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ABSTRACT

These materials were designed to provide a logical development of the number systems from the natural numbers through the real number system. The course attempts to develop pedagogical strategies which will enable naive mathematics students to cope with its content. Revised versions of these materials will be used with prospective elementary school teachers. The content of the course is predetermined but the pedagogy is experimental. Recommendations of the Cambridge Conference on School Mathematics regarding elementary school mathematics were given serious consideration in the development of the topics included in this course. [Not available in hardcopy due to marginal legibility of original document]. (FL)

THE DEVELOPMENT OF THE REAL NUMBER SYSTEM

by

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PREFACE

This course grew out of the mutual desire of the Cambridge Conference on School Mathematics and the author to present basic notions of the real number system clearly and honestly. To what extent this goal is attained I attribute to my students who provided the necessary feedback; to the Cambridge Conference for its encouragement and financial support; to John F. Martin, Jr, now at Shippensburg (Pa.) State College, for his many fine contributions in the revision of these notes; and to Professor Peter Hilton of Cornell, for his suggestions for improvement of the manuscript. I personally assume responsibility for any deficits.

J.D. Kaplan

CONTENTS

ANALYSIS COURSE

INTRODUCTION:

The main purpose of this course is to develop a logical development of the number systems from the naturals through the real number system. Starting from primitives, the material will motivate the definition of the natural numbers and their extensions to the integers, the rational numbers and then to the reals. Besides exposing students to the specific content of the course, the development hopes to enable the student to become more familiar with the spirit of a formal body of mathematics.

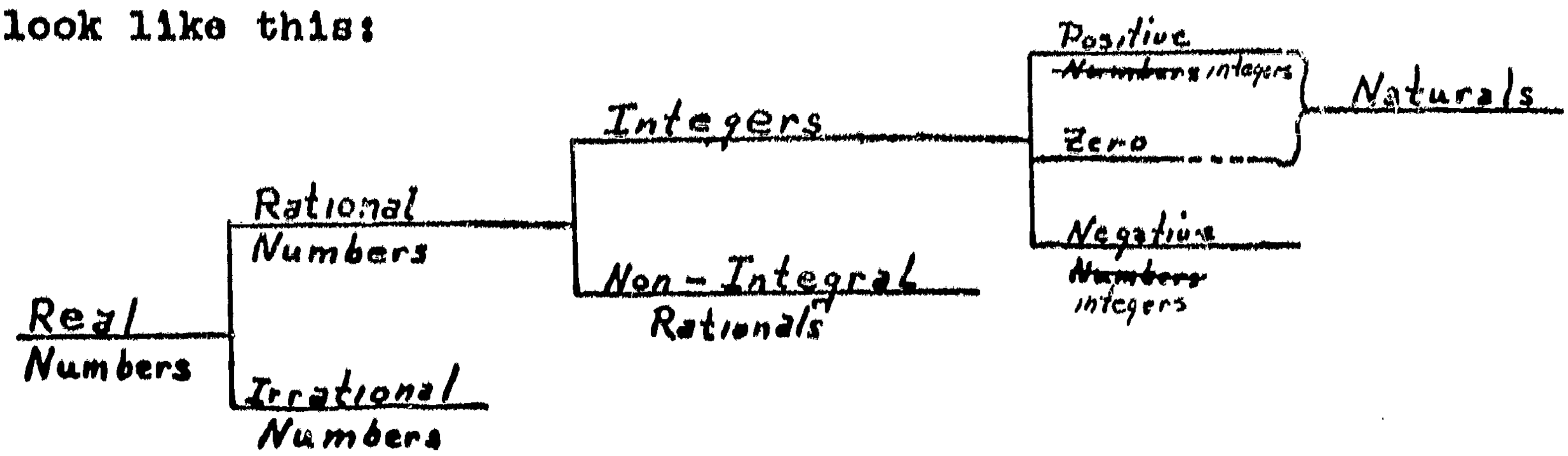
A second purpose of this course is to develop pedagogical strategies which will enable the naive mathematics student to cope with its content. For the ultimate pedagogical goal of these materials is to use the revised versions of the course with future teachers of the elementary school not only in programs offered at Teachers College but at other institutions where undergraduate teacher training is in full bloom.

For these reasons, the material developed for this course will be sensitive to the feedback of the students. While the course's goals are approximately fixed, the pace, strategy, problem sessions, and methodology are not. Whenever possible, students are urged to direct their commentary about the course directly to the instructor or the assistant. The content is more or less pre-determined; the pedagogy remains experimental. The major question in this latter area is: what methods can be devised to teach the subject matter, given the normal conditions which surround a course offering?

Given the pragmatic conditions of a course offering, what methods can be devised to enable the students to master the subject matter?

One of the main pursuits of mathematics is generalization. This pursuit has led to vast systematization and structure throughout all fields of mathematics during the past 75 years. What do these objects have in common? What are the properties that define these objects? The numbers of our daily lives offer us a good example of some very familiar objects whose properties are studies in different groups. The integer +3 and the rational number $+\frac{2}{3}$ belong to different sets with different underlying structures. We shall examine these structures as we go along for they are at the heart of the distinctions among number systems.

Our work will begin with definitions and development of the natural numbers through the basic notion of sets. Using the naturals as the basic ingredients, we shall build the set of integers; then the latter set becomes the building block upon which we erect the rational number system; and finally we extend the rationals to the real numbers. A final map will look like this:



For years the study of the set of real numbers was relegated to the senior high school where it received cursory treatment at best, certainly not in any systematic way. All that has changed during the last 10 years and, now, the development of the real numbers starts in the elementary school in some programs. A recent report[1] of the Cambridge Conference on School Mathematics proposed that serious consideration be given to the introduction of integers and rational numbers before grade 3, and irrational numbers by grade 6. Infinite sequences of real numbers, it was proposed, can be given "intuitive consideration" at grade 6.

Because of the changes which are taking place not only in our teachnological world, but also in the fields of early education, the Cambridge Conference on School Mathematics has recommended that serious emphasis be placed on the training of teachers who can deal with the new content.

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[1]The Cambridge Conference on School Mathematics, Goals for School Mathematics. Boston: Houghton Mifflin, 1963.

TABLE OF SYMBOLS

$\boxed{\div}$	division in $\mathbb{Q}$	102
$\boxed{>}$	"greater than" in $\mathbb{Q}$	104
$\geq$	greater than or equal to	105
$\langle \cdots \rangle$	a sequence	121
∽	relation between Cauchy Sequences of rational numbers	127
$[p_n]$	real number generated by $\langle p_n \rangle$	134

NUMBER SYSTEMS

$\mathbb{W} = \{0, 1, 2, 3, 4, \cdots\}$		Whole numbers
$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$		Natural numbers
$\mathbb{Z} = \{\cdots, ^-2, ^-1, 0, 1, 2, \cdots\}$		Integers
$\mathbb{Q} = \left\{\frac{a}{b} \,\middle|\, \text{a and b are elements } \mathbb{Z},\ b \neq 0\right\}$		Rational numbers
$\mathbb{R}$		Real numbers

CHAPTER 0: Elementary Logical Concepts

In mathematics we study collections of objects and their properties. To facilitate this study we must have a language to use in talking about these collections and properties. Two very important terms in this language are 'and' and 'or'. These terms have specific meanings so different individuals will receive similar meanings from interpreting a given statement.

As normally used in mathematics, the conjunction 'and' will mean "both". Consider the example;

It is cloudy today and it is raining.

The conjunction 'and' is interpreted to mean both events are occuring, i.e., "it is cloudy today" and "it is raining".

Examine the following statement;

The road is open and I drove to town.

This sentence means that both conditions, "the road is open" and "I drove to town", are satisfied.

We are frequently interested in the truth value of compound mathematical sentences. Mathematicians use a two alternative system of logic. By this we mean a given statement is either true or it is false; we have no use for 'maybe'. According to the commonly accepted definition, the compound sentence formed by two statements connected by 'and' will have a truth value of true iff both statements have truth values of true. Therefore, the statement

The road is open and I drove to town,

will have an affirmative truth value iff it is true that "the road is open" and it is true that "I drove to town". The symbol commonly used to represent 'and' is '$\wedge$'.

As normally used in mathematics the disjunction 'or' is given an <u>inclusive</u> meaning. Consider the statement;

"It is raining today or the sun is shining."

This statement will have an affirmative truth value when <u>either</u> "It is raining today" or "the sun is shining" have affirmative truth values, or both statements have affirmative truth values. The symbol commonly used to represent 'or' is '$\vee$'.

In much of everyday life 'or' is used to have the <u>exclusive</u> meaning. Consider the statement made by a mother to a child.

"You may have a piece of candy or a cookie."

The mother undoubtedly means the child can have one or the other but he can not have both the candy and the cookie. Study the two uses carefully as they will arise in the mathematics to be discussed during the remainder of these materials.

Another compound statement frequently used in mathematics is the "conditional" or "if .p., then .q." statements, when p and q denotes statements of some nature. Once again, the truth value of such statements is of upmost importance.

Consider the following conditional statement as a contract and we shall attempt to arrive at the usual truth value by deciding when the contract is upheld and when it is broken.

Example: If I go to the store, then I will buy you a coat.

Case 1) If I do indeed go to the store and I do indeed buy you a coat, the contract is certainly upheld. (Assign a truth value of true)

Case 2) If I do indeed go to the store and I don't buy you a coat, the contract is obviously broken. (Assign a truth value of false)

Case 3) If I don't go to the store and I do indeed buy you a coat, the contract wasn't broken. Nothing stated says the coat must come from the store. (assign a truth value of true)

Case 4) If I don't go to the store and I don't buy you a coat, the contract wasn't broken. (Assign a truth value of true)

To summarize these results we have for $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T.

Consider the following example;

If n divided by 4 leaves a remainder of zero, then n divided by 2 leaves a remainder of zero.

We leave it to the reader to check the truth value of this example.

Another form of a compound statement is the "biconditional". This form can be broken into two conditional statements. The notation for biconditional is $p \Leftrightarrow q$, read p if and only if q. The statement $p \Leftrightarrow q$ can be transformed into "if p, then q <u>and</u> if q, then p". A brief form commonly used is "p iff q".

The reader can use a combination of previous information to determine the truth value of a biconditional statement.

The properties discussed herein will be used extensively in the remainder of the materials.

CHAPTER I: Sets and Operations on Sets

1.1 Introduction

In this chapter we develop a number of definitions and preliminary notions necessary to the entire work. These fundamental ideas of mathematics occur and recur throughout this material. They give rise to basic structures or patterns which help to integrate and strengthen mathematical ideas. The emphasis on mathematical structure suggests that increased emphasis must be given to such basic principles and patterns as those inherent in number and numeration systems, and to the properties of operations from which we abstract generalizations. All of these are integrated by such concepts as the notion of set, notion of a number system, the notion of logical system, and the notion of a relation.

1.2 Sets

We begin by asserting that the idea of a set is familiar to all students. We shall not attempt to define set. We should use it as a primitive or undefined term to define other terms. A set of numbers, a set of letters, a collection of books in a library, a class of students, a set of dishes, are all examples of sets.

The objects making up a set are called <u>elements</u> or <u>members</u> of the set. In the examples of sets given in the previous paragraph, the individual numbers, letters, books, persons, or dishes in a set are the elements or members of the set. The elements or members belonging to a set are determined by the distinguishing characteristic of the set.

For example, consider all the books in <u>a</u> library as a set. Each book in the library may be, in some respects, different from or similar to all other books in the library. However, in this situation the property used to form the set is the fact that the book is in the library. This property gives us a <u>well-defined</u>

collection of objects. When the set's property is sufficient to determine whether or not any given object is a member of our set, the set is a well-defined collection of objects.

To indicate set membership we shall use the standard notation, i.e., $\in$; for non-membership we shall use $\notin$. The symbol '$\in$' is read, "is an element of" or "is a member of"; the symbol $\notin$ is read, "is not an element of" or "is not a member of."

Consider the following situation to demonstrate the use of these two symbols. Suppose "Bill" is a member of the mathematics class but "Valinda" is not a member of the mathematics class. If we denote the mathematics class by the symbol M, "Bill" by the symbol 'b' and "Valinda" by the symbol 'v', we can express the facts of the previous sentence in the following abbreviated form:

$$b \in M, \qquad \text{and} \qquad v \notin M.$$

These abbreviated forms are read "Bill is a member of the mathematics class", and "Valinda is not a member of the mathematics class".

Note: In almost every case in these materials sets shall be denoted by Capital letter (A,B,C, ...) while lower case letters (a,b,c,d, ...) will be used to denote members or elements of sets.

When discussing a set the question should arise as to what objects we can consider for membership in the set. The objects we can use are called permissible elements, and taken together, they form a set called the universe of discourse or simply the universe. Consider for example, the set of names of the months of the calendar year which have 'J' as their initial letter. The universe of discourse, in this instance, would be the 12 names of the months of the calendar year. The set defined would contain three names; January, June, and July.

Now that we can define sets, there is a need for a procedure to communicate efficiently this concept. We use braces $\{\ldots\}$ to include the elements of a set. Thus $\{2,4,6, \ldots\}$ is the set of positive even integers; the set $\{0,1,2,3,4\}$ is the set of all natural numbers less than 5. In the first example, note the use of '...' to indicate that the established pattern continues indefinitely. These three dots for numbers are equivalent to 'etc' in the English language.

On many occasions it is possible to specify a set by listing the names of all its elements. When all the elements are listed specifically, or indicated by '...', we are using the roster or listing method for displaying sets. For example, suppose the members of a committee are: Jane, Bob, Ted, and Mary. This set could be specified by: {Jane, Bob, Ted, Mary}. This would be read the set whose members are Jane, Bob, Ted, and Mary .

EXERCISE 1

Write in set notation using the roster method:

(1) The set of all whole numbers between 90 and 100 (do not include 90 and 100).

(2) The set of all whole numbers which when added to 10 give a sum of 17.

(3) The set of all whole numbers whose squares are greater than 9.

(4) The set of individuals who are president of your college.

(5) The set of authors of your mathematics textbook.

