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ABSTRACT

This monograph was written for the conference on the New Instructional Materials in Physics, held at the University of Washington in summer, 1965. The approach is phenomenological and microscopic, and is intended for college students who are not preparing to become professional physicists. The monograph has three sections. Section I includes a short review of the discovery of magnetic phenomena, and a discussion of the concepts of magnetic poles, magnetic field intensity, magnetic moment and magnetic flux. Section II deals with the magnetic interactions of steady currents. Basic calculus is used to reformulate the major concepts of magnetostatics in section III. Some problems for discussion are included in sections I and II. (LC)

# Electricity and Magnetism II

## MAGNETOSTATICS



MELBA PHILLIPS

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# GENERAL PREFACE

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in the summer of 1965. The general purpose of the conference was to create effective ways of presenting physics to college students who are not preparing to become professional physicists. Such an audience might include prospective secondary school physics teachers, prospective practitioners of other sciences, and those who wish to learn physics as one component of a liberal education.

At the Conference some 40 physicists and 12 filmmakers and designers worked for periods ranging from four to nine weeks. The central task, certainly the one in which most physicists participated, was the writing of monographs.

Although there was no consensus on a single approach, many writers felt that their presentations ought to put more than the customary emphasis on physical insight and synthesis. Moreover, the treatment was to be "multi-level" --- that is, each monograph would consist of several sections arranged in increasing order of sophistication. Such papers, it was hoped, could be readily introduced into existing courses or provide the basis for new kinds of courses.

Monographs were written in four content areas: Forces and Fields, Quantum Mechanics, Thermal and Statistical Physics, and the Structure and Properties of Matter. Topic selections and general outlines were only loosely coordinated within each area in order to leave authors free to invent new approaches. In point of fact, however, a number of monographs do relate to others in complementary ways, a result of their authors' close, informal interaction.

Because of stringent time limitations, few of the monographs have been completed, and none has been extensively rewritten. Indeed, most writers feel that they are barely more than clean first drafts. Yet, because of the highly experimental nature of the undertaking, it is essential that these manuscripts be made available for careful review by other physicists and for trial use with students. Much effort, therefore, has gone into publishing them in a readable format intended to facilitate serious consideration.

So many people have contributed to the project that complete acknowledgement is not possible. The National Science Foundation supported the Conference. The staff of the Commission on College Physics, led by E. Leonard Jossem, and that of the University of Washington physics department, led by Ronald Geballe and Ernest M. Henley, carried the heavy burden of organization. Walter C. Michels, Lyman G. Parratt, and George M. Volkoff read and criticized manuscripts at a critical stage in the writing. Judith Bregman, Edward Gerjuoy, Ernest M. Henley, and Lawrence Wilets read manuscripts editorially. Martha Ellis and Margery Lang did the technical editing; Ann Widditsch supervised the initial typing and assembled the final drafts. James Grunbaum designed the format and, assisted in Seattle by Roselyn Pape, directed the art preparation. Richard A. Mould has helped in all phases of readying manuscripts for the printer. Finally, and crucially, Jay F. Wilson, of the D. Van Nostrand Company, served as Managing Editor. For the hard work and steadfast support of all these persons and many others, I am deeply grateful.

Edward D. Lambe
Chairman, Panel on the
New Instructional Materials
Commission on College Physics

# MAGNETOSTATICS

## PREFACE

This fragmentary and preliminary material fits into an outline of "multi-level monographs" covering those aspects of electromagnetism which in our view an undergraduate physics major should come to know best. The approach is phenomenological and macroscopic, designed to take advantage of prior experience; we begin magnetostatics with magnets, for example. The material is planned on two levels to lead through the four fundamental empirical laws of electricity and magnetism to electromagnetic radiation as a climax. The propagation of electromagnetic disturbances with velocity c, reached in the "first course" material without use of the calculus and equivalent to the homogeneous wave equation, was written in an elementary way by Oliver Heaviside (Electromagnetic Theory, London, 1912, Vol. III, p. 3), but only recently has appeared in the regular pedagogical literature. In our treatment we have tried to stress the physical foundations of Maxwell's great synthesis, stating in words the argument corresponding to each mathematical step. This results in a considerably larger proportion of expository writing relative to mathematics than is customarily found in derivations of the wave equation from Maxwell's equations in their usual form. On the other hand, expression of the laws in differential form seems essential for tracing radiation to its sources in a physically meaningful way; the present Chapter 3 of Magnetostatics could be followed almost immediately by Chapter 3 of Monograph III, which would trace radiation fields to retardation effects. We regret having not sufficient time to write such a chapter, as well as the omission of what should have been Chapter 3 of Magnetostatics, an elementary treatment of magnetic materials.

OUTLINE OF MONOGRAPHS ON ELECTRICITY AND MAGNETISM

| | I. ELECTROSTATICS | II. MAGNETOSTATICS | III. CIRCULATION LAWS AND THEIR CONSEQUENCES |
|---|---|---|---|
| FIRST COURSE MATERIAL | 1. Electric Forces and Fields<br>2. Electric Energy and Potential<br>3. Electrical Properties of Matter | 1. Magnets and Magnetism<br>2. Interaction of Steady Currents<br>*Magnetic Properties of Matter | 1. Faraday's Law of Induction<br>2. Ampere's Law Modified<br>3. Propagation of Electromagnetic Disturbances |
| UPPER DIVISION COURSE MATERIAL | *4. Electrostatics Reformulated | 3. Magnetostatics Reformulated | *Maxwell's Equations and Plane Waves<br>*Radiation Fields |

*No textual material was prepared in the summer of 1965 for these chapters.

We have assumed no knowledge of special relativity, but have emphasized the necessity for choosing a frame of reference in which to define electric and magnetic field quantities, thus laying a foundation for the historical development of relativity theory. Unlike mechanics, vacuum electrodynamics needs no modification because of special relativity except in interpretation, so that an excursion into relativity theory could be made before or after study of the present material.

The experiments leading to the four fundamental laws are described at some length, but in use this written material should be accompanied by demonstrations and laboratory work. The basic experiments should come to be a part of genuine experience for students, but a laboratory monograph should be written as an extension of the present outline. Ohm's law and circuitry, for example, do not play an appreciable role in any other projected booklets. We cannot overemphasize the importance of laboratory work, although we were not able to undertake detailed consideration of its content.

We assume that students will have studied mechanics, that they know Newton's laws, the definition of work, the meaning of the $\Sigma$ symbol, and have a working knowledge of elementary vector algebra before our material is introduced. (We do define the vector cross product as if for the first time.) In the material designed for upper-class work we assume basic calculus. All vector calculus is developed as needed, but we attempt throughout to stress the physics, not the mathematics, and attempt no mathematical rigor.

The first chapters of Monographs I, II, and III should be studied in that order. The few discussion exercises we include can only indicate a type of problem we consider desirable. Numerical problems which we have made no effort to provide, are also necessary.

M. Phillips
R. T. Mara

# CONTENTS

# 1 MAGNETS AND MAGNETISM

Magnetic iron ore, known to us as lodostone or magnetite, is found in many parts of the world, and its property of attracting iron was noted by more than one civilization early in its Iron Age. The property remained merely a curiosity, even in the intellectual climate of the Golden Age of Greece, and the tendency of a magnet to orient itself along the earth's meridian escaped notice. These directive effects were probably first discovered in China, but there is no conclusive evidence that they were put to practical use. The origin of the mariner's compass is shrouded in mystery which may never be dispelled, but by the end of the 12th century the compass was well known in the Western world as a helpful device for sailors when the stars were obscured.

Myths, legends, and superstitions about magnets multiplied from ancient times through the Middle Ages, and even later. Magnets were employed in medicine, especially for the healing of wounds, and once the directional properties were recognized, in the occult sciences such as astrology. Yet magnetism began to be a genuine science during the Middle Ages; The first account of the magnet that we would call scientific was surprisingly early, a letter dated August 12, 1269. The epistle of Peter Perigrinus (Peter the Pilgrim), born Pierre de Maricourt in Picardy, sets forth a number of fundamental properties of magnetism.

It was Perigrinus who discovered poles and distinguished precisely two kinds. His method is of interest: Select a good piece of magnetite, shape it into a sphere and polish it. Now place on it a needle or sliver of iron, and mark on the surface of the sphere the direction taken by the needle. Repeat the procedure at many different positions on the sphere. At the end it is found that the lines "will run together in two points, just as all the meridian circles of the world run together in two opposite poles of the world" (Fig. 1.1). Only one of these poles points north if the magnet is free to turn. Thus two opposite magnetic poles were introduced, and Perigrinus noted that unlike poles attract. He went further to show that if a magnet is cut (see Fig. 1.2), two poles persist in every separated part, and that if two fragments are put together as before the new poles vanish. He did not notice repulsion. Perigrinus named the poles north and south, with the north pole that which points to the north. At that time magnetism was attributed to the <u>celestial</u> sphere, not

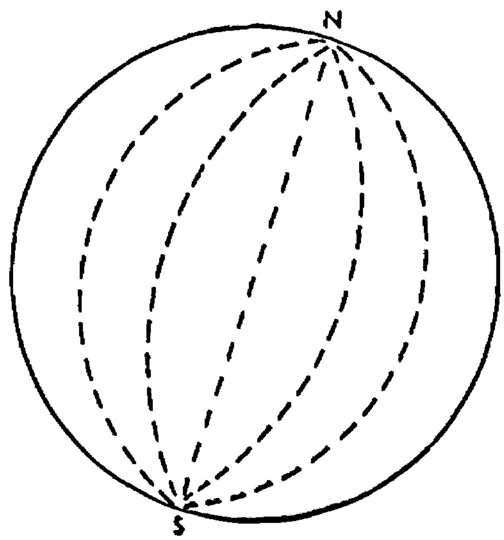


Fig. 1.1

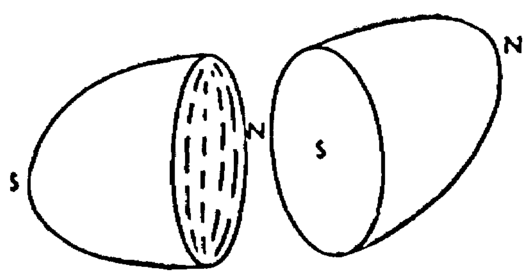


Fig. 1.2

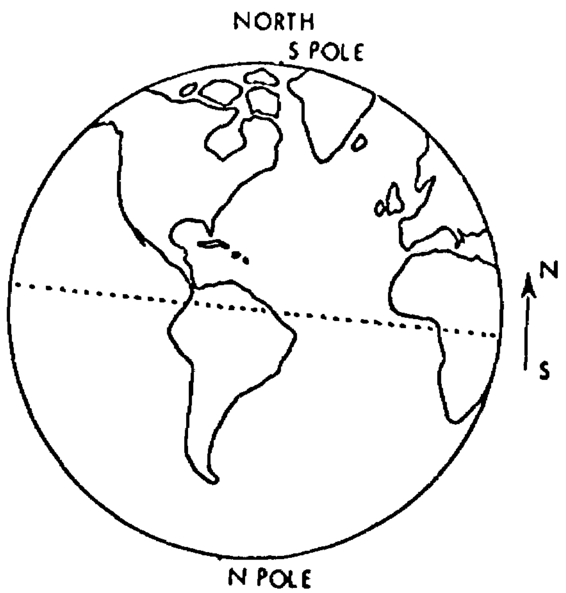


Fig. 1.3

to the earth itself, and Perigrinus did not think of the earth as a magnet.

This last step was taken by William Gilbert, physician to Queen Elizabeth, who repeated and extended the experiments of Perigrinus and others. Once the earth is taken as a magnet, the action of a compass is simply an example of the behavior of all magnets: The north magnetic pole of the earth is a "south" pole, which attracts the "north" poles of all compass needles (see Fig. 1.3). Gilbert's great accomplishment was to extract existing facts and laws of magnetism from a wealth of speculation and superstition, and to discover new properties and relations. His theories, including the supposed relation of magnetism to gravitation, need not concern us here, although we should note that they influenced Kepler, and that Newton found them suggestive in the development of his own ideas. Gilbert's book, De Magnete, published in 1600, is still a classic presentation of many qualitative aspects of magnetism.

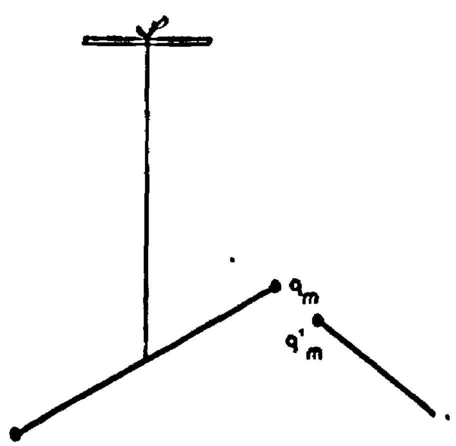


Fig. 1.4

Magnets exert forces on each other and on iron without being in contact. It was known to Gilbert and those who followed him that the effect of a magnet decreases as the distance from the magnet increases, but the quantitative relationship was first discovered by the Reverend John Michell in 1750. The same relation was found by Coulomb in 1785. The torsion balance, which facilitated these experiments, was invented independently by Michell and by Coulomb, and Michell's balance was later used by Henry Cavendish for his famous measurement of the constant in Newton's law of gravitation.

