
3	.1975	.3252	.5354	-.0007
4	-.9071	-.0168	-.0819	.7808
5	-.9341	.1090	-.0586	.8334
6	-.4188	.0184	.0940	.3311
Percent Communality	52.63	26.88	42.45	35.84

The item difficulties for the pretest, as shown in Table 51, ranged from .24 to .51 with four item difficulties below .40. The posttest item difficulties ranged from .37 to .88 with only one difficulty below .40. All of the item difficulties increased from pre- to posttest with the greatest increase being for the items involving the reflexive property. Table 52 gives the means and standard deviations of the pre- and posttest.

TABLE 51

ITEM DIFFICULTY OF CONSERVATION OF LENGTH TEST

Item	Difficulty	
	Pretest	Posttest
1	.43	.45
2	.24	.45
3	.29	.37
4	.29	.80
5	.35	.78
6	.51	.88

TABLE 52

MEANS AND STANDARD DEVIATIONS OF CONSERVATION OF LENGTH TEST

Test	Mean	Standard Deviation
Pretest	2.12	1.44
Posttest	3.75	1.46

It is noted in Table 53 that one four- and one five-year-old earned a score of five or six on the pretest. The number of four- and five-year-olds meeting the criterion on the posttest increased to six and nine respectively. The two children who met the pretest criterion did not meet the required level of performance on the posttest.

However, the proportion of students who changed from noncriterion to criterion is greater than the proportion of children who changed from criterion to noncriterion as noted in Table 54. There was also an increase from pre- to posttest in the number of children that responded correctly to all the reflexive items but did not meet the criterion. The change was from seven to twenty-one.

TABLE 53

RATIOS OF STUDENTS MEETING THE CRITERION FOR THE CONSERVATION OF LENGTH TESTS

Test and Group	Ratio
Pretest	
4-year-olds	1/18
5-year-olds	1/33
Total	2/51
Posttest	
4-year-olds	6/18
5-year-olds	9/33
Total	15/51

The classroom of which the children are members does not appear to be related statistically to the number of children in the classroom who meet criterion on the pre- and posttests of Conservation of Length as shown in Tables 55 and 56.

TABLE 54

McNEMAR TEST FOR THE SIGNIFICANCE OF CHANGES
PRE- AND POST- CONSERVATION OF LENGTH

		Posttest	
		-C	C
Pretest	C	2	0
	-C	33	15

$\chi^2 = 8.50$; $p < .005$; C - Met Criterion; -C - Did Not Meet Criterion

TABLE 55

CONTINGENCY TABLE: CRITERION BY CLASSROOM FOR
THE CONSERVATION OF LENGTH PRETEST

Classroom	Met Criterion	Did Not Meet Criterion	Total
#1	2	16	18
#2	0	15	15
#3	0	18	18
Total	2	49	51

$\chi^2 = 3.787$; Not Significant at .05

TABLE 56

CONTINGENCY TABLE: CRITERIO[illegible] CLASSROOM FOR THE CONSERVATION OF LE[illegible] POSTTEST

Classroom	Met Criterion	[illegible]ot Meet [illegible]erion	Total
#1	4	14	18
#2	6	10	16
#3	5	12	17
Total	15	36	51

χ^2 = .947; Not Significant at .05

When the distribution of total scores by the four-year-olds is considered (Table 57), it is found that the frequency distribution for both the pre- and posttests does depart statistically at the .005 level from the binomial distribution. Diagram VI contains the graphs of these distributions.

The theoretical and actual frequency distributions of scores earned by the five-year-olds on the pretest and posttest also depart statistically at the .005 level, as is indicated in Table 58. Diagram VII contains the graph of these distributions.

Table 59 contains the correlations of the pre- and posttest total scores with the variables (1) Verbal Maturity, (2) IQ, (3) Age, and (4) Social Class. All the correlations are low but the correlations between total scores and Social Class are significantly different from zero. There appears to be little, if any, correlation between Verbal Maturity, IQ, and Age and total scores.

TABLE 57

THEORETICAL AND ACTUAL FREQUENCIES OF CONSERVATION
OF LENGTH TEST SCORES: FOUR-YEAR-OLDS

Frequency	0	1	2	3	4	5	6
Pretest[1]							
Theoretical	.28	1.69	4.22	5.62	4.22	1.69	.28
Actual	3	3	5	4	2	0	1
Posttest[2]							
Theoretical	.28	1.69	4.22	5.62	4.22	1.69	.28
Actual	1	0	1	6	4	2	4

1 $\chi^2 = 32.755$, $p < .005$

2 $\chi^2 = 50.705$, $p < .005$

DIAGRAM VI

THEORETICAL AND ACTUAL FREQUENCY OF SCORES - CONSERVATION OF LENGTH PRE- AND POSTTEST: FOUR-YEAR-OLDS

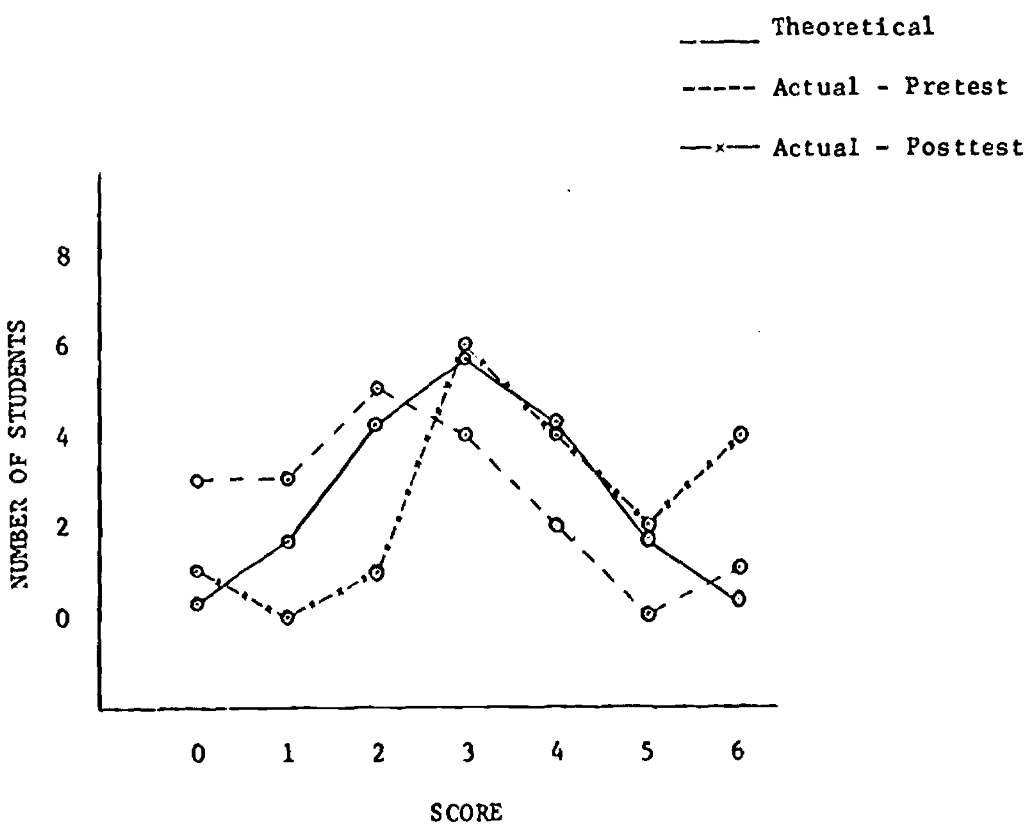


TABLE 58

THEORETICAL AND ACTUAL FREQUENCIES OF CONSERVATION OF LENGTH TEST SCORES: FIVE-YEAR-OLDS

Frequency	0	1	2	3	4	5	6
Pretest[1]							
Theoretical	.52	3.09	7.73	10.31	7.73	3.09	.52
Actual	4	8	8	9	3	0	1
Posttest[2]							
Theoretical	.52	3.09	7.73	10.31	7.73	3.09	.52
Actual	1	1	2	13	7	5	4

[1] $\chi^2 = 37.692$, $p < .005$

[2] $\chi^2 = 31.283$, $p < .005$

DIAGRAM VII

THEORETICAL AND ACTUAL FREQUENCY OF SCORES - CONSERVATION OF LENGTH PRE- AND POSTTEST: FIVE-YEAR-OLDS

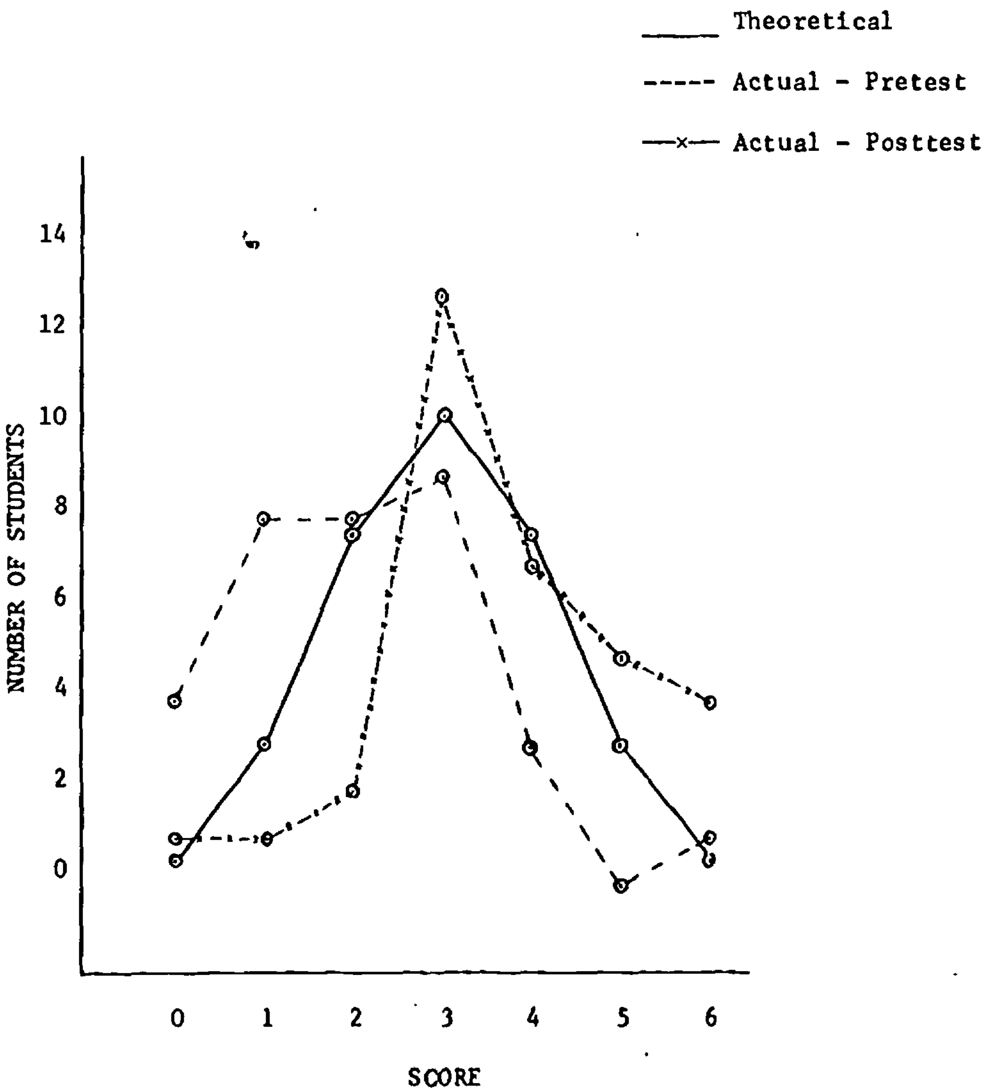


TABLE 59

CORRELATION MATRIX

STUDENT CHARACTERISTICS WITH CONSERVATION OF LENGTH PRE- AND POSTTEST SCORES

Test	Characteristics			
	Verbal Maturity	IQ	Age	Social Class
Pretest	-.03	-.07	.03	.42**
Posttest	.23	.06	.12	.40**

**Significantly greater than zero; $p < .01$

Section 5

Transitivity

The results of the internal-consistency reliability study of the Transitivity Test are given in Table 60. The reliabilities for both administrations are low. This may be expected as Table 61 reveals the existence of more than one factor in the test. For the first administration results, items involving transitivity of "shorter than" or "longer than" loaded greater than positive .5 on Factor 1. Factor 2 is a combination of transitivity of "longer than" and "same length as." The second administration Factor 1 is transitivity of "same length as." Factor 2 is transitivity of "shorter than" and "longer than."

