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ABSTRACT

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in summer, 1965. It is intended for use by college students who are non-physics majors. The approach is phenomenological and macroscopic. The monograph contains three chapters. Chapter 1 discusses Faraday's experiments on induction, the concept of an electromotive force, generators, motors, and the Betatron. Chapter 2 is concerned with Ampere's circulation law for steady currents and its applications. The propagation of an electromagnetic disturbance is the subject of chapter 3. Each chapter has a number of exercises. (LC)

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# Electricity and Magnetism III

## CIRCULATION LAWS AND THEIR CONSEQUENCES



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G E N E R A L   P R E F A C E

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in the summer of 1965. The general purpose of the conference was to create effective ways of presenting physics to college students who are not preparing to become professional physicists. Such an audience might include prospective secondary school physics teachers, prospective practitioners of other sciences, and those who wish to learn physics as one component of a liberal education.

At the Conference some 40 physicists and 12 filmmakers and designers worked for periods ranging from four to nine weeks. The central task, certainly the one in which most physicists participated, was the writing of monographs.

Although there was no consensus on a single approach, many writers felt that their presentations ought to put more than the customary emphasis on physical insight and synthesis. Moreover, the treatment was to be "multi-level" --- that is, each monograph would consist of several sections arranged in increasing order of sophistication. Such papers, it was hoped, could be readily introduced into existing courses or provide the basis for new kinds of courses.

Monographs were written in four content areas: Forces and Fields, Quantum Mechanics, Thermal and Statistical Physics, and the Structure and Properties of Matter. Topic selections and general outlines were only loosely coordinated within each area in order to leave authors free to invent new approaches. In point of fact, however, a number of monographs do relate to others in complementary ways, a result of their authors' close, informal interaction.

Because of stringent time limitations, few of the monographs have been completed, and none has been extensively rewritten. Indeed, most writers feel that they are barely more than clean first drafts. Yet, because of the highly experimental nature of the undertaking, it is essential that these manuscripts be made available for careful review by other physicists and for trial use with students. Much effort, therefore, has gone into publishing them in a readable format intended to facilitate serious consideration.

So many people have contributed to the project that complete acknowledgement is not possible. The National Science Foundation supported the Conference. The staff of the Commission on College Physics, led by E. Leonard Jossem, and that of the University of Washington physics department, led by Ronald Geballe and Ernest M. Henley, carried the heavy burden of organization. Walter C. Michels, Lyman G. Parratt, and George M. Volkoff read and criticized manuscripts at a critical stage in the writing. Judith Bregman, Edward Gerjuoy, Ernest M. Henley, and Lawrence Wilets read manuscripts editorially. Martha Ellis and Margery Lang did the technical editing; Ann Widditsch supervised the initial typing and assembled the final drafts. James Grunbaum designed the format and, assisted in Seattle by Roselyn Pape, directed the art preparation. Richard A. Mould has helped in all phases of readying manuscripts for the printer. Finally, and crucially, Jay F. Wilson, of the D. Van Nostrand Company, served as Managing Editor. For the hard work and steadfast support of all these persons and many others, I am deeply grateful.

Edward D. Lambe
Chairman, Panel on the
New Instructional Materials
Commission on College Physics

# CIRCULATION LAWS AND THEIR CONSEQUENCES

## PREFACE

This fragmentary and preliminary material fits into an outline of "multi-level monographs" covering those aspects of electromagnetism which in our view an undergraduate physics major should come to know best The approach is phenomenological and macroscopic, designed to take advantage of prior experience; we begin magnetostatics with magnets, for example. The material is planned on two levels to lead through the four fundamental empirical laws of electricity and magnetism to electromagnetic radiation as a climax. The propagation of electromagnetic disturbances with velocity c, reached in the "first course" material without use of the calculus and equivalent to the homogeneous wave equation, was written in an elementary way by Oliver Heaviside (Electromagnetic Theory, London, Benn, Vol. III, p. 3), but only recently has appeared in the regular pedagogical literature. In our treatment we have tried to stress the physical foundations of Maxwell's great synthesis, stating in words the argument corresponding to each mathematical step. This results in a considerably larger proportion of expository writing relative to mathematics than is customarily found in derivations of the wave equation from Maxwell's equations in their usual form. On the other hand, expression of the laws in differential form seems essential for tracing radiation to its sources in a physically meaningful way; the present Chapter 3 of Magnetostatics could be followed almost immediately by Chapter 5 of Monograph III, which would trace radiation fields to retardation effects. We regret having not sufficient time to write such a chapter, as well as the omission of what should have been Chapter 3 of Magnetostatics, an elementary treatment of magnetic materials.

OUTLINE OF MONOGRAPHS ON ELECTRICITY AND MAGNETISM

| | I. ELECTROSTATICS | II. MAGNETOSTATICS | III. CIRCULATION LAWS AND THEIR CONSEQUENCES |
|---|---|---|---|
| FIRST COURSE MATERIAL | 1. Electric Forces and Fields<br>2. Electric Energy and Potential<br>3. Electrical Properties of Matter | 1. Magnets and Magnetism<br>2. Interaction of Steady Currents<br>*Magnetic Properties of Matter | 1. Faraday's Law of Induction<br>2. Ampere's Law Modified<br>3. Propagation of Electromagnetic Disturbances |
| UPPER DIVISION COURSE MATERIAL | *4. Electrostatics Reformulated | 3. Magnetostatics Reformulated | *Maxwell's Equations and Plane Waves<br>*Radiation Fields |

*No textual material was prepared in the summer of 1965 for these chapters.

We have assumed no knowledge of special relativity, but have emphasized the necessity for choosing a frame of reference in which to define electric and magnetic field quantities, thus laying a foundation for the historical development of relativity theory. Unlike mechanics, vacuum electrodynamics needs no modification because of special relativity except in interpretation, so that an excursion into relativity theory could be made before or after study of the present material.

The experiments leading to the four fundamental laws are described at some length, but in use this material should be accompanied by demonstrations and laboratory work. The basic experiments should come to be a part of genuine experience for students, but a laboratory monograph should be written as an extension of the present outline. Ohm's law and circuitry, for example, do not play an appreciable role in any other projected booklets. We cannot overemphasize the importance of laboratory work, although we were not able to undertake detailed consideration of its content.

We assume that students will have studied mechanics, that they know Newton's laws, the definition of work, the meaning of the $\Sigma$ symbol, and have a working knowledge of elementary vector algebra before our material is introduced. (We do define the vector cross product as if for the first time.) In the material designed for upperclass work we assume calculus. All vector calculus is developed as needed, but we attempt throughout to stress the physics, not the mathematics, and attempt no mathematical rigor.

The first chapters of Monographs I, II, and III should be studied in that order. The few discussion exercises we include can only indicate a type of problem we consider desirable. Numerical problems, which we have made no effort to provide, are also necessary.

R. T. Mara
M. Phillips

Sections 1.4 through 1.7 can be omitted without losing the line of the argument leading to the propagation of electromagnetic waves. Nothing in Chapters 2 and 3 depends explicitly on the development in these sections.

I had hoped to complete two additional sections: Experimental Confirmation and Reference Frames. These would have appeared at the end of Chapter 3. The first would have described the experiments confirming Maxwell's predictions about the propagation of electromagnetic waves. The second would have described the interplay of $q\vec{E}$ and $q\vec{v} \times \vec{B}$ forces as we transform from one inertial frame to another, and it would have exhibited the difficulties inherent in applying Galilean relativity to Maxwell's equations. While I had no intention of introducing special relativity in any detail, this seemed the ideal place at which to set the stage. In any case, students should not be left with the impression that everything they learned about transformations in the context of Newtonian mechanics can be carried over to electromagnetic theory.

R. T. Mara

C O N T E N T S

# 1 FARADAY'S LAW OF INDUCTION

## 1.1 FARADAY'S DISCOVERY

Michael Faraday knew about Ampere's work, so he knew that a magnetic field accompanies a current. The monograph Magnetostatics gives a thorough discussion of the relationship between magnetic fields and currents. In mathematical terms,

$$\vec{B} = \sum \Delta\vec{B} = \frac{\mu_0}{4\pi}\sum_{s} \frac{I\ \Delta\vec{s} \times \hat{r}}{r^2}, \qquad (1.1)$$

where $\vec{B}$ is the magnetic induction field at a point P (see Fig. 1.1), I is the steady current in the element of length $\Delta\vec{s}$ along the circuit and $\Delta\vec{s}$ is in the direction of that current, r is the dis-distance between $\Delta\vec{s}$ a d the field point P, and $\hat{r}$ is a unit vector pointing from $\Delta\vec{s}$ toward P.

There is a circulation law closely associated with Eq. (1.1).

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 I = \mu_0 \sum_{S} \vec{j} \cdot \Delta\vec{S}. \qquad (1.2)$$

This says that the circulation of $\vec{B}$ is proportional to the net charge that passes per second through any surface S bounded by the circulation path s. There is a convention that relates the sense in which $\Delta\vec{s}$ is taken as positive around the circulation path and the direction in which $\Delta\vec{S}$ and hence I is to be considered positive. This convention is illustrated in Fig. 1.2.

The monograph Magnetostatics considers $\vec{B}$ only in cases for which the field point is in empty space, outside of matter, and Eqs. (1.1) and (1.2) are valid only for such cases. We shall also study fields in empty space in this monograph.

While Faraday did not have the mathematical conception of Ampere's work that appears in Eqs. (1.1) and (1.2), he understood the physics contained in those equations. And he had a strong hunch that there was more to the story. To understand Faraday's language and hence something of the way he thought, we quote the first paragraph of a paper he presented to the Royal Society in 1831.

> 1. The power which electricity of tension (electrostatic charge) possesses of causing an opposite electrical state in its vicinity has been expressed by the general term Induction; which, as it has been

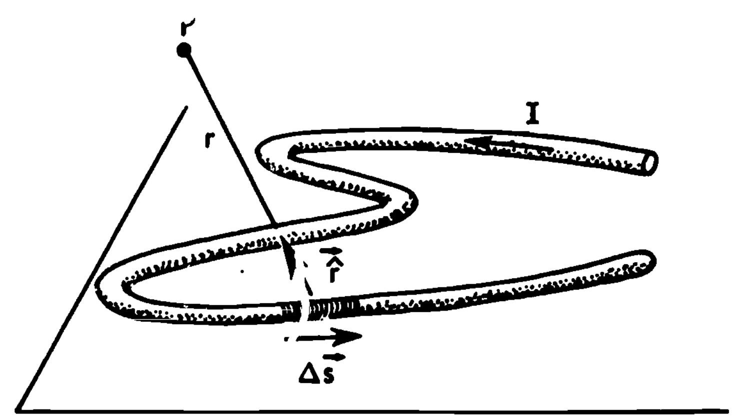



Fig. 1.1 The quantities appearing in Eq. (1.1) describing the vector $\vec{B}$ at the field point P. Only a part of the complete circuit is shown.

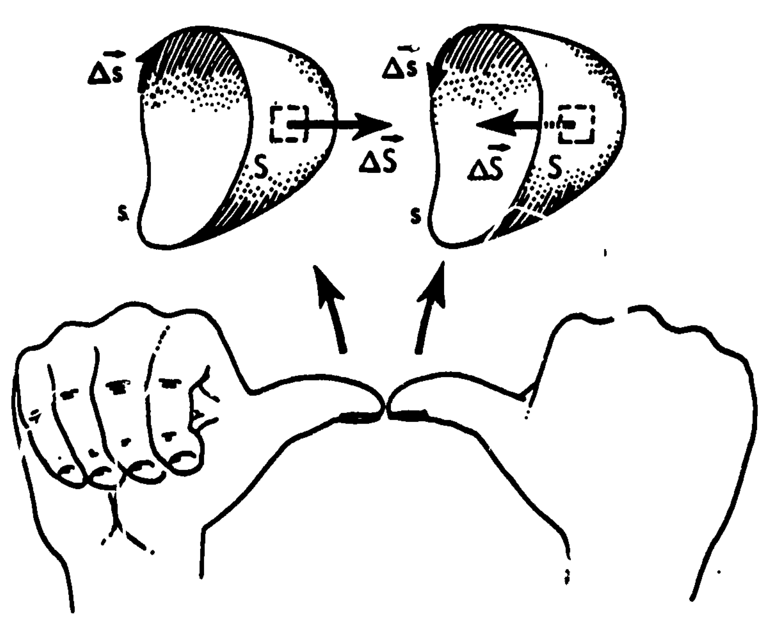



Fig. 1.2 The right-handed convention relating the sense of $\Delta\vec{s}$ and the direction of $\Delta\vec{S}$ for Ampere's circulation law.

received into scientific language, may also, with propriety, be used in the same general sense to express the power which electrical currents may possess of inducing any particular state upon matter in their immediate neighborhood, otherwise indifferent. It is with this meaning that I purpose using it in the present paper.[1]

In much the same sense that he imagines static charge induces opposite charges in nearby conductors, Faraday sees a current inducing a magnetic field in its neighborhood. Faraday continues in the same paper.

2. Certain effects of the induction of electrical currents have already been recognized and described: as those of magnetization; . . . Still it appeared unlikely that these could be all the effects which induction by currents could produce; especially as, upon dispensing with iron, almost the whole of them disappear, whilst yet an infinity of bodies, exhibiting definite phenomena of induction with electricity of tension, still remain to be acted upon by the induction of electricity in motion.

3. Further: Whether Ampere's beautiful theory were adopted, or any other, or whatever reservation were mentally made, still it appeared very extraordinary, that as every electric current was accompanied by a corresponding intensity of magnetic action at right angles to the current, good conductors of electricity, when placed within the sphere of this action, should not have any current induced through them, or some sensible effect produced equivalent in force to such current.

4. These considerations, with their consequences, the hope of obtaining electricity from ordinary magnetism, have stimulated me at various times to investigate experimentally the inductive effect of electric currents. . . .

Faraday apparently felt deeply that a current in one conductor should do something to a nearby conductor. And paragraph 4 above seems to indicate Faraday's hunch that if magnetism could result from a current, then, in some way or other, current should be obtainable from magnetism. His viewpoint is at least partly evident in experiments he performed as early as 1824. He passed a magnet through a conducting helix but noted no effect from doing it. He passed a current through one wire but found nothing in a nearby wire. Then, in ten days of experimentation, starting August 29, 1831, and ending November 4, 1831, Faraday found what he was after, although not really what he had expected. On November 24, he read his famous paper to the Royal Society, and the quotations above are taken from that paper.

In that short span Faraday discovered just about every way possible to induce current in conductors, and he formulated a law that accounted for all of them. That work is the basis for all modern electric power, from the giant turbines at power stations to the motors that drive lathes and drill presses. We do not intend to cover all the work Faraday reported in even that first paper. In any case, no one is likely to improve on Faraday's own report, so James Clerk Maxwell's advice is good,

I would recommend the student, after he has learned, experimentally if possible, what are the phenomena to be observed, to read carefully Faraday's Experimental Researches in Electricity. He will there find a strictly contemporary historical

[1]Quoted from Faraday's Experimental Researches in Electricity, Vol. I, which is the first of three volumes in which are collected all the papers Faraday published in the Philosophical Transactions in the years 1831-1838 (Richard and John Edward Taylor, London, 1839).

account of some of the greatest electrical discoveries and investigations, carried on in an order and succession which could hardly have been improved if the results had been known from the first, and expressed in the language of a man who devoted much of his attention to the methods of accurately describing scientific operations and their results.[2]

In his laboratory notebook Faraday recorded what is apparently the first experiment exhibiting the effect that now carries his name.

> I have had an iron ring made (soft iron), iron round and 7/8ths of an inch thick, and ring six inch inches in external diameter. Wound many coils of copper round, one half of the coils being separated by twine and calico; there were three lengths of wire, each about twenty-four feet long, and they could be connected as one length or used as separate lengths. By trials with a trough each was insulated from the other. Will call this side of ring A. On the other side, but separated by an interval, was wound wire in two pieces, together amounting to about sixty feet in length, the direction being as with the former coils. This side call B.
>
> Charged a battery of ten pairs of plates four inches square. Made the coil on B side one coil, and connected its extremities by a copper passing to a distance, and just over a magnetic needle (three feet from wire ring), then connected the ends of one of the pieces on A side with battery: immediately a sensible effect on needle. It oscillated and settled at last in original position. On breaking connection on A side of battery, again a disturbance of the needle.[3]

Figure 1.3 (see next page) shows Faraday's setup. It is not difficult to depict, given the clear account in the notebook. The important thing Faraday describes is that the magnetic needle is disturbed only at the instant the contact to the battery is made or broken; in between it settles down, even though there is current in A.

While this was likely the first experiment in which Faraday got a noticeable effect, it is not the first he describes in his paper. There he describes his results, "not as they were obtained, but in such a manner as to give the most concise view of the whole." We shall select a few of his experiments to explore in this chapter, but we shall not return to the one already described. For our purposes here, the iron ring is just a complication. When you have learned something about fields in magnetic materials, you will understand why the effect is considerably stronger with the iron present, and then you can make reasonable conjectures about why Faraday first noticed it this way. The first experiment Faraday described in his paper is equivalent to the one pictured in Fig. 1.3, but there is no iron present. Figure 1.4 (see next page) pictures the way Faraday's worktable might have been arranged when he performed this experiment.

He wound copper wire around a wooden cylinder, and he wound thread at the same time to keep successive turns of wire separated. He covered this layer with cloth, and wound a second layer of wire and thread on top of the first. He continued until he had a total of twelve windings, each wound in the same sense. He then connected the ends of the first, third, fifth, etc., windings to make a single

[2]Quoted from the Preface to the First Edition of James Clark Maxwell's A Treatise on Electricity and Magnetism, Vol. I, 3rd edition, 1891, republished by Dover Publications, Inc., New York (1954).

[3]Quoted from The Life and Letters of Faraday, Vol. II, 2nd edition, revised by Bence Jones; Longmans, Green, and Co. (1870).

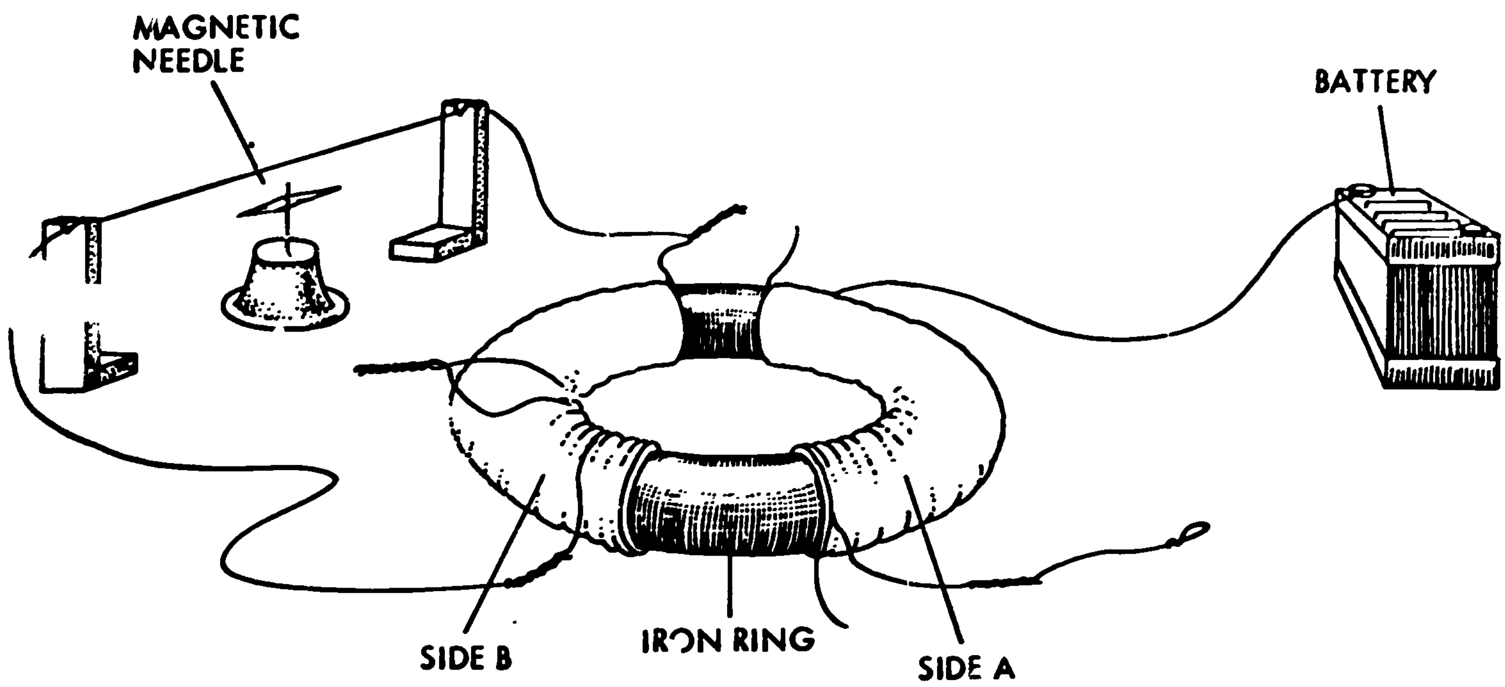



Fig. 1.3 The experimental setup when Faraday first noticed an induction effect.