In long magnetized needles, or stiff wires of hardened iron, the magnetic effect is well concentrated at the ends, or poles (Fig. 1.4). Michell established experimentally that the two poles are opposite and of equal strength for any one magnet, and that repulsion and attraction between the poles of two magnets are of equal magnitude if distances of separation are kept the same. He also found that the force exerted by the pole of a long magnetized wire is the same in all directions. He then determined the force between poles for various distances, and found its strength was inversely proportional to the square of the distance between poles.

In order to write the law of Michell and Coulomb in mathematical form, we must assume some quantitative measure of pole strength, and it is convenient to call north-seeking poles positive (+) and south-seeking poles negative (-), by analogy to the designation of electric charge as positive and negative. If we designate pole strength by $q_m$, the force between two poles of strengths $q_m$ and $q_m'$ has a magnitude given by

$$F = kq_m q_m'/r^2,$$

with k an arbitrary constant determined by the units employed. In this formula a negative force signifies attraction, but force is actually a vector quantity, and the force exerted <u>on</u> $q_m$ <u>by</u> $q_m'$ at distance r may be written

$$\vec{F} = kq_m q_m'\vec{r}/r^2,$$

where $\vec{r}$ is a unit vector in the direction from $q_m'$ toward $q_m$. The force is directed as in Fig. 1.5 if $q_m$ and $q_m'$ are of like sign.

This law has exactly the same form as Coulomb's law for the interaction of electric charges, and a formal analogy between the interaction of magnets and electrostatic interactions can be carried further. The <u>magnetic field intensity</u> may be defined at any point in the vicinity of a magnet as the force per unit positive pole at that point. The conventional designation for magnetic field intensity is $\vec{H}$, so that

$$\vec{F} = q_m\vec{H},$$

where $\vec{H}$ is the field intensity at the position of $q_m$. The field intensity produced by a single pole $q_m'$ at distance r from the pole is

$$\vec{H} = (kq_m'/r^2)\vec{r},$$

where the unit vector $\vec{r}$ is directed <u>from</u> $q_m'$ <u>to</u> the point where $\vec{H}$ is to be determined. If $q_m'$ is negative (an S pole) the field intensity $\vec{H}$ is then directed toward $q_m'$. The field intensity produced by two poles is the vector sum of the contributions of the two poles taken separately - the principle of superposition applies. The fact that for real magnets the poles are not

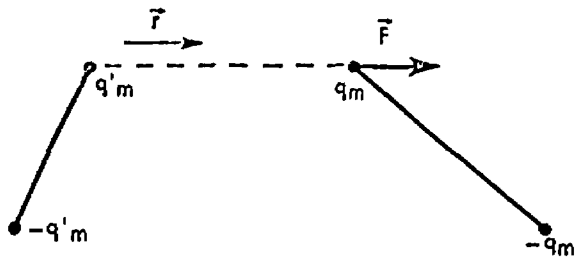


Fig. 1.5

points, that pole strength is actually distributed over the surface or volume of a magnet, causes no difficulty: All effects are additive and can be summed over elementary surfaces or volumes at various distances from the point at which the field is to be computed, just as in electrostatics. In other words, the principle of superposition applies to magnetic forces, just as it does to electric forces.

We shall find that we have little occasion to work with poles as such in the further development of magnetism, although the magnetic field intensity remains an important concept. We should note that the mks unit of pole strength is called the <u>weber</u>, so that $\vec{H}$ is in newtons per weber. (To anticipate Chapter 2: The weber is defined in terms of the ampere.) The constant k required to give the force in newtons if pole strength is measured in webers is $(10^7/16\pi^2)$ newton-meter$^2$/weber$^2$, or roughly $6 \times 10^4$. Clearly the mks system is not designed to be convenient in working with poles, but the weber remains useful in other connections. It was named for Wilhelm Eduard Weber (1804-1891), who collaborated with the great mathematician Gauss in putting the whole question of electromagnetic units on a rational basis.

In principle the magnetic field intensity in the vicinity of magnets could be computed at every point from a knowledge of the strength and position of all poles, but the numerical results would be hard to visualize. On the other hand, the direction and some notion of the strength of $\vec{H}$ can be demonstrated very easily with iron filings

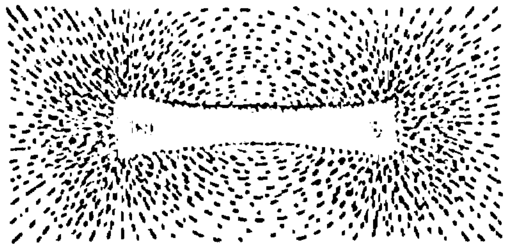

Fig. 1.6

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(Fig. 1.6) or a large number of very small compasses. By definition a compass sets itself in the direction of the field intensity. A field line is a line so drawn that its tangent is in the direction of the field at each point, and to a good approximation a field line can be traced along its length by a small compass. The number of lines drawn does not matter, but we see from Fig. 1.7 that the lines tend to converge in regions of high field intensity, and to become more widely spaced where the field intensity is weak. We are restricted to the region outside the magnet, and even then show only a two-dimensional cross section of space, but it is clear that the lines traced out in this way are smooth and continuous.

Iron filings and small compass needles are not entirely equivalent, although both are oriented along magnetic field lines. Soft iron, of which the filings are made, shows little or no residual magnetism; a small sliver

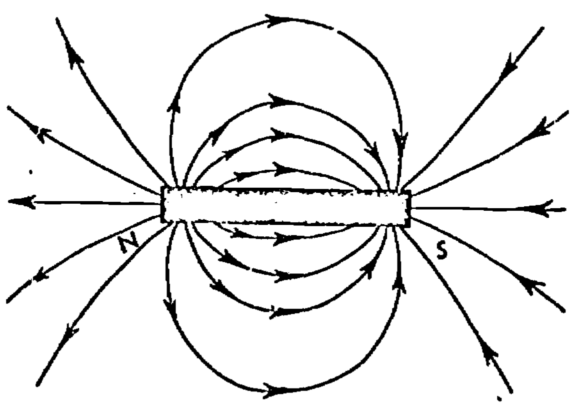


Fig. 1.7

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of iron has no poles of its own, and orients itself equally readily if turned through 180° in the field of a magnet. The magnetism a piece of iron exhibits owing to the presence of magnetite or other permanent magnets is said to be induced.

The tiny compass needles with which lines of magnetic field intensity can be traced out are each complete with two opposite poles of equal magnitude. That the tiny magnet tends to align itself along the lines of $\vec{H}$ is to be expected, since the force on the positive pole is equal and opposite to the force on the nearly coincident negative pole. But poles are well localized only on long needles such as those used by Michell and Coulomb. (Coulomb's needles were 25 inches long.) For short needles it is practically impossible to determine a point position equivalent to the actual distribution of pole strength. A more convenient property by which the strength of a magnet can be measured is its magnetic moment, or, more precisely, its magnetic dipole moment. The dipole moment can be determined even for magnets which are entirely inaccessible for direct examination.

The dipole moment of a long magnet with poles at the ends is defined as a vector directed from the negative toward the positive (north-seeking) pole, whose magnitude is the pole strength $q_m$ times the distance between the poles. Let us call the magnetic moment of such a magnet $\vec{m}$. If the magnet is placed in a uniform field, as indicated in Fig. 1.8, the forces on it are equal and opposite, but it will experience a torque of magnitude mH sin $\theta$, which tends to bring it into alignment with the lines of $\vec{H}$. (Here $\theta$ is the angle between the dipole moment m and the direction of $\vec{H}$.) For macroscopic magnets this torque can be measured even if the pole strength is so diffuse that its exact position has little or no meaning. For microscopic linear magnets the dipole moment is the only accessible measure of magnetic strength.

Even the torque may be difficult to measure for a magnet so small as to be mechanically inaccessible with a device such as a torsion balance. The dipole moment can still be determined by finding the energy involved in lining up such magnets in the direction of the field lines, or by finding the energy required to reverse their positions. We note that if the magnet is initially in the direction of $\vec{H}$, the work required to turn it through 90°, so that it becomes perpendicular to the field, is mH. (Proof of this statement is left to the problems.) It is conventional to say that the magnetic dipole has zero energy when it is perpendicular to the lines of $\vec{H}$. The energy is equal to $-mH\cos\theta = -\vec{m}\cdot\vec{H}$ for any other orientation. This is potential energy, since it is determined by the position of the magnet relative to the field, and the negative sign makes the most stable position (alignment with the field lines) that of the lowest potential energy.

The description of magnetic interactions in terms of an inverse square law of force between poles seems to go very smoothly, but it has a distinct weakness: It does not include any statement of the experimental fact that poles cannot be isolated from each other. This could, of course, be added in words. Coulomb went further than Michell: After establishing the inverse square law and the direct dependence on magnetic pole strength, he postulated that the "molecules of magnetic fluid" are themselves elementary magnets, complete with two poles of equal strength. Chains of such "molecules" would then cancel each other except at the ends, which appear as magnetic poles. This would explain the fact that two poles appear when a magnet is cut, equal and opposite in strength and each equal and opposite to the original pole still attached to it. This is very much like the modern view, except that we now view elementary magnetism as a property of matter itself, not a separate fluid.

We have no proof that isolated magnetic poles do not exist somewhere,

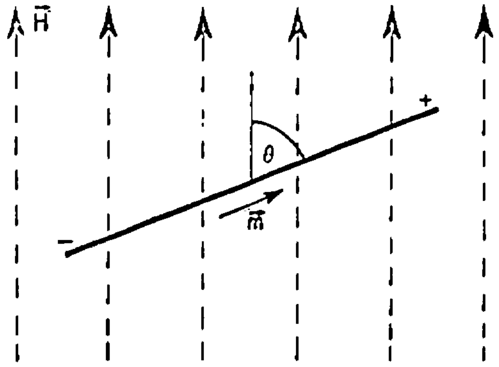


Fig. 1.8

and there are no known reasons why they should not exist. Nevertheless the mathematical description of magnetism as we find it should reflect the empirical fact that no isolated magnetic pole has ever been detected. This description would have to be amended if isolated poles are ever discovered, but in the meantime it would include an essential fact of magnetism as presently experienced.

The impossibility of separating poles can be stated by saying that in any volume cut off physically from remaining space by a bounding surface, there is no net pole strength; in cutting through a magnet, you create a pole equal and opposite to one you were trying to surround. Now in electrostatics we are able to write a simple relation between the electric field intensity and its sources within a particular volume in terms of the flux of the field intensity. We can, similarly, define the flux of the magnetic field intensity $\vec{H}$ through an element of surface $\Delta\vec{S}$ as $H\cdot\Delta\vec{S}$, and find that mathematically, as a result of the inverse square law, the total outward flux of $\vec{H}$ is

$$\sum_{s\ \text{closed}} \vec{H}\cdot\Delta\vec{S} = 4\pi k' q_m$$

where $q_m$ is the total pole strength inside the volume, as in Fig. 1.9. (We recall that in the electrical case, owing to the inverse square of the distance in the quantity to be summed, it

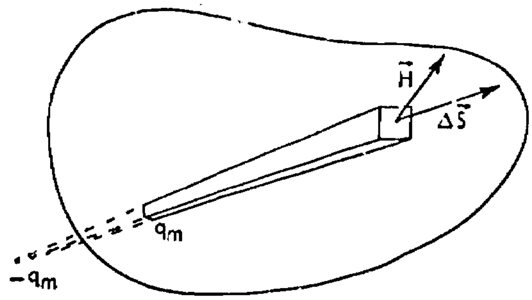


Fig. 1.9

does not matter where the charge is located within the volume, and the principle of superposition then insures that the total flux is independent of the distribution of charge within the volume.) But the surface involved in this theorem of Gauss's is mathematical; there is no more physics in the new statement than in the original law of Michell and Coulomb. The power of the Gauss form of Coulomb's law in electrostatics is that net electrical charges can be isolated with empty space completely surrounding them, that for any electric charge q there is no uniquely associated -q. As a result, Gaussian surfaces can be chosen with the same symmetry as that of the charge, so as to yield an expression for the electric field intensity. The analogous theorem for the flux of $\vec{H}$ is not so useful, since isolated spheres and lines of magnetic pole strength cannot be constructed.

And yet the concept of magnetic flux suggests a way of stating the inseparability of poles and the inverse square law at the same time. Let us assume a quantity $\vec{B}$ which is indistinguishable from the magnetic field intensity in empty space outside the magnet but defined inside the magnet by the condition that its total outward flux from every closed surface is zero whether there are magnets or not. Thus we may write

$$\sum_{S\text{ closed}} \vec{B}\cdot\Delta\vec{S} = 0,$$

for all possible surfaces. This condition is satisfied by $\vec{H}$ itself for surfaces which do not cut through magnets, but not in general for surfaces which do. The demand on $\vec{B}$ is equivalent to demanding a physical cut through the magnet, which would create a pole strength to cancel that already inside the volume, rather than the mere mathematical surface of Gauss's theorem for the field intensity. The behavior of lines of $\vec{B}$ through a magnet is shown in Fig. 1.10.

We shall see in the following chapters that $\vec{B}$ can be given an operational definition in connection with another aspect of magnetism. The only virtue in introducing it here is to state in mathematical form the inseparability of poles as we find them in nature. In the mks system of units, $\vec{B}$ and $\vec{H}$ are expressed in different units:

$$\vec{B} = \mu_0\vec{H}$$

in empty space, where $\mu_0$ is $4\pi \times 10^{-7}$. Again in empty space outside the magnet, a single pole of strength $q_m$ webers gives rise to $\vec{B}$,

$$\vec{B} = (q_m/4\pi r^2)\hat{r},$$

so that $\vec{B}$ is measured in webers per square meter. We should note the occurrence of the geometrical factor $4\pi$, just as in electrostatics. One place or another this factor is sure to enter the description.

