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ABSTRACT

This publication is a collection of articles which are related to elementary school mathematics. The authors note that the book was written primarily for elementary school principals, mathematics specialists and supervisors, and elementary school teachers. The articles discuss such topics as major issues in mathematics education, instructional procedures for teaching mathematics, inservice programs, and future directions for elementary school mathematics. (FL)

TEACHING MATHEMATICS ELEMENTARY SCHOOL

what's needed? what's happening?

NATIONAL ASSOCIATION OF ELEMENTARY SCHOOL PRINCIPALS, NEA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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INTRODUCTION

We are past the days when "the new math" was a new term—a new term that produced a wide range of reactions from administrators, from parents, and from children. Initially, resistance came from people who sincerely believed that no significant improvements in instruction and in learning would come from the changes that were being explored. Resistance came also from people who felt inadequate to cope effectively with the changes that would be required of them. And it is fair to say that in this latter category there were many who were not anti-new math but who recognized the importance of making adequate preparation before introducing a new program. Support came from the many people who were excited by the possibilities they could see for stimulating the interest of children and for improving the quality of the teaching-learning process.

In this publication, *Teaching Elementary School Mathematics—What's Needed? What's Happening?* you will find few references to the new math, but you will find a continuing concern for a good mathematics program for children, and you will find a diversity of articles dealing with many factors that influence the effectiveness of the elementary school mathematics program.

The book was planned with the elementary school principal very much in mind. What are his needs? What are his responsibilities? What kind of help can we provide for him? But the publication should be equally helpful to the elementary mathematics specialist or supervisor, and it contains much that can be helpful to teachers. It also contains some things that may well be of interest to parents. For the opening article, we have reprinted an essay written many years ago by Stephen B. Leacock, an economist and a political scientist, with a good sense of humor.

Teaching Elementary School Mathematics is a publication that has been made possible by the cooperative efforts of the National Association of Elementary School Principals, NEA, and the National Council of Teachers of Mathematics. Working together we have produced a book which we hope will contribute to the increased effectiveness of the mathematics program in the elementary school.

National Association of Elementary School Principals
National Council of Teachers of Mathematics

A, B, AND C
THE HUMAN ELEMENT IN MATHEMATICS

Stephen B. Leacock

The student of arithmetic who has mastered the first four rules of his art and successfully striven with money sums and fractions finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and, though betraying a strong family resemblance, they are not without a certain element of romance.

The characters in the plot of a problem are three people called A, B, and C. The form of the question is generally of this sort:

"A, B, and C do a certain piece of work. A can do as much work in one hour as B in two, or C in four. Find how long they work at it."

Or thus: "A, B, and C are employed to dig a ditch. A can dig as much in one hour as B can dig in two, and B can dig twice as fast as C. Find how long. . . ."

Or after this wise: "A lays a wager that he can walk faster than B or C. A can walk half as fast again as B, and C is only an indifferent walker. Find how far. . . ."

The occupations of A, B, and C are many and varied. In the older arithmetic they contented themselves with doing a "certain piece of work." This statement of the case, however, was found too sly and mysterious or possibly lacking in romantic charm. It became the fashion to define the job more clearly and to set them at walking matches, ditch-digging, regattas, and piling cordwood. At times, they

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became commercial and entered into partnership, having, with their old mystery, a "certain" capital. Above all they revel in motion. When they tire of walking matches, A rides on horseback or borrows a bicycle and competes with his weaker-minded associates on foot. Now they race on locomotives; now they row; or again they become historical and engage stagecoaches; or at times they are aquatic and swim. If their occupation is actual work, they prefer to pump water into cisterns, two of which leak through holes in the bottom and one of which is water-tight. A, of course, has the good one; he also takes the bicycle, and the best locomotive, and the right of swimming with the current. Whatever they do, they put money on it, all three being sports. A always wins.

In the early chapters of the arithmetic, their identity is concealed under the names of John, William, and Henry, and they wrangle over the division of marbles. In algebra they are often called X, Y, Z. But these are only their Christian names and they are really the same people.

Now to one who has followed the history of these men through countless pages of problems, watched them in their leisure hours dallying with cordwood, and seen their panting sides heave in the full frenzy of filling a cistern with a leak in it, they become something more than mere symbols. They appear as creatures of flesh and blood, living men with their own passions, ambitions, and aspirations like the rest of us.

A is full-blooded, hot-headed, and strong-willed. It is he who proposes everything, challenges B to work, makes the bets, and bends the others to his will. He is a man of great physical strength and phenomenal endurance. He has been known to walk forty-eight hours at a stretch and to pump ninety-six. His life is arduous and full of peril. A mistake in the working of a sum may keep him digging a fortnight without sleep. A repeating decimal in the answer might kill him.

B is a quiet, easy-going fellow, afraid of A and bullied by him, but very gentle and brotherly to little C, the weakling. He is quite in A's power, having lost all his money in bets.

Poor C is an undersized, frail man, with a plaintive face. Constant walking, digging, and pumping has broken his health and ruined his nervous system. His joyless life has driven him to drink and smoke more than is good for him, and his hand often shakes as he digs ditches. He has not the strength to work as the others do—in fact, as Hamlin Smith has said, "A can do more work in one hour than C in four."

The first time that ever I saw these men was one evening after a regatta. They had all been rowing in it, and it had transpired that A could row as much in one hour as B in two or C in four. B and C had came in dead fagged and C was coughing badly. "Never mind, old fellow," I heard B say, "I'll fix you up on the sofa and get you some hot tea." Just then A came blustering in and shouted, "I say, you fellows, Hamlin Smith has shown me three cisterns in his garden and he says we can pump them until tomorrow night. I bet I can beat you both. Come on. You can pump in your rowing things, you know. Your cistern leaks a little, I think, C." I heard B growl that it was a dirty shame and that C was used up now, but they went and presently I could tell from the sound of the water that A was pumping four times as fast as C.

For years after that I used to see them constantly about the town and always busy. I never heard of any of them eating or sleeping. After that, owing to a long absence from home, I lost sight of them. On my return I was surprised to find A, B, and C no longer at their old tasks; on inquiry I heard that work in this line was now done by N, M, and O, and that some people were employing, for algebraical jobs, four foreigners called Alpha, Beta, Gamma, and Delta.

Now it chanced one day that I stumbled upon old D in the little garden in front of his cottage, hoeing in the sun. D is an aged laboring man who used occasionally to be called in to help A, B, and C. "Did I know 'em, sir?" he answered. "Why I knowed 'em ever since they was little fellows in brackets. Master A, he were a fine-hearted lad, sir, though I always said, give me Master B for kind-heartedness-like. Many's the job as we've been on together, sir, though I never did no racing nor aught of that, but just the plain labor, as you might say. I'm getting a bit too old and stiff for it nowadays, sir —just scratch about in the garden here and grow a bit of a logarithm or raise a common denominator or two. But Mr. Euclid, he uses me still for propositions, he do."

From the garrulous old man I learned the melancholy end of my former acquaintances. Soon after I left town, he told me, C had been ill. It seems that A and B had been rowing on the river for a wager, and C had been running on the bank and then sat in a draught. Of course the bank had refused the draught and C was taken ill. A and B came home and found C lying helpless in bed. A shook him roughly and said, "Get up, C, we're going to pile wood." C looked so worn and pitiful that B said, "Look here, A, I won't stand this; he isn't to pile wood tonight." C smiled feebly and said, "Perhaps I might pile a little if I sat up in bed." Then B, thoroughly alarmed, said, "See here, A, I'm going to fetch a doctor; he's dying." A flared up and answered, "You've got no money to fetch a doctor." "I'll reduce him to his lowest terms," B said firmly, "that'll fetch him."

C's life might even then have been saved but they made a mistake about the medicine. It stood at the head of the bed on a bracket, and the nurse accidentally removed it from the bracket without changing the sign. After the fatal blunder C seems to have sunk rapidly. On the evening of the next day, it was clear, as the shadows deepened, that the end was near. I think that even A was affected at the last as he stood with bowed head, aimlessly offering to bet with the doctor on C's labored breathing. "A," whispered C, "I think I'm going fast." "How fast do you think you'll go, old man?" murmured A. "I don't know," said C, "but I'm going at any rate." The end came soon after that. C rallied for a moment and asked for a certain piece of work that he had left downstairs. A put it in his arms and he expired. As his soul sped heavenward, A watched its flight with melancholy admiration. B burst into a passionate flood of tears and sobbed, "Put away his little cistern and the rowing clothes he used to wear. I feel as if I could hardly ever dig again."

The funeral was plain and unostentatious. It differed in nothing from the ordinary, except that out of deference to sporting men and mathematicians, A engaged two hearses. Both vehicles started at the same time, B driving the one which bore the sable parallelepiped containing the last remains of his ill-fated friend. A on the box of the empty hearse generously consented to a handicap of a hundred years, but arrived first at the cemetery by driving four times as fast as B. (Find the distance to the cemetery.) As the sarcophagus was lowered, the grave was surrounded by the broken figures of the first book of Euclid.

It was noticed that after the death of C, A became a changed man. He lost interest in racing with B and dug but languidly. He finally gave up his work and settled down to live on the interest of his bets. B never recovered from the shock of C's death; his grief preyed upon his intellect and it became deranged. He grew moody and spoke only in monosyllables. His disease became rapidly aggravated, and he presently spoke in words whose spelling was regular and which presented no difficulty to the beginner. Realizing his precarious condition, he voluntarily submitted to be incarcerated in an asylum, where he abjured mathematics and devoted himself to writing the history of the Swiss Family Robinson in words of one syllable.

THE GROWING NEED FOR MATHEMATICS EDUCATION

John E. Dawson

The thesis of this article is simple: Every elementary school student must acquire a thorough foundation in modern mathematics. If this sounds like academic chauvinism, I can only say that I am not a mathematician. Nor am I asking you to accept such an assertion simply as a traditional article of faith. Our purpose here will be to suggest some of the fundamental developments in our society which —much more than any reasons of faith or tradition—make a thorough mathematics education essential.

Let us begin by recalling a few taken-for-granted propositions. The schools do not exist in a vacuum. They function to assist the young to learn, both for individual development and for the benefit of society at large. While educational programs are necessarily rooted in the past and conducted in the present, they must aim toward the future. This is a way of saying that in order to further individual development and to benefit society, education must be designed to be as consistent with future requirements as possible. The time-frame within which educational programs are conducted is not only the present (as important as this may be in the context of student motivation and perceptions of relevance) but also the future when the outcomes of educational experience primarily will be utilized.

How well have the future needs of the individual and of society been anticipated in our educational programs? How good are the choices made by school administrators and teachers in planning to assist children in their present learning for their future lives? How, when, and with what motivations do they decide that what they are doing isn't enough to meet an uncertain tomorrow—not just for themselves, but for the young they work with and for? What can we reasonably expect about the adult lives of today's children that should affect our efforts in mathematics education today?

The Traditional Rationale

In some form or another, the study of mathematics has long been entrenched as part of education. School tablets survive from the days of ancient Sumeria, encrusted with multiplication and division tables, square and cube roots, and exercises in applied geometry. In the early days of our own country, at least a minimum ability to cipher was considered essential in even the feeblest attempts to school Colonial offspring. Barring a few exceptions of theological training in ancient times and the "educational" rites of initiation in certain primitive societies, there would seem never to have been a question that some degree of mathematical ability and knowledge should be acquired by every recipient of an educational effort.

The rationale of this has varied from society to society and from time to time. Often, the impetus for teaching mathematics has been eminently practical: to promote trade, to facilitate navigation, to provide a common basis for dividing and measuring time. Mathematics has also been thought of as means to develop the logic capabilities of the mind—as a discipline of reasoning. Further, it has been considered basic to advanced intellectual pursuits in such fields as the natural sciences and engineering. For some or all of these reasons, mathematics has been established over the centuries as a key element of curriculum. It generally has been viewed as one segment of man's accumulated knowledge which should be transmitted to all members of succeeding generations.

Increased Importance of Mathematics

None of this traditional rationale for the teaching of mathematics is obsolete. On the contrary, an assessment of current society and a projection of the probable future can only reinforce the case.

Even a cursory examination of our society and of what it will become in the balance of this century reveals a widespread and intensive growth in the use of mathematics. The exponential curve of technological change, the shifting profile of occupations, and the newer applications of mathematics to problem-solving and reasoning are some of the major dimensions of this phenomenon. For the child in school today, the implications of this growth in the use of mathematics are that he will increasingly find that mathematical literacy is a prerequisite for understanding the age in which he lives, for entry into more and more occupations, and for helping him to order his own life.

In short, a sound foundation in modern mathematics will be relevant to a far greater proportion of this generation of elementary school students than of past generations. And the penalties for lacking such a grounding in mathematics will be considerably more pronounced in the future for more people. Just as an inability to use written and oral language effectively is a serious hindrance for many adults today, an inability to utilize the language and operations of mathematics will be a real handicap for many adults tomorrow.

Two aspects of the increased importance of mathematics in our society seem to be particularly relevant to the planning and conduct of mathematics programs in elementary schools: 1) the escalation of what is necessary as a minimum level of mathematical literacy; and 2) the growing use of mathematics as a thinking tool.

Need for Greater Mathematical Literacy

One of the most fascinating events of recent years, for children and adults alike, was the first landing of a man on the moon. This accomplishment, occurring an incredibly short twelve years after the first unmanned earth-orbiting vehicle, is one of the more dramatic examples of technological development. There is no need to rehearse the astounding changes that have been taking place—in nuclear energy, electronics, satellite communications, and on and on. Nor is there any need to enter into debate about whether the products of technology contribute to the progress of man. The fact is that technology is one of the major pervasive characteristics of life today—and tomorrow.

An important aspect of understanding the world in which we live requires the capacity to grasp the influence of technology. Issues of public concern—the condition of the cities, nuclear energy, pollution of the environment—tend to be at least partially the offspring of technology; and, expectedly, remedies are at least partially dependent upon the future products of technology. Mathematics is the universal language of technology and, consequently, mathematics is increasingly a prerequisite for an in-depth understanding of such issues.

Technology is also affecting the structure and composition of the labor force. It has been estimated that of all the scientists who have ever lived, half are alive today. While technological innovation starts with the scientist, the development, production, utilization, and maintenance of the fruits of science and technology occur through a constantly growing labor force in a vast array of occupational specialties. The traditional argument that mathematics education is important because of a technological age's demand for engineers and scientists remains effective, but now it is amplified in scope and breadth.

A special application of technology is generating whole new fields of human endeavor. The computer, coupled with various filming and reproductive devices, is facilitating an information explosion. Here again, new occupations, involving heavy doses of mathematics, have been created.

Advancement in most of the social sciences also increasingly requires mathematics—the use of statistics in sociological and behavioral studies; mathematical modeling, as well as statistics, in economics; the development of operations research and systems analysis as tools of management. To cite a management example: In 1947, a problem-solving technique was developed—linear programming. Today it is utilized to plan oil refinery operations, locate warehouses, select defense transportation systems. It is in these latter applications to the problems of management of private and public organizations that some of the most fascinating developments involving mathematics are occurring. Man is developing additional tools for decision making which depend upon reasoning quantitatively. While these tools are being applied to management tasks, *the concepts are applicable to individual tasks of decision making as well.* And, importantly, it is not a process of "solution by the numbers." What is involved are ways of thinking about problems.

Because of the increasing importance of mathematical aids to corporate, governmental, and individual decision making, let us examine the use of mathematics as a tool of thought more closely.

Quantitative Reasoning

Earlier it was suggested that today we need to become mathematically literate. The term "literate" is appropriate because mathematics provides an additional language, especially for the purpose of reasoning. In a word-based language, words are used to symbolize various objects and concepts for purposes of thinking and communicating. Mathematical *symbols* serve in a similar fashion for those familiar with this language. Of the two languages, the word-based system permits greater scope of expression. It can be used to convey emotional as well as logical content. It permits imprecision and ambiguity when the situation requires such communication. However, the narrower scope of mathematical language is its great strength for the uses to which it can be applied. Mathematical language demands *precision* and *explicitness*. It is intended as a vehicle for analytic thinking.

Mathematical language has its parts of speech. Variables are vaguely analogous to nouns and are used as subjects and objects in mathematical sentences. Continuing the analogy, parameters serve as adjectives as they describe, limit, or qualify variables. Finally, the operators of mathematics are the verbs, stating "is equal to" (or not equal to) and directing actions to be taken such as adding, subtracting, multiplying and dividing.[1]

The language is used to state relationships by composing mathematical sentences. A very general sentence is $y = f(x)$ which simply says that y is related in some yet unspecified way to x. John Maynard Keynes utilized this form of expression for the consumption function, $C = f(Y)$, in his *General Theory* as a way of expressing the expenditures on consumption as a function of income.[2] This proposition has become one of the most important concepts in modern economics. Such statements of relationships become ways to be very explicit about a problem. They can be used to describe symbolically a situation existing in the real world. Describing a problem mathematically requires systematic and explicit thinking—in order to specify significant variables, parameters, functional relationships. Much can be gained by simply utilizing mathematical language to write down and convey to others (as well as to oneself) what one thinks are the important parts of a problem and how they might be related. However, the gain does not end with description or clarity of communication. The process generates a symbolic representation of the real world, useful in thinking about the real world situation but differing from the real world in important respects:

1. The symbolic representation of the real world is *simpler*. To create the symbolic expression, ingredients from the real world are sorted systematically and only those items which are believed to be significant are symbolized. The complexity of the real world is analyzed so that one can focus systematically on what are perceived to be the crucial factors.

The consumption function, $C = f(Y)$, which states that consumption is a function of income, is an excellent example of this simplifying process. Many other factors also affect consumption, including changes in tastes, new product development, and advertising practices. However, by temporarily ignoring these factors, attention is concentrated on the central importance of a stable relationship between consumption and income, thereby leading to economic analyses using this function which have enlarged economic understanding.

2. If necessary, the simplified symbolic version can include a tremendously large number of factors. Brogan, a spirit akin to Parkinson, suggests that any problem with less than three variables is trivial and

1. For an excellent description of mathematics as a language utilizing the analogy briefly stated here, see Goetz, Billy E. *Quantitative Methods: A Survey and Guide for Managers.* New York: McGraw-Hill, 1965, pp. 61-79.

2. Keynes, John Maynard. *The General Theory of Employment, Interest and Money.* New York: Harcourt, Brace, 1936, p. 90.

any problem with more than eight variables can't be solved. While this caveat may seem harsh, ask yourself how many variables, doing a random mind-walk in your head, you can handle well; then ask the corollary question: How many significant variables are there in the problems you face? Mathematics provides an orderly, logical way to treat multidimensional problems. There are instances today where formal mathematical models involving hundreds of significant variables and parameters are being used successfully to assist decision-makers dealing with extremely complex problems.

3. The symbolic world can be *manipulated* mentally as a process of reasoning directed toward a better understanding of the problem and the generation of information that may clarify the feasibility and preferability of alternative courses of action. Alternatives can be tried by manipulating the symbols, rather than by the more costly and frequently impossible strategy of trying out all the various alternatives in the real world. Manipulation is performed by following the logic system of mathematics—its rules.

Thus, if one wished to examine the problem of transporting children to and from school, a mathematical model of the school transportation problem could be developed. Such a model could be used to improve the route structure in order to minimize average student travel time or to meet some other criterion. The model also could be used to try out larger or smaller buses to see whether the system would work better. Through use of the model, the alternatives can be investigated without buying buses, hiring and firing drivers, or changing bus routes. If one of the "tries" provides a considerably improved system in terms of student travel time as related to costs, then normal staff exploration of the problems involved in changing the system would appear to be worthwhile. The modeling process clarifies the problem and directs attention to the more advantageous courses of alternative action.

4. If numbers can be obtained (or estimated or approximated) for the symbols used, explicit quantitative predictions about outcomes can be developed (costs, levels of goal attainment, etc.). For example, empirical testing of Keynes' $C = f(Y)$ has occurred, and data for this function is utilized in attempts to estimate the future performance of the economy. As important as such numerical values can be in decision making, the power of the quantitative reasoning approach lies in the assistance it can give to rational inquiry—to systematic thinking.

Two characteristics of the process sketched here should be noted. First, all the process does is handle explicitly, systematically, and logically whatever one perceives or believes or interprets regarding the real world. That, of course, is doing a lot, but there is no guarantee that the input of human perceptions is accurate. What the process of quantitative reasoning attempts to provide is information that assists one to make a decision in a manner that is rationally consistent with one's perceptions of the problem.

Second, to participate directly in (or even to observe intelligently) such a decision-making process in an organization requires an increasing degree of mathematical literacy. Three years ago in an article in *Fortune,* Max Ways discussed these new approaches to decision making. He observed: *"The further advance of this new style* [in decision making] *is the most significant prediction that can be made about the next ten years. By 1977 this new way of dealing with the future will be recognized at home and abroad as a salient American characteristic."* [3] In the same article, Arjay Miller, then president of Ford Motor Company, was quoted as follows: "Hunches and cut-and-try methods are giving way to the systems-analysis approach, a whole new way of perceiving problems and testing in advance the consequences of alternative actions to solve those problems. Computers and other technical devices, including mathematical models, have extended greatly our ability to understand and cope with the complex problems we face in today's world."

I am regularly involved in executive training work which includes introducing senior executives to quantitative reasoning in decision making. It is painful to observe, for example, their discomfort in dealing with inequalities (they were schooled in equalities); their difficulties in plotting a line by use of slope (when they were taught to use pairs of x, y values); and their struggles with problems stated only in symbolic terms. If the full potential of the growing body of quantitative aids to decision making is to be realized, there must be a rapid and drastic reduction in mathematical illiteracy among the coming generation of public and private managers. And their facility in the use of symbolic language must begin in the elementary school.

All of these developments—the pervasive character of technology which demands mathematical literacy to understand more and more aspects of our world, the growing applicability of mathematics to a widening range of occupational specialties, and the utilization of mathematics as a tool for decision making—add up to the fact that an education which fails to provide a basic mathematical foundation is narrowing the life choices of many individuals. Without a thorough grounding in modern mathematics in the elementary school, a child will find many of his options for the future closed, and he will find that many significant features of the world in which he lives are difficult and perhaps impossible for him to comprehend.

3. Ways, Max. "The Road to 1977." *Fortune,* January 1967.

"Life Is Painting a Picture"

Much of what has been written here is said with some sense of personal regret. My formal education in management occurred when the major quantitative demand was a capacity to use the page numbers on the reading list. I like the sentiment of Justice Holmes that "life is painting a picture, not doing a sum." Mathematical ways of reasoning are really an adventure of my thirties rather than my youth because of *when* I went to school. Occupationally, I am like a wide receiver in professional football: I hear footsteps—except in this case they are the footsteps of the young who have had set theory from the beginning and don't know it's wrong!

Robert S. McNamara, a controversial manager identified with the willing use of systematic thought through quantitative reasoning in decision making, has said: "All the evidence of history suggests that man is indeed a rational animal but with a nearly infinite capacity for folly. His history seems largely a halting but persistent effort to raise his reason above his animality. He draws blueprints for Utopia but never quite gets it built. In the end, he slugs away obstinately with the only building material really ever at hand: his own part-comic, part-tragic, part-cussed, part-glorious nature." [4]

If, through the *ways* mathematics is taught, our ability to reason can be strengthened to aid our persistence, then perhaps our circumstances will permit Justice Holmes' desire for artistry in living to be more fully realized as well. For the educators to find and utilize the *ways* is the purpose of this publication.

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4. McNamara, Robert S. *The Essence of Security: Reflections in Office*. New York: Harper & Row, 1968, p. 158.

WHAT IS A CONTEMPORARY ELEMENTARY MATHEMATICS PROGRAM?

Harold H. Lerch

Inquire at almost any elementary school about the math program and you will most likely be told that a modern elementary mathematics program is being taught in that school system. The response will usually be supported by such statements as these:

"The system is using a recently published textbook series."