1.3 Alternative Notation

Often it is awkward to list all the members of a set. For instance, the set of all public school teachers in New York City. For convenience, we express this set and others like it in the following manner:

$$\{x \mid x \text{ is a public school teacher in New York City}\}$$

To emphasize the meaning of this expression, consider the following dissection, showing how it may be read:

$\{\cdots\}$	x	$\vert$	
the set of	all elements x	such that	x is a public school teacher in New York City.

The mathematical statement:

$\{x \mid x \in A \text{ and } x \in B\}$, may be read as follows:

"the set of all elements x such that x is an element of set A and x is an element of set B". This method of defining the membership of a set is called "descriptive method".

EXERCISE 2

Rewrite the sets of Exercise I using the descriptive method.

1.4 Finite and Infinite Sets

Throughout your mathematical endeavors you will be confronted with sets containing a great variety of elements. Some of these sets will have finite membership while others will have infinite membership. The following sets are examples of sets with finite membership:

(1) the set of players on a basketball team;
(2) the set of quarterbacks on a baseball team;
(3) the set of residents of New York City; and
(4) the set of natural numbers less than 10,000.

<u>Definition</u>: A set is <u>finite</u> if it is empty or if its members can be counted by a natural number.

Consider the set $\{2,4,6,8,10, \ldots\}$, the membership of this set can not be counted by a natural number. This set is called infinite.

<u>Definition</u>: A set is infinite, if it is not finite.

1.5 Relations between Sets

Suppose we have two sets, A and B. To be specific let A denote the set of all race horses and B denote the set of all horses. It should be rather obvious that all members of A are also members of B. That is the elements A are elements of B and A does not have elements which fail to be elements of B. When this duality of members occurs A is called a 'subset' of B.

<u>Definition</u>: A set A is a <u>subset</u> of set B, if and only if <u>every</u> element of A is an element of B.

We write $A \subset B$ to denote A is a subset of B. The statement $B \supset A$ stands for the same idea as $A \subset B$. Making use of the notation introduced to this point, the definition of subset can be stated as:

$$(A \subset B) \Longleftrightarrow (x \in A \Rightarrow x \in B).$$

Some other examples are:

1) $X = \{1,2,3,4\}$ and $Y = \{1,2,3,4,5\}$

$$X \subset Y$$

2) $X = \{$car, truck, motorcycle$\}$ and $Y = \{$truck, car, motorcycle$\}$

$$X \subset Y.$$

3) The set of squares is a subset of the set of rectangles

We write $A \not\subset B$ whenever A is not a subset of B. Another important relation that can exist between sets is equality. Let M = {house, garage, table} and N = {table, garage, house}. By examining these sets carefully the reader should observe that 1) $M \subset N$, since every element of M is an element of N; 2) $N \subset M$, since every element of N is an element of M. When both the relations occur, set M is said to be equal to set N.

Definition: Let A and B be sets. Set A equals set B if and only if set A is a subset of set B and set B is a subset of set A.

In short, $A = B \iff A \subset B$ and $B \subset A$. Another example of equality of set is the following:

M = {x| x is a state of the U.S. which borders on the Pacific Ocean}, and

N = {California, Oregan, Washington, Alaska, Hawaii}.

The reader will observe that $M \subset N$ and $N \subset M$, hence M = N. Also one can reason M = N thus M must be a subset of N and N must be a subset of M.

From time to time in discussing sets and set relation a very particular set with crucial properties will arise. For example one might mention "the set of all humans in a room who are 7′ 6" tall". Upon examining the persons in the room he finds no one satisfying the stated condition. Thus the set defined contains no elements, such a set is refer to as a 'null set' or 'empty set'.

Definition: The null set or empty set is a set which contains no elements.

The null set is usually denoted by $\emptyset$, a letter of the Swedish alphabet or { }, a pair of braces void of elements.

EXERCISE 3

(1) State the relationships among these set:

$A = \{1,2,3, \ldots 10\}$
$B = \{2,4,6,8,10\}$
$C = \{1,3,5,7,9\}$
$D = \{2,4,6\}$

(2) Is A a subset of A? Explain

(3) Show that the empty set is a subset of any set, including itself.

(4) Show that the empty set is unique.

1.6 Cartesian Products

There are several methods for combining sets to yield a new set. The first method we shall consider is fundamental in the understanding of many of the early ideas studied in elementary mathematics. The idea is very closely related to the directions given for finding a location on an ordinary road map. Suppose the index shows the location of your destination is K-4. This means you locate your destination by following the vertical boundary until you locate section K; then follow the horizontal boundary until you find section 4. Thus the pair K-4 locates a position on the map.

Given two sets, e.g., A = {Bill, Bob, Joe} and B = {movie, swimming} we can form a new set composed of all possible pairings of boys to activities. Each element of the new set will be composed of two objects or ideas. In the case under consideration the first entry in the pair will be the name of a boy and the second entry will be one of the activities. The elements for this particular case are;

{(Bill, movie), (Bill, swimming), (Bob, movie),
(Bob, swimming), (Joe, movie), (Joe, swimming)}.

This new set is called the Cartesian Product or Cross Product of sets A and B. It should be observed that the elements of the cartesian product are not elements from either set but are what we call <u>ordered pairs</u>.

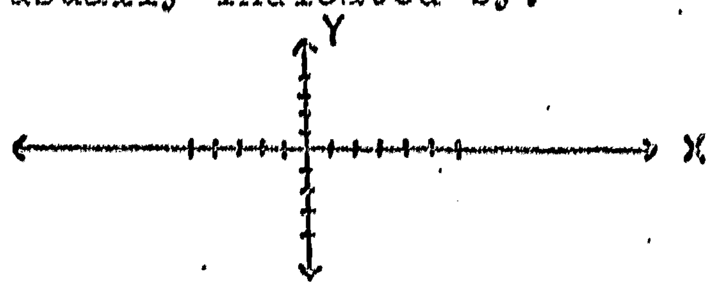
<u>Definition</u>: The <u>Cartesian Product</u> of sets A and B, symbolized by $A \times B$, is defined as the <u>set</u> of all ordered pairs (a,b) such that a is a member of set A and b is a member of set B.

Making use of our notation, we write:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

We read $A \times B$ as "A cross B."

Example 1: An example of a cartesian product that many students have had experience with is commonly called the cartesian plane. This set is usually indicated by:



where X and Y are both the set of real numbers. Some elements of the cross product would be: (2,3), (0,1), (-1,5), $(2\frac{1}{2},3)$, (2,0). In general we write: $X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$, i.e., the set of all equal pairs (x,y) such that $x \in X$ and $y \in Y$.

Example 2: $A = \{x,y,z\}$ $B = \{1,2,3\}$

$A \times B = \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3),(z,1),(z,2),(z,3)\}$

$B \times A = \{(1,x),(2,x),(3,x),(1,y),(2,y),(3,y),(1,z),(2,z),(3,z)\}$

Notice in the above example that $A \times B \neq B \times A$, in general. Also, that $A \times B$ and $B \times A$ are <u>sets</u>.

<u>Exercise 4</u>

(1) When is $A \times B = B \times A$.

(2) If A has m elements and B has n elements, how many elements are there in $A \times B$? in $B \times A$?

(3) How many elements in $\emptyset \times Z$ where Z is any set? in $Z \times \emptyset$?

(4) Let n(A) stand for the number of elements in A. Suppose $n(A) = q$; then $n(A \times A) = ?$

1.7 - Operations on Sets

As indicated in the previous section, cartesian product is not the only way of ~~using~~ Combining two sets to form a [redacted] set. In this section we shall study two more procedures, those of intersection and union.

Sometimes two sets have elements in common, for example, let $A = \{a,b,c\}$ and $B = \{c,d\}$. You should observe that 'c' is an element of set A and c is an element of set B. A new set is formed by taking all the elements in common to both sets. This new set is called the intersection of the original two sets. The symbol used to indicate the operation of forming a new set is '$\cap$'.

Definition: Let A and B be sets. The intersection of A and B ($A \cap B$) is the set of all elements that belong to both set A and B.

Symbolically, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ or $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$. $A \cap B$ is read "A intersection B" or "the intersection of A and B". The reader should observe the close relation existing between the definition of intersection and the meaning of the conjunction 'and'.

Examples: 1) $X = \{\text{orange, apple, pear, lemon}\}$
$Y = \{\text{peach, orange, plum, fig}\}$
$X \cap Y = \{\text{orange}\}$

2) $M = \{1,3,5,7,9\}$
$N = \{x / x \text{ is a positive number}\}$
$M \cap N = M$.

To emphasize the relation between intersection of sets and the meaning of the conjunction 'and', as well as placing emphasis on the interpretation of the definition of intersection, reconsider example 2) above;

$$M \cap N = \{x \mid x \in M \text{ and } x \in N\}.$$

In example 2), we observe that when x is replaced by 3,

'$3 \in M$ and $3 \in N$'

is true, therefore $3 \in M \cap N$. On the other hand, if x is replaced by 12, we have

'$12 \in M$ and $12 \in N$'

which is false. Why? Thus $12 \notin M \cap N$.
If the two sets have no common elements, they are said to be <u>disjoint</u>. This fact is often indicated by, stating that the intersection of the two sets is empty. Symbolically for sets A and B, $A \cap B = \emptyset$.

A third procedure for forming a new set from two given sets is union. The reader should see that the union of sets is closely related to the meaning of the word "or".

If set $A = \{a,b,c\}$ and $B = \{1,3,5\}$ we can form a new set by taking the element of set A, together with the elements of set B. This new set is called the union of A and B.

<u>Definition</u>: Let A and B be sets. The <u>union</u> of A and B ($A \cup B$), is the set consisting of all elements in set A or in set B.

Symbolically, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. The word 'or' in the previous definition means "inclusively", i.e., at least one of the statements, '$x \in A$', '$x \in B$', is true.

To emphasize the relation between union of sets and the meaning given to the disjunction 'or', as well as placing emphasis on the interpretation of the definition of union, consider the following example;

3) $A = \{1,2,3\}$ $\qquad$ $B = \{5,7,8\}$

$$A \cup B = \{1,2,3,5,7,8\}.$$

By the definition $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

In example 3), we observe that when x is replaced by 5,

'$5 \in A$ or $5 \in B$'

is true, therefore $5 \in A \cup B$. Also, if x is replaced by 2,

'$2 \in A$ or $2 \in B$'

is true, therefore $2 \in A \cup B$. On the otherhand, if we randomly substitute something for x, say #, we have

'$\# \in A$ or $\# \in B$'

which is false. Thus $\# \notin A \cup B$.

To emphasis a point about the notation of listing the elements of a set formed by taking the union of two arbitrary sets consider this example.

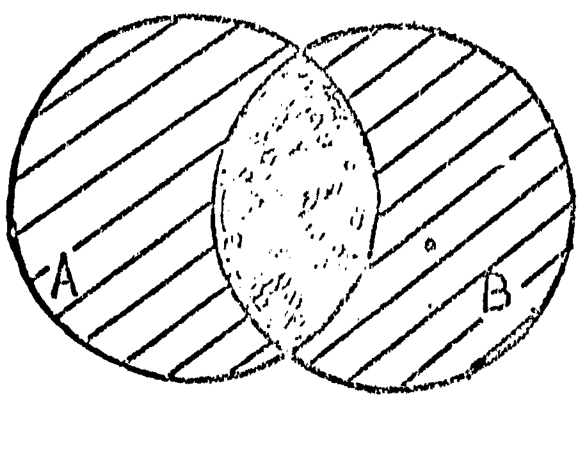
$$A = \{a, 1, 2, b\} \qquad B = \{1, 2, 3, 4\}$$

$$A \cup B = \{a, 1, 2, 3, 4, b\}$$

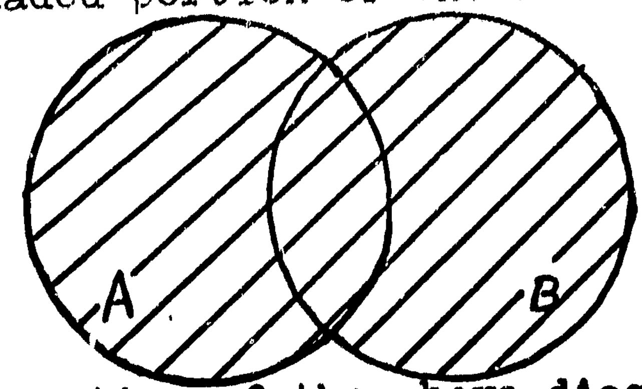
It should be observed that elements which occur in both set A and set B are not listed in a repeated fashion in $A \cup B$, i.e., $A \cup B$ is not listed as $\{a, 1, 2, b, 1, 2, 3, 4\}$. It is part of the meaning of the symbolism that we are to consider as elements of this set the objects or ideas which the symbols name.

1.8 Venn Diagrams

We consider the representation of sets as circular regions, including the boundaries:



The dark shaded portion of the above diagram represents $A \cap B$.



The shaded portion of the above diagram represents $A \cup B$.