Table 62 shows that the mean scores for the two administrations are 2.00 and 2.67, respectively. The increase in mean score from first to second administration corresponds to the increase in item difficulty for each item as indicated in Table 63. Only item 6 on the second administration has a difficulty that exceeds .5.

TABLE 60

RELIABILITIES OF TRANSITIVITY TEST
(KUDER-RICHARDSON # 20)

Administration	Reliability
1st	.50
2nd	.45

TABLE 61

FACTOR ANALYSIS OF THE TRANSITIVITY TEST

Item	First Administration 1	First Administration 2	Second Administration 1	Second Administration 2
1	.4362	.4998	.8001	-.0065
2	.2940	.4637	.5762	.3157
3	.6970	-.1967	.1468	-.4648
4	.5621	-.5222	.1204	-.5562
5	.3424	.2903	.4260	-.0658
6	.4241	-.0545	.7806	-.1921
Percent Communality	44.44	27.90	61.50	11.11

TABLE 62

MEANS AND STANDARD DEVIATIONS OF TRANSITIVITY TEST

Administration	Mean	Standard Deviation
First	2.00	1.48
Second	2.67	1.53

TABLE 63

ITEM DIFFICULTY OF TRANSITIVITY TEST

Item	Difficulty	
	First Administration	Second Administration
1	.35	.47
2	.15	.29
3	.39	.49
4	.37	.47
5	.29	.37
6	.43	.55

It is noted from Table 64 that four four-year-olds and five five-year-olds met the criterion (a total score of five or six) for transitivity on the first administration. A total of fifteen students met the criterion on the second administration of which five were four-year-olds and ten were five-year-olds. Five students that met the criterion on the first administration did not meet the criterion on the second administration. Table 65 reveals that three of these students were unable to make the necessary length comparisons upon which to base the transitive property. Therefore, only two students may have lost transitivity. The level of performance of one of these two students may involve a chance fluctuation since transitivity was exhibited three out of five times on the second administration.

TABLE 64

RATIOS: STUDENTS MEETING THE CRITERION ON THE TRANSITIVITY TEST: FIRST AND SECOND ADMINISTRATION

Group by Administration	Ratio
First Administration	
Total Group	9/51
4-year-olds*	4/19
5-year-olds	4/32
Second Administration	
Total Group	15/51
4-year-olds*	5/19
5-year-olds	10/32

*This ratio may have been obtained by guessing.

TABLE 65

LEVEL OF PERFORMANCE ON SECOND ADMINISTRATION TRANSITIVITY TEST OF STUDENTS MEETING THE CRITERION ON FIRST BUT NOT SECOND ADMINISTRATION

Student Number	Age Group	Ratio: Correct Comparisons	Correct Transitivity
12	5	1/6	1
18	4	2/6	2
23	4	4/6	1
26	5	5/6	3
51	4	1/6	0

The actual frequency distributions of scores earned by the four-year-olds, as given in Table 66, do not depart statistically at the .05 level from a binominal distribution based on guessing responses. The relations between the actual and theoretical frequency distributions are shown graphically in Diagram VIII. Since the actual frequency distribution for the four-year-olds does not depart from the theoretical distribution based on guessing responses, no four-year-olds will be considered to have the ability to use the transitive property of the length relations involved. In the calculation of the theoretical binominal distribution based on guessing responses, a probability p for correct responses is .30. This value is based on an efficiency level of .78 as calculated from the Length Comparison Application Test, first administration.

It can be seen from Table 67 that the actual frequency distributions of scores earned by the five-year-olds on the first and second administrations do depart statistically from a binominal distribution at the .01 and .005 levels, respectively. The main departure for the first administration scores is the number of 0 and 1 scores as is shown in Diagram IX. An increase in frequency of total scores in the range of 3 to 6 is noted for the second administration results.

An analysis of Table 68 reveals small differences between the mean Verbal Maturity scores for the children not meeting the criterion and those meeting the criterion on both the first and second administrations. The same is true for the mean IQ scores. There appears to be little, if any, difference between the mean age for the two levels of performance of any one age group.

TABLE 66

THEORETICAL AND ACTUAL FREQUENCIES OF FIRST AND SECOND ADMINISTRATION TRANSITIVITY TEST SCORES: FOUR-YEAR-OLD GROUP

Frequency	0	1	2	3	4	5	6
First Administration[1]							
Theoretical	1.88	5.30	6.24	3.92	1.38	.26	.02
Actual	2	7	3	3	3	1	0
Second Administration[2]							
Theoretical	1.88	5.30	6.24	3.92	1.38	.26	.02
Actual	3	4	4	3	4	1	0

1 $\chi^2 = 6.478$; Not significant at .05

2 $\chi^2 = 9.101$; Not significant at .05

Table 69 supports that there appears to be little, if any, relation between the variables (1) Verbal Maturity, (2) IQ, and (3) Age and the levels of performance. Also, the social classes in which the students are members do not appear to be related to their level of achievement.

TABLE 67

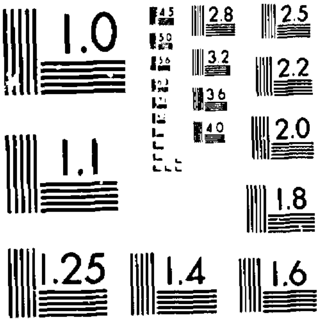
THEORETICAL AND ACTUAL FREQUENCIES OF FIRST AND SECOND ADMINISTRATION TRANSITIVITY TEST SCORES: FIVE-YEAR-OLD GROUP

Frequency	0	1	2	3	4	5	6
First Administration[1]							
Theoretical	3.16	8.93	10.51	6.59	2.33	.44	.03
Actual	8	3	10	6	3	2	0
Second Administration[2]							
Theoretical	3.16	8.93	10.51	6.59	2.33	.44	.03
Actual	2	4	4	12	5	4	1