"Units of work dealing with sets, geometry, systems of numeration with bases other than ten, properties of number, and so forth are being taught at various instructional levels."

"Teachers have recently attended an inservice workshop in modern mathematics."

"The system has invested a substantial amount of money in instructional materials and aids."

But after analyzing the program on the basis of desirable characteristics of contemporary mathematics programs, you may well reach the disturbing conclusion that the program is contemporary only in the sense that it currently exists. Physical attributes—which might include a new textbook series, units of work dealing with content new to the elementary curriculum, inservice workshops for teachers, and instructional materials and aids—contribute to the *possibility* of a contemporary mathematics program. They do not, however, assure that there is a contemporary program.

What, then, is a contemporary mathematics program? It is impossible to offer a definition in a few sentences or in a few paragraphs. Each of the articles in this volume defines and describes important aspects of the program and states or implies their interrelationships. One might respond to the question, however, by discussing two related questions: How should current programs differ from programs of the past? What type of involvement—on the part of pupils, teachers, and administrators—is required if the mathematics program is to serve its objectives effectively?

Differences and Similarities

Contemporary mathematics programs and the arithmetic programs of yesterday are more different than they are similar. Similarities exist only in some of the objectives and in the arithmetic content. Contemporary programs differ from programs of the past in these ways:

Psychological foundations. Changing views of how children learn have influenced the selection, organization, and sequence of content, as well as the instructional procedures used.

Program objectives. Contemporary programs retain the objectives of earlier arithmetic programs. These are objectives concerned with developing understanding of our system of numeration, concepts of quantity and quantitative relationships, skill in computation, and ability to think in quantitative situations and to solve the problems of present and future everyday living. In addition, however, modern programs strive to help each child understand the structure of mathematics, its laws and principles, its sequences and order, and the way in which mathematics as a system expands to meet new needs. They also try to develop independent thinking processes and to help each child prepare for the next steps in mathematical learning that are appropriate for him in terms of his potential and his future educational requirements.

Scope of content. The scope of the content of elementary mathematics programs has been expanded to include mathematics that has not previously been taught at the elementary level. Thus, pupils are now expected to gain a knowledge of the real number system and the main ideas of geometry in the elementary grades. This new content is included in contemporary programs not as fragmented parts or merely because children *can* learn it. Rather, it is included to help children begin to develop an understanding of the structure of mathematics and to give them a basis for understanding and utilizing the relationships of the integrated field. Emphasis is on major mathematical ideas rather than on isolated facts, rules, and manipulations.

Organization of content. In modern programs, major concepts are introduced at earlier levels and spirally developed at later stages. As the topic arises with increasing complexity at later levels, provision is made for each child to expand his understanding, knowledge, and application of the concept. Emphasis is upon the search for patterns and upon understanding relationships between concepts. Content is organized around topics, concepts, and relationships of importance or enduring value, including number and numeration, properties of number, operations, geometry, and measurement.

Instructional procedures. There is current consensus that the inductive approach, or the discovery method, is more conducive to developmental learning and to retention than is an instructional process based on presentation of rules and facts by the teacher. In programs of the past, the teacher's role was to explain and demonstrate; the pupil's role was to memorize facts, rules, and manipulations. Pupils were passive recipients of information, while teachers emphasized "telling." All this has changed in good contemporary programs to a teaching-learning process in which pupils are guided by teachers to discover for themselves the basic structural elements, concepts, and interrelationships of mathematics. As a result, the role of the teacher has now become one of structuring learning situations, guiding, questioning, diagnosing, and encouraging active pupil participation in the learning process. The role of the pupil now stresses observing, recording and collecting evidence, checking hunches, organizing relevant data, and formulating conclusions. A modern instructional approach includes, among other things, developing and using the language of mathematics; forming and exploring concepts through the use of concrete and illustrative materials; using different approaches when topics are reintroduced at successive levels of difficulty; adjusting instruction to the learning styles, pace, and achievement of individual pupils; and accepting from pupils various possible correct solutions to quantitative problems.

Applications. In the past, arithmetic programs sought to develop in pupils the skills and knowledge that would enable them to solve everyday problems and to handle the quantitative problems they encountered in their other academic endeavors. While these applications are still considered to be valid for a contemporary program, the skill and knowledge requirements for making such applications have changed. New and developing programs in the sciences and social studies, for example, often demand a kind of mathematics quite different from the mathematics pupils were taught in the old arithmetic. Contemporary programs must take these new requirements into account if they are to be applicable to pupils' needs. In a contemporary program, applications should pertain to present and future everyday life situations, to other academic disciplines, and to the utilization of mathematical concepts within the field of mathematics itself.

Evaluation. The evaluation of modern elementary mathematics programs cannot be based upon pupils' ability to recall facts and rules and to perform operations mechanically. Previously described changes in the program objectives and teaching-learning processes require modifications in the nature of evaluation. Evaluation procedures must be directed toward ascertaining pupils' understanding of facts, generalizations, and operations; their ability to discern patterns; their knowledge and utilization of relationships; and their ability to apply mathematical skills and concepts in situations removed from the mathematics class. In regard to the total program, consideration must be given to how well it succeeds in achieving the broad objectives established for pupils and to how it compares with alternative types of programs in meeting these objectives.

Need for Involvement

The possibilities of what can be accomplished in a contemporary elementary mathematics program are directly related to the nature and extent of the involvement of pupils, teachers, and administrators. Modern elementary mathematics programs are *active* programs. Activity is not limited to pupil participation in learning experiences of an inductive or discovery nature and to the use of more and more instructional aids and materials. It extends to the involvement of administrators and teachers in efforts to expand their knowledge of content, instructional procedures, and organizational techniques and to their continuous efforts to improve the program.

Active participation in a modern program can give administrators insight into the unique instructional problems of their school or system. Knowledge and understanding of such problems will assist them in 1) providing appropriate physical facilities and materials for the classroom, organizational arrangements, and educational opportunities for the faculty; and 2) giving professional leadership in evaluating the program and in planning and implementing desired changes. The administrative leadership necessary in a modern elementary mathematics program cannot be most effective if administrative participation is limited to the external contributions of organizing an inservice workshop for teachers, occasionally visiting mathematics classes, appointing a mathematics curriculum committee, and approving requisitions for new mathematics textbooks or materials.

Teacher involvement is also critical. The elementary mathematics program is in the hands of classroom teachers; if it falls, it will be at their feet. Moreover, it is not inappropriate to consider what a mathematics program can do for teachers to help them become better teachers. The role of the teacher in a modern program is not a passive one of standing back and letting children discover. The teacher can learn more about the unique characteristics of his pupils and be better prepared to adjust instruction to those differences through active participation in selecting appropriate learning experiences, structuring learning situations, questioning and guiding, and working with individuals. Many teachers may learn more mathematics through teaching in a modern program or participating in inservice programs than they thought they were capable of learning—or interested in learning. Their attitudes toward mathematics and the teaching of mathematics may change in a favorable direction. Professional growth in knowledge of the mathematics content of the total program and of appropriate instructional procedures and materials can result from active involvement in a modern program.

Although many recent publications concentrate on what contemporary mathematics programs can do for talented children, modern programs should be directed toward *all* pupils. Unless the program helps all children to learn mathematics better and more efficiently than they would in a traditional approach, it can hardly be called modern. One of the most self-degrading characteristics we can give to the adult of the future is to make him dependent upon others for interpreting quantitative data necessary to his occupational endeavors and to the decisions he must make as an active citizen in a technological society. Modern programs can and should provide opportunities for all pupils to achieve appropriate individual educational and vocational objectives. The content being studied and the goals for pupils may differ, but the extent of their active participation in the learning experiences of the program should be similar. In a good contemporary program, all pupils will be exploring, observing, testing hypotheses, and generalizing in their development of the language, concepts, structures, and techniques of mathematics.

An integral part of the contemporary elementary mathematics program is planned change. It is reasonable to believe that further modification will occur in the content and instructional procedures of existing and future programs. The sources of possible change are as numerous as the individuals—pupils, teachers, administrators, consultants, researchers, theorists—who are actively involved. Beware of the contemporary program as it becomes settled or stationary. The stationary program of today becomes the inadequate traditional program of tomorrow.

PREPARING AND USING INSTRUCTIONAL OBJECTIVES

Elizabeth J. Boisclair

Before teachers undertake a contemporary mathematics program, they need assurance that they are traveling the way of sound educational policy—that this is not just a fad or a dizzy trip on a merry-go-round. And if they are to feel that they are making progress along the way, they must be supported in their practical desire to apply their new learnings in today's classrooms. The rewards and reinforcements of a new approach cannot wait for some future class that teachers may face in a dim tomorrow.

An old proverb says, "A journey of a thousand miles begins with a single step." Our job is to make it possible for elementary school teachers to take that single step with safety and success, and through that single step to learn our destination, what direction we are going, how we plan to prevent getting off the path, and how we will know where we are with reference to our destination.

The route of an effective educational program is marked by clearly defined landmarks. These are:

Educational goals. Our goals in education are based upon our assumption that the individual needs to function in a worthwhile way in our present society and in our anticipated future society. It is on this assumption that we attempt to determine the present and potential needs of our students and thereby formulate our goals in elementary education.

Instructional objectives. In order to attain our educational goals, we must translate them into more specific objectives of instruction. This requires that we determine what kind of learning performance on the part of students will best lead toward attainment of the goals.

Program of instruction. The specific instructional objective. selected for different students to pursue will vary with their individual requirements. Having decided upon the specific instructional objectives, we can then plan learning experiences that will enable individual students to meet the requirements stated in the objectives.

Evaluation. A continuing analysis of the total educational program is essential if it is to remain functional and flexible enough to adjust to necessary changes.

Stating Instructional Objectives

As we consider these four basic components of a sound educational program and their implications for teaching modern mathematics, we begin to recognize the critical importance of how we select and state our instructional objectives. Our objectives must be appropriate both in terms of general goals and in terms of students, and they must serve as useful guides in planning the program of instruction.

In the area of mathematics, there seems to be general agreement about certain broad educational goals. These include the need in our present and future society for individuals who possess knowledge and skill in:

- using a problem-solving approach in varied situations;
- using symbolic reasoning in daily life;
- using computational skills effectively and accurately;
- making practical use of basic mathematical ideas;
- reading and interpreting the language of mathematics;
- showing an awareness of spatial relationships.

The stumbling block lies in how we translate these big ideas related to mathematics into learning experiences for elementary school children. It is to meet this problem that new curriculum materials have been produced, new programs initiated, new textbooks published, and new manipulative devices and visual materials introduced. All of these are helpful aids in making the transition from general goals to specific objectives. The important step, however, must be taken by the staff of a particular elementary school in terms of the children enrolled and the resources available to the teachers. Sometimes we set our sights on the development of understanding, the use of thought processes, the multiplicity of approaches, the awareness of basic concepts—all in a "cloud nine" atmosphere. Too often, the result has been confusion, fogginess, and frustration for both teachers and pupils. More than that can be expected from a subject that is synonymous with systematic thought and practicality.

Recent developments in stating objectives in behavioral terms offer clear guidelines to those interested in modern programs of elementary mathematics. It is all very well to have as our goal "a good understanding of . . . ." or "a concept of . . . ." But we must be able to determine how well the student understands, and this demands that we have a basis for judging by what he says or does in the classroom. A student's understanding can only be communicated in a way that is directly observable by the teacher.

The pupil's behavior, which demonstrates his understanding, may be manifested in different ways depending upon the grade level, the conditions under which the behavior is performed, the response expected, and the child himself. If an objective is truly specific and is stated in behavioral terms, it must clearly describe what the learner must be able to do in order to demonstrate his accomplishment of the objective. A well-written instructional objective must:

- identify the pupil for whom the objective is written;
- describe the conditions under which the pupil will perform;
- specify the exact nature of the performance expected;
- state the extent to which the performance is expected to occur as indication of attaining the objective.

In order to specify pupil behavior, instructional objectives must use action verbs—words which describe observable behavior. A number of such words are recommended by specialists in this field.[1]

1. *Identify*—(Pick up or choose one from many to identify objects, properties of objects, or specified changes in objects.)
2. *Distinguish*—(Select one from others that have properties or characteristics that are potentially confusing in identification.)
3. *Construct*—(Make or finish making a drawing or construction that identifies a particular object or set of conditions.)
4. *Name*—(Say or write the name of the object or property of a class of objects.)
5. *Order*—(Arrange according to a specified category—time, importance, size, etc.)
6. *Describe*—(Write or tell enough that the identification of the object, property, or event is considered to be sufficiently communicated.)
7. *State a rule*—(State it not necessarily in technical terms but so that the generalization or conclusion is given accurately, naming all objects or events relevant to the situation.)
8. *Apply a rule*—(Use a rule to solve a problem in a new situation, employing a systematic process to arrive at the solution.)
9. *Demonstrate*—(Show how to use all the steps needed in the application of a rule.)

1. *Science—A Process Approach.* American Association for the Advancement of Science. University of Maryland, 1964.

10. *Interpret* (Apply a series of rules as a means of explanation.)

Let's look at some sample instructional objectives related to "developing a good understanding of a base ten numeration system," as they might be written for different grade levels. The following do not cover all of the possible objectives related to the topic; they simply suggest the differences that might be found when stating the conditions under which the pupil will perform, the nature of the performance, and the extent of performance.

Grade 1. Given three different sets of objects, each set containing more than ten but less than thirty objects, the pupil should, with 100 percent accuracy, be able to *show* each set grouped into subsets of tens and *say* how many tens and how many single ones are in each set.

Grade 2. Given three different 2-place numerals, the pupil should, with 75 percent accuracy, be able to *write* the answers to *oral questions* (similar to the following) about the meaning of each numeral: 1) How many tens have been counted? 2) How many ones were left over after grouping the tens? 3) How many ones were counted altogether?

Grade 3. Given three different 2-place numerals, the pupil should, with 90 percent accuracy, be able for each number represented to do the following: *name* the number orally; *show* how each number may be represented on a computing abacus; *show* how the same number could be represented on a computing abacus after changing 1 ten to 10 ones; *rename* the number orally; *tell* why the number is the same.

Grade 4. Given ten open sentences related to renaming numbers (similar to the following), the pupil should, with 100 percent accuracy, be able to *complete* each sentence in writing and *demonstrate* with manipulative materials why the statement is true.

Sample:

$$34 = 2 \text{ tens} + \_\_ \text{ ones}$$
$$51 = \_\_ \text{ tens} + 11 \text{ ones}$$
$$421 = 3 \text{ hundreds} + \_\_ \text{ tens} + 1 \text{ one}$$
$$4 \text{ hundreds} + 13 \text{ tens} + 3 \text{ ones} = \_\_$$
$$7 \text{ tens} + 17 \text{ ones} = \_\_$$
$$1 \text{ hundred} + 15 \text{ tens} + 14 \text{ ones} = \_\_$$

Grade 5. Given five pairs of numerals, each pair using the same digits but in different places, the pupil should, with 80 percent accuracy, be able to *state* in writing the relative value of a digit as used in different places.

Sample: (530, 305): The first "5" represents 100 times the value represented by the second "5."

(235, 325): The first "3" represents 1/10 the value represented by the second "3."

Grade 6. Given five numbers, each to be multiplied and divided by 10, 100, and 1,000, the pupil should, with 100 percent accuracy, be able to *write* the appropriate products and quotients without computation. He should also be able to *state* in his own words *the rule* being used in the short-cut method and *demonstrate* its application to decimal fractions as well as to whole numbers.

Sample: $10 \times 25 = 250$ (If a number is multiplied by 10, the digits in the multiplicand move one place to the left, giving a product which is 10 times the value of the multiplicand.)

$25 \div 10 = 2.5$ (If a number is divided by 10, the digits in the dividend move one place to the right, giving a quotient which is 1/10 the value of the dividend.)

Instructional objectives stated in this specific manner take the general goal out of the clouds and bring it down to the realities of the classroom. Well-written objectives enable us to be explicit about what we expect pupils to learn and give us a basis for evaluating pupil progress.

To evaluate learning, pupils should be tested in terms of the conditions stated in the objectives. The criterion test—the nature and extent of the performance expected—is implicit in the objective and should be prepared at the time the objectives are stated. Keep in mind that the test items should be so constructed and identified as to allow opportunity for the child to demonstrate his ability to perform in a variety of ways, not just those limited to pencil and paper performance or those solely dependent upon skill in reading.

It is also important in preparing instructional objectives to consider levels of intellectual functioning. For example, do we want the pupil to be able to *recall* knowledge or to be able to *apply* a concept? In the *Taxonomy of Educational Objectives: The Classification of Educational Goals. Handbook I: Cognitive Domain,* Bloom categorizes six levels of intellectual tasks:

1. Knowledge

To recall specific facts and information already discussed and developed. This level of learning is dependent primarily upon the process of remembering and may include knowledge of terminology, of sequential steps, of a learned structure, or of criteria for evaluating.

2. Comprehension

To understand the meaning. This involves the ability to interpret symbolism or to translate verbal material into symbolic statements. It also includes the ability to explain or summarize, to predict or to determine continuing patterns.

3. *Application*

To apply abstract ideas in specific and concrete situations. This involves using generalizations and applying rules in situations that have *not* previously been discussed and developed.

4. *Analysis*

To analyze a statement so that the organization and the relation of its parts become clear. This involves the ability to pick and choose, to select statements that support a generalization or to discard those not applicable.

5. *Synthesis*

To select, classify, and organize random materials in such a way that a conclusion or a theory is seen more clearly. This level usually requires a creative approach.

6. *Evaluation*

To evaluate and judge the validity of material for a specific purpose and to justify decisions.

Planning Instruction

If a teacher has written a behavioral objective and has prepared a criterion test, certain aspects of instruction begin to fall into place. Each objective assumes that certain prerequisite behaviors have already become part of the pupil's experiences. Let's examine the sample objectives listed earlier in this article. Can you discover what assumptions have been made in stating each objective? The assumed behaviors can be expressed in terms of pupil performance and serve as a guide to diagnostic procedures. This step is vital. If the instruction is based on certain assumptions and some parts of the pattern are missing, the objective will not be reached. The pupil either will not be able to perform as expected at the end of the instruction or he will have learned to give automatic responses, which will defeat our goal of developing understanding. While determining readiness, let's move one step farther and make an effort to assess pupil performance on the behavior referred to in the objective. This will give us an opportunity to diagnose strengths and weaknesses in relation to the new learning, and we may discover that some children have already learned what we are going to teach.

Our diagnostic procedures must be varied enough in assessing kinds of pupil performance to give us some information about the learning skills of various individuals in a class. For example:

"This child seems to respond best at the manipulative, verbal level with respect to this idea."

"These children will need some transitional experience to help them move from the concrete level to the symbolic level of expressing this idea."

"This child demonstrates a mature grasp of the idea and should try applying his understanding to different and new situations."

It is easy to see that the above readiness and diagnostic procedures offer clues to the learning prescriptions to be planned by the teacher. Obviously, there will need to be a multiplicity of paths to follow and a variety of materials and media to be used if we are to meet the needs, abilities, and interests of the individuals in the class. For example, the program of instruction might include:

- teacher-presented developmental lessons through lecture, discussion, demonstration, visual aides;
- opportunity for guided exploratory experiences with manipulative materials or with symbolic graphic materials where the children can make discoveries, classify ideas, make tentative generalizations, test them out, verify findings;
- independent assignments based on textbooks, workbooks, seatwork prepared by the teacher to allow pupils to progress at varying rates and through different levels of intellectual tasks;
- opportunities to share ideas, helping children to verbalize their understanding and relate it to the language of mathematics.

Under this kind of instructional program, the teacher's role is to diagnose, develop, confer, guide, direct, redirect, correct, reinforce, and refine related concepts, skills, and abilities.

Evaluation

After the pupils have followed the options offered by various learning paths, the class is ready to be tested. Did we arrive? If performance on the criterion test indicates that the objective was not reached, the whole structure will need to be analyzed:

- Was the objective geared to the level of the pupils?
- Did the learning options offer appropriate paths for the children?
- Were the options offered in a sequence that would provide for success?
- Was the criterion test based on the objective?

This kind of evaluation and feedback makes it possible for us to know immediately if retracing some steps is necessary. If not, we can take another step forward and move to the next objective.

Planning a Series of Objectives

The analysis here has been in terms of one specific objective. As teachers work with one objective and consider the prerequisite and related behaviors, they usually find it more convenient to plan a series of objectives which suggest a possible sequential learning pattern. This plan becomes the foundation of a small unit of work without losing sight of the main concepts which flow through and unify all mathematics. The unit plan, in summary, may follow this order:

1. Identify what large concept in mathematics is being developed.
2. Write a set of related specific behavioral objectives selected for the class and list them in order of sequential development.
3. Write the criterion test based on the objectives and identify the levels of intellectual functioning expected.
4. Determine the prerequisite behaviors and establish readiness for the new behaviors by diagnosing related needs, abilities, and interests of the individuals in the class.
5. Plan and provide multiple-option learning paths based on the diagnosis of individual and group needs.
6. Administer the criterion test to evaluate effectiveness of the instruction.
7. Follow the feedback information from the test, reteaching and retesting as indicated.

Groups of teachers can follow the above plan and write, try out, revise, and share small units of work. By doing this, they will gradually create a functional program which is uniquely suited to the pupils of their school. They will be making use of the curriculum materials, textbooks, and manipulative and visual aids available to them; they will be able to evaluate progress along the way. Here, then, is a procedure for improving instruction that combines the talents of classroom teachers and mathematics specialists—and all to the benefit of children in the classrooms *today*.

FOUR MAJOR ISSUES IN MATHEMATICS INSTRUCTION

Beatrice Ward

During the past decade, the elementary mathematics program has undergone major changes. As a result, most classroom teachers are now using a mathematics program that is different in content from what was commonly found in elementary schools ten years ago. In many cases, however, changes in the teaching and learning process have not been as fundamental or as widespread as the shift in content. This is not surprising since implementation of the content changes has demanded a great deal of our effort and since changes in the teaching-learning process are usually more difficult to accomplish than is introduction of new curricular content. Nevertheless, it is essential that we attend to the requirements of mathematical teaching and learning.

A recent summary of research in mathematics education concludes that comparative studies of traditional and new or experimental mathematics programs reveal no significant differences in student learning.[1] While there may be a number of causes for this finding, both statistical and pedagogical, it would certainly appear that we need to review our instructional procedures to determine if we can achieve better outcomes through improvement of the teaching-learning process in the classroom.

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1. Romberg, Thomas A. "Current Research in Mathematics Education." *Review of Educational Research* 39: 473-92; October 1969.

This article discusses four basic aspects of the instructional process that principals and teachers need to consider in relation to mathematics education: 1) the individual—as contrasted with the group—as the focus of instruction; 2) emphasis on mastery of mathematical content as opposed to coverage; 3) the extent to which conceptual understanding, as contrasted with computational skill, is sought; and 4) the use of concrete versus abstract learning experiences in teaching mathematics.

Individual vs. Group Focus

Throughout the history of education, both theorists and practitioners have continually stressed the importance of the individual student in the classroom. Current thought continues to emphasize this theme. For example, Goodlad proposes that human variability demands alternative forms of instruction.[2] Siegel and Siegel have suggested that students are differentially involved in learning, even when they are exposed to an apparently identical instructional experience.[3] And Bolvin has stated that it is the individual who learns and that thus the plans for learning should be prepared for him, not for the group.[4]

Obviously, from a theoretical viewpoint, the individual should be the focus of instruction. This has never been a point of argument. From a practical, day-to-day operational viewpoint, however, attainment of an individualized instructional program has proved to be a difficult undertaking. The real question, therefore, is not whether instruction *should* focus on the individual instead of the group, but rather *how* we can individualize mathematics instruction in the elementary school.

