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ABSTRACT

Reported is an experiment designed to (1) test the effect of teaching certain concepts of logic on verbalization of discovered mathematical generalizations, (2) prepare a research population which has demonstrated the ability to verbalize newly discovered mathematical generalizations with precision, and (3) test the effect of an ability to verbalize discovered mathematical generalizations upon the ability to use that generalization. The sample consisted of eighth grade students of Chiddix Junior High School in Normal, Illinois. They were enrolled in seven mathematics classes taught by three teachers. Students in Phase I completed a programmed unit "Sentence of Logic." Phase II consisted of discovery programs using nonverbal awareness, verbalization on the part of the text, and verbalization on the part of the student. Phase I of the experiment yielded "normal" population to verbalize discovered mathematical generalizations with precision. Evidence from Phase II indicated that those students with high verbalization ability could better transfer the mathematical generalizations which they discovered. (RP)

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U.S. DEPARTMENT OF
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TABLE OF CONTENTS

SUMMARY

Mathematics educators are agreed that a student should discover as many mathematical generalizations as possible. Some would not have these discoveries verbalized immediately. Hendrix advocated that the teacher delay verbalization of discovered generalizations on essentially two grounds:

1. There is a common evidence that a student does not have the linguistic capacity to state his discovery with precision. Imprecise verbalization may have undesirable effects.

2. There is research evidence that a student who immediately attempts to state his discovery is less able to use that discovery than one who possesses the discovery on a nonverbal awareness level.

However, Ausubel argues that the verbalization of a subverbal insight is an integral part of the thinking process; he suggests teachers not leave a discovery in a nonverbal awareness level because this would abort the thinking process.

Henderson pointed out that teachers do encourage a student to verbalize his discoveries and the length of delay before verbalizing depends on the student's facility with language. Retzer, using a college capable and gifted population, provided research evidence which weakened the first of Hendrix' arguments. Studying some selected concepts of logic enabled the treatment group to verbalize discoveries with precision.

Phase I of this experiment yielded evidence that studying logic concepts was sufficient to enable students in a normal population to verbalize discovered mathematical generalizations with precision. Thus, teachers may choose to include concepts of logic in the curriculum with a view to strengthening linguistic ability and lessening the need for a nonverbal awareness teaching strategy. Since studying sentential logic increased the verbalization ability for students with higher general ability levels more than those with lower abilities, a teacher might feel it more crucial to include logic in the curriculum for college capable and gifted students.

Evidence from Phase II indicated that those students with high verbalization ability could better transfer the mathematical generalizations which they discovered. Researchers may want to test this finding with additional studies, and teachers may begin to feel that what they teach which enhances ability to verbalize precisely may also enhance the ability to use mathematical discoveries. Whether discoveries were immediately verbalized by the textbook or the student, or whether they were left on a nonverbal awareness level, seemed to make no difference. This evidence runs counter to a finding of research by Hendrix, and may weaken the recommendation that discoveries be left on a nonverbal awareness level because of anticipated inability to transfer.

BACKGROUND FOR THE STUDY

The structure of any branch of mathematics contains axioms and theorems which may be stated as generalizations. Thus, generalizations occupy an important central position in the structure of mathematics. While some contemporary mathematics educators feel that students should discover for themselves as many mathematical generalizations as possible (3, 20-26)*, opinions differ on the desirability of immediately verbalizing these discoveries.

Hendrix (6, 290-292) pioneered the thought of delaying verbalization based on a series of experiments she conducted in 1946 and 1947 and on her experience in helping student teachers acquire skill in inductive teaching. One may paraphrase her thoughts with the remainder of the statements in this paragraph. It is wrong to consider 'generalizing' as synonymous with 'composing a sentence which states the generalization involved'; the separation of the discovery phenomena from the process of composing sentences which express those discoveries is the big breakthrough in pedagogical theory. As far as transfer power is concerned, the whole thing is there as soon as the nonverbal awareness has dawned. Not only did the learners who completed correct verbalization of the discovery do no better on transfer tests than those for whom the teaching was terminated at the nonverbal awareness stage, but the verbalization of the discovery seemed actually to diminish the power of some persons to apply the generalization. Teachers often call for statements of generalizations when the students do not possess the vocabulary and rules of sentence formation necessary for a precise verbalization of the generalization. The teacher must either ignore an incorrect spontaneous generalization or stick with it until students see the need to discard it. Pressing for a correct verbalization usually demands a long, laborious digression. If the verbalization is postponed until a later lesson, the linguistic formulation can be undertaken as an end in itself.

Ausubel (1, 22-23) agrees that subverbal awareness exists and says:

> The principal fallacy in Gertrude Hendrix' line of argument, in my opinion, lies in her failure to distinguish between the labeling and process functions of language in thought. . . Verbalization, I submit, does more than verbally gild the lily of subverbal insight; it does more than just attach a symbolic handle to an idea so that one can record, verify, classify, and communicate it more readily. It constitutes, rather, an integral part of the very process of abstraction itself. When an individual uses language to express an idea, he is not merely encoding subverbal insight into words. On the contrary, he is engaged in a process of generating a higher level of insight that transcends by far---in clarity, precision, generality, and inclusiveness---the previously achieved stage of subverbal awareness.

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*The first numerals in these ordered pairs refer to sources in the bibliography in Appendix VII and the second numerals refer to page numbers.

Here, then, seems to be an important issue in mathematics education. Shall we, as Hendrix suggests, delay the verbalization of a newly discovered generalization (since the students do not have the linguistic ability to state it precisely and since one may expect a loss in transfer power) or shall we, as Ausubel seems to indicate, immediately verbalize to keep from aborting the thinking process?

Henderson (5,287) seems to identify the crux of the problem of delaying the verbalization when he states that even though the verbalization may be postponed, ultimately the teacher encourages the student to state his dis-overy as a generalization. Whether the teacher seeks to have the students immediately state the generalization in precise form depends on their facility with language. The more facile they are the sooner the verbalization can be attempted.

Retzer (9, 4) suggested that instead of asking, "How soon after discovering generalization should a student verbalize?", perhaps we should be asking, "What is the student's facility with the language he needs to precisely verbalize his discovery?" With this in mind he hypothesized what knowledge about language is sufficient for a student to be able to write a generalization which he has discovered and incorporated this knowledge in a linearly programmed unit entitled Sentences of Logic. An experiment was performed using this unit with college capable junior high school students in which the treatment group did significantly better in verbalizing discoveries precisely than the control group. This experiment provided evidence that linguistic ability is a factor that can be manipulated for experimental purposes. Thus, a teacher who wants students to verbalize generalizations as these are discovered and also avoid the undesirable consequences of imprecise verbalization may teach knowledge about universal generalizations per se and anticipate the verbalization called for will be more precise.

One may question the wisdom of generalizing the results of this experiment to all students since the research population consisted of college capable subjects. Phase I of this experiment was designed to replicate Retzer's original experiment with a cross section of junior high school students with a wide range of abilities using the same treatment and evaluation instrument. The knowledge gained thereby could influence those teachers who delay verbalizations of newly discovered mathematical generalizations in anticipation that they will get one which is so imprecise as to require an undesirable digression to correct it.

Another important objective of Phase I was to prepare a research population for an entirely separate study (which was Phase II of this experiment). Based on the contingency that the results of Phase I of this experiment would be approximately the same as with the original experiment, there would exist a research population whose members have demonstrated their ability to verbalize newly discovered generalizations with precision. Then, Phase II was designed to investigate this ability on transfer power (i.e. ability to use a discovered generalization).

Hendrix (6, 290-292) has stated that as far as transfer power is concerned, the whole thing is there as soon as the nonverbal awareness has dawned; verbalization of the discovery seemed to diminish the power of some persons to apply the generalization. One may question if the loss of transfer power, which was concomitant with imprecise verbalization, would also be experienced if the student could state his discovery with precision. In a bulletin devoted to research in mathematics education, Hendrix (7, 58-59) places this question on a list of critical unanswered questions, "Would immediate linguistic formulation of an individual's discoveries have the same detrimental effects if he had sufficient linguistic power to avoid making incorrect trial sentences?" Finally, Becker and McLeod (2, 105) summarize the current (1967) state of research in this area and related areas,"...the role which verbalization plays in transfer of mathematics learning remains unclear. Consequently, specific additional research is needed in these areas."

Hendrix' original research in this area has been criticized by Becker and McLeod (2, 102) because of lack of evidence of statistical controls and the limited scope of the transfer tested and by Ausubel (1, 52) on the basis of control problems, small research population, untenable randomization assumption with respect to the influence of uncontrolled variables, and a low level of significance. However, Neuhouser performed an experiment in which he compared three teaching methods under more defensible experimental conditions; among his conclusions he states (8, 61), "Some people may believe that stating a rule after discovery is a better method of teaching for transfer than not ever stating the rule. This hypothesis could be rejected at the .05 level." His findings confirm those of Hendrix.

Neuhouser granted permission to use his experimental materials to carry out Phase II of this experiment. These involved programmed materials on some of the laws of exponents as well as a test on transfer whose reliability had been established as a result of its use in his experimentation.