EXERCISE 5

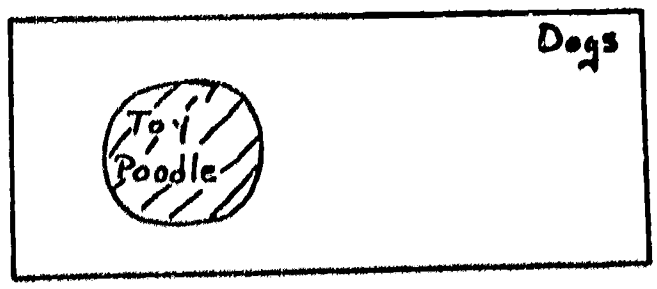
Characterize each of the following as true or false. Make sure you make an honest check either from the definitions, or from Venn diagrams, or from other representations:

(1) $A \cup B = B \cup A$
(2) $A \cap B = B \cap A$
(3) $A \subset (A \cap B)$
(4) $B \subset (A \cap B)$
(5) $(A \cup B) \subset A$
(6) $A \supset (A \cap B)$
(7) $A \subset (A \cup B)$
(8) $A \supset B \Rightarrow A \cap B = A$
(9) $A \supset B \Rightarrow A \cap B = B$
(10) $A \supset B \Rightarrow A \cup B = A$
(11) $A \supset B \Rightarrow A \cup B = B$
(12) $(A \cap B) \cap C = A \cap (B \cap C)$
(13) $(A \cup B) \cup C = A \cup (B \cup C)$
(14) $A \cup \emptyset = A$
(15) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(16) $A \cap \emptyset = A$
(17) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$
(18) $A \cup A = A$
(19) $A \cap A = A$

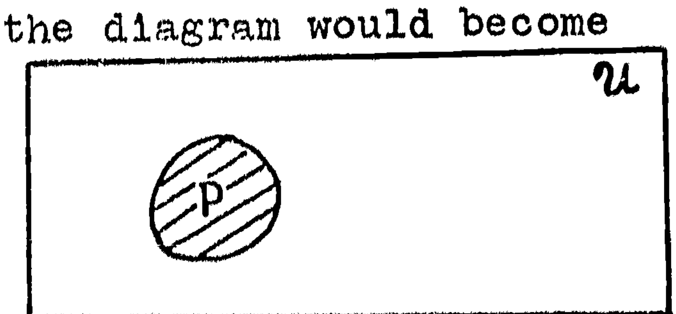
1.9 - Complement

If one has a fixed set of objects to which the discussion is limited and all sets to be discussed are subsets of this fixed set, this overall or fixed set is refer to as the Universe. As mentioned in section 1.2 this fixed set is frequently called the Universe of discourse, since it is subject to change as different problems and situations are considered.

Let's consider the situation where the universe of discourse is the set of all dogs. A very popular breed of dogs is the toy poodle. This special collection of dogs would form a subset of the entire collection of dogs. We can use a Venn diagram to illustrate the situation.



It is rather common practice to represent the universe by a rectangular region and particular subsets by circular regions. For example if we let $\mathcal{U}$ denote the set of all dogs and P denote the set of toy poodles the diagram would become



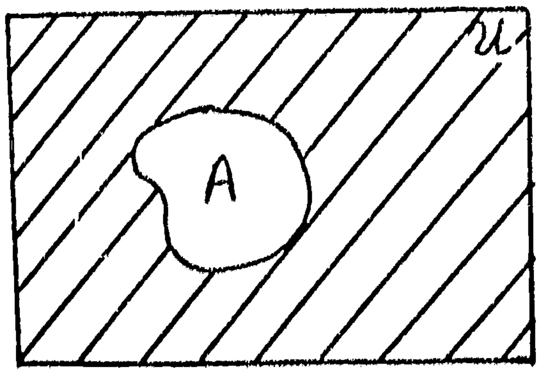
One should observe that not all dogs are toy poodles, hence we can form a set of all dogs which are not toy poodles. This new set is the complement of P with respect to the Universe($\mathcal{U}$).

Definition: If $A \subset X$, the complement of A in X(A' or $X - A'$) is the set of all elements in X which are not in A.

Symbolically we can write

$$X - A' = A' = \{x \mid x \in X \text{ and } x \notin A\}$$

The shaded region in the following diagram represents A';



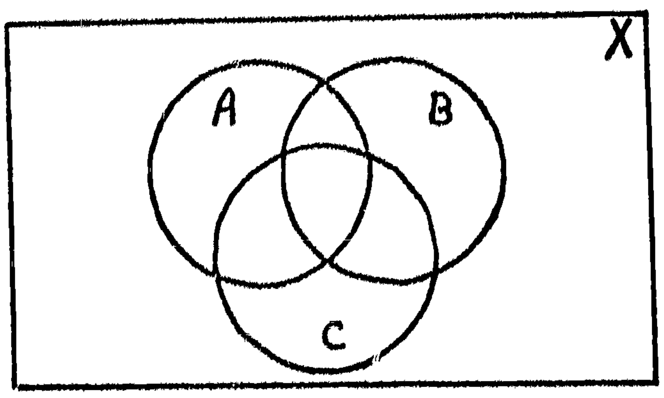
Example: 1) Let $X = \{1,2,3,4,5\}$
$A = \{1,3\}$
$X-A = A' = \{2,4,5\}$

2) Let $X = \{x \mid x \text{ is a whole number}\}$
$A = \{x \mid x \text{ is an even whole number}\}$

$X-A = A' = \{x \mid x \text{ is an odd whole number}\}$

Note: Sometimes the expression X-A is defined as the complement of A relative to X. The reader should observe that the complement of a given set is always relative to <u>some</u> Universe of definition.

<u>EXERCISE 6</u>



Using the above model for each exercise, shade the following sets:

(1) A'
(2) $(A \cup B)'$
(3) $A' \cup B'$
(4) $(A \cap B)'$
(5) $A' \cap B'$
(6) $(A' \cap B') \cap C'$
(7) $\emptyset^{A}$
(8) X'

CHAPTER II: Relations and the Equivalence Theorem

2.1 Relations

Another very powerful and useful concept in mathematics is that of relation. Its power comes from its simplicity while its usefulness comes from its generality. We introduce the concept of relation early because special instances of relations are helpful in procuring a clear understanding of the abstract idea of number. Two very fundamental relations which will facilitate the development of the concept of number are equal and equivalence relations. Before discussing these relations in particular, we shall attempt to give a general notion of what we mean by a relation.

You are already familiar with a number of relations which you use regularly. For example, "is better than" is a relation you might use to compare the abilities of players of a football team, the content of textbooks, the instruction of professors, etc. Although, your meaning of the word better may differ from your friends, you are pairing objects of some set by making the various comparisons. Another relation "is the cousin of" might be a relation you can employ to compare members of your family. Some other common relations are:

1) "is longer than" in comparing rooms, tables, etc.
2) "is taller than" in comparing buildings, people, horses, etc.
3) "is brother of" } in comparing people, dogs, etc.
4) "is father of" }
5) "is a subset of" in comparing sets.

As you should begin to see there have been many instances in your pass experiences where you have used the concept of a relation. You should also observe that these relations were comparisons between objects of a set or possible objects of two different sets. These sets, for the most part, are well-defined. For example, when considering the relation "is taller than" the set of people under consideration might be the enrollment of your math class. In most instances, this set is well-define.

Let's look at a relation which is not new to most of you and which we will consider in greater detail later in the book. The relation is "is greater than" for the set of whole numbers, $W = \{0,1,2,3, \ldots\}$. We can state that 7 is greater than 4, 9 is greater than 2, and many additional statements comparing two whole numbers with this relation. For brevity, we can symbolize this relation by 'G', i.e., G represents the relation "is greater than". Then we can write 7G4, 9G2, and so on. These statements will be read 7 is greater than 4, 9 is greater than 2, respectively.

In general when given a relation R we can write xRy, for x and y members of the set(s) under consideration, and we shall mean that x is related to y by the relation R.

Another way to symbolize a relation is by using ordered pairs. Consider again the relation "is greater than" defined on the set of whole numbers, $W = \{0,1,2,3,4, \ldots\}$. Mathematically we can define "is greater than" (denoted by G) as a set of ordered pairs (x,y) such that for x and y members of W, x is greater than y. Symbolically, $G = \{(x,y) \mid xGy, \text{ for } x, y \in W\}$. Observe that the ordered pair (9,2) is a member of G, i.e., $(9,2) \in G$; similarly, $(7,4) \in G$, $(14,5) \in G$. But, $(2,9) \notin G$ and $(10,20) \notin G$. As indicated above we can state $(x,y) \in G$ whenever x and $y \in W$ <u>and</u> x is greater than y.

In summary, given a set X and a relation R defined on set X, the relation will be a subset of the cartesian product of the set cross itself. We can designate the members of this relation R by either of the previously mentioned methods:

1) for x and y members X, xRy or

2) for x and y members X, $(x,y) \in R$.

More generally, a relation compares objects from two sets X and Y (such as X a set of galvanized pipe, and Y a set of boards for the relation "has the same length as"). In this case the relation is a subset of $X \times Y$. Much of the work in this book will rest on relations which are subsets of the Cartesian Product of the same set.

2.2 Equivalence Relation

To this point we have been discussing relations in general, but many relations have characteristics or properties which allow categorization of the relations according to the properties they possess. Among the important properties of relations is the reflexive property.

Definition: Let X be a set. A relation R on X is reflexive if and only if xRx, for all $x \in X$.

Alternative Definition: A relation R on X is reflexive if and only if $(x,x) \in R$, for all $x \in X$.

What this definition states is that every element of the given set has the given relationship with itself. The relation "the same age as" defined on the set of students in your math class clearly satisfies the reflexive property. For example if Jack is a member of your math class, he is obviously the same age as himself. A similar statement can be made about every member of the class. Thus, the relation "the same age as" defined on the roster for your math class possess the reflexive property. We say the relation is reflexive on the defined set.

Some other examples of reflexive relations are:

1) "attends the same school as" on the set of U.S. citizens
2) "congruent" on the set triangles of a plane
3) " is as strong as" on the set of horses
4) "is equal to" on the set of whole numbers
5) "is subset of" on a collection of sets

There are many more which could be listed, but there are also some relations which are not reflexive. To establish when a relation is not reflexive we must produce an element of the set which does not have the gwen relation with itself. Consider the relation "is the father of" defined on the set of living and dead human beings. George Washington is obviously an element of the set but clearly George Washington is not his own father. Some other relations which are not reflexive are:

1) "is greater than" on the set of whole numbers
2) "is taller than" on the membership of your math class
3) "is perpendicular to" on the set of line in a plane

The symmetric property is another common characteristic of many relations.

Definition: Let X be a set. A relation on X is symmetric if and only if when xRy then yRx.

Alternative Definition: A relation on X is symmetric if and only if when $(x,y) \in R$ then $(y,x) \in R$.

What this definition states is that whenever two elements are paired in one order they must also be paired in the reverse order. Suppose you have a relation which has two elements, one of which is (Jim,Bill) if the relation is to be symmetric the other element must be (Bill, Jim). Remember, there is nothing in the definition stating that x and y must be distinct.

Some examples of relations which are symmetric are:

1) "has the same birth date as" } ...... defined on the members of your math class
2) "the same age as" }
3) "is parallel to" } .... on the set of lines in a plane
4) "is perpendicular to" }
5) "is equal to" .... on the set of whole numbers

This listing is not ~~conclusive~~ exhaustive but there are also relations which are not symmetric. Some examples of these relations are:

1) "is subset of" on a collection of sets
2) "is older than"
3) "is taller than"

   } ------ defined on the members of your math class

4) "is less than" on the set of whole numbers

The third property of extreme importance is that of **transitivity**.

<u>Definition</u>: Let X be a set. A relation R on X is <u>transitive</u> if and only if when xRy <u>and</u> yRz then xRz.

<u>Alternative Definition</u>: A relation R on X is <u>transitive</u> if and only if when $(x,y) \in R$ <u>and</u> $(y,z) \in R$ then $(x,z) \in R$.

A warning remark is also in order pertaining to the meaning of this definition. There is nothing in the definition requiring that the three elements be distinct, i.e., x might equal y or z. What is stated is that <u>when</u> you find two elements of the relation which are of the form (± , *), (* , /) then in order for the relation to be transitive you must find the element (± , /) in your relation. The interpretation of this property rest heavily on the logical of implication (conditional) statements. It may be useful to review the brief discussion in chapter 0.

Some examples of relations which are transitive are as follows:

1) "is a subset of" ------- on a collection of sets
2) "is younger than"
3) "seated in the same row as"

   } on the members of your math class

4) "less than"
5) "equal to"

   } ---- on the set of whole numbers

Examples of relations which are not transitive:

1) "has a different height than"
2) "has a different first initial"

   } on the members of your math class

3) "perpendicular to" on the set of lines in a plane

4) xRy iff $x-y<5$ on the set of whole numbers.

Finally, relations may possess all three of the aforementioned properties. One of the most common relations which possesses all three is equality. Any relation which possesses all three is called an equivalence relation.

Definition: Let X be a set. A relation R on X is an equivalence relation if and only if R is reflexive, symmetric, and transitive.

EXERCISE 7

(1) Give an example of a reflexive, symmetric and transitive relation.

(2) Give an example of a relation that is reflexive, symmetric but not transitive.

(3) Give an example of a relation that is reflexive, transitive but not symmetric.

(4) Give an example of a relation that is transitive but neither reflexive nor symmetric.

(5) Let X and Y be disjoint sets, and $R \subset X \times Y$. Can R be an equivalence relation? Explain your answer.

(6) Let R be the relation: $ARB \Longleftrightarrow A \cap B \neq \emptyset$. What properties does R possess?

(7) Let R be the relation $ARB \Longleftrightarrow A \cap B = \emptyset$. What properties does R possess?

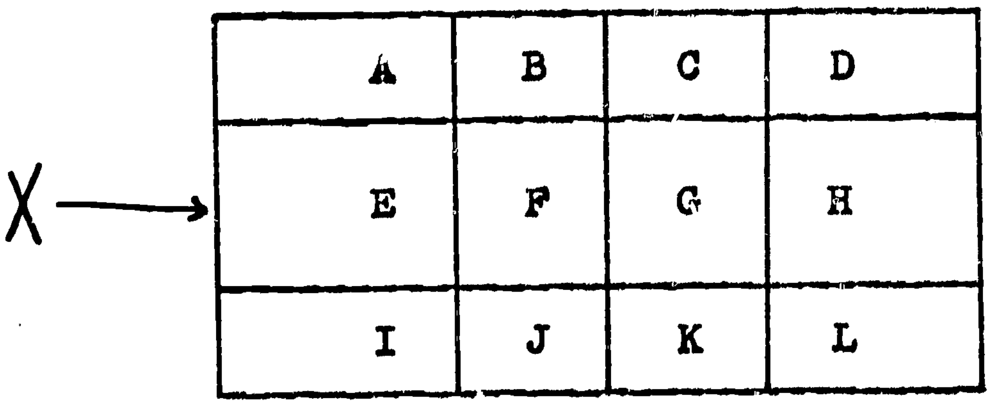
2.3 Equivalence Theorem

We shall now examine a basic property of an equivalence relation. This property will be used at several significant points in the development of the real numbers. While this property may seem rather obvious, we must convince ourselves that it has this "obvious" behavior. We first need a preliminary definition.