During the latter part of the 19t several major efforts were made to design programs of individualized instruction. These include Individually Prescribed Instruction (IPI), developed by the Learning Research and Development Center at the University of Pittsburgh; the program developed by the Duluth, Minnesota, public schools; Project PLAN (Program for Learning in Accordance with Needs), developed by the American Institutes for Research in Palo Alto, California; and the League of Cooperating Schools, developed by John Goodlad under the sponsorship of the Kettering Foundation's IDEA. Each of these programs provides one or more ways of adapting

2. Goodlad, John I. "Diagnosis and Prescription in Educational Practice." Santa Barbara: Center for Coordinated Education, University of California at Santa Barbara, 1966. (Mimeographed)

3. Siegel, Laurence, and Siegel, Lila C. "The Instructional Gestalt." *Instruction: Some Contemporary Views.* (Edited by Laurence Siegel.) San Francisco: Chandler Publishing Company, 1967.

4. Bolvin, John O. "An Overview of the Program: Individually Prescribed Instruction." Pittsburgh: Learning Research and Development Center, University of Pittsburgh, 1966. (Mimeographed)

mathematical instruction to the individual learner. Teachers and principals can obtain ideas from these efforts which may be useful in moving toward more individualized instruction in their own schools. For example, IPI suggests ways to handle classroom management problems and offers a sequenced series of learning objectives for elementary mathematics. The Duluth project also provides help with management problems and includes extensive independent learning activities.[5]

Any attempt to individualize mathematics instruction must begin by determining on what bases instruction will be differentiated—in terms of rates of learning, in terms of student interests, and/or in terms of styles of learning and levels of intellectual functioning. Each of these three bases for individualization is discussed in the following paragraphs, along with some suggestions for classroom implementation.

Rates of learning. One type of individualized instruction organizes a set of learning materials so that different students may proceed through them at different rates. For example, a series of math units—these may be individual work sheets or they may be study outlines which refer students to particular pages in a textbook or to some concrete activity—may be prepared and made available for the student to complete as rapidly as possible. Under this arrangement, all students in a class will be expected to complete all the units at some time; but on a given day, each student would probably be working on a different unit.

This approach to individualized instruction necessitates a substantial amount of advance planning and preparation of materials. Before attempting to initiate such a program, careful thought should be given to whether you have the time, monetary resources, and staff to support the development of the requisite materials and to operate the program. Both IPI and Project PLAN, which particularly emphasize differentiation in terms of rates of learning, have invested several years and large amounts of money in developing the necessary materials. However, some schools, working as individual local units, also have succeeded in establishing programs that permit differentiation on the basis of learning rate. Generally, they have been able to do so because:

1. The central administration, the principal, and at least several teachers have been totally committed to the idea.

2. Assistance has been received from people involved in similar projects. Assistance is required to develop instructional materials, to establish classroom management procedures, and to help teachers refine the requisite teaching skills.

5. Direct involvement in these programs is available to only a few schools selected by the project directors. However, all the projects make descriptions of their programs available and, in some cases, arrangements may be made to obtain sample instructional sequences and materials.

3. On-site visits have been paid to schools that are operating such a program—for example, one of the schools engaged in the official testing of IPI or Project PLAN.

4. The district is willing and able to invest money in materials and teacher planning time.

Student interests. Another approach to individualization of instruction adapts curriculum content to the interests of a particular student. When teachers employ this form of individualized instruction, they frequently utilize independent study projects. These projects allow students to pursue their unique areas of interest in addition to participating in group learning activities.

For independent study projects of this type to be effective, it is essential that the teacher work with the student *prior* to the initiation of the project. In this respect, six points are important.

1. The teacher should ask the student, "What do you expect to learn as a result of completing this project?" When a student is unable to provide an adequate response, the teacher should help him formulate a comprehensive statement. It should be clear in the student's mind that he is undertaking the project not just to do something but in order to learn something.

2. The student should be asked to specify how he will demonstrate that he has acquired the skills and knowledge that he intends to learn through the project. Having a student make this statement before he begins his independent activity establishes a definite goal toward which he can work.

3. The teacher should help the student to specify the activities or tasks he will undertake to accomplish his learning goal. As a student does this, he will tend to emphasize books as a major learning tool. This is not surprising since the majority of his school learning experiences probably have been book related. However, one of the major advantages of independent study projects is that they enable students to use manipulative materials and other learning devices which may not be available in large enough quantities for an entire class. Teachers, therefore, will want to help students identify many unique and creative tasks for inclusion in independent study projects.

4. Before beginning an independent project, a student should agree with the teacher on the checkpoints at which he will verify the accuracy and appropriateness of his learning activities. These checkpoints may involve discussion with the teacher, or they may utilize other means of verifying the acceptability of his activities. If, for example, an independent project in measurement requires the student to develop his own measurement instrument, he may need to show the product of his work to the teacher. On the other hand, there are numerous occasions

in a mathematics project when a child can verify his progress by completing and self-correcting an exercise in a textbook that will serve as a review test. From a managerial standpoint, checkpoints requiring interaction with the teacher should be utilized only when absolutely essential for insuring that the student can proceed accurately with subsequent tasks.

5. The teacher and student should establish a definite deadline by which the project is to be completed. Generally, elementary school children require teacher guidance in setting reasonable deadlines. Once established, a deadline provides the teacher with a simple and effective monitoring tool. When a student fails to meet his deadline, the teacher is alerted that a problem may exist and he can take appropriate action. Failure to meet a deadline may mean that the student lacks the skills to complete a particular activity and needs teacher help in acquiring them. Or the learning step may be more involved than originally thought, and the student may merely need more time. Or the student may have lost interest in what he is doing and need help in redirecting his independent learning efforts.

6. The next activity which a student may anticipate doing should also be specified before a project begins. If the student selects this activity himself, it serves two major purposes. First, it provides an interim task for the student between finishing his project and meeting with the teacher to develop a new independent learning activity or participating in a group activity. This is helpful in managing a classroom of thirty students engaged in independent projects. Second, designation of an anticipated activity provides motivation for completing the project.

Several guidelines may be used by teachers in helping students select anticipated activities:

- The activity should be selected by the student.
- The activity may be taken from any subject area, regardless of the area of emphasis of the independent project.
- The activity should be one which can be completed in whole or in part in a short period of time. For example, a student may choose to work with clay; this activity can be terminated at the end of a single work period. Or he may choose a task such as building a model; in this case, work on the model would be the anticipated activity for several independent projects with various parts of the model being built at the end of each project.
- The activity should involve a minimum of noise and confusion so it will not distract other students.

Styles of learning. A third basis for individualizing instruction differentiates in terms of styles of learning and levels of intellectual functioning. Thus, a group of students may be working on assignments concerned with the same general topic or concept, but they will be approaching the task in different ways. An example of this form of individualized instruction may be seen in a lesson in which students are asked to find the sum of fractional numbers with unlike denominators. Some students will not need to use manipulative materials to derive the solution and will be able to explain the reasons for each step they took in reaching a solution. Other students will need to use geometric figures which model the various fractional portions of a whole. Since these students will probably still be trying to develop a firm meaning of a fractional number, they may be unable to verbalize the steps in the solution process. Both groups of students, while having significantly different learning experiences, can be given the same work sheet and can be studying the same broad concept area of mathematics. The individual differences are represented by the levels of complexity and abstraction with which the concept is dealt.

Mastery vs. Coverage

A second major aspect of the instructional process which merits our attention is the extent to which we are concerned with mastery of mathematical content as contrasted with coverage of content. While this concern will be discussed separately from the preceding consideration of individualization, the two are actually interrelated. As soon as mastery of content becomes the goal, individualization of instruction must follow. Likewise, to determine when a student has gained mastery and, concurrently, where individualized instruction should begin, a teacher must be able to determine what the child already knows and can do. This process, commonly referred to as diagnosis, again interrelates with both mastery and individualization.

As is the case with individualization, mastery versus coverage of content is really a moot issue. Any teacher or principal has mastery as a goal of instruction. Yet we know that we fall short of this goal. In fact, Bloom suggests that we expect only about one-third of the students in a classroom to learn adequately what is taught—to achieve mastery. Another third we expect to fail or just get by and the remaining third to learn a good deal but not enough to really achieve mastery. Nevertheless, most students—perhaps over 90 percent in Bloom's judgment—can achieve mastery if they are given the proper learning environment.[6]

What, then, is an appropriate learning environment if mastery is our goal? Bloom proposes that a learning environment which facilitates

6. Bloom, Benjamin S. "Learning for Mastery." *Evaluation Comment,* May 1968. Published by the Center for Study of Evaluation of Instructional Programs, University of California at Los Angeles.

mastery must take into account three fundamental learner characteristics:

1. *The student's ability to work independently.* For example, some students learn well through open-ended, independent learning activities. Others need highly structured, continually monitored teaching-learning situations in which they receive constant supervision from the teacher.

2. *The student's ability to understand instruction that emphasizes verbal communication.* In this regard, differences in student capability to comprehend written directions and explanations are particularly important.

3. *The student's perseverance—the time he is willing to spend in learning.* Students appear to approach different learning tasks with varying amounts of perseverance. Accommodating these differences often necessitates radical changes in our traditional practice of designating a given hour of the day for mathematics instruction. Thus, a student who is highly motivated and involved on a given day should be allowed to pursue his mathematical tasks for the major portion of that particular school day. On the other hand, a student who on that same day finds mathematics to be unstimulating should be permitted to devote only a short period of time—perhaps only ten or fifteen minutes—to mathematical activities. Admittedly, the latter situation could lead eventually to limited mastery, but there are solutions to this problem. The student might be asked to engage in a number of ten- to fifteen-minute work periods throughout a given day. Or his mathematics lesson might be planned so it includes a sequence of learning steps, each of which requires about ten to fifteen minutes to complete. Between each of these steps, the student could engage in another area of work.

The teacher who attaches a high degree of importance to students' mastering mathematical concepts and skills will also find that he requires precise information about each child's mathematical progress. To obtain this information, he must use more than paper-and-pencil tests as sources of information. In particular, the teacher will need to have individual discussion sessions with the student and to observe him while he is working.

In an individual discussion session, for example, a teacher might check a child's concept of number by saying, "Show me six." If the child takes six objects, circles them with his hands, and says, "That's six," the teacher has a fairly good indication that the child understands the concept of number. However, if the child counts the six objects—one, two, three, four, five, six—and then points to the last object and says, "That is six," the teacher has a good idea that the child can count but does not understand number.

Classroom observation gives the teacher additional diagnostic information. For example, by watching as a child counts the number of red, blue, and green beads in a jar, he can assess the child's understanding of and ability to use addition. If the child is able to put together subgroups of red beads as he takes them from the jar and then combines these groups to find the total number of red beads, he probably understands addition. But if he waits until he has taken all of the red beads from the jar and then counts the total group, he probably is not yet using the addition operation.

Holt has suggested several other behaviors that a teacher can observe to determine whether a student understands a particular mathematical concept.[7] For example, a student may demonstrate his understanding of the concept of commutativity by: 1) stating the concept in his own words; 2) recognizing $4 + 3 = 3 + 4$ as an example of the concept; 3) providing additional examples; 4) using commutativity to derive the solution to 9×6 when he knows that $6 \times 9 = 54$; and 5) stating an example of the opposite of the concept—explaining that a number system would not have the commutative property if $4 + 3$ equals 7 but $3 + 4$ does not equal 7.

Another important requirement when mastery is the goal of instruction is a record of the concepts and skills the student has mastered. This record might take the form of a series of sheets, each of which states a key concept and lists behaviors indicating mastery of the concept. For example:

Name: ____________________

Concept: Identity of Equivalent Sets

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1. Is able to match members of sets in one-to-one correspondence.

   ______Yes ______No

2. Is able to determine if one set contains more members than another set.

   ______Yes ______No

3. Is able to determine whether one set has fewer numbers than another set.

   ______Yes ______No

4. When given several sets of objects, can identify the sets that are equivalent.

   ______Yes ______No

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7. Holt, John. *How Children Fail.* New York: Dell Publishing Co., 1964.

By using record sheets of this nature, the teacher will have specific information about each student's progress toward mastery of key concepts.

Conceptual Understanding vs. Computational Skill

A third concern about the teaching-learning process in mathematics is whether instruction should stress attaining computational skill or conceptual understanding. Of course, this is not an either-or question. As Lloyd Scott, University of California at Berkeley, has suggested, teaching children to understand mathematical concepts should not detract from their ability to compute and solve quantitative problems. Rather, it should enhance this ability. Thus, if a student understands the operation of addition as the combining of two sets to make a new set, he should also be able to compute the results of that operation.

Within the context of contemporary mathematical programs, however, the link between the concept and the computational algorithm sometimes is not established. We often separate the conceptual experience from the computational experience, rather than building one upon the other. This parallels our tendency to provide children with all inductive or all deductive experiences. When teaching multiplication, for example, we may provide a wide variety of inductive experiences—construction of arrays, repeated addition involving the manipulation of beans, soda straws, and the like. After the students have had several opportunities to carry out such processes, the teacher asks them to use these experiences to explain the concept of multiplication. Some students are able to do so, and others are not. Yet we frequently fail to follow up with a deductive experience for those students who still have not mastered the concept. A better approach would be for the teacher to take the students who do not understand the concept and present them with the definition of multiplication followed by demonstrations of the concept and a reiteration of the wide variety of applications. In this way, the demonstrations of the concept, the definition of the concept, and the computational outcome and application of the concept can be linked together for all students.

The question of emphasis between conceptual development and computational skill also relates to the practicality of real-world use of elementary mathematical learning. Ability to utilize an algorithm quickly and accurately to determine a quantitative amount is generally recognized as essential to productive employment in the adult world. Thus, many educators wonder at what point skill in using the algorithm should supersede the development of conceptual understanding. The solution may be found in the area of applied mathematics in which a student uses mathematical operations in a science experiment, social science investigation, or some measurement task. In these instances, some form of counting may help both to explain the concept and to establish the computational skill.

When reviewing a mathematics program, principals and teachers may well want to look at the ways in which they are helping students to apply their mathematical skills, rather than focusing on a decision of when they should stress computational skills over conceptual understanding. Both are essential, and appropriate applications of mathematics can provide a vehicle for supporting both goals.

Concrete vs. Abstract Experiences

A fourth aspect of the instructional process that we need to consider is the extent to which mathematical learning utilizes concrete learning experiences as compared with abstract learning experiences. The work of Piaget has important implications in this regard.

Piaget identifies four stages of development: the sensory-motor stage, the preoperational or representational stage, the concrete operations stage, and the formal operations stage.[8] The child reaches the stage of concrete operations at about age 7 and remains in it until he is approximately 11 or 12 years old—a period of time which encompasses most of the elementary school years. During this stage of development, concrete operations—such as classification, serialization, reversibility, transitivity, and all the fundamental operations of the logic of classes—derive from the child's interaction with his environment. Through the use of these processes, the child can shift back and forth between part-part and part-whole relationships. He can form and describe classes and subclasses. Thought becomes independent of the child's own focus; it is loosened from its ties with perceiving and acting. Yet the child still shows capability of using these operations primarily in relation to objects he already has manipulated or in relation to experiences that come from manipulative experiences that occur as the child formulates his thoughts. When asked to perform the same operations through verbal propositions, the child generally is unable to do so unless he has had the previous concrete experience.

The role of manipulative experiences in providing the concrete learning framework implied by Piaget's theories is obvious. Manipulative materials are particularly important in mathematics because many of the major accomplishments of the period of concrete operations are mathematically oriented. For example, a child generally cannot establish a part-to-part and part-to-whole relationship (and therefore would find it difficult to understand division) until he has reached the stage of con… …erations. Similarly, it is not until a child reaches the level

8. Inhelder, Bärbel, and Piaget, Jean. *The Growth of Logical Thinking.* New York: Basic Books, 1958.

of concrete operations that he can reverse what he has done—that he can return to the point of origin. Until he reaches this stage, the statement, "If $3 + 5 = 8$, then $8 - 5 = 3$," would have little meaning to him. In order to achieve the initial capability to perform the operation of reversibility, the child requires concrete manipulative experiences in which he performs and then undoes a number of different actions.

Another example of the importance of manipulative experiences for elementary mathematics instruction may be found in the fact that a child can manipulate symbols that represent the environment during the period of concrete operations, but the symbols have meaning for him as a result of his interaction with the environment. Thus, before the symbolic numeral 3 can be meaningfully manipulated, the quantitative value of 3 must be acquired through interaction with the environment—specifically, through manipulative experience. As the child progresses through the period of concrete operations, he increasingly differentiates between the signifier (the numeral 3) and the significates (the quantity which the numeral 3 represents). This allows him to perform meaningful computations during the later elementary years, so long as he acquired this differentiation through concrete experience.

Manipulative materials, therefore, should be used extensively in a child's early mathematical experiences so he may be actively involved with a concrete learning experience. And, regardless of age, a child's initial experience when a new concept is introduced will be enhanced by manipulative activities. Equally fundamental, it is extremely important that the sequence of mathematical learnings be planned so that a child has reached the developmental stage that will allow him to comprehend what is being presented. If this is not done, our teaching of mathematics will be ineffective.

AN INSTRUCTIONAL SETTING FOR MATHEMATICS EDUCATION: A MATHEMATICS RESOURCE CENTER

William D. Hedges Edith McKinnon

The instructional setting for a modern mathematics program should be drastically different from what we commonly find in elementary schools today.

Imagine the following scene. About 100 upper elementary students, each pursuing a somewhat different program of study in mathematics, are working in a large open space surrounded by small rooms for seminars, special projects, and teacher planning. The children are taking considerable responsibility for their own learning, guided by their teachers who have specific, current information about each student's progress. Many of the students are working individually or in small groups with all sorts of learning materials—everything from worksheets to equipment for conducting a complex measurement project. All are involved in a task which is individually challenging but for which they have the prerequisite skills and knowledge. In one of the small seminar rooms a teacher is helping eight or ten students who have a specific problem in common. Another teacher and an aide are answering questions and checking progress at the various work centers in the open space area. A third teacher is preparing for a large-group demonstration and lecture to be given later in the day.

"Impossible," you say. Not so. Actually, programs which contain many of these elements are operating today in a number of elementary schools. But this kind of mathematics program is not common. Mathematics is normally taught with little more than a chalkboard, a few textbooks and workbooks, a protractor, compass, and rule, and maybe a few simple counters and geometric shapes—all in a group-paced program followed by periodic group examinations. And this is the case even in the newer mathematics programs.

Such a setting for teaching mathematics is unacceptable because it violates some of the basic requirements for effective teaching and learning. Children must have an active role in their learning with opportunities for many exploratory experiences which foster discovery, critical thinking, and the search for alternatives. They need to have direct, concrete experiences first, then develop intuitive ideas, and finally strive to describe their experiences in words and symbols. All of this requires a learning environment with a relatively informal atmosphere and with provision for a wide range of individual and small group activity. Equally important, it requires an instructional setting which is oriented to the individual learner rather than to the group, and which can cope effectively with the differences in learning style and pace among students.

Such a learning environment may be provided through a Mathematics Resource Center (MRC)—the type of instructional setting which we imagined at the beginning of this article. Let's examine more closely what such a center would be like and how it might be organized and operated in an elementary school.

Physical facilities. Basic to the learning environment we are describing is a drastic rearrangement of space so that the physical setting will facilitate the necessary learning activities. Obviously, the old 30′ x 30′ classroom with 30 chairs is obsolete for this kind of mathematics program. The old-fashioned classroom was designed for teacher lecture-demonstrations at the chalkboard, student recitations, and some work at the chalkboard by students. There was a minimum of interaction among the students; most of them worked quietly at their seats, doing their assignments after an explanation by the teacher.

The new program requires that students get up and move around. Thus, a major physical characteristic of the MRC is flexible space. In addition, acoustics must be excellent, lighting must be superior, and the area must be large enough to accommodate a large group—perhaps about 100 students. If feasible, it is helpful if small work rooms can be partitioned off around the periphery of the large central area. These provide teachers with space for preparing materials and individual exercises and for meeting with small groups of students. They also may be used by students for some special projects which should be left undisturbed for a period of time or projects which produce noise or confusion and should not be located in the central area.

Some schools which have set up an MRC have had available a large open area which could be converted for the purpose or even one or two extra classrooms which could be opened up with another room. In other cases, two portable classrooms might be used as one large area. If none of these alternatives is available, it is possible in some schools to remove part of the walls between three classrooms and to use the central room as a materials center.

Materials and equipment. A wide range of materials and equipment should be provided in the MRC.

First, there must be tables arranged in alcoves or centers where students can conduct experiments. In one grouping, there might be balances with a variety of weights with which the students could experiment. Some of these experiments would be inductively designed so that students could reach certain generalizations themselves.

Another grouping might contain a number of materials for independent exploration, using teacher-prepared directions for guided discovery. Examples of such materials are the Madison Project independent exploratory shoe boxes, attribute boxes, creature cards, mathematical models, and solids.

Still another work center would provide a variety of measurement equipment: cuisenaire rods, multibase arithmetic blocks, containers for pouring and measuring, an abacus, geoboards, caliphers, trundle wheels, inifix cubes, and place value cubes. Also available should be yardsticks, metric rules, foot rulers, steel tape, map measurer, irregular figures for area measurement and estimation, clocks and timers, pan balance scales, and calibrated and uncalibrated centigrade and Fahrenheit thermometers. Simple hand-cranked calculators are valuable, as well as simple adding machines and slide rules, pocket calculators, compasses and protractors, number lines and pegboards. Marked cards on which to wind different thicknesses of knitting yarn for comparisons and fraction development also can be utilized.

The MRC should contain, too, a center for mathematical games and puzzles—for example, checkers, dominoes, Tower of Hanoi, chess, Go, and dice games. Mathematically based games can bring fun and excitement to learning mathematics—something that is too often lacking.

There will be a reading center supplied with materials on the history of mathematics, biographies of great mathematicians, science fiction, articles from contemporary magazines on popular mathematics (including the electronic computer), and reference books for mathematics research.

Another center will be devoted to listening and viewing. Here there will be listening stations with earphones and files of tapes containing lessons prerecorded by teachers and indexed by topic which can be checked out by students. These tapes will provide three major types of instruction: 1) basic explanation of a mathematical concept, 2) specific comment about a particular diagram in a given reference—the kind of tape which talks to the student while he looks at pictorial materials; and 3) drill types of lessons. Also in this center may be five- and ten-minute film clips (along with earphones and a small screen) which students may have the option of seeing once, twice, or as often as necessary. A number of carefully selected filmstrips on specific skills or topics, such as the division of fractions, should also be easily available from the teacher or an aide for student use.

There must also be a large storage center which shelves a multiplicity of arithmetic exercises. These will include: 1) short units, worked out in advance, with pretests, assignments to read, problems, and posttests; 2) programed materials cross-indexed by topic and level of difficulty; and 3) hundreds of carefully planned individual exercise sheets. Located here will also be a number of well-constructed teaching machines such as the Cycloteacher.

Obviously, we have mentioned only a few of the possibilities for equipping a Mathematics Resource Center. Many other activities with supporting materials might be provided. There is virtually no limit—given funds and imagination.

Work spaces. Implict in the preceding discussion of materials and equipment is the need for space for a variety of uses. There should be: 1) space for small group and individual activity; 2) a large amount of storage space with shelves and cabinets; 3) open floor space for drawing and measuring, for rhythm games involving geometric patterns, and the like; 4) space for teacher-centered activity, including a chalkboard, flannel board, and seats for students; 5) some individual student carrels equipped with small screens, earphones, and desks; 6) an area for exhibition of students' work; and 7) at least one virtually noise-free area where a great deal of normal conversation can take place without disturbing others.