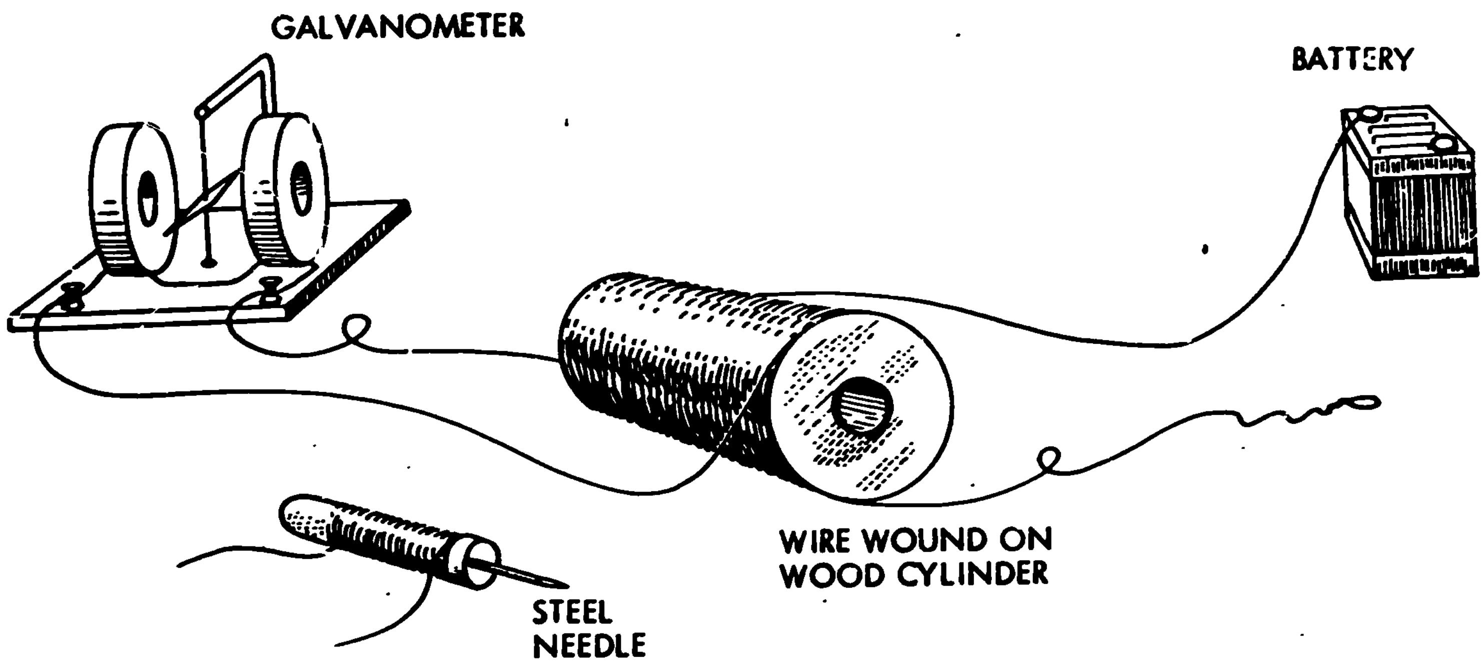



Fig. 1.4 Faraday's arrangement for detecting induced currents.

conductor. He did the same with the second, fourth, sixth, etc. In this way "two principal helices were produced, closely interposed, having the same direction, not touching anywhere, and each containing one hundred and fifty-five feet in length of wire."

He connected one principal helix across a galvanometer and the other across "a voltaic battery of ten pairs of plates four inches square, with double coppers and well charged; yet not the slightest sensible deflection of the galvanometer needle could be observed." He did the same thing with other metals used for the wire. No difference. He then made a bigger version of the whole affair. He increased the length of wire in each principal helix to two hundred feet and the battery to "one hundred pairs of plates four inches square, with double coppers, and well charged." This time something did happen.

When the contact was made, there was a sudden and very slight effect at the galvanometer, and there was

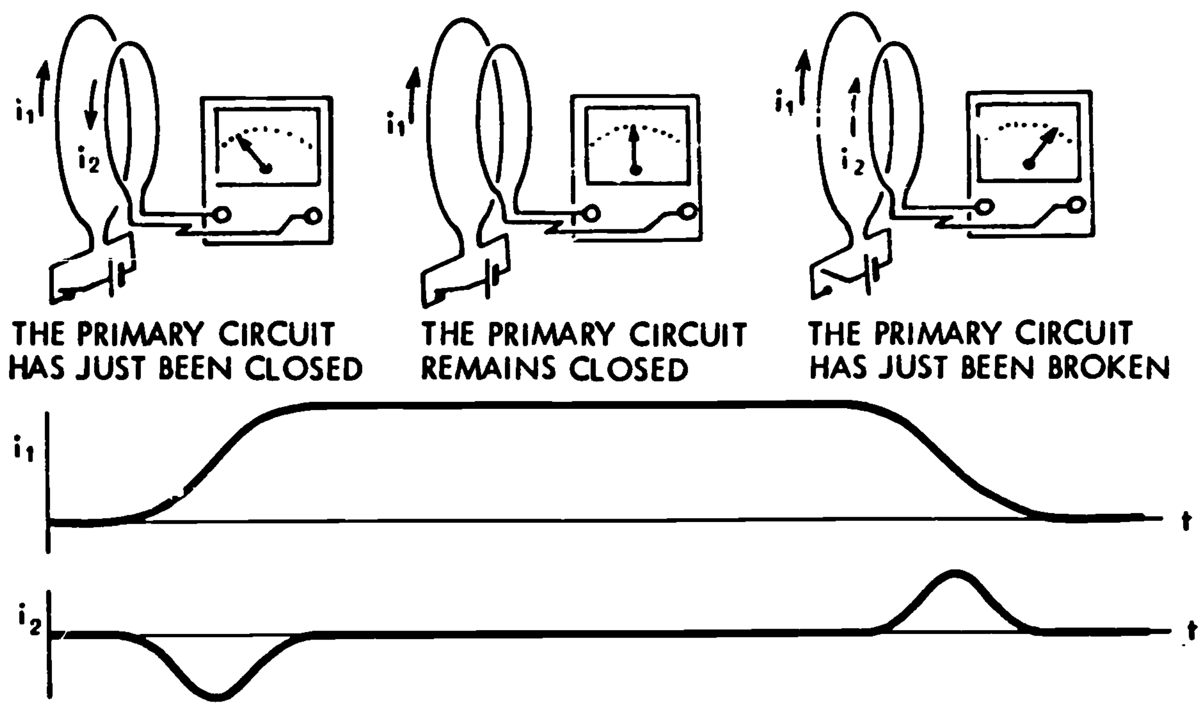



Fig. 1.5 Changing the current in a nearby circuit can induce a current.

> also a similar slight effect when the contact with the battery was broken. But whilst the voltaic current was continuing to pass through the one helix, no galvanometer appearances nor any effect like induction upon the other helix could be perceived, . . . .

Faraday was certain that the same effect must have been present when he had used the smaller coils and weaker battery. He suspected that his galvanometer was simply not sensitive enough to detect it, so he devised an ingenious substitute for the galvanometer. He wound a coil around a glass tube and placed a steel needle inside the tube (see Fig. 1.4). If a current passes through that coil, then there will be a magnetic field inside the tube that will magnetize the needle. He substituted this arrangement for the galvanometer and repeated the experiment.

With the needle originally unmagnetized, he made the connection to the battery in the primary circuit. Before breaking that connection he pulled out the needle and found it to be magnetized. Now he put a second, unmagnetized needle into the tube and then broke the primary circuit. The second needle was also magnetized, but its polarity was the reverse of that found in the first needle.

When he put in an unmagnetized needle before closing the primary circuit and left it in until after that circuit was broken, he found "little or no magnetism." When he closed the primary circuit before putting the needle in the tube, and then removed the needle before breaking that circuit, he found no magnetism in the needle.

These results can be explained only on the basis that the current in the induced, or secondary, circuit is in one direction for a short time when the primary has just been closed and in the opposite direction for a short time when the primary has just been broken, and that there is no current in the secondary during the intervening time. That is, current appears in the secondary only when the current in the primary is changing.

Further, Faraday discovered that "The [galvanometer] deflection on making a battery contact always indicated an induced current in the opposite direction to that from the battery; but on breaking the contact the deflection indicated an induced current in the same direction as that of the battery."

In Fig. 1.5 we have a summary of the experimental results. The experi-

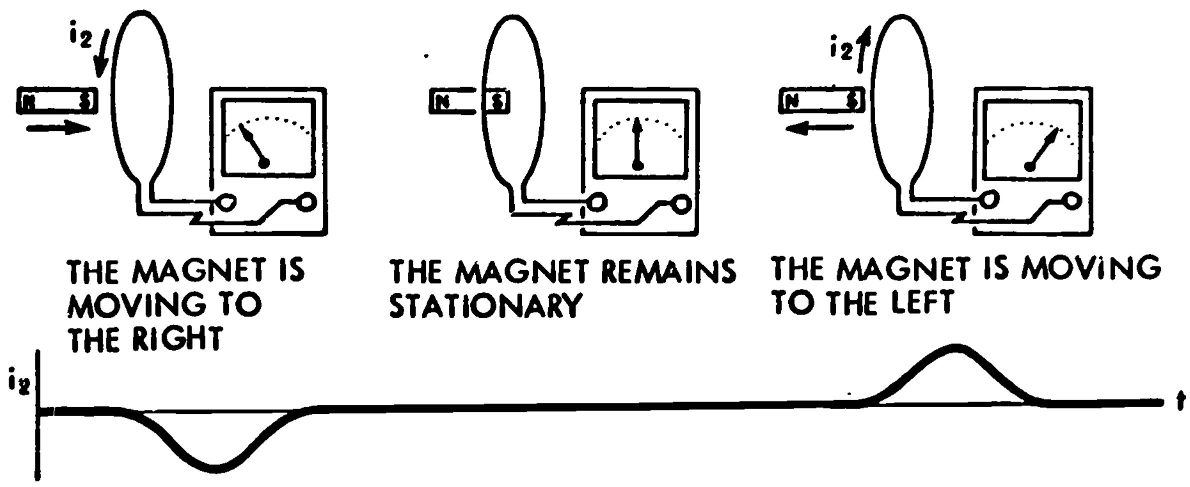



Fig. 1.6 Moving a nearby magnet can induce a current.

mental setup is a simplified version of Faraday's, so that the effects can be illustrated more clearly.

What is it that is changing at the secondary that brings about the induced current? Well, as the current builds up in the primary, a magnetic field is building up, too. Can it be that a changing magnetic field causes the induced current? There is a simple way to find out.

The experimental setup shown in Fig. 1.5 can be modified as shown in Fig. 1.6. A movable magnet substitutes for the primary circuit. When the magnet moves, the magnetic field at the secondary certainly changes. If a changing magnetic field is responsible for induced currents, a current should appear in the secondary while the magnet is moving. And it does, as Faraday discovered in just about this way.

Now we have an interesting question. Suppose that in the experiment depicted in Fig. 1.6 we arrange to have the conducting loop move and the magnet remain at rest (see Fig. 1.7). If

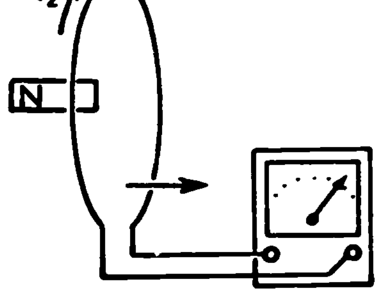


Fig. 1.7 Moving the loop in a fixed magnetic field can induce a current.

we can extend what we have learned in mechanics into the realm of electricity and magnetism, then the physics should be the same, no matter which moves. And that turns out to be correct. The galvanometer response is the same whether the magnet moves to the left and the loop is stationary or the magnet is stationary and the loop moves to the right. And Faraday discovered that, too.

At this point, Faraday had his hands on what he wanted. Things had not turned out to be quite what he had imagined when he began his Researches, but his hunch was correct. His hope that a "current should be obtainable from magnetism" was realized.

Note:

The experiments described in this section are relatively easy to do, and reading about them is not really the same as doing them or, at least, seeing them done. Maxwell in the Preface to his Treatise on Electricity and Magnetism, says, ". . . before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's Experimental Researches in Electricity. . . . I would recommend the student, after he has learned, experimentally if possible, what are the phenomena to be observed, to read carefully Faraday's Experimental Researches in Electricity."

This chapter is certainly no

satisfactory substitute for the Researches, and Maxwell's admonition about experimental understanding is even more germane if a student is relying largely on what is written here. It is possible to reproduce much of what Faraday did with inexpensive equipment, and it is well worth while doing just that.

## 1.2 FARADAY'S LAW OF INDUCTION

Before we proceed to the general statement of Faraday's law of induction, we want to give some simplified order to the results of Faraday's experiments. In essence, he found three ways to produce induced currents, and these are represented in Fig. 1.8. He produced an induced current in a loop by changing the current in a nearby circuit. An idealized version of that experiment is depicted in Fig. 1.8a, where the current is being changed by changing the resistance in the rheostat R. He also found that an induced current can appear in a loop when the loop is moved in a magnetic field. That experiment is depicted in Fig. 1.8b, where a circuit with constant current is used as the source of the magnetic field. And finally, Faraday found that an induced current can appear in a stationary loop when the source of the magnetic field is moved (see Fig. 1.8c).

We intend to discuss all three of these experiments of Faraday's in this chapter, but we start with a special case of the first. Suppose we put a loop inside a long coil in which we can vary the current (see Fig. 1.9). If the current in the coil rises from zero to some steady value $I_0$ in the time $\Delta t$, then the galvanometer needle will be deflected during that time. Suppose we make up a set of loops identical in size, but constructed of different known nonmagnetic conductors. Now we use one loop at a time, and note the galvanometer deflection for each as the current in the coil rises to $I_0$. In general, the deflections are not the

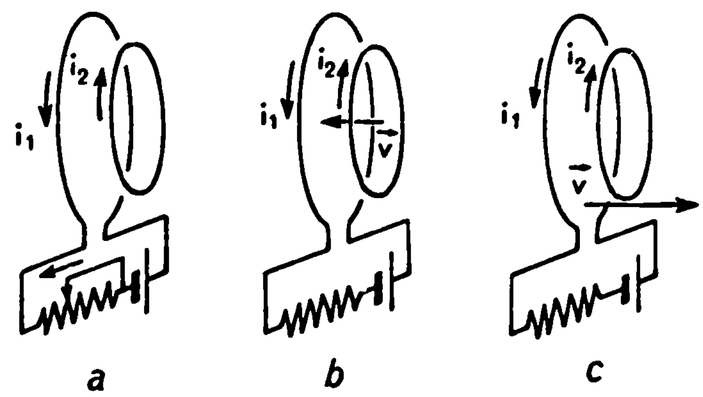



Fig. 1.8 Three ways to produce an induced current.

same, the deflection being greatest for those conductors whose resistance is least. This means that while the geometric arrangement is the same in each case, and while the current rises in the coil in the same way in each case, the induced current in the various loops is not the same.

We make now a different set of loops, these to have different radii but the same resistance. Again the galvanometer deflections are different, the loop with the larger radius giving the greater deflection. And finally, for a given loop, the more rapidly we increase the current in the coil the larger induced current we find in the loop. (If you try to perform this set of experiments, you will likely do better if you substitute an oscilloscope for the galvanometer.)

If we can use Ohm's law to describe what is going on in the loop, we see that while there is an induced

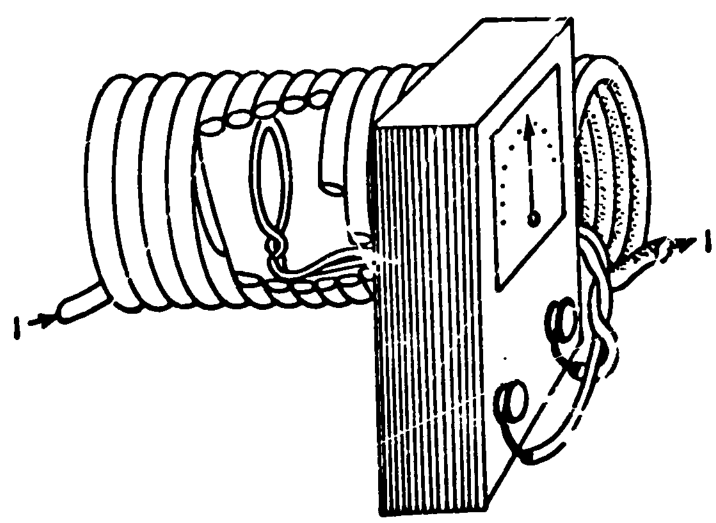



Fig. 1.9 Experimental arrangement for studying induction.

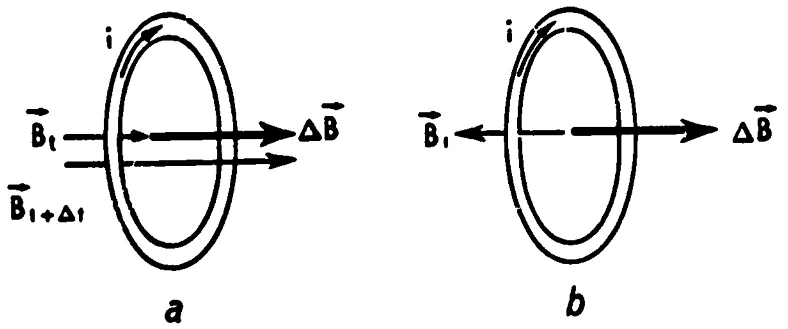



Fig. 1.10 Illustration of Lenz's law. (a) $\vec{B}_t$ is the magnetic induction field at the time t, $\vec{B}_{t+\Delta t}$ at the time t + Δt, and i the induced current. (b) $\vec{B}_i$ is the magnetic induction field created by the induced current.

current in the loop there must also be an induced electromotive force. That is,

$$\mathcal{E} = iR, \qquad (1.3)$$

where $\mathcal{E}$ is the electromotive force, i the current, and R the resistance of the loop.[4] Recall that the electromotive force (almost always abbreviated as emf) is defined as being the work done on a unit charge as that charge travels once around the circuit. It is too bad that this is called a force when it really is work, but historical origins of words often color our language in strange ways.

Up to this point, we have been concentrating on the detection and measurement of induced currents in describing experiments. But it turns out that the relationship between what goes on in the loop and what is happening to the environment in which that loop sits is most directly described in terms of the induced emf. The mathematical statement of that relationship is simply

$$\mathcal{E} = -\frac{\Delta\Phi_B}{\Delta t}, \qquad (1.4)$$

where $\mathcal{E}$ is the induced emf in the loop, and $\Phi_B$ is the flux of $\vec{B}$ threading that loop, i.e.,

$$\Phi_B = \sum_S \vec{B} \cdot \Delta\vec{S}, \qquad (1.5)$$

where S is any surface bounded by the loop. In words, this says that the induced emf $\mathcal{E}$ in a loop is equal to the negative of the time rate of change of the magnetic flux $\Phi_B$ threading the loop.

While Maxwell was the first to put this simple but powerful concept into mathematical form, Faraday was the first to discover the effect. It is most often called Faraday's law of induction. Later on we shall call it his circulation law; that it is in fact a circulation law will soon be clear.

We should look closely at Faraday's law as given by Eqs. (1.4) and (1.5), so that some important details become clear. First, we look at the reason for the minus sign in Eq. (1.4). We want the convention relating the sense of the emf and the direction of the flux change to be the same as that used in Ampere's law, where the sense in which a circulation about a closed path is related to the direction in which the current threading that path is considered to be positive (see Eq. (1.2) and Fig. 1.2). In Fig. 1.10(a), we show a stationary loop through which $\Phi_B$ is changing. The direction of that change is the direction of $\Delta\vec{B}$. The corresponding sense of the induced emf and of the induced current appear, too, as they are experimentally determined. But the relationship between these is just the opposite of our convention, so we need the minus sign in Eq. (1.4) to make that fact explicit.

There is another way to remember the sense of the induced current and, hence, of the induced emf. This way has a sound physical basis. The current induced in the loop is always in the sense such that the magnetic induction $\vec{B}_i$ that it creates is in the

[4] It is certainly true that Ohm's law is really valid only for steady currents, and the induced currents are not steady here. That means that what consequences of Ohm's law we use will in detail be incorrect, because we have not taken account of the magnetic field created by the induced current itself. Those interested in this matter can, after finishing this section, look up the meaning of the word self-inductance.

direction that opposes the change of $\Phi_B$ threading the loop. Figure 1.10(b) illustrates this relationship, which is often called Lenz's law. We can see that the world must behave that way when we consider what would happen if the induced current went the other way.

If the induced current were such as to create a $\vec{B}_i$ that increases the inducing $\Delta\Phi_B$, then that increase would bring about a yet greater induced current, which would further increase $\Delta\Phi_B$, which would again increase the induced current, and so on without end. That certainly cannot happen, so the induced current must create a magnetic field that reduces the flux change $\Delta\Phi_B$. That is, Lenz's law is a special consequence of the requirement that energy be conserved. And so we understand the appearance of the minus sign in Eq. (1.4).

We continue our close look at details of Faraday's law. Eq. (1.5) is just a definition of the flux of $\vec{B}$ through a surface S, and that should give us no difficulties. But in Faraday's law we can apparently use any surface S that is bounded by the loop. We are to understand this in precisely the same way we understand that in Ampere's law we can calculate the current through any surface bounded by the circulation path. In the case of Faraday's law, we are justified in the claim that any surface will suffice, provided it is bounded by the loop, by the fact that

$$\sum_{S \text{ closed}} \vec{B} \cdot \Delta\vec{S} = 0. \qquad (1.6)$$

That is, the flux of $\vec{B}$ through any closed surface is zero. This is nothing but the statement that there are no separable magnetic poles (see monograph Magnetostatics). Given (1.6), we can easily see why we can use any surface in Eq. (1.5), provided, of course, that surface is bounded by the loop.

That there are no separable magnetic poles means that the lines of $\vec{B}$ have no beginning and no ending. That happens sometimes because the line of $\vec{B}$ is closed, but it need not be closed. If we start to trace along a line of $\vec{B}$,

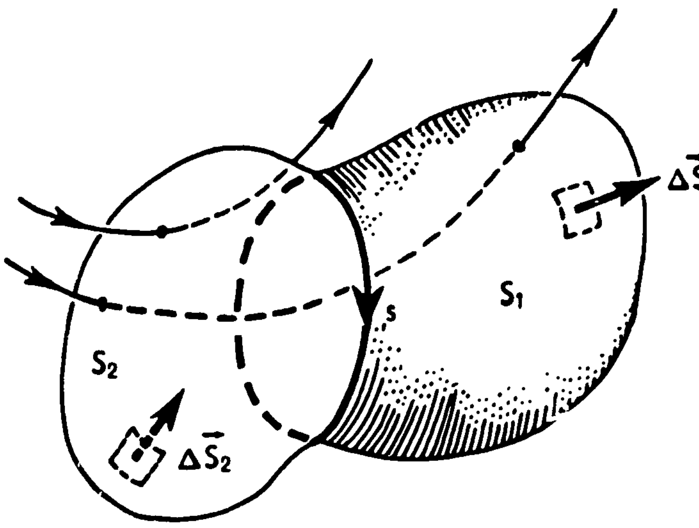



Fig. 1.11 Surfaces $S_1$ and $S_2$, each bounded by the curve s, together form a closed surface. The vectors $\Delta\vec{S}_1$ and $\Delta\vec{S}_2$ are shown in the directions considered to be positive when the sense of the circulation shown about the curve s is considered positive.

we will never reach a point at which it is terminated. This means that if a line of $\vec{B}$ passes through one of the surfaces shown in Fig. 1.11, it must do one of two things. It must turn around and pass back through that same surface, or it must pass through the second surface in the same direction it passed through the first. In the first case, it contributes nothing to the flux through either surface. In the second, its contribution to the flux through one surface is the same as its contribution to the flux through the other. Thus, we are certain that the flux through a surface bounded by the loop is the same as the flux through any other surface bounded by the same loop.