[1] $\chi^2 = 17.185;\ p < .01$

[2] $\chi^2 = 74.852;\ p < .005$

ED 451



MICROCOPY RESOLUTION TEST CHART

DIAGRAM VIII

THEORETICAL AND ACTUAL FREQUENCY OF SCORES: TRANSITIVITY TEST
FIRST AND SECOND ADMINISTRATIONS
FOUR-YEAR-OLDS

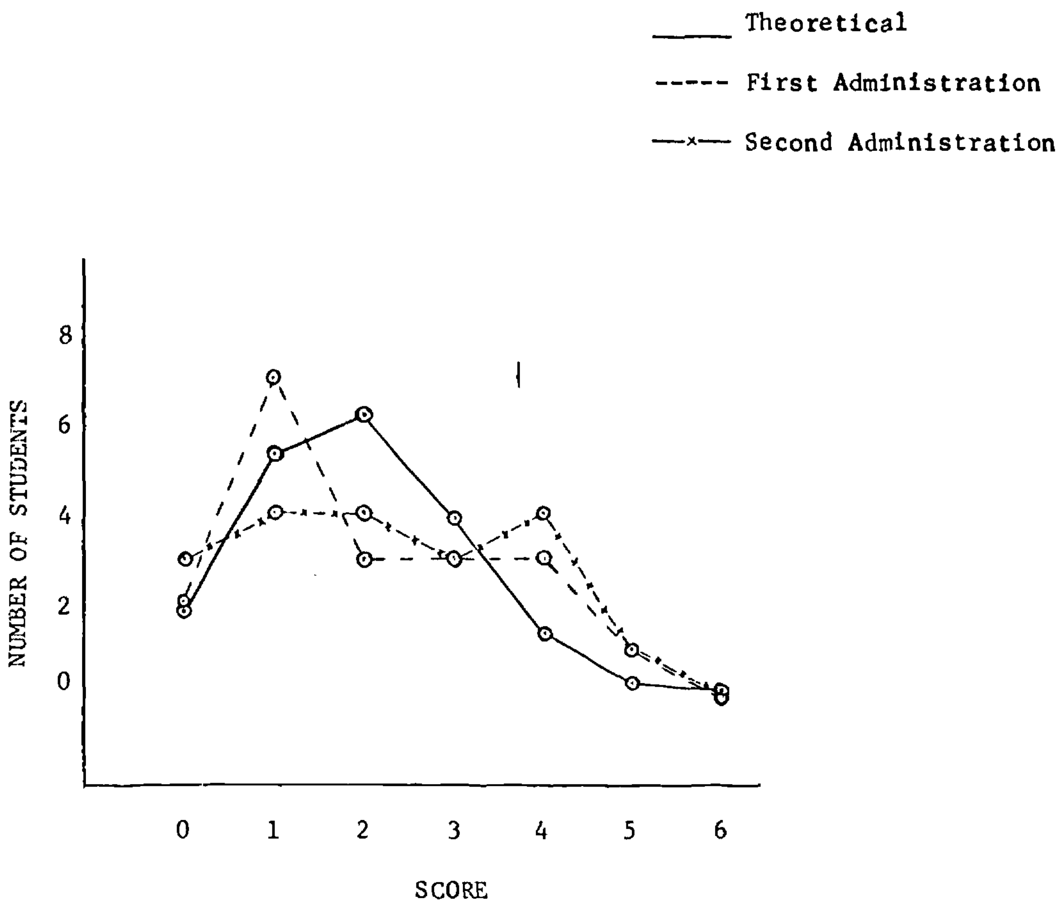


DIAGRAM IX

THEORETICAL AND ACTUAL FREQUENCY OF SCORES: TRANSITIVITY TEST
FIRST AND SECOND ADMINISTRATIONS
FIVE-YEAR-OLDS

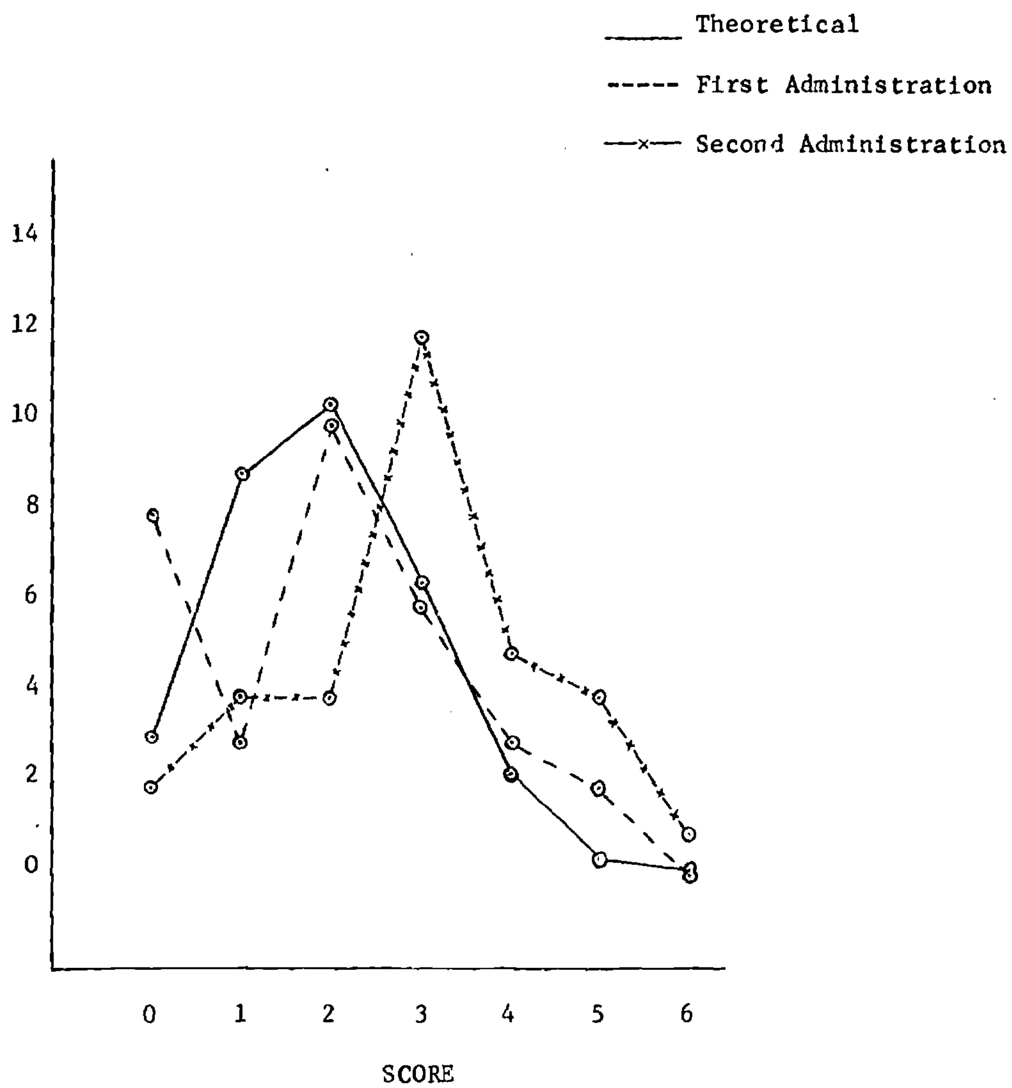


TABLE 68

CHARACTERISTICS OF STUDENTS BY LEVEL OF PERFORMANCE ON THE TRANSITIVITY TEST: FIRST AND SECOND ADMINISTRATION

	Mean Verbal Maturity	Mean IQ	Mean Age
First Administration - Criterion			
Total Group	103.9	115.8	59.6
4-year-olds	101.2	116.2	53.5
5-year-olds	106.0	115.4	64.4
First Administration - Not Criterion			
Total Group	99.5	112.9	60.6
4-year-olds	102.4	120.6	53.5
5-year-olds	97.9	108.7	64.5
Second Administration - Criterion			
Total Group	101.2	114.7	60.9
4-year-olds	103.0	118.0	53.4
5-year-olds	100.3	112.7	64.7
Second Administration - Not Criterion			
Total Group	99.6	113.4	60.2
4-year-olds	101.9	120.3	53.6
5-year-olds	98.1	108.9	64.5

TABLE 69

CORRELATION MATRIX

STUDENT CHARACTERISTICS WITH TRANSITIVITY TEST SCORES

Administration	Characteristics			
	Verbal Maturity	IQ	Age	Social Class
First	.10	.06	-.11	-.08
Second	-.05	.00	.30	.17

Section 6

Conservation and Transitivity Relationships

Prior to formal experiences in conserving length, 2 students met criterion on the Conservation of Length Test, as Table 70 reveals, neither met criterion on the Transitivity Test nor criterion for Conservation of Length Relations (Level I and Level II). One student met criterion only in the case of Conservation of Length Relations (Level I).

After formal experiences, only 1 student out of the 14 who met criterion on the Conservation of Length Test met criterion on the Transitivity Test. This student did not meet the criterion for Conservation of Length Relations (Level I or Level I and Level II). However, 7 students did meet criterion for Conservation of Length Relations: Level I and Level II, and 3 students met criterion only in the case of Conservation of Length Relations: Level I.

It is noted from Table 71 that of the 4 students who met criterion for Conservation of Length Relations (Level I but not Level II) prior to formal experiences in conservation of length or length relations,

TABLE 70

ANALYSIS OF PERFORMANCE ON CONSERVATION OF LENGTH RELATIONS AND TRANSITIVITY TESTS BY STUDENTS MEETING CONSERVATION OF LENGTH CRITERION

Conservation of Length:

Student Number	Level I Conservation of Length Relations	Level I and Level II Conservation of Length Relations	Transitivity
Pretest			
11	X		
(13)			
Posttest			
(1)	X		
4	X		
6			
(8)	X		
(23)			
25			X
26		X	
29		X	
(30)		X	
31			
37		X	
38		X	
39		X	
(50)		X	
(54)			

X Met the criterion ( ) Four-year-old

TABLE 71

ANALYSIS OF CRITERION PERFORMANCE ON CONSERVATION OF LENGTH AND TRANSITIVITY TESTS BY STUDENTS MEETING CRITERION FOR CONSERVATION OF LENGTH RELATIONS LEVEL I BUT NOT LEVEL II

Conservation of Length Relations: Level I but not Level II Student Number	Conservation of Length Test	Transitivity Test
Pretest		
(8)		
9		X
11	X	
16		X
Posttest		
(1)	X	
4	X	
5		
(8)	X	
12		
(13)		
(15)	X	
33		
41		
46		

X Met the criterion ( ) Four-year-old

2 met criterion on the Transitivity Test and 1 met criterion on the Conservation of Length Test. The 2 students meeting the transitivity criterion did not meet the conservation of length criterion.

Only 4 out of 10 students who met criterion for Conservation of Length Relations (Level I but not Level II) after formal experiences met criterion in the case of the Conservation of Length Test. None of the ten met criterion on the Transitivity Test.

Table 72 reveals that before formal experiences in Conservation of Length Relations, only 1 out of the 6 students who met criterion for Conservation of Length Relations (Level I and Level II) met criterion on the Transitivity Test. Not any of the 6 students met the criterion on the Conservation of Length Test.

Table 73 shows that 7 of the 19 students meeting the criterion for Conservation of Length Relations (Level I and Level II) met criterion on the Conservation of Length Test. Seven different students met criterion on the Conservation of Length Test. Five students were not able to conserve length or use the transitive property.

As noted in Table 74, before having formal experiences in Conservation of Length or Length Relations, only 2 or 5 students who met criterion on the Transitivity Test met criterion only for Conservation of Length Relations (Level I). One of these five students met Criterion for Level I and Level II. Not any of these 5 students met criterion on the Conservation of Length Test.

TABLE 72

ANALYSIS OF CRITERION PERFORMANCE ON CONSERVATION OF LENGTH AND TRANSITIVITY TESTS BY STUDENTS MEETING CRITERION FOR CONSERVATION OF LENGTH RELATIONS: LEVEL I AND LEVEL II PRETEST

Student Number	Conservation of Length	Transitivity
(3)		
5		
12		
(14)		X
17		N.S.
(30)		

X Met criterion N.S. = No score ( ) Four-year-old

On the posttest, 7 of the 10 students who met criterion on the Transitivity Test also met criterion for Conservation of Length Relations (Level I and Level II). Only 1 student met criterion on the Conservation of Length Test.

TABLE 73

ANALYSIS OF CRITERION PERFORMANCE ON CONSERVATION OF LENGTH AND TRANSITIVITY TESTS BY STUDENTS MEETING CRITERION FOR CONSERVATION OF LENGTH RELATIONS: LEVEL I AND LEVEL II POSTTEST

Student Number	Conservation of Length Test	Transitivity Test
Posttest		
2		X
(3)		
9		X
10		X
(14)		
17		
26	X	
27		
29	X	
(30)	X	
34		X
35		X
37	X	
38	X	
39	X	
40		X
44		X
(50)	X	
52		

X Met criterion ( ) Four-year-old

TABLE 74

ANALYSIS OF CRITERION PERFORMANCE FOR CONSERVATION OF LENGTH RELATIONS: LEVEL I OR LEVEL II AND CONSERVATION OF LENGTH TESTS BY STUDENTS MEETING CRITERION ON THE TRANSITIVITY TEST

Student By Test	Conservation of Length Relations: Level	Conservation of Length Relations: Level I and Level II	Conservation of Length
First Administration			
9	X		
12		X	
16	X		
26			
44			
Second Administration			
2		X	
9		X	
10		X	
16			
25			X
34		X	
35		X	
40		X	
44		X	
49			

X Met criterion

CHAPTER IV

CONCLUSIONS, DISCUSSION AND IMPLICATIONS

Section 1 and 2

Length Comparison

Before or after formal experiences in qualitative comparisons, the level of performance of four-year-old children in establishing a relation between two curves is not different from that of five-year-olds. These children do not perform at a different level when each of the relations of "longer than," "shorter than," and "same length as" are separately considered.

It appears that four- and five-year-old children easily learn the relation "longer than" from informal experiences in their environment, or testing facilitates learning of this relation. Heilen did find that testing facilitates first-grade children's solution of measurement tasks. The children's informal interaction with their environment does not seem to be sufficient for them to learn to compare objects in terms of "shorter than" and "same length as."

Formal experiences in qualitative length comparisons does significantly improve the ability of both four- and five-year-old children to make length comparisons. The formal experiences utilized in this study involved a continuous interplay between language and manipulation of objects as Bruner recommends. This was an endeavor to eliminate experiences solely dependent upon language and not real practical action which Adler considers to be a failure of formal education. Also, the continuous interaction between language and action on materials may have aided the children in not responding on a perceptual basis as has been suggested by Wohlwill.

Moreover, small group instruction in qualitative length comparison significantly improves the ability of four- and five-year-old children to establish each of the relations "shorter than" and "same length as" between two curves. The formal experiences appear to have the greatest influence on the children's level of performance with length comparison involving the relation "shorter than."

The ability of four- and five-year-old children to make length comparisons involving the relations "longer than," "shorter than," and "same length as" is not limited to the situations in which they learned to use these relations. These children have the ability to use the relations in novel length comparison situations. The formal experiences with concrete materials may have been sufficient for the majority of the children to reach an overt operational level with the qualitative length relations. This level of performance was retained over the several months this study was in progress.

There appears to be little, if any, relation between the variables of Verbal Maturity, IQ, Age, and Social Class and the ability of four- and five-year-old children to use the qualitative length comparisons of "longer than," "shorter than," and "same length as." This is similar to

Beilen's finding that IQ was not a factor in first grade children's learning to measure length.

Section 3

Conservation of Length Relations

The definitions given for length relations on a qualitative basis and conservation of these relations (i.e., that the relation obtains regardless of the proximity of the curves) seem to have been supported by the results of the study. On the Length Comparison Application Test, first administration, the mean score was 78 percent with a standard deviation of only 3.89. At this point in time, the children in the study were able to associate a relational term with an overt comparison of curves in such a way that they were able to discriminate among the comparisons denoted by "longer than," "shorter than," and the "same length as." The particular relation a child established on the first administration of the application test through overt comparison was a function of the proximity of the curves involved. This is supported by the fact that at most two children could be classified at only Level I and at most four children could be classified at Level I and Level II (those four who met criterion on both the first and second administration for Conservation of Length Relations). With the exception of these last four children and possibly the former two, there is no evidence that at the time of the first administration of the Application Test an overt comparison constituted a logical-mathematical experience for the child making the comparison. The overt comparison was certainly not sufficient for the child (using Piaget's terms) to disengage the structure of the relation he/she established. It certainly may be the case that the relation for the child not only was a function of the proximity of the curves but was a function of the external physical situation so that he/she did not think about the relation in the absence of the external situation. In Bruner's terms, the child had not internalized the relation; or in Lovell's terms, the child was not aware of the significance of his actions in the overt comparison of the curves.

The definitions of Level I and Level II were well supported by the factor analysis on the pretest. These analyses show that the items written at Level I and Level II involve differential abilities. In particular, for the pretest, the items at Level II which involved the asymmetrical property of "longer than" or "shorter than" loaded on Factor 2 as well as an item involving a logical consequence of "the same length as." On the posttest, the items written at Level I were much less difficult than those items written at Level II which certainly may contribute to the factors present in the factor analysis.

Level I items were constructed to measure the extent to which the children realize that the qualitative length relation they established between two curves is independent of the proximity of the curves. As noted, before the administration of Units II and III, only about 12 percent of the children could be categorized at Level I. After the administration of Units II and III, however, the evidence indicated that about 57 percent of the children could be categorized at that level.