Instructional procedures. The critical characteristic of instructional procedures in the kind of program we are discussing is that they are designed to promote appropriate individual involvement in learning. This means that group instruction is used when, and only when, it is appropriate for all the students in the group. This might be, for example, when several students are having difficulty with the same concept or skill. However, group instruction is not utilized when students can learn more effectively through differentiated activities.

One of the advantages of a Mathematics Resource Center is that it enables a teacher to assign individual students different tasks for learning which they can carry out with considerable independence. It permits instruction to be appropriate for the student who profits most from study in complete isolation and for the student who works best in a small group that allows for mutual help. It permits accommodation to the needs of a student who achieves a great deal in working with programed learning materials as well as to the needs of a student who benefits from listening to an explanation of concepts from a tape recorder. It allows a student who needs a great deal of opportunity to view filmstrips and other visual aids to have this experience, while another can be working through some of the content in a textbook. And it provides for the student who needs to manipulate concrete materials as well as for the one who gains most from discussion and active verbal participation. All of these differences in styles of learning can be accommodated simultaneously in a well-designed Mathematics Resource Center.

Similarly, differences in learning pace are taken into account. Since many activities are conducted in small groups or individually, it is not necessary that all students learn the same thing at the same time. Each student may proceed through learning tasks at his own rate. He may take a mastery test when he decides that he is ready. And these tests are diagnostic in nature so that both the student and the teacher know specifically the areas in which he is strong and those in which he needs further work. On the basis of this information, the student may either be reassigned to work which will help him to master concepts and skills in which he is weak or he may undertake a new topic.

Organization and management. At first glance, an instructional setting such as we are proposing seems to present enormous problems of organization and classroom management. Most teachers and principals who have not worked in such an environment and have never seen one operate successfully are somewhat staggered by the prospect. This attitude of caution actually is wise—there is no point in establishing a Mathematics Resource Center if it can't be operated successfully. However, many schools have already proved that this type of instructional setting can be managed effectively.

First, it is perfectly obvious that a team of teachers is required (perhaps three), along with a teacher aide and at least one clerk. An alternative staffing pattern might be a master teacher, several beginning teachers, and two or three clerks. However, the most necessary ingredient is teachers who are sincerely interested in such an approach and who are willing and able to work together. Teachers absolutely must be able to work with each other, to compromise, and to support and reinforce each other's efforts. In addition, the importance of clerks should not be minimized. It is impossible for even the most competent teachers to function well in an environment of this kind without really adequate clerical support to prepare materials, file, keep records, and the like.

Second, complete and up-to-date records must be maintained for each student. The teachers must have readily available in written form a profile of the skills and concepts which each student has already mastered and of the activities in which he is currently engaged. This is essential for diagnosing problems, for planning subsequent activities, for knowing when to check on a student's progress in an independent project, and for determining when a group of students engaged in a similar unit of work might well be pulled out into a small group for a teacher-led discussion and presentation. The goal is not simply to have students involved in a variety of stimulating activities; the purpose is to foster learning, and this requires knowledgeable adult direction.

Third, students should be permitted and encouraged to take as much responsibility for their own learning activities as possible. This is desirable both as an end in itself for the students and as an aid to classroom management. Most upper elementary students are quite capable of this type of self-direction and will respond favorably when they are given a measure of responsibility for their own behavior. In fact, a number of schools have discovered that behavioral problems are lessened, rather than magnified, when students are removed from the highly controlled, teacher-dominated traditional classroom. Students working in a Mathematics Resource Center can, for example, pick up their own folders each day to check for teacher instructions. They can check the bulletin board to see whether their name is listed for a special group activity. They can self-correct many diagnostic and review tests. And they can take considerable initiative for seeking help when they have difficulty.

There are many techniques which may be used to facilitate organization and management of the MRC. Principals and teachers who are interested in exploring the possibilities of this type of instructional setting would do well to contact and, if possible, visit other schools which are successfully operating such a program.

The principal's role. A program such as we are describing has little chance without the principal's wholehearted support. By being indifferent, even if not overtly hostile, a principal can make the whole undertaking impossible. Without his help, who is going to press for the increased budget for materials and equipment with the superintendent and board of education? Who is going to help plan the new facilities or ways to modify the old ones? Who will coordinate the inservice and curriculum work during the school year and perhaps in the summer? Who will be on the alert for schools with similar programs which might provide helpful assistance? Who will push for laboratory clerks and aides? Who will encourage teachers to share ideas and to work together?

A Mathematics Resource Center has real potential for improving mathematics instruction in the elementary school. The principal is the key to whether this potential can be realized—the person who can make the biggest difference in successful planning and implementation.

HELPING TEACHERS IMPROVE MATHEMATICS INSTRUCTION

Elizabeth J. Boisclair

In this day of specialization, we in the elementary school have almost lost sight of a very important fact: A qualified elementary classroom teacher is, in his own right, a specialist—a specialist in elementary education. This statement is sometimes implied, often ignored, and frequently denied. I want to state it forthrightly and offer it as a premise for this article.

More and more, educators are coming to appreciate the vital contribution the elementary school makes in developing basic concepts and establishing firm foundations in every subject matter area. Demands increase and pressures mount as specialists in every field dictate the role of the classroom teacher in relation to particular subjects. But seldom is this person, the teacher, identified and analyzed. Who is this qualified classroom teacher on whom so many depend?

1. He is a person who has a genuine concern for children and feels personally involved in the progress of each child in the classroom.

2. He has a background of knowledge and experience about human growth and development and knows how children are alike and how they are different.

3. He is knowledgeable in the psychology of learning and can apply this knowledge to elementary education.

4. He possesses a fundamental background of knowledge in basic subject matter areas.

5. He is a person who is interested in personal growth, is aware of the necessity for continued learning experiences for himself, and accepts change as part of growth.

Because there has been a shortage of qualified classroom teachers for many years, some of our classrooms are manned by persons who do not meet these qualifications. It is at this point that our vision is blurred and we are tricked into focusing improperly on the role of the unqualified classroom teacher in relation to a particular subject matter area. If the teacher is not a specialist in elementary education, we must recognize first of all our responsibility to develop some of the characteristics of a good elementary teacher. This may mean trying to change behavior in two areas important to good teaching: 1) basic attitudes, and 2) basic skills and knowledge.

This reeducation cannot take place in a vacuum; it must occur through some practical classroom experience. The program of instruction in elementary mathematics may serve as the vehicle. But let's remember that, in this instance, the mathematics program becomes the *means* through which the teacher's ability as a classroom teacher may be improved. The related inservice program should emphasize elementary education and show how its guiding principles are used in mathematics instruction; otherwise, the problems of the unqualified classroom teacher will be compounded.

In contrast, when we are discussing the role of the qualified classroom teacher, the program of instruction in mathematics becomes the *end*. We can assume that certain attitudes, skills, and knowledges related to elementary education are present and that the goal of the inservice program is directly related to the improvement of instruction in elementary mathematics.

What can we say to *this* teacher—the qualified specialist in elementary education—about the modern approach to elementary mathematics? How can we prepare him for anticipated changes in curriculum and method?

Improving Attitudes Toward Mathematics

First of all let's consider the teacher's own attitude toward mathematics. Does he like it? Is he afraid of it? What image is evoked in his mind when mathematics is mentioned? What proportion of the day does he devote to mathematics instruction? What proportion of his planning time does he devote to the subject? Questions of this nature, when discussed openly and frankly, can be very revealing to the teacher. He will become more aware of his own attitude and may recognize that it probably reflects the type of mathematics instruction to which he has been exposed.

The attitude of a teacher and the responding attitudes of the pupils have a critical effect on a mathematics program. A conventional, traditional approach to instruction may produce a feeling that mathematics is one of the necessary evils of life—a dull, rigid collection of facts to be memorized and symbols to be manipulated according to innumerable rules. This method passes my father's test for determining whether a medicine is "good for you"—it must either hurt or taste terrible. A modern approach to mathematics, one which is not characterized by rote memorization and endless drill, may create quite different attitudes toward mathematics. It can help teachers and pupils to view mathematics as a body of knowledge that represents man's dynamic response to his quantitative needs in a changing world, as a creative product of man's unique ability to think abstractly. This view of mathematics suggests a teaching method which is just as "good for you," but less painful and more productive for both teachers and pupils.

To improve attitudes toward mathematics we also need to assure teachers that their own fund of knowledge in mathematics is recognized and will become the basis for their continued learning in the field. Let's begin where the teacher is. Teachers who have never been exposed to the "whys" of all the "hows" they have mastered are spellbound as this door is opened. They are eager to satisfy their own curiosity about many questions that may have been hammering in the backs of their heads for a long time. Often understanding will break through very suddenly. I remember my own experience as a teacher attending a meeting many years ago when someone said casually, "Division can be related to repeated subtraction." This unassuming little statement has become so commonplace now that it is fast becoming hackneyed. But on that day, I couldn't believe it! Instead of taking notes, I surreptitiously jotted down:

$$\begin{array}{r} 28 \\ \underline{-7} \\ 21 \\ \underline{-7} \\ 14 \\ \underline{-7} \\ 7 \\ \underline{-7} \\ 0 \end{array}$$

I counted the sevens—and sure enough, there were four of them! I don't know what else happened at that meeting, but I had experienced the thrill of discovery and revelation. If teachers are given the opportunity to explore what they already know and to experience the excitement of discovery for themselves, two things will happen: They will want to expand their own knowledge, and they will want to share this kind of experience with their pupils.

Providing Assistance in Planning Instruction

Once teachers have developed positive attitudes toward mathematics, more specific guidance can be given in planning learning experiences for students. Throughout this process, a clear distinction must be made between what is offered as background for the teacher and what is recommended as part of the elementary mathematics curriculum.

The first step is to help teachers to see elementary mathematics in relation to all of mathematics—to see the basic structure and the fundamental mathematical concepts that flow from preschool to graduate courses. Once the relationship between what is taught on the elementary level and what is learned later is clearly seen, it is easier to avoid the common pitfalls. No teacher wants to spend his time and talents teaching children something that will have to be unlearned later.

Second, attention should be given to how teachers can meet the responsibility they feel for teaching the basic facts and providing practice in skills. All too often, the impression is given that this is an old-fashioned and discredited idea. Practice and drill are integral parts of every mathematics program, and we need to assure teachers that these will remain important in elementary school. A modern program simply offers a means whereby facts can be effectively taught and skills really improved.

For example, every teacher has heard a child's plaintive cry, "I'm stuck!" Loosely translated, this means, "I've forgotten. What do I do now?" There's not much help in the teacher's brightly encouraging response, "Think!" If meanings have been developed and relationships established, if the habit of looking for different paths to a solution has been encouraged, there might be a better answer to that plea. To be most effective, practice must be provided according to sound educational principles. Understanding and meaning must precede practice; practice must carry a built-in safety device insuring reasonable success and must provide for a pattern of steady, cumulative, sequential, successful responses. Elementary teachers earnestly desire to plan such a program. They are vitally concerned with the answers to such practical questions as: How can readiness for a new mathematical idea be determined? What are the steps in a natural, sequential development of a mathematical concept? What varied and stimulating experiences related to the concept can be provided for elementary school children?

Third, we should emphasize with teachers that contemporary mathematics is for *all* children. A modern program is built upon a meaningful approach—meaningful for each child. The thrust is toward the development of critical thinking at every level of maturity. As opposed to all children learning "the one right way," the teacher and children together explore different ways of solving a problem or a variety of methods that can be used to record computation; they analyze statements to determine if they are true or false; they classify and organize facts to determine relationships. Each individual has the opportunity to respond to an idea on his own level of understanding; the teacher, meeting him there, can lead him into an ever-widening experience. An emphasis on the mainstreams of mathematical thought makes it possible for the child to meet the same basic idea over and over again in many different situations until he becomes aware of the fundamental nature of the idea and can apply it to new situations. This variety in techniques, procedures, and materials makes it possible for a teacher to adapt the program to meet the needs of all his pupils. And it is to this kind of program that teachers will respond.

Fourth, one facet of the modern approach to mathematics that has more relevance to the skills of an elementary teacher than to those of any other educational specialist is the reemphasis given to the concept that mathematics is a language. Who can understand better than elementary teachers the following sequence for developing ability in the use of a language?

a. providing opportunities for conceptual experiences

b. relating the conceptual experience to oral language

c. associating the oral language with the related written symbols

d. providing opportunities for the development of skills in reading, writing, and interpreting the symbols.

This is what the modern approach to mathematics is all about. The so-called traditional approach reversed the order; it began with the symbolism and worked backward toward meaning. Usually the step relating to comprehending and interpreting the symbolism was completely ignored.

Once an elementary teacher can begin to understand mathematics as a language, a significant element of an inservice program has made its impact. The teacher is on sure ground here. He is dealing with his specialty—the psychology of learning as it relates to young children. Since mathematics is concerned with abstract ideas, experiences must be provided that will lead children from the concrete, physical world to the abstract mathematical world. Exploratory experiences with concrete materials build readiness for a meaningful introduction to the abstract concept. Since emphasis is on understanding, not on the use of technical terms, children can discuss the idea orally in their own words.

When the related symbolism has been introduced, the same procedures and techniques used in developing skill in reading can be applied to developing skill in reading, comprehending, interpreting, analyzing, and organizing ideas expressed in the language of mathematics. Interest in this phase usually runs high because it builds on skills which the teachers already have—and because it is an exciting way to teach. How dull it can be to teach a child to look at $2+3=5$ and parrot, "Two plus three equals five." How rewarding it is to develop an understanding of numbers, relate this understanding to numerals, introduce different ways of naming numbers, find out how to express ideas about number in number sentences—and then hear a child explain in his own words that $2+3$ is the same number as 5 and see him demonstrate that this sentence tells us something that is true!

It is at this point that teachers begin to be enthusiastic about mathematics. As one teacher said to me, "Now I see how I can be as creative in my teaching of mathematics as I can be in other areas. I always hated mathematics because I didn't really feel that I was functioning as a teacher. I was just feeding children facts. Now I can help children learn to think." If, as in this case, the teacher's own background of mathematical experiences has been limited to naming and identifying mathematical symbols and manipulating them to "come out right," an incentive is established for learning precise meanings and correct ways to read and interpret mathematical ideas.

Fifth, as we work with teachers, let's highlight the importance of the classroom teacher as a teacher of all subjects. We can increase awareness of the relationship between mathematics and other areas of the curriculum, thereby broadening the teacher's own understanding and helping to make his total program more dynamic, more effective, and more practical. Consider the implications of the following questions for elementary school teachers and for mathematics specialists interested in improving instruction in elementary mathematics:

- How can we keep abreast of current developments in our technological world without an increased understanding of mathematics? Take the front page of any daily newspaper and cut out or cover over every direct or indirect reference to numbers, space, size, time, measurement. What is left of the page?

- How can we interpret tables, charts, diagrams, and graphs in science and social studies without an understanding of estimation, relative sizes, comparison of numbers, the association established between numbers and points? Think of discussing history without a concept of time, or geography without an understanding of spatial relationships.

• How can we use maps intelligently if we are unaware of the imagery of geometry? Is the dot really the city? Is the city located where its name is written or where the dot is placed? How do we contrast a state with a state boundary?

• How can we make full use of the problem-solving approach recommended for most curriculum areas if we are not familiar with the logic of deductive and inductive proof? How do we arrive at generalizations and test them out?

• How can we solve the quantitative problems of our daily life if we cannot structure a physical situation which is described in words so that we can state it in mathematical terms?

Such questions as these illustrate the point that learning basic mathematical concepts and skills cannot be confined to one period of the day. To be meaningful and to be applied in various contexts, the ideas must be developed and used "where the action is." Here lies the advantage of the elementary school teacher who is responsible for instruction in several subject matter areas.

To summarize, then, when we are working with qualified elementary school teachers to develop a contemporary program of mathematics instruction, we should:

1. Explore our feelings about mathematics and consider the influence of attitudes on our teaching and on our pupils.
2. Use what knowledge we have about mathematics in developing increased understanding of basic concepts and greater facility with mathematical operations.
3. Recognize the relationship between what we teach in the elementary school and what pupils will learn later in mathematics.
4. Develop skills of providing productive practice and drill in basic mathematical facts and skills.
5. Emphasize that an effective mathematics program will meet the needs of *all* children.
6. Point out that mathematics is a language and apply teaching-learning principles which we use in a language arts program to the teaching of skills in mathematical thinking and recording.
7. Relate mathematical concepts and skills to other instructional areas.

We take such an approach in full confidence that the qualified classroom teacher can acquire the knowledge and skills that will enable him to teach mathematics effectively in the elementary school. This does not deny the need for specialists in an elementary mathematics program. Elementary teachers need and desire the services of consultants who can share the results of research in mathematics teaching, can help them grow in knowledge of mathematics content, and can introduce new materials and techniques into the classroom. Teachers need guidance in defining objectives and evaluating the instructional program. With this kind of resource help, teachers will be receiving an ongoing inservice mathematics program.

Where Might We Err?

Hindsight is better than foresight—if we can look back, learn from our mistakes, and move forward. The guidelines discussed in this article are the result of ten years of labor in inservice programs in elementary mathematics. As we tried to cope with change, where did we err?

Have we given lip service to the importance of attitude and then set up courses that were destined to make teachers feel inadequate and insecure, that convinced them that only mathematics specialists could teach elementary mathematics?

Have we overwhelmed teachers with too much, too fast, and neglected the necessity of meeting teachers where they are?

Have we overemphasized new content too soon instead of encouraging a new look at old content?

Have we become so entranced with the theory of mathematics that we have lost sight of important practical applications of mathematics?

Have we been so engrossed with broad, fundamental concepts that we have neglected to call attention to the very specific objectives that lead to understanding of the concept?

Have we given insufficient attention to methodology and the application of basic principles of learning?

Have we given teachers new content to present and new ways of teaching it but left them with old, inappropriate, and unrelated means of evaluating what was supposed to have been taught?

The answers to many of these questions might well be, "Yes." If so, each one represents a danger signal as we plot new paths to our goal of helping classroom teachers to define, understand, and practice their new role as mathematics teachers in the elementary school.

CHARACTERISTICS OF GOOD INSTRUCTIONAL PROCEDURES

Jesse D. Dillon, Jr.

Sound instructional procedures do not just happen. They are the result of careful, thoughtful planning. Good instruction is deliberately designed to support the objectives of instruction and to be consistent with the learning requirements of students.

The student behaviors which instruction seeks to produce directly affect teaching strategies. For example, in a contemporary mathematics program, students are expected to demonstrate an understanding of basic mathematical concepts and an ability to apply those concepts in discovering new generalizations. Instructional procedures which support such objectives are very different from those that might be used in a program which sought only immediate recall of basic computational skills. Similarly, a modern mathematics program seeks to promote the student's ability to think critically and to apply mathematical concepts independently in problem solution. The processes of teaching and learning which are most appropriately employed in developing these student behaviors are again different from those used in a program concerned only with the ability to reproduce facts and skills presented by the teacher.

Good instructional procedures are also consistent with the learning needs of students. In mathematics, as in all areas, this implies attention to differences in ability and previous achievement and suggests that instructional procedures must be differentiated for individual pupils.

It also implies that knowledge about how children learn mathematical concepts and skills should be utilized in planning instruction.

An elementary mathematics program which employs instructional procedures consistent with contemporary learning objectives and with student needs will evidence certain characteristics. Eight such characteristics are discussed below along with illustrations of appropriate teaching techniques.

Good Instructional Procedures Develop Understanding

There was a time when memorizing certain processes constituted the mathematics program. By rote method, the student learned how to add, subtract, multiply, and divide. It mattered little whether he understood what he was doing or why so long as he used, step by step, the prescribed processes.

The old rote memory approach to teaching mathematics has now given way to a process that develops an understanding of what is being done. This is a more practical way of doing things because a youngster will remember the processes he understands and tend to forget those which he memorizes without understanding.

I remember visiting a classroom where the students had just been working with the concept of area. They had come to a definite understanding that area was a surface measurement and understood how the area of regularly shaped objects—such as squares, rectangles, and triangles—could be determined and expressed in square units. While these youngsters were summarizing their findings, one pupil asked the teacher, "Can we find the area of a circle?" Rather than telling the class that there was a formula for solving this problem, the teacher suggested that the youngsters apply what they had already learned about area to what they could observe about the circle. When the students had determined that the area of a circle could be computed, the next question was, How? At this point, the teacher gave the class a definite assignment—to experiment with various circles to see if they could come close to finding the area.

The next day, the students returned with a variety of approaches to the problem. Some had tried to determine a circle's area by estimating number of square units. Others, however, came armed with the formula, $A = \pi r^2$, which their fathers had given them. The teacher did not simply accept this formula but reviewed again with the class some of their rules for solving problems. One of these rules was not to accept any solutions they did not understand. Since no one knew what *pi* was, the next problem was to discover its meaning.

At this point, the teacher instructed the students to bring to class cans of all sizes and a piece of string to go around each one. The following day, as the students began measuring the circumference and diameter of the cans they had brought, excitement mounted. The measurements were listed on the board, and the students were asked to look at them, crude though they were, to see if they could observe something. It soon became very obvious to the class that in every case the circumference was a little more than three times the diameter. "Could it be," one boy asked, "that *pi* is the relationship of the circumference of a circle to its diameter?" The class seemed to think this was the case, and a little more reading would confirm their belief. But the important point was that the students discovered the relationship for themselves.

The teacher continued to use similar instructional procedures to help the youngsters discover how the relationship they had found could be applied to finding the area of a circle. As a result, the students understood the process in a way which would have been impossible had the teachers simply told them how to find a circle's area. While this kind of teaching is time consuming, it is time well spent when it achieves the desired results. If anyone were to ask these students some time later how to find the area of a circle, they would be able to explain the method with complete confidence. This kind of knowledge, based on understanding, can be applied in other situations and will remain much longer than the mere memorization of a formula.

When evaluating instructional procedures, the principal would do well to ask himself, "Is this particular class developing an understanding of the mathematical concepts presented?" If it is a good mathematics program, the answer will be "yes."

Good Instructional Procedures Utilize the Method of Discovery

When children are encouraged to discover for themselves the solution to a problem, they understand better not only the problem itself but also the ways in which it can be solved. Children need to know that in many instances there is no one right way to solve a problem, that generally there are many ways in which a problem may be solved. By sharing each others' discoveries, students may determine that some ways of solving a problem are better than others, both in terms of time and in terms of the accuracy of results.

Suppose a third-grade teacher asks, "How many chairs are there in the room?" and encourages each student to find the answer in his own way. Some youngsters may need to get up and count every chair; others may count the number of chairs in each row or at each table and add the subtotals. Still other youngsters may quickly arrive at the shortcut of counting the number of rows and the number of chairs in each row, and then multiplying one by the other—assuming, of course, the same number of chairs in each row. By encouraging each child to solve the problem in his own way and then to discuss his method, the teacher helps students gain greater understanding.

In the same way, a first-grade child may be asked to make piles of a certain number of popsicle sticks, bottle caps, buttons, or other concrete objects. By so doing, he discovers the concept of numbers and will be better equipped for more sophisticated problems later.

There are other examples of the discovery method. A first-grade youngster, when confronted with a problem of equally dividing his materials with a classmate, soon discovers that this can be accomplished by simply stating over and over again until the supply is depleted, "One for you and one for me; one for you and one for me." By equally dividing what he has with another classmate, he has established a second and equivalent set and discovers the concept of equivalents.

One of the most exciting lessons in addition combinations that I have seen occurred in a first-grade class where the children worked in small groups and discovered the addition facts by the union of groups or sets of objects. A variety of addition combinations were derived as the children worked in the groups and then in pairs and individually. Later, I observed these same children working at an abstract level with great understanding because of their previous experience in discovering various addition combinations through the union of sets.