OBJECTIVES

The objectives of Phase I and Phase II of the present experiment are as follows:

1. To test the effect of teaching certain concepts of logic on verbalization of discovered mathematical generalizations using a normally distributed research population of junior high school students.

2. To prepare a research population which has demonstrated the ability to verbalize newly discovered mathematical generalizations with precision.

3. To test the effect of an ability to verbalize discovered mathematical generalizations upon the ability to use that generalization.

STATISTICAL DESIGN

The research design used for Phase I is a two-by-two analysis of variance, each of the factors having two levels. The dependent variable is the ability of junior high school students to verbalize precisely mathematical generalizations which they discovered. The independent variables are the following factors listed with their respective levels.

Factor A: Study of selected logical concepts, or lack thereof.

A_1: Having completed the programmed unit, Sentences of Logic.
A_2: Not having studied the programmed unit, Sentences of Logic.

Factor B: Ability level.

B_1: College capable (I.Q. 116 and above).
B_2: Not college capable (I.Q. 113 and below).

The ability levels used in factor B were chosen to help determine if the study of the treatment unit might be more profitable for the college capable students than those who are not.

The hypotheses to be tested in Phase I follow.

H_1: Completion of the Sentences of Logic unit has no effect on the ability of junior high school students to verbalize discovered mathematical generalizations.

H_2: The ability level of junior high school students has no effect on their ability to verbalize discovered mathematical generalizations.

H_3: The effect of the completion of the Sentences of Logic unit on verbalization ability is independent of the ability level of junior high school students.

Thirty students were assigned to each cell in the research design so that each of the main effects will be tested by comparing two groups of 60.

The research design used for Phase II is a three-by-two analysis of variance; the first factor has three levels and the second, two. The dependent variable is the ability to use mathematical generalizations which have been discovered. The independent variables are the following factors listed with their respective levels.

Factor A: Verbalization of discovered generalizations or lack thereof.

A_1: Having completed exponent program B which contains no verbalization of discoveries.

A_2: Having completed exponent program C in which the text verbalizes the generalizations after the student discovers them.

A_3: Having completed exponent program D in which verbalization of the discoveries are elicited from the students.

Factor B: Ability to verbalize discovered generalizations with precision.

B_1: Above the median score for precision of verbalization of the generalizations discovered in phase I.

B_2: Below the median score for precision of verbalization of the generalizations discovered in phase I.

The hypotheses tested in Phase II follow.

H_1: Verbalization of discovered mathematical generalizations has no effect on the ability of junior high school students to use the generalizations.

H_2: The ability to state discovered mathematical generalizations with precision has no effect on the ability of junior high school students to use the generalizations.

H_3: The effect of verbalizing discovered mathematical generalizations on ability of junior high school students to use the generalizations is independent of the ability to state the generalizations with precision.

Ten students were assigned to each cell of this research design. Thus, the main effect of factor A was tested with three groups of 20 subjects each and the main effect of factor B was tested with two groups of 30. The original research plans were to use twenty students in each cell to take advantage of the dependability of statistics comcomitant with large research populations, but as the students' subjects with verbalization scores were accumulated from Phase I, there were only slightly over 60 nonzero verbalization scores.

STATISTICAL TOOLS

Most of the tools used in the experiment are in the form of programmed booklets which appear in copyrighted theses materials. They are not part of the final report for this reason and because of their bulk, but a description of them and information on where they may be located seems appropriate.

Sentences of Logic used in Phase I is a linearly programmed unit written by the experimenter for use in a former experiment (9, 45-90). It was written after it was observed that it is possible to analyze universal generalizations into logical components. The possibility exited that if students understood these logical components of generalizations, they might be able to precisely verbalize the mathematical generalizations which they have discovered. Sentences of Logic is intended to contain the knowledge sufficient for a student to be able to write a discovered generalization. For the purpose of these experiments, it was hypothesized that this knowledge consists of the concepts of a variable, an open sentence, a universal quantifier, an instance of a generalization, and knowledge of the conditions under which a universal generalization is true or false and how to convert a true singular statement into a true universal generalization.

The evaluation instrument used in Phase I is another programmed unit entitled A Short, Short Story about Vectors. It, too, was used in the former experiment (9, 91-114) and was written to lead the subject to discover three generalizations about vectors and to write an expression of those discoveries. The three generalizations, in the order in which they appeared in the evaluation unit, may be expressed in the following three sentences. For each vector (a,b), for each vector (c,d), (a,b) + (c,d) = (a+c, b+d). For each vector (a,b), (0,0) + (a,b) = (a,b). For each scalar m, for each vector (a,b), m(a,b) = (ma, mb). These generalizations were chosen because they are not ordinarily taught at the junior high level and, thus, are new to the student, because they are sufficiently simple that the student has the mathematical background needed to be led to discover them, and because they are universal generalizations of sufficient variety sufficiently close together so that the time to study intervening concepts is not prohibitive. Notice that these universal generalizations contain two variables bound over the same universal set, one variable bound over a universal set, and two variables bound over different universal sets, respectively.

In Phase II of the experiment, subjects, whose verbalization ability scores were known from Phase I, were led by branching programmed units to discover the following generalizations:

For each real number x, for each natural number m, for each natural number n:

1. $x^m x^n = x^{m+n}$
2. if $m < n$ and $x \neq 0$, then $\frac{x^n}{x^m} = x^{n-m}$

3. if $n < m$ and $x \neq 0$, then $\frac{x^n}{x^m} = \frac{1}{x^{m-n}}$

4. $(x^m)^n = x^{mn}$.

Exponent programs A, B, and C represent three teaching methods in Neuhouser's thesis experiment (8, 109-329). Program A was written in a didactic style and was not used in Phase II. Programs B, C, and D represent different types of discovery methods. Program B uses a <u>nonverbal awareness method</u> since after sufficient correct responses indicate the student has discovered one generalization, he is led to discover the next one. Program C is identical to program B except for some pages which have been inserted after each discovery in which the textbook verbalizes for the student the discoveries he has made; thus, it uses a <u>verbalized discovery method</u> in which the textbook does the verbalizing. Program D is also identical to program B except for insertion of pages after each discovery. With these insertions the student is led to write the generalizations he has just discovered, so that Program D also uses a <u>verbalized discovery method</u>. In this case the student does the verbalizing. Since the insertions needed to transform Program B into Program D are not included in Neuhouser's thesis, they are included as Appendix I of this report.

A copy of the pretest used to make sure subjects in Phase II had no prior knowledge of the laws of exponents is appended to this report as Appendix II, and a copy of the transfer test used to determine if the students would use their new discoveries in a computation situation is appended as Appendix III. It is worth noting that the transfer test measured the extent to which a student <u>would</u> transfer his discoveries rather than the extent he <u>could</u> use them. The only instructions given the students appear in the test itself, and there was no hint that the discoveries they had recently made might help their computation.

METHODS

A total of 202 eighth grade students of Chiddix Junior High School, Normal, Illinois, worked with the research related materials of this experiment. They were enrolled in seven mathematics classes taught by three teachers. In the statistics for Phase I, 120 of them comprised the research population and 60 of the 120 were the population of Phase II. Teachers whose classes represented a cross section of ability level were approached and cooperated without exception. Randomness existed in assignment of students to classes to the extent it does in a school which groups mathematics students according to general ability levels without having a specific track system. All students were assigned a coded I.D. Using the I.D. and a table of random numbers, the research population was narrowed to 120 and each subject was assigned to a cell in the research design for Phase I after first eliminating those students whose I.Q. scores were not available or who were absent for part of the experiment. In a similar fashion, the number of students with nonzero verbalization scores was narrowed to 60 and assigned to cells of the design for Phase II.

The inserts for exponent program D were written and field tested with small groups of 5th and 7th grade pupils from Metcalf Elementary School on the ISU campus in Normal. At the same time other pre-existing experimental materials were being duplicated and assembled.

The treatment group for Phase I completed the programmed unit Sentence of Logic. After 3 week's interim the entire population completed the programmed unit entitled A Short, Short Story about Vectors in which they were led to write their discoveries about vector operations. The experimenter, Professor Wilson P. Banks and Professor Hal M. Gilmore served as independent judges of the precision of the verbalization attempts of each subject using what was termed the strict scoring guidelines developed for Retzer's original experiment. These guidelines called for explicit expression of logical components of generalizations rather than interpretable incorrect expressions. More details on scoring verbalization and a copy of the guidelines appear as Appendix IV. A 2 x 2 analysis of variance was used to test the hypotheses. These findings are discussed in a later section of this report.

Each subject in Phase II completed exponent program B, program C, or program D which are discovery programs using nonverbal awareness, verbalization on the part of the text, and verbalization on the part of the student, respectively. Over a month intervened between the end of Phase I and this part of Phase II. Immediately after completion of the programmed units the students were given two short posttests, one for manipulative ability and the other for understanding. After a week the test for transfer ability, used in this experiment, was administered. After a month a posttest was given to test retention. Only the third posttest bears relevance to this experiment and, for the purpose of this experiment, the other three tests served as placebos to put the transfer test in the context of a complete testing followup of the experimental work. All of these tests appear in Neuhouser's thesis; a copy of the transfer test appears in Appendix III.