If this were all there were to magnetism, there would be no connection with electricity, and the two would be considered as separate subjects. It

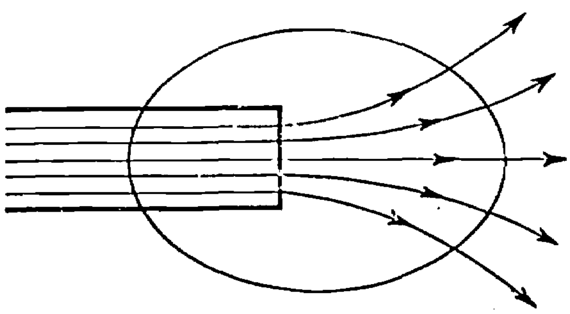

Fig. 1.10

was, in fact, one of Gilbert's achievements that he distinguished clearly between magnetism, as produced by magnetite and magnetized iron, and static electricity as produced by rubbing glass with silk. But there is current electricity which consists of a net flow of charge, whether as free charge or in electrical conductors, and magnetism is connected with the motion of charge, as we shall see in the next chapter.

## PROBLEMS

1.1 Suppose you were confronted with two iron bars that look identical in every respect, but one has been "permanently" and strongly magnetized with its poles well localized at the two ends, and the other not. Without any additional equipment whatsoever, how could you determine which is a permanent magnet and which has no residual magnetism of its own? Describe in detail the operations you could perform and the conclusions you would draw at each step.

1.2 When one end of a magnet is brought close to one end of an initially unmagnetized nail, the nail itself becomes a magnet and will attract other nails. The effect is even more pronounced if the end of the magnet is actually brought into contact with the end of the nail. (If you have not observed such phenomena you can easily do so with toy magnets obtained at a variety store.) What electrical phenomenon is this analogous to? Is the analogy complete, i.e., in what way do the phenomena differ? (Hint: What would happen in the electrical case after the two objects came in contact with each other?)

1.3 Make a detailed list of ways in which electrostatic and magnetostatic phenomena clearly differ from each other. (Note such aspects as the nature of materials that exhibit relevant properties, phenomena of conduction, phenomena of polarity, etc.)

# 2 THE MAGNETIC INTERACTIONS OF STEADY CURRENTS

The electrical effects first studied were those of static charges produced by rubbing amber with cloth, but no quantitative results were obtained until the properties of conductors were distinguished from those of insulators. In electrostatics a conductor is an object whose surface has an equilibrium distribution of charge: There is no difference of potential between any two points on the surface of a conductor. This follows from the definition of a conductor as something in which charge is free to move.

But if a difference of potential can be maintained between two points of a conductor, there will be a flow of charge from one point to the other. We no longer have a static situation, but we may have a steady flow. Let us consider a linear conductor such as a straight wire, and assume that some external device can maintain a constant difference of potential between the ends (see Fig. 2.1). This device will need to supply charge, but the charge does not build up anywhere - it leaves the wire at the same rate that it enters, and the conductor need have no net charge. The result is a steady flow of charge in the wire. The amount of charge per unit time which passes any position P is the current:

$$I = \frac{\Delta q}{\Delta t} \frac{\text{coulombs}}{\text{second}},$$

and one coulomb/second is called an ampere. The direction of the current is that of positive charge flow. A flow of negative charge is equivalent to a current whose direction is opposite to the motion of the charge.

Many practical uses of conductors involving the transfer of charge are treated in the laboratory monograph of this series. Here we shall not be concerned primarily with any particular relation between the magnitude of cur-

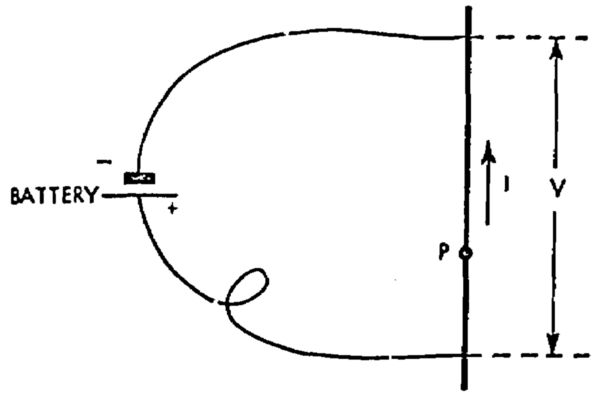


Fig. 2.1

rent in a conductor and the applied potential difference, but demonstration of the effect under immediate consideration does involve setting up electrical circuits, for which it usually suffices to know Ohm's law: For many conductors the current I of Fig. 2.1 is directly proportional to the applied potential difference denoted by V in the figure. Mathematically,

$$V = RI,$$

where R, called the resistance, depends on the material of which the conductor is made - whether copper or aluminum, for example. For a uniform wire the resistance is directly proportional to the length and inversely proportional to the cross-sectional area of the wire. Ohm's law is important as describing the behavior of many conductors, but it is not a fundamental law of electricity and magnetism. The magnetic effects of electric currents do not depend on the applicability of Ohm's law.

The first currents observed were those obtained by discharging conductors which had been charged electrostatically, but such currents are usually small and sporadic. Production of fairly large steady currents became possible only after Volta's invention of the chemical battery at the beginning of the nineteenth century. Development of the battery as a practical device facilitated many kinds of electrical experiments including the effects of current electricity.

Even earlier (1752), Benjamin Franklin had demonstrated that lightning is an electrical discharge. Observation of occasional erratic behavior of compass needles during a thunderstorm suggested to Hans Christian Oersted of Denmark some connection between electricity and magnetism, and led to his remarkable discovery in 1819 that current electricity is accompanied by magnetic effects. Oersted's experiment consisted of setting a long straight portion of an electric circuit above and parallel to a compass needle, and finding that the needle is deflected from its original north-south orientation when the circuit is closed (see Fig. 2.2). This is not a temporary effect: The deflection is maintained so long as the current is maintained. With a strong current the needle is very nearly at right angles to the line of the current, and the deflection of the needle is reversed when the current is reversed. If the compass is held above the wire instead of below, and the direction of the current is unchanged, the deflection of the needle is again reversed.

We have seen that a compass aligns itself along the direction of the magnetic field intensity, and that field lines can be traced out with a small compass. The "sense" or direction of the arrow on a field line is that of the dipole moment of the compass: from negative to positive, or from S pole to N pole. The field lines so traced for a long straight wire carrying a current are circles, directed in accord with a right-hand rule: If the wire is grasped with the right hand, the thumb pointing in the direction of positive charge flow, the direction of the curved fingers is that of the field lines. (Check this rule with Fig. 2.3.)

A quantitative study of the magnetic field intensity accompanying a long straight linear current was undertaken by Biot and Savart in Paris, immediately after hearing of Oersted's discovery. They found that the magnitude of the magnetic field intensity $\vec{H}$ at any point is directly proportional to the strength of the current, and inversely proportional to the shortest distance from the point to the wire. Quantitatively,

$$\vec{H} = I/2\pi r \qquad (2.1)$$

for a long straight wire carrying current of magnitude I, in mks units. The current is measured in amperes and r in meters. There is no explicit arbitrary constant (for a change!), because

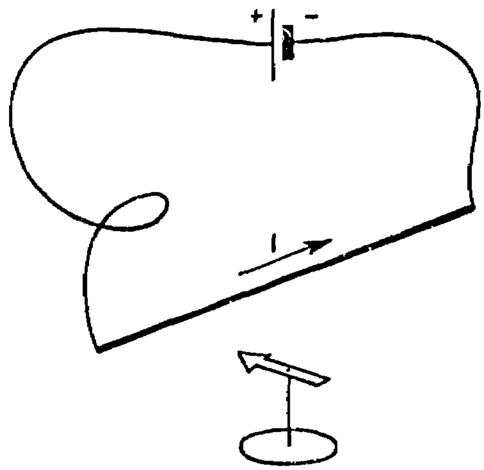


Fig. 2.2

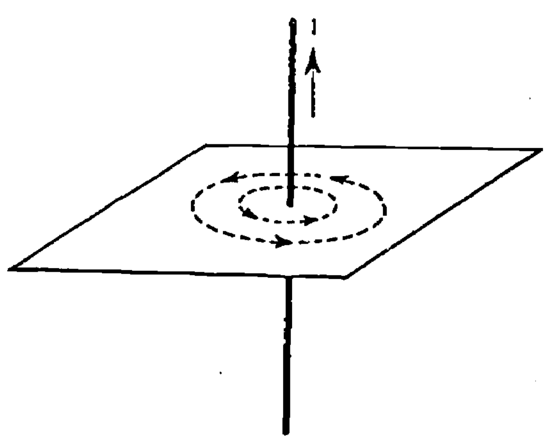


Fig. 2.3

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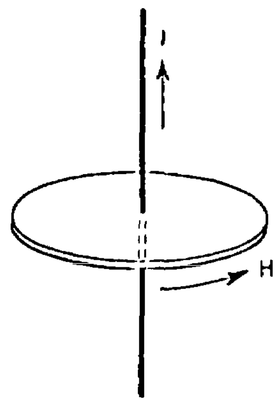


Fig. 2.4

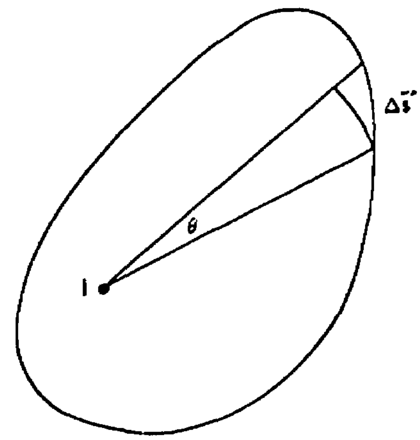


Fig. 2.6

the ampere and the weber, which we have already encountered in Chapter 1 as a unit of magnetic pole strength, are so defined that the only factor in this formula is $2\pi$, the ratio of the circumference of a circle to its radius. Clearly the magnetic field intensity $\vec{H}$ may be expressed in amperes per meter instead of newtons/weber, and it is usually so expressed in mks units. In fact, $\vec{H}$ can be defined by this equation, with the magnitude of $\vec{H}$ at a distance of one meter from a long straight wire carrying a current of one ampere being equal to $(1/2\pi)$ amperes/meter. The equation as it stands, however, does not give the direction of the magnetic field intensity, and the right-hand rule must be kept in mind as well as the relation of magnitudes.

The form of the Biot-Savart law given above, together with the right-hand rule, suggests another way of putting the relation between I and the

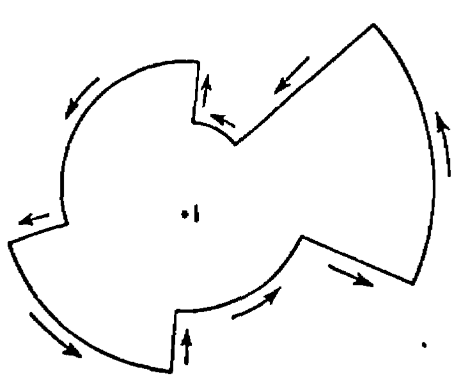


Fig. 2.5

magnitude of $\vec{H}$. Let us define what is called the circulation of $\vec{H}$: Consider a closed path, s, in the field; for every part of the path multiply the element of length $\Delta s$ by the component of $\vec{H}$ parallel to $\Delta\vec{s}$, and sum the product over the entire path. The result is called the circulation of $\vec{H}$ about the path chosen. For a circular path in a plane perpendicular to the current whose line passes through the center of the circle (Fig. 2.4), this process is easily carried out. The field intensity is everywhere in the same direction as the path, the entire path length is $s = 2\pi r$, and the circulation is simply

$$\sum_{s \text{ closed}} \vec{H} \cdot \Delta\vec{s} = 2\pi r\, \vec{H} = 2\pi r\, I/2\pi r = I, \qquad (2.2)$$

just the current in the wire. But let us evaluate the circulation of $\vec{H}$ over a somewhat more complicated path consisting of concentric circular arcs connected with radial lines as in Fig. 2.5. The length of each arc is proportional to both its radius and the angle it subtends, but the field intensity is inversely proportional to the radius. The radial portions of the path contribute nothing, since they are perpendicular to $\vec{H}$, so that the circulation about this path is I, just as before. Even a slant element of path contributes to the circulation only $\theta/2\pi$, where $\theta$ is the angle it subtends as shown in Fig. 2.6. Thus the cilation of $\vec{H}$ is equal to I if the area bounded by the closed path has I passing through it, and equals zero if the current circuit does not link through the loop. Otherwise, the shape of the closed path does not matter, nor does the exact position of the current; the only consideration is whether the current "threads" the loop, that is, whether there is flow of charge through the area bounded by the loop (Fig. 2.7).

That this result is perfectly general follows from its independence of the shape of the loop for a line current, and from the principle of superposition for $\vec{H}$. We may write

$$\text{circulation of } \vec{H} = \sum_{s \text{ closed}} \vec{H} \cdot \Delta\vec{s} = I, \tag{2.3}$$

where I is the total (net) current threading the path of the circulation. The circulation of $\vec{H}$ about two equal and opposite currents is zero, even though the currents are displaced from each other and the magnetic field intensity itself may have quite appreciable values at various points along the path.