We can make the same proof in a more formal fashion. In Fig. 1.11 the surfaces $S_1$ and $S_2$ form a closed surface when combined. We can use Eq. (1.6) on that closed surface, so that

$$\sum_{S_1} \vec{B} \cdot \Delta\vec{S}_1 - \sum_{S_2} \vec{B} \cdot \Delta\vec{S}_2 = 0,$$

where we have taken due care to have the outward direction positive. From this,

$$\sum_{S_1} \vec{B} \cdot \Delta\vec{S}_1 = \sum_{S_2} \vec{B} \cdot \Delta\vec{S}_2 .$$

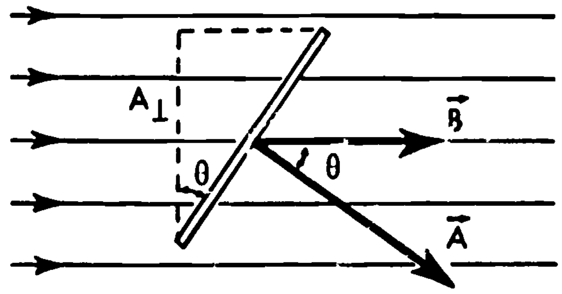



Fig. 1.12 Edgewise view of a plane loop in a uniform magnetic field $\vec{B}$ when the loop is tilted. The projection of the loop's area on a plane perpendicular to $\vec{B}$ is $A_\perp$.

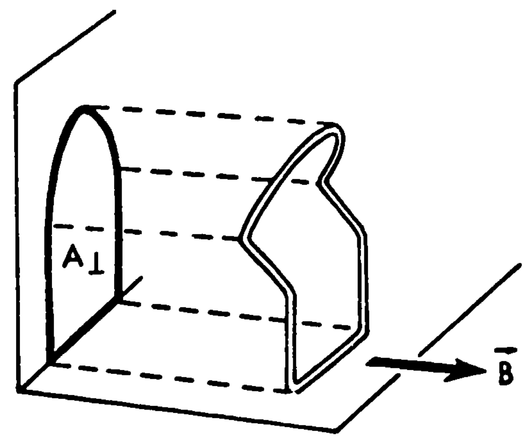



Fig. 1.13 The projected area of a nonplanar loop. Note that this is a special case of $A_\perp$ for the loop shown.

Since $S_1$ and $S_2$ here can be any two surfaces bounded by the loop, we have proved our contention.

With Faraday's law in hand we can understand the results of the experiments using the long coil shown in Fig. 1.9. If the coil is tightly wound and is long enough, and if we stay near the center of the coil, then $\vec{B}$ inside the coil is very nearly the same at all points. Its magnitude is given by

$$B = \mu_0 n I,$$

where n is the number of turns per unit length of the coil measured along the coil's axis, and I is the current in the coil. Further, $\vec{B}$ is directed parallel to the coil's axis. This result is derived using Ampere's law in the monograph Magnetostatics. If we are sure that the loop is oriented with its plane perpendicular to the coil's axis, then we can find the flux $\Phi_B$ threading the loop of area A. Since B is the same everywhere, then choosing S to be the plane surface of the loop, we can write

$$\Phi_B = \sum_S \vec{B} \cdot \Delta\vec{S} = \vec{B} \cdot \left(\sum_S \Delta\vec{S}\right) = \vec{B} \cdot \vec{A}$$

$$= BA.$$

Now putting in the magnitude of B,

$$\Phi = \mu_0 n A I.$$

Then, by Faraday's law, we can say that

$$\mathcal{E} = -\mu_0 n A \frac{\Delta I}{\Delta t},$$

where $\Delta I/\Delta t$ is the time rate of change of the current in the coil. If we put this result into Ohm's law, we get

$$i = -\frac{\mu_0 n A}{R} \frac{\Delta I}{\Delta t}, \qquad (1.7)$$

where i is the current induced in the loop, and R is the resistance in the circuit containing the loop and the galvanometer.

All the results reported earlier for this experimental arrangement are contained in Eq. (1.7). If we believe Eq. (1.7) to be correct, then we can predict what will happen when we change the loop's area or change the resistance R. And we can predict how the result will depend upon the time rate of change of the current in the coil; in particular we would predict the behavior shown in the plots appearing in Figs. 1 4 and 1.5.

We can generalize Eq. (1.7) to take care of cases when the loop is not oriented with its plane perpendicular to $\vec{B}$. When calculating $\Phi_B$ we are really concerned with the projection of the coil's area on the plane perpendicular to $\vec{B}$ (see Fig. 1.12). That projection $A_\perp$ is just equal to A cos θ; and that cos θ factor appears when we calculate $\Phi_B = \vec{B} \cdot \vec{A}$, since the angle between $\vec{B}$ and $\vec{A}$ is also θ. Then for any orientation of the loop, Eq. (1.7) is generalized to

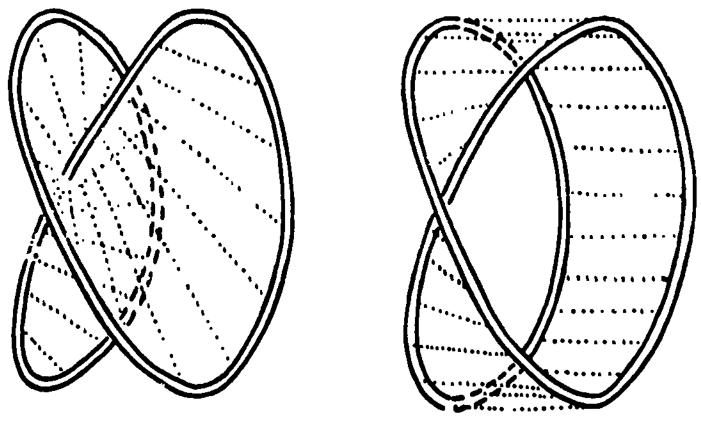



Fig. 1.14 Two different surfaces "bounded" by the same closed loop.

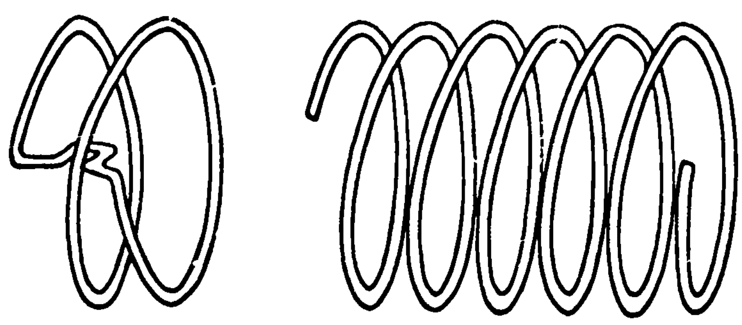



Fig. 1.15 (a) A closed loop of two turns. Compare with Fig. 1.14(a). (b) Now we have a coil. It is not a loop; it isn't closed.

$$i = -\frac{\mu_0 nA \cos\theta}{R}\frac{\Delta I}{\Delta t}. \qquad (1.8)$$

In fact, there is nothing in our development that depends upon the shape of the loop; it need not be circular as we have been picturing it. Furthermore, the loop need not even lie in a plane, if we interpret $A_\perp$ to be the area enclosed by the projection of the loop onto a plane perpendicular to $\vec{B}$ (see Fig. 1.13). Of course, this last statement can be true only if no line of $\vec{B}$ passes twice or more through the surface bounded by the loop.

There are cases in which the phrase "the surface bounded by the loop" might seem ambiguous. Such a case is shown in Fig. 1.14. The surface we want is the one shown in Fig. 1.14(a). The surface in Fig. 1.14(b) is really not bounded at all. We can see this by noting that this surface does not have two sides; it is the famous Möbius strip. You can make a Möbius strip for yourself by laying out a strip of paper, giving it a half twist about the long axis, and then pasting the two free ends together. Mathematics students sometimes like to plague younger brothers by asking them to color one side of this strip blue and the other red. If you have never seen this tried, you might make the effort in the secrecy of your room.

QUESTION: See Fig. 1.14(a). Assume the presence of a uniform magnetic field $\vec{B}$ directed from left to right. How would you calculate the flux $\Phi_B$ through this loop? Include a statement of any assumptions you have made to cover information not explicitly given to you.

If you have thought carefully about the Question, you should be able to conclude that for the purpose of calculating $\Phi_B$, the loop shown in Fig. 1.14(a) is identical with that in Fig. 1.15(a); and that each is equivalent to two circular loops. From this argument we would expect that had we used a loop of two turns in the experiment depicted in Fig. 1.9, we would have doubled the $\Phi_B$ and thus doubled the value of $\Delta\Phi_B/\Delta t$ and of the induced emf $\mathcal{E}$. And experimental results agree with that expectation. If we use a loop with N turns, then we would need to modify Eq. (1.8) to read

$$i = -\frac{\mu_0 NnA \cos\theta}{R}\frac{\Delta I}{\Delta t}. \qquad (1.9)$$

QUESTION: What is N for the loop shown in Fig. 1.15(b)? Does it matter how the ends are connected?

## 1.3 FARADAY'S LAW AS A CIRCULATION LAW.

In the case of the stationary loop, an induced emf can appear only if there exists an electric field intensity $\vec{E}$. The emf in a loop is defined as being the work done on a unit charge as that charge traverses the loop once, i.e.,

$$\mathcal{E} = \sum_{s \text{ closed}} \frac{\vec{F}}{q} \cdot \Delta s,$$

from the definition of work. But what is $\vec{F}/q$ in this case? Nothing but the definition of $\vec{E}$, the electric field intensity. That means that for this case, we can write Faraday's law as

$$\sum_{s \text{ closed}} \vec{E} \cdot \Delta\vec{s} = -\frac{\Delta}{\Delta T}\left(\sum_{S} \vec{B} \cdot \Delta\vec{S}\right), \tag{1.11}$$

which is a circulation law. Further, since the loop is stationary, all flux changes are due to changes in $\vec{B}$. In that case, we can write Eq. (1.11) as

$$\sum_{s \text{ closed}} \vec{E} \cdot \Delta\vec{s} = -\sum_{S} \frac{\Delta\vec{B}}{\Delta t} \cdot \Delta\vec{S}, \tag{1.12}$$

where the surface S is any surface bounded by the circulation path s - here the loop. We are forced to a startling conclusion that has enormous consequences: If we have a changing magnetic field, i.e., one in which $\vec{B}$ is time dependent, then there is an associated electric field! That is, there can be an electric field even though the charge density is zero everywhere.

Maxwell was the first to express Faraday's law of induction as a circulation law. But he generalized the law. He imagined that the law as expressed in Eq. (1.11) or (1.12) holds in empty space, so that the circulation of $\vec{E}$ need not be interpreted as the induced emf in a material, conducting loop. If we assume that Maxwell was correct in his assumption, and he was, then we can state Faraday's law in the following way.

The circulation of $\vec{E}$ about a closed path s at rest in the frame in which $\vec{B}$ is measured is equal to the negative of the time rate of change of the flux of $\vec{B}$ passing through any surface S bounded by that circulation path.

According to this circulation law, the circulation of $\vec{E}$ is not necessarily zero in time-dependent circumstances. For time-independent cases, i.e., static cases, the circulation of $\vec{E}$ is always zero, no matter what circulation path we choose (see monograph Electrostatics). That is, static electric fields are conservative; not so for electric fields associated with changing magnetic fields. Nonetheless, the circulation law given by Eq. (1.11) is true in general, i.e., when the electric field intensity $\vec{E}$ has contributions from both charge densities (static field $\vec{E}_s$), and changing magnetic fields (induced field $\vec{E}_i$). If we write that $\vec{E} = \vec{E}_s + \vec{E}_i$, then the circulation of $\vec{E}$ is

$$\sum_{s \text{ closed}} \vec{E} \cdot \Delta\vec{s} = \sum_{s \text{ closed}} (\vec{E}_s + \vec{E}_i) \cdot \Delta\vec{s}$$

$$= \sum_{s \text{ closed}} \vec{E}_s \cdot \Delta\vec{s} + \sum_{s \text{ closed}} \vec{E}_i \cdot \Delta\vec{s}$$

$$= \sum_{s \text{ closed}} \vec{E}_i \cdot \Delta\vec{s}.$$

The circulation of $\vec{E}_s$ is zero, leaving only the circulation of $\vec{E}_i$, which is equal to $-\Delta\Phi_B/\Delta t$ as required by the circulation law. Then we see that the circulation law is true in general, because the static contribution to the total field $\vec{E}$ does not contribute to the circulation of $\vec{E}$.

We see clearly from this argument that knowing the circulation of $\vec{E}$ does not mean that we know much about $\vec{E}$ itself. In fact, we can use this circulation law to calculate $\vec{E}$ only in special, highly symmetrical circumstances. The experimental arrangement shown in Fig. 1.9 is one of these special cases. We need not consider the probe loop shown, if we accept Maxwell's generalization to empty space. Again we imagine that we are near the middle of a long, tightly wound coil, so that we may assume that $\vec{B}$ is uniform and directed parallel to the coil's axis. There are no charge densities around, so there is no static field contribution to $\vec{E}$ inside the coil.

Figure 1.16 is a view looking along the coil's axis. We choose as the circulation path a circle of radius r, centered on the coil's axis, and oriented so that its plane is perpendicular to the coil's axis. As shown, r is less than the inside radius of the coil. If the current in the coil is changing, then the magnitude of $\vec{B}$ is changing, and so is the flux $\Phi_B$ that passes through a surface bounded by the circulation path. Then the circulation of $\vec{E}$ around that path is not zero. Can we calculate what it is? Yes, we can use the right-hand side of the circulation law.

$$\sum_{s\ \text{closed}} E \cdot \Delta s = -\frac{\Delta\Phi_B}{\Delta t} = -\left(\frac{\Delta B}{\Delta t}\right)(\pi r^2).$$

Now since we have good cylindrical symmetry, the magnitude of $\vec{E}$ at one point on the circular path must be the same as it is at every other point on the path. If there is no charge density around, then $\vec{E}$ at each point on that circle is tangent to the circle, and we can write the circulation law as

$$(E)(2\pi r) = -(\pi r^2)\left(\frac{\Delta B}{\Delta t}\right).$$

And now we have that

$$E = -\frac{r}{2}\left(\frac{\Delta B}{\Delta t}\right). \qquad (1.13)$$

For this highly symmetric situation we have used the circulation law to get an expression for E that should be valid at all points inside the coil and far from its ends. If, as we have assumed, $\Delta B/\Delta t$ is the same everywhere in that region, then we see that E is zero on the coil's axis and increases linearly with the distance from that axis.

EXERCISE

Using Gauss's law, show that there cannot be a radial component to $\vec{E}$ in the example if there are no charge densities in the vicinity.

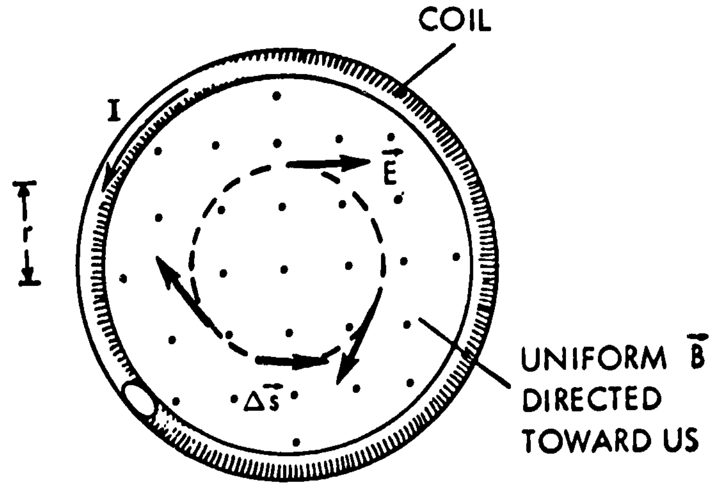
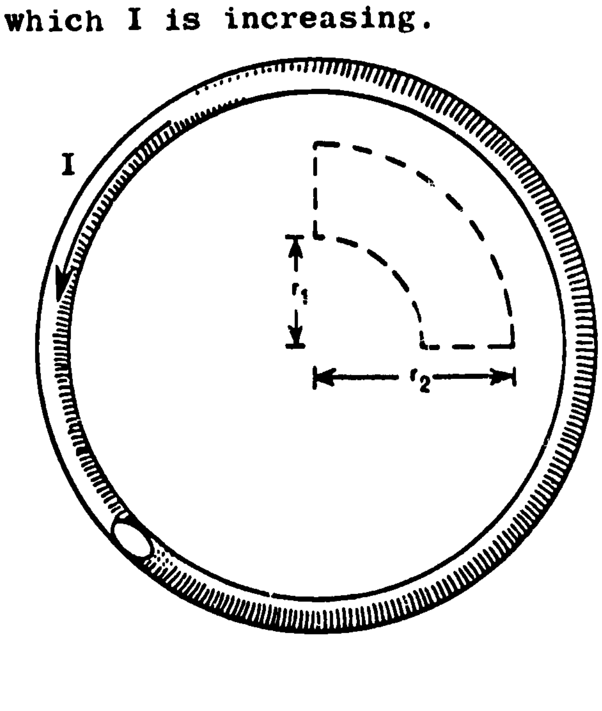



Fig. 1.16 View along the axis of a long coil. With the sense of $\Delta\vec{s}$ shown, the positive flux $\Phi_B$ is directed toward us. The directions of $\vec{E}$ on the circular path are for the case of increasing $\Phi_B$.

The minus sign in Eq. (1.13) tells us the direction of $\vec{E}$. At each point on the circular path, $\vec{E}$ must have the direction just opposite to that of $\Delta\vec{s}$ at that point, provided that $\vec{B}$ is increasing so that $\Delta B/\Delta t$ is positive. That is the situation depicted in Fig. 1.16. If the current I decreases, then $\Delta B/\Delta t$ is negative, and the direction of $\vec{E}$ is opposite to that shown in the figure.

EXERCISES

Using Eq. (1.13), calculate the circulation of $\vec{E}$ directly for the path shown. Arrange your result so that you can see clearly that it is equal to the negative of the time rate of change of $\Phi_B$. Choose the case in which I is increasing.



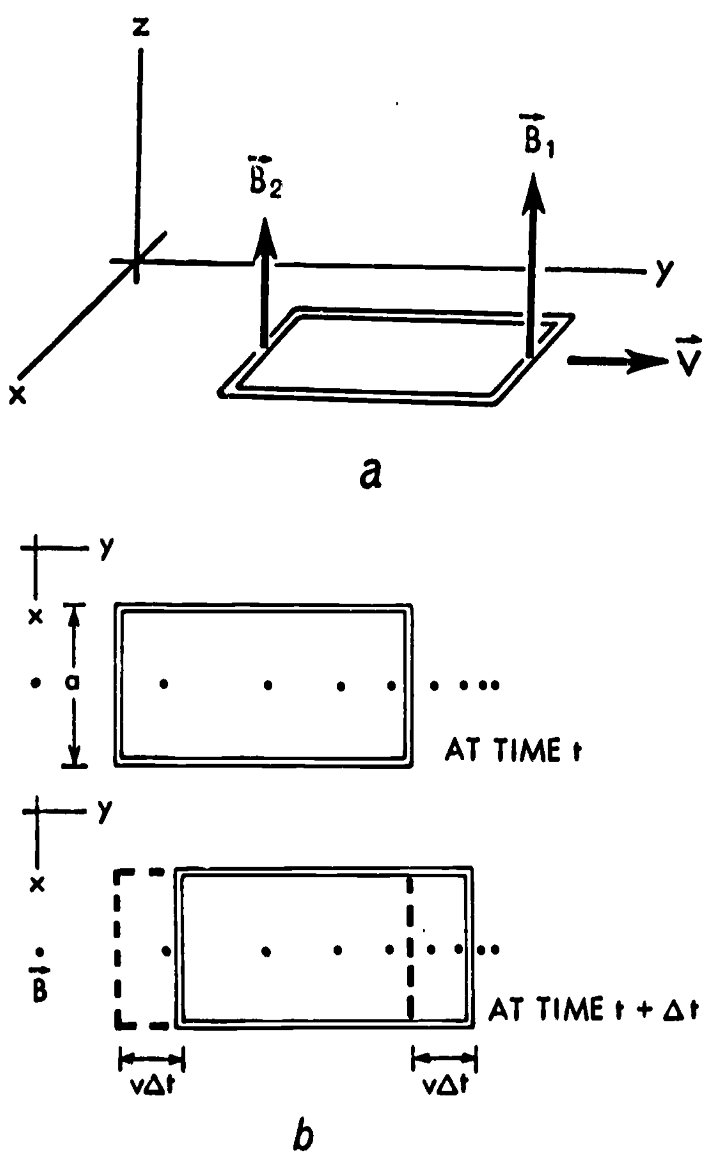



**Fig. 1.17** A loop moves through a nonuniform, time-independent, magnetic field. The flux $\Phi_B$ is changing.

Suppose that in the above Exercise we add a long wire along the coil's axis and let this wire carry the linear charge density λ coulomb/meter. Now calculate the circulation of $\vec{E}$ for the path shown. In what way does your result differ from that you got when no charged wire was present? Why? Does your answer to this last question depend upon the fact that the static electric field due to the line charge is radial? Explain.

QUESTION. In either of the above two exercises, what meaning can we assign to the idea of "potential difference between two points in the field" while the current in the coil is changing?