A 3 x 2 analysis of variance was performed with the scores and the outcome appears in a later section of this report.

RESULTS OF THE EXPERIMENT

Records were kept on the time each subject spent during consecutive class periods to complete work on experimental materials. For the Sentence of Logic unit the time ranged from 57 to 153 minutes with a median time of 117 minutes. Time ranged from 47 to 81 minutes for the Short, Short Story about Vectors with a median of 67 minutes. Vector programs B, C, & D ranged from 68 to 167 minutes, 64 to 133 minutes, and 71 to 134 minutes respectively with respective medians of 97, 101, and 100 minutes.

The research population for Phase I had a median I.Q. of 116 with a range from 49 to 143. The cells A_1B_1, A_1B_2, A_2B_1, and A_2B_2 had median I.Q.'s of 131, 94, 122, and 103 respectively with respective ranges of 116 to 143, 49 to 113, 116 to 139, and 70 to 113.

For Phase II the median I.Q. was 121 with a range of 70 to 143. The cells A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, and A_3B_2 had median I.Q.'s of 128, 105, 125, 117, 134, and 112 respectively with respective ranges of 116 to 143, 78 to 139, 112 to 140, 79 to 122, 111 to 139, and 70 to 127.

An examination of the vector response books in Phase I indicated that no student in the research population failed to discover the three generalizations about operations with vectors which the vector unit was designed to lead him to discover. This was determined by the fact that each subject was getting correct answers as he did the vector operations he had been led to discover.

Each attempt to verbalize each discovery was assigned precision points as outlined in the Precision Point Scoring Key which is attached to this report as explanation 3 of appendix IV. These precision points were assigned by three judges working independently. They were three member of the Department of Mathematics at Illinois State University. A copy of the instructions given to each judge is attached as explanation 4 of appendix IV. The total precision points assigned to each attempt were multiplied by a predetermined weight designed to give the greater total score points to those who could verbalize with a given degree of precision with a fewer number of hints.

The form of the sentence or the words and symbolism used by the student did not matter. Precision was based on the appearance of complete and correct information--not on symbolism or form.

It may be well to mention, however, that the word "generalization" was interpreted in this experiment in such a way that words such as "each", "every", or "all" would need be used within a declarative sentence to provide evidence that the student intended to speak generally. Implications or imperative sentences were not interpreted as generalizations.

After each judge assigned precision points independently, a comparison of the precision points was made to locate crude scoring errors, and each judge corrected these scoring errors. Of course, no score was changed which represented a difference of opinion among the judges.

A test using analysis of variance to estimate reliability of measurements was used to estimate the reliability of scores assigned by the three judges. This test is described in Winer's <u>Statistical</u>

Principles in Experimental Design (10, 124). This statistical test indicated that if the experiment were to be repeated with another random sample of three judges, but with the same subjects, the correlation between the mean ratings obtained from the two sets of data on the same subjects would be .99.

A reliability coeffecient (derived from comparison of two 95 point subtests, use of the Spearman-Brown formula, and use of the constant 195/95) was computed to predict the reliability of the entire 195 point evaluation unit. This coefficient was .96.

An examination of the assumptions underlying use of the analysis of variance was made. The assumption that the population for each factor was randomly selected from a normal population was met to the extent that students are randomly assigned to mathematics classes. The range of abilities varied from I.Q. of 49 to 143 with a median of 116. Analysis of variance assumes homogeneity of variance of the samples. Using Hay's (4, 318 - 379) authority that this can be violated without serious risk if the number of cases in each sample is the same, no attempt was made to use a test for homogeneity of variance. There were exactly 30 in each cell for Phase I. A third assumption, that of independent observations, was met; each student's score was determined independently of the others.

The mean scores for each of the four cells are listed in the following table.

TABLE 1

MEAN SCORES IN THE EXPERIMENT

PHASE I

Factor	B_1	B_2	Row
A_1	29.8	3.6	16.7
A_2	13.3	1.2	7.3
Column	21.6	2.4	

FINDINGS FOR PHASE I

A two way analysis of variance was run on the scores of the experimental population. The hypotheses of Phase I were formulated as relating to the main effects of each of the factors and to the interaction between factors; and the result of the statistical analysis is discussed in terms of these hypotheses.

Hypothesis H_1 was completion of the Sentences of Logic unit as no effect on the ability of junior high school students to verbalize discovered mathematical generalizations. The F test for this factor was significant at the .05 level and pemmits us to reject this hypothesis.

Hypothesis H_2 was the ability level of junior high school students has no effect on their ability to verbalize discovered mathematical generalizations. The F test for this factor was significant at the .05 level and permits us to reject this hypothesis. It is interesting to note that the F value of both sets of factors was sufficiently large that the probability of chance occurrences under hypotheses H_1 and H_2 were less than .005; thus, H_1 and H_2 could have been rejected at the .005 level.

Hypothesis H_3 was the effect of completion of the Sentences of Logic unit on verbalization ability is independent of the ability level of junior high school students. This hypothesis may be rejected at the .05 level.

The scores assigned to each subject are listed in table 1 of appendix V. Tables summarizing the statistical analysis of these findings follow.

TABLE 2

ANALYSIS OF VARIANCE FOR TESTING THE HYPOTHESES (PHASE I)

Source	SS	df	MS	F	
Between					
Factor A'	2,688.53	1	2,688.53	8.923	$P < .005$
Factor B	10,975.87	1	10,975.87	36.426	$P < .005$
Interaction	1,490.70	1	1,490.70	4.947	$P < .05$
Within	34,952.87	116	301.317		
Total	50,107.97	119			

Each person in the population of 60 for Phase II successfully completed one of the exponent programs. In the context of two posttests given immediately after the program completion and one a month later, the transfer test used in this experiment was adminstered a week after the completion of the exponent units.

The same considerations were made concerning the extent to which the assumptions underlying the use of analysis of variance on the scores of the transfer test as were made in its use in Phase I, the same conclusions were reached. A major reservation about the statistics is that one of the main effects were tested among three groups of 20 each since there were only 10 available for each cell.

The mean scores for each of the six cells is listed in the following table.

TABLE 3

MEAN SCORES IN THE EXPERIMENT
PHASE II

Factor	B_1	B_2	Row
A_1	11.2	5.3	8.3
A_2	8.8	4.8	6.8
A_3	9.3	4.0	6.7
Column	9.8	4.7	

FINDINGS FOR PHASE II

A 3 X 2 analysis of variance was run on the scores of the transfer test. The hypothese of Phase II were formulated as relating to the main effects of each of the factors and to the interaction between factors; the result of the statistical analysis is discussed in terms of these hypotheses.

For Phase II, Hypothesis H_1 stated that the verbalization of discovered mathematical generalizations has no effect on the ability of junior high school students to use the generalizations. It was impossible to reject this null hypothesis at the .05 level chosen for use in this experiment.

Hypothesis H_2 stated that the ability to state discovered mathematical generalizations with precision has no effect on the ability of junior high school students to use the generalizations. This null hypothesis may be rejected at the .05 level and could have been rejected at the .005 level.

Hypothesis H_3 stated that the effect of verbalizing discovered mathematical generalizations on ability of junior high school students to use the generalizations is independent of the ability to state the generalizations with precision. It was not possible to reject this null hypothesis at the .05 level.

The transfer scores of each subject are listed in table I of Appendix VI. A table summarizing the statistical analysis of Phase II follows.

TABLE 4

ANALYSIS OF VARIANCE FOR TESTING THE HYPOTHESES
PHASE II

Source	SS	df	MS	F	
Rows	31.24	2	15.62	.744	$P \nless .05$
Columns	385.07	1	385.07	18.353	$P < .005$
Interaction	9.43	2	4.715	.224	$P \nless .05$
Error (within cells)	1,133.00	54	20.981		
Totals	1,558.74	59			

CONCLUSIONS AND IMPLICATIONS FOR PHASE I

It should be rememberd that the experiment which comprised Phase I was an essential replication of Retzer's earlier experiment with a research population of a different ability range. The original experiment involved only college capable students and compared those with I.Q.'s of 116-128 with those of I.Q. of 135 and above. The I.Q. range for Phase I was 49-113 compared to 116-143. The findings, however, were quite similar and increases confidence that one might generalize the conclusions reached in Phase I.

Those students who completed the programed Sentences of Logic unit did significantly better in verbalizing newly discovered universal generalizations than those who did not. This conclusion is supported by the facts that the mean verbalization score for the treatment group was higher than that of the control group and that it was possible to reject the first hypothesis that completion of the Sentences of Logic unit has no effect on the ability of junior high school students to verbalize discovered mathematical generalizations.

If the standards of measuring verbalization ability developed for this experiment are acceptable to the mathematics education community, one can now say that there is additional experimental evidence of a relationship between studying logical components of universal generalizations and the ability to express a discovered universal generalizaation with precision.