The circulation law for $\vec{H}$ is extremely useful in finding the field intensity associated with all current configurations which have cylindrical symmetry. An important example is the solenoid, a coil of insulated wire wound in a close helix or spiral on a hollow cylinder, or having the shape of of a cylinder. Let us consider a very long coil of this kind, of n turns per unit length, eacy carrying current I. We may investigate the field intensity well away from both ends by taking a circulation path partly inside and partly outside the coil, as in Fig. 2.8, barely including the current carrying wires. For each turn of wire the magnetic field in the plane of the turn is at right angles to the plane, and if the contributions of the loops above and below are considered in pairs it can be seen that the whole field is parallel to the axis of the solenoid, so long as we stay far from the ends.

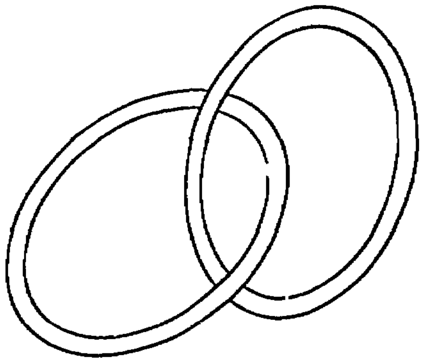

Fig. 2.7

Thus the short sides of the circulation rectangle contribute nothing, and

$$H_{inside}\,\ell - H_{outside}\,\ell = n\ell I,$$

where $\ell$ is the long dimension of the circulation rectangle. But this same equation holds if the circulation path is changed to the dotted line inside the coil, or for any other position of the inside leg of the rectangle. We can therefore conclude that the field in-

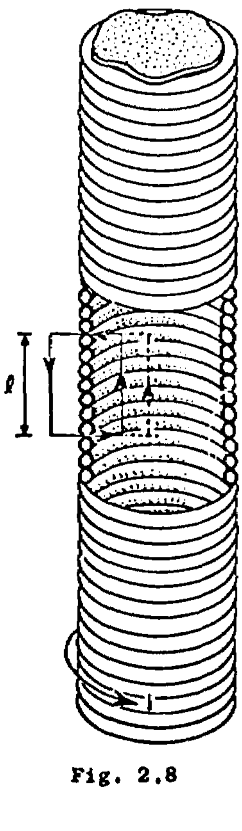


Fig. 2.8

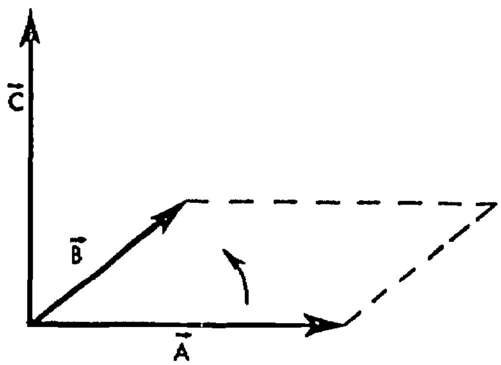


Fig. 2.9

tensity inside the cylinder is uniform, having the same strength and direction over the entire cross section. Exactly the same argument can be made for the leg of the rectangle outside the cylinder; the sides which are short in Fig. 2.8 may be made as long as we please. The field intensity outside the solenoid, in a plane that cuts the cylinder far from the ends, is also uniform. But the field intensity outside must be very small indeed; full justification of this statement is left to a problem. Therefore:

$$H_{inside} - H_{outside} = nI = H_{inside} \quad (2.4)$$

to a very good approximation.

Except in current configurations of cylindrical symmetry, the circulation law is not so very useful in finding the magnetic field intensity accompanying a current. To express the field intensity at a point in terms of the current in a circuit of arbitrary geometry we shall need the vector product of two other vector quantities.

The prototype vector is almost literally the directed line segment in three-dimensional space by which other vector quantities such as force and velocity are represented. We are familiar with the scalar product of two vectors as it occurs in $\vec{F} \cdot \vec{s}$ = work, or $\vec{E} \cdot \vec{s}$ = electrical potential difference, giving a scalar quantity which can be expressed as a single number. The prototype cross product of two vectors is the area of the parallelogram defined by two directed line segments, represented in a direction perpendicular to that area. That two lines specify a definite parallelogram in a plane is shown in Fig. 2.9. By definition

$$\vec{C} = \vec{A} \times \vec{B} = AB \sin\theta \, \hat{c},$$

where $\hat{c}$ is a unit vector at right angles to the plane of $\vec{A}$ and $\vec{B}$. The sense or sign of $\vec{A} \times \vec{B}$ is determined by a right-hand rule: With the plane of your hand at right angles to the plane of $\vec{A}$ and $\vec{B}$, let the fingers of your open right hand point in the direction of the first named vector ($\vec{A}$) with the hand oriented so that the parallelogram is in front of your palm - partial closing of the fingers would bring them parallel to the second vector ($\vec{B}$); the direction of your extended thumb is that of $\vec{A} \times \vec{B}$.

Vector cross multiplication is not commutative: It is readily seen that

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}).$$

The cross product of $\vec{A}$ and $\vec{B}$ vanishes if $\vec{A}$ and $\vec{B}$ are parallel, and has its maximum magnitude if they are at right angles to each other. We note that an area is represented as normal to its plane, but that the sign of the normal is chosen by an arbitrary rule. Many physical quantities share this characteristic. Cross products may be represented by directed line segments, and, for most purposes, such as addition and multiplication, they behave like ordinary vectors. Actually a vector product is not exactly the same kind of quantity as at least one of its vector factors - that a directed area is not quite like a directed line segment is shown in one of the problems. In the problems the cross product is also expressed in terms of the components of $\vec{A}$ and $\vec{B}$ in Cartesian coordinates.

From other experiments of Biot and Savart and of Ampere (to whose further work we shall turn our attention shortly) on circular circuits and those

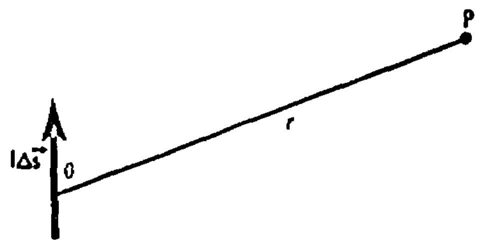


Fig. 2.10

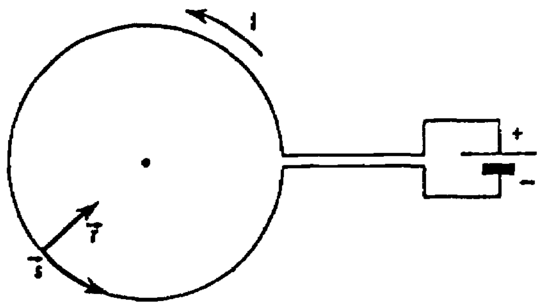

Fig. 2.11

which combine circular arcs and radial circuit elements, it became evident that magnetic effects are always directly proportional to the magnitude of the current, and that the effect of each element of current at a particular point P depends not only on the distance of the point, but also on the orientation of the current with respect to the line between it and the point at which $\vec{H}$ is to be determined. With reference to Fig. 2.10, the contribution of a length of circuit $\Delta s$ to the field intensity at a point whose distance is r is given by

$$\Delta H = I\Delta s \sin \theta / 4\pi r^2, \qquad (2.5)$$

where $\theta$ is the angle between $I\Delta\vec{s}$ (taken positive in the direction of the current) and the line between the current element and the point. The direction of $\vec{H}$ is at right angles to both $I\Delta\vec{s}$ and r, and in this instance into the page. All this information is conveyed more simply by means of the cross product

$$\Delta\vec{H} = \frac{I\Delta\vec{s} \times \hat{r}}{4\pi r^2}, \qquad (2.6)$$

where $\hat{r}$ is a unit vector along r, directed from the current element to the point in space. This formula has to be inferred from experiments with complete circuits, for which the field intensity is correctly given by

$$\vec{H} = \sum_{\text{s closed}} \frac{I\Delta\vec{s} \times \hat{r}}{4\pi r^2}. \qquad (2.7)$$

Let us apply this last formula to find the field intensity at the center of a circular loop of radius r, carrying current I. The lead wires from the battery produce no effect. (Why?) Since $\Delta\vec{s}$ is perpendicular to the radius of the loop, and the distance r is the same for all elements of current, the sum over all parts of the circuit can be evaluated at once:

$$H = \frac{I}{4\pi r^2} 2\pi r = \frac{I}{2r} \qquad (2.8)$$

at the center of a circular loop. The direction of $\vec{H}$ is that of $\Delta\vec{s} \times \hat{r}$, out of the page for the current indicated in Fig. 2.11.

Ampere, who learned of Oersted's discovery at the same time as did Biot and Savart, reasoned that there should be forces between two current circuits if both produce magnetic effects, since two magnets interact with each other. Within a week he had shown that two parallel wires carrying currents in the same direction (see Fig. 2.12) attract each other, and repel each other if the currents are in opposite directions. The magnitude of the force between the wires is directly proportional to both currents. As the result of a remarkable series of experiments performed during the next three years,

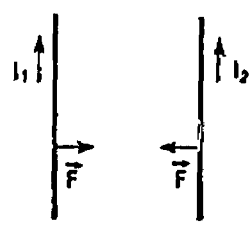


Fig. 2.12

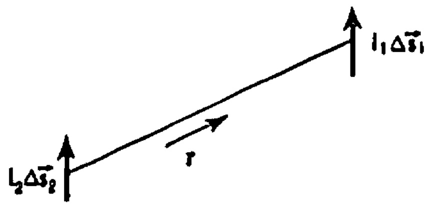


Fig. 2.13

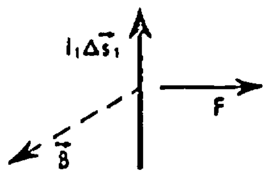


Fig. 2.14

Ampere was able to infer that the force between two parallel current elements $I_1\Delta s_1$ and $I_2\Delta s_2$ is given by

$$\Delta F = k'\ I_1 I_2 \sin\theta\ \Delta s_1\ \Delta s_2/r^2$$

$$= \frac{\mu_0}{4\pi}\frac{I_1 I_2\ \Delta s_1 \Delta s_2 \sin\theta}{r^2},$$

where $\mu_0/4\pi$ is the constant of proportionality in mks units (see Fig. 2.13). The value of $k'$ is taken arbitrarily as $10^{-7}$ newton/ampere$^2$, so that $\mu_0 = 4\pi \times 10^{-7}$ newton/ampere$^2$. The size of the ampere (and thus also the coulomb) is in fact determined by taking the constant of exactly this magnitude.

Again the presence of $\sin\theta$ suggests a cross product. Still another angle must be taken into account if the two current elements are not parallel, and the formula begins to look even more complicated:

$$\Delta\vec{F}(\text{on } I_1\Delta\vec{s}_1) = \frac{\mu_0}{4\pi}\frac{I_1\Delta\vec{s}_1 \times (I_2\Delta\vec{s}_2 \times \hat{r})}{r^2}$$

$$= I\Delta\vec{s}_1 \times \left(\frac{\mu_0}{4\pi}\frac{I_2\Delta\vec{s}_2 \times \hat{r}}{r^2}\right), \qquad (2.9)$$

where $\hat{r}$ is a unit vector directed from $\Delta\vec{s}_2$ toward $\Delta\vec{s}_1$. For the total force on current element $I_1\Delta\vec{s}_1$ exerted by circuit 2, we must sum over all $I_2\Delta\vec{s}_2$:

$$F(\text{on } I_1\Delta\vec{s}_1) = I_1\Delta\vec{s}_1 \times \left[\frac{\mu_0}{4\pi}\sum_{s_2\text{ closed}}\frac{I_2\Delta\vec{s}_2 \times \hat{r}}{r^2}\right].$$

But this way of writing the force suggests a simplification, since the term in parentheses is, apart from the constant $\mu_0$, the magnetic field intensity found by Biot and Savart as a force per unit pole. Moreover, it is found that a current element experiences a force when placed in the vicinity of an ordinary magnet, as indeed we might expect from the fact that a current exerts forces on a magnetic compass needle. The force on our current element $I_1\Delta\vec{s}_1$ may be written as

$$\vec{F} = I_1\Delta\vec{s}_1 \times \mu_0\vec{H} = I_1\Delta\vec{s}_1 \times \vec{B}, \qquad (2.10)$$

where

$$\vec{B} = \sum_{s_2\text{ closed}}\frac{\mu_0}{r\pi}\frac{I_2\Delta\vec{s}_2 \times \hat{r}}{r^2} \qquad (2.11)$$

if $\vec{B}$ is produced by current $I_2$. In general throughout empty space

$$\vec{B} = \mu_0\vec{H}, \qquad (2.12)$$

whatever the sources of magnetic field intensity (see Fig. 2.14).

In Chapter 1 the field $\vec{B}$ was defined so that its net flux through the surface enclosing any volume of space vanishes, although the net flux of $\vec{H}$ might be different from zero. The lines of both $\vec{B}$ and $\vec{H}$ as produced by currents are without beginning or end, and the net flux of both quantities through the surface enclosing any volume is zero. To see this we need consider only an increment of field arising from a single current element $I\Delta\vec{s}$, and remember that the entire field at a point is the sum of such increments. Let us examine

$$\Delta\vec{B} = \mu_0\Delta\vec{H} = \frac{\mu_0 I}{4\pi}\frac{\Delta\vec{s} \times \hat{r}}{r^2}.$$

The direction of $\Delta\vec{B}$ is perpendicular to both $\Delta\vec{s}$ and $\hat{r}$, and its magnitude depends on $r$. If we choose any point and move along the direction of $\Delta\vec{B}$, wo shall trace out a circle in a plane at right angles to the direction of $\Delta\vec{s}$, which would yield no net flux from the surface of a volume such as shown in Fig. 2.15. Increments of $\vec{B}$ arising from other current elements have the same property: The circles of $\Delta\vec{B}$ they contribute may lie in different planes, but the lines of each are continuous. Thus for $\vec{B}$ as a whole, the summation of the flux over any closed surface,

$$\sum_{S \text{ closed}} \vec{B} \cdot \Delta\vec{S} = 0 = \sum_{S \text{ closed}} H \cdot \Delta S$$

if $\vec{H}$ is produced by currents.