Good Instructional Procedures Are Inventive

An inventive classroom teacher is constantly searching for ways to make lessons more meaningful and interesting to children. He is always on the lookout for better ways to illustrate important concepts and recognizes that children respond positively to fresh approaches to learning. The inventive teacher, then, brings a freshness, an air of excitement, to the instructional process.

A good illustration of inventive teaching is a fraction party which was held for a fifth-grade class of rather low-achieving, but highly motivated, youngsters.

The basic materials for the fraction party were four large cakes iced on top so that they were marked off into fractional parts. One group of children was given a cake marked in thirds and another cake marked in ninths; a second group had a pair of cakes marked in halves and eighths. The youngsters could not share in the cakes until each pair was equally divided. Along with the cakes, the teacher had provided chocolate bars that were to be equally divided and juice to be measured in a variety of pint, quart, and cup containers.

Of course, the party was a great success. The youngsters enjoyed solving the problems both because they were presented in an interesting way and because they knew full well the reward would be a piece of cake, so many pieces of chocolate, and so much juice. The children dealt readily with the concept of common denominators, and I was amazed by their understanding of how to cope with the problems they were given.

Some time later, I saw these same youngsters computing with fractions, finding common denominators, reducing to lowest terms—all at the abstract level. I was sure that their fraction party and other ingenious activities which the teacher had created had given them the understanding necessary for handling abstractions.

An inventive teacher can utilize many such techniques in mathematics instruction. In so doing, he has two primary concerns—that the activities will be appropriate for the particular group of students and that they will be relevant to instructional objectives. With these criteria in mind, a teacher will be able to find many techniques for improving the teaching-learning process in mathematics.

Good Instructional Procedures Make Learning Enjoyable

One rather obvious benefit in departing from traditional teaching of mathematics is that the new programs, if well taught, are enjoyable for both students and teachers. Instead of the drudgery of rote learning day after day, good contemporary programs make mathematics stimulating and enjoyable to learn.

A recent visit to a third-grade classroom made this difference especially vivid for me. I recalled that when I was a boy, my mother and teachers insisted that I chant the multiplication facts backwards and forwards in what was almost verse rhythm. This was their idea of how to teach multiplication. By contrast, the class I was visiting was engaged in an enjoyable experience as they worked on multiplication by 9. Far from a rote process of memorization, the instruction was planned to develop understanding and to involve the youngsters in the excitement of learning.

The teacher had asked the children to write on the board a series of factual notations about the nine's table: $1 \times 9 = 9$, $2 \times 9 = 18$, and so forth through $10 \times 9 = 90$. Then he said, "Let's look at the chart and find as many things that are unique about this table as possible."

One hand went up. "I notice in the one's column," the child said, "that the numbers start out 9, and then 8, then 7, all the way down to 1. And in the ten's column it starts off 0 and then gets bigger and bigger—1, 2, 3, 4." Another youngster said, "Say, I see that when you add the ten's column to the one's column, the answer is always 9: 3 plus 6 is 9, 4 plus 5 is 9." Finally, another youngster said, "You know, I notice that the number you are multiplying 9 by is always one more than the number in the ten's column. So, 1 times 9 would be nine and there are zero tens; 3 times 9 is 27 and there are two tens, one less ten than the number we are multiplying 9 by."

After a discussion, the pupils readily constructed a formula for knowing their nine's facts. This formula was based on the fact that when 9 is multiplied by any number from 1 through 10, the sum of the digits in the product will be 9 and the number in the ten's column of the product will be one less than the number by which 9 is multiplied. As I reviewed my own experience in trying to memorize the nine's facts, I thought how much more difficult it was compared with the enjoyment these youngsters were having.

Similarly, an understanding of the commutative property can cut in half the work of learning the multiplication facts. Thus, if we know that $3\times2=6$, we also know that $2\times3=6$. I don't remember having realized this as a boy, because we didn't work with figures that way. I do recall that it was very easy for me to remember that 6×8 was 48 because it seemed to rhyme, but 8×6 was a very difficult concept. Had I discovered the cummutative property of multiplication, I would not have had the same problems.

Good Instructional Procedures Provide Sequential Development

In a good mathematics program, the teacher leads students through a careful step-by-step development which reveals the reasoning behind mathematical processes. If this is not done, the results can be disastrous.

I recall once taking a new teacher to observe a mathematics lesson which greatly impressed him. A few days later, I asked the new teacher how things were going. He said that there were problems and asked me to visit his class. When I went to his room to observe, I saw a teacher imitating another teacher. The lesson fell flat because the students had not been prepared for the concept which he was introducing. He had failed to take into account the sequential nature of the learning process.

Generally speaking, good instructional procedures in mathematics emphasize the inductive method in which children work a great number of problems, using varied methods, and eventually find an algorithm which defines how such problems may best be solved. This process should be carefully sequenced so that each step builds on the next.

Thus, one method for introducing a new concept is to begin with readiness development. This readiness should be largely motivational, but at the same time, it should imply readiness for the specific content to be learned. The children should have a time of experimentation during which they explore a variety of experiences and life situations. Out of these experiences, they discover methods for solving problems which directly involve the concept being learned. This exploration period quite often is at the concrete, manipulative level; however, it may include semi-concrete activities in which figures and shapes are drawn on paper or the chalkboard to symbolize the concrete. These steps then lead in an orderly sequence to forming an explicit statement of the abstraction itself, and youngsters are given more complex problems to which they can apply their discoveries. These discoveries are refined into an algorithm which can be used in the future as a shortcut for solving problems. At this point, the students evaluate the process they have evolved, and the teacher reinforces their understanding of the concept and processes they have learned.

Good Instructional Procedures Allow for Individual Differences

Any group of youngsters will differ widely in ways that affect the teaching-learning process. For this reason, the teacher should plan his instructional procedures so that he can deal with these individual differences. For example, when introducing and developing new concepts, a teacher may find that some youngsters must constantly work at the concrete level, that they cannot move on to the semi-concrete and certainly not the abstract levels. Other youngsters may not be able to learn the facts as we know them, but they can manipulate the number line and, therefore, function reasonably well with semi-concrete materials. These are illustrative of some of the differences among students which should be accommodated in the instructional process.

When I enter a room in which youngsters are working with facts of multiplication, I know that some of them will understand and be able to apply these facts in solving various problems. I also know that other youngsters will need to refer to a muliplication table of some sort, while still others will be using the number line or counting objects in order to reach solutions. It disturbs me, then, if I find all the youngsters working at the abstract level—or attempting to do so—because an algorithm has already been shown them by the teacher. Invariably, in the back of the room is a Sally Jones who has no idea what she's doing and whose answers, therefore, don't make much sense. Frequently, I hear the teacher say to such a student, "But I've told you many times, don't forget to carry the one." The teacher sometimes might look at me with exasperation and say, "She just can't learn a thing." And I might be tempted to say, "Oh, yes, she can; she has just learned to hate mathematics."

Similarly, any homework assignments should be given with differences among students in mind. Students should have complete understanding and mastery of the concepts involved in the assignment. When practicing an understood algorithm, any student should be assigned only enough examples to fill the purpose of the assignment, given his particular needs. The teacher should not feel that every youngster in the class has to receive the same assignment. Again, good instructional procedures in mathematics allow for individual differences.

Good Instructional Procedures Provide Oral as Well as Written Activity

Probably one of the most important activities in any day's math session is the time spent on oral computations. A teacher should give oral problems to the class for five or ten minutes and let the children solve them mentally without paper and pencil. After all, we solve many simple, everyday problems in real life without writing them out.

A very effective oral game that I've seen played in classrooms is called, "What's My Rule?" During one such game, a pupil stood in front of the room and said, "Give me a number." One student gave the number "five." The one who was "it" said, "The answer is ten." Another youngster said, "Seven." The leader replied, "The answer is fourteen." After several sessions like this, one child gave the rule correctly and became "it."

Youngsters enjoy these oral computational periods; at the same time, these sessions give them practice in using the processes and concepts they have been learning.

Good Instructional Procedures are Constantly Evaluated

Evaluation is necessary in any program, whatever the subject area. Evaluation should be concerned not only with the outcomes of instruction but also with the processes we are using. No matter how good we think our procedures may be, they should be regularly evaluated to determine how well they are contributing toward students' attaining our learning objectives.

A teacher can use a number of techniques to check the value of his instructional procedures. One means is readily available in every classroom—the pupils themselves. The minute a teacher sees a perplexed look on a yongster's face, he has useful evaluative information. The student is saying by the knotted forehead, "I don't understand." The teacher then realizes that his instructional procedures have not yet enabled that child to grasp what is being taught and he knows that some sort of change is in order.

Another, somewhat more formal, means of evaluating instructional procedures involves comparing the results of different means of teaching a particular concept or skill. Given a specific, behavioral objective which the student should be able to achieve as a result of an instructional sequence, the teacher can compare student performance when alternative procedures are used for teaching and learning the lesson. This type of comparison may be made either between two groups of students in the same class who are introduced to the concept in different ways, or it may be made over time between two different classes of students. A similar evaluation process might be used to assess the appropriateness of an instructional procedure for different kinds of students—students who are highly motivated and those who are not, students who have evidenced mastery of preceding instructional sequences and those whose achievement is less satisfactory, and so forth.

Ultimately, continuing evaluation may well be the most important of all the characteristics of good instructional procedures. Evaluation helps us to make teaching and learning more effective and this, after all, is what we are seeking when we plan procedures of instruction.

USING MANIPULATIVE MATERIALS IN MATHEMATICS INSTRUCTION

Sandra J. Margolin Mary Y. Nesbit James R. Pearson

Mathematics is often thought of as a highly structured, rigid body of knowledge which cannot be adapted to the varying needs of those who learn. Nothing could be further from the truth. To the contrary, the infinite variety, flexibility, and objectivity of mathematics make it easily adaptable to the wide range of man's learning patterns and abilities.

Why, then, are there so often problems in teaching mathematics? One reason is that seldom is enough effort put forth to determine the abilities, learning patterns, and previous skill and knowledge levels of individual learners. And even if a teacher does make such a diagnosis to a satisfactory degree, the requisite materials for adapting instruction often aren't available, or, if they are, a teacher's lack of training in how to use them may obscure their value.

As one strategy for attacking such problems, periodic workshops should be held as part of a well-organized in-school, inservice program. In these workshops, the structural approach to teaching mathematics should be emphasized. Teachers should learn fundamental mathematics concepts and become familiar with a wide range of instructional aids and techniques to use in helping pupils understand these concepts. Throughout this process, teachers should be helped to put into practice the philosophy of a student-centered, rather than teacher-centered, classroom.

This article is concerned with one important means of adapting mathematics instruction to the varying learning requirements of students—the appropriate use of manipulative materials. This type of instructional aid should be used throughout the elementary mathematics program. The notion that manipulative materials are appropriate only in the primary grades is pure fantasy. At each level of development, mathematics concepts should be presented so that learning activities ranging from the concrete to the abstract are included. This is essential if instruction is to be adapted to the learning patterns or styles of individual pupils. Further, manipulative materials are an important means of involving the pupil in the learning process and of promoting a discovery approach. In *The Conditions of Learning,* Robert M. Gagné underscores the significance of this kind of pupil involvement: ". . . The key to achievement motivation is the *conception of action* on the part of the student. He must want to be able *to do something.*" Manipulative materials are an effective means of stimulating this desire "to do something."

Many of the materials and activities presented on the following pages are basic to learning mathematics concepts. They are highly motivational in nature and, with adaptation, may be used to good effect throughout the elementary program.

One important reminder: In using these, and other instructional, materials, one should not assume that an entire classroom of children must necessarily be doing the same thing at the same time. There are, of course, often occasions when it is appropriate to introduce or explain a concept to the entire class. But there are many occasions when this is not appropriate, given differences in learning pace and style. Even within the constraints of a self-contained classroom with one adult and thirty children, an imaginative teacher may devise numerous ways to utilize these manipulative materials so that small groups or individuals within the class may simultaneously be working on different activities. For example, many of the games lend themselves to being played independently by children with tape-recorded instructions and with immediate feedback through individual answer sheets. Or colored slides in an automatic carousel-type projector, operated by the child, may be used to present and answer some of the problems. Another possibility is the use of pupil tutors. With older elementary pupils, an additional technique is to provide written instructions to guide independent or small-group learning. And, of course, there are often times when the teacher may work with one group while other pupils are engaged in more abstract paper-and-pencil activities.

Geometric Shapes

These games and activities help children to build techniques for "seeing" patterns and solutions. The basic materials required are large geometric shapes of various colors. These may be made of construction paper, cardboard, or oaktag—materials that are available in most schools.

Four shapes are used—a circle, triangle, square, and rectangle. For each of these shapes, make two large and two small forms in various colors. Be sure each child has his own set of shapes (a total of 16 forms in a set) in an envelope.

○	◯	△	△	□	□	▭	▭
ORANGE RED	GREEN BLUE	BLUE GREEN	RED ORANGE	GREEN ORANGE	BLUE RED	RED BLUE	GREEN ORANGE

From the chart it may be seen that any given shape may differ by no more than two properties—size and color. Thus, the circles differ by size (large and small) and by color (orange, red, green, and blue). When two different shapes are used, the forms can be distinguished by three properties—size, color, *and* shape.

With these materials, many concepts necessary for early number skills, such as classification, can be developed and practiced in a game atmosphere. In addition, games may be played which lead to set and subset recognition. Also included is the development of skill in distinguishing among the four colors. Several activities and games of this nature are now described. None of these should be thought of as one day's work, since more time may be required for a complete presentation.

"What Kind?"

Have the children raise their hands while you approximate the following dialogue: "Hold your hand in the air and move it around like this." (Make circular motion with your hand.) "What are we drawing in the air?" (Ball, round things, circle, etc.) "This is like a circle—kind of round." (Have the children lower their hands.)

"Look at the colored shapes you have in your envelope. Do you see any shapes that are like the shape we drew in the air?" After the children begin to answer to your satisfaction, tell them that the "round" things are "circle shapes." Hold up a circle shape (e.g., the large blue circle) and say: "This is a blue circle shape. Can you find a blue circle shape in your envelope? Hold it up and I will walk around while you show me." Walk around the room and be sure the right circle has been chosen. Continue this way for the large green circle shape and the two smaller orange and red circle shapes.

Next, hold up the large blue circle shape and ask, "What shape is this?" (Circle or circle shape.) "What color is this circle shape?" (Blue.) Hold up the small red circle shape and ask the same questions. Now ask, "Which is larger? This (red circle shape) or this (blue circle shape)?" Do this for the other two circle shapes.

Play a simple game in which each row or table of children is a team. Have two children from each team stand. As you suddenly reveal a large circle shape, ask, "What kind?" The child first calling out "large" gets a point for his team. Similarly, points may be scored by naming the correct color when asked. Two points may be scored by calling out the correct size and color all at once, no matter which property is asked for. As soon as you believe the children are generally aware of the size and color of the circle shapes, ask them to take out their four circle shapes. A member of team 1 will reveal a chosen circle to a member of team 2, who will call out what he sees. Whether or not he is correct is determined by the member of team 1. Naturally, if neither child is correct, no points are scored and someone from team 2 challenges team 3, and so forth.

Introducing the Square

After reviewing the four circle shapes, show the children the large red square shape. Place it over the large blue circle to demonstrate that the "corners" of this new shape stick out around the edge of the circle. Count the corners out loud to show there are four of them. (Although young children are not expected to understand the rationale for their counting, it is imperative that they hear counting being used. *It is a great error to assume that number and numeral must be introduced simultaneously.* This is not true. In fact, counting—and therefore number—existed in tribes with no written language.) Take the square shape to the board and trace along the length of one side. Place each of the other sides beneath the tracing and help the children to discover that all sides are "just as long" or "the same" or "the same length" or "the same size" or any similar conclusion that means the four sides are equal in length.

Introducing the Rectangle and Triangle

The rectangle may be shown as a four-sided shape with two sides congruent but shorter than the other two congruent sides. The triangle may be introduced as having three sides and three corners. The construction paper triangles are isosceles, so be sure to generalize the "threeness" of a triangle by drawing many different three-sided figures on the board.

Further development of these geometric shapes continues in the following activities. Also included will be development of spatial concepts and positional and comparative vocabulary.

Selecting Shapes

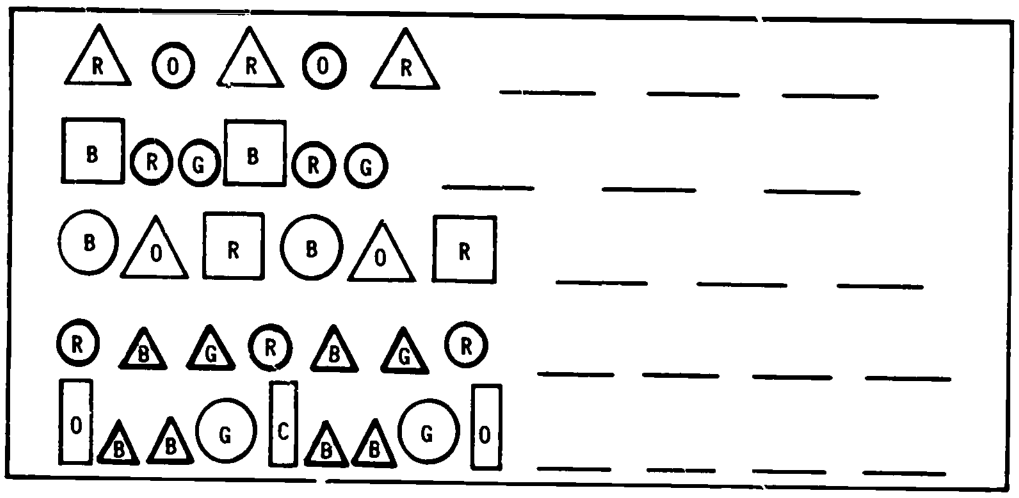
Ask each child to select one shape and put all the other forms in his desk. Then say: "All children with large green triangle shapes dance." Or, "All children with small red circle shapes stand." Continue in this fashion.

Then have the children march around the room holding one shape they have selected. Say: "All children with green triangle shapes walk like ducks." Or, "All blue circle shapes walk like elephants," and so forth. Continue in this fashion, and have children take turns giving instructions to the whole class.

Other techniques for practicing selection of shapes involve having the children select a particular shape from the complete array placed on their desks. Thus, you may say: "Show me a large square shape. Show me a small triangle shape. Put the green triangle shape beside your desk. Put the green triangle shape inside the blue circle shape. Place the blue rectangle shape between the two large circle shapes." These requests may be directed either toward the entire class or toward individuals.

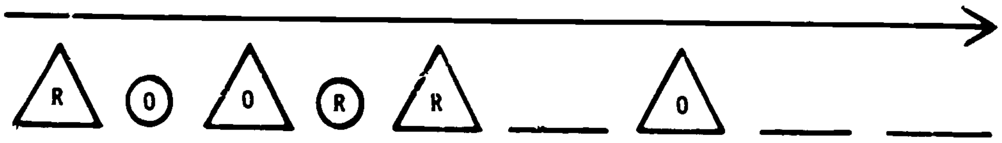
Completing the Patterns

Place the following shapes in the chalk tray as shown and ask for volunteers to put the next shape in the series. Also have children create original patterns. (The letters suggest the colors: R, red; O, orange; B, blue; G, green.)



Before and After Concepts

Using the same patterns, develop before and after concepts. These are important in number development. To do this, set up a pattern on the chalk tray as follows and draw an arrow above it to indicate direction.



Work with the children in this fashion: "The arrow tells us which way we're going. Now, look at this red triangle." (Point to the first red triangle.) "What comes *after* the large red triangle?" (Small orange circle.) Continue through the next three shapes until the blank, using the word "after" in each case. The term "before" may be introduced in a similar manner. Continue this same procedure with other patterns of your own.

Feel Box

Cut holes large enough for a hand to fit through in the sides of a covered cardboard box. Place shapes in the box and ask the children to remove the shapes you describe. Each child gets a point for removing the correct shape. (For fun, ask the children if they can find a red shape by feeling.)

A variation of the game can be played by having children select any shape from the feel box. Points may be earned by correctly identifying the selected shape before they remove it from the box and see it.

Sticks and Counters

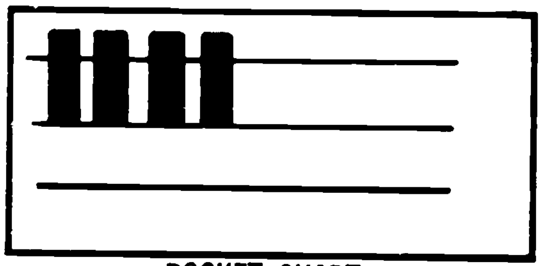
At all levels, children need to use counting materials to reinforce basic learnings in mathematics—for example, classification, place value, counting, computation, and number theory. A wide range of materials might be used for this purpose: seashells, stones, blocks, buttons, bottle caps, clothespins, coffee stirrers, or tongue depressors.

One-to-One Matching (Match the Object)

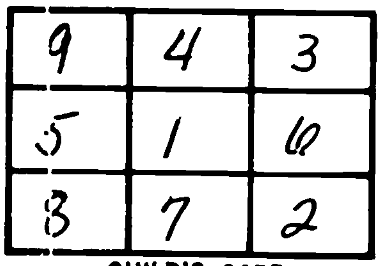
Place several objects on a table. Then have the children place one stick or counter on their desks for each object displayed on the table.

Number-Numeral (Numbo Game)

Construct cards which are divided into nine squares. Place a numeral from 0 to 9 in each square, using a different order on each card. Each child should be given a card along with nine counters. To play the game, the teacher places a given number of sticks in a pocket chart. Each child who has the numeral on his card corresponding to the number of sticks then covers it with a counter. The child whose card is covered first wins. (In a large group, several children may win simultaneously. Several variations are possible by making the winner the first child to cover the vertical center row or to complete some other pattern.)



POCKET CHART



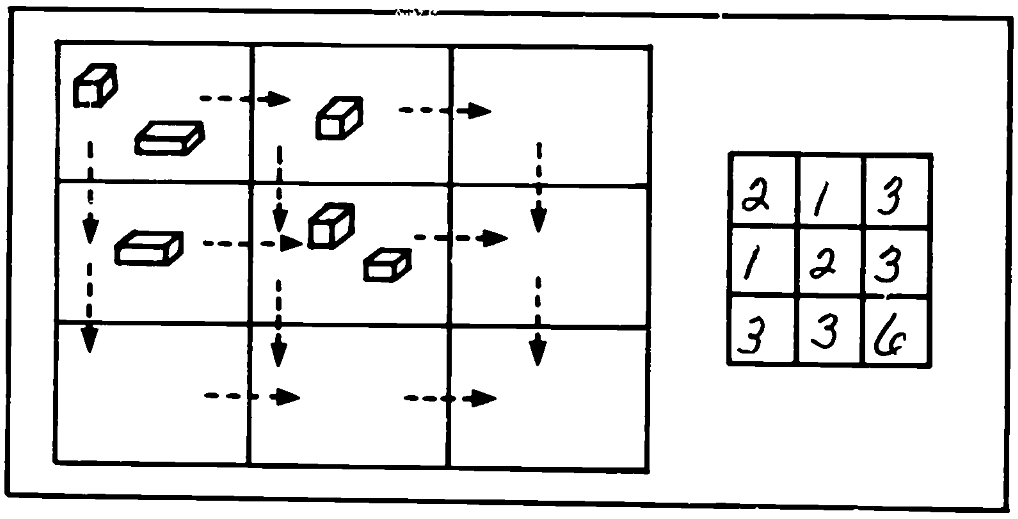
CHILD'S CARD

Addition, Subtraction (Wishing Box)

Using an open box and several counting disks, the teacher places three disks in the box. "I wish I had seven in the box. Who can make my wish come true?" Or, "There are eight counters in the box. I wish there were three. Who can make my wish come true?"