This result further weakens a major reason used to support the case for leaving a discovered generalization on a non-verbal awareness level--that of avoiding the undesirable consequences of imprecise verbalization caused by an undeveloped linguistic facility.

Those who support the idea of delaying the verbalization of a generalization will, undoubtedly, want to do further research to compare the effects of leaving a generalization on the nonverbal level with the effects of asking for an immediate verbalization and receiving a precise sentence. This experiment helps make such a research problem plausible.

For those who believe that discovery of a generalization without verbalizing it is an abortion of the thinking process, this experiment provides further evidence that under certain conditions it is possible to ask for immediate verbalization for a newly discovered generalization and get a precise one.

The rejection of the hypothesis that the ability level of junior high school students has no effect on their ability to verbalize discovered mathematical generalizations will come as a surprise to very few people. A comparison of the means within the two levels of the ability factor of the experiment lends the additional evidence necessary to conclude that college capable students (I.Q. 116-143) did precisely verbalize newly discovered mathematical generalizations significantly better than non-college capable students (I.Q. 49-113). The ability to verbalize is one of the standards by which educators classify students as college capable. I.Q. tests which are used to classify students as college capable contain a substantial amount of material which tests verbalization ability. Thus, when one supplies evidence that college capable students verbalize better than other students, he is nearly in the position of supplying empirical evidence to support a tautological truth.

The interaction between the two factors in this experiment provides interesting and challenging results. The question has been raised as to whether there are mathematically related materials which might be more profitably used with college capable students than with other students. The third null hypothesis concerning the interaction between the two main effects was designed to help answer this question. This hypothesis stated that the effect of completion of the Sentences of Logic unit on verbalization ability is independent of the ability level of a junior high school student.

This third null hypothesis was rejected. This indicates that the difference in verbalization ability between the students who used the

the Sentence of Logic unit and those who did not was independent of the ability level of the students. An examination of the mean scores for each cell indicates that the difference between means of the treatment group as compared with the control group was considerably greater than the same comparison for the noncollege capable group. This gives evidence that the verbalization of college capable students was aided more by the logic unit than that of the other students.

Teachers may hesitate to delay verbalization of a generalization because they believe that verbalization is an integral part of the thinking process or because they want to use verbalization as a means of determining whether a student has discovered the generalization to which he was led. Yet some theoreticians suggest that they should delay verbalization because an inadequate linguistic facility would lead to an imprecise statement which, in turn, has undesirable consequences for the learner; therefore a teaching strategy would be to leave the generalizarion on a non-verbal awareness level until maturation or educational experiences of the student equip him with the linguistic ability to express generalizations precisely. The results of this research suggest an alternative. Students who studied logical components of generalization such as variables, quantifiers, and open sentences were ab le to verbalize three different kinds of universal generalizations more precisely than those who did not. Thus, teachers could teach logical components of generalizations as an explicit part of the curriculum; then they could choose between delaying verbalization of a discovered universal generalization and asking for immediate verbalization with the increased expectancy that the students will be able to respond precisely. The positive results of this experiment reduce the cogency of the argument that in order to avoid the difference which may result from imprecise verbalization one should delay verbalization until the linguistic skills are acquired as an implicit part of the "usual" curriculum.

Characteristics and outcomes of this experiment have implications for further research also. This experiment assentially tested the efficacy of the programed Sentences of Logic unit. The generalizations discussed in this unit were universal ones with one quantified (bound) variable, two variables quantified over a single doman, and two variables quantified over separate domains. These are kinds of generalizations which appear quite frequently in mathematics. But experience with still other kinds may help equip a student of mathematics to better express the entire range of mathematical generalizations he encounters.

Mathematics contains several other types of generalizations, for example, existential generalizations and ones with a varied combination of universal and existential quantifiers. Positiive results with the kinds of generalizations used in this experiment offer encouragement for additional research in which students may be provided with experience with other kinds of generalizations and in which its effect on verbalization is tested.

A major outcome of this experiment, in the view of the experimenter, is that evidence continues to mount that, regardless of ability level, the ability to precisely state discovered mathematical generalizations is a factor that can be manipulated for educational purposes. It suggests that a teacher does not need to wait until linguistic ability is "picked up" implicitly within the mathematical and other aspects of the curriculum; a teacher may make formation of linguistic ability an explicit part of the curriculum. Some concepts of logic may be explicitly taught and then the teacher may choose the teaching strategy of asking for immediate verbalization of discovered mathematical generalizations and reasonably expect to get a precise one.

CONCLUSIONS AND IMPLICAITIONS FOR PHASE II

In Phase II the only null hypothesis which could be rejected at the .05 level was H_2 which stated that the ability to state discovered mathematical generalizations with precision has no effect on the ability of junior high school students to use the generalizations. This result together with an examination of the means enables one to conclude that one who has the ability to state generalizations precisely is better able to transfer what he has discovered.

Some may react that since this type of linguistic ability is somewhat commensurate with I.Q., this result simply brings out that a person who is talented in one direction has talent in another. That, in itself, adds to what is already known about transfer. However, in view of the evidence of Phase I that teaching some concepts of logic enhances this linguistic ability, a teacher may have even stronger reason for making logic a part of his curriculum. Not only may a student increase his ability to state his discoveries but he may increase his ability to use his discovery as well. This is the direction this evidence points and further research along this line may help indicate the extent to which this evidence may be generalized.

Inability to reject hypothesis H_1 of Phase II tends to make one cautious about supporting one discovery strategy over another in an attempt to teach for transfer. H_1 stated that verbalization of discovered mathematical generalizations has no effect on the ability of junior high school students to use the generalizations. The difference in the means of the transfer scores for students who left their discoveries on a nonverbal awareness level, for students who were asked to verbalize their discoveries, and for students for whom the textbook verbalized were so slight that superiority is not indicated for any one of the strategies. This evidence tends to weaken the argument that calling for immediate verbalization may lessen transfer power.

Finally, the lack of interaction between major factors of Phase II lends no evidence to dispute the hypothesis that as far as transfer power is concerned ability to state a newly discovered mathematical generalization with precision is independent of the three discovery strategies used in Phase II.

APPENDIX I

EXPONENT PROGRAM D

8^{45}. You have discovered a rule for simplifying expressions like $8^{20} \cdot 8^{45}$. We want to lead you to state this rule using correct language. In the blank at the bottom of this page, write a sentence which expresses this rule you have discovered. Write the most precise sentence you can. Leave the blank empty only if you have no idea of what to write and cannot even make a guess.

GUESS 1.____

80A

No answer is listed here because we want to try a kind of guessing game. Unless you feel your first guess was a perfect statement of the rule you have discovered, we will try to lead you, step by step, to what you may need to consider in writing a precise sentence expressing what you have discovered.

HINT 1. In simplifying expressions like $8^{20} \cdot 8^{45}$, which one of the following operations did you use on the exponents? Put the letter corresponding to the correct answer in the appropriate blank.

________ a. addition
b. subtraction
c. multiplication
d. division
e. tracheotomy

80B

ans.____a____

Now if the fact that you use addition on the exponents helps you write a better guess than GUESS 1, then write the better sentence in the space below. Leave the space blank only if you have no idea what to write and cannot even make a guess. Write the rule for simplifying expressions like $8^{20} \cdot 8^{45}$ now.

GUESS 2.____

80C

Again, no answer is listed because we want you to improve your answer with each guess until you have a statement that mathematicians would say is complete and correct. Of course, once you get a sentence that you feel cannot be improved upon, you don't need to change it following any specific hints. Change the sentence you write in any GUESS blank only if the hint has helped you write a better sentence.

HINT 2. This hint will cover three pages and will try to show you how mathematicians use patterns to express sentences. For example:

If Sheryl has $5 and earns $3 more, she then has $8 (or 5+3).
If Kent has $10 and earns $2 more, he then has $12 (or 10+2).
If Mr. Martin has $25 and earns $10 more, he has $35 (or 25+10).
Thus,

If a person has m dollars and earns n dollars more, this person has m+n dollars.

This last statement gives us a formula or pattern for all situations of this type. It expresses that no matter what numbers m and n, are considered, we would add these numbers. The particular letters used are not important. For example, if the letters x and y had been used, the statement would have been the following:

If a person has x dollars and earns y more dollars, he has ______ dollars.

80D

ans. x+y

HINT 2. (Continued)

Now let us look at another type of situation and its pattern. If Martin has 5 apples and gives 2 apples away, he then has 3 apples (or 5-2).
If Vernee has 10 candy bars and gives 3 away, she then has 7 candy bars or (10-3).
If Mrs. Brown has 25 cookies and gives 10 cookies away, she then has 15 cookies or (25-10).
This type of situation might be symbolized as follows:

If a person has m objects and gives n of them away, this person has ________ objects.

80E

ans. m - n

HINT 2 (Continued)

Let us look at one more situation and a pattern used to symbolize it.
If there are 5 pills in each of 4 bottles, there are 20 pills in all. (20 or 5 · 4)
If there are 10 marbles in each of 10 bags, there are 80 marbles in all. (80 or 10 · 8)
If there are 6 toys in each of 3 boxes, there are 18 toys in all. (18 or 6 · 3)

This situation may be symbolized as follows:

If there are m objects in each of n containers, there are ______ objects in all.