## 1.4 MOTIONAL emf

To this point in our study of Faraday's law, we have concentrated upon emf's induced in a stationary loop as a consequence of a time-varying magnetic field. In such a case, the flux $\Phi_B$ changes, because $\vec{B}$ changes with time, at least over some region of space. But there are other ways to make $\Phi_B$ change for a given loop, and we are going to look at one of them now.

Consider the following simple case for which we can make some calculations. We have a piece of wire bent into a rectangle, so that it forms a closed conducting loop. We place the loop in the xy plane, as shown in Fig. 1.17(a). There is a magnetic field present such that $\vec{B}$ points in the +z direction at every point in the region of interest. Further, the magnitude of $\vec{B}$ increases with increasing y, but $\vec{B}$ at each point is constant in time.

Now we imagine that the loop is moving with the velocity $\vec{v}$ in the direction of increasing y while it remains entirely in the xy plane. The flux $\Phi_B$ threading the loop is then changing, and, if Faraday's law is correct, we should expect that an emf will be induced in a loop which moves in such an environment.

We should realize that this is a quite different situation from that discussed in the two previous sections. This corresponds to the case shown in Fig. 1.8(b). The flux change is being created by the motion of a loop in a nonuniform, time-independent, magnetic field. We shall refer to the emf induced in such a fashion as a <u>motional</u> emf.

We can calculate the time rate of change of $\Phi_B$ for the simple case cited. The locations of the loop at the times t and t + Δt are shown in Fig. 1.17(b). If we let $\Phi_B$ be positive when it is in the +z direction, we can get the corresponding $\Delta\Phi_B$ by calculating the gain in $\Phi_B$ at the front edge and the loss of $\Phi_B$ at the back That is,

$$\Delta\Phi_B = +(B_1 \text{ av } \Delta t) - (B_2 \text{ av } \Delta t)$$

$$= (B_1 - B_2) \text{ av } \Delta t, \qquad (1.14)$$

where $\vec{B}_1$ and $\vec{B}_2$ are the magnetic induction fields at the front and back edges of the loop. We have assumed that $\Delta t$ is so small that $\vec{B}$ does not vary appreciably over either of the areas (av $\Delta t$) at the front and back edges. Since we let $B_1 > B_2$, then $\Delta\Phi_B$ is positive; $\Phi_B$ is increasing.

From Eq. (1.14),

$$\frac{\Delta\Phi_B}{\Delta t} = (B_1 - B_2) \text{ av}, \qquad (1.15)$$

so that the emf induced in the loop is, by Faraday's law,

$$\mathcal{E} = -(B_1 - B_2) \text{ av}, \qquad (1.16)$$

and the induced current is in the clockwise sense around the loop as we view the loop in Fig. 1.17(b).

If the magnetic field is uniform, a translation of the loop will not produce an induced emf, since there will be no flux change. The quantities $B_1$ and $B_2$ in Eq. (1.16) will be equal. But if we rotate the loop, even in a uniform magnetic field, then we can find an induced emf in the loop. Figure 1.18 shows a rectangular loop rotating about an axis which is perpendicular to a uniform field $\vec{B}$. In this case the flux $\Phi_B$ is changing because the projected area $A_\perp$ is changing. At the instant shown, there is an induced emf in the clockwise sense around the loop as viewed from the xy plane. If $\vec{B}$ is uniform, then the flux $\Phi_B$ is simply $BA_\perp$ for the case shown. Then, since $\vec{B}$ is constant in time,

$$\mathcal{E} = -\frac{\Delta\Phi_B}{\Delta t} = -B\frac{\Delta A_\perp}{\Delta t}. \qquad (1.17)$$

QUESTION. Is there an axis about which we can rotate the loop in Fig. 1.18 without inducing an emf?

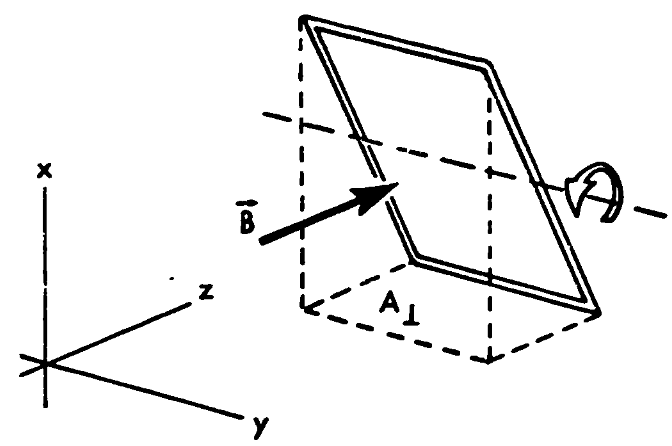



Fig. 1.18 Inducing an emf in a loop by rotating the loop in a uniform magnetic field.

If there is, does it matter what the loop's orientation is with respect to that axis?

The loop in Fig. 1.18 need not be a rectangle; it was drawn that way for simplicity. The loop can have any shape; it could even be a coil with many turns. In any case, we have here the fundamental concepts of the generator, which we shall discuss in some detail in section 1.6.

We see now that an induced emf can appear if we move a loop in a time-independent magnetic field. The needed change of the flux $\Phi_B$ is brought about in two ways: The loop moves through a region in which $\vec{B}$ is not uniform (Fig. 1.17), and the loop's orientation relative to $\vec{B}$ changes (Fig. 1.18). Of course both these could be going on at the same time; calculating $\Delta\Phi_B/\Delta t$ could become a tricky and messy business. But it turns out that in many cases such a calculation is not difficult at all. We shall make an assertion here, and then we shall prove the assertion. There is no physics in what we shall say; it is simply a consequence of some mathematical reasoning. And, for the moment, we shall view the assertion as an aid to calculation. The assertion:

If $\vec{B}$ is independent of time, then

$$\frac{\Delta\Phi_B}{\Delta t} \equiv \frac{\Delta}{\Delta t}\sum_S B \cdot \Delta\vec{S}$$

$$= -\sum_{s \text{ closed}} (\vec{v} \times \vec{B}) \cdot \Delta\vec{s}, \qquad (1.18)$$

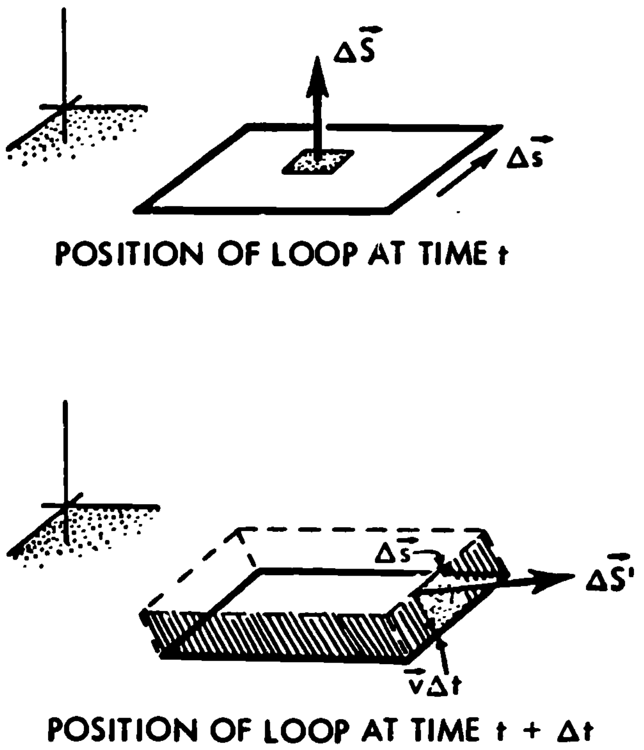



Fig. 1.19 A moving loop.

where S is any surface bounded by the curve s (our loop), $\vec{v}$ is the velocity of the element $\Delta\vec{s}$, and $\vec{B}$ in the right-hand sum is the field at each $\Delta\vec{s}$. The velocity $\vec{v}$ is measured in the same frame of reference in which $\vec{B}$ is measured.

Equation (1.18) tells us that under certain circumstances the time rate of change of a flux can be written as a circulation. The particular circumstances for which Eq. (1.18) is a valid mathematical statement will appear as we develop the proof, which we now proceed to do.

Figure 1.19 shows the positions of a loop at the times t and t + Δt. At the time t, the flux is simply

$$\Phi_B(t) = \sum_S \vec{B} \cdot \Delta\vec{S}.$$

To find the flux at the time t + Δt, we can use any surface bounded by the loop, provided only that the lines of $\vec{B}$ do not terminate. Or what is equivalent, provided the flux of $\vec{B}$ over any closed surface is zero. (This condition is certainly satisfied by the magnetic induction field.) Then suppose we choose as the surface at t + Δt the original surface S plus the edge surface S' that has been generated by the moving loop. Then

$$\Phi_B(t + \Delta t) = \sum_S \vec{B} \cdot \Delta\vec{S} + \sum_{S'} \vec{B} \cdot \Delta\vec{S}'.$$

If $\vec{B}$ at each point in space is the same at the time t + Δt as it was at the time t, then the first term on the right side is just Φ(t). Then the change in the flux $\Phi_B$ that has occurred in the time Δt is

$$\Delta\Phi_B = \Phi_B(t + \Delta t) - \Phi(t) = \sum_{S'} \vec{B} \cdot \Delta\vec{S}'.$$

But from Fig. 1.19, $\Delta\vec{S}' = (\vec{v}\Delta t) \times \Delta\vec{s}$ where we have been careful to keep the directions of $\Delta\vec{s}$, $\Delta\vec{S}$, and $\Delta\vec{S}'$ consistent. If we substitute $\Delta t(\vec{v} \times \Delta\vec{s})$ for $\Delta\vec{S}'$ in the sum, then we need to sum over all the $\Delta\vec{s}$ rather than over the $\Delta\vec{S}'$. Then we can write

$$\Delta\Phi_B = \Delta t \sum_{s\ \text{closed}} \vec{B} \cdot (\vec{v} \times \Delta\vec{s}).$$

We can see that if we are going to be able to complete our proof at all, we must be close to doing so now. Only some juggling remains. We can divide both sides by Δt to bring the left-hand side into order.

$$\frac{\Delta\Phi_B}{\Delta t} \equiv \frac{\Delta}{\Delta t}\sum_S \vec{B} \cdot \Delta\vec{S} = \sum_s (\vec{v} \times \Delta\vec{s}) \cdot \vec{B}.$$

We have also changed the order of the dot product on the right-hand side, but that doesn't change anything. Comparison of what we now have with Eq. (1.18), what we are trying to prove, shows that we have only one thing left to do. The proof that

$$(\vec{v} \times \Delta\vec{s}) \cdot \vec{B} = -(\vec{v} \times \vec{B}) \cdot \Delta\vec{s}$$

is really rather simple, but it is a bit long. So that it will not clutter up our work here, the proof has been put at the end of this section. It is really rather nice, and you might enjoy looking at it.

With this final bit of juggling we have completed our mathematical proof. We are now prepared to state the mathematical equality given in Eq. (1.18). It is true for any vector field $\vec{B}$ satisfying the two conditions that we imposed in our proof. First, the flux of $\vec{B}$ over any closed surface must be zero, or, put in another way, the lines of $\vec{B}$ must have no beginning and no ending. Second, $\vec{B}$ must not change with time. It may be different at different points in space, but at any given point in space $\vec{B}$ remains constant.

Equation (1.18) contains no new physics at all. But we can apply it to the case of motional emf, because the two conditions on $\vec{B}$ are satisfied. The vector $\vec{B}$ always satisfies the first of these, and since we are considering loops that move in time-independent magnetic fields, $\vec{B}$ satisfies the second in this instance.

If we use Eq. (1.18) in Faraday's induction law, we get

$$\mathcal{E} = + \sum_{\text{s closed}} (\vec{v} \times \vec{B}) \cdot \Delta\vec{s}, \qquad (1.19)$$

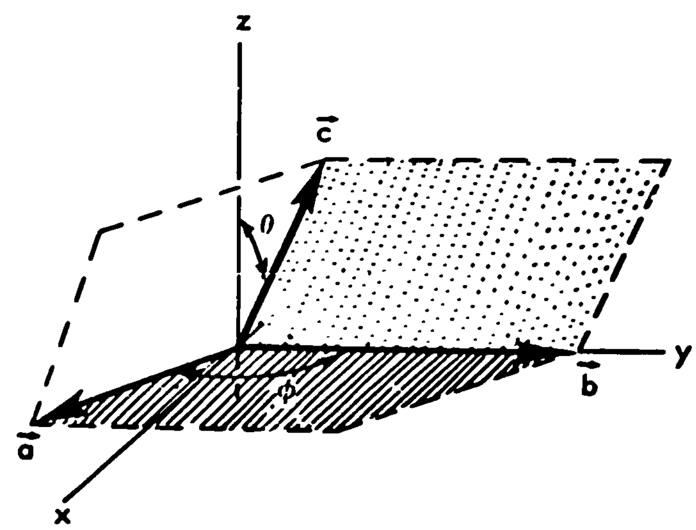
where the right-hand side is just the negative of $\Delta\Phi_B/\Delta t$ for this special case.

Going back over our development, we see that we did not really assume any particular shape for the loop. Equations (1.18) and (1.19) are good for any loop. Further, there is nothing in our proof of Eq. (1.18) requiring all parts of the loop to have the same velocity, so that Eq. (1.19) is valid for a loop which is rotating.

We can apply Eq. (1.19) to any case of motional emf. The right-hand side tells us to evaluate a circulation about the loop, but we know that we are really calculating the negative time rate of change of $\Phi_B$, as is required by Faraday's law.

EXERCISE

Use Eq. (1.19) to find the induced emf in the loop shown in Fig. 1.17. Check your result with Eq. (1.16).




This is the proof promised earlier. Consider the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ shown in the diagram. For ease of reference we have put the vectors $\vec{a}$ and $\vec{b}$ in the xy plane, but nothing we do will depend upon any particular frame.

The vector product $\vec{a} \times \vec{b}$ gives a vector in the +z direction (perpendicular to the plane containing $\vec{a}$ and $\vec{b}$). The magnitude of this vector, ab sin $\Phi$, is just the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. The scalar product of $\vec{a} \times \vec{b}$ and $\vec{c}$ gives the volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$, and $\vec{c}$. That is, it gives the area of the base times the vertical height c cos $\theta$.

$$\text{Volume} = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

If we take the vector product the other way around, i.e., as $\vec{b} \times \vec{a}$, then we get a vector in the $-z$ direction. The scalar product of that vector with $\vec{c}$ gives a negative number, just the negative of the volume. So if we want the volume to come out as a positive quantity, we need the cross product in the order $\vec{a} \times \vec{b}$. Then the angle between the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$ is less than $\pi/2$.

Of course it cannot matter which face of the parallelepiped we choose to be the base. We have used the one defined by $\vec{a}$ and $\vec{b}$, but we could as well start with the one defined by $\vec{b}$ and $\vec{c}$ or the one defined by $\vec{c}$ and $\vec{a}$. We just have to keep track of the order in which we take each vector product, so that we are sure we get a positive result for the volume. Then it must be that

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}.$$

We have shown this result for a particular set of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, but a little thought should convince you that the conclusion is valid for any set. If the vectors we choose are such that $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is positive, then the other two arrangements will be positive too. If $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is negative, then the other two arrangements will also be negative. The result we have here is independent of the labeling and relative orientation of the three vectors.

Now we use the first and third of these, reversing the order of the vector product in the third and putting in a minus sign to take care of that reversal,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}.$$

Since $\vec{a}$, $\vec{b}$, and $\vec{c}$ are any vectors, this completes the proof that

$$(\vec{v} \times \Delta\vec{s}) \cdot \vec{B} = -(\vec{v} \times \vec{B}) \cdot \Delta\vec{s},$$

which is what we asserted earlier.

## 1.5 MAGNETIC FORCE ON A MOVING CHARGE.

Something rather interesting happened in the last section, but we did not pay any attention to it at the time. Now we want to take a closer look at Eq. (1.19).

$$\mathcal{E} = \sum_{\text{s closed}} (\vec{v} \times \vec{B}) \cdot \Delta\vec{s} \qquad (1.19)$$

We got the right-hand side of this by seeing what $-\Delta\Phi_B/\Delta t$ was for a special case: a loop moving in a time-independent magnetic field. That is, Eq. (1.19) is just Faraday's law for the special case where $\mathcal{E}$ then is the resulting motional emf.

Now the emf is the work done on a unit charge as that charge traverses the circuit one time. But what is the force doing that work in our special case? There are no charge densities around to give a static electric field. And in the frame in which $\vec{v}$ is measured the magnetic induction field $\vec{B}$ is not changing with time; i.e., $\Delta\vec{B}/\Delta t = 0$, so that in that frame there is no induced electric field either. Then how shall we account for the resulting emf?

Using the definition of $\mathcal{E}$ in mathematical terms, i.e.,

$$\mathcal{E} = \sum_{\text{s closed}} (\vec{F}/q) \cdot \Delta\vec{s}. \qquad (1.20)$$

We can write Eq. (1.19) as

$$\sum_{\text{s closed}} \frac{\vec{F}}{q} \cdot \Delta\vec{s} = \sum_{\text{s closed}} (\vec{v} \times \vec{B}) \cdot \Delta\vec{s}, \qquad (1.21)$$

where both sums are over the same closed loop.

There is no logical or mathematical basis for equating the bracketed terms on the two sides of Eq. (1.21). That is because it might be that

$$\frac{\vec{F}}{q} = \vec{v} \times \vec{B} + \vec{C},$$

where $\vec{C}$ is a vector field whose circulation is always zero, as, for example, in the static electric field. But if for our special case we do set the bracketed terms equal to each other, i.e., if we just set $\vec{F}/q$ equal to $\vec{v} \times \vec{B}$ at every point along the loop, then we get that

$$\vec{F} = q(\vec{v} \times \vec{B}) \qquad (1.22)$$

at every point on the loop. But this is just the equation for the magnetic force on a moving charged particle (see monograph Magnetostatics).

Suppose we move a conductor in a magnetic field. There are charges that are free to move inside the conductor, and they do so move when they experience the $q(\vec{v} \times \vec{B})$ force. In a conductor such as copper, electrons are the particles that can move around. When a copper wire is given a velocity $\vec{v}$, the electrons have that velocity too. Then the electrons experience the $q(\vec{v} \times \vec{B})$ force that may result in an emf in the loop. Figure 1.20 is meant to illustrate what happens. In the case shown, the $\vec{v} \times \vec{B}$ force is along the wire, and it can thus contribute to an emf. Since q is negative for the electron, the force $\vec{F}$ is directed opposite to $\vec{v} \times \vec{B}$. The conventional current I, however, is still in the direction of $\vec{v} \times \vec{B}$.

If we try to trace the motion of one of these electrons, if we try to keep track of the forces on it as it moves inside the conductor, we can get ourselves into a great tangle. As soon as the electron gains a component of velocity along the wire, then the force it feels is no longer that shown in Fig. 1.20. That is because its velocity is no longer $\vec{v}$. The velocity $\vec{v}$ appearing in Eq. (1.19), and thus in Eq. (1.22), is the velocity of the wire, not the velocity of the electron when the electron travels inside that wire. Further, we certainly know that the electron experiences an enormously complex force field as it travels through the conductor. Nevertheless, if we assume the simple force law given by Eq. (1.22), we come up with the right result for the emf. That certainly seems strange, but it is true.

With the interpretation of $\vec{v} \times \vec{B}$ as a force per unit charge, the right side of Eq. (1.19) takes on a new physical meaning. We need not think of it as being the negative of the time rate of change of $\Phi_B$, which it certainly is. We can think of it as being the direct calculation of the emf using the defining Eq. (1.20) with $\vec{v} \times \vec{B}$ being the force per unit charge.

All of this leads us to the following point of view: A charged particle can experience a force that is velocity independent, and we write this force as $q\vec{E}$. The electric field intensity $\vec{E}$ can be the consequence of a distribution of charge density or of a time-varying magnetic field. A charged particle can also experience a velocity-dependent force $q(\vec{v} \times \vec{B})$, if there is a magnetic field present. If we put these together, we can say in general that the electromagnetic force on a particle with charge q is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \qquad (1.23)$$

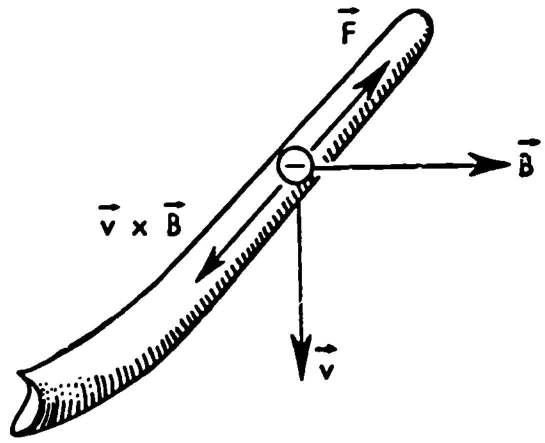



Fig. 1.20 A segment of a conducting wire which has the velocity $\vec{v}$ while moving in the field $\vec{B}$. $\vec{F}$ is the force on an electron in the wire.

This has come to be called the Lorentz force. Its application to the motions of free particles in empty space is often more straightforward than it is when applied to particles which themselves move around in moving materials. Since $\vec{v}$ is the velocity of the material, not that of the particle inside the material, it seems as if the particle is in a field $\vec{v} \times \vec{B}$ that exists inside materials whenever they move in magnetic fields.

The strange nature of a velocity-dependent force is discussed in a later section. Sooner or later, we must look into the difficulties encountered when we try to reconcile the concept of a velocity dependent force field with the principle that all inertial frames of reference are equivalent.