The equation $\vec{F} = I\Delta\vec{s} \times \vec{B}$ is often taken as the definition of the vector field $\vec{B}$, which is called the magnetic induction field. The units of $\vec{B}$ are newtons/ampere-meter. It is left to the problems to show that these units are consistent with those given in Chapter 1. In empty space, where our equations hold, the field quantities $\vec{B}$ and $\vec{H}$ are really indistinguishable, although their units are arbitrarily different in the mks system. We shall investigate further the equivalence of currents and magnets, and how this equivalence depends on the absence of separable magnetic poles, but let us first look again at the role of $\vec{B}$ in the interaction of two currents.

We have investigated the magnitude and direction of $\vec{B}$ (although we sometimes called it $\vec{H}$) in relation to its sources in some simple cases. If we know $\vec{B}$ at any point we can immediately find the force on a current element I $I\Delta\vec{s}$ placed at that point by computing $I\Delta\vec{s} \times \vec{B}$. In terms of Ampere's experiments, the interaction between two currents is thus for convenience considered in two steps, the production of a field $\vec{B}$ by one circuit and the action of the field $\vec{B}$ on the other circuit. In view of the complicated dependence of the forces on the angles involved, this procedure has great advantages, since we need now consider only one angle at a time. Furthermore, the contributions to $\vec{B}$ from different sources, be they currents or magnets, are additive, so that simultaneous interactions may be considered in a relatively simple way. Even so, it should be remarked that the greatest advantages of the field concept become apparent only when the sources, and therefore the fields, are permitted to vary in time. The subject of time varying fields is reserved for another monograph in this series, but we should note that light and other electromagnetic radiation can be simply understood only on the basis of electric and magnetic field quantities.

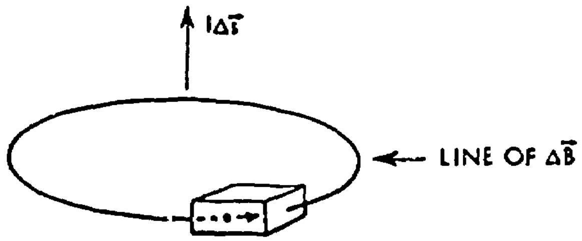


Fig. 2.15

If we consider only forces on current elements, we need only one magnetic field quantity, that defined to give the force per unit current at right angles to the direction of the field, namely, $\vec{B}$. The necessity for considering a second field quantity, the magnetic field intensity $\vec{H}$, does not then arise until we consider magnetic materials, within which $\vec{B}$ and $\vec{H}$ are different. But can we confine our attention exclusively to currents? It was Ampere's hypothesis that all magnetic interactions can in fact be traced to currents, whether they occur in macroscopic circuits or are assumed to exist in the most elementary form of matter. To establish the basis for this hypothesis we must consider the forces on a loop of current in a field $\vec{B}$.

Let us take a rectangular loop of wire cdef, as shown in Fig. 2.16, having dimensions a and b, placed initially so that its plane is parallel to a field $\vec{B}$ which is uniform in space and constant in time. The current in the loop is I. We may compute the force on each straight section of the loop from the formula $\vec{F} = I\Delta\vec{s} \times \vec{B}$. With the current as indicated, we see that there

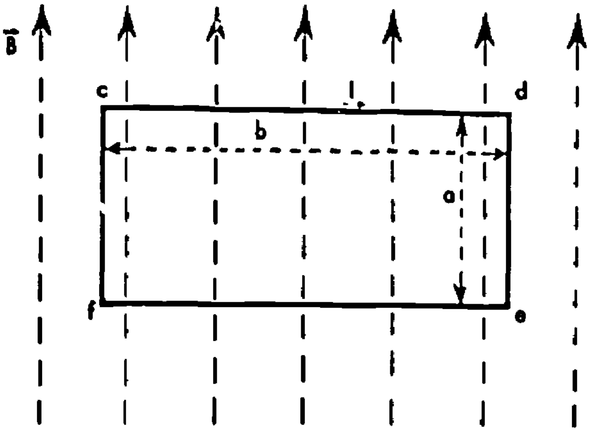


Fig. 2.16

is a force IbB directed out of the plane on the wire cd, a force IbB directed into the plane on the wire ef and no forces on fc and de since these wires are in the same direction as the lines of $\vec{B}$. The net force on the loop is zero, but there is a torque of magnitude abIB tending to turn it out of the plane, into such a position that what is now the front of the loop faces down, with the plane of the loop perpendicular to the lines of $\vec{B}$. This is exactly what would happen to a rectangle of magnetic material whose front face is a negative (south-seeking) pole and whose back face is positive (north-seeking). The loop is thus equivalent to what Ampere called a magnetic shell, a flat magnet of magnetic moment proportional to the product of its area and the current on its boundary.

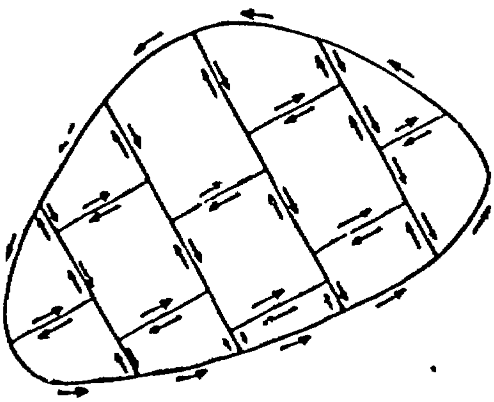

Fig. 2.17

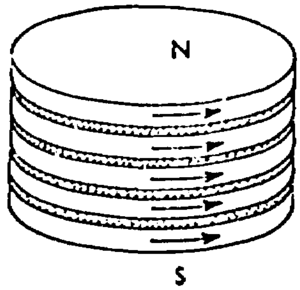


Fig. 2.18

We may recall that the torque on a magnet of magnetic moment $\vec{m}$ in a region of field intensity $\vec{H}$ is mH when the direction of the moment is at right angles to $\vec{H}$. The torque on our loop is IAB, where $A = a \times b$ is the area of the loop. If we want to keep the same units as before for magnetic moment, we may ascribe to the loop a moment of magnitude $\mu_0 IA$, directed perpendicular to the plane of the loop, and positive toward a right-hand thumb whose curved fingers point along the current. For other orientations of the loop the torque is IAB $\sin\theta$, where $\theta$ is the angle between the magnetic moment and the field lines, in agreement with the torque mH $\sin\theta$ on a magnet in a field intensity $\vec{H}$.

The magnetic moment of a current loop does not depend on the shape of the loop. A circular loop is equivalent to a magnetic disk whose faces are of equal and opposite polarity, and whose magnetic moment is again $\mu_0 IA$, with A the area of either face. Ampere went further; according to his hypothesis, all magnets are current configurations, which exist on a submicroscopic scale. A "magnetic shell" would consist of an indefinite number of tiny current whirls, all oriented in the same sense, so that the net current is zero except at the boundary of the shell, or loop; internally the currents of contiguous whirls cancel each other as is evident in Fig. 2.17. A helix of wire with adjacent turns of current is thus equivalent to a stack of magnetic disks, as in Fig. 2.18; the net effect as a magnet is a positive (north-seeking) pole at one end of the helix and a negative pole at the other. The impossibility of separating poles is then just the impossibility of separating the faces of a disk.

The modern view of magnetism is not very different from this simple picture, and Ampere's hypothesis has been accepted in principle. Because of the absence of separable magnetic poles, all magnetism is traced to currents, even when the physical currents are not accessible to measurement. The neutron, for example, is an uncharged particle, but it does have a magnetic moment; in this sense it behaves like a circulating negative charge. Recent experiments have shown that the neutron does behave more like an infinitesimal current whirl than like an infinitesimal linear magnet, although its current is quite inaccessible for detailed investigation.

Search for the isolated magnetic pole continues, but all the evidence in hand supports the view that $\vec{B}$, a field which acts on currents and can be traced to moving charge, is a more basic concept than that of the magnetic field intensity $\vec{H}$, simply because electric charges and currents do exist. Nevertheless we continue to explore magnetic fields with iron filings and compass needles, and the concept of magnetic field intensity $\vec{H}$ can hardly be avoided in the description of magnetic materials.

Einstein has pointed out in his "scientific autobiography" that concepts used to describe the physical world are in truth intellectual creations, devised by scientific imagination but not freely: The discipline imposed by observation and experiment is very strict. The concepts of electromagnetic theory are very intimately related to one another, reflecting an enormous body of diverse but related experimental results. More than one of the concepts plays a dual role, and experiments can often be described in more than one way. Yet we shall see in Monograph III that there are only four fundamental empirical laws of electricity and magnetism, which can be described in a most elegant and simple way in terms of field quantities. We have already considered three of these laws: Coulomb's law for the interaction of two static charges, the law of Michell and Coulomb for the interaction of two magnetic poles plus a statement of the inseparability of poles, and the law describing the magnetic effect of currents or the interaction of two currents, usually called Ampere's law. The remainder of this booklet will be devoted to further development of this third law. Before undertaking a more mathematical and thus more powerful formulation of the law, however, let us examine the effect of magnetic fields on individual charges.

The currents we have considered thus far were assumed to exist in uncharged conductors, but a current is defined as a flow of charge. The question of whether the mechanical motion of a charged body produces magnetic effects was first tested experimentally by Henry Rowland of Johns Hopkins University in 1878. Rowland electrostatically charged the rim of an insulating disk, rotated the disk, and found that magnetic effects were indeed produced. The experiment is very difficult to perform because the currents produced in this way are small, but the result is unambiguous. That moving charges experience a force in a magnetic field is much easier to demonstrate: Streams of electrons in a cathode ray tube produce a visible glow on the glass envelope of the tube which is shifted very readily by even a small magnet. In fact, it was concluded that cathode rays are charged particles as a result of their deflection by both electric fields and magnetic fields.

The correct formula for the force on a charge moving in a magnetic field can be obtained from that for the force on an element of current in a conductor. Let us assume that a conductor has $N$ movable charges per unit volume, each charge of magnitude $q$. These movable charges may undergo very complicated motions, but if there is a net flow of charge in one direction, we may ascribe

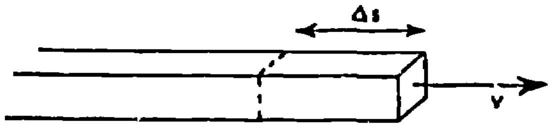

Fig. 2.19

to them an average "drift" velocity v. A linear conductor of unit cross section, a segment of which is indicated in Fig. 2.19, would then carry a current Nqv - this is the amount of charge crossing the face shown per unit time. The amount of charge per unit time flowing through a portion of the face whose area is A is then NqAv, and the general formula is I = NqAv. For a current element of length $\Delta s$,

$$I\Delta s = NqA\Delta sv.$$

But $A\Delta s$ is just the volume of the current element and $NA\Delta s$ is the total number of charges involved. Thus the force per charge q is

$$\vec{F} = \frac{I\Delta\vec{s} \times \vec{B}}{NA\Delta s} = q\vec{v} \times \vec{B}. \qquad (2.13)$$

This derivation of $\vec{F} = q\vec{v} \times \vec{B}$ involves too many assumptions to be rigorous, but the result is entirely correct. For a charge q moving with velocity $\vec{v}$, $q\vec{v}$ is equivalent to a current element. The force $q\vec{v} \times \vec{B}$ is called the Lorentz force, first derived rigorously by the famous Dutch physicist H. A. Lorentz in 1892. It is often taken as a fundamental equation, and the force on a current element derived from it. It can be taken as the defining equation for the magnetic field quantity $\vec{B}$: $\vec{B}$ is that field which gives a velocity dependent force on a charge q, in accord with the equation for the Lorentz force, as distinguished from the electric field intensity $\vec{E}$, which produces a force which is independent of the velocity. We note that both $\vec{E}$ and $\vec{B}$ are defined in a particular frame of reference, that in which the velocity of the charge is $\vec{v}$.

One of the most interesting properties of the Lorentz force is that it is incapable of changing the speed or kinetic energy of a moving charge, since the force (and therefore the acceleration) is at right angles to the velocity. In a uniform magnetic field which is itself perpendicular to the velocity, the motion of a charged particle is circular.

## PROBLEMS

2.1 Show that if $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the direction of increasing x, y, and z in a right-handed Cartesian coordinate system,

$$\hat{x} \times \hat{x} = 0 = \hat{y} \times \hat{y} = \hat{z} \times \hat{z}$$
$$\hat{x} \times \hat{y} = \hat{z}$$
$$\hat{y} \times \hat{z} = \hat{x}$$
$$\hat{z} \times \hat{x} = \hat{y}$$

Show also that in terms of Cartesian components, the vector product of $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$ may be written in the form of a determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

2.2 Suppose you have a long cylindrical conductor of radius $r_0$, carrying total current $I_0$. The current is distributed uniformly over a cross section of the conductor, so that $I_0 = \pi r_0^2 j$, where j is the current density in amperes/meter$^2$.