80F

ans. $m \cdot n$

HINT 2 (Continued - for the last time!)

Up to now we have seen how several mathematical situations can be symbolized with a pattern of formula. Now see if you can use this information to help write a pattern for symbolizing the discovery you have made about exponents. For example, you know that

$$8^2 \cdot 8^3 = 8^5$$

and

$$10^4 \cdot 10^9 = 10^{13}$$

then

$$5^m \cdot 5^n = 5^{\_\_\_\_}$$

80G

ans. $5^m \cdot 5^n = 5^{m+n}$

Now if HINT 2 helped you write a better statement of your discovery, write it in the blank below. See if you can make a correct statement of your discovery.

GUESS 3. ____

80H

HINT 3.

You may have used a pattern something like

$$5^m \cdot 5^n = 5^{m+n}$$

in your statement in GUESS 3. While that is close to a complete pattern for your discovery, the pattern needs to be symbolized further. For example, you know that

$$5^x \cdot 5^y = 5^{x+y} \text{ (The letters we use don't matter.)}$$

$$7^x \cdot 7^y = 7^{x+y}$$

$$19^x \cdot 19^y = 19^{x+y}$$

We know this will work for any base so we could make a complete pattern as follows:

$$b^x \cdot b^y = \_\_\_\_\_\_\_\_$$

80I

ans. $b^x \cdot b^y = b^{x+y}$

Now see if this hint helps any by putting the best statement of your discovery that you can write in the blank below.

GUESS 4. ____

80J

Suppose you did use a pattern like $b^x \cdot b^y = b^{x+y}$ in the statement of your discovery. This would give you a part of what you need for a complete correct sentence which expresses your discovery, but it still leaves something to be desired. By itself this pattern does not tell you what would be proper replacements for the letters in the pattern. The names of what kind of numbers should be used to replace the letters b, x, and y in the pattern if you want to make a true sentence from the pattern? HINT 4 will help you determine the answer to this question.

HINT 4. There are several different sets of numbers, and you will study these different sets as you continue to study mathematics. Rather than try to tell you what these different sets of numbers are, we will list the sets with some sample numbers in each set. Three dots will indicate numbers not listed.

80K

Natural numbers	$N = \{0, 1, 2, 3, 4, \ldots\}$
Counting numbers	$C = \{1, 2, 3, 4, 5, \ldots\}$
Integers	$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$R = \{\frac{-5}{2}, \ldots, -2, \ldots, \frac{-4}{3}, \ldots, -1, \ldots, 0, \ldots, \frac{3}{4}, \ldots, 1, \ldots, \frac{16}{15}, \ldots, 2, \ldots\}$
Real numbers	$R\# = \{-\pi, \ldots, -3, \ldots, \frac{-1}{2}, \ldots, 0, \ldots, 1, \ldots, \sqrt{2}, \ldots, \frac{17}{4}, \ldots\}$

Perhaps it is a little difficult for you to understand what the set of rational numbers and real numbers are just from the above listing of sample elements of the sets. Some people would loosely say that the set of rational numbers contains all whole numbers together with all the fractions between the whole numbers---both positive and negative. The real numbers contain all of these rational numbers plus all irrational numbers like π, $\sqrt{2}$, etc. The real

80L

numbers are all the numbers you know of that exist--at least until you study complex numbers in about your junior year in high school.

Now we return to the question which is the most important question in HINT 4. If you use the pattern $b^x \cdot b^y = b^{x+y}$ to express your discovery, what kind of numbers do b, x, and y refer to? You may want to use your answer to this question as you make another guess at a correct verbalization of your discovery. Make that attempt now.

GUESS 5. ____________________

80M

HINT 5. If you made a correct guess to the answer of the question about correct replacements for b, x, and y in the pattern $b^x \cdot b^y = b^{x+y}$, you said that b could be replaced by a name for a real number and x and y should be replaced by names for counting numbers. So far we have felt free to use any kind of number for bases but have never used zero, negative numbers, or fractions for exponents.

Now we turn to the question of which real numbers can be used as bases and which natural numbers can be used as exponents. See if you can include your answer to this point in another guess at a correct sentence. Write it now.

GUESS 6. ____________________

80N

HINT 6. You could have used several ways of expressing which real numbers could be used as bases and which counting numbers could be used as exponents because there are no limitations on which ones can be used. For bases, you could say, "For each real number," "For all real numbers," or any other words that mean the same thing; for exponents you could say, "For each counting number," or any other phrase that means the same thing. Use this hint to make one final attempt at stating your discovery precisely. Remember to include the pattern, an indication of what sets of numbers the letters in the pattern refer to, and an indication of which real and counting numbers they refer to.

GUESS 7. ____________________

ans. (finally!) For each real number b, for each counting number x, y, $b^x \cdot b^y = b^{x+y}$.

Any statement that you made that conveys exactly the same information as the sentence above would also be correct. We are going to call this statement RULE I. Here are some instances of RULE I for you to complete.

1. $5^2 \cdot 5^6 = 5^8$
2. $7^5 \cdot 7^4 =$ ______
3. $2^{10} \cdot 2^{15} =$ ______
4. $9^{100} \cdot 9^{52} =$ ______

80P

ans. 2. 7^9 3. 2^{25} 4. 9^{152}

Before looking at another kind of problem involving exponents, let us review the meaning of exponents.

10^2 is an abbreviation for 10·10

10^3 is an abbreviation for 10·10·10

10^4 is an abbreviation for ______________

80Q

ans. 10·10·10·10

2^3 is equal to which of the following? ________________

(a) 5
(b) 6
(c) 8
(d) 9
(e) none of these

143A

You have now discovered a second rule for exponents because you know how to simplify expressions like $8^{45}/8^{20}$. We want to lead you, like we did before, to write a sentence which expresses this discovery precisely. Write the best sentence you can which expresses the discovery in the blank below.

GUESS 1. ____

144A

No answer is given here because we want to go through another series of hints and guesses to help you write the best sentence you can for the discovery. On any guess, change your previous guess only if the hint helped you write a better sentence. No matter what the hint is, always write the best sentence you can on each guess.

HINT 1. In simplifying expressions like $8^{45}/8^{20}$ which one of the following operations is used on the exponents? ________

a. addition
b. subtraction
c. multiplication
d. division
e. appendectomy

144B

ans. b.

HINT 1. (Continued) Now that you know which operation is used on the exponents, see if you can use this information within a pattern or formula to express your discovery about simplifying expressions like $\frac{9^{17}}{9^{10}}$.

GUESS 2. ____

144C

HINT 2. Consider the following examples and see if you can write a pattern for simplifying these expressions.

$$\frac{9^{14}}{9^{3}} = 9^{14-3} = 9^{11}$$

$$\frac{10^{8}}{10^{3}} = 10^{5}$$

Now see if you can write the pattern for problems of this type.
Use x for the base and m and n for the exponents.
Now answer the following questions.

1. Which numbers are used as bases (replacements for x)?
2. Which numbers are used as exponents (replacements for m and n)?

144D

ans. $\frac{x^m}{x^n} = x^{m-n}$

If knowing what the pattern is helps you write a better statement of your discovery, use it to write your discovery in the blank below. When you write your statement, you may also want to include information about what kinds of numbers x, m, and n represent.

GUESS 3. ____

144E

<u>HINT 3.</u> Now you know a pattern you can use in writing your sentence, but you may not be sure what numbers x, m, and n represent. To help you with this, examine the following list of kinds of numbers.

Natural numbers $N = \{0, 1, 2, 3, 4, \ldots\}$

Counting numbers $C = \{1, 2, 3, 4, 5, \ldots\}$

Integers $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers $R = \{\frac{-5}{2}, \ldots, -2, \ldots, \frac{-4}{3}, \ldots, -1, \ldots, 0, \ldots, \frac{3}{4}, \ldots, 1, \ldots, \frac{16}{15}, \ldots, 2, \ldots\}$

Real numbers $R\# = \{-\pi, \ldots, -3, \ldots, \frac{-1}{2}, \ldots, 0, \ldots, 1, \ldots, \sqrt{2}, \ldots, \frac{17}{4}, \ldots\}$

Now answer the following questions.

1. Which numbers are used as bases (replacements for x)?
2. Which numbers are used as exponents (replacements for m and n)?

144F

ans. 1. <u>real numbers</u> 2. <u>counting numbers</u>

<u>HINT 3. (Continued)</u> This information may help you write a better statement of your discovery of a rule for simplifying expressions like $\frac{9^{16}}{9^4}$. You may want to include in your statement an indication of <u>which</u> of these numbers may be used making replacements in the pattern.

GUESS 4. ____

144G

<u>HINT 4.</u> You may not be sure <u>which</u> real numbers may be used as a base and <u>which</u> counting numbers may be used as exponents in a pattern like

$$\frac{x^m}{x^n} = x^{m-n}$$

There are no restrictions on which real numbers and on which counting numbers may be used. Each real number could be a base. And each counting number could be an exponent--provided we consider a restriction about the comparative size of the two counting numbers. If we tried to use the pattern listed above on $\frac{8^3}{8^5}$, we would get 8^{3-5} which doesn't make sense at the present time. So, we may use this pattern if one of the following is true.