## 1.6 GENERATORS AND MOTORS

Modern electrical technology began with Faraday's discovery, for then engineers had the ideas they needed to permit the design of machines to create, deliver, and use electric energy. When

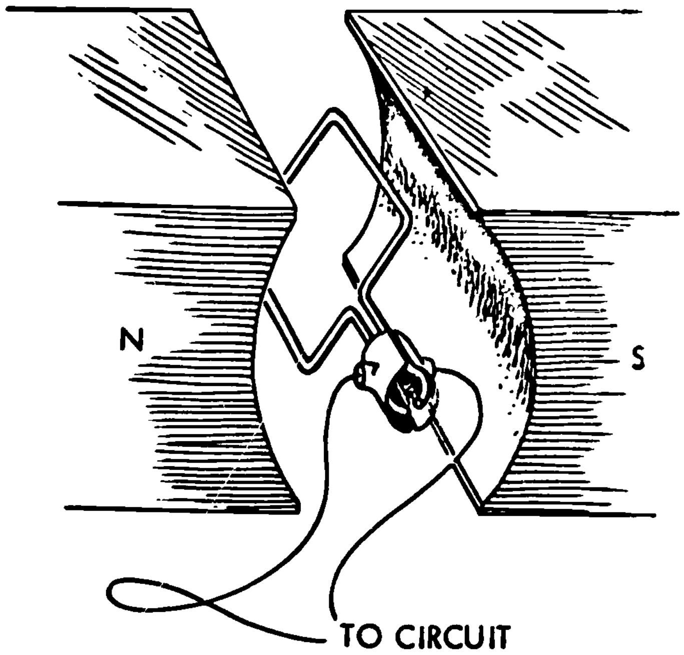



Fig. 1.21 A generator.

they were restricted to batteries, electric energy was so limited that engineers were unable to make much use of it. Anyway, batteries run down. Now electricity can light cities, turn millions of wheels, heat homes, and carve mountains - all because Michael Faraday wondered about the ways that magnets and currents were related.

Faraday had to suffer officials who visited his laboratory while he was working, and he was often called upon to give public lectures on his work. There is no way to know how many times he was asked, "What's the use of all this?" But he was ready with a reply when Gladstone, then Chancellor of the Exchequer, interrupted him impatiently, "But, after all, what use is it?" Faraday fired back, "Sir, you may one day be able to tax it." Such opportunities come rarely, even to men of Faraday's stature.

Benjamin Franklin responded to questions like that with a question of his own. "What is the use of a baby?" What exactly is this baby that grew up to be taxed? Its essence is in Fig. 1.18 which is meant to show a loop rotating in a magnetic field.

Suppose we arrange, by some means or other, to keep a loop rotating in a magnetic field. And suppose, too, that we arrange to make this loop part of a larger electric circuit. Figure 1.21 shows what we have in mind, although it doesn't show how we intend to support the loop or to keep it rotating. But while that loop is rotating, it is a source of emf for the circuit, and, using Eq. (1.19), we should be able to calculate that emf.

A look at Fig. 1.22 should help us to calculate the circulation of $\vec{v} \times \vec{B}$ around the rotating coil. If the axis of rotation of the loop passes through the loop's center and is parallel to two of the edges, then each of these edges has the same speed v. In terms of the notation on the figure, $v_1 = v_2$. Further, if we have a uniform magnetic field, then the magnitude of $\vec{v}_1 \times \vec{B}$ is equal to the magnitude of $\vec{v}_2 \times \vec{B}$ since $\sin(\pi - \theta) = \sin\theta$. If we take the sense of the circulation about the loop to be along the directions of both $\vec{v}_1 \times \vec{B}$ and $\vec{v}_2 \times \vec{B}$, then these two sides of the loop contribute

$$2vBL \sin\theta$$

to the circulation.

The two other sides contribute nothing at all to the circulation, because at every point on them $\vec{v} \times \vec{B}$ is perpendicular to $\Delta\vec{s}$. Then we already have the circulation of $\vec{v} \times \vec{B}$ around the loop, so we can say that the emf induced in the loop is

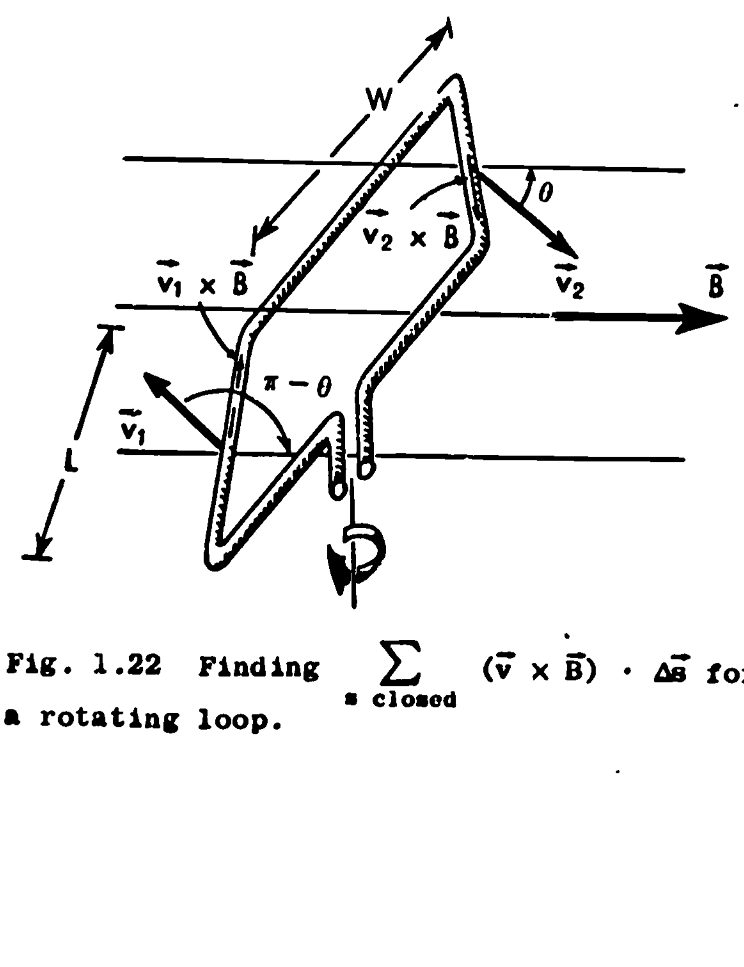



Fig. 1.22 Finding $\sum_{\text{closed}} (\vec{v} \times \vec{B}) \cdot \Delta\vec{s}$ for a rotating loop.

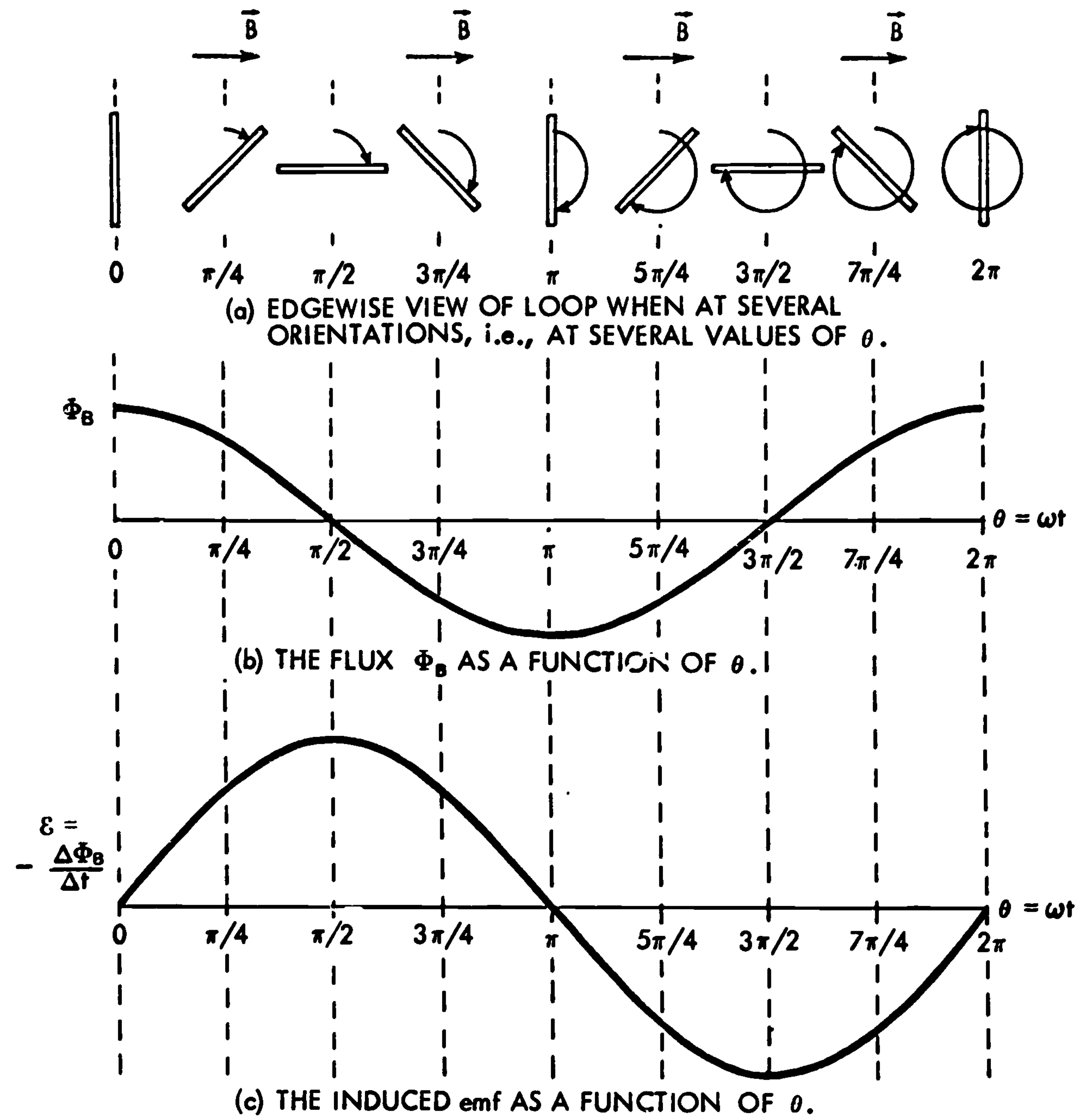



**Fig. 1.23 A plane loop rotates with constant angular velocity in a uniform magnetic field.**

$$\mathcal{E} = 2vBL \sin \theta, \qquad (1.24)$$

where we remember that v is the speed of these sides which are always perpendicular to $\vec{B}$. If we introduce the other dimension of the loop; i.e., w, we see that each point on these two sides travels in a circle of radius w/2. Then we can write that $v = (w/2)\omega$, where $\omega$ is the angular velocity of the loop; i.e., $\omega$ is just $\Delta\theta/\Delta t$ measured in radians/second. With this substitution, Eq. (1.24) becomes

$$\mathcal{E} = BA\omega \sin \theta, \qquad (1.25)$$

where we have substituted A, the loop's area, for the product wL.

We want another calculation on hand to help in our discussion of the physical content of Eq. (1.25). When the loop is in the position shown in Fig. 1.22, then the flux $\Phi_B$ that the loop intercepts is

$$\Phi_B = BA \cos \theta. \qquad (1.26)$$

At the instant shown in Fig. 1.22, the flux $\Phi_B$ is positive but it is decreasing. When $\Phi_B$ is decreasing, then $\Delta\Phi_B/\Delta t$ is negative. Since a minus sign appears in Faraday's law itself, then the emf should be positive at that instant.

Figure 1.23 shows all the pertinent relationships for the rotating loop over one complete rotation. Figure 1.23(a) displays the physical position of the loop, the second position corresponding roughly to that shown in Fig. 1.22. For the plots in Fig. 1.23, we have let $\theta$ be zero at $t = 0$, so that for constant angular velocity, $\omega$ we have that $\theta = \omega t$. Comparing the plots in (b) and (c) of Fig. 1.23, we see that $\mathcal{E}$ has a maximum magnitude when $\Phi_B$ is zero, and $\mathcal{E}$ is zero when $\Phi_B$ has its maximum magnitude. Well, that is really what we expect, since Faraday's law says that the induced emf is proportional to the time rate of change of $\Phi_B$. And that time rate of change $\Delta\Phi_B/\Delta t$ is just proportional to the slope of the $\Phi_B$ vs. $\theta$ curve in Fig. 1.23(b). And certainly that slope has its maximum magnitude at $\theta = \pi/2$ and $\theta = 3\pi/2$, the same place that $\mathcal{E}$ has its maximum magnitude. Since $\Delta\Phi_B/\Delta t$ is negative at $\theta = \pi/2$, then $\mathcal{E}$ is positive there. Since $\Delta\Phi_B/\Delta t$ is positive at $\theta = 3\pi/2$, then $\mathcal{E}$ is negative there. And since $\Delta\Phi_B/\Delta t$ is zero at $\mathcal{E} = 0$ and $\theta = \pi$, then $\mathcal{E}$ is zero at those value of $\theta$.

EXERCISE

Using Eqs. (1.25) and (1.26) along with Faraday's law, convince yourself that

$$\frac{\Delta \cos(\omega t)}{\Delta t} = -\omega \sin(\omega t),$$

where $\omega$ is a constant.

Figure 1.23 tells us that the emf in the loop is in one sense when $0 < \theta < \pi$ and in the opposite sense when $\pi < \theta < 2\pi$. But the contact that the rotating loop makes with the circuit can be arranged so that the emf in the circuit is always in the same direction. That sort of contact is shown in Fig. 1.21; it is called a split-ring commutator. If the commutator is arranged as shown in Fig. 1.24, then the emf in

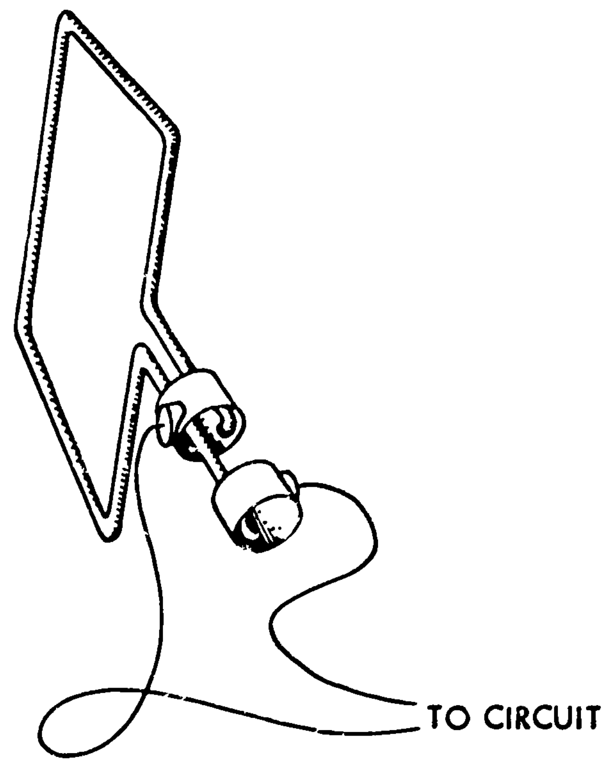



Fig. 1.24 Electrical connection for an a-c generator.

the circuit is in one sense for half a cycle of the loop, and in the opposite sense for the other half cycle. In that case, we say that we have an a-c (alternating current) generator.

EXERCISE

Plot the emf in the complete circuit when a split-ring commutator is used such as is illustrated in Fig. 1.21. At what values of $\theta$ do the contacts pass through the "splits"? Why?

Now we have the fundamental principle of a generator. There are certainly refinements that we could make. For instance, we could use a loop of many turns, so that the emf would be increased provided we can maintain a sufficiently high angular velocity. But we still have an important question to answer. How do we keep the loop rotating? We need an answer, because we get an emf only when the loop rotates.

One way to crank this system is to attach a paddle wheel to the loop and then put it at the bottom of a waterfall. The water is going to lose all that potential energy anyway, so it may as well fall on the paddle wheel and, thus, spin the loop. The large, specially designed, paddle wheels located at places such as Niagara Falls are called turbines. If there is no nearby waterfall, someone might build a high dam on a river, so that giant pipes running down the inside of the dam provide an artificial waterfall. Lacking a dam, the turbine can be driven by steam at high pressure. In that case, we need to boil water, which requires large quantities of coal or else a nuclear power reactor. In any case, the goal is to rotate a turbine.

Engineers have spent a lot of time designing electric power plants. And they have solved an enormous number of complex technical problems so that they can operate these plants at the highest efficiency. Our short description does not do justice to what they have accomplished. We have just looked at the basic scientific law that is the heart of the matter.

If the electrical energy developed at a plant is to be delivered to some place that is far away, there are lots of other interesting problems to solve. But we cannot go into the transmission problems here. We are going to see what we can do with this electric energy once it has been delivered to us. We are normally provided with two electrical contacts; in houses the usual potential difference between one of these contacts and the other alternates, taking on all values between about +155 volts and −155 volts, the root mean square (rms) value being about 110 volts. If you want to run washing machine and drier you will likely need a pair at 220 volts rms potential difference. A much higher potential difference is maintained in transmission lines, because energy losses turn out to be less along the way when the voltage is high. But the voltage is reduced by a series of transformers that are located between the transmission lines and our houses.

What can we do with this potential difference? Well, we can do some obvious things. We can put that potential difference across some resistance so that we get heat. The resistance can be in thin wire embedded in ceiling plaster; that will heat a room. Or the resistance can be in a coil on the top of an electric range; that will boil water and cook food. If the resistance is in the filament of a light bulb, then we can heat that filament; that will light our rooms.

Of course, we can use that potential difference to run a radio or television transmitter and to activate radio and television receivers. We shall not go into the modern technology of electronics that has become so important. We are going to leave out all those marvelous gadgets used for communications, for calculations that go on inside a computer, and for detecting the presence of subatomic particles. We are going to study something that seems much more prosaic: the electric motor.

Without the electric motor, the industrial revolution would surely have fizzled. The electric motor does a very large fraction of the work that needs doing in an industrialized, technological society. And anyway, the electric motor is easy to understand, once we understand the electric generator. The electric motor is just the electric generator driven backward. Even though true, that doesn't tell us much, so we want to look a little closer at the way such a motor works.

Suppose we have a loop sitting still in a magnetic field. Figure 1.21, used in the discussion of a generator, pictures the situation. But now suppose that the loop is part of an electric circuit that has a source of emf in it, a battery or a generator. As soon as we complete that circuit by closing a switch, current appears in the loop. According to Ampere's law, a current-carrying wire experiences a force in the presence of a magnetic field. Let's look at the forces on that loop.

The force on a current element $I\Delta\vec{s}$

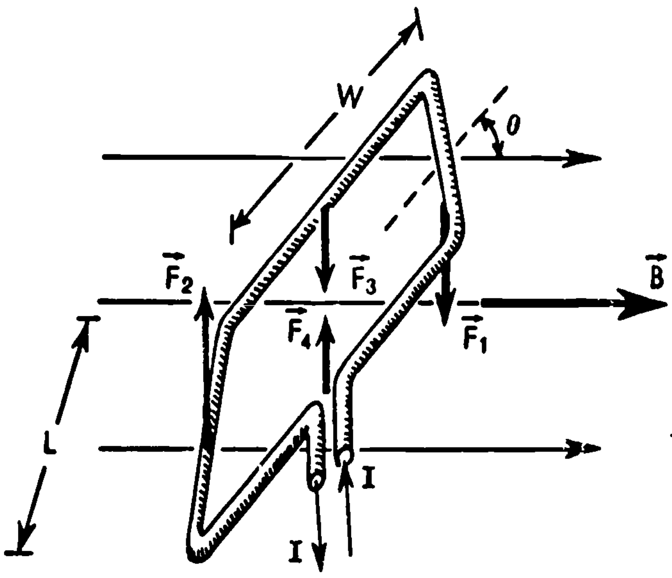



Fig. 1.25 Forces on a current-carrying loop in a uniform magnetic field.

is just $I\Delta\vec{s} \times \vec{B}$, so we need to sum the forces on all the elements in the loop. In Fig. 1.25, $\vec{F}_1$ is the force on one side of the loop, and its magnitude is just IBL. The force $\vec{F}_2$ on the opposite side has the same magnitude. Since these two forces are in opposite directions, their sum is zero.

The force $\vec{F}_3$ has the magnitude IBw sin $\theta$, and so does the force $\vec{F}_4$. Therefore $\vec{F}_3 + \vec{F}_4 = 0$, since they, too, are oppositely directed. We have ignored the little gap in side 4, but we can certainly make that gap as small as we want.

Then the net force on the loop is zero. But it certainly won't stay at rest. There is a net torque on the loop created by the forces $\vec{F}_1$ and $\vec{F}_2$. The magnitude of that torque is

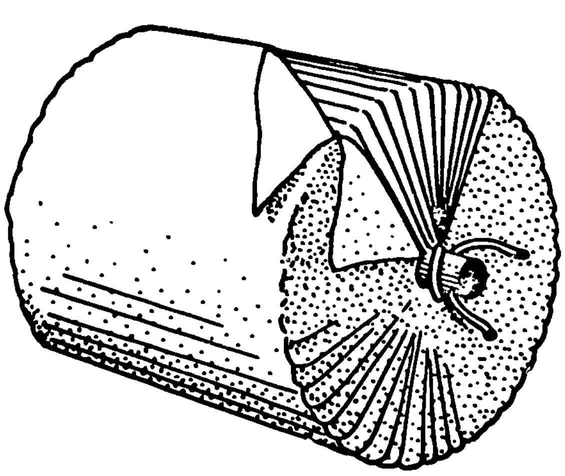


Fig. 1.26 A motor loop wound to give a nearly constant torque.

$$\tau = IBLw \cos\theta .$$

We can write this in vector form

$$\vec{\tau} = I\,\vec{A} \times \vec{B}, \qquad (1.27)$$

where $\vec{A}$ is perpendicular to the plane of the loop and has the magnitude Lw, the loop's area. We have also made use of the fact that the angle between $\vec{A}$ and $\vec{B}$ is $(\pi/2 + \theta)$ and that cos $\theta$ = sin $(\pi/2 + \theta)$.

Now we see that there is a torque on the loop, so that the loop will have an angular acceleration. If we connected a pulley to the loop, that torque would turn the pulley and lift a weight hanging from it. That is exactly what an electric motor does: It turns a shaft to which pulleys or other devices can be attached.

So it is true: An electric motor is just a generator operated backward. If we run a current through a loop that sits in a magnetic field, we get a torque on the loop. And we can use that torque to do work.

We have described a very rudimentary motor. It will operate better if we wind the loop as shown in Fig. 1.26. Then the torque is very nearly constant and it is always greater than the torque on a loop of a single turn. Most motors have more refinements, but they all operate on the principle that tells us that a magnetic field produces a force on a current-carrying wire.

Faraday's law determines the way a generator works and Ampere's law determines the way a motor works. It is all wrapped up in Fig. 1.27. Faraday's baby, now full grown, has changed the face of the planet.