<u>Definition</u>: A <u>partition</u>, P, of a set X is a collection of subsets of X: A,B,C, ..., such that each element of X is an element of one and only one of the subsets.

This means that <u>any</u> two of the subsets A,B in the partition, P, are disjoint, i.e., $A \cap B = \emptyset$; and X is the union of the subsets in the partition.

Here's a diagram of <u>a</u> partition:



Note: We shall use the notation $\sim$ to denote an equivalence relation.

Let $[a] = \{x \mid x \in X \text{ and } x \sim a\}$. Thus $[a]$ is the set of all elements in X which are related to a.

<u>Definition</u>: $[a]$ is called the <u>equivalence class generated by a</u>.

For the relation "is congruent to" and for the specific triangle $\triangle ABC = a$, $[a]$ is the set of all triangles which are congruent to $\triangle ABC$. The equivalence class generated by $\triangle ABC$ is the set of all triangles congruent to $\triangle ABC$.

In an attempt to make the previous notion more realistic let's consider a relation defined of the members of your class. Let R be the relation "has the same birth month as"; You should check whether R is an equivalence relation. If we apply this relation to the class we find that the class has been divided into at most twelve subsets. Further more, these subsets are disjoint and every member of the class is a member of some one of the subsets. Hence if you review the definition of partition you should observe that this relation partitioned the class.

Now let's find the equivalence class which 'you' generate. Recall by definition this will be the set of all members of the class whose birthday falls in the same month as yours. Suppose Betty is another classmate and her birthday is in the same month as yours, what is the relationship between the equivalence class generated by 'you' and the class generated by 'Betty'? What is the relationship between the class generated by 'you' and the subsets contained in the partition.

Before attacking the key theorem we show what relationship holds between equivalence classes generated by elements of X which are related by an equivalence relation R. It is claimed that for a and $b \in X$ $a R b$ iff $[a] = [b]$. Recall from Chapter 0 this biconditional statement can be written as two conditional statements connected by a conjunction.

Thus we have a) $aRb \Rightarrow [a] = [b]$ and

b) $[a] = [b] \Rightarrow aRb$.

Let's consider b) first:

By hypothesis $[a] = [b]$ and by definition $a \in [a]$.

Since $[a] = [b]$, by definition of equality of sets $a \in [b]$. Hence it follows by definition of equivalence class aRb.

Secondly, we wish to show that $[a] = [b]$. Recall that to prove these two sets are equal one must show that $[a] \subset [b]$ and $[b] \subset [a]$. Let c be any element of $[a]$; Then $c \in [a]$, we have that cRa by definition of equivalence class. Now aRb by hypothesis. Taking cRa and aRb since R is an equivalence relation we use the transitive property which implies that cRb. Again by definition of equivalence class $c \in [b]$. Each element of $[a]$ is an element of $[b]$, so by definition of subset, $[a] \subset [b]$.

In a similar fashion we prove $[b] \subset [a]$. Let d be <u>any</u> element of $[b]$. Hence dRb. By hypothesis R is an equivalence relation so R is symmetric. Thus since aRb we have bRa. Now dRb and bRa using the transitive property of R we conclude dRa. Therefore $d \in [a]$. Hence, each element of $[b]$ is an element of $[a]$. So $[b] \subset [a]$.

$$[a] \subset [b] \text{ and } [b] \subset [a] \Rightarrow [a] = [b].$$

These concepts are often stated formally in what is know as the Equivalence Theorem.

Theorem: If X is a set and R is an equivalence relation defined on X, then X is partitioned into non-overlapping equivalence classes, and conversely.

We are assuming that R is an equivalence relation. on set X. We would like to show 1) every element of X is in some equivalence class of X which is created by the relation R, and 2) the equivalence classes are either equal or disjoint.

Let's consider condition 1) first.

Proof of 1):

Let $x \in X$, x will belong to an equivalence class. In particular, $x \in [x]$ since by definition $[x] = \{y \in X \mid xRy\}$ and R is given as an equivalence relation, hence R is reflexive, i.e., xRx.

Now the second part of the problem; that of showing "two equivalence classes, are either identical or disjoint, i.e., for a and $b \in X$, $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Suppose $[a] \cap [b] \neq \emptyset$ then there exists $x \in X$ such that $x \in [a]$ and $x \in [b]$. But then xRa and xRb this implies aRb (why). Now since aRb we can conclude that $[a] = [b]$ (why?).

We state the converse of the equivalence theorem without giving its proof.

Theorem: If a set X is partitioned into non-overlapping classes, then there exists precisely one equivalence relation for which the given classes are equivalence classes.

EXERCISE 8

(1) Consider the set of all integers: $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and the relation R: $x - y$ is divisible by 3 where x and y are integers. For example, 7R4 since $7 - 4$ is divisible by 3. Also, (13,10), (25,22), (60,30) $\in R$. Show that R is an equivalence relation.

(2) What are the equivalence classes of (1)?

(3) Show that the equivalence theorem is satisfied for example (1).

(4) Let X = (a,b) a,b are integers . Define R on X by $(a,b)\ R\ (c,d) \Leftrightarrow a+d = b+c$.

(a) Show that R is an equivalence relation.

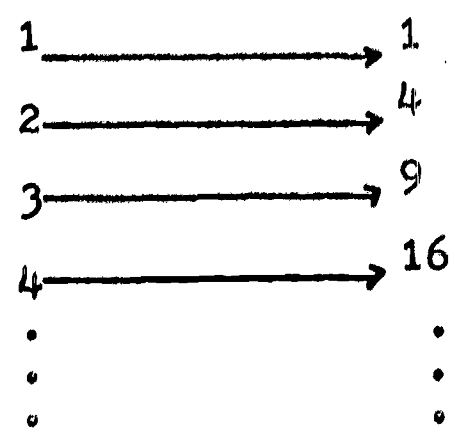
(b) What are the equivalence classes?

CHAPTER III: Functions

We now turn our attention to relations between sets X and Y. As you will observe, the sets X and Y need not be the same.

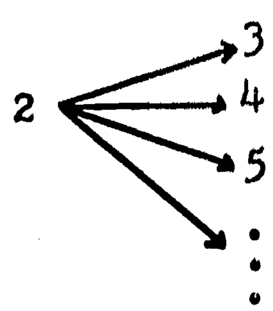
We will give the definition and an elementary treatment of one of the most important concepts in modern mathematics - the function. Think of a relation as a way of relating the elements of two sets and you almost have the idea of a function, except that a function is a special kind of a relation. When we defined a relation R we said that R was a set of ordered pairs. For example, if R is the relation "is less than" over the set of whole numbers, then $(3,4) \in R$, $(9,10) \in R$ and $(14,400) \in R$.

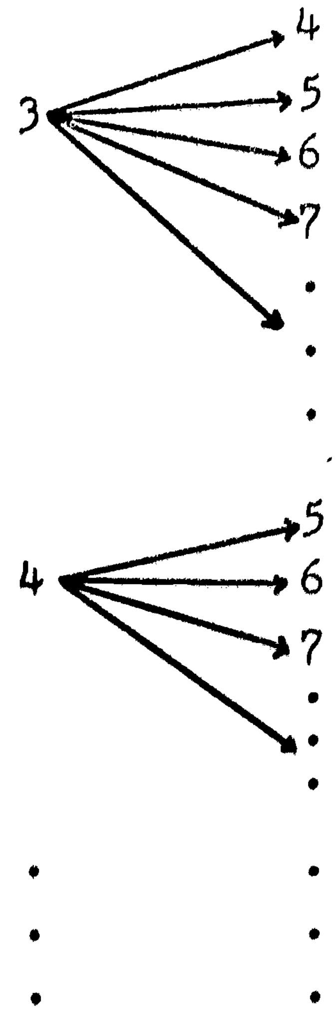
Consider this relation: the number "squared". This relation is made up of $\{(1,1),(2,4),(3,9),(4,16), \dots\}$. We can draw a "map" of this relation:



The arrows in this map indicate that 1 corresponds to 1, 2 corresponds to 4, 3 corresponds to 9, etc. What we have done is to match elements of one set with elements of another set to form ordered pairs.

If we were to draw a map of the relation "is less than," part of the map would look like this:





One important difference marks the two relations shown above. In the case of the squaring relation each of the first numbers corresponds to one and only one second number, while for the "is less than" relation, each of the first numbers corresponds to more than one second number (in fact, an infinite number). The first relation is a function; the second relation is not because of this difference.

We shall define a function as a special subset of $X \times Y$.

<u>Definition</u>: A <u>function</u>, f, from X into Y is a subset of $X \times Y$ such that: 1) $\forall\, x \in X \Rightarrow \exists\, y \in Y \ni (x,y) \in f$,
2) $(x,y), (x,z) \in f \Rightarrow y = z$.

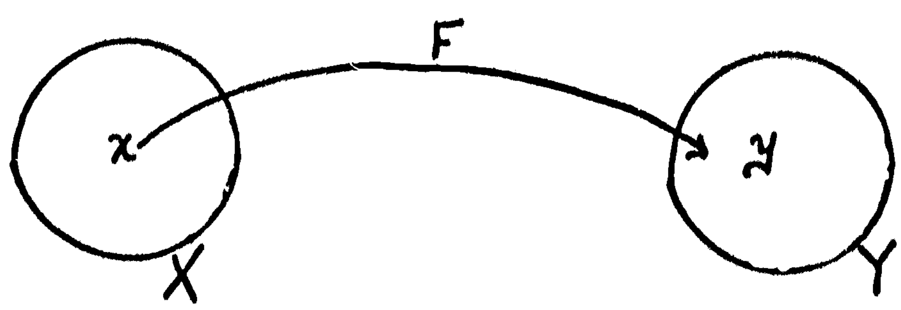
As a relation, a function is a subset of $X \times Y$ where the first elements of the ordered pairs of the function are from X and the second elements are from Y. Observe the important fact that a function is a relation such that no two distinct ordered pairs have the same first elements. Often X and Y are equal.

Example: Let $X = \{1,2,3,4,5\}$
$Y = \{2,3,4,5,6\}$
$F = \{(1,4),(2,5),(3,6),(4,2),(5,5)\}$ is a function
while $R = \{(1,4),(1,6),(2,4),(3,4)\}$ is not a function,
however, it is a relation.

Often, the idea of matching elements from two sets is determined by a <u>rule</u> such as the squaring rule above or the rule that associates each whole number with the number tripled. Then we think of a function displayed as a map, we often call the function a <u>mapping</u>. Thus, the rule: associating each whole number with itself tripled results in this mapping:

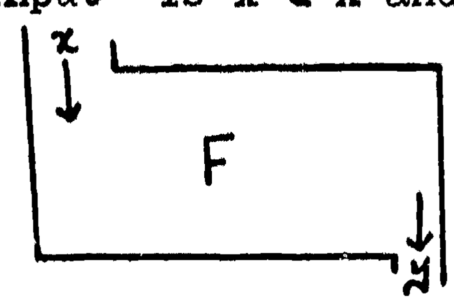
$1 \longrightarrow 3$
$2 \longrightarrow 6$
$3 \longrightarrow 9$
$4 \longrightarrow 12$
$\vdots \qquad\qquad \vdots$

In general, a mapping from set X to set Y looks like this:



F consists of all ordered pairs (x,y), where $x \in X$ and $y \in Y$.

Another way of demonstrating a function is to portray a "machine" whose "input" is $x \in X$ and whose "output" is $y \in Y$:



In this portrayal, F is supposed to be acting on x to produce y.

Again, the ordered pair $(x,y) \in F$.

In the example discussed, X is called the domain of F and Y is called the co-domain of F. A very special subset of Y is often used in discussing a mapping or function. This subset is called the range of F.

Definition: If F is a function from X into Y, the range of F is the set of all $y \in Y$ such that $(x,y) \in F$ for some $x \in X$.

The definition of "function" states that we have a method of determining exactly one element of the range whenever we choose any element of the domain. The method may be given by a clear rule (the square of each element of the domain as shown above) or a formula (e.g., $A = \pi r^2$), but nothing in the definition of "function" requires that it be specified by a formula. What should be clear is that given an element of the domain, then one can easily identify the corresponding element of the range.

If (a,b) is an element of a function f, then we usually write $b = f(a)$. We read these symbols as "b is f of a" or "b is f at a".

Let X and Y be the domain and co-domain of f, respectively. We sometimes write $f: X \rightarrow Y$ to indicate the function f from set X into set Y.

Examples: (1) $X = \{0,2,4,6\}$

$Y = \{1,3,5,7\}$

$f = \{(0,3),(2,1),(4,7),(6,5)\}$

$f(0) = 3$

$f(2) = 1$

$f(4) = 7$

$f(6) = 5$

Domain f = X

Range f = Y = co-domain f

(2) $X = \{0,2,4,6\}$

$Y = \{1,3,5,7\}$

$f = \{(0,1),(2,5),(4,5),(6,7)\}$

Domain of $f = X$

Range of $f = \{1,5,7\} \subset Y$

Co-domain of $f = Y$

<u>Definition</u>: The set of elements of the range is often called the set of <u>images under</u> f.

In the first example above, 3 is the image of 0 under f, 1 is the image of 2 under f; and so forth.

Example (3): $X = Y = \{1,2,3, \ldots\}$

$f: X \rightarrow Y$

$f = \{(a,b) \mid b = 2a + 1\}$

$f = \{(1,3),(2,5),(3,7),(4,9), \ldots\}$

$f(1) = 3$; $f(2) = 5$; etc.

Range of $f = \{3,5,7,9, \ldots\}$

Co-domain of $f = Y$

Range of $f \subset Y$

33 is the image of 16; 101 is the image 50.

<u>Definition</u>: A function, f, from X to Y is <u>onto</u> $\Leftrightarrow$ the range $f = Y$.

Symbolically, $f: X \xrightarrow{\text{onto}} Y$

<u>Definition</u>: A function, f, from X into Y is <u>one-to-one</u> $\Leftrightarrow \forall u, v \in X$, $u \neq v \Rightarrow f(u) \neq f(v)$.