Which one? ____________________

a. m is larger than n
b. n is larger than m
c. m = n

144H

ans. ___a.___

Now make a final try to state your discovery about exponents. Write it in the blank.

GUESS 5. ____

144I

ans. For each real number x, for each counting number m and n such that m is greater than n, $\frac{x^m}{x^n} = x^{m-n}$. We will call this statement RULE II.

If your guess contained exactly the same information as the sentence above, your guess was a correct statement of the generalization. Some examples of the use of this rule follow. Complete the unfinished ones.

1. $\frac{6^8}{6^2} = 6^6$

2. $\frac{3^{12}}{3^{10}} =$ ____________

3. $\frac{2^{28}}{2^{12}} =$ ____________

4. $\frac{10^{100}}{10^{80}} =$ ____________

144J

ans. 2. 3^2 3. 2^{16} 4. 10^{20}

144K

We have been working problems like $\frac{2^5}{2^2} = 2^3$ where the exponent in the numerator is larger than the exponent in the denominator. Now we are going to work problems which look very much like these but differ in a small but important way. In these, the exponent in the denominator is larger than the exponent in the numerator. Complete this chain of reasoning.

$\frac{2^5}{2^2} = 2^3$ so we know that $\frac{2^3}{2^5}$ does not equal 2^3.

But $\frac{2^2}{2^5} = \frac{2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

so $\frac{2^2}{2^5} = \frac{1}{2 \cdot 2 \cdot 2}$

so $\frac{2^2}{2^5} =$ ______

144L

ans. $\frac{1}{2^3}$ (since $\frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{2^3}$)

$$\frac{2^3}{2^5} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

$$\frac{2^3}{2^5} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \cdot \frac{1}{2 \cdot 2}$$

$$\frac{2^3}{2^5} = \frac{1}{2 \cdot 2}$$

$$\frac{2^3}{2^5} = \frac{1}{2\text{---}}$$

205A

You have now discovered a third rule for exponents. Some examples of it are:

$$\frac{9^3}{9^{14}} = \frac{1}{9^{11}} \qquad \frac{10^3}{10^8} = \frac{1}{10^5}$$

Write the best statement of this rule that you can in the blank below.

GUESS 1.____

206A

Again, no answer is given because we want to give you a series of hints to help you write the best sentence you can which expresses your discovery.

HINT 1. As in previous sentences you may want to use a pattern or formula that expresses what you do with the exponents in order to simplify the expressions. Your pattern may look something like the following. What would you fill in the blank to make the pattern complete?

$$\frac{x^m}{x^n} = \frac{1}{x^{\_\_\_\_}}$$

206B

ans. n-m Check this answer carefully because m-n would not be correct.

If this hint helps you write a better statement of your discovery, write it in the blank below. You may also want to include information about what numbers x, m, and n represent and which numbers in these sets could be used to make true sentences from the pattern.

GUESS 2.____

206C

HINT 2. This is the third generalization you have discovered in this programed text so far. It might be helpful if you compared a correct statement of the first two discoveries with the statement you want to write. We will call your third discovery RULE III and let you fill in a blank that would make a statement of it complete.

RULE I. For each real number x, for each counting number m and n, $x^m \cdot x^n = x^{m+n}$.

RULE II. For each real number x, and for each counting number m and n such that m is greater than n, $\frac{x^m}{x^n} = x^{m-n}$.

RULE III. For each real number x and for each counting number m and n such that ____________________, $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$.

206D

ans. n is greater than m (or m is less than n)

If this hint helps you write a good sentence which expresses your discovery, write it in the blank below. It will be the final guess.

GUESS 3. ____

206E

RULE III. For each real number x and for each counting number m and n such that n is greater than m, $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$.

So, for example, 1. $\frac{5^8}{5^{12}} = \frac{1}{5^4}$

2. $\frac{4^{22}}{4^{33}} =$ __________

3. $\frac{9^{15}}{9^{35}} =$ __________

4. $\frac{2^{66}}{2^{89}} =$ __________

206F

ans. 2. $\frac{1}{4^{11}}$ 3. $\frac{1}{9^{20}}$ 4. $\frac{1}{2^{23}}$

Now we want to investigate another kind of problem.

In the expression $(2^2)^3$ the 3 is an exponent with 2^2 as the base. That is $(2^2)^3 = 2^2 \cdot 2^2 \cdot 2^2$ and since $2^2 = 2 \cdot 2$ then

$$(2^2)^3 = (2 \cdot 2) \cdot (2 \cdot 2) \cdot (2 \cdot 2)$$

$$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

$$= 2^{\_\_\_\_}$$

206G

ans. ___6___

$$(10^3)^4 = (10^3)\cdot(10^3)\cdot(10^3)\cdot(10^3)$$

$$= (10\cdot10\cdot10)\cdot(10\cdot10\cdot10)\cdot(10\cdot10\cdot10)\cdot(10\cdot10\cdot10)$$

$$= 10^{\_\_\_\_\_}$$

242A

Now you have discovered the fourth (and final) rule for exponents. For example, you know that $(8^3)^4 = 8^{12}$. Try to state this discovery in the most precise sentence you can write.

GUESS 1. ____

243A

You may have had enough experience in writing these sentences that the sentence you used for GUESS 1 is the best possible sentence. In case it isn't, we will go through one final series of hints. Always write the best sentence you can each time you guess.

HINT 1. In simplifying expressions like $(8^3)^4$ which one of the following operations is used on the exponents?

a. addition

b. subtraction

c. multiplication

d. division

e. tonsillectomy

244A

If this information helps you write a better guess, write it in the blank. You may want to include in your statement a pattern or formula for expressions of this type.

GUESS 2. ____

245A

HINT 2. Examine the following examples and see if they help you write a pattern for simplifying problems of this type.

$$(9^{11})^3 = 9^{11 \cdot 3} = 9^{33}$$

$$(10^4)^{20} = 10^{80}$$

Complete the pattern. $(x^m)^n = x^{\_\_\_\_\_}$

246A

ans. $(x^m)^n = x^{m \cdot n}$

You may want to use this pattern in the sentence which expresses the discovery you made about exponents. You may also want to include information about what numbers may be bases, what numbers may be exponents, and which numbers from these sets of numbers make the pattern into true statements. This is the final guess.

GUESS 3. ____

247A

RULE IV. For each real number x, for each counting number m and n, $(x^m)^n = x^{m \cdot n}$.

Some examples to complete.

1. $(10^3)^8 = 10^{24}$
2. $(7^{10})^5 =$ __________
3. $(2^{11})^8 =$ __________
4. $(5^6)^9 =$ __________

248A

ans. 2. 7^{50} 3. 2^{88} 4. 5^{54}

USE RULES I, II, III, and IV to complete the following examples.
For each real number x and for each counting numbers m and n,

1. $x^m \cdot x^n = x^{m+n}$ (so $2^{10} \cdot 2^3 = 2^{\_\_\_}$)
2. If m is larger than n, then

   $\frac{x^m}{x^n} = x^{m-n}$ (so $\frac{2^{10}}{2^3} = 2^{\_\_\_}$)

3. If n is larger than m, then

$\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$ (so $\frac{2^3}{2^{10}} = \frac{1}{2^{\_\_\_}}$)

4. $(x^m)^n = x^{m \cdot n}$ (so $(2^3)^{10} = 2^{\_\_\_\_}$)

249A

ans. 1. 13 (or 10+3) 2. 7 (or 10-3) 3. 7 (or 10-3) 4. 30 (or 3·10)

Use the following patterns to fill in the blanks below.

1. $x^m . x^n = x^{m+n}$ 2. If m is larger than n, then $\frac{x^m}{x^n} = x^{m-n}$

3. If n is larger than m, then $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$ 4. $(x^m)^n = x^{m \cdot n}$

1. $\frac{8^{13}}{8^{60}} =$ ______ 2. $5^9 \cdot 5^{10} =$ ______ 3. $(27)^4 \cdot (27)^{10} =$ ______

2. $(48^6)^3 =$ ______ 5. $\frac{9^{32}}{9^{16}} =$ ______ 6. $(4^{10})^5 =$ ______

7. $\frac{10^{60}}{10^{15}} =$ ______ 8. $10^6 . 10^8 =$ ______ 9. $\frac{3^4}{3^{20}} =$ ______

250A

ans. 1. $\frac{1}{8^{47}}$ 2. 5^{19} 3. 27^{14}

4. 48^{18} 5. 9^{16} 6. 4^{50}

7. 10^{45} 8. 10^{14} 9. $\frac{1}{3^{16}}$

Here is one last question before you hand this booklet in to the teacher.