At the same two points in time, the percentages were 8 and 37 with regard to Level I and Level II, which was a statistically significant change. It must be emphasized that the children in this 37 percent not only were able to establish a relation between two curves and retain the relation regardless of the proximity of the curves but were able to use the asymmetric property and logical consequences of the relations under consideration. It is certainly true that the experiences contained in Units II and III did not readily increase the children's ability to use logical consequences of the relation they were able to establish.

The data suggest that the mean IQ for the five-year-old children who met criterion for Level I and Level II is greater than the mean IQ for those who did not meet criterion. The correlation of total scores for Level I and Level II with the variables of Verbal Maturity, IQ, Age and Social Class are not significant with the possible exception of a low correlation between IQ and Level II posttest scores.

Section 4

Conservation of Length

Very few four- and five-year-old children are able to conserve length prior to formal experiences in conservation of length. Conservation of length referred to here involves both the reflexive property of "the same length as" and nonreflexive property of "longer than" or "shorter than." Elkind apparently would classify this type of conservation as conservation of identity even though he did not subdivide conservation of identity with regard to the reflexive and nonreflexive properties. An effort is made not to confuse conservation of length with conservation of length relations which Elkind refers to as conservation of equivalence.

Some four- and five-year-old children have the ability to conserve length involving the reflexive property but not the nonreflexive property. Informal experiences appear to be sufficient for these children to exhibit this type of conservation of length. Before formal experiences, 14 percent of the sample used conservation of length involving the reflexive property compared to four percent who conserved length using both properties.

Selected experiences significantly increase the ability of four- and five-year-old children to conserve length involving both properties. After the formal experiences, 41 percent of the sample conserved length involving only the reflexive property and 30 percent of the sample conserved length involving both. Only 29 percent of the sample did not have the ability to conserve length involving the reflexive or nonreflexive properties.

The above conclusions substantiate Piaget's Theory that experience is necessary but not sufficient for the development of logical thought since all the children received the same selected experiences.

The data substantiate that the ability to use the reflexive property is different from and precedes the ability to use the nonreflexive property. It appears that reflexive situations are not sufficient to determine a child's ability to use conservation of length.

Conservation of length is not unitary in nature relative to the reflexive and nonreflexive properties.

There appears to be little, if any, relation between the student variables Verbal Maturity, IQ, Age, and Social Class and scores earned by four- and five-year-old children on conservation of length items involving the reflexive or nonreflexive property. Only correlations involving Social Class were significantly different from zero, but these correlations were low. These variables seem to have very little effect on the ability of four- and five-year-olds to benefit from formal or informal experiences in conserving length.

Section 5

Transitivity

Few five-year-old children were able to use the transitive property after only formal experiences in establishing length relations. At this point in time, only 16 percent of the five-year-olds used the transitive property. The distribution of total scores for the four-year-olds did not statistically depart from a distribution based on guessing.

The experiences in establishing length relations do not appear to be sufficient for qualitative transitivity. Some children performed poorly due to their inability to establish the two initial comparisons. Smedslund considers this as a reason for the failure of some yound children to use the transitive property.

Formal experiences in establishing length relations, conserving length relations, and conserving length do increase the ability of five-year-olds to use the transitive property. The percentage of five-year-olds able to use the transitive property increased to 31. These same experiences do not seem to increase the ability of four-year-old children to use the transitive property since again the distribution of total scores for the four-year-olds did not statistically depart from a distribution based on guessing. The number of five-year-olds that used qualitative transitivity of relations is below that found by Braine but above that found by Smedslund. It appears that these experiences were not logical-mathematical experiences that readily increase children's ability to use the transitive property. All the children may not have had a mental structure sufficient to allow assimilation of the information as is emphasized by Piaget.

The mean Verbal Maturity and IQ of five-year-old children who are able to use the transitive property appears to be slightly higher than for those who do not use this property. However, the correlations between these two variables and transitivity scores earned by the total sample was not statistically different from zero. Also, there appears to be little, if any, relationship between the variables Age and Social Class and the ability of four-and five-year-old children to use the transitive property.

Section 6

Conservation and Transitivity Relationships

In this section, the interrelationships of conservation of length, conservation of length relations, and transitivity of length relations will be discussed on each of the first and second administrations.

As noted earlier, the Conservation of Length Test involved both the reflexive property of "the same length as" and the nonreflexive property of "longer than" and "shorter than." On the first administration (pretest), only two children met criterion on this test so that a discussion of interrelationships is not appropriate. However, on the posttest, 30 percent of the children met criterion. Of this 30 percent, only one child met criterion on the Transitivity Test. Since there were 10 children who met criterion on the Transitivity Test, it is quite apparent that the ability to conserve length as measured here is not a necessary nor a sufficient condition for the ability to use transitivity of length relations. This observation is quite consistent with the fact that the reflexive property of "the same length as" does not imply the transitive property of "the same length as" nor does the nonreflexive property of "longer than" or "shorter than" imply the transitive property of these two relations on a logical basis. Conversely, the transitive property of "longer than" or "shorter than" does not imply the nonreflexive property of these two relations. Since, on a logical basis the reflexive property is a consequence of the symmetric and transitive properties, and since some children could use the reflexive property but not the transitive property, there may be factors which enable children to use the reflexive property before they are able to use transitivity (e.g., spatial imagery or the definition of "the same length as"). In fact, the results indicate that the reflexive property may be necessary for transitivity or is not related to transitivity but easily obtained by children. It may be that the use of the reflexive property is more of a "learned response" than a logical-mathematical process.

It also appears that conservation of length involving both the reflexive and nonreflexive properties is not a necessary nor sufficient condition for conservation of length relations: Level I and Level II. Of the 30 percent who met criterion for conservation of length, only seven children met criterion for conservation of length relations: Level I and Level II. These observations are also consistent with the logical interrelationships of the properties of the relations. However, the data do not contradict the fact that conservation of length involving only the reflexive property may precede conservation of length relations Level I and, therefore, Level II. If, as Elkind stated, Piaget's aim is to explain conservation of identity (interpreted here as conservation of length), then the data of this study support the fact that conservation of length is not unitary in nature nor can one argue that conservation of length involving the nonreflexive property is a necessary condition for conservation of length relations at either of the Levels I or II as Elkind seems to suggest. On a logical basis and on a psychological basis, when one considers "conservation" problems, it is necessary to consider the properties of the relation involved.

For those 19 children who met criterion for conservation of length relations: Level I and Level II, 7 met criterion on the Transitivity Test. Since only 10 children met criterion on the Transitivity Test, it seems that conservation of length relations: Level I and Level II is necessary for qualitative transitivity. The fact that 2 of 3 children who met criterion on the Transitivity Test but not for length relations: Level I and Level II, did not meet criterion for Level I or for conservation of length, indicates an inaccurate assessment. The above data are consistent with Smedslund's observation that what he calls conservation of length is a necessary condition for what he calls transitivity.

The study involves many implications for further research and development. Among these implications, the following are relevant;

(1) With the exception of the transitive property, it may be highly important to introduce first the properties, interrelationships, and consequences of the relations involved at the point in time in which the children are first able to associate a relational term with an overt comparison and before perceptual conflict is introduced. The children could then observe, with perceptual support, the properties, etc., involved. If the children were thus able to learn that the relation(s) they establish is (are) not a function of the proximity of the curves involved, they may be able to use the properties, etc., in the absence of perceptual support, and indeed, even in the presence of perceptual conflict. (2) The relation "as many as," "more than," and "fewer than," and their properties are basic in the development of the cardinal numbers. For this reason, an analogous study as suggested in (1) above is important. (3) If children are able to learn particular equivalence or order relations and their properties, interrelationships and consequences, are they able to transfer this knowledge to other such relations given knowledge of that relation. (4) On a logical basis, the relations involved in this study are basic to measurement. Moreover, the relation of "more than," "fewer than," and "as many as" are basic to cardinal numbers. Are the relations basic also on a psychological basis?

BIBLIOGRAPHY

1. Adler, Marilynne J. Some Educational Implications of the Theories of Jean Piaget and J. S. Bruner. Canadian Educational and Research Digest, IV (1964), 291-305.

2. Almy, Milley, et.al. Young Children's Thinking: Studies of Some Aspects of Piaget's Theory. Teachers College Press, Teachers College, Columbia University, New York, 1966.

3. Beilin, Harry and Franklin, Irene C. Logical Operations in Area and Length Measurement: Age and Training Effects. Child Development, 1962, 33, 607-618.

4. Beilin, Harry. Learning and Operational Convergence in Logical Thought Development. Journal of Experimental Child Psychology, 1965, 2, 317-339.

5. Beilin, Harry. Perceptual-Cognitive Conflict in the Development of an Invariant Area Concept. Journal of Experimental Child Psychology, 1964, 1, 208-226.

6. Braine, Martin D.S. Development of a Grasp of Transitivity of Length: A Reply to Smedslund. Child Development, 1964, 35, 799-810.

7. Braine, Martin D.S. The Ontogeny of Certain Logical Operations: Piaget's Formulation Examined by Nonverbal Methods. Psychological Monographs: General and Applied. No. 475, 1959.

8. Bruner, Jerome S. and Kenney, Helen J. Representation and Mathematics Learning. In Mathematical Learning, Monographs of the Society for Research in Child Development. Serial No. 196, 50-59.

9. Bruner, Jerome S., Olver, Rose S. and Greenfield, Patricia M., et. al. Studies in Cognitive Growth, New York: John Wiley and Sons, Inc., 1966.

10. Buck, R. Creighton. Advanced Calculus. New York: McGraw-Hill Book Company, Inc., 1956.

11. Elkind, David. Piaget's Conservation Problems. Child Development, 1967, 38, 15-27.

12. Feller, William. An Introduction to Probability Theory and Its Applications. New York: John Wiley and Sons, Inc. 1957.

13. Greenhouse, Samuel W. and Geisser, Seymour. On Methods in the Analysis of Profile Data, Psychometrika, 1959, 24, 95-112.

14. Harper, E. Harold and Steffe, Leslie P. The Effects of Selected Experiences on the Ability of Kindergarten and First Grade Children to Conserve Numerousness. Technical Report No. 38, Research and Development Center for Cognitive Learning, C-03, OE 5-10-154. Madison: University of Wisconsin.

15. Kelly, Francis J. and Kaufman, Burt A. Needed Research in Mathematics Curriculum: Reaction Paper. Journal of Research and Development in Education, 1967, 1, 123-128.

16. Lovell, Kenneth. Concepts in Mathematics: Analysis of Concept Learning. eds. Herbert J. Klausmier and Chester W. Harris. New York: Academic Press, 1966, 207-222.

17. Lovell, K., Healy, D. and Rowland, A. D. Growth of Some Geometrical Concepts. Child Development, 1962, 33, 751-767.

18. Piaget, Jean and Inhelder, Barbel. The Child's Conception of Geometry. New York: Harper and Row, 1964.

19. Piaget Rediscovered. A Report of the Conference on Cognitive Studies and Curriculum Redevelopment. eds. Ripple, Richard E. and Rockcastle, Verne E. Cornell University, School of Education, March, 1964.

20. Rosenbloom, Paul C. Implications of Piaget for Mathematics Curriculum. Improving Mathematics Education for Elementary School Teachers: A Conference Report. ed. W. Robert Houston. Michigan State University, College of Education, August, 1967.

21. Romberg, Thomas A. and DeVault, M. Vere. Mathematics Curriculum: Needed Research. Journal of Research and Development in Education, 1967, 1, 95-112.

22. Rosskopf, Myron F. Needed Research in Mathematics Curriculum: Peaction Paper. Journal of Research and Development in Education, 1967, 1, 115-116.

23. Roydan, H. L. Real Analysis. New York: The Macmillan Company, 1964.

24. Seigel, Sidney. Nonparametric Statistics for the Behavioral Sciences. New York: McGraw-Hill Book Company, Inc. 1956.

25. Smedslund, Jan. Concrete Reasoning: A Study of Intellectual Development Monographs of the Society for Research in Child Development. Serial No. 93, 1964.