<u>Alternate Definition</u>: A function, f, from X into Y is <u>one-to-one</u> $\Leftrightarrow \forall u, v \in X, f(u) = f(v) \Rightarrow u = v$.

Examples of one-to-one and onto functions are found on pages 31 and 32.

Notice that Example (1) on page 31 is both one-to-one and onto, while Example (2) on page 32 is neither one-to-one or onto. The example (3) on page 32 is one-to-one but not onto.

Note that if f is one-to-one we say that f sets up a one-to-one correspondence between the domain and range of f. Under a one-to-one function, each element of the domain is paired with exactly one element of the range; and each element of the range is paired with exactly one element of the domain.

Suppose we have two sets A and B. We say set A is in a one-to-one correspondence with set B if there exists a one-to-one function from A onto B; f: $A \xrightarrow[\text{onto}]{1-1} B$. The relation one-to-one correspondence is a relation whose domain and range are collections of sets; one-to-one function is a relation whose domain and range are set of elements.

Example: $A = \{a,b,c\}$

$B = \{x,y,z\}$

A is in a one-to-one correspondence with B because there exists a one-to-one function, f, whose domain is A and whose range is B. Note that this f is not unique. One of these functions, f, is:

$$f = \{(a,x),(b,y),(c,z)\}.$$

The existence of one function is all that is required. Another function, f′, which would also suffice is:

$$f' = \{(a,y),\ (b,z),(c,x)\}.$$

If a function is one-to-one then it is possible to interchange within each order pair the domain element with the range element and obtain a new function. Notice that this interchange with functions which are not one-to-one produces relations which are not functions.

Examples: $X = \{1,3,5,7\}$ and $Y = \{2,4,6,8\}$ with $f : X \longrightarrow Y$.

(1) $f_1 = \{(1,2),(3,4),(5,6),(7,8)\}$

f_1 is one-to-one

Interchanging the coordinates of each of the ordered pairs in f_1 produces this relation:

$$\{(2,1),(4,3),(6,5),(8,7)\}$$

which is also a function.

(2) $f_2 = \{(1,2),(3,4),(5,4),(7,8)\}$

f_2 is a function which is not one-to-one.

Interchanging the coordinates of each of the ordered pairs in f_2 produces this relation:

$$\{(2,1),(4,3),(4,5),(8,7)\}$$

which is <u>not</u> a function.

In the following example the domain and co-domain will be changed.

(3) $f_3 = \{(x,y) \mid y = 7x\} = \{(1,7),(2,14),(3,21),(4,28), \ldots\}$

f_3 is a one-to-one function if the domain is the set of counting numbers.

Interchanging the coordinates of each of the ordered pairs in f_3 produces this relation:

$$\{(7,1),(14,2),(21,3),(28,4), \ldots\} = \{(x,y) \mid y = \frac{x}{7}\}$$

which is also one-to-one from the <u>range</u> of f_3 onto domain of f_3.

This example demonstrates an important mathematical concept.

<u>Definition</u>: The <u>converse</u> of a function is the relation which results when the elements of the domain and range are interchanged.

<u>Definition</u>: If the converse of a function is also a function we call the converse the <u>inverse</u> of the function.

<u>Alternative Definition</u>: $f: X \rightarrow Y$ has an <u>inverse</u> iff f is one-to-one.

In the examples above, f_3 has an inverse, but f_2 does not. Notice that the inverse of f_3 contains those elements obtained by dividing, while f_3 itself contains elements obtained by multiplying. We shall have more to say about this relationship later.

We denote the converse of f by f^{-1}

In the examples, above, for instance:

$$f_2^{-1} = \{(2,1),(4,3),(4,5),(8,7)\}$$

$$f_3^{-1} = \{(x,y) \mid y = \frac{x}{7}\}$$

3.2 Composition of Functions

We now examine the important property of functions. Suppose we have a function, f, from X onto Y and another function, g, from Y into Z:

A natural question to ask would be; Is there a single function from X into Z which has the same effect as f and g? The answer to the question is yes.

To gain some insight as to why the answer is affirmative, let us consider a straight forward example:

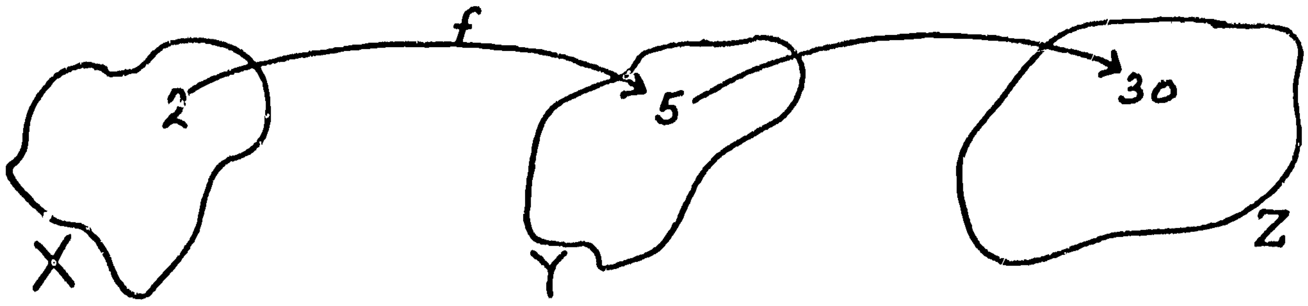
Let X = Y = Z = positive integers

Define: f: $X \to Y$ by $f(x) = x + 3$

Define: g: $Y \to Z$ by $g(y) = 6y$

Now consider $2 \in X$. Observe that $f(2) = 5$. Since g is defined for each element of Y, we evaluate g at 5: $g(5) = 30$.

To represent this pictorially we have:



The element 2 in X corresponds under the "combination" of f and g to 30 in Z. Likewise, each element of X can be shown to correspond to a particular element of Z. We use this example to generalize about the notion of "combined" functions.

<u>Definition</u>: If f is a function with domain X and range Y, and g is a function with domain Y and co-domain Z, then the <u>composition</u> of f and g, denoted by $g \circ f$, is defined on the domain X and $[g \circ f]\ (x) = g(f(x))$.

Observe that the composite of two functions is only defined if the range of the first function to be applied is the domain of the second function to be applied.

What is the single function that will take you from X to Z in the example on the previous page?

$f(x) = x + 3$ and $g(y) = 6y$

$[g \circ f]\ (x) = g(f(x))$

Hence, $g(f(x)) = g(x + 3) = 6(x + 3) = 6x + 18$.

$\therefore\ [g \circ f]\ (x) = 6x + 18.$

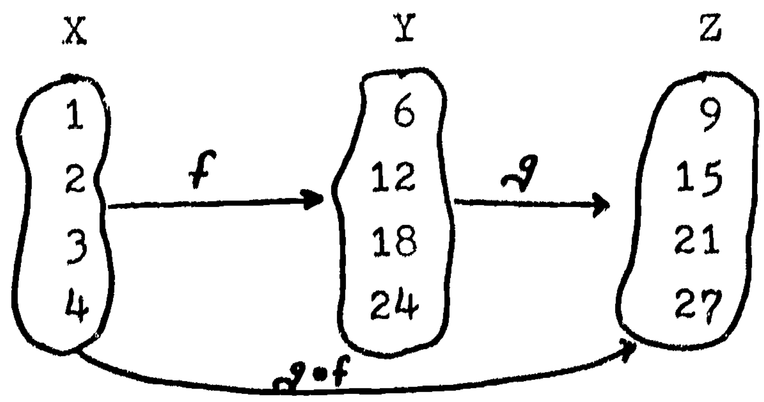
Note that $[g \circ f]\ (x)$ does not map X onto Z, but it is 1-1.

Suppose $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are <u>each</u> 1-1 and onto; then $g \circ f$ will also be 1-1 and onto. This fact will not be proven in general but let's consider a very simple example to help visualize this fact.

Let $X = \{1,2,3,4\}$, $Y = \{6,12,18,24\}$, $Z = \{9,15,21,27\}$

Define $f: X \longrightarrow Y$ by $f(x) = 6x$

$g: Y \longrightarrow Z$ by $g(y) = y + 3$



Notice $g \circ f$ is a 1-1 and onto function from X to Z.

EXERCISE 9

Let $X = \{1,2,3\}$, $Y = \{a,b,c\}$ and $Z = \{a,b,c,d\}$

$f_1 = \{(1,a), (2,a), (3,a)\}$

$f_2 = \{(1,a), (1,b), (2,b), (3,b)\}$

$f_3 = \{(1,a), (2,b), (3,d)\}$

$f_4 = \{(1,b), (2,a)\}$

$f_5 = \{(1,a), (2,b), (3,c)\}$

(1) Which of the above relations re functions?

(2) For those which are, find their domains, ranges and co-domains?

(3) Which are one-to-one functions?

(4) Which functions are onto?

(5) Which functions have inverses? Represent the inverse

(6) Which of the above are functions from X into Y? X into Z?

(7) Let $f_6: X \longrightarrow X$ be defined by $f_6 = \{(x,y) \mid y = 2x + 7\}$

where X is the set of all counting numbers.

(a) Is f_6 one-to-one?

(b) Is f_6 onto?

(8) Let f_7 be defined in the same way as f_6 in (7) except let

X be the set of all integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

(a) Does $f_6 = f_7$? Why? Or why not?

(b) Is f_7 one-to-one?

(c) Is f_7 onto?

(9) Let g: $X \longrightarrow X$ be defined by $g = \{(x,y) \mid y = x^2 + 7\}$ where $X = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(a) Is g one-to-one?

(b) Is g onto?

(10) Define <u>equality</u> for two functions.

CHAPTER IV: Operations

4.1 Binary Operations

We give a special name to a familiar class of functions such as addition, subtraction, multiplication and division. We call these functions operations. In these operations what is generally occurring is that two numbers are paired together and associated with a third number. For example, in addition of whole numbers, 2 and 3 are paired together and associated with 5; in multiplication, 2 and 3 are paired together and associated with 6.

For addition we can form an ordered pair (2,3) and associate it with 5:

$$(2,3) \xrightarrow{+} 5$$

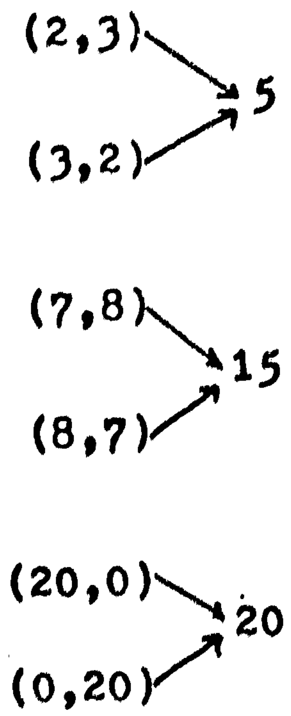
The pair (2,3) is an element of the domain of addition and 5 is an element of the range of addition. For multiplication, one example of the rule can be portrayed as follows:

$$(2,3) \xrightarrow{\times} 6$$

Here, (2,3) is an element of the domain of multiplication and 6 is an element of the range of multiplication. These examples lead to this definition.

Definition: A binary operation on a set A is a function whose domain is a subset of $A \times A$ and whose range is some set, B. If $B \subset A$ then A is closed under the operation, and the operation itself is said to be closed.

If A is the set of whole numbers, then $A \times A$ is the set of all ordered pairs of whole numbers. For addition, the domain consists of $A \times A$ completely. A map of addition, in part, looks like this:



Suppose G is the operation addition. We can write such statements as G((2,3)) = 5, G((7,8)) = 15, and G((0,20)) = 20. But a better symbol for G is "+" and if we drop the use of double parentheses our statements become: +(2,3) = 5, +(7,8) = 15, and +(0,20) = 20. You should be aware that a partial domain is (2,3), (7,8), and (0,20) (ordered pair) and that the images are 5, 15, and 20, respectively (single elements). The word "binary" in the definition above indicates that the operation acts on pairs of numbers which makes up each element of the domain. Of course, conventionally, we use the symbol "+" in this manner: 2 + 3 = 5.

An operation * on a set {a,b,c,d} can be completely specified by a table. By examining the table we can see how the operation acts on any two elements of the set:

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

a * a = a	b * b = c
a * b = b	b * a = b
a * c = c	c * a = c
a * d = d	d * a = d
c * c = a	d * d = c
c * b = d	b * c = d
b * d = a	d * b = a
c * d = b	d * c = b

Another, less abstract example of an operation that is often exhibited by a table is addition of the natural numbers. It should be observed that it is not practical, for the sake of space, to specify completely the result of addition of any two natural numbers, but a clear idea of this operation can be given by examining a finite number of entries:

+	0	1	2	3	4	.	.	.	b ...
0	0	1	2	3	4	.	.	.	0+b
1	1	2	3	4	5	.	.	.	1+b
2	2	3	4	5	6	.	.	.	2+b
3	3	4	5	6	7	.	.	.	3+b
4	4	5	6	7	8	.	.	.	4+b
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
a	a+0	a+1	a+2	a+3	a+4	.	.	.	a+b
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

EXERCISE 10

(1) Convince yourself that "+" is an operation.

(2) Convince yourself that W, the set of whole numbers, is closed under "+".

(3) Convince yourself that $+(a,b) = +(b,a)$ for all $a, b \in W$.

(4) a) Consider ordinary subtraction on the whole numbers, W, such that for $x, y \in W$, $x - y$ exists only when $x \geq y$. Show that it is an operation on W.

b) What is the domain, range, and co-domain of "-"?

c) Is W closed under "-"?

4.2 Properties of Operations

We now consider a set X of objects which has a binary operation defined on it. This section is concerned with defining some of the properties of the operation which permits manipulations of the elements of the set. The first property of an operation is inherit from the definition given in section one of this chapter. This property is the "uniqueness follows from the fact that an operation is defined as a function. Why does this require uniqueness of the results?

We shall list three important and frequently used properties of a mathematical system. In general, an operation will be denoted by "$*$".

Definition: A binary operation $*$ whose domain is $A \times A$ is commutative if $*(a,b) = *(b,a)$ for all $a, b \in A$. (Alternately, $a*b = b*a$.)

This is saying that the result of the operation is independent of the element considered first.