Addition, Subtraction (Cross Number Puzzle)

Construct a large cross number puzzle on the floor with masking tape. Children can use a variety of counters to solve problems by "sweeping" objects across and down.



Multiplication (Counter Arrays)

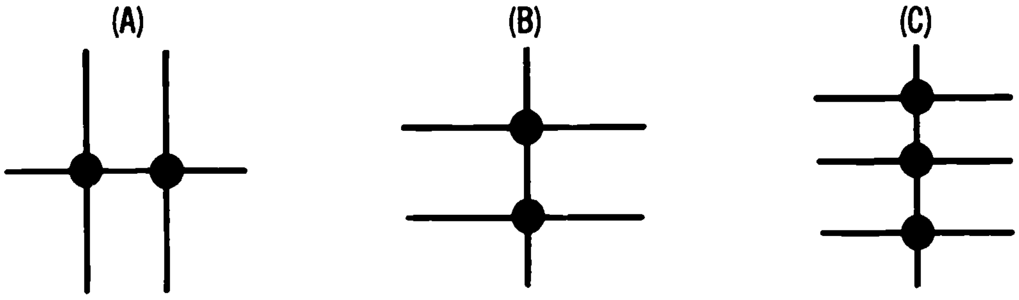
After establishing that an array is an orderly arrangement of objects in rows and columns, have the children make their own arrays using counters. Thus, given 6 counters, they might make arrays of 6 counters in one horizontal row, 6 counters in one vertical row, 6 counters in two horizontal rows of 3 each, and 6 counters in two vertical rows of 3 each. The object is to have the children discover the various orderly arrangements which are possible with a given number of counters.

Multiplication (Stick Matching)

As an introduction to multiplication, children can use sticks to show lines and crossing points. The teacher should construct the following chart on the blackboard:

	A	B	C
— LINES	1	2	3
| LINES	2	1	1
	2	2	3

The children should then work at their desks to construct the patterns indicated on the chart. Pieces of yarn, as well as sticks, can be used.



Joining Sets (Red-Blue Game)

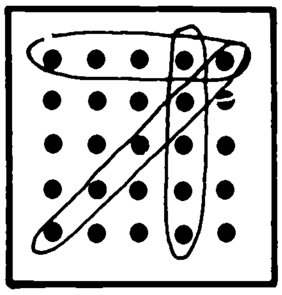
Each child receives 10 blue squares and ten red squares cut from construction paper. (Cubical counting blocks might also be used.) On the blackboard, the teacher draws the chart shown below, and the children use their paper squares to represent the answers.

R	B	R + B
3	2	?
4	0	?

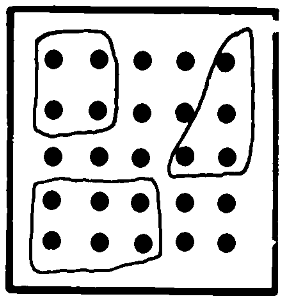
Geoboards

Geoboards are helpful in teaching direction, visual perception, area, properties of geometry, and rational numbers. On a square board, 9″ by 9″, place 25 nails in five horizontal and five vertical rows at $1\frac{1}{4}''$ intervals. Use rubber bands of various colors to reproduce figures and lines. Use the boards as suggested on the next two pages.

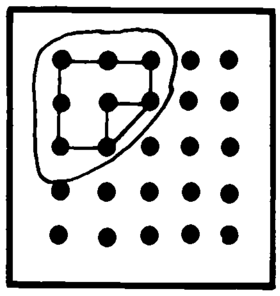
Reproducing horizontal, vertical, and diagonal lines.



Reproducing geometric shapes.

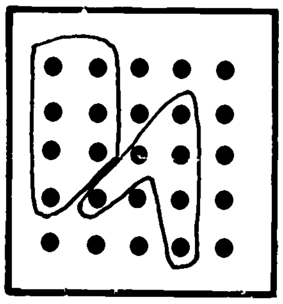


Finding area and discovering fractional parts.



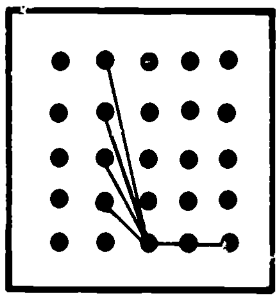
3½ units

Adding fractional parts to find combined area.

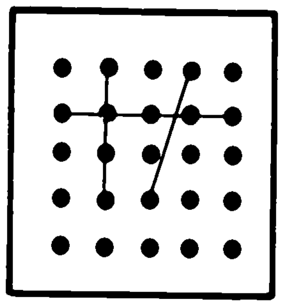


A + B = ___ units

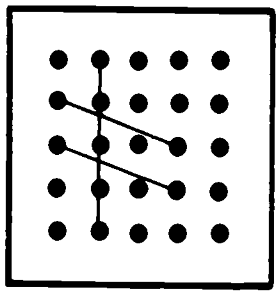
Discovering properties of angles. (We use two segments to form an angle, placed so they have one point in common—vertex.)



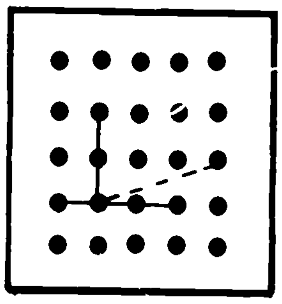
Corresponding and alternating angles.



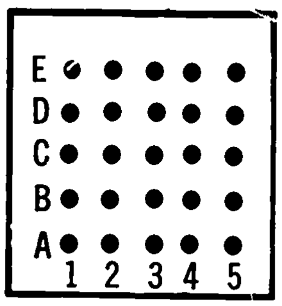
Parallel lines where there are corresponding angles.



Perpendicular lines where there are right angles and complementary angles.



Finding coordinates. (Mark boards with letters and numerals.)



In introducing the board marked with coordinates, explain: "Notice that the rows are marked with letters. Run your finger along the E row. Notice that up and down the board we have columns. These are marked with numerals. Run your finger along the 1 column. Along the 4 column." Then have the children place their rubber bands in accordance with instructions: "Let your yellow rubber band go from E 1 to E 5; from E 5 to A 5; from A 1 to E 5; from C 2 to E 4," and so forth.

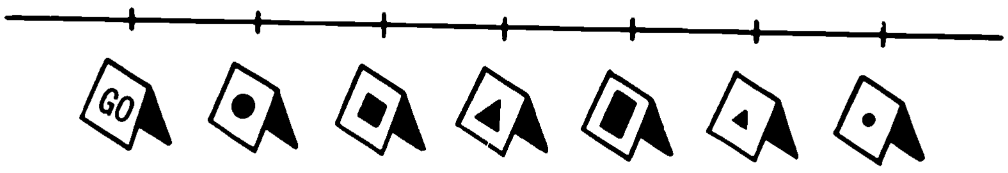
Number Lines

Some form of the number line should be used constantly in grades K - 6 to help make counting, number sequence, adding, subtracting, multiplying, and dividing more understandable. In the early grades, the number line is used as a means of helping the child understand new concepts and check computation. In the upper grades, it is used to assist in understanding common fractions, decimals, and negative integers. After number line activities are introduced, each child should have an individual desk number line.

Shape and Size Recognition (Magic Carpet)

This game uses a walk-on number line marked off on the floor with chalk or masking tape. (Oilcloth can be used instead.) Stopping places are marked at one-foot intervals, and large stand-up cards, marked on both sides with geometric shapes in different sizes, are placed beside the stopping places. The first card should be labeled "Go."

The player stands at "Go" and follows directions which lead him to a particular stopping place: "Go to the large triangle"; "Go to the stopping place between the large circle and the large triangle."



Counting, Natural Order of Numbers (How Many?)

Using a walk-on number line with stand-up cards marked with numerals, have children answer questions by walking to the appropriate station on the number line. For example: "How many wheels on a tricycle?" "How many days in a week?" "How many fingers on one hand?"

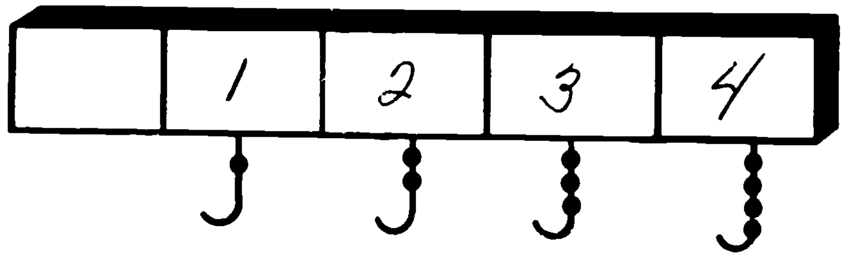
Counting, Natural Orders of Numbers (Non-Verbal Number Line)

Draw a line on the board and begin to label the points, making no comment:

Now point to the fourth dot and get ready to label it. Stop. Look at the children. Offer them the chalk, suggesting that someone might know what comes next. If there is no response, go back and *tap* each of the tallies. Then offer the chalk again. Repeat this if necessary. Soon, someone will know what comes next.

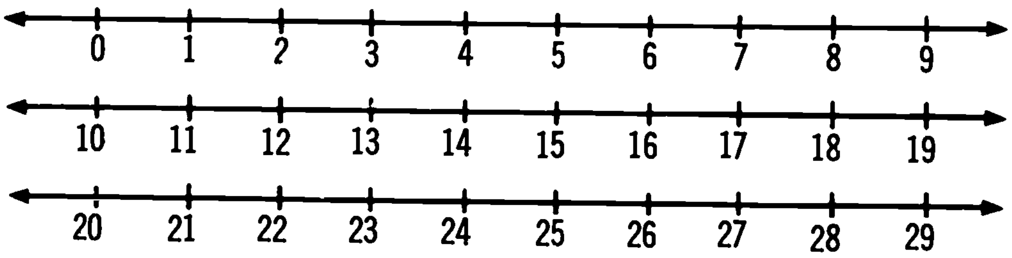
Natural Order (Bead Number Line)

This number line helps to develop concepts of "before," "after," and "one more." Prepare a board similar to the one shown below. Attach wire or pipe cleaners so that beads can be added as each number is developed. Later, when zero is introduced, you can attach an empty wire to show this idea.



Addition, Subtraction (Number Line Cricket)

Draw a number line chart, as shown below, on oaktag or on the board. It can then be used to show that it is easily possible to jump from 3 to 13 or from 5 to 25. Once they discover a pattern, the children do not need to count. Each jump down is the same as adding 10. A jump down and 2 right is the same as adding 12. A guessing game can be played with the chart: "I'm thinking of a number. Start at 2, jump 2 rows down and 4 to the right. What is the number?" (26)

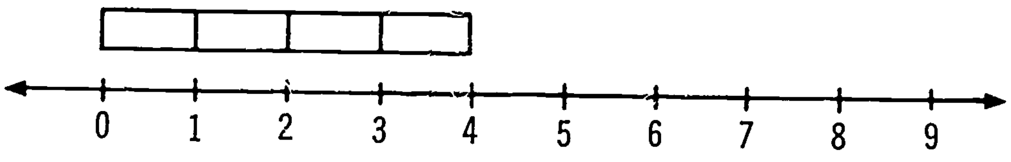


Multiplication (How Far Does It Go?)

For this game, which builds readiness for multiplication, the materials required are a number line drawn on the board and pieces of cardboard backed with masking tape so they will stick to the board. The number line should go from 0 through 25, and the distance between each numbered point should be 3 inches. The pieces of cardboard should be 1 inch wide. The following pieces should be included:

A. Some pieces 3 inches long
B. 12 pieces 6 inches long
C. 8 pieces 9 inches long
D. 6 pieces 12 inches long
E. 5 pieces 15 inches long
F. 4 pieces 18 inches long
G. 3 pieces 24 inches long

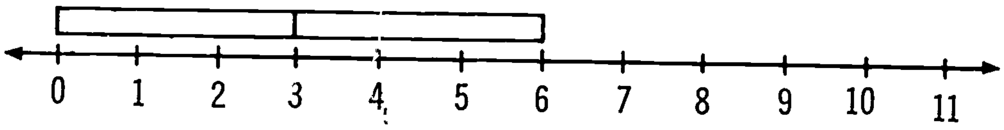
Begin by demonstrating that an "A" piece fits between 0 and 1 on the number line. Also show that piece "A" can be placed end-to-end 2 times between 0 and 2, 3 times between 0 and 3, and so forth. Agree



that piece "A" will be called "1," and record the information you have discovered on a chart. (See the "Piece 1" row in the chart below.)

NUMBER OF TIMES	1	2	3	4	5	6	7	8	9	10
PIECE 1	1	2	3	4	5	6	7	8	9	10
3	3	6	9	12	15	18				

Ask a child to pick any piece and find out "how far it goes." If a child picks piece C, for example, he would find that it "fits between 0 and 3." This piece may then be called "3" in the same way piece A was called "1." On the chart above, enter "3" in the appropriate space in the "Piece" column. Then point to the chart and say, "When the 3 is placed 1 time (point to the 1 in the top row labeled "number of times"), how far does it go?" "To the 3." Place a 3 in the chart and show that 1 "3" piece goes to the 3. Ask a child to find another "3" piece and add it to the number line:



Ask, "How many times?" "2 times." "How far do they go?" "To the 6." Record this information on the chart. Continue with the pieces until you find an appropriate vocabulary being used with the chart, which is really a multiplication table for products through 25.

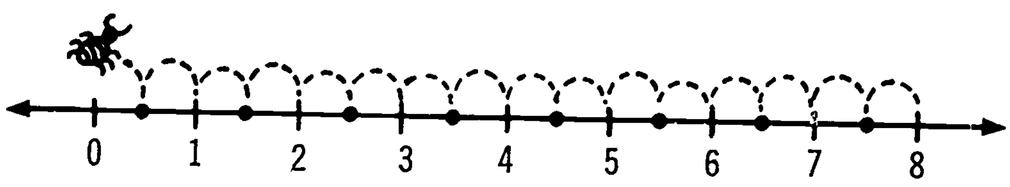
Multiplication (The Jumping Bug)

Draw number lines on the chalkboard for this activity. Cut a small bug from construction paper and paste it on the end of a pointer. Use the bug to show different size jumps, and ask the children to show what a "one bug" does on the number line, a "two bug," and so forth.

Rational Numbers (The Jumping Bug)

Using the same procedure as described above, name the bug a "one-half bug." Children should try to name the points that the "one-half bug" lands on:

$$\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}.$$



This activity can be extended to include a "one-fourth bug," a "one-third bug," and so forth. Later, try:

$$\frac{2}{3}, \frac{2}{5}, \frac{3}{4}.$$

Show Me Cards

"Show Me Cards" permit every child to participate in various activities. They can be purchased or easily made. The cards should be about 1½″ wide and 3″ long. Each child should have 20 cards, 2 numbered 0, 2 numbered 1, and so on through 9. A pocket for displaying the cards can be made by using a 9″ by 6″ piece of cardboard and turning up the bottom 1 inch and stapling. This can be divided into smaller pockets by stapling to form subdivisions.

With all his show me cards spread in order on top of the desk, a child can quickly show his answer to a variety of problems.

Place Value

The show me cards are ideal for place value practice since the children literally "place" a numeral according to the value it has been given. Thus, if "8 tens" are requested, the pocket second from the right is used for the numeral "8." Similarly, the teacher might ask: "Show me 4 tens and 2 ones. Show me 146. Show me a simpler name for 7 tens and 13 ones. Show me a simpler name for 80 + 15."

Addition, Subtraction, Multiplication,
Division (Follow the Sign)

The teacher should make a chart such as the one shown below. He then points to a space, and the children use their cards to show the answer. For variation, children might write answers on pre-cut squares and paste them on the chart.

	−3	÷2	×2	÷4	−16	+7
20						
32						
16						
40						
36						

Abacus

Since the abacus is a semi-abstract device, it is desirable for pupils, as well as teachers, to manipulate it individually in order to understand fully the concept of place value. Activities may be supplied on task cards or can be presented by the teacher. The abacus should never be used to introduce place value to young children. But after introductory work, it may be utilized effectively to reinforce and extend the concept of place value.

Introduction of Place Value on the Abacus

Before any work is done with the abacus, the device should be introduced to the children. First, identify the ones wire and the tens wire. Then begin counting the ones and slipping them individually over the wire. After 9 ones have been counted aloud by the teacher and class, ask the class what should be done after ten beads are on the ones wire. There may be a variety of answers, but accept any response close to "Make one on the tens wire." Remove the ten ones, slipping one bead onto the tens wire to "stand for" the ten ones.

Once the idea of sliding the ten ones back and sliding one bead onto the tens wire is established, the teacher can continue counting with the instructions that the class should call "Stop" when it is time to replace the ten ones with one ten. This counting may also be done backwards, except the children should call "Stop" when it is necessary to get "change" for the ten.

Children then should practice showing numbers on the abacus. For example, 1 ten, 0 ones; 4 tens, 6 ones; $30 + 2$; $(200 + 5) + (20 + 3)$.

Using the Abacus

Children can be given a wide variety of problems to solve on the abacus.

1. Place value, counting, adding by endings. "Show the number that is 1 more than 99. Show the number that is 3 more than 75."

2. Place value, addition, subtraction. Show on the abacus:

$5 \text{ tens} - 2 \text{ tens} = 3 \text{ tens}$

$3 \text{ tens} + 4 \text{ tens} = 7 \text{ tens}$

3. Regrouping using multiples of 10, exponential notation. Show on the abacus:

$12 \text{ hundreds} + 14 \text{ tens} + 11 \text{ ones}$

$(3 \times 100) + (4 \times 10) + (6 \times 1)$

$(5 \times 10^2) + (2 \times 10^1) + (6 \times 10^0)$

4. Multiplication. Find the products on the abacus:

$2 \times (100 + 30 + 4)$

4×200

These are a few of the many instructional materials that the classroom teacher may develop for use in teaching mathematics. There are also many fine commercial materials available—notably the cuisenaire rods and Dienes blocks. The most important consideration in using any materials, commercial or teacher-made, is the extent to which they involve the child in his own learning process. One of the great values of the kinds of manipulative materials which have been described here is that they promote pupil involvement in learning and help the child to discover basic mathematics concepts for himself.

The importance of this and the effect it has on children's attitudes toward mathematics were never more apparent than on a children's television program a few years ago. After a little girl had told the master of ceremonies that math was her favorite subject in school, he commented that kids used to hate it. She retorted that new math was different, and the MC countered: "But two plus two is still four."

"Yes," she answered, "but we're finding out *why* it's four!"

THE SEARCH FOR VALUES WITH A FOCUS ON MATH

Merrill Harmin Howard Kirschenbaum Sidney B. Simon

This is a crazy world we live in. We realize just how crazy it is when we notice those clever advertising lies, read the bewildering newspaper headlines, witness the obvious contradictions between what adults preach and what adults do. We notice how frequently our political leaders act with easy duplicity, how seldom our citizens find joy when they achieve affluence. We try to reconcile the suffering, rioting, and even killing with the technology, culture, and—occasionally—wisdom.

Somehow today's youth seem especially aware of those ills and as a consequence have more difficulty than we did in giving our society their allegiance. Whereas earlier generations slipped through a stage of idealism into an acceptance of the existing order, more and more of today's youth are not accepting their society. In simplest terms, it would seem that traditional values are just not very palatable to vast numbers of today's youth.

The result, it seems to us, is that more and more of our young people are becoming valueless, aimless, empty of commitment. That is a frightening state to be in, for it is not easy to live in aimless drift, feeling that life has no purpose. This sometimes leads to psychological damage (a feeling of self as a meaningless unit) and sometimes to more visible damage (a frustrated, acting out of rage against the institutions that helped produce such emptiness). Much of this is behind the recent student outbursts. Note the negative tone of those outbursts: Students seem so much more aware of what they are against than what they are for.

What has all of this to do with mathematics? Well, we believe that teachers—regardless of what subject matter they are teaching—can help young people make some sense out of the confusion that surrounds all of us. Every teacher can help students find commitments and a positive sense of self in society. This we call teaching for value clarification, and we believe strongly that there are few educational objectives more vital.

In general, the way to help students clarify their values is to give them practice in grappling with values issues so they can work through the many conflicting pressures they receive from society and can develop their own skills for evaluating and resolving those pressures.

There are many mathematically based problems that may be used to give students this sort of practice in dealing with values issues. Let's look at a number of examples. Some of these examples, we should point out, are more appropriate for some groups of children than for others, either because of the difficulty of the mathematics or because of the relevance of the subject matter. We've purposely included a wide range of suggestions in order to indicate the many possibilities.

Word Problems with Value Implications

If mathematics lessons would include a few word problems which touch on value issues, and if these issues were discussed briefly either before or after the computations were performed, the search for value clarity would be advanced. Look at these word problems, for example.

1. *Problem:* Bill bought a sting-ray type bike for $35. Three years later, he sold it for $15. While he owned the bike, he spent $9 on repairs. How much did it cost Bill to use his bike each year? What percent of the purchase price did he lose when he sold it?

Value questions: Under what conditions would you tell a buyer everything that was wrong with something you were trying to sell him? What if he didn't ask? Would you lower the price to someone who was really poor? If you didn't sell your old playthings, what else might you do with them?

2. *Problem:* Look at this chart showing the number of fires this year in various parts of our city. This second chart lists the number of houses and apartment buildings (called "dwellings") in each fire district. How many dwellings in each district did *not* have a fire? How many dwellings in each district did have a fire? What percentage (or fractional part) of the dwellings had fires? Make a

graph showing the districts that had the most fires, the second most, and so forth.

Value questions: Do you know how the fires started? Were the buildings in poor condition? Were people careless? How close have you ever come to setting fire to your house or apartment? What did you learn from that narrow escape? Have you ever put out a fire? What was it like? How did you feel? Little children set a lot of unnecessary fires. What would you do to prevent them from playing with fire? Would you like the job of a fireman? How do you know?

3. *Problem:* June works 1 hour once a week washing the kitchen floor for her mother. She also dries the dishes every evening, but that takes only 15 minutes a night on the average. She spends about 30 minutes a week picking up her clothes and playthings. How much time does June spend in working around the house each week? Each year? What proportion of her work time does she spend drying dishes?

Value questions: Do you think that June should be paid for her housework—for all of it, part of it, or none of it? If she is paid for her housework, then do you think June should help to pay for her food and clothing? If you have your own children, how will you handle chores around the house? What can you do now to improve the way work is done around your house?

4. *Problem:* The average hot lunch in the school cafeteria costs 45¢, including milk. If you buy lunch outside of school, you can buy a hot dog for 25¢; a wedge of pizza, 20¢; a bag of potato chips, 10¢; soda pop, 10¢; ice cream on a stick, 12¢; candy bars, 6¢; apples, 2 for 25¢; and oranges, 11¢ each.

Make up a list of the various combinations of food you can buy outside of school for 45¢. How much would it cost to bring a sandwich and a piece of fruit from home? What does it cost for a typical week if you eat in school, if you eat outside of school, if you bring a bag lunch from home?

Value questions: Where do you usually eat lunch now? Have you tried one of the other alternatives? If you could save $1 a week on lunch, how would you spend that $1? If you were given $1 a week more for lunch money, how would you spend it? Could you go without lunch one day to save more? Would you be just as happy if someone invented a pill which would give you the same food value as a lunch? Explain. What other purposes does lunch serve in your life—besides giving you food?

Value discussions such as these hold students' attention. But is this kind of work really mathematics? How about the time it takes away from regular classroom study of math?

These are the questions we hear most frequently when we suggest that values issues be woven into mathematical word problems. Our reply is that students need to learn how to deal with complicated values issues. Indeed, there can be terribly serious consequences for not knowing how to confront the values problems which surround our lives. We say that if a teacher is willing to take a little time away from pure mathematics (no more than he feels comfortable about), he will be providing an important service to his students and, perhaps, to society. We also point out that teachers who use this kind of an approach usually find that their students are happier in class and in a better frame of mind for their more formal work in mathematics. They enjoy the change of pace that occasional values discussions provide.