26. Smedslund, Jan. Development of Concrete Transitivity of Length in Children. Child Development, 1963, 34, 389-405.

27. Steffe, Leslie P. The Performance of First Grade Children in Four Levels of Conservation of Numerousness on Three IQ Groups When Solving Arithmetic Addition Problems, Technical Report No. 14, Research and Development Center for Cognitive Learning, C-03, OE5-10-154. Madison, University of Wisconsin.

28. Van Engen, Henry. Needed Research in Mathematics Curriculum: Reaction Paper. Journal of Research and Development in Education, 1967, 1, 10-114.

29. Wohlwill, J. F. A Study of the Development of the Number Concept By Scalogram Analysis. Journal of Genetic Psychology, 1960, 97, 345-377.

30. Zimiles, H. The Development of Differentiation and Conservation of Number. Monograph of the Society for Research in Child Development, 31, No. 6, 1966.

APPENDIX I

Student Characteristics

Student Number	Age (Months)	IQ	Verbal Maturity	Social Class
1	53	128	116	IV
2	59	103	81	III
3	56	118	105	III
4	69	81	79	V
5	62	128	103	III
6	68	98	88	III
7	65	111	108	II
8	55	116	119	III
9	62	128	120	IV
10	65	114	85	IV
11	60	119	100	IV
12	63	122	117	III
13	57	109	100	III
14	57	130	103	III
15	47	98	98	III
16	63	95	97	V
17	69	116	96	III
18	50	120	103	I
19	67	112	91	II
20	67	89	55	V

Student Number	Age (Months)	IQ	Verbal Maturity	Social Class
21	59	103	86	I
22	49	117	103	II
23	54	116	107	II
24	67	103	107	II
25	62	115	117	III
26	66	122	102	IV
27	62	117	114	I
28	60	101	70	II
29	62	130	116	III
30	57	107	105	II
31	62	115	103	III
32	54	118	112	III
33	68	107	112	III
34	69	120	90	III
35	68	121	98	II
36	56	145	102	I
37	60	126	112	III
38	67	111	85	IV
39	69	87	79	V
40	67	120	108	II

Student Number	Age (Months)	IQ	Verbal Maturity	Social Class
41	63	97	105	I
42	57	136	105	III
43	48	134	105	III
44	68	110	94	II
45	67	96	97	IV
46	64	107	98	II
47	52	113	92	IV
48	66	85	97	III
49	64	101	113	III
50	52	132	114	III
51	53	122	90	I
52	57	114	97	II
53	51	103	82	IV
54	56	116	95	IV

APPENDIX II

Instructional Units

INSTRUCTIONAL UNIT I

LENGTH COMPARISON

Lesson 1

Activities

1. Select two girls of different heights. (We will call the taller girl Mary and the shorter girl Amy.) Place Mary and Amy so that they are standing back-to-back on the floor.

   Ask:

   A. "Who is taller?" (If incorrect response, tell pupils the correct response.)

   B. "Who is shorter?" (If response is incorrect, tell pupils the correct response.)

2. Now, have Amy stand on a chair and Mary stand by the chair. The girls should be back-to-back.

   Ask:

   A. "Who is taller when Amy is standing on the chair?"

   B. "Who is shorter when Amy is standing on the chair?"

3. Again, have the two girls stand back-to-back on the floor. Ask or comment:

   A. "Who is taller?" (If the response is incorrect, tell pupils correct response.)

   B. "Who is shorter?" (If incorrect response, tell pupils correct response.)

   C. "Mary and Amy both have their feet on the floor. Mary is taller than Amy. Amy is shorter than Mary."

4. Instruct Amy to stand on the chair.

   A. "Who is higher?"

   B. "But, who is taller?" (If any incorrect responses, Ask:

   "Are Amy's feet on the floor?" (Answer your own question.)

5. Ask Amy to put her feet on the floor. Comment:

   A. "See, Mary is taller. Amy is shorter."

   B. "Amy was higher because her feet were on the chair."

6. Again, have Amy stand on the chair. Be sure the two girls are standing back-to-back.

   Ask:

   A. "Now, who is higher?"

   B. "Who is taller?"

   C. "Who is shorter?"

7. Instruct Mary to stand back-to-back with Amy on the chair. Ask or comment:

   A. "Who is taller?"

   B. "Who is shorter?"

   C. "Mary is taller than Amy. Amy is shorter than Mary."

   Repeat the instructions 1-7 using different children. Be sure that the children in any pair are of different heights.

8. Select two boys. (We will call the taller boy Tom and the shorter Dick.) Place Tom standing on the floor with Dick lying at his feet. Ask or comment:

   A. "Is Tom taller than Dick?"

   B. "Is Tom shorter than Dick?"

C. "Let us find out who is taller."

D. "What can we do to find out who is taller?"

(If any pupils do not know the correct response, tell them.)

9. Instruct Tom and Dick to stand back-to-back on the floor.

Ask or comment:

A. "Who is taller?"

B. "Who is shorter?"

C. "Tom and Dick are both standing on the floor. Tom is taller than Dick. Dick is shorter than Tom."

10. Select two boys and have them lie on the floor. They need not be in parallel positions. (We will call the taller boy Jack and the shorter boy Bill.)

Ask or comment:

A. "Do we know who is taller?"

B. "How can we find out who is taller?"

C. "Let us find out who is taller without Jack and Bill standing on the floor."

11. Instruct Jack and Bill to lie on their backs with their feet against the rectangular block.

Ask or comment:

A. "Who is taller?"

B. "Who is longer?"

C. "Jack is taller than Bill. Jack is also longer than Bill."

12. Have Jack and Bill stand back-to-back on the floor.

Ask or comment:

A. "Now, who is longer?"

B. "When Jack and Bill are both lying on the floor, Jack is longer than Bill. When they are both standing, Jack is still longer than Bill."

Repeat instructions 10-12 utilizing different children. Be sure that the children in any pair are of different heights.

Lesson 2

Materials

2 3-ft. boards (1 red, 1 white); 2 2 1/2-ft. boards (1 red, 1 white);
1 2-ft. 11-in. board (red); 1 3-ft. board (white)

Activities

1. Place one red 3-foot board and one white 2 1/2-foot board so that they are touching each other and also perpendicular to the floor.

   Ask:

   A. "Which board is longer?" (If incorrect response, tell pupils the correct response.)

   B. "Which board is shorter?" (If incorrect response, tell pupils the correct response.)

2. Place the white board on a chair. Place the red board next to the chair with one end on the floor.

   Ask:

   A. "Now, which board is longer?"

   B. "Which board is shorter?"

3. Again, stand the boards on the floor as in instruction #1.

   A. "Which board is longer?" (If incorrect response, tell pupils the correct response.)

   B. "Which board is shorter?" (If incorrect response, tell pupils the correct response.)

C. "Both boards are on the floor. The red board is longer than the white board. The white board is shorter than the red board."

4. Place the white board on the chair. Place the red board next to the chair with one end on the floor.

   Ask or comment:

   A. "Which board is higher?"

   B. "But, which board is longer?" (If any incorrect responses, ask: "Are both boards on the floor?" Answer your own question.)

5. Place the white board on the floor.

   Comment:

   A. "See, the red board is longer."

   B. "The white board is shorter."

   C. "The white board was higher because it was on the chair."

6. Again, place the white board on the chair.

   Ask:

   A. "Now, which board is higher?"

   B. "Which board is longer?"

   C. "Which board is shorter?"

7. Place the red board next to the white board on the chair.

   Ask or comment:

   A. "Which board is longer?"

   B. "Which board is shorter?"

   C. "The red board is longer than the white board. The white board is shorter than the red board."

8. Lay the two 2 1/2-ft. boards on the floor. Do not be concerned about their positions.

   Ask or comment:

   A. "Which board is longer?"

   B. "How can we find out which board is longer?"

   C. "Let us stand the boards on the floor."

9. Stand the boards next to each other on the floor.

   Ask or comment:

   A. "Is one board longer?"

   B. "Is one board shorter?"

   C. "The boards are the same length. The red board is the same length as the white board. The white board is the same length as the red board."

10. Lay the two 3-foot boards on the floor. Do not be concerned about their position.

   A. "Which board is longer?"

   B. "How can we find out which board is longer?"

   C. "Could we put one end of each board against a block of wood?"

11. Place an end of each of the 3-foot boards against the block of wood.

   Ask or comment:

   A. "Is one board longer?"

   B. "Is one board shorter?"

   C. "The boards are the same length. The red board is the same length as the white board. The white board is the same length as the red board."

12. Lean the 2-foot 11-inch (red) and 3-foot (white) boards against a desk or some other object. Be sure that the top end of each board is at the same level. Do this before the activity begins.

Ask or comment:

A. "Everyone look at these two boards."

B. "Can we tell if these boards are the same length by looking?" (Answer your own question.)

C. "We must do something with the boards to find out if they are the same length."

13. Hold the two boards in positions that do not allow the pupils to compare their lengths. Have one end of each board touching the floor. Ask:

A. "Now can we tell if the boards are the same length?" (Answer your own question.)

14. Stand the two boards next to each other in a vertical position on the floor.

Ask:

A. "Now can we tell if the boards are the same length?" (Answer your own question.)

B. "Are the boards the same length?" (Answer your own question.)

C. "Which board is longer?"

Lesson 3

Materials

Four pencils for each pupil (3 of equal length, 1 a quarter-inch shorter); one stick for each child

Activities

1. Pair off the pupils. Give each partner of each pair a stick of different lengths. Ask each child separately the following:
   - A. "Which stick is longer?" (If they do not know how to find the answer to this question, give individual help until they understand.)
2. Give each pupil two pencils of different lengths, i.e., one a quarter-inch shorter than the other. Say:
   - A. "Hold up the longer pencil." (Check each pupil; if necessary, give individual help in comparing.)
   - B. "Hold up the shorter pencil." (Check each pupil; if necessary, give individual help in comparing.)
3. Give each pupil two pencils of the same length. Ask:
   - A. "Is one pencil longer?" (Give individual help to pupils who do not compare the two pencils.)

Lesson 4

Materials

One bag of ten sticks for each pupil

Game

Pair off the pupils. Give each member of every pair one bag of sticks (one of red sticks and one of green sticks per pair).

Game Instructions

Each pupil is to take a stick out of his bag and compare the length of the stick with that of his partner's stick. The pupil who has the longer stick will be allowed to keep his partner's stick. (Be sure that pairs compare the sticks properly.) After a few minutes have the pupils stop the game and determine the winner in each pair by matching. Then change partners. Repeat the game but this time allowing the pupil with the shorter stick to keep his partner's stick.

Lesson 5

Materials

1 block of wood; 1 6 1/2-ft. rope; 1 6-ft. board; 1 4-ft. board; 1 3 1/2-ft. rope; 6 ropes and one stick per pupil

Activities

1. Place a 6-foot board and a 6 1/2-foot coiled rope on the floor.

   Ask:

   A. "Which is longer, the rope or board?" (Do not be concerned with answers.)

   B. "How can we find out which is longer?" (Regardless of the suggestions, have the pupils compare the board and rope by placing one end of each against a block of wood. Be sure pupils know that the rope must be uncoiled.)

   C. "Is the rope longer than the board?" (Be sure every pupil knows the correct answer.)

2. Place a 4-foot board and a 3 1/2-foot coiled rope on the floor.

   Ask:

   A. "Is the rope or board longer?" (Do not be concerned with the answers.)

   B. "How can we find out which is longer without using the wood block?" (Regardless of the suggestions, have the pupils compare the board and rope by placing one end of the rope adjacent to one end of the board.)

C. "Is the rope shorter than the board?" Be sure every pupil knows the correct answer.)

3. Give each pupil six ropes and a stick. While making the following comments, be sure that the pupils are using correct procedures.