Definition: A closed binary operation $*$ whose domain is $A \times A$ is associative if $*(*(a,b),c) = *(a,*(b,c))$, (Alternately, $(a*b)*c = a*(b*c)$), for all $a,b,c \in A$.

Definition: A closed operation, $\circ$, is distributive with respect to a second closed operation, $*$, $\Leftrightarrow$ both have domain $A \times A$ and $a\circ(b*c) = (a\circ b)*(a\circ c)$.

For the whole numbers multiplication is distributive with respect to addition; e.g., $6\cdot(2 + 4) = 6\cdot 2 + 6\cdot 4$.

EXERCISE 11

(1) Examine "+", "-", "x", and "÷" for commutativety over W.

(2) Examine which of the operations +, -, x, ÷ over W are associative.

(3) Convince yourself that multiplication is distributive over addition.

(4) Find two operations such that one is not distributive over the other.

(5) If a*b represents a, does * represent an operation? What properties if any does it possess?

(6) Consider the operation addition in set W, again the set of whole numbers. Is this function one-to-one? Why, or why not?

(7) Again, study the operation of addition in W. Does this operation have an inverse? Why, or why not?

(8) Consider ∩ and ∪ as relations on the collection of sets. What properties does each of these relations possess?

4.3 Unary Operations

Not all operations which confront us are binary operations. One such operation is demonstrated in the following map from W into W.

W		W
0	⟶	4
1	⟶	5
2	⟶	6
3	⟶	7
.		.
.		.
.		.

Clearly, the range elements of this function are obtained by adding 4 to each element of the domain. We might call this function "adding 4." Observe that it is not a binary operation, but a unary operation (the elements of the domain are single elements, not ordered pairs.) We may write this function in this way:

$$+4 = \{(0,4), (1,5), (2,6), (3,7), \ldots\}$$

EXERCISE 12

(1) Is +4 a one-to-one correspondence from its domain to its range?

(2) Does +4 have an inverse? (Demonstrate y ur answer.)

(3) What conclusions can you draw about the relations +4 and -4?

(4) Consider the operation "x4" in W. Does it have an inverse? What can you conclude about the relations "x4" and "+4"?

(5) Define a relation "*" in W by:

$x*y = x^y$, x and y are not simultaneously zero.

Does this define an operation in W? What are some of its properties (closed, commutative, associative, etc.)?

(6) Define * in W by:

$$x*y = x^2 + y^2.$$

Does this define an operation in W?

(7) Define * in $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (the set of integers) by:

$$x*y = 1$$

Is * a binary operation in the set of integers?

CHAPTER V: Natural Numbers

Consider the set of fingers on a normal right hand; the set of toes on a normal left foot: the set u,v,x,y,z and the set of players on a basketball team. All of these sets are equivalent, i.e., they are in a one-to-one correspondence with each other. For example, there is a one-to-one function from the set of fingers onto the set $\{u,v,x,y,z\}$ hence, a one-to-one correspondence between the two sets.

There are many sets which are in a one-to-one correspondence with any one of the sets above. (Can you think of any? Avoid using the word "five.") All of these sets have one thing in common which distinguishes them from other sets - they are all in a one-to-one correspondence with any one of the sets above; for instance, the set of fingers on a normal left hand. We are going to define the cardinal number five as the class of all these sets.

Recall that a one-to-one function from A onto B creates a one-to-one correspondence between A and B. In the collection of sets, $\mathcal{U}$, we define a relation, r, to mean a one-to-one correspondence. To show that r is an equivalence relation on we have to prove that:

i) ArA, for all sets $A \in \mathcal{U}$.

ii) $ArB \Rightarrow BrA$, for any two sets A and $B \in \mathcal{U}$.

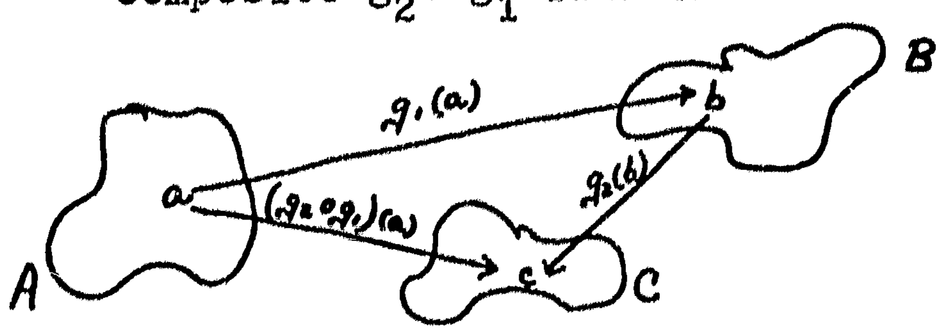
iii) ArB, $BrC \rightarrow ArC$, for any three sets A,B, and $C \in \mathcal{U}$.

Proof: For each of these properties all that is necessary is to exhibit a one-to-one function from the first set onto the second set.

i) In the case of reflexivity, we see that the identity function, f_1, such that $f_1(a) = a$, for each element a in A, is a one-to-one function from A onto A.

ii) By hypothesis, ArB means there exists a one-to-one function, g, from A onto B. Thus, for each $a \in A$, there exists $b \in B$ such that $g(a) = b$ and b, of course, is unique. (Why?) Since g is one-to-one, and onto, g^{-1} is a one-to-one function from B onto A. g^{-1} is the one-to-one, onto function that will serve to show that BrA.

iii) ArB means there exists a one-to-one function, g_1, from A onto B; BrC means there exists a one-to-one function, g_2, from B onto C. For each $a \in A$, there exists $b \in B$ such that $g_1(a) = b$; and for each $b \in B$, there exists $c \in C$ such that $g_2(b) = c$. All we have to do is consider the composite $g_2 \circ g_1$ from A onto C.



If we examine $[g_2 \circ g_1]$ (a) for each element $a \in A$, we note that this image yields a unique element c of C. One way to see that this composite function from A to C is one-to-one onto C is to prove that the converse is a function. But g_2^{-1} is a function and so is g_1^{-1}. So the composite function $g_1^{-1} \circ g_2^{-1}$ acting on $c \in C$ yields a unique element a of A. Notice that $g_1^{-1} \circ g_2^{-1}$ is the inverse function of $g_2 \circ g_1$. Also, if ArB, we say simply that A and B are <u>equivalent sets</u>.

<u>EXERCISE 13</u> (Use set equivalent to u,v,x,y,z to answer the following questions)

(1) Show by means of a mapping diagram a one-to-one correspondence between any two of the sets above.

(2) How many different one-to-one correspondences are there between any two of the sets described above?

The relation "one-to-one correspondence" defined on the collection $\mathcal{U}$ of sets is an equivalence relation (see page 45) Application of the equivalence theorem will partition this collection of sets into non-overlapping equivalence classes. Each of the equivalence classes will consist of all sets which are in a one-to-one correspondence with each other, and no others. Every set will be in exactly one of the equivalence classes. Thus the equivalence theorem creates an array of classes which may be portrayed as follows:

ϕ	$\{-\}$	$\{\Delta, \square\}$	$\{\pi, \delta, \Delta\}$	. . .
	$\{*\}$	$\{\rho, \sigma\}$	$\{\ominus, \oplus, +\}$	.
	$\{\sigma\}$	$\{\cdot, \circ\}$	$\{\cdot, -, \times\}$	.
	$\{\square\}$	$\{-, -\}$	$\{*, \alpha, \rho\}$	.

The universe of sets has been "neatly" categorized by the relation one-to-one correspondence. We shall define each one of these categories or classes as a cardinal number. However, we will actually confine our attention for the time being to finite sets.

We begin by defining what is meant by a standard set which will serve as a reference set for each class.

__Definition__: The __immediate successor__ of set A is $A \cup \{A\}$.

Example: The immediate successor of $\{1\}$ is $\{1\} \cup \{\{1\}\} = \{1 \cup \{1\}\}$; the immediate successor of $\{1\}$ is the union of $\{1\}$ and $\{\{1\}\}$. There are __two__ elements in the successor of $\{1\}$.

The effect of the immediate successor of a set is to "add" an element to a set.

__Definition__: We define __standard sets__ as follows:
(1) The empty set is a standard set.
(2) The immediate successor of $\emptyset$ is a standard set.
(3) Any set which is obtained from the empty set by repeated application of the immediate successor operation is a standard set.

From this definition, let's examine some standard sets.

$\emptyset$ is a standard set by (1) of the definition.

By (2) of the definition the immediate successor $\emptyset$ is a standard set.

The immediate successor of $\emptyset$ is $\emptyset \cup \{\emptyset\}$ which is equal to $\{\emptyset\}$. So $\{\emptyset\}$ is a standard set. $\{\emptyset\}$ contains one element, the empty set. For convenience let $a = \{\emptyset\}$.

By (3) of the definition, the immediate successor of 'a' is a standard set. The immediate successor of a is $\{a \cup \{a\}\}$.

Let $b = \{a \cup \{a\}\}$. The successor of b is also a standard set. This set is $\{b \cup \{b\}\}$. In this manner we generate a series of standard sets:

$\emptyset$
$a = \{\emptyset\}$
$b = \{a \cup \{a\}\}$
$c = \{b \cup \{b\}\}$
$d = \{c \cup \{c\}\}$
$\vdots \qquad \vdots$

Each set of this series contains one more element that its immediate predecessor.

Consider all the sets which are in a one-to-one correspondence with each of the standard sets. We end up with this array which is identical to the previous array:

∅	a	b	c	d	e	. . .
	{-}	{−, o}	{*, σ, ρ}	.	.	
	{*}	{*, ⊛}	{−, ⊙, o}	.	.	
	{σ}	{σ, *}	{σ, △, □}	.	.	
	{□}	{·, ⊙}	{◇, ⊕, ▽}	.	.	
	.	.	.	.	.	
	.	.	.	.	.	
	.	.	.	.	.	

<u>Definition</u>: Each of these <u>classes</u> is a cardinal number

<u>Definition</u>: The cardinal number of a set A is the <u>class</u> which contains A.

Example: Zero is the cardinal number of the empty set.

One is the cardinal number of {*}

Two is the cardinal number of {*, ⊛}

Three is the cardinal number of {*, σ, ρ}

And so forth.

<u>Definition</u>: A <u>finite</u> set is one which can be put in a one-to-one correspondence with a standard set.

Definition: A natural number is the cardinal number of a finite set. Let $n(A)$ stand for the cardinal number of set A.

Definition: If $n(A) = p$ and $n(B) = q$, then $p = q \Leftrightarrow$ A is in a one-to-one correspondence with B.

The numerals 0,1,2,3, ... are common ways of naming the natural numbers. Several ways of naming the natural number five are: 5, V, 𝍸 , and 'five'. These symbols are not the number five. These numerals for five are ways of representing the idea or abstraction of five.

If m = n as in the definition above, we interpret this statement to mean that two different symbols (or numerals) m and n stand for the same idea, the natural number associated with a particular equivalence class.

Definition: A is a proper subset of $B \Leftrightarrow A \subset B$ and $B \not\subset A$.

Definition: If $n(A) = a$ and $n(B) = b$, then a is greater than $b \Leftrightarrow$ B is equivalent to some proper subset of A. The symbol is read "is greater than" while the symbol $<$ is read "is less than". $a > b \Leftrightarrow b < a$.

CHAPTER VI: Operations on Natural Numbers

6.1 Addition

Children are taught to add two numbers, say 2 and 3, by means of a number of examples in which they witness or manipulate the combining of two sets. They observe two sets, a set of 2 objects and a set of 3 objects, and after these sets are joined together, they are asked to specify the total in the new set. These children are taught how to add by means of the concept of union of sets, in a manner which is identical to the definition of addition of two natural numbers. Of course the two sets must have no elements in common. The student should notice that any two naturals can always be represented by disjoint sets. (Why?)

Definition: If $n(A) = a$ and $n(B) = b$, where $A \cap B = \emptyset$, then 'a + b', the sum of a and b, is the natural number of $A \cup B$. In short, $a + b = n(A \cup B)$. This operation is called addition.

(We assume, in the definition, that the sets A and B are finite.)

Because of the sequential development of the materials the student should be able to prove most of the commonly accepted properties of the natural numbers. So that the student has one example of a proof we shall write out in detail the proof of commutativity of addition. All the needed properties and definitions have been studied earlier in the text. Thus the problem is one of organizing the proper information to formulate a proof of commutativity of addition.

Problem: Prove the commutative property of addition on the set of natural numbers. i.e., for a and b natural numbers, $+(a,b) = +(b,a)$ or $a+b = b+a$.

Proof: Since a is a natural number there exists a finite set A such that $a = n(A)$. Similarly there exists a finite set B such that $b = n(B)$.

Now we would like to prove the equality of two numbers (natural numbers, problem 4 in the exercises). The Questions you should be asking yourself is "when are two natural numbers equal"? and "what are the two natural numbers"?

The two numbers are "a+b" and "b+a". They are received from the definition of addition of natural numbers. Namely $a + b = n(A \cup B)$ and $b + a = n(B \cup A)$.

These two numbers will be equal according to the definition on page 50 iff $A \cup B$ is equivalent to $B \cap A$. But we know $A \cup B = B \cup A$ by commutativity of union of sets proven in CHAPTER I. Hence we can conclude $A \cup B$ is equivalent to $B \cup A$, and so $n(A \cup B) = n(B \cup A)$ or $a+b = b+a$. Thus the problem is completed.

This proof can be written in fewer words but at this point we feel it is important for the student to observe the analysis of the problem simultaneous with the writing of the proof. The reader should attempt to shorten the proof, but be sure every step follows logically from its predecessors.

EXERCISE 14

(1) Show that $2 + 3 = 5$

(2) What does "=" mean in the statement "$2 + 3 = 5$"?

(3) The definition of addition (above) produces an operation '+'. Since an operation is a function, then each element of the domain (a,b) must be associated with one and only one element of the range, (a + b):

$$(a,b) \longrightarrow a+b = n(A \cup B).$$

How do we know that we get one and only one element in the range for each element of the domain?

(4) Is $\mathbb{N} = \{0,1,2,3, \ldots\}$ the set of natural numbers, closed with respect to +?