(a) The field lines of H outside the cylinder are circles centered at the axis. Would this also be true of the magnetic field lines within the cylinder? Give your reasons.

(b) What is the magnetic field intensity on the axis of the cylinder?

(c) What is the magnitude of H at $r = \frac{1}{2}r_0$?

(d) Plot the magnitude of H against the distance r from the axis of the cylinder, both inside and outside the cylinder. At what distance from the axis is this magnitude greatest?

2.3 Let P be a point outside the long solenoid of Fig. 2.8, whose distance from the axis is large compared with the radius of the solenoid. According to Eq. (2.6), each element of current contributes to the field intensity $\Delta\vec{H} = I\Delta\vec{s} \times \hat{r}/r^2$.

(a) Consider qualitatively the contributions to the magnetic field intensity from current elements $I\Delta\vec{s}$ parallel to $\vec{r}$ and those perpendicular to $\vec{r}$ from a single turn of the coil in the plane of P perpendicular to the axis. Do they tend to reinforce or to cancel each other?

(b) Make the same qualitative estimate of contributions $\Delta\vec{H}$ for a point inside the solenoid.

(c) Consider, again qualitatively, the contributions $\Delta H$ from two turns, one above and one below the plane of P perpendicular to the axis.

(d) What are your conclusions concerning the validity of Eq. (2.4)?

2.4 If a charged body moves with velocity $\vec{v}$ at right angles to a uniform field $\vec{B}$, its path is circular, since $\vec{F} = q\vec{v} \times \vec{B}$, and thus its acceleration is perpendicular to $\vec{v}$. What sort of path would result if $\vec{v}$ were in the same direction as the lines of $\vec{B}$? What sort of path would be produced if the angle between $\vec{v}$ and $\vec{B}$ were neither 0° nor 90°?

# 3 MAGNETOSTATICS REFORMULATED

Ampere's law is all there is to magnetostatics if we confine our attention to the magnetic effects of steady currents, but some aspects of the subject become apparent only if more elegant mathematical methods are available. Mathematics is more than a powerful tool for the solution of problems: Relations between physical quantities are often revealed by mathematical analysis, so that more physics emerges clearly. The danger lies in a tendency to substitute mathematical formalism for physical thought, to overlook or neglect the physical content of a mathematical equation or a line of mathematical reason, instead of being guided by it. Let us try to keep the physics, or sometimes only geometry, firmly in mind in our further analysis of magnetostatics. For this chapter we shall only assume knowledge of basic calculus, that branch of mathematics invented by Newton not only to make hard problems easier but also to sharpen and clarify his ideas concerning the physical world.

Let us begin by rewriting the relations considered earlier in terms of differentials and integrals. Ampere's law has appeared in essentially two different forms. The direct expression of the force on a current element $I_1 d\vec{s}_1$ is

$$\vec{F}(\text{on } I_1 d\vec{s}_1) = I_1 d\vec{s}_1 \times \vec{B}, \qquad (3.1)$$

where

$$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{I_2 d\vec{s}_2 \times \hat{r}}{r^2}, \qquad (3.2)$$

where $\hat{r}$ is a unit vector from $I_2 d\vec{s}_2$ to the point at which $\vec{B}$ is computed. The substitution of $d\vec{s}$ for $\Delta\vec{s}$ is routine, but the circle at the middle of the integral sign is shorthand notation for the "s closed" below the summation sign of Chapter 2. Here it should be read "the integral over the entire circuit s" of the integrand as written. This integral is, of course, a vector, every element contributing to $\vec{B}$ in a direction at right angles to both $d\vec{s}_2$ and to the line from $d\vec{s}_2$ to the position of $I_1 d\vec{s}_1$ where $\vec{B}$ is to be determined to give the correct force.

We have also noted that $\vec{B}$ so defined yields a flux such that the total outward flux of $\vec{B}$ through any closed surface is zero. If $d\vec{S}$ is an element of surface, directed out from the volume inclosed by the surface,

$$\int_{\text{closed}} \vec{B} \cdot d\vec{S} = 0 \qquad (3.3)$$

Unfortunately one must write in words that this surface integral is closed. A little sphere at the middle of the integral sign would be a convenient shorthand notation except for the fact that a sphere is indistinguishable from a circle in two dimensions. In the booklet on electrostatics we have considered a function corresponding to such a closed surface integral which describes a property of the integrand vector at each point in the inclosed space. We shall consider this property of $\vec{B}$ at a later stage of the discussion.

We have seen that the physical content of Ampere's law may be written alternatively as a circulation law for $\vec{B}$, which may also be written as an integral. The field $\vec{B}$ is a vector quantity defined at all points $(x,y,z)$ in the region of interest. The line integral of $\vec{B}$ on a path C from point P to point P' is a scalar, written as $\int_P^{P'} \vec{B} \cdot d\vec{s}$, where ds is an element of length in the direction of the local tangent to the path, and the integrand is the product of $d\vec{s}$ and the component of $\vec{B}$ parallel to $d\vec{s}$ (see Fig. 3.1). (In terms of magnetic poles this integral would represent $\mu_0$ times the work done by the field in moving a unit pole from P to P'.) The circulation of a vector implies a closed path, not necessarily in a plane, and the integral sign can again be written with a circle at the middle to indicate that the initial and final points are coincident:

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I \qquad (3.4)$$

is simply a neater way of writing the circulation law for $\vec{B}$.

We have thus far considered almost exclusively linear currents such as those carried by thin wires, but the total current through a surface bounded by the circulation loop may be distributed over the surface and may vary from one part of the surface to the next. We may readily take account of such variations if we express the total current I in terms of the current density $\vec{j}$. For a conductor of cross-sectional area A in which the density of current is uniform, $I = jA$. As a vector quantity, $\vec{j}$ is the current per unit area of a plane normal to the direction of charge flow. Since an element of surface can be represented by a vector dS normal to its surface,

$$I = \int \vec{j} \cdot d\vec{S} \qquad (3.5)$$

is the total current through a surface over which the integral is evaluated (see Fig. 3.2). (In this monograph we confine our attention to steady currents, for which the integral of $\vec{j}$ over a closed surface would net zero.) Therefore the circulation law for $\vec{B}$ may be written

$$\oint_c \vec{B} \cdot d\vec{s} = \mu_0 \int \vec{j} \cdot d\vec{S}, \qquad (3.6)$$

where the integral on the right is to be carried out over a surface, in fact any surface, which is bounded by the curve C. We should recall that the positive direction for $d\vec{S}$ is chosen by a right-hand rule; for a curve traversed counterclockwise in the plane, $d\vec{S}$ is positive out of the page; if clockwise, $d\vec{S}$ is positive into the page. Thus far we have hardly changed the form of the equations relating magnetic fields to their sources: $\vec{B}$ is written as an integral over all portions of a linear current, instead of the corresponding sum, and the line integral of $\vec{B}$ is related to current through a surface, as was the sum of $\vec{B} \cdot \Delta\vec{s}$ in Chapter 2. Is it possible to relate the magnetic field to its source strength at each point, much as we found in the Electrostatics monograph that we could relate the electric field intensity to the charge density at each point? If there were only magnets with poles, and no magnetic effects of currents, the answer would be completely analogous to the electrostatic relations. It is indeed true that the net outward flux of the magnetic field intensity $\vec{H}$ (but not that of $\vec{B}$) from a closed volume is the total pole strength within the volume, and that H is related to density of

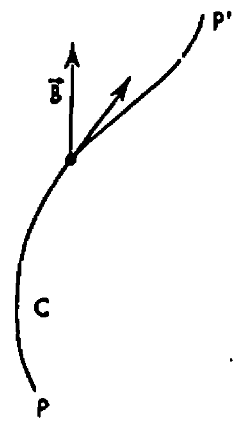


Fig. 3.1

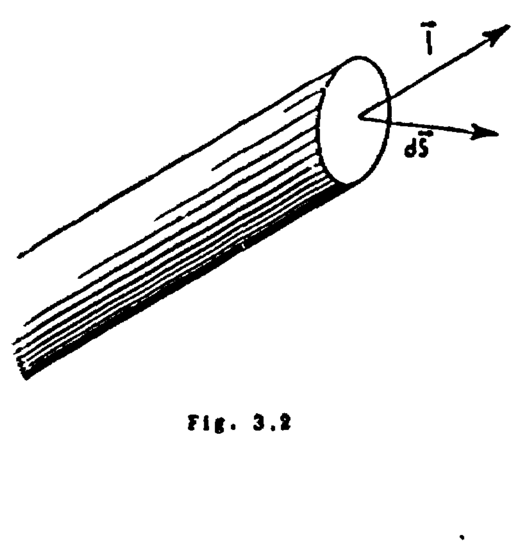


Fig. 3.2

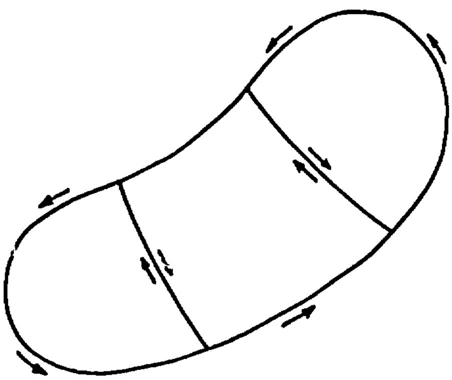

Fig. 3.3

pole strength in the same way that the electric field intensity is related to the charge density. But we have seen that the net flux of both $\vec{H}$ and $\vec{B}$ from a closed volume is zero if these fields are produced by currents. This is a very interesting and important property of magnetic fields, but it is of no immediate help in relating fields to the currents which produce them.

The answer comes from the circulation law for $\vec{B}$ (or for $\vec{H}$, since the two fields are the same in empty space, except for an arbitrary constant factor $\mu_0$). It will be necessary to examine circulation more closely and to develop further the mathematics associated with it. This mathematics applies to all vector fields, but we shall call the field $\vec{B}$, and feel free to apply the mathematical consequences to other vector fields as the need arises.

In electrostatics we were guided to the appropriate mathematical theorem by Gauss's law: Beginning with a relation between the surface integral of the electric intensity $\vec{E}$ over a closed surface and the charge within the enclosed volume, we found that the function of $\vec{E}$ which can be identified with the charge density at each point in space is the divergence of $\vec{E}$ (div $\vec{E}$). The circuital form of Ampere's law relates a closed line integral of $\vec{B}$ to the integral of the current density over a surface bounded by the line. By analogy we should expect to find a function of $\vec{B}$ at each point of space such that its integral over any surface bounded by a line is equal to the circulation of $\vec{B}$ about the perimeter of the surface. If it is to be identified with the current density, it must be a vector quantity. In other words, we seek a vector $\vec{C}$ which is a mathematical function of $\vec{B}$ such that

$$\oint \vec{B} \cdot d\vec{s} = \int \vec{C} \cdot d\vec{S} = \mu_0 \int \vec{J} \cdot d\vec{S}. \tag{3.7}$$

If the second equality is to be true for any surface bounded by the curve of of the line integral, it follows that the integrands must be the same, and $\vec{C} = \mu_0 \vec{J}$.

Consider any simple closed curve in a region where there is a field $\vec{B}$. An area bounded by a closed curve, whether plane or not, can be divided into two or several areas by lines between two points on the boundary. In Fig. 3.3 there are three areas, around each of which we may take the circulation of $\vec{B}$ in the counterclockwise sense indicated. In the sum of these circulations all the interior boundaries are traversed twice, in opposite directions. Thus

$$\oint \vec{B} \cdot d\vec{s} = \oint_1 \vec{B} \cdot d\vec{s}_1 + \oint \vec{B} \cdot d\vec{s}_2 + \oint \vec{B} \cdot d\vec{s}_3. \tag{3.8}$$

The sum of counterclockwise circulations of $\vec{B}$ about all the subdivisions is just the circulation about the whole, and this result is independent of whether the surface, or the curve bounding it, is plane. The same is true of clockwise circulations, of course, but the sense of all the circulations in the sum must be the same for cancellation of adjacent interior boundaries. The rule is equally justified for ten subdivisions, or a hundred. In general

$$\oint \vec{B} \cdot d\vec{s} = \sum_{\text{all } i} \oint \vec{B} \cdot d\vec{s}_i. \tag{3.9}$$

(We are here reminded of Ampere's hypothesis on the equivalence of a closed

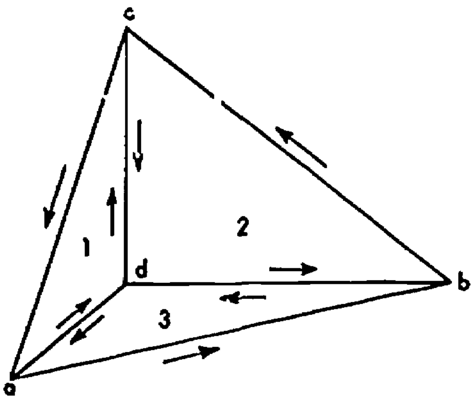


Fig. 3.4

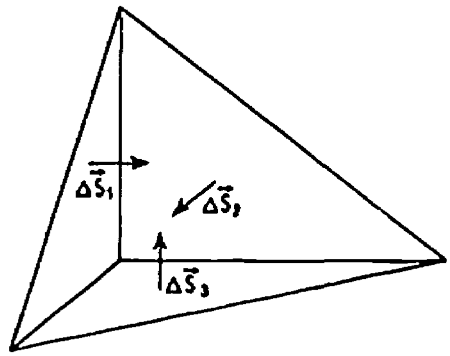


Fig. 3.5

current and a shell of magnetic dipoles, which involves submicroscopic current whirls; there is indeed a similarity, which we shall pursue later.)