   A. "Find the ropes that are longer than the stick."

   B. "Find the ropes that are shorter than the stick." (Check each pupil.)

   C. "Find the ropes that are the same length as the stick." (Check each pupil.)

Lesson 6

Materials

1 3-ft. rope; 1 3 1/2 ft. rope; 1 4-ft. rope; 1 4 ft. 2-in. rope; 1 ball of string; 1 pair of scissors

Activities

1. Place a 3-ft. rope and a 3 1/2 ft. rope on the floor. Ask:
   A. "Which rope is longer?" (Do not be concerned with the answer.)
   B. "How can we find out which rope is longer?" (Regardless of the suggestions, have the pupils compare the ropes by placing one end of each against the block of wood.)
   C. "Which rope is longer?" (Be sure all know correct answer.)
2. Place a 4-ft. rope and a 4-ft. 2-in. rope on the floor. Ask:
   A. "Which rope is shorter?" (Do not be concerned with the answers.)
   B. "How can we find out which rope is shorter without using the wood block?" (Regardless of suggestions, have pupils compare ropes, placing ends adjacent to each other.)
   C. Pair off the pupils. Give each pair a ball of string and a pair of scissors. Have the pupils in each pair cut off one piece of string. Then have them cut off a piece of string that is longer than the first piece.

   Repeat this activity using the phrases "shorter than" and "the same length as." This exercise may be repeated several times.

Lesson 7

Materials

1 bag of ropes per three pupils; 1 board per three pupils

Activities

Group the pupils by threes. Give each group one bag of ropes. Place one board on the floor by each group. The pupils in each group must separate their ropes into "longer than," "shorter than," or "the same length as" the board.

INSTRUCTIONAL UNIT II

CONSERVATION OF LENGTH

Lesson 1

Materials

For each child: 1 green stick 5-in. long, 1 red stick 6-in. long

For teacher: 1 6-in. candlestick in box with lid, 1 cylinder; 1 5-in. strip of flannel; a flannel board; 1 red stick 5-in. long, 1 green stick 6-in. long

Activities

1. Give each child a 5-in. green stick. Place the 5-in. felt strip horizontally (hereafter called a strip) on the flannel board. Have each child compare his stick with the strip.

   Say:

   "(John), see if your stick is the same length as this strip."

   Now, place the strip vertically on the flannel board.

   Ask:

   "Now, is the strip still the same length?"

   Regardless of the answers, say,

   "Let us find out if it is still the same length."

   Now, have each child compare his stick with the strip to ascertain if his stick is still the same length as the strip. Say,

   "(John), see if your stick is still the same length as this strip."

   Ask:

   "Did moving the strip change its length?" (The children may respond "yes", "no," or not respond at all. Don't force answers.)

Teacher now demonstrates as follows:

(1) Place the strip horizontally on the flannel board.

(2) Compare a red stick (the same length as the strip) with the strip.

Say:

"This red stick is the same length as the strip."

(3) Move the strip to a vertical position. Compare the stick with the strip again.

Say:

"See, the strip hasn't changed in length."

2. Give each child a 6-in. red stick. Hold one 6-in. green stick in your hand. Have the pupils compare their sticks to your 6-in. green stick.

Say:

"(John), see if your stick is the same length as mine." (Do this for each child, giving affirmative reinforcement to each child).

Then place about three inches of your stick in the available cylinder. Ask:

"Is my green stick shorter than it was?"

Regardless of answers, say,

"Let's find out if it is shorter than it was."

Remove it from the cylinder and have each child compare their sticks with it again. Say:

"(John), see if your stick is still the same length as mine."

Ask:

"Did putting the green stick in here change its length?"

Regardless of the answers, the teacher makes a demonstration as follows:

(1) Compare your green stick with a red stick of the same length. Say:

"These two sticks are the same length."

(2) Put the green stick in the cylinder.

Ask:

"Is the green stick shorter than it was?"

(3) Say, while removing the green stick from the cylinder and comparing it with the red stick,

"No, because these are still the same length."

3. Give each child one 6-in. green stick. Have each child compare his green stick with the 6-in. candlestick. Say,

"(John), see if your stick is the same length as this candlestick."

Now place the candlestick in the box. Be sure the lid is on the box. Ask:

"Now that the candlestick is in the box, is it longer?"

Regardless of answers, say,

"Let us find out if it is longer in the box."

Take the lid off the box and have the children compare their sticks with the candlestick while it is in the box. Say:

"See, the candlestick is still the same length as your stick.

It is not longer when it is in the box."

Teacher now demonstrates as follows:

(1) Take a 6-in. green stick and compare it with a 6-in. candlestick. Say,

"The green stick is the same length as the candlestick."

(2) Place the candlestick in the box. Say,

"See, the candlestick is now in the box."

Compare the green stick again with the candlestick while it is in the box. Say:

"See, the candlestick hasn't changed length."

Lesson 2

Materials

One set of M-L lines; masking tape; one 6-in. flannel strip and flannel board;

For each child: 1 4-ft. piece of string; 1 6-in. flannel strip; 1 kite stick; 1 6-in. stick

Activities

1. Give each child a 4-ft. piece of string. With your assistance, have the children place on the floor a piece of masking tape the same length as the piece of tape on the floor. Have pupils compare their string with the masking tape. Give each child a kite stick and say,

   "Roll your string up on this kite stick."

   Ask:

   "Is your string shorter now than it was before you rolled it up?"

   Regardless of the answers, say,

   "Let us find out if it is still the same length."

   Have each child compare his string with the tape to ascertain that the string is still the same length. Say:

   "(John), see if your string is still the same length."

Ask:

"Did rolling the string change its length?"

Regardless of the answers, the teacher now demonstrates as follows:

(1) Take a string the same length as the tape and compare them.

Say:

"The string and the tape are the same length."

(2) Roll the string up on the kite stick. Say:

"I wonder if this string is the same length as it was before I rolled it on this stick?"

Ask:

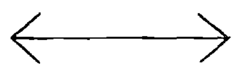
"How could I be sure?"

(3) Compare the string with the tape. Say:

"See, rolling the string on a stick doesn't change the length."

2. Place a 6-in. strip on the flannel board. Say:

"Look at this strip."

Then place the Müller-Lyer lines at each end of the strip so the total figure looks as follows:  Ask:

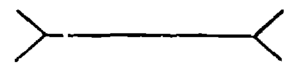
"Is the strip shorter than it was?"

Regardless of answers, remove it from the M-L lines and have each child compare it with a 6-in. stick. Sa

"See if your sticks are the same len the strip."

Replace the strip in the M-L lines and h he children again compare it with the same 6-in. sticks. Say

"See, the strip is still the same l your sticks. Its length hasn't changed."

3. Place the same strip as in (2) in M-L lines in such a way that the configuration looks as follows:  Ask:

   "Is the strip now longer than it was?"

   Again have the children compare it with the 6-in. strip. Say:

   "Your sticks are still the same length as the strip. The length of the strip hasn't changed."

4. Repeat (3) with the teacher demonstrating all activities.

5. Place the strip at different positions on the flannel board, each time asking,

   "Is the strip the same length? Why?"

Lesson 3

Materials

One 7-in. string for each child. One box of 7-in. pipe cleaners.

Activities

1. Give each child a piece of string 7-in. long. Place a pipe cleaner in view of the children. Ask:

   "Is your string the same length as this pipe cleaner?"

   Have each child compare his string to the pipe cleaner to find out. Then bend the pipe cleaner as follows:

   Bend I

   Again have the children compare their string to the pipe cleaner. After comparing, ask,

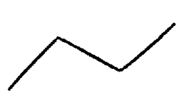
   "Is the pipe cleaner as long as it was before I bent it?"

   Regardless of the answers, straighten the pipe cleaner out and say,

   "Your string is the same length as the pipe cleaner when it is like this (straight) and like this (rebend it)."

2. Using different pipe cleaners, repeat (1) above utilizing the following bends:

Bend II



Bend III

After (1) and (2) are completed, ask,

"Does bending a pipe cleaner change its length?"

Regardless of the answers, the teacher should say, while bending a pipe cleaner,

"See, no matter how I find it, we can always straighten it out However I bend it, it is always the same length."

INSTRUCTIONAL UNIT III

CONSERVATION OF LENGTH RELATIONS

Lesson 1

Materials

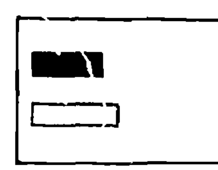
Flannel board, box of flannel strips

Activities

1. Put a 5-in. strip on a flannel board. Give the children a box of strips. Have a child find a strip longer than the one on the flannel board. Say:

   "(Mary), find a strip in the box longer than this strip."

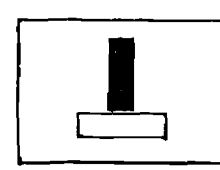
   Arrange them so they look as follows:

   Pointing to the appropriate strips, say,

   "This strip is longer than this strip."

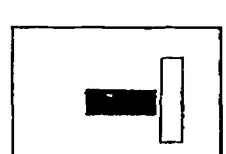
   Rearrange the strips so they look as follows:

   Have each child point to the strip he/she thinks is now longer. Say:

   "(John), now point to the strip you think is longer."

   Repeat with the other children. Do not correct the children if they point to the wrong strip. After they have all answered, place the shorter strip beside the longer one to establish that the longer one is in fact longer. Then, arrange the strips as follows:

Ask:

"Now which one do you think is longer?"

Have the children point to the one they think is longer. If any respond incorrectly, compare the strips again and say,

"See, this one is still longer."

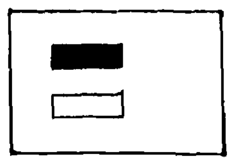
Now, move the longer strip to various positions, each time asking,

"Is this one (pointing to the longer one) still longer?"

2. Put a 5-in. strip on the flannel board. Have a child find a strip that is the same length as the one on the flannel board. Say:

"(Mary), find a strip in the box the same length as this strip."

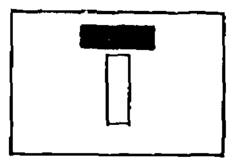
Arrange them so they look as follows:



Pointing to the appropriate strips, say,

"This strip is the same length as this strip."

Rearrange the strips so they look as follows:



Ask:

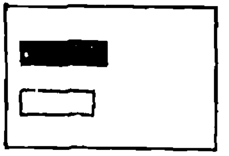
"Now, are the strips still the same length?"

Regardless of the answers, compare the strips again to establish that the strips are still the same length. Repeat the above procedure using different positions.

3. Put a 5-in. strip on the flannel board. Have a child find a strip shorter than the one of the flannel board. Say:

"(Mary), find a strip in the box shorter than this strip."

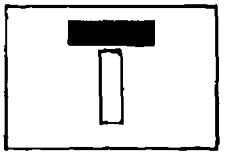
Arrange them so they look as follows:



Pointing to the appropriate strips, say,

"This strip is shorter than this strip."

Rearrange the strips so they look as follows:



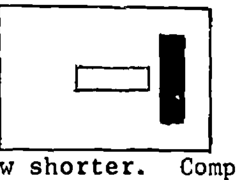
Have each child point to the strip he/she thinks is now shorter. Say:

"(John), now point to the strip you think is shorter." (Repeat with the other children).

Do not correct those children who are wrong. After they have all answered, place the shorter stick beside the longer one to establish which one is in fact shorter. Say:

"See, this one is still shorter."

Then, arrange the strips as follows: Have each child point to the



strip he/she thinks is now shorter. Compare strips if necessary. Repeat, moving the shorter strip, each time asking, "Is this one (pointing to the shorter one) still longer?"

Lesson 2

Materials

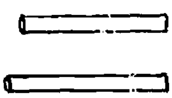
2 6-in. straws (one red and one green); 2 5 3/4-in. straws (one red and one green); 2 6 1/4-in. straws (one red and one green); 1 7-in. straw; 1 7-in. straw; 2 6 3/4-in. pipe cleaners; 2 7-in. pipe cleaners; 2 7 1/4-in. pipe cleaners

Activities

1. Display the straws to a group of children. Using the green 6-in. straw, ask a child to find a red straw longer than that green straw. Ask:

   "(John), find a red straw longer than this green straw."

   When (John) has found a correct straw, be sure the straws are arranged as diagrammed.

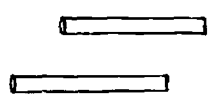
   green straw

   red straw

   Have each child identify the straw which is longer by touching it. Say:

   "(Mary), touch the straw that is longer."

   Now, move the red straw to a position as diagrammed.

   green straw

   red straw

   Have a student now point to a straw. Say:

   "(Peter), point to the straw that is longer."