(5) Prove the associative property for +:

$$(a + b) + c = a + (b + c).$$

6.2 Multiplication

The way multiplication is defined is related to this example: Think of two boys and three girls at a party. A rule of the night is that each boy must dance at least once with every girl. How many different dancing couples are there? Let the set of boys be {Jim, Mike} and the set of girls be {Carol, Jane, Beth}. Couples for dancing can be formed in the following manner: (Jim, Carol), (Jim, Jane), (Jim, Beth), (Mike, Carol), (Mike, Jane), (Mike, Beth). Thus, we find there are 6 couples for dancing.

Definition: If $n(A) = a$, $n(B) = b$, where A, B are finite, then $a \cdot b$, the product of a and b, is $n(A \times B)$. This operation is called multiplication.

Definition: If a, b, and c are natural numbers and if $a \cdot b = c$, then c is a multiple of a or b; a and b are factors of c.

Definition: The binary operation subtraction, "-", is defined as follows: For natural numbers, a and b, with a b if there exists a natural number c such that $c + b = a$, then $a - b = c$.

Definition: The binary operation division, "÷", is defined as follows: For natural numbers, a and b, if there exists a natural number d such that $d \cdot b = a$, then $a \div b = d$ $b \neq 0$.

Both subtraction and division are binary operations in $\mathbb{N}$ because we have already observed that if c,d (above) exist they are unique. However their domains are not equal to the whole set, $\mathbb{N} \times \mathbb{N}$, but proper subsets of $\mathbb{N} \times \mathbb{N}$.

Subtraction and division are restricted operations. For example,

$$(4,1) \in \text{"-"}$$

but

$$(1,4) \notin \text{"-"} ;$$

furthermore,

$$(12,4) \in \text{"+"}$$

but

$$(12,5) \notin \text{"+"} , \quad (4,12) \notin \text{"+"}.$$

EXERCISE 15

(1) Prove $4 \cdot 6 = 24$.

(2) See EXERCISE 14, above, example 2. Answer the same question for multiplication.

(3) Prove: $a \cdot b = b \cdot a$ (Alternately, $\cdot(a,b) = \cdot(b,a)$). This is called the commutative property of multiplication.

(4) Is $\mathbb{N}$, the set of naturals, closed with respect to $\cdot$?

(5) Show that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(6) Show that $a \cdot (b + c) = a \cdot b + a \cdot c$.

(7) Prove the cancellation property for +:

$$a + b = a + c \Longrightarrow b = c.$$

(8) Prove the cancellation property for multiplication, $a \neq 0$:

$$a \cdot b = a \cdot c \Longrightarrow b = c.$$

(9) Prove the converse of the cancellation property for +.

(10) Prove the converse of the cancellation property for $\cdot$.

(11) Prove: $0 + x = x$, for any $x \in \mathbb{N}$.

(12) Prove: $0 \cdot x = 0$, for any $x \in \mathbb{N}$.

(13) Prove: $x \cdot 1 = x$, for any $x \in \mathbb{N}$.

6.3 Theorem

We now prove a theorem of extreme importance in mathematics. This algebraic property of our number system is that a product of two numbers can only be zero if at least one of the factors is zero. The student uses this fact constantly, perhaps without realizing it. This property is used when you solved a polynomial equation (High School Algebra I) by means of factoring. For instance, the quadratic equation $x^2 - 5x + 6 = 0$ can be expressed as $(x - 3)(x - 2) = 0$ by factoring the polynomial $x^2 - 5x + 6$. From this we can conclude that $x - 3 = 0$ or $x - 2 = 0$. Hence, the possible values for x are 2 and 3.

Theorem: If a and b are natural numbers, and if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The statement:

If a and b are natural numbers, and if $a\,b = 0$, then either $a = 0$ or $b = 0$,

is logically equivalent to the statement:

If a and b are natural numbers, $a \neq 0$ and $b \neq 0$, then $a\,b \neq 0$.

We shall prove the alternate statement of the theorem which, in turn, proves the original statement since the two are logically equivalent.

Proof: $a = n(A)$ for some set A

$b = n(B)$ for some set B

Since $a \neq 0$ and $b \neq 0$, then $A \neq \emptyset$ and $B \neq \emptyset$

Now $a \cdot b = n(A \times B)$; but $A \neq \emptyset$ and $B \neq \emptyset \Longrightarrow A \times B \neq \emptyset$.

Thus, $n(A \times B) \neq 0$.

Hence, $a \cdot b \neq 0$.

At this point we shall not generate any more theorems about the natural numbers. Suffice to say that given the definitions and statements already proved (especially those in the exercises!), we are in a position to prove all the well-known theorems or "facts" about the natural numbers. We conclude with several definitions and more exercises.

EXERCISE 16

(1) Prove the statement: if $a, b, \in \mathbb{N}$ then $a > b \Leftrightarrow \exists\, c \neq 0 \in \mathbb{N} \ni a = b + c$.

(2) Prove: $a > b$ and $b > c$, then $a > c$.

(3) Prove: $a > b \Rightarrow a + c > b + c$.

(4) Prove: $a > b \Rightarrow a \cdot c > b \cdot c,\ c \neq 0$.

THE INTEGERS

CHAPTER VII

7.1 Introduction to the Integers

We developed the set of natural numbers by starting from fundamentals-sets and operations on sets. By means of the relation of one-to-one correspondence between sets, we defined each natural number. We whowed that one-to-one correspondence is an equivalence relation and applied the equivalence theorem. By means of the equivalence theorem we were able to sort sets into different classes. The equivalence classes which resulted became the natural numbers.

The operations of addition and multiplication of natural numbers were defined as the natural number which resulted from operations on sets. For addition, union was employed; for multiplication, Cartesian product. Subtraction and division were also defined, in terms of addition and multiplication, respectively. The major part of the development of the natural numbers was concluded with an inspection of the structural properties of the operations.

The reader has seen that the naturals are closed with respect to addition and multiplication; that both of these operations are commutative and associat.ve, and that multiplication is distributive over addition. Furthermore, the reader will observe that the set of naturals has an identity element for each of the operations - addition and multiplication.

Zero is the identity element for addition because $x + 0 = x$, for all $x \in \mathbb{N}$, the set of natural numbers. One is the identity element for multiplication because $x\ 1 = x$, for all $x \in \mathbb{N}$. Both 0 for addition and 1 for multiplication do not "affect" the identity of the natural number x when the operate on x.

The reader already has observed that the system of natural numbers has certain restrictions - subtraction and division are not closed operations in $\mathbb{N}$. Another way of saying that subtraction is restricted is to assert that the ordered pair $(1,4)$, for example, does not have an image in the natural numbers under subtraction. Or, to put this specific example in another form, the equation

$$x + 4 = 1$$

has no solution in $\mathbb{N}$. More generally, the ordered pair (a,b) does not have an image whenever $a < b$; or the equation

$$(1) \qquad x + b = a$$

cannot be solved in $\mathbb{N}$ whenever $a < b$. The restriction on division is that (m,n) has no image if m is not a multiple of n. As an equation, this restriction is translated into the insolvability of

$$(2) \qquad n \cdot x = m$$

in the set of natural numbers whenever m is not a multiple of n.

To be able to solve equation (1) above, we must "enlarge" $\mathbb{N}$. This "enlargement" is the set of integers. A further "enlargement" of the integers will result in the set of rational numbers which will enable us to solve equation (2).

Throughout the development of the natural numbers, we asked the reader to rely on his prior acquaintance with those numbers to help him move through the abstract treatment of familiar territory. Again, we ask the reader to make use of his experience, this time with the integers, to assist him in reading through this chapter, In constructing the integers, we shall again define the objects (integers) of the system, define two operations (addition and multiplication) on them, and prove some fundamental properties of these operations. We take the set of natural numbers as our point of departure, for this set is all we know up to this point.

7.2 Definition of Integers

Consider the set $\mathbb{N} \times \mathbb{N} = \{(a,b) \mid a,b \in \mathbb{N}\}$, the Cartesian product of $\mathbb{N}$ with itself. We define the relation ⊜ among these ordered pairs by this equation:

$$(a,b) ⊜ (c,d) \Longleftrightarrow a + d = b + c.$$

Note that ⊜ is a relation between ordered pairs of <u>natural numbers</u>.

As an aid to understanding the motivation behind the use of ordered pairs to <u>define</u> integers, the reader should think of (a,b) as 'a − b' and (c,d) as 'c − d'. Thus, (7,2) can be thought of as '7 − 2' and (2,7) as '2 − 7'.

<u>Example</u>: $(2,4) ⊜ (5,7) \Longleftrightarrow 2 + 7 = 4 + 5$

In the above definition, $a,b,c,d \in \mathbb{N}$ and + is the operation, addition already defined for $\mathbb{N}$. As the note above hints, ordered pairs will represent integers, and ordered pairs such as (2,4) and (5,7) will represent the same integer, $^{-}2$ (negative 2). The pairs (2,4) and (5,7) will be equivalent pairs or elements because they will belong to the same equivalence class.

The temptation is great to ask the reader to develop the system of integers with these hints, by using only the available machinery at his command. By "develop" we mean define the set of integers, the operations on integers, and the properties of these operations. We shall resist the temptation, but perhaps the reader can try, as a mental exercise, to anticipate the next few pages. We shall assume that these meager hints serve as a map of where we are going, not as a device for you to do our work.

Again we shall use the equivalence theorem to get the "right" ordered pairs of $\mathbb{N} \times \mathbb{N}$ into classes. Thus, we must first prove that the relation ⊜ for $\mathbb{N} \times \mathbb{N}$ is an equivalence relation. This result, part of Exercise 17, is the reader's contribution to this development.

Application of the equivalence theorem to $\mathbb{N} \times \mathbb{N}$ produces classes of non-overlapping sets of ordered pairs - equivalence classes of ordered pairs - equivalence classes of ordered pairs of natural numbers. Each element of each class is equivalent to all other elements in that class.

For standard elements of these classes we choose these ordered pairs: (0,0), (1,0), (2,0), (3,0), (4,0), ... and (0,1), (0,2), (0,3), (0,4), ...

Analogous to the portrayal of the classes for the natural numbers we have this table for the integers:

...	(0,3)	(0,2)	(0,1)	(0,0)	(1,0)	(2,0)	(3,0)	...
	(1,4)	(1,3)	(1,2)	(1,1)	(2,1)	(3,1)	(4,1)	
	(2,5)	(2,4)	(2,3)	(2,2)	(3,2)	(4,2)	(5,2)	
	(3,6)	(3,5)	(3,4)	(3,3)	(4,3)	(5,3)	(6,3)	
	(4,7)	(4,6)	(4,5)	(4,4)	(5,4)	(6,4)	(7,4)	
	.	.	.	.	.	.	.	
	.	.	.	.	.	.	.	
	.	.	.	.	.	.	.	

Each of these columns represents an equivalence class generated by ⊜ . Not that each column is headed by a standard ordered pair. As before, each of these classes will define a number, in this case an integer. The column headed by (0,0) will be called the integer 0; the classes to the right of 0 are $^{+}1$, $^{+}2$, $^{+}3$, ... (positive 1, positive 2, positive 3, ...); the columns to the left of 0 are $^{-}1$, $^{-}2$, $^{-}3$, ... (negative 1, negative 2, negative 3, ...).

These classes define the set of integers, $\mathbb{Z}$:

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots \ .$$

The standard elements which we chose are the ones which seem to be the simplest.

Of course we saved ourselves a good deal of detailed work by having the equivalence theorem available for pa titioning of the ordered pairs of $\mathbb{N} \times \mathbb{N}$ into the requisite classes. The relation $\circledcirc$ was created by mean of hindsight - we knew where we wanted to go and designed $\circledcirc$ accordingly.

7.3 Operations on Integers: Addition

An integer has been defined as an equivalence class of ordered pairs. Let $[a,b]$ be the equivalence class containing (a,b) and let $\mathbb{Z}$ stand for the set of integers. We drop the parenthesis inside the brackets to simplify notation.

Definitions: Let p be the integer generated by (p_1, p_2) i.e., $p = [p_1, p_2]$, and let q be the integer generated by (q_1, q_2), or $q = [q_1, q_2]$. $p \oplus q$ is the equivalence class generated by $(p_1 + q_1, p_2 + q_2)$, or $p \oplus q = p_1 + q_1, p_2 + q_2$.

According to this definition, the operation $\oplus$ is an operation on equivalence classes of ordered pairs of natural numbers. The reader should observe that addition of integers is symbolized by $\oplus$ to distinguish it, temporarily, from addition on the natural numbers.

In order to insure that the above definition is well-defined,, we must guarantee that $[p_1 + q_1, p_2 + q_2]$ is an equivalence class, i.e., an integer. That $[p_1 + q_1, p_2 + q_2]$ is an integer follows from the facts that p_1, p_2 , q_1, and q_2 are all natural numbers and so are $p_1 + q_1$ and $p_2 + q_2$. (Why?)

We must also show that the name (ordered pair) used for the generator of the equivalence class does not affect the results of the operation. There is if $p = [p_1', p_2']$ and $p = [p_1, p_2]$ while $q = [q_1', q_2']$ and $q = [q_1, q_2]$ then the sum obtained $p+q$ is exactly the same class no matter which generators are used in the operation. When we say $p = [p_1', p_2']$ and $p = [p_1, p_2]$ what is really being said is that $(p_1', p_2') \circledequal (p_1, p_2)$. Likewise, $q = [q_1', q_2']$ and $q = [q_1, q_2]$ implies that $(q_1', q_2') \circledequal (q_1, q_2)$. Now using the definition of $\circledequal$ we have $p_1' + p_2 = p_2' + p_1$ and $q_1' + q_2 = q_2' + q_1$.

These are statements of equality between natural numbers, so we can use any property developed for the set of natural numbers. If we look ahead at the desired result, we would like to say that $p \oplus q = [p_1' + q_1', p_2' + q_2']$ as well as equaling $[p_1 + q_1, p_2 + q_2]$, i.e., $(p_1' + q_1', p_2' + q_2') \circledequal (p_1 + q_1, p_2 + q_2)$.