Circulation itself is a scalar quantity, but any plane surface has an orientation in space which can be specified by a vector normal to the plane, and any well behaved surface can be broken up into elements sufficiently small to be considered plane. (We shall exclude surfaces with infinite peaks.) Let us see what the orientation of a surface has to do with the circulation of a vector about its boundary.

Consider a small triangular plane boundary abc, Fig. 3.4, and two surfaces bounded by it, one the plane abc, the other a surface made up of segments of three planes chosen at right angles to each other. According to the sum rule,

$$\oint_{abc} \vec{B} \cdot d\vec{s} = \oint_{adc} \vec{B} \cdot d\vec{s}_1 + \oint_{bcd} \vec{B} \cdot d\vec{s}_2 + \oint_{abd} \vec{B} \cdot d\vec{s}_3 . \qquad (3.10)$$

It is most unlikely that the three terms on the right contribute equally to the total circulation, for several reasons. Each small circulation specifies a different area and a different length of perimeter, and of course $\vec{B}$ itself and its variation in space is quite independent of the surfaces we happened to choose. There is a relation between the surfaces themselves, which follows from the theorem that the vector sum of all the outward surfaces of a polyhedron equals zero. (The proof of this theorem is left to a problem.) We note that the four triangles bound a tetrahedron (Figs. 3.4 and 3.5), for which the right-hand rule for circulation leads to positive inward directions for surfaces 1, 2, and 3, and a positive outward surface for the triangle abc. If $\Delta\vec{S}_1$, $\Delta\vec{S}_2$, $\Delta\vec{S}_3$, and $\Delta\vec{S}$ represent these four surfaces, then

$$\Delta\vec{S} = \Delta\vec{S}_1 + \Delta\vec{S}_2 + \Delta\vec{S}_3 . \qquad (3.11)$$

But does this vector relation between surfaces have anything to do with the scalar relation of the circulations? If there is a vector related to the circulation about each surface such that its scalar product with the surface itself would net the circulation the answer would certainly be affirmative.

A vector related to the circulation about an elementary plane surface can be constructed by multiplying the circulation of $\vec{B}$ about its boundary by a unit normal to the surface, and dividing by the magnitude of the surface: Let us write $(\oint \vec{B} \cdot d\vec{s}_1/\Delta S_1)\vec{n}_1$ as a vector in the direction of the unit normal $\vec{n}_1$ to $\Delta\vec{S}_1$. The scalar product of this vector with $\Delta\vec{S}$ is just $\oint \vec{B} \cdot ds_1$, since $\vec{n}_1 \cdot \Delta\vec{S} = \Delta S_1$. In fact, if we take the vector

$$\vec{C} = \frac{\oint \vec{B} \cdot d\vec{s}_1}{\Delta S_1} \vec{n}_1 + \frac{\oint \vec{B} \cdot d\vec{s}_2}{\Delta S_2} \vec{n}_2 + \frac{\oint \vec{B} \cdot d\vec{s}_3}{\Delta S_3} \vec{n}_3 , \qquad (3.12)$$

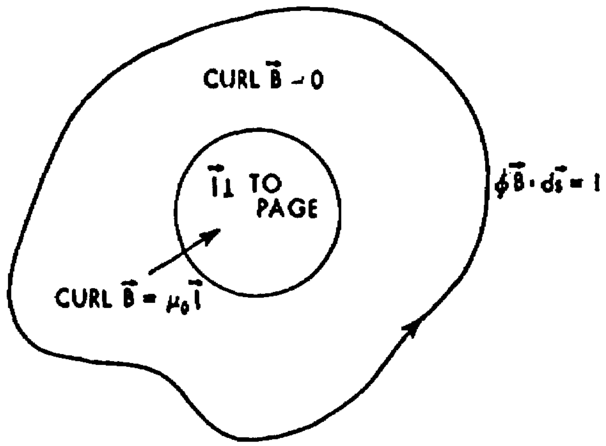


Fig. 3.6

then $\vec{C} \cdot \Delta\vec{S}$ is the original sum of circulations. We note that $\vec{C}$ is not in general the same as $(\int \vec{B} \cdot d\vec{s}/\Delta S)\vec{n}$, where $\vec{n}$ is the unit vector normal to the surface abc, although $\vec{C} \cdot \Delta\vec{S} = \int \vec{B} \cdot d\vec{s}$. The quantity $\int \vec{B} \cdot d\vec{s}/\Delta S$ is the component of $\vec{C}$ normal to the surface $\Delta\vec{S}$.

The components of $\vec{C}$ are thus far defined in relation to plane surfaces, which may be oriented in three mutually perpendicular and thus independent directions. Our surfaces $\Delta\vec{S}_1$, $\Delta\vec{S}_2$, and $\Delta\vec{S}_3$ are themselves components of $\Delta\vec{S}$. Now the surface $\Delta\vec{S}$ may be taken as small as we please, and in the limit of small $\Delta S$ we define a vector which is called the curl of $\vec{B}$, written curl $\vec{B}$. For the component of curl $\vec{B}$ normal to a surface element $\Delta S$ whose orientation is $\vec{n}$,

$$(\text{curl } \vec{B})_{\vec{n}} = \lim_{\Delta s \to 0} \frac{\int \vec{B} \cdot d\vec{s}}{\Delta S} \qquad (3.13)$$

with the line integral taken around the boundary of $\Delta S$. The integral form of the circulation sum is then

$$\int \vec{B} \cdot d\vec{s} = \int \text{curl } \vec{B} \cdot d\vec{S} \qquad (3.14)$$

where the surface integral extends over the surface (any surface) bounded by the closed curve of the line integral, the positive direction of $d\vec{S}$ being determined in relation to the circulation path by the right-hand rule.

The vector curl $\vec{B}$ is then a function of $\vec{B}$ at every point in space which is mathematically related to the circulation of $\vec{B}$ by this formula.

The mathematical relation between the line integral of a vector $\vec{B}$ about a closed path and the surface integral of curl $\vec{B}$ was derived by George Gabriel Stokes, and is known as Stokes' theorem. There is no physics in it. But we have seen that if $\vec{B}$ represents the magnetic field,

$$\int \vec{B} \cdot d\vec{s} = \mu_0 \int \vec{J} \cdot d\vec{S}, \qquad (3.15)$$

where $\vec{J}$ is the current density. Therefore, for any surface $\int \text{curl } \vec{B} \cdot d\vec{S}$ must equal $\int \mu\vec{J} \cdot d\vec{S}$, a demand which is impossible to satisfy unless

$$\text{curl } \vec{B} = \mu_0 \vec{J} \qquad (3.16)$$

at every point. This is the physical relation we have sought between $\vec{B}$ and the current at any point in space.

The name curl suggests going around, and we have arrived at the idea of curl by considering circulation, but of course curl is not identical with circulation. Consider, as a simple example, a long conducting circular cylinder in which there is a uniform current density $\vec{J}$, as indicated in the cross-section diagram (Fig. 3.6). Curl $\vec{B} = \mu_0\vec{J}$ has nonvanishing value only within the cylinder, but $\int \vec{B} \cdot d\vec{s} = I$, the total current threading the circulation loop, even if every point on the loop is outside the cylinder. There is, of course, a magnetic field $\vec{B}$ both outside and inside the cylinder, and it is $\vec{B}$ itself of which one takes the circulation.

The magnetic field intensity itself can be mapped out by a compass needle, a single small magnetic dipole free to orient itself along the field lines; the curl of the field intensity can be demonstrated with a magnetic "quadrupole" - two small permanent magnets with like poles cemented together - but an "octopole" is more stable and convenient. The negative (south-seeking) poles of four small permanent magnets can be cemented to a wire as indicated in Fig. 3.7, so that their positive or N poles are the four tips of a cross at right angles to the wire. A cork attached to the wire will make the whole contrivance float in a solution of NaCl, for example. If the electrodes are arranged so that the current flows vertically in a cylinder of electrolyte, the "curl-$\vec{H}$-meter" will rotate continuously, the sense of rotation depending on the direction of the current. Outside the cylinder of current a dipole will show the presence of a magnetic field, but the "curl-$\vec{H}$-meter" does not rotate in the absence of current in the solution.

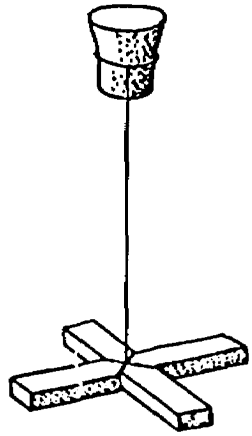

Fig. 3.7

We have noted in electrostatics that the line integral of the electric field intensity $\vec{E}$ between any two points is independent of the path connecting the points and thus $\oint \vec{E} \cdot d\vec{s} = 0$ around any closed curve. It is now seen from the definition of the curl of a vector that this absence of circulation corresponds to the statement that curl $\vec{E} = 0$ at every point in an electrostatic field. We also found that the point by point relation satisfied by $\vec{E}$ so as to express the physical content of Coulomb's law is div $\vec{E} = \rho/\epsilon_0$, where $\rho$ is the electric charge density. From the definition of the divergence of a vector and the fact that the net flux of the magnetic field $\vec{B}$ from any closed volume is zero, it follows that div $\vec{B} = 0$. All these relations can be summarized:

| Electrostatics | Magnetostatics |
|---|---|
| div $\vec{E} = \rho/\epsilon_0$ | div $\vec{B} = 0$ |
| curl $\vec{E} = 0$ | curl $\vec{B} = \mu_0 \vec{J}$ |

These are the basic equations of electrostatics and magnetostatics. Their physical content is Coulomb's law and Ampere's law.

In electrostatics we went further, and found that the determination of the field $\vec{E}$ corresponding to any configuration of static charges was much facilitated by the introduction of a scalar potential function $\phi$. The properties of the static field included the condition that the line integral of $\vec{E}$ from one point to another is given by

$$-\int_1^2 \vec{E} \cdot d\vec{s} = \phi_2 - \phi_1, \qquad (3.17)$$

the difference of potential between the two points. The possibility of relying on $\phi$ to obtain $\vec{E}$ depends on the condition that curl $\vec{E} = 0$. It is left to the problems to show that the curl of the gradient of any scalar function of position vanishes identically.

It is clear that we cannot depend on a scalar potential to obtain $\vec{B}$ if the magnetic field owes its existence to currents, since the circulation of $\vec{B}$ does not in general vanish. But the divergence of $\vec{B}$ does vanish - the lines of $\vec{B}$ never begin or end. These conditions suggest that the magnetic field may be written as the curl of another vector, for it can be shown that the divergence of any vector which is itself a curl is identically zero. To show this let us again consider a finite volume (Fig. 3.8), on the surface on which there is a closed curve that divides the surface into $S_1$ and $S_2$. For any vector field $\vec{A}$ the circulation of $\vec{A}$ about the closed curve is

$$\oint \vec{A} \cdot d\vec{s} = \int \text{curl } \vec{A} \cdot d\vec{S}_1$$
$$= -\int \text{curl } \vec{A} \cdot d\vec{S}_2$$

by the right-hand rule, since we have taken $d\vec{S}$ positive outward from the en-

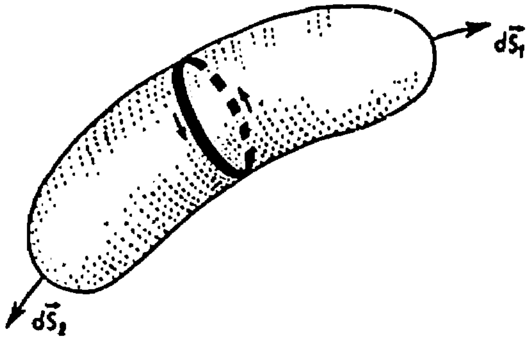


Fig. 3.8

closed volume for both surfaces. But the total flux of curl $\vec{A}$ out of the volume is

$$\int \text{curl } \vec{A} \cdot d\vec{S}_1 + \int \text{curl } \vec{A} \cdot d\vec{S}_2 =$$

$$= \int_{S \text{ closed}} \text{curl } \vec{A} \cdot d\vec{S} =$$

$$= \oint \vec{A} \cdot d\vec{s} - \oint \vec{A} \cdot d\vec{s} = 0.$$

This is true for any volume, and for any closed curve on the surface of the volume, and must therefore be true in the limit:

$$\text{div curl } \vec{A} =$$

$$= \lim_{\Delta V \to 0} \frac{\int_{S \text{ closed}} \text{curl } \vec{A} \cdot d\vec{S}}{\Delta V} = 0.$$

Thus writing $\vec{B}$ as the curl of another vector insures that its divergence is zero, a condition imposed physically on the magnetic field $\vec{B}$ if it is produced by currents, and defined as a property of $\vec{B}$ as produced by magnets to include the inseparability of magnetic poles along with Coulomb's law for magnets.