If the student points to the wrong straw, say,

"Let's find out if you pointed to the straw that is longer."

(Move the straws to their original position).

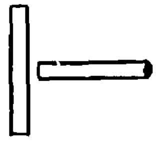
Say:

"Did (Peter) point to the longer straw?"

If the student pointed to the red straw, say,

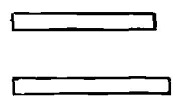
"That is right," (while comparing the two straws again).

2. Now select the 5 3/4-in. red straw. Ask a child to find a green straw longer than that red straw. Repeat activity (1) moving the green straw as diagrammed after the initial comparison:

green straw  red straw

3. Display the pipe cleaners to a group of children. Using the 7-in. straw, ask a child to find a pipe cleaner longer than the straw. Ask:

"(John), find a pipe cleaner longer than this straw."

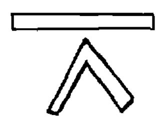
When the child finds the pipe cleaner, be sure the straw and pipe cleaner are arranged as follows:  straw

pipe cleaner

Have each child touch the longer one. Say:

"(Mary) touch the one that is longer."

Now, bend the pipe cleaner so the arrangement looks as diagrammed.



straw

pipe cleaner

Have a student now point to the longer one.

Say:

"(Peter), point to the one that is longer."

If the student points to the straw, say,

"Let us see if you pointed to the one that is longer."

(Bend the pipe cleaner straight and compare it with the straw).

Say:

"Did (Peter) point to the one that is longer?"

If the student points to the pipe cleaner, say,

"That is right" (while bending the pipe cleaner straight and comparing it with the straw).

4. Repeat 3 but use the following bend:

straw

pipe cleaner

Lesson 3

Materials

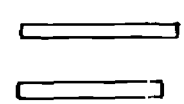
2 6-in. straws (one red and one green); 2 5 3/4-in. straws (one red and one green); 2 6 1/4-in. straws (one red and one green); 1 7-in. straw; 2 6 3/4-in. pipe cleaners; 2 7-in. pipe cleaners; 2 7 1/4-in. pipe cleaners

Activities

1. Display the straws to a group of children. Using the green 6-in. straw, ask a child to find a straw shorter than that green straw. Ask:

   "(John), find a red straw shorter than this green straw."

   After (John) has found a correct straw, be sure the straws are arranged as diagrammed.

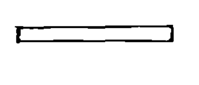
   green straw

   red straw

   Have each child identify the straw which is shorter by touching it. Say:

   "(Mary), touch the straw that is shorter."

   Now, move the shorter (red) straw to a position as diagrammed:

   green straw

   red straw

   Have a student now point to a shorter straw. Say:

   "(Peter) point to the straw that is shorter."

If the student points to the wrong straw, say,

"Let us find out if you pointed to the straw that is shorter."

(Move the straws to their original positions).

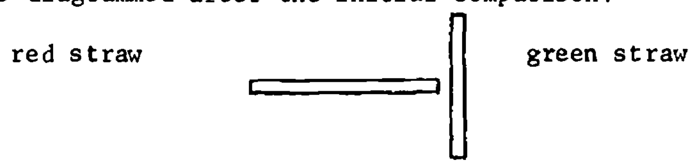
Say:

"Did (Peter) point to the shorter straw?"

If the student did point to the shorter straw, say,

"That is right" (while comparing the two straws).

2. Now select the 6-in. red straw. Ask a child to find a green straw shorter than the red straw. Repeat activity (1) moving the green straw as diagrammed after the initial comparison:

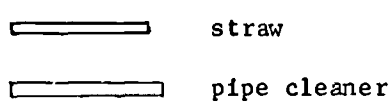


3. Display a pipe cleaner to a group of children. Using the 7-in. straw, say,

"(John), see if this straw is shorter than a pipe cleaner."

Compare the straw with the pipe cleaner to see which is shorter. (The child may need some assistance to find a pipe cleaner longer than the straw.)

After the child finds the pipe cleaner, be sure the straw and pipe cleaner are arranged as follows:



Have each child touch the shorter one. Say:

"(Mary), touch the one that is shorter."

Now, bend the pipe cleaner so the arrangement looks as diagrammed.

straw

pipe cleaner

Have a student now point to the shorter one. Say:

"(Peter), now point to the one that is shorter."

If the student points to the pipe cleaner, say,

"Let us see if you pointed to the one that is shorter."

Bend the pipe cleaner straight, and compare it with the straw. Then say:

"Did (Peter) point to the one that is shorter?"

If the student points to the straw, say,

"That is right" (while bending the pipe cleaner straight and comparing it with the straw).

4. Repeat 3 but use the following bend:

straw

pipe cleaner

Lesson 4

Materials

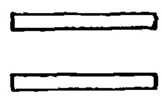
(Same as Lesson 3)

Activities

1. Display the straws to a group of children. Using the green 6-in. straw, ask a child to find a red straw the same length as the green straw. Ask:

   "(John), find a red straw the same length as this green straw."

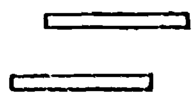
   After (John) has found a correct straw, be sure the straws are arranged as diagrammed:

   green straw

   red straw

   Now, move the red straw to a position as diagrammed:

   green straw

   red straw

   Ask:

   "Now, is one straw longer?"

   If a child says, "yes," have him touch the straw he/she thinks is longer.

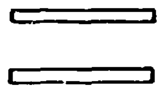
   Say:

   "Point to the straw you think is longer."

   Say:

   "Let us see if that straw is longer."

Rearrange the straws as diagrammed:



green straw

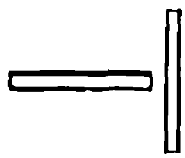
red straw

Say:

"Was it longer, or did it just look longer?"

2. Now, select the 6-in. red straw. Ask a child to find a green straw the same length as the red straw. Repeat activity (1) except move the green straw as diagrammed after the initial comparison:

red straw

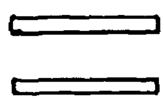


green straw

3. Display the pipe cleaners to a group of children. Using the 7-in. straw, say:

"(John), find a pipe cleaner the same length as this straw."

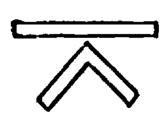
After the child finds the pipe cleaner, be sure the straw and pipe cleaner are arranged as follows:



straw

pipe cleaner

Now, bend the pipe cleaner so the arrangement looks as diagrammed:



straw

pipe cleaner

Ask:

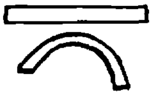
"Is the pipe cleaner now shorter than the straw?"

If some students say, "yes," bend the pipe cleaner straight and compare it again with the straw. Ask:

"Is the pipe cleaner shorter than the straw?"

"Does bending the pipe cleaner make it shorter than the straw?"

4. Repeat 3, but use the following bend:



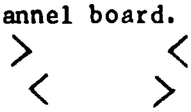
Lesson 5

Materials

Flannel board; box of red flannel strips; M-L lines made of flannel; 1 5-in. green flannel strip; 1 5-in. blue flannel strip

Activities

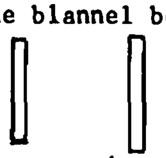
1. Put the M-L lines on the flannel board.



   Place a green flannel strip in a vertical position to the right of the M-L lines. Give the children a box of red strips. Have a child find a strip longer than the one on the flannel board (by measuring). Say:

   "(Mary), find a strip in the box longer than this green strip."

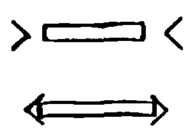
   Arrange the strips on the blannel board as follows;

   green  red

   Pointing to the appropriate strip, say,

   "This strip is longer than this strip."

   Arrange the strips within the M-L lines so that they look as follows:



green

red

   Now, have each child point to the strip he/she thinks is longer. Say:

"(John), now point to the strip you think is longer." Repeat with the other children. Do not correct the children if they point to the wrong strip. After all have taken a turn, say,

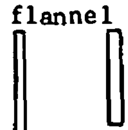
"The red strip is still longer than the green strip. I will show you that the red strip is still longer than the green strip."

Move the green strip beside the red strip to establish that the red strip is in fact longer than the green strip.

2. Place a blue flannel strip in a vertical position to the right of the M-L lines. Give the children a box of red strips. Have a child find a strip shorter than the one on the flannel board (by measuring). Say:

"(John), find a strip in the box shorter than this blue strip."

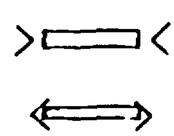
Arrange the strips on the flannel board as follows:

blue  red

Pointing to the appropriate strip, say,

"This strip is shorter than this strip."

Arrange the strips within the M-L lines so that they look as follows:



red

blue

Now, have each child point to the strip he/she thinks is shorter. Say:

"(John), now point to the strip you think is shorter."

Repeat with the other children. Do not correct the children if they point to the wrong strip. After all have taken their turn, say,

"The red strip is still shorter than the blue strip. I will show you that the red strip is still shorter than the blue strip."

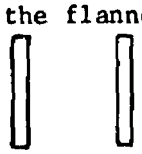
Move the red strip beside the blue strip to establish that the red strip is in fact shorter than the blue strip.

3. Place a green flannel strip in a vertical position to the right of the M-L lines. Give the children a box of red strips. Have a child find a strip the same length as the one on the flannel board (by measuring).

Say:

"(Mary), find a strip in the box the same length as this green strip."

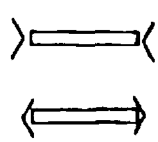
Arrange the strips on the flannel board as follows:

green  red

Pointing to the appropriate strip, say,

"This strip is the same length as this strip."

Arrange the strips within the M-L lines so that they look as follows:



red

green

Now, have each child point to the strip he/she thinks is longer.

Say:

"(Mary), if you think one of the strips is now longer, point to that strip."

Repeat with the other children. Do not correct the children if they point to the wrong strip. After all have tried, say,

"The red strip is still the same length as the green strip. I will show you that the red strip is still the same length as the green strip."

Move the red strip beside the green strip to establish that the red strip is in fact the same length as the green strip.

APPENDIX III

Measuring Instruments

INSTRUMENT I

Length Comparisons Test

Material Set I

Materials:

One green stick; 3 pieces of white string, one being longer than, one shorter than, and one the same length as the green stick

Directions:

Item 1. Using 3 pieces of string, find a piece longer than this green stick.

Item 7. Using these pieces of string, find a piece shorter than this green stick.

Item 14. Using these pieces of string, find a piece the same length as this green stick.

Material Set II

Materials:

One green stick; 3 red sticks, one being longer than, one shorter than, and one the same length as the green stick

Directions:

Item 2. Using these red sticks, find a stick longer than this green stick.

Item 11. Using these red sticks, find a stick shorter than this green stick.

Item 18. Using these red sticks, find a stick the same length as this green stick.

Material Set III

Materials:

One piece of black string; 3 pieces of white string, one being longer than, one shorter than, and one the same length as the black string

Directions:

Item 3. Using these pieces of white string, find a piece longer than this black string.

Item 12. Using these pieces of white string, find a piece shorter than this black string.

Item 13. Using these pieces of white string, find a piece the same length as this black string.

Material Set IV

Materials:

One red stick; 3 green sticks, one being longer than, one shorter than, and one the same length as the red stick

Directions:

Item 4. Using these green sticks, find a stick longer than this red stick.

Item 10. Using these sticks, find a stick shorter than this red stick.

Item 17. Using these green sticks, find a stick the same length as this red stick.

Material Set V

Materials:

One piece of white string; 3 pieces of black string, one being longer than, one shorter than, and one the same length as the white string

Directions:

Item 5. Using these pieces of black string, find a piece longer than this white string.

Item 9. Using these pieces of black string, find a piece shorter than this white string.

Item 16. Using these pieces of black string, find a piece the same length as this white string.

Material Set VI

Materials:

One red string; 3 pieces of white string, one being longer than, one shorter than, and one the same length as the red stick

Directions:

Item 6. Using these pieces of white string, find a piece longer than this red stick.

Item 8. Using these pieces of white string, find a piece shorter than this red stick.