## 1.7 THE BETATRON

Most of the practical applications of induced emf are instances of motional emf rather than examples of induced emf as the result of a time-varying magnetic field. That is because it is easier to move a loop in a controlled way than it is to change con-

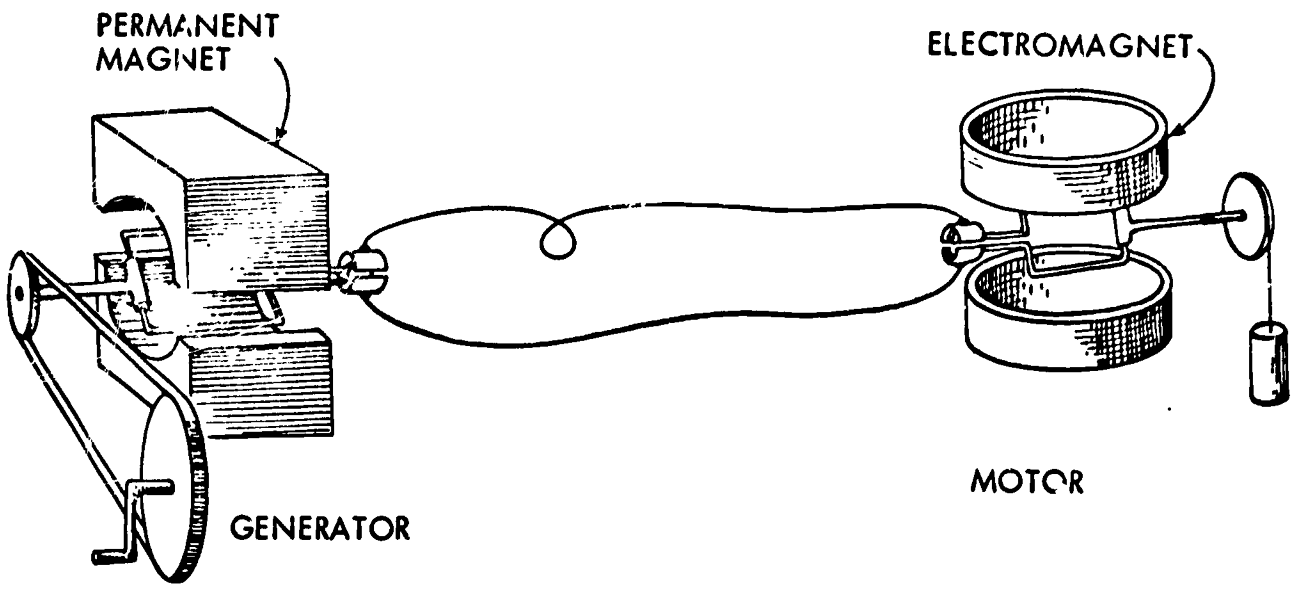



Fig. 1.27 Crank the handle to spin the generator loop to create an emf to send a current through the motor loop to establish a torque to lift the weight.

tinuously a magnetic field. But there are uses for the time-varying field approach, and one of the more striking is in the operation of a betatron.

The betatron is a machine designed to accelerate electrons to quite high velocities. Electrons are often called beta particles when they are not bound to atoms, thus the name betatron. All accelerators, including betatrons, have lots of electronic gear attached to them. There are many technical problems that must be solved before a betatron will perform satisfactorily, but we are going to ignore most of the problems and concentrate on the way a changing magnetic field is used to induce an electric field which in turn accelerates the electrons.

The heart of a betatron is a hollow toroidal affair usually made of ceramic. Figure 1.28 is a schematic picture of one of these. You can see why those in the business call this the doughnut. We pass over a lot of hard work and assume that we can evacuate the doughnut and that we have arranged to feed free electrons into it. Also, we somehow supply a magnetic field which is directed perpendicular to the doughnut's horizontal median plane at all points on that plane and which is cylindrically symmetric about the axis of the doughnut. Further, we arrange to have that field be variable in time.

Suppose we have an electron moving inside the doughnut in the median plane and in a circular orbit of radius r concentric with the axis of the doughnut, that radius being determined by the electron's velocity $\vec{v}$ and the magnetic induction field $\vec{B}_0$ at the electron's position. Now we suddenly increase the magnetic field, so that there is a $\Delta\Phi_B/\Delta t$ through the electron's orbit. We can use Faraday's law in empty space, as Maxwell suggested,

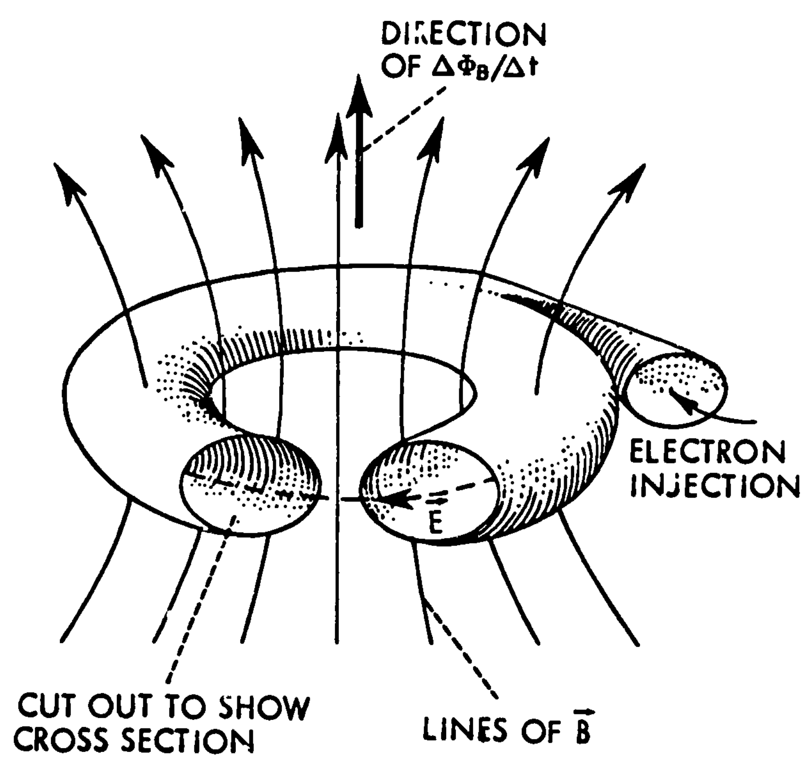



Fig. 1.28 The doughnut of a betatron.

so that if we take the circulation path to be the electron's orbit,

$$2\pi r E = -\frac{\Delta\Phi_B}{\Delta t}, \qquad (1.28)$$

since the magnetic induction field is cylindrically symmetric about the doughnut's axis. The induced electric field $\vec{E}$ at the orbit is tangent to the orbit, so that if we have the electron moving in the proper direction to begin with, the force $q\vec{E}$ will increase its speed. We can use Eq. (1.28) along with Newton's second law of motion to give

$$-eE = +\frac{e}{2\pi r}\frac{\Delta\Phi_B}{\Delta t} = \frac{\Delta(mv)}{\Delta t},$$

where $m\vec{v}$ is the electron's linear momentum, and $q = -e$ is the electron's charge. From the last two terms, we have

$$\Delta(mv) = \frac{e}{2\pi r}\Delta\Phi_B, \qquad (1.29)$$

where $\Delta(mv)$ is the change in the electron's momentum that occurs over the time it takes to change the flux by $\Delta\Phi_B$.

In getting to Eq. (1.29), we have assumed that the electron continues to move in the same circle of radius r as the flux increases. How can we arrange to accomplish that? Using Newton's law again and the fact that the electron experiences the centripetal force $-e(\vec{v} \times \vec{B})$, we get

$$evB_0 = mv^2/r,$$

where $v^2/r$ is the electron's centripetal acceleration, and $B_0$ is the magnetic induction field at the electron's position; i.e., at the circular orbit. From this, we get $mv = erB_0$ and thus

$$\Delta(mv) = er\Delta B_0, \qquad (1.30)$$

where $\Delta B_0$ is the change in the magnetic induction field at r that accompanies the change in the electron's momentum $\Delta(mv)$.

The momentum change in Eq. (1.29) is the same as that in Eq. (1.30), so it must be that

$$\Delta B_0 = \frac{1}{2}\frac{\Delta\Phi_B}{\pi r^2}. \qquad (1.31)$$

We arranged the terms in this way because we can say that

$$\Delta\Phi_B = \Delta(\pi r^2 B_{av}) = \pi r^2\,\Delta B_{av},$$

where $B_{av}$ is the average magnetic induction field that exists over the area of the circle. If we put this into Eq. (1.31), the $\pi r^2$ terms cancel and we are left with

$$\Delta B_0 = \tfrac{1}{2}\Delta B_{av}. \qquad (1.32)$$

Then the condition for keeping the electron in a circular orbit is simply that the change in $\vec{B}$ at the orbit must equal just one-half the change in the average $\vec{B}$ over the area ringed by that orbit. One way to accomplish that is to have $B_0$ itself equal to half of $B_{av}$ at all times, but there are certainly other ways.

We have not described how we go about assuring that the electrons do not wander away from the median plane, or how to synchronize the injection of electrons into orbit with the changing magnetic fields, nor do we intend to. The arts of designing, building, and operating particle accelerators are complex and mysterious for the uninitiated.

But we have seen how two very important ideas are used in the betatron. When there is a time-dependent magnetic field, then the circulation of $\vec{E}$ can be different from zero, even in empty space. A time-varying magnetic field induces an electric field, and that electric field is just what accelerates the electron in its orbit. Also, when a charged particle has a velocity in a magnetic field, then it experiences the force $q(\vec{v} \times \vec{B})$. And that force is just what keeps the electron in the circular orbit. In short, we need to use the full Lorentz force $q(\vec{E} + \vec{v} \times \vec{B})$ to describe the appropriate behavior of electrons in a betatron.

# 2 MODIFICATION OF AMPERE'S LAW

## 2.1 AMPERE'S CIRCULATION LAW FOR STEADY CURRENTS

Ampere's circulation law for steady currents is developed and explained in the monograph Magnetostatics. The law was applied only to cases in which the circulation path is entirely in empty space, and we shall continue that restriction here.

The law says that the circulation of the magnetic induction field $\vec{B}$ about any closed path is proportional to the current I passing through any surface bounded by that path. We can express this statement in mathematical terms:

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 I. \qquad (2.1)$$

The current I passing through a surface is just the flux of the current density $\vec{J}$ through that surface, i.e.,

$$I = \sum_{S} \vec{J} \cdot \Delta\vec{S}. \qquad (2.2)$$

If the circulation path over which we sum $\vec{B} \cdot \Delta\vec{s}$ does not encircle a current, then that circulation is zero. This does not imply that $\vec{B}$ itself is zero at all the points on such a path; it implies only that the positive and negative contributions to the circulation cancel exactly.

Figure 2.1 shows a few kinds of circulation paths and two sorts of surfaces bounded by circulation paths. These serve as reminders of the way Ampere's circulation law works. You should pay particular attention to the relationship between the sense in which the circulation path is traversed and the direction in which the current is considered positive. This convention is the same as the one we used in Faraday's law when connecting the sense of the circulation of $\vec{E}$ with the positive direction of $\Delta\Phi_B/\Delta t$. So far as Eqs. (2.1) and (2.2) are concerned, this is the convention relating the sense of $\Delta\vec{s}$ with positive direction of $\Delta\vec{S}$; the same one we used throughout Chapter 1 and which was illustrated in Fig. 1.2.

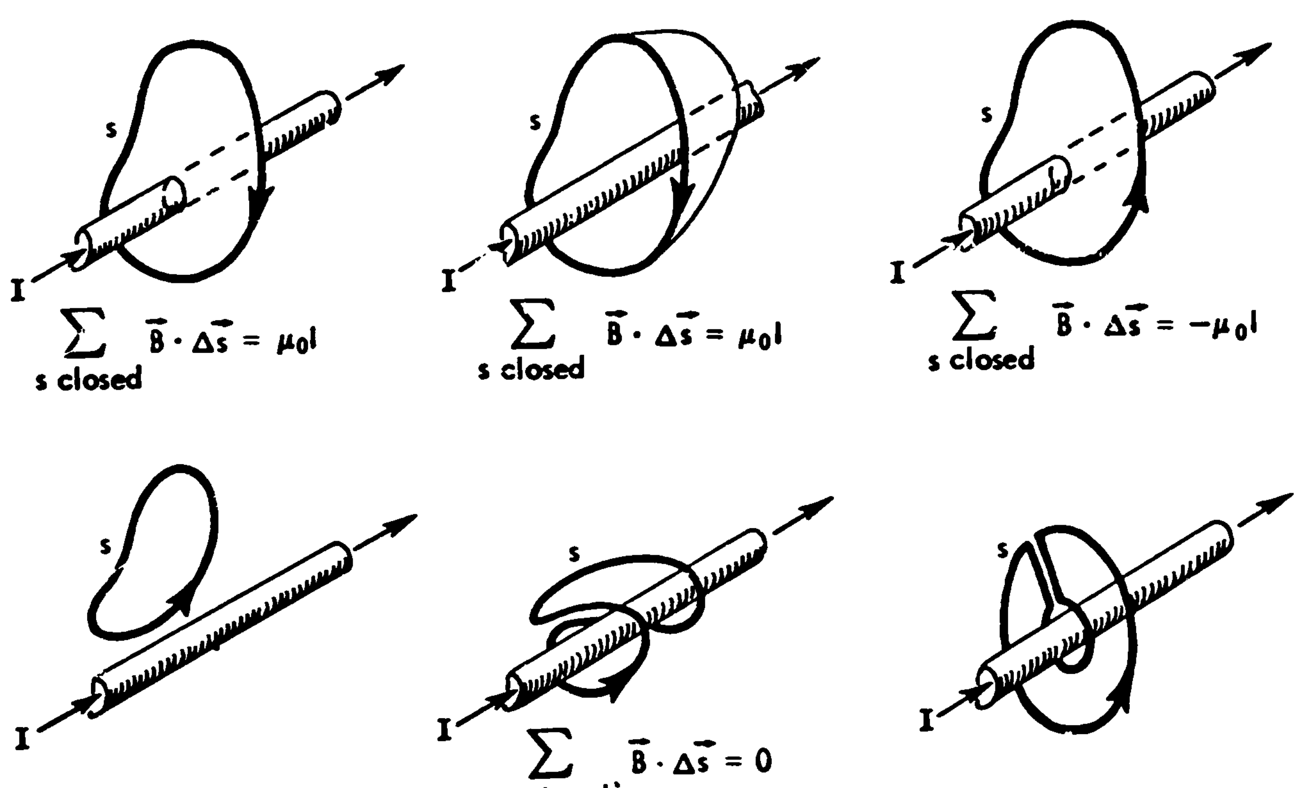



**Fig. 2.1 Reminders of the way Ampere's circulation law works.**

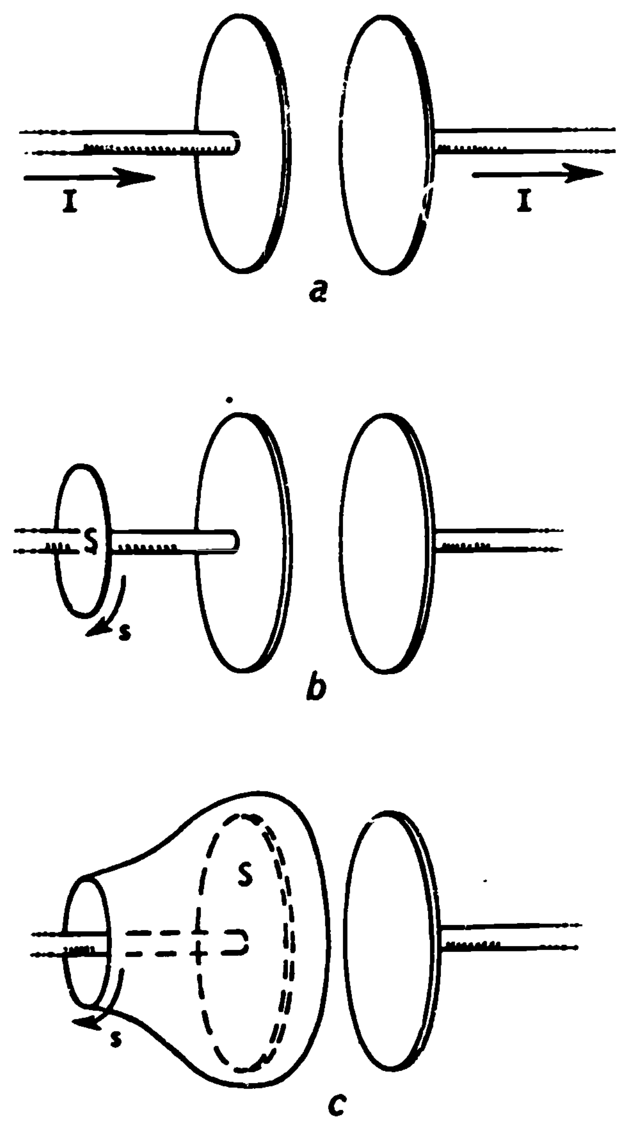



Fig. 2.2 A charging capacitor, a time-dependent situation.

There is a simple way to decide when a closed path is such that the circulation of $\vec{B}$ around it will be zero. While we understand that currents are not always in wires, assume for the moment that the current is in a wire as it is shown in Fig. 2.1. Imagine now that you have taken a length of string, laid it along the circulation path, and knotted the loose ends. Now, in your imagination, of course, if you could pull the string away from the wire, then the circulation of $\vec{B}$ around that path will be zero. In that case, we would say that the path, or the string, did not encircle the current. You might check this with the cases shown in Fig. 2.1.

From the discussions in the monograph Magnetostatics, we are convinced that Ampere's circulation law is valid for time-independent situations, i.e., when the currents are steady and the charge densities are constant. What about time-dependent situations? Will the law be valid when current or charge densities are changing? It turns out that the law is not applicable in such cases, at least not as it stands in Eq. (2.1). In the next section we are going to look at a rather common situation for which Ampere's circulation law certainly does not work.

## 2.2 A TIME-DEPENDENT SITUATION

As an example of a time-dependent situation, we are going to investigate the state of affairs depicted in Fig. 2.2(a). We suppose that we are charging a capacitor consisting of two conducting plates that are circular and parallel. The figure shows the long straight wires leading to these plates, and we assume that the rest of the circuit is so far away that it does not affect what goes on in the region we are investigating.

This is a time-dependent situation. We know that at least one thing is changing - the charge on the plates. We shall find a fundamental contradiction if we apply Ampere's law here. That contradiction will convince us that Ampere's circulation law is not generally applicable to time-dependent situations.

Let's see how the law behaves. Suppose we pick the circulation path s which is a circle of radius r and which is concentric with the current-carrying wire. That path is shown in Fig. 2.2(b). We can choose the plane surface bounded by s to be our surface S, and that is also shown in Fig. 2.2(b).

Since we have a nice symmetrical arrangement, we are quite sure that at any instant the magnitude of $\vec{B}$ is the same at every point on the circular path s. If we assume further that the angle between $\vec{B}$ and the circle's tangent is the same at every point, then we can calculate the circulation that appears on the left-hand side of Ampere's law. We get

$$\sum_{s \text{ closed}} \vec{B} \cdot \Delta\vec{s} = 2\pi r B_t, \qquad (2.3)$$

where $B_t$ is the component of $\vec{B}$ tangent to the circle, assumed to be the same everywhere on the circular path. We have some confidence in this result, although we should know what the currents are like in the capacitor plates before we give it too much credence. For our purposes here, the detailed correctness of Eq. (2.3) is not important. Anyone too disturbed by the many assumptions made in reaching this result can just substitute the average value $\bar{B}_t$ for $B_t$, and then the result will be true by definition.

We can calculate the right-hand side of Ampere's law too. Using the plane surface shown in Fig. 2.2(b), we get

$$\mu_0 \sum_S \vec{j} \cdot \Delta\vec{S} = \mu_0 I, \qquad (2.4)$$

where I is the current in the wire at the same instant at which we calculated the circulation to get Eq. (2.3).

If we were to equate the results in Eqs. (2.3) and (2.4), as we can when Ampere's law is correct, then we would get a result for $B_t$ that looks like what we got for an unbroken, infinitely long, straight current (see Monograph II, Magnetostatics). And we would expect that result to be reasonably good, particularly if the circulation path is far away from the capacitor or if the capacitor plates are small and close together.

Of course, Ampere's law tells us that we may choose any surface bounded by s, so we could have chosen the vase-like one shown in Fig. 2.2(c). That one is also bounded by s, but it passes between the capacitor plates. Using that S to calculate the right-hand side of Ampere's law, we get

$$\mu_0 \sum_S \vec{j} \cdot \Delta\vec{S} = 0, \qquad (2.5)$$

since no charge at all passes through that surface; i.e., the current density $\vec{j}$ is zero everywhere on that surface.

Now we can see the contradiction. We certainly cannot equate Eq. (2.3) to Eq. (2.4) and then turn around and equate Eq. (2.3) to Eq. (2.5) too. Since $\vec{B}$ cannot be both not zero and zero at the same time, we can expect trouble when we try to use Ampere's law in time-dependent circumstances. Of course, no one ever told us that we could use the law when there are time variations in currents or charge densities. We just wanted to try extending the range of application of the law, and we were not successful.

If we persist in trying to extend the applicability of Ampere's law, we shall need to be more careful - and thoughtful. Maybe we should first understand why the law does not work in time-dependent situations. That is what we look into in the following section.

## 2.3 CHARGE CONSERVATION AND AMPERE'S LAW

We are going to see why Ampere's circulation law fails in time-dependent situations. Once we understand that, we can investigate the possibilities of generalizing the law so that it will work. We begin by returning to a fundamental principle, the conservation of charge (see Monograph I, Electrostatics).

Consider a volume in space bounded by the closed surface S. If a net current passes through that surface, either into or out from the volume, then the net charge existing in the volume must change. And the magnitude of that change must be just equal to the magnitude of the net charge carried across that surface. This is simply another way of saying that we cannot create or annihilate net charge; we can just move charges around. And we can keep track of them.