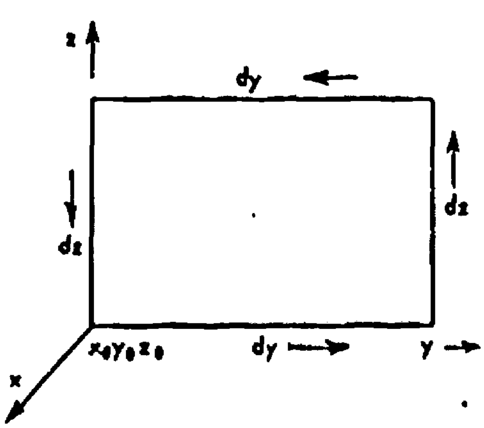


Fig. 3.9

In view of the limiting process by which the curl was defined it is not surprising that it resembles the gradient and the divergence in being a differential operator with respect to coordinates in three-dimensional space. In order to make quantitative use of the concept we must write it in terms of coordinates, although the physical quantity it represents is quite independent of the particular coordinate system chosen. For our purposes, the familiar Cartesian coordinates will suffice, particularly if we remember to choose the origin of coordinates and the orientation of the axes so as to make the description of the physical problem as simple as possible.

Let us consider the x component of curl $\vec{A}$ in a right-handed Cartesian coordinate system at the point $(x,y,z)$. In Fig. 3.9, dy and dz are shown as finite increments in the direction of increasing y and z; eventually we shall let dy and dz become as small as we please. By definition,

$$(\text{curl } \vec{A})_x = \underset{dzdy \to 0}{\text{limit}} \frac{1}{dzdy} \oint \vec{A} \cdot d\vec{s}_1, \quad (3.19)$$

with the line integral taken around the boundary of the small rectangle shown. The vector $\vec{A}(x,y,z)$ must vary with changing y or z (or both) if the line integral is to be different from zero, and we must allow for this variation to first order in dy and dz. For the legs of the rectangle adjacent to the point $x_0, y_0, z_0$ $\vec{A}$ is $\vec{A}(x_0, y_0, z_0)$, but all of the leg dz on the right is at $y_0 + dy$, and the leg dy at the top is at $z_0 + dz$. The line integral is then

$$A_y(x_0,y_0,z_0)\,dy + A_z(x_0,y_0+dy,z_0)\,dz$$

$$- A_y(x_0,y_0,z_0+dz)\,dy$$

$$- A_z(x_0,y_0,z_0)\,dz,$$

the last two terms being negative because the path is traversed in the direction of decreasing y and z, respectively. To take into account the fact that in one term $A_z$ is evaluated at $y_0 + dy$, we write

$$A_z(x_0, y_0 + dy, z_0) = A_z(x_0, y_0, z_0) + \left(\frac{\partial A_z(x_0, y, z_0)}{\partial y}\right)_{y_0} dy$$

where $(\partial A/\partial y)_{y_0}$ is the slope of $A_z$ plotted against y, evaluated at the point $y_0$, the other two coordinates remaining unchanged. This does not imply that $A_z$ is a linear function of y. If dy were to remain a finite length we should have to worry in more detail about the dependence of $A_z$ on y. Similarly,

$$A_y(x_0, y_0, z_0 + dz) = A_y(x_0, y_0, z_0) + \left(\frac{\partial A_y(x_0, y_0, z)}{\partial z}\right)_{z_0} dz.$$

When these expressions are substituted in the closed line integral all terms which do not involve derivatives cancel, and we are left with

$$(\text{curl } \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad (3.20)$$

where the coordinates need not be written explicitly.

The other components of curl $\vec{A}$ can be derived in the same way, but it is equally valid to invoke the symmetry of a right-handed Cartesian coordinate system and obtain the y and z components by cyclic permutation of x,y,z:

$$(\text{curl } \vec{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$(\text{curl } \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad (3.21)$$

The result is reminiscent of the form of the cross product of two vectors. If x, y, and z are unit vectors in the direction of increasing x, y, z, we recall that

$$(\vec{A} \times \vec{B}) = (A_yB_z - A_zB_y)\vec{x} + (A_zB_x - A_xB_z)\vec{y} + (A_xB_y - A_yB_x)\vec{z}.$$

In writing the gradient and the divergence in Cartesian coordinates we have already made use of a vector differential operator $\vec{\nabla} \equiv (\vec{x}\partial/\partial_x) + (\vec{y}\partial/\partial_y) + (\vec{z}\partial/\partial_z)$, and have found it convenient to write grad $\phi = \vec{\nabla}\phi$, and div $\vec{E} = \vec{\nabla}\cdot\vec{E}$. Here we may write curl $\vec{A} \cdot \vec{\nabla} \times \vec{A}$, and the determinantal form o the cross product of two vectors s again a helpful mnemonic device:

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad (3.22)$$

The symbol $\vec{\nabla}$ (del) is useful in the manipulation of mathematical relations between physical concepts, since it can be treated as an ordinary vector so long as its role as a differential operator is kept firmly in mind, but the meaning is more apparent if we say "curl" instead of "del cross" in reading a formula.

We have seen that the magnetic field $\vec{B}$ can be written as the curl of some other vector which is also defined for all points of space. The relation between a vector and its curl is clarified by consideration of some simple examples. Let us take

$$B = \text{curl } \vec{A} \qquad (3.23)$$

where $\vec{A} = B_0(\vec{y}x - \vec{x}y)$. It is easily found that curl $\vec{A} = B_0\vec{z}$, a uniform field parallel to the z axis of coordinates, but what about the lines of $\vec{A}$? It is a simple exercise to show that they are concentric circles, lying in planes perpendicular to the z axis.

As a second example consider $\vec{A} = (\mu_0 j/4)(x^2 + y^2)\vec{z}$ parallel to the z axis but depending symmetrically on

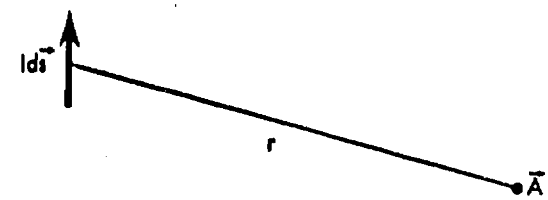


Fig. 3.10

x and y. (The first factor is constant and a scalar.) Find $\vec{B}$ = curl $\vec{A}$. Now find curl $\vec{B}$. The answer has been anticipated in writing the constant factor in the expression for $\vec{A}$, but the relation between the succession of vectors found by taking the curl is interesting.

The vector $\vec{A}$ of which the magnetic field $\vec{B}$ is the curl is called the "vector potential." The scalar potential in electrostatics was defined as work per unit charge, measured in volts, and could be traced to its sources by summing the effects of all charges giving rise to the field intensity $\vec{E}$. We saw that its gradient is just the electric field intensity (except that we change the sign), that is, the physically observable force per unit charge is derived from the scalar potential by means of the operator "del." The relation between work and force is then quite apparent, and the measurement of $\vec{E}$ in volts per meter reinforces the connection. The role of the vector potential in magnetic fields is more complicated, largely because the force per unit current element is not in the direction of the field $\vec{B}$ but at right angles to it. The simplest justification for using the word potential here is that $\vec{A}$ represents a quantity whose derivative (its curl, this time) is a physically measurable field, namely, $\vec{B}$.

Just as the scalar potential $\phi$ can be traced to electric charges we should expect that $\vec{A}$ can be traced to currents. We shall write down the correct relation between the vector potential and linear current sources, then show that this relation is compatible with the dependence of $\vec{B}$ on these same currents, as known from Ampere's law. Before doing so, however, we should note that $\vec{A}$ is not completely determined by the demand that its curl give the correct magnetic field; any vector whose curl is zero could be added to $\vec{A}$ without affecting $\vec{B}$ at all. We have noted in the description of the fields $\vec{E}$ and $\vec{B}$ that it is necessary to know both the curl and the divergence to specify a vector. Since $\vec{A}$ has been introduced only so that its curl represents $\vec{B}$, nothing has been said about its divergence, which may be anything. It is customary in magnetostatics to require that div $\vec{A}$ = 0, but this restriction is arbitrary. The fact that the vector potential is not completely defined by requiring that its curl give the right magnetic field is reminiscent of the ambiguity of the scalar potential of electrostatics, to which any arbitrary constant could be added.

It is possible to show that the expression for $\vec{B}$ in terms of a current, Eq. (3.2), may be written as the curl of some other vector quantity which can then be identified as the vector potential. It is somewhat simpler, mathematically, to write down a formula for the vector potential and show that its curl gives Eq. (3.2). Let us put

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{s}}{r} \qquad (3.24)$$

where $Id\vec{s}$ is an element of current, as usual, and r is the distance from $Id\vec{s}$ to the point where $\vec{A}$ (and hence $\vec{B}$) is to be computed, which we may call the field point (see Fig. 3.10). The curl of $\vec{A}$ is to be taken at the field point, and depends on the coordinates of that point, not those of the source. (After all, the same field $\vec{B}$ at some point could be produced equally well by a variety of source configurations.) Moreover, owing to the principle of superposition, it makes no difference whether we take the curl at each point of the integrand and then sum, or first sum over all parts of the circuit Ids and then take the curl at the field point. In other words,

$$\text{curl } \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{I d\vec{s}}{r} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{I d\vec{s}}{r}\right)$$

$$= \frac{\mu_0}{4\pi} \int \vec{\nabla}\left(\frac{1}{r}\right) \times I d\vec{s}$$

where in the last term we have made explicit use of the fact that the del operator does not act on the coordinates of the current element. The field point is involved in the integrand only through the factor $1/r$, where r is the distance from $I d\vec{s}$ to the field point, and thus depends both on the variable of integration ds and those at which the vector derivative is taken.

In writing curl $\vec{A}$ in the final form above we have taken advantage of the fact that the operator del behaves like a vector as well as a differential operator, but the integrand now reads differently: $\vec{\nabla}(1/r)$ = grad $(1/r)$, and we have the cross product of a gradient of a scalar and $I d\vec{s}$. The gradient of $1/r$ is very familiar from electrostatics, since the electrostatic potential of a point charge is proportional to $1/r$, where r is the distance from the point charge to the point at which we take the gradient to find $\vec{E}$, the electric field intensity. (Of course, we could also simply compute it again.) If we take the source point, the position of $I d\vec{s}$, as the origin of coordinates, grad $(1/r) = -\hat{r}/r^2$, where $\hat{r}$ is a unit vector directed from $I d\vec{s}$ to the field point. With this substitution, and a change of order in the cross product which changes its sign,

$$\text{curl } \vec{A} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{s} \times \hat{r}}{r^2}. \qquad (3.25)$$

which is identical with Eq. (3.2). Thus our expression for $\vec{A}$ is justified.

The vector potential, like the scalar potential, is defined at the field point, but we see that it is very closely related to the current. In fact, for each element of current

$$\Delta\vec{A} = \frac{\mu_0}{4\pi} \frac{I d\vec{s}}{r} \qquad (3.26)$$

and thus each increment of $\vec{A}$ is in the same direction as the current element which produces it. From the definition of the curl we see again that the field $\vec{B}$ is at right angles to $\vec{A}$, as well as to $I d\vec{s}$.

The vector potential is sometimes useful in solving problems in magnetostatics, but it does not play nearly so practical a role in determining $\vec{B}$ from steady currents as does the scalar potential in electrostatics problems. On the other hand, it is almost indispensible in relating fields to nonsteady currents, the fields produced by time varying currents. We shall return to this point in another chapter. But before leaving the subject, let us note an interesting relation between the vector potential and the flux of the magnetic field $\vec{B}$ through a surface. Thus far we have considered the circulation of $\vec{B}$ in relation to the current through a surface bounded by the circulation path, and have noted that the flux of $\vec{B}$ through a closed surface always vanishes, but we can now derive a new circulation law. Consider the flux of $\vec{B}$ through a surface bounded by a closed path. By definition this flux is the integral of $\vec{B} \cdot d\vec{S}$ over the surface. But $\vec{B}$ = curl $\vec{A}$. Therefore the flux through the surface is

$$\phi_B = \int \vec{B} \cdot d\vec{S} = \int \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{s}, \qquad (3.27)$$

just the line integral of $\vec{A}$ round the boundary of the surface. In the early chapters of Monograph III you will have learned that a changing flux of $\vec{B}$ through a surface is accompanied by a circulation of the electric field intensity $\vec{E}$. Thus

$$\frac{d\phi_B}{dt} = \int \frac{\partial B}{\partial t} \cdot d\vec{S} = -\oint \vec{E} \cdot d\vec{s} = \oint \frac{\partial \vec{A}}{\partial t} \cdot d\vec{s}, \qquad (3.28)$$

and therefore an electric field which has a circulation is related to the vector potential: $\vec{E} = -(\partial\vec{A}/\partial t)$.

The reformulation of magnetostatics in terms of vector calculus has in itself added nothing to the physical content of Ampere's law; in fact, the physics of curl $\vec{H} = \vec{j}$ may be less transparent than $\oint \vec{H} \cdot d\vec{s} = I$, except for the "curl-H-meter" which works only in fluid conductors. As for the vector potential, $\vec{A}$ is sometimes, but not always, useful for solving problems, but it is not even directly observable by means of classical currents or magnets. The power of the differential formulation of the laws of electricity and magnetism is fully realized only when variations of the fields in time are taken into account. If $\vec{E}$ and $\vec{B}$ (or $\vec{H}$) are permitted to vary in time, as they are bound to do, a whole new set of electromagnetic consequences are observed. The differential forms of Ampere's and Faraday's laws helped Maxwell to conclude that "light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws." Even Maxwell did not succeed in tracing electromagnetic radiation to its sources; to accomplish this in an unambiguous way requires the vector potential, or something equivalent to it, a single quantity to which both the electric field and the magnetic field are related.