Item 15. Using these pieces of white string, find a piece the same length as this red stick.

INSTRUMENT II

Conservation of Length Relations (Application Test)

Conservation of Length Relations

Level I--Longer Than

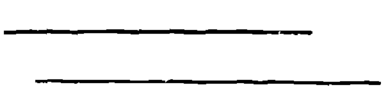
1. Materials:

One green straw; 3 red straws, one being longer than, one shorter than, and one the same length as the green straw

Statement:

Using these red straws, find a straw longer than this green straw.

Transformation:



green

red (move the red straw)

Question:

"Is this red straw still longer than this green straw?"

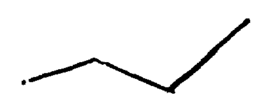
2. Materials:

One red pipe cleaner; 3 white pipe cleaners with one longer than, one shorter than, and one the same length as the red pipe cleaner.

Statement:

Using these white pipe cleaners, find a pipe cleaner longer than this red pipe cleaner.

Transformation:



(bend the white pipe cleaner)

Question:

"Is this white pipe cleaner still longer than this red pipe cleaner?"

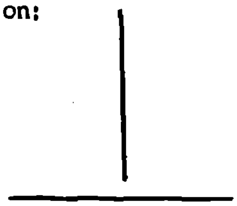
3. Materials:

One red straw; 3 green straws with one longer than, one shorter than, and one the same length as the red straw

Statement:

Using these green straws, find a straw longer than this red straw.

Transformation:



red

green (move the red straw)

Question:

"Is the green straw still longer than this red straw?"

Level I--Shorter Than

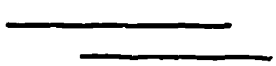
4. Materials:

One red straw; 3 green straws, one being longer than, one shorter than, and one the same length as the red straw

Statement:

Using these green straws, find a straw shorter than this red straw.

Transformation:



red

green (move the green straw)

Question:

"Is this green straw still shorter than this red straw?"

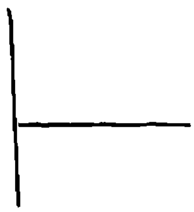
5. Materials:

One green straw; 3 red straws, one being longer than, one shorter than and one the same length as the green straw

Statement:

Using these red straws, find a straw shorter than this green straw.

Transformation:



green
red (move the red straw)

Question:

"Is this red straw still shorter than this green straw?"

6. Materials:

One red pipe cleaner; 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the red pipe cleaner

Statement:

Using these white pipe cleaners, find a pipe cleaner shorter than this red pipe cleaner.

Transformation:

red
white (bend the red pipe cleaner)

Question:

"Is this white pipe cleaner still shorter than this red pipe cleaner?"

Level I--Same Length As

7. Materials:

One red pipe cleaner; 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the red pipe cleaner

Statement:

Using these white pipe cleaners, find a pipe cleaner the same length as this red pipe cleaner.

Transformation:

red

white (bend this white pipe cleaner)

Question:

"Is this white pipe cleaner still the same length as this red pipe cleaner?"

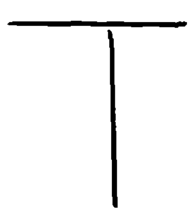
8. Materials:

One red straw; 3 green straws, one being longer than, one shorter than, and one the same length as this red straw

Statement:

Using these green straws, find a straw the same length as this red straw.

Transformation:



red

green (move the green straw)

Question:

"Is this green straw still the same length as the red straw?"

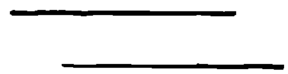
9. Materials:

One green straw; 3 red straws, one being longer than, one shorter than, and one the same length as the green straw

Statement:

Using these red straws, find a straw the same length as this green straw.

Transformation:



green

red (move the red straw)

Question:

"Is this red straw still the same length as this green straw?"

Conservation of Length Relations

Level II--Longer Than

1. Materials:

One green straw; 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the green straw

Statement:

Using these pipe cleaners, find a pipe cleaner longer than this green straw.

Transformation:

green straw

pipe cleaner (move the green straw)

Question:

"Now is the green straw longer than the pipe cleaner?"

2. Materials:

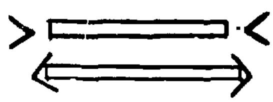
One 6-in. green flannel strip; 3 red flannel strips, 5 7/8, 6, and 6 1/8 in. long; one M-L board

Statement:

Find a red strip longer than this green strip.

Transformation:

Place strips as follows:



green strip

red strip

Question:

"Now, is the red strip shorter than the green strip?"

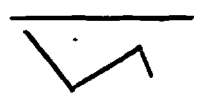
3. Materials:

One green straw; 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the green straw.

Statement:

Using these pipe cleaners, find a pipe cleaner longer than this green straw.

Transformation:



green straw

pipe cleaner (bend pipe cleaner)

Question:

"Now, is the green straw longer than the pipe cleaner?"

Level II--Shorter Than

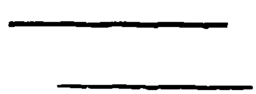
4. Materials:

One green straw; 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the green straw

Statement:

Using these pipe cleaners, find a pipe cleaner that is shorter than this green straw.

Transformation:



green straw

(move the pipe cleaner)

Question:

"Now, is the green straw shorter than the pipe cleaner?"

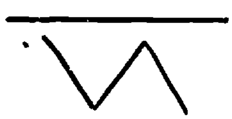
5. Materials:

One red pipe cleaner; 3 green straws, one being longer than, one shorter than, and one the same length as this pipe cleaner

Statement:

Using these green straws, find a straw shorter than this pipe cleaner.

Transformation:



green straw

(bend the pipe cleaner)

Question:

"Now, is the pipe cleaner shorter than the green straw?"

6. Materials:

One 6-in. green flannel strip; 3 red flannel strips, 5 7/8, 6, and 6 1/8 in. long; one M-L board

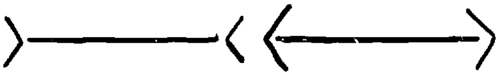
Statement:

Find a red strip shorter than this green strip.

Transformation:

Place the strips as follows:

red strip green strip



Question:

"Now, is the red strip longer than the green strip?"

Level II--Same Length As

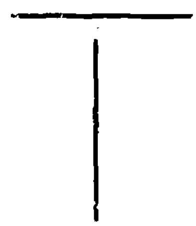
7. Materials:

One green straw; 3 white pipe cleaners with one longer than, one shorter than, and one the same length as the green straw

Statement:

Using these pipe cleaners, find a pipe cleaner the same length as this green straw.

Transformation:



green straw

Pipe cleaner

(move the pipe cleaner)

Question:

"Now, is the pipe cleaner longer than the green straw?"

8. Materials:

One red pipe cleaner; 3 green straws with one longer than, one shorter than, and one the same length as the pipe cleaner.

Statement:

Using these straws, find a straw the same length as this red pipe cleaner.

Transformation:

green straw

(bend the pipe cleaner)

Question:

"Now, is the pipe cleaner shorter than the green straw?"

9. Materials:

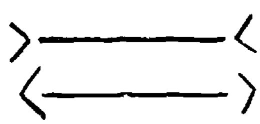
One 6-in. green flannel strip; 3 red flannel strips, 5 7/8, 6, and 6 6 1/8 in. long; one M-L board.

Statements:

Find a red strip the same length as this green strip.

Transformation:

Place strips as follows:



red strip

green strip

Question:

"Now, is the red strip longer than the green strip?"

INSTRUMENT III

Conservation of Length

1. Materials:

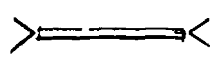
1 cardboard with M-L Diagram. 1 6-in. flannel strip

Statement:

Look at the length of this strip.

Transformation:

Look at the strip here 

Look at the strip here 

Question:

"Now, is the strip longer?"

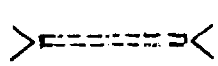
2. Materials:

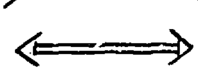
1 cardboard with M-L Diagram; 1 6-in. flannel strip.

Statement:

Look at the length of this strip.

Transformation:

Look at the strip here 

Look at the strip here 

Question:

"Now, is the strip shorter?"

3. Materials:

1 7-in. cylinder; 1 7-in. stick

Statement:

Look at the length of this stick.

Transformation: 

Question:

"Now, is the stick shorter?"

4. Materials:

1 6-in. pipe cleaner

Statement:

Look at the length of this pipe cleaner.

Transformation: 

Question:

"Now, is the pipe cleaner the same length?"

5. Materials:

1 12-in. string

Statement:

Look at the length of this string. (Straighten string on table.)

Transformation:  (coil the string)

Question:

"Now, is the string the same length?"

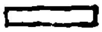
6. Materials:

1-flannel strip

Statement:

Look at the length of this strip.

Transformation:

Original  Final 

Question:

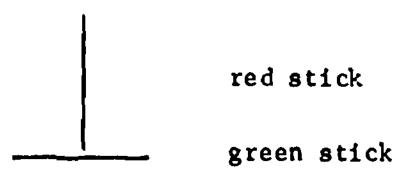
"Now, is the strip the same length?"

INSTRUMENT IV

Transitivity Test

1. Materials:

A red stick and a green stick of the same length attached to a cardboard as follows:



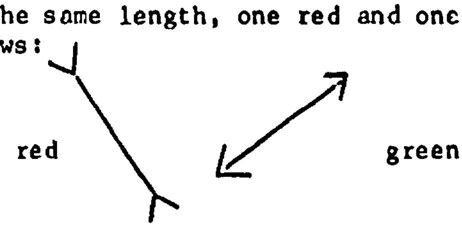
A white stick the same length as the red and green sticks for the child's use.

Questions:

(a) "Is the red stick the same length as your stick?"
(b) "Is the green stick the same length as your stick?"
(c) "Is the green stick shorter than the red stick?"

2. Materials:

Two flannel strips of the same length, one red and one green attached to a cardboard as follows:



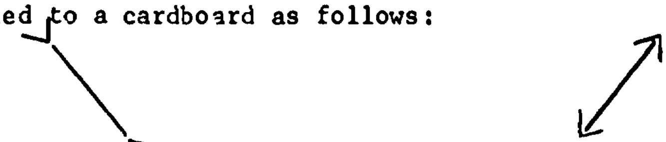
A blue straw the same length as the two flannel strips for the child's use.

Questions:

(a) "Is the red strip the same length as your straw?"
(b) "Is the green strip the same length as your straw?"
(c) "Is the red strip longer than the green strip?"

3. Materials:

A red straw and a green straw (the red straw must be shorter than the green) attached to a cardboard as follows:



red straw green straw

A yellow straw for the child's use of length between the lengths of the red and green straws.

Questions:

(a) "Is the red straw shorter than your straw?"
(b) "Is your straw shorter than the green straw?"
(c) "Is the red straw shorter than the green straw?"

4. Materials:

A red straw and a blue straw (the length of the red straw is greater than the length of the blue straw) attached to a cardboard as follows:

blue straw

red straw

A yellow pipe cleaner for the child's use of length between the lengths of the red and blue straws.

Questions:

(a) "Is the red straw longer than your pipe cleaner?"
(b) "Is your pipe cleaner longer than the blue straw?"
(c) "Is the red straw longer than the blue straw?"

5. Materials:

A green and white pipe cleaner of the same length, displayed on a cardboard as follows:

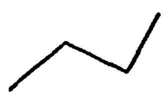
white pipe cleaner green pipe cleaner

A red pipe cleaner the same length as the green and the white pipe cleaners for the child's use.

Questions:

(a) "Is the white pipe cleaner the same length as your pipe cleaner?"
(b) "Is the green pipe cleaner the same length as your pipe cleaner?"

Transformation:

white pipe cleaner  green pipe cleaner

(c) "Is the green pipe cleaner the same length as the white pipe cleaner?"

6. Materials:

A green stick and a white stick of the same length attached to a cardboard as follows:

green stick

white stick

A red stick the same length as the green and white sticks for the child's use.

Questions:

(a) "Is the green stick the same length as your stick?"
(b) "Is the white stick the same length as your stick?"
(c) "Is the white stick longer than the green stick?"