Which one of the following best expresses your answer to the following question: Do you enjoy studying mathematics this way better than the usual classroom procedures? ________

a. yes, very much better
b. yes, a little better
c. about the same
d. no, not quite as well
e, no, not nearly as well

APPENDIX II

TRANSFER TEST FOR PHASE II

Pretest

Name______________________________

I.D. Number________________________

Date______________________________

This is a test to see how fast you can compute. There will probably be some questions with which you are not familiar. If there are, you should just omit them and go on to the next questions. Also, if you think it would take you very long to obtain the answer to any particular question, then it would probably be better for you to omit it and go on to the next question. You may do any scratch work right on the test paper. You will have four minutes to work on this test.

Pretest I.D._________ Name____________________

1. What is the sum of 13 and 15? ________
2. 16 x 20 = ________
3. 38 x 20 = ________
4. 4 x 5 x 2 = ________
5. What is the product of 49 and 100? ________
6. What is the sum of 8, 16, and 12? ________
7. What is the product of 8^{15} and 8^{10}? ________
8. 45 divided by 15 is ________
9. 5 x 13 = ________
10. 10^3 = ________
11. 13 x 8 = ________
12. 13 + 8 = ________
13. 64 - 16 = ________
14. 64 ÷ 16 = ________
15. $\frac{7^{90}}{7^{30}}$ = ________
16. $\frac{7^{30}}{7^{90}}$ = ________
17. (6 + 9) ÷ 3 = ________
18. 24 ÷ (4 + 2) = ________
19. 43 + 57 = ________
20. 43 + 99 + 57 = ________
21. $(6^8)^{11}$ = ________

(over)

22. $\frac{330}{30} =$ ____________

23. $\frac{3}{5} \times \frac{10}{7} =$ __________

24. $\frac{3}{5} + \frac{10}{7} =$ __________

25. 5 x 93 x 2 = __________

26. 64 x 16 = ___________

27.
```
  452.5
  167.4
   32.0
  105.9
    8.0
1,432.3
```

28. Subtract:
```
52.431
49.754
```

29. Divide: 492 $\overline{)125460}$

30. Multiply:
```
42,918
 6,034
```

APPENDIX III

TRANSFER TEST

FOR

PHASE II

POSTTEST III

Name____________________

I.D.____________________

Date____________________

Sometimes in mathematics it is necessary to be able to look up values in a table. You have a table before you which gives the values of 2^1, 2^2, and so on, up to 2^{15} (e.g. 2^{14} = 16,384). In order to answer some of the questions on this test you will need touse this table. You may use the table in any way that will help you to answer the questions. For every question there will be a blank in which you are to write the answer. The final answer which you write in the blank should be a numeral which does not contain an exponent even though there are exponents in the question. This is a timed test, so the object will be for you to answer the questions as quickly as possible. You will have six minutes to work on this test.

You will be given plenty of space to do any computation that may be necessary. Any computation that you need to use pencil and paper for, should be done in the space provided.

1. $\frac{2^{14}}{2^{10}}$ = ________

2. What is the product of 16 and 2^8? ________

3. What is the product of 2^4 and 2^6? ________

4. What is 2^{15} divided by 2^{12}? ________

POSTTEST III I.D.________ Name ____________________ 3.

5. $\frac{2^{15}}{2^{7}} =$ ________

---

6. What is 2^{14} divided by 256? ________

---

7. What is the product of 2^5 and 2^8? ________

---

8. What is the product of 32 and 2^9? ____________________

(over)

POSTTEST III I.D.________ NAME ________________________________ 4.

9. 8^4 = ____________

---

10. What is the product of 2^6 and 2^8? ________

---

11. What is 16,384 divided by 2^{11}? __________

---

12. What is the product of 128 and 2^7? _________

13. What is 4096 divided by 128? __________

14. $(2^7)\cdot 64 =$ __________

15. $4^6 =$ __________

16. What is 2^{10} divided by 8,192? __________

(over)

POSTTEST III I.D. __________ NAME ______________________________ 6.

17. $(2^4)^3$ = __________

---

18. $(32)^3$ = __________

---

19. What is the product of 512 and 64? __________

---

20. What is 2^{14} divided by 1,024? __________

REFERENCE CHART FOR POSTTEST III

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

$2^6 = 64$

$2^7 = 128$

$2^8 = 256$

$2^9 = 512$

$2^{10} = 1024$

$2^{11} = 2048$

$2^{12} = 4096$

$2^{13} = 8192$

$2^{14} = 16,384$

$2^{15} = 32,768$

A P P E N D I X IV

M E A S U R I N G P R E C I S I O N
O F
V E R B A L I Z A T I O N

APPENDIX IV

Explanation 1.

METHOD AND RATIONALE FOR MEASURING PRECISION OF VERBALIZATION

Two questions are crucial in attempting to measure a student's ability to verbalize precisely newly discovered mathematical generalizations. How can we be sure the discovery has taken place so that a possible inability to verbalize will not be caused by an inability to discover? How can we measure the ability of the student to precisely verbalize these generalizations?

The first question was answered in this experiment by providing the cues that lead to a discovery of the generalization within a linear program; since the student worked individually through his booklet, an examination of the number of correct responses in using the generalization determined if he had discovered a generalization and possessed it on, at least, a sub-verbal awareness level.

Secondly, a measure of the ability to verbalize <u>precisely</u> was determined by scoring the evaluation unit according to a scoring key for generalizations developed for use in this experiment which contained the criteria used to measure <u>precision</u>. And the <u>ability to verbalize</u> with precision was determined by asking for a verbalization of the discovery and then asking several more times with a hint designed to aid verbalization placed before each subsequent attempt. Precision scores on each attempt to verbalize were weighted so that the student who verbalized with a certain degree of precision with a fewer number of hints was considered to have the greater verbalization ability.

An example may illustrate how the ability to judge with precision, as outlined in the previous paragraph, was determined. For example, one of the generalizations each student discovered could be stated, "For each scalar m, for each vector (a,b), m(a,b) = (ma, mb)." We know each student had discovered the generalization because he was getting correct computational results when multiplying scalars by vectors. One of the students wrote the following verbalization of his discovery, "The number outside the ordered pair times the number inside gives you the answer." When the student turned to the next frame in the programmed booklet there was nothing to indicate whether his verbalization was correct or not. He was told that he would encounter a series of hints to help him, if possible, write a better expression of his generalization. The first hint outlined several common errors students make in attempting to state a generalization precisely, and he was asked to write a second expression of his generalization which would be different from his first expression only if the hint helped him write a better sentence. A second hint, given after he wrote his second attempt, gave an overview of the information contained in a complete correct statement of a generalization, and he was asked to make a third attempt. If he was unable to profit from this second hint, a third hint attempted to lead him, step by step, to a correct expression of a part of a complete, correct sentence, and subsequent hints attempted to lead him to other parts. In all he was confronted with six hints and made seven attempts to write a correct expression of his generalization.

Each sentence he wrote was assigned a precision score using the standards outlined in a precision point scoring key, and this number served as an indication of how precisely the sentence was worded. Then, in order to discriminate better between students with different levels of ability to verbalize with precision, each precision score was multiplied by a weight which gave the higher score to the sentence which was written with the fewer number of hints.

Going back to the example at hand, if the student stated the information contained in the expression, "For each scalar m, for each vector (a,b), m(a,b) = (ma,mb).", then he would receive the maximum of 5 precision points. If he did this without a hint, this 5 points would have been multiplied by a weight of 5, so his first sentence would contribute 25 points to his total score. Yet, if he got the maximum precision points after two hints had been given, the appropriate weight would have been 3 so that this sentence would contribute 15 points to his total. And the same sentence written after the entire series of hints had been encountered would have been worth 5 points because the appropriate weight used would have been 1.

Each attempt to verbalize each discovery was assigned precision points as outlined in the Precision Point Scoring Key. These precision points were assigned by three judges working independently. They were three members of the Department of Mathematics at Illinois State University. The total precision points assigned to each attempt were multiplied by a predetermined weight designed to give the greater total score points to those who could verbalize with a given degree of precision with a fewer number of hints. For example, one student had discovered the generalization "For each scalar m, for each vector (a,b), m(a,b) = (ma,mb).". He made a total of seven attempts to verbalize this generalization precisely and was given hints between each of these attempts. Examining his first, fourth, and sixth attempts we find the following three expressions: "S(a,b) = (Sa,Sb).", "S = scalars, (a,b) = vectors, S(a,b) = (Sa,Sb)" and S = all scalars, (a,b) = all vectors, S(a,b) = (Sa,Sb)". One judge assigned precision points of 1, 3 and 5 to these respective sentences. The score of 1 was multiplied by 5, 3 by 2, and 5 by 1; the multiplier in each case was the appropriate predetermined weight.

The form of the sentence or the words and symbolism used by the student did not matter. Precision was based on the appearance of complete and correct information--not on symbolism or form.

The whole number nearest the average score assigned by the three judges was used for each subject's score in the statistical analysis of the experiment.

Explanation 2.

PRECISION POINT SCORING KEY

1 (os) point for a correct open sentence of the generalization. Any complete* sentence(s) (even an imperative sentence(s)) which would give sufficient information (or instructions) to enable one to get a correct result of the operation would merit this point. No point is to be given to a specific example in which specific vectors or scalars are named. An example, however, should not lessen a score; it should be ignored and the awarding of the point should be decided on the basis of what is on the paper along with the example. In general, ignore sketches, set notation, symbols, and other extraneous matter.