Many other value-laden word problems can be constructed. Each problem, of course, should utilize appropriate mathematical skills and should involve values issues which are relevant to the children and which they can deal with at a suitable level. Here are a few examples of such topics. Each topic is accompanied by a very brief indication of the mathematics related to the topic which might be formed into a word problem and then, in parentheses, by some illustrative value questions which could be discussed.

Planning a birthday party. Given a certain number of guests, how much food will be needed? How much will it cost? What proportion of time might be spent in eating, playing games, and so forth? (Value questions: How do you decide whom to invite to your party? Should you ever invite someone who doesn't have many friends? What would happen if someone came to your party without a gift? What is the purpose of giving a gift?)

Mail-order buying. Examine a mail-order catalogue and then figure out the shipping charges and insurance costs for the things you would like to buy. Consider differences between express and parcel post costs and between shipping many things together and separately. (Value questions: How important are material things to you? If you didn't spend the money for the mail-order goods, what else could you do with it? How do you decide what you want to buy? Do you usually want the things your friends have?)

Writing letters to the city council. If each member of the class writes a letter to all of the council members, how many letters will be sent in all? How much will the postage cost for these letters? (Value questions: Is it important to express your ideas to elected public officials? What other ways than letters might people use to communicate with public officials? How can you make your voice heard? What do you think are the best ways to express your views? Why?)

School statistics. Looking at the appropriate tables, find out how many adults in our country have not finished high school. How many have not finished elementary school? Make a graph showing the changing proportion of adults in the United States who had not finished high school as reported each decade since 1920. (Value questions: What do you think some of the reasons might be for people not wanting or not being able to finish high school? How important is education to you? What problems might you have in getting all of the education you want? What might you be able to begin doing right now to solve these problems?)

These are only a few suggestions of the kinds of word problems which a teacher can develop to explore a variety of mathematical problems that have value implications. Teachers who are interested in this technique will find the daily newspaper an excellent source of facts and figures that relate to values issues.

PERSONALIZED MATHEMATICS PROBLEMS

One reason students may not become interested in math problems is that the situations described in textbooks often generate an air of unreality. An urban youngster, for example, is unlikely to be captivated by the problem of how many bushels of wheat Farmer Jones can produce. On the other hand, if a student can work with figures and situations which are related to his own experience and daily life, the chances are that he will become quite involved and be more motivated to solve problems accurately.

With this in mind, some teachers have their students keep *time diaries* and *budgets*. The time diary is simply a chart with a column for each day of the week, divided into half-hour blocks. *All* activities (eating, sleeping, talking on the telephone, watching TV, doing homework, hanging around, and so forth) are included.

The budget, of course, is an accurate record, down to the penny, of where the student's money comes from and where it goes. Students are asked to keep these records up-to-date and to bring them to class every day. Problems utilizing the student's own statistics are easy to frame and naturally lead to values issues. Here are some examples of problems and values questions which could be drawn from children's time diaries and budgets. Again, the values issues are in parentheses.

1. What percentage of your waking hours is spent with others? What percentage is spent alone? (Do you like this balance? Is it right for you? For everyone? Why do some people always need people around them? In what ways do you spend time differently from other children?)

2. What fraction of your time is spent doing things you really and truly enjoy? (How can you make this fraction bigger? Is your goal in life to enjoy yourself? If not, what is it?)

3. Using the total time you slept last week as an average, how many hours per year do you sleep? What is the ratio of your sleeping time to the time you spend watching TV? (Do you get enough sleep to satisfy you? What activities might you cut out if you wanted more time to sleep? What is the best time of day for you to do difficult things? How does not getting enough sleep affect your behavior?)

4. On the average, how much free time a day do you have? What percentage of this time is spent watching TV? (How do you handle conflicts with other members of your family over what to watch? The last time your television wasn't working, what did you do with your free time?)

5. If you put 20 percent of your monthly income into a savings account and received 5 percent interest compounded quarterly, how much would you have in 5 years—assuming your income remains the same? (How much do you actually save a month? Is there anything you want so much that you are saving for it? Do you think your savings habits are good?)

6. What percentage of your income do you spend primarily for the benefit of others? (When you are older, what percentage of your income would you like to donate to worthy causes? What are some causes you simply wouldn't give a penny to? Do you ever send anonymous gifts? When was the last time you were given too much change when you bought something? What did you do?)

7. A researcher reported that when a family gets into financial trouble, the last thing that changes is its standard of living. (If your income were cut in half, what things would be the first to go out of your budget? What would be the last to go?)

EXTRA ASSIGNMENTS WITH VALUES IMPLICATIONS

Some teachers use out-of-class assignments that combine thinking about values issues with mathematics experiences. If a teacher can thus stimulate value thinking (without being judgmental), students typically get themselves involved in value discussions and thoughts, even if no class time is allocated for that purpose.

The following examples will suggest the kinds of assignments students might be given for out-of-class work. In every case, of course, it is important that the teacher be sure that the assignment will be appropriate to the students. The values questions in these examples are placed in parentheses.

1. Take a poll of students' attitudes toward several things that you think are important. Then put your findings in graph form. (Discuss your reasons for selecting these attitudes to survey, and comment on your findings.)

2. Using U.S. Census reports, find the per capita income of white and non-white persons in rural and urban areas in your section of the country. (Present your data and discuss what you think might be done to eliminate any inequities you uncover.)

3. Find advertisements that illustrate faulty logic and unfounded claims. Look especially for advertisements which use percentages and phrases such as "More people use Brand X." (Discuss the role of advertising in influencing what you buy.) [1]

4. Imagine that you have $1,000,000 to use any way you wish to improve your school. How would you spend it? Be specific. Obtain facts from the principal, business office, and so forth. (Keep in mind that how you spend money usually indicates what you value.)

5. Observe a water faucet at home which is leaking or which you allow to drip slowly for a few minutes. Try to estimate how much water is wasted in a given amount of time. What is the cost per month? Per year? (Whose water is it? Would you repair a leaky faucet for a neighbor?)

6. Look at the supermarket advertisements in your newspaper. Make up a list of basic groceries used in your home and see which would be the cheapest place to buy these items. (Would it be worth going to that store to buy these things? Explain.)

7. Here is a flyer from a discount store listing several sale items. Pretend that you are a checker. How much change would you have to give a person if he bought each of these items and gave you a $5 bill? A $20 bill? (Do you think you might be able to buy some of these items less expensively in a Goodwill or Salvation Army store? Would you rather pay less in one of these stores or would you prefer to buy in a regular discount or department store? Why?)

8. Compare the annual cost of subscribing to various magazines you like to read with the cost of buying them each month on the newsstand. (What magazines do you read? How do you decide what to read? Do you think you will want to read different magazines when you are older?)

9. Figure out how much several different brands of cereal cost per ounce. Compare the cost per ounce when you buy each of these cereals in different sizes of boxes. (What do you eat for breakfast? How do

1. For a discussion of ways students can analyze advertising, see Simon, Sidney B., and Lieberman, Phyllis. "Analyzing Advertising: An Approach to Critical Thinking." *National Elementary Principal* 46: 16; September 1966.

you decide what to eat? Would you be willing to prepare breakfast for the family? What would you need to do in order to prepare breakfast? Why should you do this? How do you act on mornings when you don't have breakfast?)

These assignments might well be modified with more or less mathematics and with more or less stimulation of value-oriented thought. With a little experimentation, a teacher will soon find a form that suits him and the students he is teaching Most students seem to enjoy this kind of exercise even when it is assigned in addition to regular school work.

In Summary

It should be clear that we believe that students need to learn more than subject matter. They also need to learn how to think through the values confusion that so characterizes our age, and they need to find more effective ways of relating to the world around them. Those who do not learn these things are likely to suffer from apathy, blind conformity, or irrational rebelliousness; those are the most frequent alternatives to clear personal values.[2]

Values-clarifying practice *can* be woven into mathematics lessons. We suggest the construction of word problems that lead to values discussions, efforts to personalize the data of mathematics study, and the use of out-of-class assignments that combine values and mathematics.

It must be emphasized that the purpose of these exercises is not to get students to accept the teacher's values. On the contrary. Students already have plenty of adults telling them what to think. But students get very little help in learning the skills necessary for choosing on their own from among the many conflicting values they hear about.

Nonprofessionals typically argue for a particular set of values; only a professional educator is likely to master the strategies necessary to help students learn to think for themselves in this confusing realm. This does not mean, however, that the teacher must be silent about his own values. Not at all. It is wise to show students that teachers face the same values issues as do students, and when a teacher states his position, he should say that his value judgment is merely one alternative among many and that it is offered only for further consideration.

Clearly, our objective is not to have students pretend to hold values that they will not, in fact, live by. Our aim is to help the student

2. For a detailed look at the theory that supports this point of view and for more examples of classroom activities, see Raths, Louis E.; Harmin, Merrill; and Simon, Sidney B. *Values and Teaching*. Columbus, Ohio: Charles E. Merrill Books, 1966.

develop skill and confidence in dealing with values confusion—and that takes lots of practice. All around will be the pressures to join a conforming or a dissenting majority. Our dream is of students who march to their own drummers. Mathematics teachers who appreciate the significance of this goal can take some time to help students search for that drum beat.

INSERVICE PROGRAMS IN ELEMENTARY MATHEMATICS

Harold H. Lerch

Inservice education is an essential part of a contemporary mathematics program. It is essential because a mathematics program which seeks to meet the future, as well as the present, needs of students must constantly be modified, and this in turn requires that teachers and principals be continually involved in efforts to improve the teaching of mathematics. Inservice programs are also necessary because they provide an orderly, planned basis for implementing the mathematics curriculum and for working toward its improvement. If no organized inservice opportunities are provided, the implementation and improvement of the mathematics program depends on the whims and sporadic efforts of individuals. In such instances, even action based upon the best motives may obtain negative results. Inservice programs provide a foundation for change and the means by which plans for teaching elementary mathematics are successfully implemented.

The term "inservice mathematics program" has been ascribed to a great variety of efforts directed toward helping teachers improve the instructional program. Some of these efforts have been quite limited in content, scope, and time, while others have been extensive and comprehensive. The nature of inservice activities depends upon the requirements of the local district—upon the immediate and future objectives of the inservice program and upon the district's assessment of the steps necessary to attain those objectives. Nevertheless, there are several characteristics of an effective inservice program which are generalizable to all school districts. These are concerned with the basic needs of teachers and principals which should be taken into account and with the content and procedures of the program.

Needs of Teachers and Principals

In planning inservice activities, attention should be given to three categories of personnel who are involved in the daily implementation of the mathematics program: experienced teachers, inexperienced teachers, and principals.

Experienced classroom teachers—those who have taught for many years—are often maligned for being incompetent in teaching modern mathematics. We seem to have forgotten that their preservice training programs did not prepare them for teaching a contemporary mathematics program. And our locally sponsored inservice efforts have done little to alleviate the problem. Most experienced teachers recognize that they are not thoroughly familiar with the content of contemporary mathematics programs. From the texts and other instructional materials they have been asked to use, they are aware that the organization of the subject matter and the mathematical concepts to be emphasized have changed. They also realize that the new programs demand modification of their instructional techniques, but they are often unsure about how to proceed. Further, their own experiences as students may have engendered negative attitudes toward mathematics, and their recent experiences with the new mathematics programs—which they were not adequately prepared to teach—may have left them somewhat less than enthusiastic.

Given this combination of circumstances, inservice programs for experienced teachers must attempt both to change attitudes toward contemporary mathematics programs and to provide the required knowledge and skills. More specifically, our efforts for experienced teachers should be directed toward dispelling their fears and apprehensions and developing more favorable attitudes, increasing their knowledge of the subject matter, developing and refining their instructional materials used in new programs, and extending their familiarity with the instructional materials used in new programs.

Inexperienced teachers may also encounter problems with mathematics. Reports from the Committee on the Undergraduate Program in Mathematics (CUPM) and from other professional sources indicate that we cannot necessarily assume that simply because a teacher has recently completed preservice training he is prepared to perform effectively in a mathematics program.

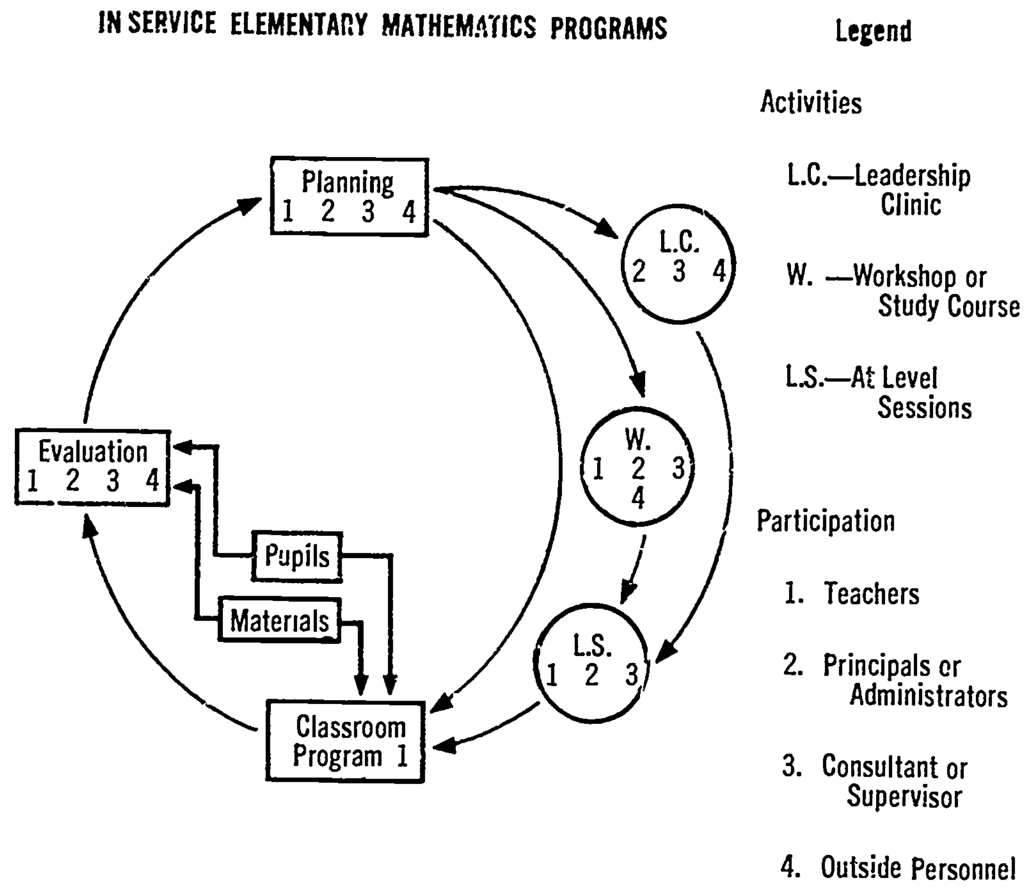
For example, inexperienced teachers may be knowledgeable about the content of elementary mathematics programs, but they may not be effective in teaching mathematics to elementary school children. It is one thing to understand the identity element of multiplication and quite another to help children learn to use that principle in renaming fractional numbers, dividing with common fractions, and dividing with decimal fractions. New teachers probably will have at least a superficial understanding of modern instructional procedures and materials, but their actual experience in using these procedures and materials with pupils will be limited. In the classroom, without help and perhaps in desperation, they may very well revert to teaching in the traditional manner by which they themselves were taught. In addition, these new teachers may have some of the problems that we recognize in experienced teachers. Through participation in inservice mathematics programs, inexperienced teachers can learn more about techniques of working effectively with their pupils. Inservice activities also are a means of acquainting these teachers with the specific nature of the local program, thus allowing them to become active contributors to its improvement.

Finally, the requirements of principals for assistance in implementing and improving the elementary mathematics program should be heeded in designing a local inservice program. There is little reason to believe that principals are better prepared for their leadership role in a contemporary mathematics program than teachers are for their instructional role. Principals need to have a comprehensive understanding both of the content of the program and of how it should be taught. Although they may not require the same degree of knowledge and skill as a master teacher or mathematics consultant, they do need more than a nodding acquaintance with the subject and with the associated instructional procedures and materials. This is essential if principals are to have a knowledge base from which to provide leadership.

Principals' participation in inservice mathematics programs should take two forms: They should be involved as learners and as leaders. In those parts of the program which deal intensively with mathematics content and instructional procedures, principals should be participant learners along with teachers. In other aspects of the inservice program, they should be leaders in planning, organizing, and implementing. In addition, special inservice sessions for principals can be directed toward improving their performance in leadership roles.

Program Content and Procedures

The objective of an inservice program is planned continuous improvement of the instructional program for children. Continuous improvement requires a long-term, comprehensive inservice program, rather than individual workshops or study sessions. In such a comprehensive effort, three major steps are involved: 1) evaluation of the mathematics program; 2) planning, including the conduct of study sessions for teachers and principals; and 3) planning, organization, and implementation of improvements in the classroom program. These three steps occur in a continuing cycle in which evaluation leads to planning which leads to implementation of plans in the classroom program which, in turn, should be evaluated, and so on. This process is shown in Figure 1.



Evaluation: Evaluation of the mathematics program is an integral part of inservice activities. The first step is to review the objectives of the program and to determine whether or not they are appropriate and are stated in terms which will permit an assessment of how well they are being attained. Then attention may be directed toward pupils' achievement and attitudes as they relate to mathematics objectives. During this process, consideration should be given to the scope and organization of curriculum content and to the use of instructional procedures and materials. Participants in the inservice program should attempt to identify possible causes for any shortcomings in the program as revealed by failure to attain objectives for learners. Throughout this process, teachers and administrators should work cooperatively. If desired, personnel from outside the district may be utilized to assist in evaluation.

Another aspect of evaluation should be concerned with assessing the effectiveness of the inservice program itself. This evaluation, involving all of the participants, should provide an ongoing basis for modifying inservice activities so that they will be of maximum benefit.

Planning for Change. Evaluation of the mathematics curriculum should result in plans for change. These plans for curriculum improvement, which should encompass both the changes desired and the means for bringing them about, should be a shared responsibility of teachers and administrators. Outside personnel may assist in these efforts, but local initiative should be the guiding force.

Some plans for curriculum improvement may be implemented directly in the classroom. Others may require intermediary activities such as a general elementary mathematics workshop, special sessions for teachers at specific instructional levels, or leadership clinics for principals, consultants, and supervisors. To be most effective, these workshops and study sessions should possess certain characteristics related to organization, content, and procedures.

First, the workshops or study sessions should be organized to establish a climate favorable to learning. This is as important in inservice activities as it is in the elementary classroom. Sessions should be conducted at a time which is convenient for teachers and when they are most capable of active participation. Released time and some form of credit or financial reward may serve as professional inducements and help to emphasize the importance of inservice programs. When the workshop leader is a member of a university faculty, college credit might be arranged. Best results are generally obtained when the leader is from outside the school system and is qualified both in the field of mathematics and in elementary education.

Second, the content of the workshop should take account of the particular needs of the participants in terms of their knowledge of mathematics, methods of teaching, and instructional materials. As a general rule—although the specifics may vary from one situation to another—participants should acquire some familiarity with the language of sets and set relations and with their use in the elementary mathematics program. Emphasis should be upon the underlying structural properties, developing the main ideas of mathematics, rather than upon computational details. Suggestions about procedures for developing pupils' abilities to think in the language of mathematics through inductive or discovery methods should parallel the development of mathematical topics. References to appropriate instructional materials can be made, and participants should be encouraged to begin collecting these materials. It is very important that the content of the workshop be as interesting, fascinating, and applicable for teachers as the content of the classroom program is to be for pupils.

Third, the procedures used in inservice workshops should be similar to those used in elementary classrooms. The workshops should, in effect, model the type of teaching which is desired in the classroom. A variety of learning experiences may be utilized: lectures, demonstrations, role-playing situations, audiovisual presentations, discussions, experimentations with manipulative materials, and simulated classroom situations involving pupils. Emphasis upon one mathematical topic at a time, based on an integrated view of subject matter and teaching approach, will be most meaningful to teachers. While some of the content and approaches gained from the workshop may be used immediately in the classroom, teachers should be cautioned against introducing ideas or teaching procedures which they do not completely understand. In general, outside assignments for future workshop sessions should be kept to a minimum.

The preceding suggestions about organization, content, and procedures apply to the various types of study sessions which may be conducted as part of an inservice program in mathematics. Usually, these sessions will deal with the elementary mathematics curriculum across multiple levels of instruction. However, effective inservice programs also provide opportunities for teachers from a given instructional level to work cooperatively on the specific problems and possibilities for program improvement at that level. They also include special leadership sessions for principals, consultants, and supervisors to strengthen their skills for leadership in improving mathematics instruction.

Implementation in the classroom. Since the purpose of an inservice program is to improve the instructional program for children, the translation of plans into actual classroom practice is extremely critical. Implementation of desired changes should be planned cooperatively by teachers, consultants, supervisors, and administrators.

Principals and consultants must provide effective professional leadership at all stages of the implementation process. They should facilitate teachers' efforts by helping to solve problems of obtaining materials and of scheduling; by securing assistance when it is required; and by helping teachers to self-critique and modify their procedures when this is necessary. Without this support, the results of even the best inservice programs may be far less than satisfactory.

Inservice programs in elementary mathematics may succeed or fail because of the depth of the initial plunge. Effective programs require careful planning of each activity. Some portions of the program may better be initiated before others. Some, such as an intensive study session, are more appropriate at one time of the year than another. Thus, the timing of inservice activities, as well as the nature of the work undertaken, is strategic.

The strength and value of inservice activities will increase as improvements in the classroom program become visible. As teachers gain knowledge and skill, the program will be benefitted by their growing contributions to the conduct of the work and to other teachers. Those teachers who may not have participated initially will become interested, and this wide participation will help to support the total effort. Further, teachers' activity in inservice programs will tend to be accompanied by increased participation in professional organizations, greater attention to the professional literature, and self-directed study.

There is no point at which an inservice mathematics program should end. Some specific activities may eventually become unnecessary, but a continuing, orderly effort will continue to be important in providing a foundation for improving instruction.

AN INSERVICE MATHEMATICS PROGRAM FOR PRINCIPALS

Alton L. Reynolds

"The principal is the educational leader of the school." This oft-quoted statement holds many challenges for the elementary school principal. How shall he meet his responsibilities as educational leader? How can the principal be thoroughly informed about the curriculum of the school—about content, methodology, teaching techniques, and underlying philosophy and theory?

Some students of administrative theory have advocated that a principal specialize in one subject matter area. He could then utilize this knowledge to provide strong leadership for curriculum development in that one field and thereby gain respect from teachers. Involvement of other specialists would provide a well-rounded team for curriculum improvement.

Other administrative theorists have emphasized the importance of principals' skillful use of techniques of group leadership. Rather than attempting to provide leadership through extensive knowledge about the curriculum, the principal would facilitate the work of other people who do have such knowledge.

Another alternative is found in the practice of many principals who have participated in various inservice programs for teachers and have learned along with the faculty.

All of these approaches have real merit, but each has some shortcomings as well. In curriculum development, the principal is all too often caught in the middle. It is he who answers to parents, to unenthusiastic teachers, and to the central administration. If he is to cope with these cross currents and provide direct and continuing leadership in the improvement of instruction, the principal should be as well informed as possible in all aspects of the school program. None of the approaches to meeting the principal's responsibility for leadership which we have described, therefore, seems completely adequate.

Another approach to the problem, one which seems to have real merit, was used in Wellesley, Massachusetts. Here, with principals in a very active leadership role, the elementary schools developed and implemented a strong program in contemporary mathematics. The result has been a program which has been operated successfully and which has continued to improve through the active, supportive involvement of elementary principals.