The mathematical statement of the conservation of charge principle is

$$I = \sum_{S \text{ closed}} \vec{j} \cdot \Delta\vec{S} = -\Delta Q/\Delta t, \qquad (2.6)$$

where we have expressed the principle

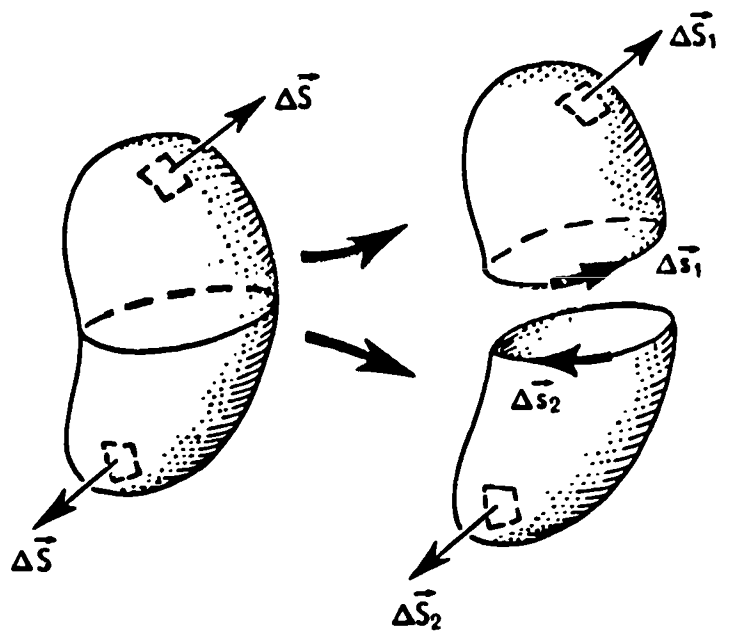



Fig. 2.3 Breaking a closed surface into two surfaces, each of which is bounded by a closed path.

in terms of the time rate of change of charge instead of just the change of charge. Here $\vec{j}$ is the current density on the surface S, and Q is the net charge in that volume enclosed by S. We need the minus sign in Eq. (2.6), because I is considered positive when the net conventional current is outward through the bounding surface.

For steady currents and constant-charge densities the quantity $\Delta Q/\Delta t$ is zero for any volume, for whatever currents pass into a volume must pass outward, too. If a current passed into a volume without leaving, then positive charge would accumulate in that volume. In that case, the charge density would be changing at some place in the volume, and that is contrary to the assumption that charge densities are constant; i.e., time-independent. Then the mathematical statement of the principle, or law, of charge conservation for the special case when everything is independent of time is just

$$\sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S} = 0. \qquad (2.7)$$

Now we must be careful. Equation (2.7) is not the right-hand side of Ampere's law given by combining Eqs. (2.1) and (2.2),

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 \sum_{S} \vec{j} \cdot \Delta\vec{S}. \qquad (2.8)$$

In Eq. (2.7) the surface S is closed, and it completely bounds a volume. In Eq. (2.8) the surface is, in general, not closed, but it is itself bounded by the closed path s. Nevertheless, charge conservation as given by Eq. (2.7) is contained in Ampere's law as given by Eq. (2.8). And that is just what assures us that Ampere's law cannot possibly work in time-dependent situations.

To support this last assertion we want to show that the special case of charge conservation, Eq. (2.7), is contained in Ampere's law, Eq. (2.8). Consider a closed surface S which in our mind's eye we imagine to be cut into the two surfaces $S_1$ and $S_2$, as shown in Fig. 2.3.

We can certainly say that the current passing out from the closed surface S is just that passing "out" through $S_1$ plus that passing "out" through $S_2$. (Why are the two "outs" in quotes?) We put this in mathematical form:

$$\sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S} = \sum_{S_1} \vec{j} \cdot \Delta\vec{S}_1 + \sum_{S_2} \vec{j} \cdot \Delta\vec{S}_2 .$$

There is not really any physics in this mathematical statement, but now we can apply Ampere's law to each of the terms on the right-hand side. If we do put physics into the equation this way, we get

$$\sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S} = \frac{1}{\mu_0}\left[\sum_{s_1\ \text{closed}} \vec{B} \cdot \Delta\vec{s}_1 + \sum_{s_2\ \text{closed}} \vec{B} \cdot \Delta\vec{s}_2\right].$$

To be sure that we are calculating the currents "out" from $S_1$ and $S_2$, we must make sure that we traverse the paths $s_1$ and $s_2$ in the right senses. You should check to see that $\Delta\vec{s}_1$ and $\Delta\vec{s}_2$ are correctly shown in Fig. 2.3.

But $s_1$ and $s_2$ are really just the same path. In the first circulation of $\vec{B}$ we are traversing that path in one sense, and in the second we are traversing the same path in the opposite sense. The second circulation then just cancels the first, for surely

$$\sum_{s_1 \text{ closed}} \vec{B} \cdot \Delta\vec{s}_2 = - \sum_{s_2 \text{ closed}} \vec{B} \cdot \Delta\vec{s}_1 .$$

With this we are left with the result that

$$\sum_{S \text{ closed}} \vec{j} \cdot \Delta\vec{S} = 0$$

if Ampere's law is valid. And this is just Eq. (2.7), the conservation of charge principle for the special case of time-independent currents and charge densities.

Well, then, we have proved what we set out to prove: namely, Ampere's law contains within it the statement of time-independent charge conservation. No wonder it doesn't work for time-dependent cases! In particular, we see why it led to a contradiction in the previous section when we tried to apply it to the charging capacitor: Equation (2.7) simply is not true for any surface that encloses a single plate during charge or discharge.

Now that we know at least one good reason that Ampere's circulation law works only for time-independent situations, we go back to the question of what we can do to generalize it - if we can do anything.

## 2.4 MODIFIED CIRCULATION LAW, DISPLACEMENT CURRENT.

Good experimenters do not simply rummage about in a random way, hoping by chance to fall upon new information about the way nature behaves. Faraday, for instance, had something in mind when he began his Researches. His early vague thoughts were more hunches than ideas, but he followed them doggedly. In Chapter 1, we traced Faraday's progress from his first halting steps through the inspired series of experiments that searched out nature's behavior, and finally to the time when he could formulate a new fundamental physical law. The process was: hunch, experiment, theory. That is certainly an oversimplification, but it describes in a rough way a process that has often led to new knowledge. But that is not the only process in the scientific enterprise.

Sometimes the order is turned around: hunch, theory, confirming experiment. That such an order often leads to advances in scientific understanding is a surprise to some people. They are usually the ones who think that "the scientific method" is the gathering of data until that data forces the recognition of an important order, or law. What we discuss next should serve to bury that narrow conception of what "the scientific method" is.

We are going to follow Maxwell's reasoning as we try to generalize Ampere's law so that it will be valid in time-dependent situations. We shall introduce no new experimental evidence. We shall not really deduce anything. We are going to use our imagination as we try to "fix up" the law, being careful to avoid introducing relationships we already know to be wrong.

Let's review the state of our understanding. We know that Ampere's law is valid for steady currents and constant-charge densities; i.e., for time-independent situations. Further, we know that Ampere's law contains the charge-conservation principle for time-independent situations, and that alone is enough to assure us that Ampere's law cannot be applicable when there are changing currents or charge densities present.

We might ask ourselves the following question: Since Ampere's law contains the special case of charge conservation, can it be that Ampere's law is itself just a special case of a general circulation law that contains the general principle of charge conservation?

If the arrow below means "implies that," then we can write that

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 \sum_{S} \vec{j} \cdot \Delta\vec{S} \rightarrow$$

$$\sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S} = 0, \qquad (2.9)$$

for time-independent cases. For our sought-after general circulation law, we might write that

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = ? \rightarrow \sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S}$$

$$+ \Delta Q/\Delta t = 0. \qquad (2.10)$$

We will have made some progress, if for the ? in Eq. (2.10) we can put the flux of something through a surface bounded by the circulation path. That would be nice, because if we can write the general circulation law as

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \sum_{S} \vec{C} \cdot \Delta\vec{S}, \qquad (2.11)$$

then, even though we don't yet know what $\vec{C}$ is, we are certain that

$$\sum_{S\ \text{closed}} \vec{C} \cdot \Delta\vec{S} = 0. \qquad (2.12)$$

That Eq. (2.12) follows from Eq.(2.11) can be shown by exactly the same argument we used to show that Eq. (2.7) follows from Eq. (2.8); that is, the same argument that let us put the "implies that" arrow in Eq. (2.9). That is a purely mathematical argument; there is no physics in it.

Now if we can choose $\vec{C}$ such that Eq. (2.12) is the general statement of charge conservation, then we will indeed be making progress. Namely, the general circulation law will imply the general statement of charge conservation. The trouble is that the general statement of charge conservation; i.e.,

$$\sum_{S\ \text{closed}} j \cdot \Delta S + \Delta Q/\Delta t = 0,$$

has not come to us in the form of Eq. (2.12). The first term is in the right form; it is a flux through a closed surface. But the second term is not. Can we cast that term in the form of a flux of something through a closed surface? We need a relationship between the net charge contained in a volume and the flux of something through the surface enclosing that volume. We do know of such a relationship: Gauss's law (see Monograph I, Electrostatics).

Gauss's law says that the flux of the electric field intensity $\vec{E}$ through a closed surface is proportional to the net charge encompassed by that surface. In mathematical terms,

$$\sum_{S\ \text{closed}} \vec{E} \cdot \Delta\vec{S} = Q/\varepsilon_0, \qquad (2.13)$$

where $\vec{E}$ is the electric-field intensity on the closed surface S, and Q is the net charge in the volume enclosed by S. Using Gauss's law, we get that

$$\frac{\Delta Q}{\Delta t} = \varepsilon_0 \frac{\Delta}{\Delta t}\left( \sum_{S\ \text{closed}} \vec{E} \cdot \Delta\vec{S} \right),$$

or, if the surface S is held fixed in space,

$$\frac{\Delta Q}{\Delta t} = \sum_{S\ \text{closed}} \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t} \cdot \Delta\vec{S}. \qquad (2.14)$$

If we make this substitution, we can write the general principle of charge conservation as

$$\sum_{S\ \text{closed}} \vec{j} \cdot \Delta\vec{S} + \sum_{S\ \text{closed}} \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t} \cdot \Delta\vec{S}$$

$$= \sum_{S\ \text{closed}} \left( \vec{j} + \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t} \right) \cdot \Delta\vec{S} = 0. \qquad (2.15)$$

We have been trying to get the general principle of charge conservation into the form of Eq. (2.12), and we have done it. The term inside the parentheses in Eq. (2.15) is just the $\vec{C}$ we have been after. Then following our lead in Eq. (2.11), we write the proposed general circulation law as

$$\sum_{S\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 \sum_{S} \left( \vec{j} + \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t} \right) \cdot \Delta\vec{S}. \qquad (2.16)$$

We have put a $\mu_0$ in front of the right-hand side of Eq. (2.16), so that the general law will satisfy another condition that we certainly must impose. We want the general circulation law to reduce to Ampere's law for time-independent situations, since Ampere's law is correct in those cases. And Eq. (2.16) now does just that; $\Delta\vec{E}/\Delta t$ being zero if everything is independent of time.

Well, we have fixed up the circulation law in such a way that it has some nice features:

(a) For time-independent situations; i.e., steady currents and constant-change densities, we get Ampere's law back again.

(b) The law has the general, as well as the special, statement of charge conservation built into it.

(c) In a case such as the charging capacitor, Fig. 2.2, the right-hand side of Eq. (2.16) is not zero, even for a surface that passes between the plates. And we get the same result for that right-hand side no matter what surface we choose, so long as the surface is bounded by the circulation path.

EXERCISE.

Can you prove that the second sentence in (c) above is true? If you cannot, you probably do not understand the conservation of charge principle as it is given by Eq. (2.6) or by Eq. (2.10).

In developing the general circulation law, we were concerned about certain criteria we knew must be satisfied, if such a law were to exist at all. And we built in the general conservation principle too. Once begun, everything went along surprisingly well. Something like knocking over the first in a line of toy soldiers. It is encouraging that all went so smoothly, but the fact is that we have been playing a game. Aside from the requirement that we not do violence to anything already known to be true, we set our own rules for the game. We still need to answer the important question: Is our proposed general circulation law true; i.e., does it check with experiment? This is the ultimate question asked of all theory, no matter how pretty that theory may seem. The wonderful thing about what we have done here is that the general law is true. It does describe the way nature works.

It is not at all clear why our procedure did lead to a physically valid result. We set out to save Ampere's law, but we had no truly guiding experimental results to lead us. Nor did Maxwell. Equation (2.16) is a statement about the physical world. It says that a certain arrangement of measurable things is invariably equal to another arrangement of some other independently measurable things. We did not deduce this relationship from known principles, and so we have no guarantee that the relationship is true. Nevertheless, it is true. But the experimental verification came after Maxwell had proposed it.

We have, of course, presented this development in an artificial way, not at all as its creator likely did it for the first time. Textbook writers are supposed to know where they are going, so we went right on a beeline from the posing of the question to the statement of the proposed answer. And we made nary a wrong turn along the way. Our development comes closer to what might appear in a scientific journal, and that sort of thing disguises or hides all questions, mental gymnastics, and false starts. Nobody reports in a journal how many reams of paper he threw away after running down hunches that did not work out. The report that goes into a journal is usually cleaned up, so that it appears logical and straightforward.

The way we went about getting the general circulation law is a distortion in another sense. Maxwell had in mind much more than just Ampere's law, although that law was vitally important to his entire scheme. He was trying to construct what we today call a field theory for all of electromagnetism. He was using the conceptualizations of Faraday, putting them in manageable mathematical form, extending and generalizing the known physical laws, and putting it all together into a concise and consistent whole. And the whole was indeed more than the sum of its parts. We shall see Maxwell's full theory all in one place in Chapter 4.

In Chapter 3 we shall see how Maxwell's extension of Ampere's law is verified. We shall see that the term $\varepsilon_0 \, \Delta\vec{E}/\Delta t$ that we added to the regular conduction current density $\vec{j}$ turns out to be necessary for the description of electromagnetic fields. Maxwell called that added term the displacement current. That is probably not very appropriate today, but its root lies in Maxwell's model around which he built much of his theory. The model has lost its cogency, but the label continues.

# 3 PROPAGATION OF AN ELECTROMAGNETIC DISTURBANCE

## 3.1 THE CIRCULATION LAWS IN EMPTY SPACE.

We now have two circulation laws. The first, Faraday's law, tells us that the emf around any closed path is equal to the negative of the time rate of change of the magnetic flux through any surface bounded by that path.

$$\mathcal{E} \equiv \sum_{s \text{ closed}} \frac{\vec{F}}{q} \cdot \Delta s = -\frac{\Delta\Phi_B}{\Delta t}$$

$$= -\frac{\Delta}{\Delta t} \sum_{S} \vec{B} \cdot \Delta\vec{S}. \qquad (3.1)$$

The second, Ampere's law as modified by Maxwell, tells us that the circulation of $\vec{B}$ around any closed path is proportional to the sum of two fluxes through any surface bounded by that path: the conduction-current density and the displacement-current density.

$$\sum_{s \text{ closed}} \vec{B} \cdot \Delta s = \mu_0 \sum_{S} \left(\vec{j} + \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t}\right) \cdot \Delta\vec{S}. \qquad (3.2)$$

As written here, Eq. (3.2) already assumes that the circulation path s is being held fixed in space. That is not the case for Eq. (3.1), but we shall be dealing only with those circulation paths which are stationary. Further, Eq. (3.2) is written for a circulation path that is in empty space, i.e., everywhere outside material media. That is not true for Eq. (3.1), for which the circulation path can be within a material loop. But Eq. (3.1) is certainly valid when the circulation path is entirely in empty space, and we shall be considering only such paths.

Suppose now that we consider circulation paths which are stationary in empty space, and suppose further that in the region we are investigating there are no charge densities and no conduction currents. If a circulation path is stationary, then $\vec{F}/q = \vec{E}$ in Eq. (3.1), and we can write

$$\sum_{s \text{ closed}} \vec{E} \cdot \Delta\vec{s} = -\frac{\Delta\Phi_B}{\Delta t} = -\frac{\Delta}{\Delta t} \sum_{S} \vec{B} \cdot \Delta\vec{S}. \qquad (3.3)$$

and if there are no conduction currents around, then $\vec{j} = 0$ in Eq. (3.2), and we can write

$$\sum_{s \text{ closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 \varepsilon_0 \frac{\Delta\Phi_E}{\Delta t}$$

$$= \mu_0 \varepsilon_0 \frac{\Delta}{\Delta t} \sum_{S} \vec{E} \cdot \Delta\vec{S}. \qquad (3.4)$$

Equations (3.3) and (3.4) show the intimate relationships that exist between electric and magnetic fields in time-dependent circumstances. In such cases, we no longer have two separate areas of interest: electric fields and magnetic fields. When things are changing it is difficult and often misleading to think of the two fields independently. Faraday's law tells us that there is always an electric field associated with a time-dependent magnetic field, even in the absence of charges. The Maxwell-Ampere law tells us that there is always a magnetic field associated with a time-dependent electric field, even in the absence of conduction currents. Equations (3.3) and (3.4) tell us the interdependence of $\vec{E}$ and $\vec{B}$ and allow us to keep them both in mind at once. When we do that, we say that we are talking about an electromagnetic field, the description of which requires the descriptions of both $\vec{E}$ and $\vec{B}$.

In Chapter 1, when we discussed Faraday's law, we sometimes said that a changing magnetic flux creates or brings into being an electric field. In Chapter 2, we thought of a changing electric field as being a cause of a magnetic field. As we now look at Eqs. (3.3) and (3.4), we see that this kind of thinking does not make sense. The laws do not tell us which causes which, they only tell us that in time-dependent situations we get both an $\vec{E}$ and a $\vec{B}$. We know only that there are two associated field vectors: $\vec{E}$ and $\vec{B}$. It makes no sense to say that a changing $\vec{B}$ creates an $\vec{E}$, which if it changes creates a $\vec{B}$. We would need a time sequence of events, if we were to think in this way. And the laws provide no such sequence.

Nor do Eqs. (3.3) and (3.4) tell us about a mechanism through which the two fields sense each other's existence. This seeming deficiency has caused conceptual difficulties for very able physicists. We shall discuss that problem later on when it is more appropriate. First we are going to investigate a very remarkable consequence of the interdependence displayed in Eqs. (3.3) and (3.4).

## 3.2 THE VELOCITY OF PROPAGATION.

Maxwell came to a startling conclusion when he combined the physics contained in our Eqs. (3.3) and (3.4). He convinced himself that light is an electromagnetic disturbance - that light consists of a varying electric field and an associated magnetic field directed perpendicular to each other and both perpendicular to the direction in which light travels.

We are going to develop the evidence that was most convincing to Maxwell, although we shall not do it in quite the way he did. We want to show that a disturbance in an electromagnetic field travels in empty space with the speed of light. We are going to use the two circulation laws as written for empty space.

We direct our attention to an empty region of space. There are no material objects there and thus no charges and no currents. We imagine that there is a changing field some-

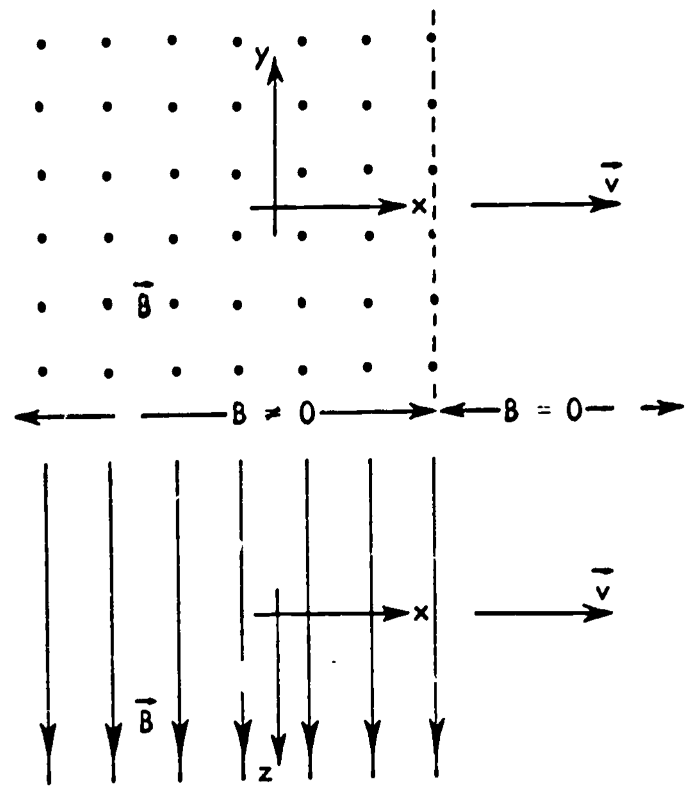



Fig. 3.1 The uniform field $\vec{B}$ is to the left of the moving plane and points in the direction of increasing z.

where in that region, and we begin with a particular kind, suiting our purpose. Suppose we have a magnetic field such that $\vec{B}$, pointing in the +z direction, is uniform everywhere on one side of a very large plane surface parallel to the yz plane and that $\vec{B}$ is zero everywhere on the other side of the surface. We introduce the changing character of the field by letting this plane travel in the +x direction with the speed v. Then, at every instant, the magnetic field is changing at every point on the moving plane. Figure 3.1 shows this arrangement.

We are assuming that this disturbance is traveling along unabated. We do not want to say anything about how it was produced. At this point in our study we have not yet convinced ourselves that such a disturbance can be produced. We just imagine that the disturbance, as described, exists and that we have no knowledge of its distant history. We know what it is doing now, and we assume that it will continue in the same way. And we ask what the characteristics of such a disturbance would be.

We first use the circulation law given by Eq. (3.3). We select the stationary rectangular circulation path in the xy plane as shown in Fig. 3.2. The left side of the rectangle is inside the region where $\vec{B} \neq 0$, i.e., to the left of the moving plane. Figure 3.2(a) shows the location of the moving plane at the time t, and Fig. 3.2(b) shows it at the time t + Δt. We have made certain that the rectangle's length L is large enough, so that the moving plane will not have reached the right side of the rectangle by the time t + Δt.

We can calculate the change in the the magnetic flux through a surface bounded by this rectangular path. In the time interval Δt the change in that flux is

$$\Delta\Phi_B = Bwv\ \Delta t,$$

so that the time rate of change of the magnetic flux is

$$\frac{\Delta\Phi_B}{\Delta t} = Bwv. \qquad (3.5)$$

We shall call this quantity positive when the flux is increasing in time in the +z direction, i.e., when its increase is as shown in Fig. 3.2.