No point should be given to a sentence which gives correct but incomplete instructions; for example, a sentence purporting to explain how to add vectors may say to add both first components and to add both second components but not explain that these sums are the components of the "answer" vector.

1 (u) point for each indication of a correct universal set.For the first two generalizations this universal set is the set of vectors. For the third generalization there are two universal sets - the set of vectors and the set of scalars (or real numbers); one "u" point is to be given for each universal set mentioned. This information need not be included in a formal quantifier. If an open sentence is used, an indication of what the variables represent is sufficient to merit a "u" point. An os point is a necessary prerequisite for awarding the (u) points.

1 (∀) point for each indication that the generalization is universal. If the expression is somewhat formal, the symbol '∀' or one of the words 'each', 'every', 'all', or 'any' could be used in such a way as to convey this information. However,imperative sentences and sentences which begin with 'If', 'Where', or 'When' are not likely to receive this point. The article 'a' will not be considered a

* A sentence is to be judged complete if it has no more than three symbols which have been omitted or used incorrectly. For example,if 'a,b' is written where '(a,b)' is intended, this counts as two errors. The lack of a period at the end of a sentence would count as one error. An entire word is counted as a single symbol; therefore, an error in spelling is counted as one error. An abbreviation is not counted as an error if the context makes clear what is being abbreviated; e.g. "C" can be used to abbreviate "component".

a universal quantifier.

A point should be granted if there is something to the effect that what they have written "has no exceptions" or "works in all cases". In the case of the third generalization where there are two different universal sets, one point should be given for each universal quantifier. An os point is a necessary prerequisite for awarding the (V) points.

Table 3

WORKSHEET FOR SCORING GENERALIZATIONS

Student's
I.D. Number

JUDGE'S NAME____________________

GENERALIZATIONS

	Pages 10-11i					Pages 16-17h			Pages 24-25f						
	GUESS NO.					GUESS NO.			GUESS NO.						
	1	2	3	4	5	1	2	3	1	2	3	4	5	6	7
os															
u															
$\forall$															
S. Tot															
Wts.	5	4	3	2	1	3	2	1	5	4	3	2	2	1	1
Total															

SCORE____________________

EXPLANATION 4. INSTRUCTIONS TO ~~JUDGES FOR SCORING GENERALIZATIONS~~

The purpose of scoring the universal generalizations in the unit on vectors is to determine the ability of each student to precisely state each generalization he has discovered. These generalizations involve vector operations.

We assume that each student has discovered, in turn, three different universal generalizations. We can verify this assumption by checking to see if he got correct answers when he performed the vector operations.

Each guess a student writes is to be graded for a degree of precision according to standards outlined in the PRECISION POINT SCORING KEY.

In an attempt to help each student state his discoveries more precisely, a series of hints is given after his first guess in each case. It is assumed that a student with a greater amount of ability to state his discoveries precisely will gain more precision points with a fewer number of hints than a student with less ability. The precision points awarded to each guess will be totaled to form a subtotal. Each subtotal will be multiplied by a weight to reflect the number of hints that have been given. And the totals for each guess will be added to determine the total score.

APPENDIX V

SCORES FOR PHASE I

APPENDIX V
TABLE 1
SCORES OF PHASE I

A_1B_1			A_1B_2		
Exp.	I.D.	Verb.	Exp.	I.D.	Verb.
C	1FW3-22	7	D	4FW3-11	40
C	2FW3-22	123	D	1FW7-60	2
B	3FW3-33	18	C	3FW7-20	0
B	6FW3-91	32	B	8FW7-30	0
D	8FW3-82	20	D	10FW7-60	1
C	9FW3-73	16	C	12FW7-21	24
D	10FW3-33	6	B	14FW7-29	6
C	11FW3-71	13	D	3MW7-31	6
C	12FW3-43	24	C	7MW7-97	3
D	13FW3-93	51	D	14MW7-99	18
D	14FW3-53	24	D	2FB7-68	0
D	15FW3-63	38	C	3FB7-78	0
C	1MW3-33	3	B	4FB7-88	0
B	2MW3-03	15	D	5FB7-19	0
C	3MW3-04	22	C	6FB7-70	0
D	4MW3-63	18	B	7FB7-28	0
B	5MW3-92	44	B	8FB7-87	6
D	6MW3-04	15	B	9FB7-50	3
C	7MW3-52	36	D	1MB7-29	0
B	10MW3-92	9	C	2MB7-79	0
D	11MW3-43	24	B	3MB7-20	0
B	13MW3-13	22	D	4MB7-49	0
B	14MW3-34	42	C	5MB7-10	1
D	15MW3-53	117	C	6MB7-39	0
B	16MW3-32	90	B	7MB7-68	0
D	2FW7-72	4	C	9MB7-94	0
C	6FW7-32	18	B	10MB7-29	0
C	11FW7-53	14	D	11MB7-89	0
B	2MW7-61	25	C	12MB7-68	0
B	11MW7-02	6	B	13MB7-99	0

The first column indicates the exponent program each subject used in Phase II. The second column lists the I.D.'s . The final column contains the verbalization scores.

APPENDIX V (CONT.)
TABLE 1
SCORES OF PHASE I

A_2B_1

D	2FG5-81	11
C	4FG5-71	10
B	6FG5-71	11
B	9FG5-82	33
B	10FG5-02	17
B	12FG5-62	24
D	16FG5-22	20
C	9MG5-83	18
D	10MG5-32	39
C	12MG5-12	0
C	2FG2-22	34
C	7FG2-91	1
C	8FG2-61	28
B	6MG2-42	0
D	7MG2-32	0
D	12MG2-03	30
B	9FG7-81	60
B	11FG7-02	13
C	3MG7-92	0
C	8MG7-02	3
D	15MG7-43	15
D	16MG7-32	2
B	3FB3-72	0
B	7FB3-22	2
C	8FB3-12	12
D	12FB3-12	0
C	5MB3-72	0
B	9MB3-93	9
B	13MB3-62	0
D	14MB3-02	8

A_2B_2

D	5FG5-31	0
C	13FG5-99	0
B	15FG5-59	0
D	2MG5-80	0
C	6MG5-29	1
D	7MG5-21	2
C	8MG5-40	0
C	1FG2-10	3
B	3FG2-69	7
B	4FG2-30	0
C	5FG2-68	5
B	1MG2-09	3
D	2MG2-80	0
C	3MG2-39	0
B	4MG2-30	0
D	5MG2-80	0
B	8MG2-39	4
C	9MG2-48	0
B	10MG2-60	0
D	11MG2-50	0
B	14MG2-10	1
C	2FG7-01	0
B	4FG7-50	0
D	8FG7-80	5
D	10FG7-91	1
D	1MG7-40	1
C	7MG7-90	0
D	9MG7-07	4
B	13MG7-00	0
D	14MG7-60	0

APPENDIX VI

SCORES

FOR

PHASE II

APPENDIX VI

TABLE 1
SCORES FOR PHASE II

A_1B_1

16MW3-32	90	18
9FG7-81	60	17
5MW3-92	44	12
14MW3-34	42	16
9FG5-82	33	8
6FW3-91	32	16
2MW7-61	25	3
12FG5-62	24	6
13MW3-13	22	11
3FW3-33	18	5

A_1B_2

10MW3-92	9	9
9MB3-93	9	13
3FG2-69	7	0
11MW7-02	6	8
14FW7-29	6	3
8FB7-87	6	1
8MG2-39	4	5
9FB7-50	3	0
7FB3-22	2	12
14MG2-10	1	2

A_2B_1

2FW3-22	123	11
7MW3-52	36	9
2FG2-22	34	2
8FG2-61	28	14
12FW3-43	24	12
12FW7-21	24	6
3MW3-04	22	3
6FW7-32	18	6
9MG5-83	18	14
9FW3-73	16	11

A_2B_2

8FB3-12	12	16
4FG5-71	10	14
1FW3-22	7	7
5FG2-10	5	0
7MW7-97	3	3
8MG7-02	3	1
1FG2-10	3	3
5MB7-10	1	0
7FG2-91	1	3
6MG5-29	1	1

A_3B_1

15MW3-53	117	11
13FW3-93	51	14
4FW3-11	40	13
10MG5-32	39	6
15FW3-63	38	13
12MG2-03	30	8
14FW3-53	24	6
11MW3-43	24	12
8FW3-82	20	2
16FG5-22	20	8

A_3B_2

3MW7-31	6	11
8FG7-80	5	3
2FW7-72	4	9
9MG7-07	4	5
1FW7-60	2	3
16MG7-32	2	3
7MG5-21	2	4
10FW7-60	1	1
10FG7-90	1	3
1MG7-40	1	5

The columns in each cell contain I.D. numbers, verbalization scores, and transfer scores respectively.

APPENDIX VII

BIBLIOGRAPHY

APPENDIX VII

BIBLIOGRAPHY

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