In the early 1960's, the Wellesley Public Schools were very much aware of a real need to make extensive changes in the elementary mathematics program. They also recognized that principals would be strategic to the success or failure of efforts to revise the teaching of mathematics. The principal, in effect, could make the program go or subtly kill it with lack of enthusiasm and understanding. In the final analysis, he would be the person primarily responsible for administering and supervising the program, and this meant that he needed to be committed to it and knowledgeable about philosophy, content, and instructional methods and materials.

To give principals this kind of background, a rather extensive effort is required. Further, principals themselves should be involved in planning the kind of program they need. Thus, late one fall, a committee of three elementary school principals was formed and charged with the responsibility of organizing a four-week workshop for principals to be followed in turn by extensive inservice programs for teachers before the new mathematics curriculum was implemented.

As it was finally developed, the principals' workshop dealt with four major areas: 1) an overview of contemporary mathematics; 2) an analysis of various new and experimental programs; 3) instructional materials, especially textbooks; and 4) plans for teacher involvement and preparation. About a week was spent on each area.

Overview. The purpose of the overview portion of the workshop was to better acquaint principals with the basic characteristics of a contemporary mathematics program in the elementary school and to discuss several basic issues.

Two outstanding mathematics consultants were employed to present their views on contemporary mathematics. Both of these consultants were independent authorities who were not tied directly to any publisher of a textbook series or to any particular experimental program.

This was important in giving us a wide-ranging and balanced viewpoint.

The meetings were informal. The group met outside of the schools in a local hotel, and the sessions often extended beyond the normal workday and well into the evening. Topics covered in considerable detail included not only what was currently happening in the field but also what would probably occur in elementary mathematics during the next ten years. We also discussed such questions as these: What aspects of experimental projects will probably find their way into published material? How have methods of teaching mathematics changed? What vocabulary is necessary for the new programs? How do we provide for reinforcement and maintenance of skills? How do we evaluate progress? What kind of experimentation should we become involved with? What is our concern for basic literacy in mathematics? What kinds of instructional materials seem to be most useful in what kinds of programs?

This week was very productive for all of the participants. The informality led to a real give-and-take among the principals and consultants and enabled us to gain insights that could be obtained in no other way. The basic understanding of contemporary mathematics programs which was gained as a result of this overview week made the principals much more comfortable about beginning a major curriculum revision effort. We became convinced that a change was necessary—an important first step. And we also began to develop a good idea of some of the specific ways in which our mathematics program should change.

New programs. Following the overview week, the next portion of the workshop reviewed a number of new and experimental projects in elementary mathematics. The principals themselves took responsibility for these sessions. The planning committee had identified several principals who were particularly oriented toward the teaching of mathematics and who had been experimenitng in their own buildings with some of the newer programs. These principals led the second week's sessions.

The programs analyzed were those of the School Mathematics Study Group (SMSG), the Madison Project developed by Professor Robert Davis, the work of Professor David Page at the University of Illinois, and the efforts at the Miquon School in Pennsylvania. Each of these programs was examined carefully with particular attention to new content and methods of teaching and learning.

The real strength of these discussions was the sharing of ideas in a close, intimate group. No one hesitated to make a point or to ask questions. And for the principals who conducted the sessions, it was an opportunity to sharpen their leadership skills in preparation for the inservice program for teachers.

Textbooks. While we recognized that a good contemporary mathematics program utilizes far more in the way of instructional materials than textbooks, our primary attention in the third week was to alternative textbook series. A number of our schools had been engaged in preliminary experimental work in teaching mathematics and had available many related instructional materials with which we were familiar. But we felt that a good basic text might best serve our immediate needs in launching a new program. We could use this text as a basic guide and supplement it wherever necessary.

Six publishers who were revising their elementary mathematics series in ways that seemed in keeping with what we wanted were asked to send authors or consultants to meet with the principals. We questioned the publishers' representatives about the philosophy behind their texts and about the content. Each series was carefully analyzed as well for methodology, for clarity for both pupils and teachers, and for flexibility of use in a multi-level approach. We also were particularly concerned about the comprehensiveness and usefulness of the accompanying teacher manuals.

During the summer workshop, the principals did not select a textbook series. Instead, it was decided to purchase several copies of each series we thought might be appropriate and make them available for teachers to test in the classroom and to review during their inservice work.

Planning for teacher involvement. Significant changes in instruction can occur only when teachers are directly involved in the process of planning for change. This kind of teacher involvement builds an understanding of the nature of the changes to be made and the reasons behind them, and it permits teachers to contribute their ideas to the curriculum effort. The fourth week of the principals' workshop, therefore, was spent in planning the inservice program which would be conducted for teachers.

We decided that the teachers would benefit from going through many of the same steps as the principals. In addition, we were concerned that the inservice program for teachers be planned so that:

- Teachers would take some of the leadership roles in teaching their peers and in sharing their ideas.
- Principals would play an active part—some in teaching and others in providing support and assistance for teachers with special interests or problems.
- Qualified outside consultants would participate in the program and would be asked to involve their audiences actively in the learning effort.
- The diverse interests and special abilities of teachers would be encouraged.

• Instructional materials consistent with the new mathematics curriculum would be developed and would be made readily available for teachers' use.

• Specific attention would be given to helping teachers plan instructional procedures and materials which would be appropriate for pupils with a wide range of ability, learning rate, and interest in mathematics.

Wellesley's experience with this four-week workshop for principals suggests some basic guidelines for inservice programs for elementary school principals.

1. When an inservice program for principals is intended to facilitate extensive changes in a curricular area such as mathematics, enough time and resources must be allocated to the task. A short workshop of only a few days or even a series of meetings held once a week throughout the school year would not have been as effective as our concentrated, four-week program. While such an extensive effort does involve expense and and takes principals away from other tasks—including other types of inservice work—it would seem logical to make this kind of an investment if the curriculum change has high priority. Indeed, the costs of not making such an investment in inservice preparation can be high indeed—high in terms of an ineffectual instructional program.

2. Principals themselves should participate in planning their inservice sessions. This will help to assure that the program is relevant to their needs.

3. Simultaneously, outside consultants are invaluable in introducing ideas and information which may not be readily available within the district. Use of outside personnel need not involve large expenditures of money. In Wellesley's case, for example, we utilized consultants supplied by several publishers in addition to independent authorities. The latter, of course, may often be found quite close to home.

4. It is very important that principals be well informed about the philosophy behind a new curriculum—in this case, contemporary mathematics. Our overview week provided a basic framework which made the subsequent, more specific sessions much more meaningful.

5. In preparing for the introduction of a new program, principals should have an opportunity to examine several alternative curriculums. The same, of course, is true for teachers. Involvement in this process builds an understanding of the program which is finally adopted and helps to clarify some of its unique characteristics.

6. The procedures used in the inservice effort should support the development of principals' leadership skills. Thus, Wellesley's program was designed to give principals practice in presenting ideas to a group and in leading discussions. Just as an inservice program for

teachers should demonstrate desirable instructional procedures for use in the classroom, so should an inservice program for principals utilize procedures which are appropriate in working with teachers.

Wellesley's inservice program in mathematics for elementary principals is illustrative of one way in which principals can gain some of the knowledge and skill required to be effective educational leaders. Whether this type of program or some other is used, it is clear that a principal who wants to meet the challenge of providing strong, innovative leadership must make the effort to become as well informed as possible about the various aspects of the elementary school curriculum.

BLOCKS TO SUCCESSFUL MATHEMATICS PROGRAMS OR HOW TO INSURE FAILURE BY REALLY TRYING

Harold H. Lerch

Any school system can do a very effective job of ruining an elementary mathematics program. It's really very simple. Just erect a barrier to success with blocks readily available in any school district —principals, parents, teachers, consultants, inservice programs, and the content of the math program itself. Of course, you could use all of these blocks to pave the way to a successful program, but with care, they can be readily adapted to ruining it. One needn't use all of the blocks we suggest to make certain that the program fails; concentrating on developing one or two extremely effective blocks will usually do the job. But as in other endeavors, we can be more certain of a truly unsuccessful program if our efforts are comprehensive and complete.

Principals. The principal can almost single-handedly insure the failure of a math program. He can begin by making everything as difficult as possible for teachers. For example, he should resolutely refuse to provide desirable physical facilities, instructional materials, and organizational arrangements. This is especially important if some teacher suggests an idea for improving the program. The principal can stop this sort of nefarious threat simply by refusing to let the teacher make any changes in the physical arrangement or equipment of the classroom or by transferring twenty more children (preferably discipline problems) into the room—or whatever specific technique will do the job of squelching the teacher's ideas. He also should be sure that none of the faculty has a chance to acquire any information about teaching elementary mathematics. This can be acomplished quite easily by forbidding teachers to exchange ideas among themselves and by precluding all educational opportunities.

The principal's comments to teachers and parents about modern mathematics programs should always be derogatory. Be sure to emphasize that there really is nothing new or different about either the content or procedures. An occasional comment that "we were all taught arithmetic the old way and look how successful we are" will help things along—especially with parents.

In appraising teaching, the principal should base his assessment on the quietness of classrooms, whether all the children have completed every page of their workbooks, and the pupils' scores on standardized tests (preferably tests which measure knowledge and skills which are not taught in the program). A few unscheduled visits to classrooms when math lessons are in progress, followed by devastating criticism in front of the children, will also help to demoralize teachers and insure complete failure.

Parents. Parents can be effective blocks to a successful mathematics program if they are sufficiently antagonized. This really isn't at all difficult. Keep them completely uninformed about the nature of the program—its objectives, content, and instructional procedures. Remind them regularly that they are incapable of learning the kind of modern mathematics which is being taught in the elementary school and be sure their children have assignments to do at home which neither the parents nor the children can understand.

Remember that parents' attitudes greatly influence the attitudes of their children and that negative attitudes will go a long way toward insuring failure.

Inservice programs. One might think of inservice programs as a means of improving the teaching of mathematics, but a little planning will make them very efficient in insuring a total debacle.

First, be sure that all inservice activities consist of no more than short intensive study sessions which begin and end on specific dates. That way no one will get so involved that they learn much of anything.

Second, keep the mathematical content of the programs at as sophisticated and complex a level as possible. It will help to get instructors for the inservice courses who are pure mathematicians or—if you can't find any of those—just use secondary school mathematics teachers. In any event, be sure the instructors awe the participants with their knowledge of the beauty of mathematics and say absolutely nothing about how to teach it. At every opportunity, the dismal ignorance of the elementary teachers should be established.

Third, require every teacher to attend all of the inservice courses with administrators appearing only to make spot-checks of attendance. (The administrators might sit right outside of the room and loudly discuss how terrible the teachers are.)

Fourth, schedule the course at an hour when teachers are most tired and unresponsive. Then be sure that the entire procedure is as deadly as possible and that the teachers are not allowed to participate in any way.

If all else fails, one final hint may save the day. Ignore those four instructions about inservice programs and do just the opposite. Make the programs as effective as possible, but spend the entire time promoting every bad practice you can think of in teaching elementary mathematics. You might even be able to find a math textbook from around 1860 to give the teachers as a curriculum guide.

Consultants. Mathematics consultants or supervisors can do a great deal toward ruining a program if they will follow a few basic principles. They should work diligently to convince teachers that they don't know enough (and certainly can't learn enough) to teach mathematics. Instructional materials should be stored under lock and key in a central warehouse at least five miles from the school. Requisitions for these materials should be completed with ten carbons and submitted one month in advance to the supervisor.

After taking all of these steps, consultants can complete the job by being conspicuously absent when important problems arise which require immediate solutions. If a teacher happens to buttonhole a consultant, however, all requests for help should be answered with gimmicks, tricks, and lengthy philosophical discourses laced by biting criticism of everything the teacher is doing. Above all, teaching demonstration lessons should be avoided at all times.

Teachers. Teachers have a distinct advantage in insuring the failure of the program and their pupils because, regardless of the reason for the failure, teachers will receive much of the credit. They can earn this credit quite easily. For example, even with the most modern and appropriate materials right in the classroom, teachers can skip important parts of the content, ignore the relationships between concepts, deny pupils any chance of participating in a discovery approach, and emphasize the rote memorization of assorted rules. It also helps if teachers expect exactly the same thing from every pupil and assign homework with no regard for the time it will take and no concern for differences in pupils' backgrounds.

As an added insurance of failure, teachers may even discard all the modern materials they have in favor of the old books and drill sheets with which they felt so comfortable.

Content. With some careful planning, the content of the math program can be selected, organized, and presented so it will block success. Begin by retaining the exact content and organization of the traditional program. For maximum effect, scatter here and there units of work which outwardly appear to be modern and which will thoroughly confuse everyone. For example, intersperse a few units on sets and operations with sets, systems of numeration with bases other than ten, along with a little geometry, logic, and properties of number. Just be sure that these units are totally abstract and unrelated to anything else in the program. As a general rule, failure will be most likely if the content is always slightly inappropriate for the pupils in terms of their previous learning and their abilities.

It shouldn't be too difficult to find other blocks to a successful mathematics program. The main goal is to do whatever will make it impossible for children to acquire understanding and skill in mathematics.

One final suggestion. Construct a checklist of all the blocks for insuring failure and specify how they should be used. This will give you an evaluative instrument for determining whether each block is really helping to cause failure and will permit quick corrections if it should turn out that one of your blocks is inadvertently creating a successful program.

FUTURE DIRECTIONS FOR ELEMENTARY SCHOOL MATHEMATICS

William J. Linville

There are those who would have us believe that the millennium in elementary school mathematics has at last come to pass. There are those who would tell us that all the needed changes have been made and that we are finally ready to journey our merry way toward a state of mathematical bliss in which each morning will bring us a class of enthusiastic, energetic students eager to "discover mathematics." We may even be told that we now hold the key that will unlock those mystical structures of mathematics and reveal its beautiful, orderly properties; that we hold the means for insuring that all the Johnnys and Marys in our charge will attain complete understanding of the fundamental operations.

There are those who would tell us that all these things have come to pass. *Don't you believe it!*

A curriculum or teaching method that will solve all of the woes of any subject area belongs in the same category as the fountain of youth, perpetual motion, and the universal solvent. It is highly improbable that it will ever be found. After all, we are working with children who have intellects, emotions, and personalities as different as their appearances. Each individual is much unlike the others around him, and this being the case, it does not seem likely that any given method of instruction, per se, will ever suffice for all students.

Even though we have not arrived at the ultimate means of teaching elementary mathematics, great strides have been taken. As a result of the combined efforts of scholars in the field and teachers in the classroom, newer mathematics programs have now passed from the experimental stage (although much research is still being done) to the point of actually being carried out in elementary classrooms. Some sense of perspective on what has been achieved is suggested by William Brownell's predictions in 1959 about the changes which would have taken place in elementary mathematics by 1970.[1] Brownell forecast that:

1. Much of the traditional content would be taught earlier.

2. Greater emphasis would be placed on the mathematical aspects of the subject and on the reasons or "why's" behind mathematical operations.

3. Practical means would be found to accommodate effectively the differences in learning ability.

4. The program in the higher grades would include much that is new, with a great deal of the "social arithmetic" disappearing from the higher grades possibly to be replaced by such topics as statistics, approximation, mental computation, very large and very small numbers, intuitive geometry, and algebraic concepts.

While many of these predictions have been realized, others are sure to follow. And because further changes most certainly are ahead, we need to question what they might be and what will be their effects.

Changes in Program Content

Future programs in elementary mathematics will probably be characterized by changes in the scope and placement of mathematical content. In particular, greater provision will be made for both gifted and below-average students. Gifted students will be offered greater breadth and depth of content. They will devote less and less time to working with the four fundamental operations of arithmetic and more and more time to the pursuit of topics that were previously reserved for the later years. As a result, they will gain a deeper understanding of the structures which carry through the entire study of mathematics and be better prepared for further study of the subject.

Similarly, compared to present programs, more provisions will be made for teaching the below-average student. While most authorities agree that the newer approaches to teaching mathematics in the elementary school offer the slower student a better chance of achievement than could ever be hoped for with more traditional approaches,

1. Brownell, William A. "Arithmetic in 1970." *National Elementary Principal* 39: 42-45; October 1959.

much still remains to be done to supply teachers with more concrete means of individualizing instruction for these pupils.

It is highly possible, too, that further shifts will be made in the grade placement of topics in elementary mathematics. Contrary to much current thinking, some of these shifts may well turn out to be upward as well as downward. In light of the best knowledge available about elementary school children and how they learn, educators will seek not only the point in time at which children can learn a given concept but also the point at which it should be learned. It may eventually be found that merely to demonstrate that a concept can be learned at a given age is not enough. All of the evidence on this question is not yet in.

Improved Textbooks and Materials

Textbooks will continue to reflect research findings in the area of elementary mathematics and will become modern in content as well as in title. Coincidental to this, many textbooks will provide a more detailed manual for teachers, including in some cases what would almost amount to an inservice training unit for those teachers who feel inadequately prepared to teach the newer programs. They will also suggest enrichment materials which will allow superior students to progress horizontally, to delve into given topics more deeply and to apply fundamental understandings as the basis for making generalizations about a particular mathematical area.

Along with textbooks that present mathematics in a different and more structured manner, manipulative materials for students and demonstration materials for teachers will be available in ever-increasing amounts. Many textbooks will include suggested lists of materials which may be secured to enhance the teaching of a particular program. While the teacher may use such lists as guides, the mathematics specialist, who will be an integral member of the consultant staff in many school systems, will also be available to help in selecting the teaching materials. Many of the materials which require considerable storage space or which are used only in presenting certain specific topics—geometric models, units of measurement, and the like—will be kept in an instructional materials center where they will be available to any teacher in the school.

Changes in Teacher Preparation

Further chan[illegible] almost certainly come in the area of teacher preparation. It [illegible]erally conceded that the ability to compute, using the four fundamental operations, can no longer be considered adequate preparation for teaching mathematics in the elementary school. It is already a prime necessity for teachers to have a basic understanding of the structures of mathematics and the fundamental principles on which the arithmetical phase of mathematics is based. This will be so to an even greater degree in the future.

In order to provide prospective elementary school teachers with a better foundation in mathematics, changes will have to be made in the curriculum of teacher training institutions. John L. Marks believes that few, if any, colleges now have the manpower to provide the twelve-unit sequence of college mathematics proposed as desirable in the preparation of elementary school teachers. Since it is unlikely that this manpower shortage will become less acute in the future, other sources for this training will have to be found. One logical place would seem to be the elementary education departments of teacher training institutions. It does not seem mandatory that all of the instruction given prospective teachers be carried out by mathematics departments when, in fact, a great deal of the emphasis must be directed toward means of implementing the curriculum and actually teaching the material. Most probably, members of elementary education departments could be found who are familiar with the basic premises of the newer mathematics programs and, perhaps equally important, who know elementary school children and how they learn.

More Extensive Inservice Programs

Inservice training in mathematics for elementary teachers will be much more extensive in the future than it is today. Already, school systems across the country are recognizing the fact that many of their teachers are poorly prepared to teach the newer mathematics programs. Inservice programs that will provide teachers with instruction in the basic structures of mathematics as they are applicable to the elementary school will be carried out by mathematics specialists on the regular staff of the school system, by consultants from nearby colleges or universities, or by both working together. Some of the developmental efforts of regional educational laboratories—for example, the minicourses for teachers being prepared by the Far West Laboratory—could also be useful for inservice training.

In these inservice sessions, more emphasis will be given to the teaching processes that are appropriate in presenting various topics to children of a particular age, as well as to uses of manipulative materials and audiovisual aids. In a radical departure from many current practices, inservice education will be conducted during released time for teachers and not involve after-school, evening, or Saturday morning sessions.

As further support of inservice education, colleges and universities will provide more summer session courses in elementary mathematics, and extension centers will offer the experienced teacher an opportunity to gain the fundamental knowledge background so necessary to effective teaching of newer mathematics programs. Such offerings can help immeasurably to allay the fears and frustrations of those teachers who do not feel that they are adequately prepared to handle the newer programs.

Curriculum research centers, located strategically around the country, will be available for teachers to visit. At these centers, they will see demonstrated the newer methods of instruction and creative ways of using materials as aids in presenting mathematical concepts and skills. It is also highly probable that personnel from these centers will be on call so that they may visit school systems to teach demonstration lessons with materials which may be adopted by the schools.

New Evaluation Instruments

We will most certainly see changes in the evaluative instruments used to measure achievement gains of students in elementary mathematics. Currently, many teachers and school administrators have been disappointed that students taught in a modern mathematics program have failed to show more significant gains on achievement tests than they did under a more traditional program. Because the newer programs direct more emphasis toward broad understandings and the ability to relate fundamental structures to each other as opposed to the student's ability merely to compute with accuracy, the achievement tests we have been using do not adequately measure student gains. With different objectives in teaching elementary mathematics, our evaluation instruments must be changed so they will measure relevant outcomes. In the future, assessment instruments will be devised which will reveal the degree to which students understand basic mathematical concepts and interrelationships as well as their computational accuracy. Only when are are able to measure more effectively what we are teaching will we have a basis on which to appraise the newer programs accurately.

Increased Action Research

The literature in the field of elementary mathematics is currently overflowing with the findings and recommendations of various experimental programs being developed and tested across the country. These studies are far from being completed. In the meantime, findings will continue to be consolidated, and suggestions will be made for even more effective approaches to teaching various topics in the classroom.

The experimental work now being carried on by Patrick Suppes at Stanford University with computer-assisted instruction for elementary mathematics could quite possibly open many exciting avenues for the elementary school. Perhaps it is through this type of instruction that the elementary teacher will best be able to cope with the problems connected with enriching the program for the gifted student, as well as providing more individual, self-paced instruction for the less able pupil. The potential implications for the elementary school of this type of experimentation alone seem virtually limitless.

As the newer programs are more widely adopted, the amount of action research by individual teachers and teams of teachers will undoubtedly increase at a tremendous rate. As school systems become increasingly involved in setting forth specific objectives for their particular situations, elements of the various experimental programs will be tested separately and in combinations to find the curriculum and teaching-learning procedures that best suit the needs of the local system. Much of this action research will probably be directed toward finding more efficient means of individualizing instruction, using the locally adopted program materials.

In the final analysis, the efforts of the past several years in elementary mathematics have just begun to bring about the changes which Brownell predicted in 1959. While many of the mathematics programs used today in elementary classrooms are "modern" in name only, the combined forces which have been discussed here will contribute to bringing about real changes in the topics studied, methods of instruction, and materials used. The mathematics programs of elementary schools will truly become different from traditional approaches, in fact as well as in name, as textbooks are written with better guides for teachers, as courses of study are prepared with more specific objectives in mind, as more useful materials are available, as teachers are more adequately prepared at the preservice and inservice levels, as better evaluative instruments are developed so that program results may be appraised more effectively, and as the findings of research from experimental programs and from action research at the local level are made available.

It may be reasonably concluded that the emerging curriculum of mathematics in the elementary school is just that—*emerging*. The topics presented, the manner of presenting them, and the materials which may best be used are by no means final. Nor can one reasonably expect that perfect solutions will be found in the near future. What we have accomplished thus far in improving the teaching of mathematics in the elementary school cannot be construed as a panacea for all of our problems in mathematics education. As long as we are dealing with people in the processes of education, we shall always be seeking means of enhancing the teaching-learning process. This, of course, is as it should be.

Nevertheless, there is strong evidence that through the cooperative endeavors of many persons, the future will hold hope for even more dynamic, interesting, and effective means of teaching mathematics to elementary school pupils. There are indications, also, that we can look forward to teachers who are better prepared and who are supported by appropriate instructional materials and expert assistance to a far greater degree than ever before. As we move toward these goals, we will increasingly achieve the kind of mathematics education which will give children a rational and ordered power over the knowledge and skills of the field.

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