Now Faraday's law, Eq. (3.3), says that the circulation of $\vec{E}$ around that same rectangular path must equal the negative of $\Delta\Phi_B/\Delta t$. Negative, that is, when we traverse the rectangle in the counterclockwise sense as seen in Fig. 3.2. Then there must be an electric field intensity $\vec{E}$ somewhere. Where is it, what is its magnitude, and in what direction is it pointing?

As we calculate the circulation of $\vec{E}$ around the rectangular path, we go along the lower side in one direction parallel to the x axis and along the upper side in the opposite direction. These two contributions surely cancel each other, since the field $\vec{E}$ at each point on one of these sides must be equal to $\vec{E}$ at the corresponding point on the other. And we can put the right side of the rectangle so far away from the moving plane that we are

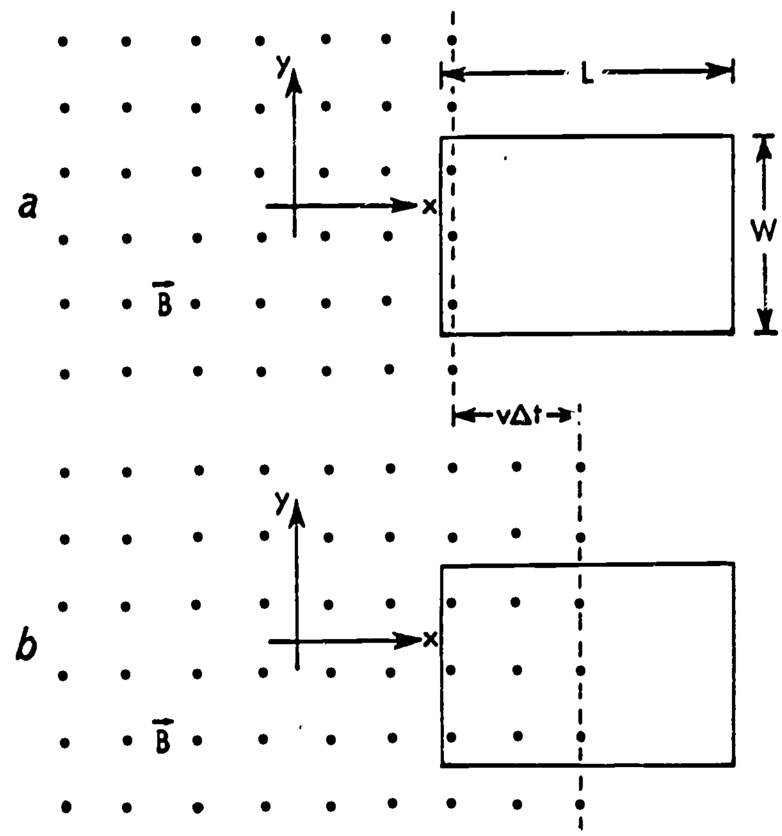



Fig. 3.2 (a) The location of the moving plane at the time t. (b) The location of the moving plane at the time t + Δt.

certain that no fields exist there, and, thus, there can be no contribution to the circulation along that side. Then only the left side remains to provide a contribution to the circulation of $\vec{E}$, and it must carry the full burden.

The contribution to the circulation along this left-hand side must be −Ew, as we see when we recall that we are going in a counterclockwise sense around the path. That is, for Faraday's law to be correct, $\vec{E}$ must be in the +y direction at all points on the left side of the rectangular circulation path (check this with Lenz's law). Equating the circulation of $\vec{E}$, i.e., −Ew, with the negative of Eq. (3.5), we get for the magnitude of $\vec{E}$ along the left side of the rectangle

$$E = vB. \qquad (3.6)$$

We have assumed that the only component of $\vec{E}$ along the left side of the rectangle is the y component. That is the only component that contributes to the circulation. But when we are

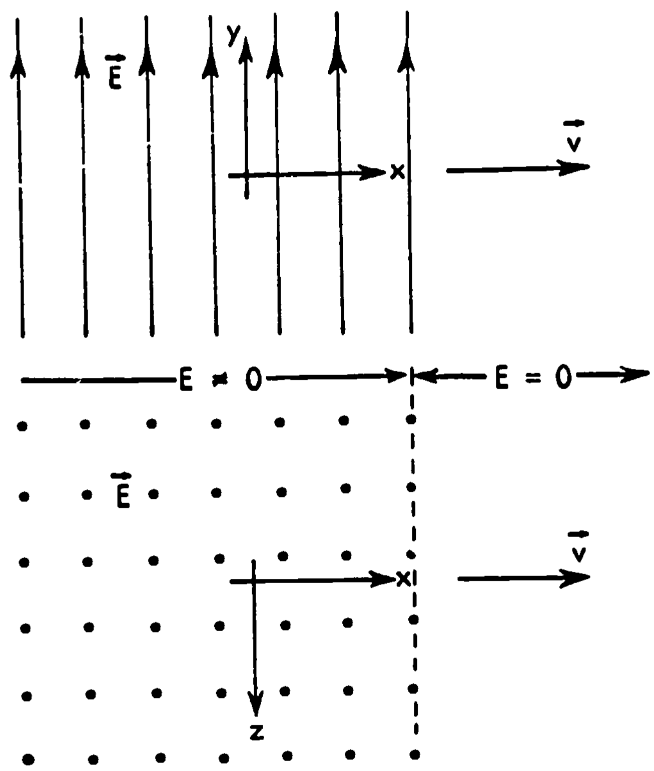



Fig. 3.3 The uniform field $\vec{E}$ is to the left of the moving plane and points in the direction of increasing y.

finished with this development, it will be clear that $\vec{E}$ has only a y component there.

We can move the left side of the rectangular circulation path anywhere to the left of the moving plane without changing the argument that leads to Eq. (3.6). Then there must be an electric field $\vec{E}$ at each point to the left of that moving plane. And the magnitude of that $\vec{E}$ must everywhere be just vB, and $\vec{E}$ must everywhere point in the +y direction. Then corresponding to the magnetic field shown in Fig. 3.1, there must be an electric field as shown in Fig. 3.3.

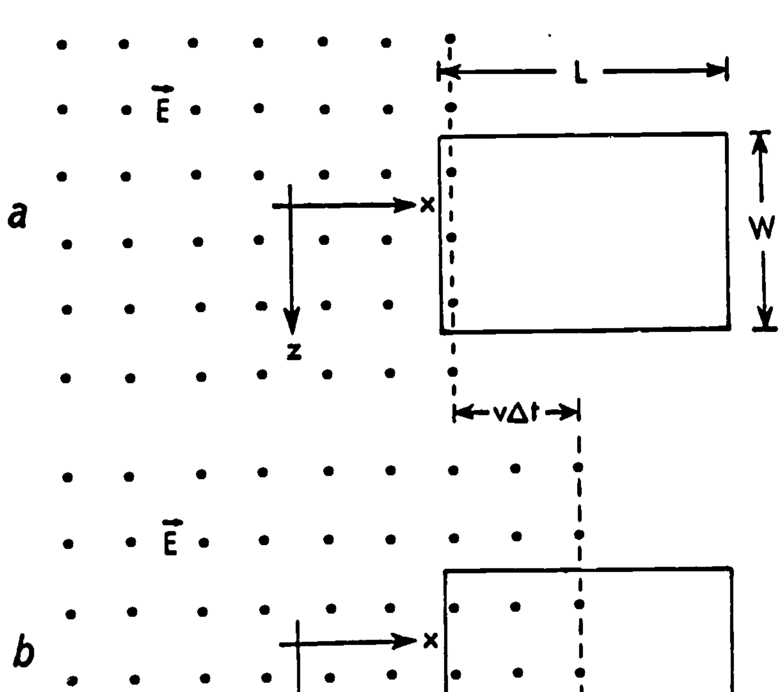



Fig. 3.4 (a) The location of the moving plane at the time t. (b) The location of the moving plane at the time t + Δt.

EXERCISE

Show that there can be no y component to any electric field intensity that exists at any point to the right of the moving plane.

Of course we can use the circulation law Eq. (3.4), too. We now choose the rectangular circulation path in the zx plane as shown in Fig. 3.4, the left side of which is inside the region where $E \neq 0$. In the time interval $\Delta t$ the change in electric flux is

$$\Delta\Phi_E = Ewv\ \Delta t,$$

so that the time rate of change of the electric flux is

$$\frac{\Delta\Phi_E}{\Delta t} = Ewv. \qquad (3.7)$$

We shall call this quantity positive when the flux is increasing in the +y direction, i.e., when its increase is shown in Fig. 3.4.

According to Eq. (3.4), the circulation of $\vec{B}$ around the rectangular path should equal $\varepsilon_0\mu_0$ times $\Delta\Phi_E/\Delta t$, when we traverse the path in a counterclockwise sense as seen in Fig. 3.4. As before, we get a contribution to the circulation only along the left side of the rectangle. If we let $\vec{B}_1$ be the magnetic induction field there, then the circulation gives us just $B_1 w$ where $\vec{B}_1$ points in the +z direction.

(Check this direction using the convention relating the sense of the circulation and the direction of positive flux.) Equating this circulation to $\varepsilon_0 \mu_0$ times Eq. (3.7), we get that

$$E = \frac{1}{\varepsilon_0 \mu_0 v} B_i. \quad (3.8)$$

Let's review what we have found up to this point. We used Faraday's circulation law on the originally given $\vec{B}$ field, and we found an induced electric field $\vec{E}$ that accompanies $\vec{B}$ everywhere behind the advancing plane. Further, $\vec{E}$ and $\vec{B}$ are perpendicular to each other in such directions that $\vec{E} \times \vec{B}$ is in the direction of the velocity $\vec{v}$. Equation (3.6) gives the relations between the magnitudes of $\vec{E}$ and $\vec{B}$.

Next, we used the Maxwell-Ampere circulation law on the electric field $\vec{E}$, and we found that an induced magnetic induction field $\vec{B}_i$ accompanies $\vec{E}$ everywhere behind the advancing plane. Further, $\vec{B}_i$ is in the same direction as the original $\vec{B}$, so that $\vec{E} \times \vec{B}_i$ is in the direction of the velocity $\vec{v}$. Equation (3.8) gives the relation between the magnitudes of $\vec{E}$ and $\vec{B}_i$.

To see what this means, we must remember an important point about the circulation laws, i.e., about Eq. (3.3) and (3.4). If we have time-varying electric and magnetic fields, then the field vectors $\vec{E}$ and $\vec{B}$ must satisfy both equations. Then the $\vec{B}_i$ we have been talking about in connection with the Ampere-Maxwell law cannot be some new magnetic field. It must be the same one we had when we used Faraday's law. That is, vectors $\vec{E}$ and $\vec{B}$ must satisfy Eqs. (3.3) and (3.4) simultaneously. Not only is the $\vec{B}$ in Eq. (3.3) equal to the $\vec{B}$ in Eq. (3.4), the $\vec{B}$ in (3.3) is the $\vec{B}$ in (3.4).[5]

How can we build in the fact that $\vec{B}_i = \vec{B}$? Well, $\vec{B}_i$ and $\vec{B}$ are in the same direction, so we just need to arrange things so that their magnitudes are the same. That means that the $B_i$ in Eq. (3.8) must be made to equal the B in Eq. (3.6), which is possible only if

$$v^2 = 1/\varepsilon_0 \mu_0. \quad (3.9)$$

That seems strange. The advancing plane cannot have just any velocity, it must have a particular velocity. We can find out what that v is by using the known values for $\varepsilon_0$ and $\mu_0$ (see Monographs I and II, Electrostatics and Magnetostatics). You may remember them in the forms

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ newton-m}^2/\text{coulomb}^2,$$

$$\frac{\mu_0}{4\pi} = 10^{-7} \text{ newton/amp}^2.$$

A little rearranging gives the result $1/\varepsilon_0 \mu_0 = 9 \times 10^{16}$ m²/sec², or

$$v = (\varepsilon_0 \mu_0)^{-1/2} = 3 \times 10^8 \text{ m/sec}. \quad (3.10)$$

But this is not just an interesting velocity, it is the velocity of light![6] Who can imagine the thoughts flashing through Maxwell's mind when this came to him?

We should be careful about what we have and have not done. We have shown that if there is a self-sustaining electromagnetic disturbance that travels through empty space, then the velocity of propagation of that disturbance is c, the velocity of light in empty space. We have not explained how to create such a disturbance. We have not shown how the disturbance is related to sources, i.e., what the sources must do to create this kind of disturbance.

[5] In fact, we could have begun with the electric field $\vec{E}$ shown in Fig. 3.3 instead of with the magnetic field. We would have used the Maxwell-Ampere circulation law on that $\vec{E}$ field, and then we would have used Faraday's circulation law on the associated magnetic field. We would have ended with the same relationships.

[6] In vacuum, which is fine since $\varepsilon_0$ and $\mu_0$ are supposed to be determined for vacuum, too. Actually, $c \approx 2.998 \times 10^8$ m/sec.

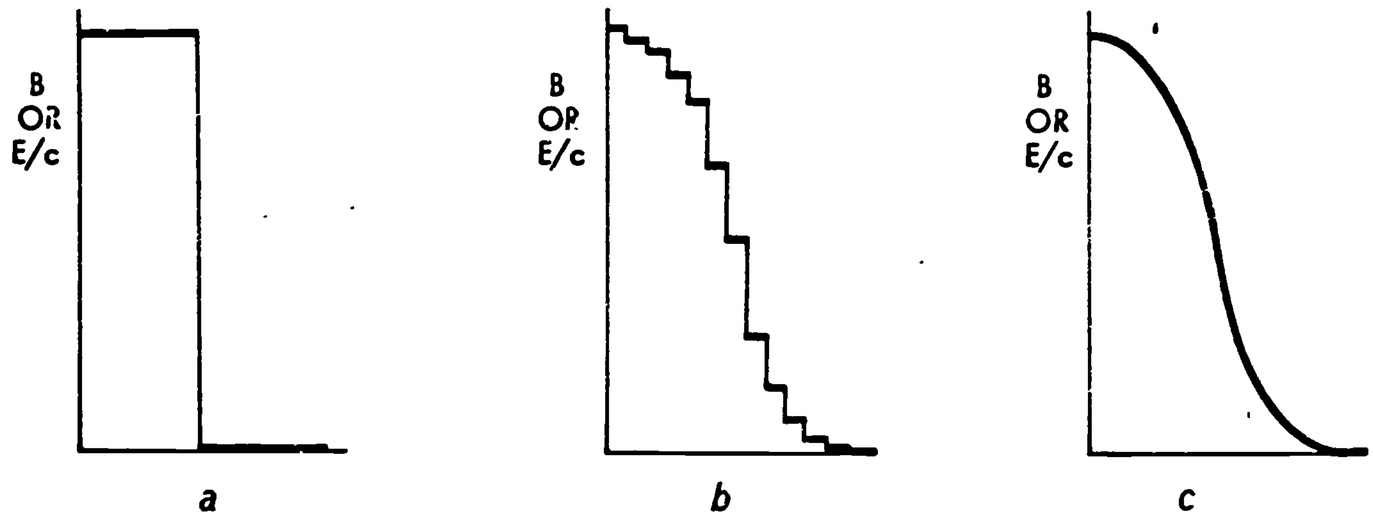



Fig. 3.5 Spatial profiles of electromagnetic disturbances: (a) a single abrupt, discontinuous rise; (b) a series of smaller discontinuous rises; and (c) a continuous rise.

This is the evidence Maxwell considered most convincing. He reported his conclusion in confident, though guarded, language, "This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations, if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."[7]

Today there is no doubt; light is electromagnetic in character. And by "light" we mean the entire electromagnetic spectrum: radio waves, microwaves, infrared radiation, visible light, ultraviolet light, rays and $\gamma$ rays. The evidence is now overwhelming. Each of these is an electromagnetic disturbance, the frequency increasing in the order in which they are listed. But conclusive experimental evidence did not come until more than twenty years after Maxwell's work (see section 3.3).

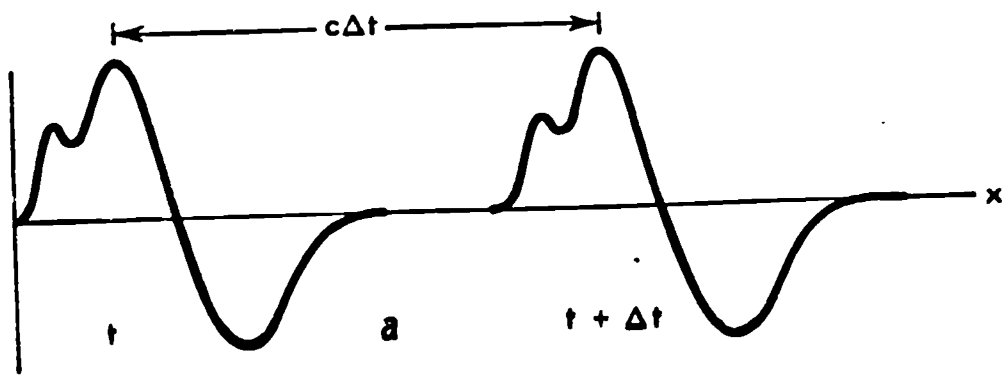
We chose a particular kind of disturbance for our development, but the conclusions are really independent of that choice. We have taken an extreme case in which the field goes from zero to a finite value abruptly at the moving plane front, Fig. 3.5(a). We could have chosen to build up the field in space at any instant by a set of smaller steps, Fig. 3.5(b). These steps need not be very widely separated in space, because we can always choose the length L of the rectangular circulation paths to be shorter still. Of course, the corresponding $\Delta t$ would need to be smaller, too. In any case, we could treat each boundary between successive steps just as we treated the single one before.

And while we are not yet prepared to give a rigorous proof, it is true that the results are the same when the field has any continuously changing profile in space, Fig. 3.5(c).

That is, no matter what the shape of the spatial profile of the electromagnetic disturbance, at every point $E = cB$, and the disturbance travels through empty space while maintaining its shape (see Fig. 3.6). All the traveling disturbances we have described are called plane waves, each point in the disturbance traveling in the same direction. We shall restrict ourselves to plane waves in this monograph, but you can likely figure out for yourself some of the characteristics of cylindrical and spherical waves.

We are now in position to see the

[7] Maxwell's paper, "A Dynamical Theory of the Electromagnetic Field," Philosophical Transactions, Vol. 155, 1865. The quote is taken from A Source Book in Physics by William Francis Magie, McGraw-Hill Book Co., New York, 1935.



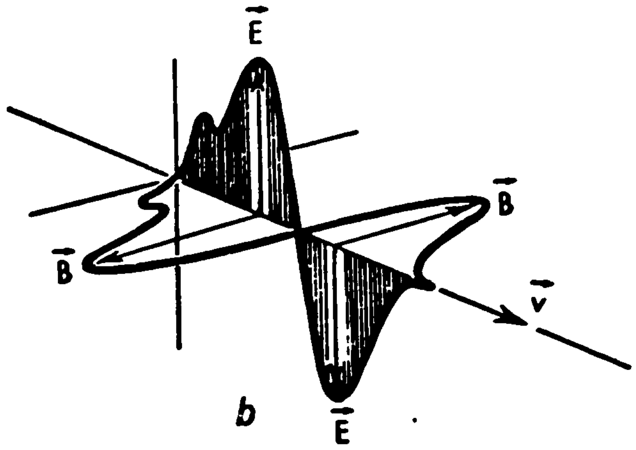




Fig. 3.6 The characteristics of a plane electromagnetic wave in empty space. (a) The spatial profile is unchanged as it travels with the speed c. (b) $\vec{E}$ and $\vec{B}$ are perpendicular, and at each point $\vec{E} \times \vec{B}$ is in the direction of the wave's velocity. At each point E = cB.

consequences of what Maxwell did to Ampere's circulation law (see Chapter 2). He not only saved the law for time-dependent situations, he predicted the character of electromagnetic radiation. The displacement current $\mu_0 \varepsilon_0 (\Delta\vec{E}/\Delta t)$ that he added, see Eqs. (3.2) and (3.4), is just what makes that prediction possible. This is surely the strongest evidence we have for Maxwell's modification of Ampere's law.

Further, we begin to see how the entire theory fits together. We now have four independent statements about electromagnetic fields in empty space:

$$\sum_{S\ \text{closed}} \vec{E} \cdot \Delta\vec{S} = Q/\varepsilon_0, \qquad (a)$$

$$\sum_{S\ \text{closed}} \vec{B} \cdot \Delta\vec{S} = 0, \qquad (b)$$

$$\sum_{s\ \text{closed}} \vec{E} \cdot \Delta\vec{s} = -\sum_{S} \frac{\Delta\vec{B}}{\Delta t} \cdot \Delta\vec{S}, \qquad (c)$$

$$\sum_{s\ \text{closed}} \vec{B} \cdot \Delta\vec{s} = \mu_0 \sum_{S} \left(\vec{J} + \varepsilon_0 \frac{\Delta\vec{E}}{\Delta t}\right) \cdot \Delta\vec{S}, \qquad (d)(3.11)$$

where the circulation paths are assumed to be at rest in the frame in which $\vec{E}$ and $\vec{B}$ are measured. These four equations along with the Lorentz force law contain all of electromagnetic theory. You should be able to say, in words, what each of these tells us about the way nature behaves. And you should be able to explain the physical basis for each and give an example or two of situations that each describes. Further, you should now see that, taken together, these equations tell us something that they did not when taken one at a time.

These are the famous Maxwell equations, here written for the case when the field point is in empty space. You will meet them again in Chapter 4 [not yet completed]. At that time we shall be able to use a bit more sophisticated mathematics to help us, but we already have all the physical concepts we need.

Before leaving this section, we should make one further point. Starting with the monograph Electrostatics, following with Magnetostatics, and now in this monograph, we have been proceeding as if the two constants $\varepsilon_0$ and $\mu_0$ were independent of each other and were, thus, independently defined and measured. But if we take the point of view, which we now do, that c is a universal constant, then we have just one of these constants to define. The choice is to define $\mu_0$ as being $4\pi \times 10^{-7}$ newton/ampere$^2$ exactly. Then $\varepsilon_0$ is no longer an independently defined quantity; it is given by $(c^2\mu_0)^{-1}$.