Now using the facts that $p_1' + p_2 = p_2' + p_1$, $q_1' + q_2 = q_2' + q_1$, and the well defined property of addition of naturals we obtain $(p_1' + p_2) + (q_1' + q_2) = (p_2' + p_1) + (q_2' + q_1)$. Using commutativity and associativity of addition of naturals, this statement can be written as $(p_1' + q_1') + (p_2 + q_2) = (p_2' + q_2') + (p_1 + q_1)$. Now examine the definition of $\circledequal$ we conclude that $(p_1' + q_1', p_2' + q_2') \circledequal (p_1 + q_1, p_2 + q_2)$.

Since these ordered pairs are equivalent they generate the same equivalence class $p \oplus q$, thus completing the proof. What has been shown is that the same sum will be received no matter what names for a number are being used, i.e., the numbers are important when performing the operation, not the names.

<u>Examples</u>: (1) $[2,3] \oplus [3,1] = [5,4]$.

(2) $[20,10] \oplus [15,20] = [35,30]$.

In both of these examples, we could have represented each integer in simpler terms by using standard ordered pairs or elements to represent the classes. Example (1) could read:

$$(1') \quad [0,1] \oplus [2,0] = [2,1] = [1,0].$$

Example (2) would then become:

$$(2') \quad [10,0] \oplus [0,5] = [10,5] = [5,0].$$

In conventional notation, these examples become:

$$(1'') \quad {}^{-}1 \oplus {}^{+}2 = {}^{+}1$$

$$(2'') \quad {}^{+}10 \oplus {}^{-}5 = {}^{+}5$$

By the definition of addition, the integers are closed under addition (see Exercise 17). Furthermore, the operation of addition is commutative and associative; the integer $[0,0]$ is the <u>zero</u> or <u>identity</u> element for $\mathbb{Z}$ (see Exercise 17).

In any mathematical system, we define the <u>inverse of an element</u> by using the identity element. Suppose z is the identity element for a general mathematical system, $*$ is an operation, and 'a' any element of that system. If there is an element of the system, s, such that

$$s * a = a * s = z$$

then s is called the <u>inverse</u> of a with respect to $*$.

For the integers, we know that

$$^{+}4 \oplus {}^{-}4 = 0$$

so that $^-4$ is the <u>inverse</u> of $^+4$ with respect to $\oplus$. Since $\oplus$ is commutative,

$$^-4 \oplus {}^+4 = 0$$

and $^+4$ is the <u>inverse</u> of $^-4$ with respect to $\oplus$. As equivalence classes the integers $[a,b]$ and $[b,a]$ are inverses of each other with respect to $\oplus$ because:

$$[a,b] \oplus [b,a] = [a + b, b + a] = [0,0]$$

<u>Example</u>: For the integer $[7,3]$ the inverse is $[3,7]$ because $[7,3] \oplus [3,7] = [10,10] = [0,0]$. Of course, the inverse of $[3,7]$ with respect to addition is $[7,3]$.

If an element s is the inverse of a with respect to addition, we often call s the <u>additive inverse</u> of a.

In simpler form, every integer can be expressed in one of these three forms: $[a,0]$ or $[0,a]$ or $[0,0]$, where $a \neq 0$. It should be clear that the inverse under $\oplus$ of $[a,0]$ is $[0,a]$ and vice-versa; and the inverse of $[0,0]$ is itself.

7.4 <u>Operations on integers: Multiplication</u>

We remind the reader that he should translate the symbol $[4,2]$ to either $^+2$ or '$4 - 2$' in thinking about this number. In general, $[a,b]$ can be thought of as '$a - b$'.

<u>Definition</u>: Let $p = [a, b]$ and $q = [c,d]$ where a,b,c,d are natural numbers. $p \odot q$ is the equivalence class generated by $(ac + bd, ad + bc)$, or $p \odot q = (ac + bd, ad + bc)$.

<u>Example</u>: $[6,2] \odot [3,4] = [6 \cdot 3 + 2 \cdot 4, 6 \cdot 4 + 2 \cdot 3]$

$$[6,2] \odot [3,4] = [26,30] = [0,4]$$

This example asserts, in conventional notation, that $(^+4) \odot (^-1) = (^-4)$. We have used a different symbol for multiplying integers than we used to multiply natural numbers. This distinction is only a temporary one.

The definition of multiplication of two integers depends solely upon the natural numbers and operations on them. The product in the definition, $p \odot q = [ac + bd, ad + bc]$, is an equivalence class of ordered pairs of natural numbers, and hence an integer. We observe this since the ordered pair $(ac + bd, ad + bc)$ is a result of products and sums of natural numbers, and $\mathbb{N}$ is closed with respect to + and $\cdot$.

We must also show that if $(a', b') \circledequal (a, b)$ and $(c', d') \circledequal (c,d)$, then $p \odot q = [a' \cdot c' + b' \cdot d', a' \cdot d' + b' \cdot c']$; in short we must also show that:

$$(a' \cdot c' + b' \cdot d', a' \cdot d' + b' \cdot c') \circledequal (ac + bd, ad + bc).$$

This equivalence results directly from the definition of $\circledequal$. The student should study this brief proof carefully; use the one for addition to supply needed assistance.

As with addition, the operation is a binary operation, and is commutative and associative, the set $\mathbb{Z}$ has an identity element with respect to $\odot$. It is $^{+}1$, or $[1,0]$ because $[a,b] \odot [1,0] = [a \cdot 1 + b \cdot 0, a \cdot 0 + b \cdot 1] = [a, b]$.

For an element, m, of the integers to have an inverse with respect to $\odot$ there must exist an element $n \in \mathbb{Z}$ such that: $n \odot m = m \odot n = {}^{+}1$. It is clear that $^{+}4$, for instance, does not possess an inverse with respect to $\odot$.

In general, integers do not have inverses with respect to multiplication, i.e., do not have multiplicative inverses. The only exceptions are $[1,0]$ and $[0,1]$.

To show why the integer $[4,0]$ does not have a multiplicative inverse, let us assume that it does and show that this assumption leads to a contradiction. Suppose that the multiplicative inverse of $[4,0]$ is $[c,d]$, i.e.,

$$[4,0] \odot [c,d] = [1,0].$$

If such an integer $[c,d]$ existed, then $[4\cdot c + 0\cdot d,\ 4\cdot d + 0\cdot c] = [1,0]$, which means that:

$$[4c,\ 4d] = [1,0], \text{ or}$$

$$(4c,\ 4d) \circledequal (1,0) \text{ or}$$

$$4c + 0 = 4d + 1, \text{ or}$$

$$4c = 4d + 1, \text{ which is impossible for natural numbers } c,d.$$

It is impossible to solve this equation because it states that a multiple of 4 is equal to one more than a multiple of 4. So, no such $[c,d]$ can exist as an inverse of $[4,0]$. As a problem, we ask the reader to prove this result for any integer except $[1,0]$ and $[0,1]$ (See EXERCISE 17.)

As with the set of natural numbers, there is a distributive property for the integers involving multiplication and addition. Here's a statement of the distributive property for the integers:

$$[a,b] \odot \left([c,d] \oplus [e,f]\right) = \left([a,b] \odot [c,d]\right) \oplus \left([a,b] \odot [e,f]\right)$$

where $[a,b]$, $[c,d]$ and $[e,f]$ are integers. The proof of this property involves merely the application of the two definitions of $\oplus$ and $\odot$, and this, too, is left to the reader to complete (see EXERCISE 17).

<u>EXERCISE 17</u>

(1) Does the set of natural numbers have an identity element for subtraction and division?

(2) Prove that $\circledequal$ is an equivalence relation.

(3) Prove that $\mathbb{Z}$ is closed under addition.

(4) Prove that $\oplus$ is commutative and associative.

(5) Prove that $[0,0]$ is the identity element for $\mathbb{Z}$ under $\oplus$.

(6) Prove that every integer has an additive inverse.

(7) Does <u>any</u> natural number have an additive inverse?

(8) Show that $\oplus$ and $\odot$ are both binary operations.

(9) Show that $\mathbb{Z}$ is closed under $\odot$.

(10) Prove that $\odot$ is commutative and associative.

(11) Prove that $[1,0]$ is the identity element for $\mathbb{Z}$ under $\odot$.

(12) Prove that $\odot$ is distributive over $\oplus$ for the set of integers.

(13) Prove that every integer except $[1,0]$ and $[0,1]$ does not have a multiplicative inverse.

CHAPTER VIII

8.1 Notation for Integers

At this point, having defined ⊕ and ⊙ , we shall drop the cumbersome notation for the integers. (This is not to infer that the student can not use the ordered pair notation, as a matter of fact, we shall use it to prove some properties.) Instead of writing [a,0] we shall write ^{+}a. Thus, [20,0] becomes $^{+}20$ and [4,0] becomes $^{+}4$. Instead of writing [0,a] we shall write ^{-}a. [0,20] becomes $^{-}20$; [0,3] becomes $^{-}3$. [0,0] is simply denoted by 0.

The integers $^{+}1$, $^{+}2$, $^{+}3$, ... are called the positive integers; $^{-}1$, $^{-}2$, $^{-}3$, ... are negative integers; 0 is the integer zero. Notice that the symbol 0 is the integer zero. Notice that the symbol 0 is the same for both the natural number zero and the integer zero. No confusion will result as long as the context is clear. Observe that every integer is both an additive inverse and has an additive inverse: $^{+}10$ and $^{-}10$ are additive inverses of each other as are $^{+}20$ and $^{-}20$. We can write $^{+}10 \oplus (^{-}10) = 0$ and $^{+}20 \oplus (^{-}20) = 0$; in general $^{+}a \oplus (^{-}a) = 0$.

We shall now use the ordered pair notation for integers to prove that the additive inverse of any integer is unique. As should be expected by "unique", we mean there is one and only one additive inverse for a given integer. Let [x,y] be an arbitrary integer and suppose [a,b] and [c,d] are distinct integers each of which is an additive inverse of [x,y] . We shall show that our assumption that [a,b] and [c,d] are distinct is in error and actually these integers are the same.

Since [a,b] is assumed to be an additive inverse of [x,y] we have

$$[a,b] \oplus [x,y] = [0,0].$$

Hence

$$[a + x, b + y] = [0,0].$$

But since $[c,d]$ is also assumed to be an additive inverse of $[x,y]$, similarly we have;

$$[c,d] \oplus [x,y] = [0,0].$$

Hence

$$[c + x, d + y] = [0,0].$$

Since '=' is an equivalence relation we can conclude

$$[a + x, b + y] = [c + x, d + y].$$

Now if two integers are equal the generator must be related by the relation ⊜. Thus

$$(a + x, b + y) ⊜ (c + x, d + y).$$

Therefore

$$(a + x) + (d + y) = (b + y) + (c + x).$$

This is a state about natural numbers thus any properties of can be applied. Using commutativity, associativity, and cancelation of $+$, we have

$$a + d = b + c.$$

Thus implies (a,b) ⊜ (c,d). ∴ Those pairs generate the same integer, hence

$$[a,b] = [c,d] .$$

Furthermore, since ^{-}a is the additive inverse of ^{+}a, and vice versa, we write $^{-}(^{-}a)$ to mean the additive inverse of ^{-}a. So, since the additive inverse is unique we can conclude that, $^{-}(^{-}a) = {}^{+}a$. And if $x \oplus y = 0$, where x and y are integers then $x = {}^{-}y$ and $y = {}^{-}x$.

When we write that x and y are integers we mean that x and y may be positive, negative, or zero. We deliberately omit any indication of their signs. But, if x is positive, then ^{-}x is negative; if x is negative, then ^{-}x is positive; and if x is 0, then ^{-}x is 0. In short, the symbol ^{-}x means the <u>opposite</u> (or negative, or additive inverse) of x and does not necessarily stipulate that ^{-}x is a negative integer. This can best be summarized by writing:

$$x \oplus {}^{-}x = 0, \text{ for all } x \in \mathbb{Z} .$$

8.2 Some Additional Properties of Integers

We know that $^{+}1 \oplus (^{-}1) = 0$ and that $0 \odot a = 0$ $(a \in \mathbb{Z})$. Putting these two results together yields:

$$(1) \quad \left[{}^{+}1 \oplus ({}^{-}1)\right] \odot a = 0.$$

By the right distributive property (see EXERCISE 18), (1) becomes:

$$\big(({}^{+}1) \odot a\big) \oplus \big(({}^{-}1) \odot a\big) = 0, \text{ or}$$

$$(2) \quad a + \big(({}^{-}1) \odot a\big) = 0$$

Equation (2) states that a and $[(^{-}1) \odot a]$ are additive inverses of each other. Therefore,

$$({}^{-}1) \odot a = {}^{-}a.$$

Another well-known result for integers is this one:

$${}^{-}(a \oplus b) = ({}^{-}a) \oplus ({}^{-}b)$$

for any integers a and b. This fact states that the negative of a sum is the sum of the negatives. The proof uses the previous result, namely:

$${}^{-}(a \oplus b) = ({}^{-}1) \odot (a + b) ,$$

which becomes, by application of the distributive property:

$$^{-}(a \oplus b) = [(^{-}1) \odot a] \oplus [(^{-}1) \odot b],$$

which results in the following because of the result above:

$$^{-}(a + b) = {}^{-}a \oplus {}^{-}b .$$

Other common results employing the additive inverse are easily obtainable. These would include (See EXERCISE 18, Number 2):

(1) $(^{-}a) \odot b = a \odot (^{-}b) = {}^{-}(a \odot b)$

(2) $(^{-}a) \odot (^{-}b) = a \odot b$.

By virtue of the presence of an additive inverse for each integer, it is possible to solve this equation:

$$x \oplus a = b,$$

for <u>any</u> two integers a and b. Recall that the comparable equation is not generally solvable over the naturals. In fact, the insolvability of the analogous equation for certain naturals is a limitation or restriction in that system of numbers. The integers do not have <u>that</u> limitation.

<u>EXERCISE 18</u>

(1) Prove that the <u>right</u> distributive property holds for the integers:

$$(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$$

(2) Prove these results:

(a) $(^{-}a) \odot b = a \odot (^{-}b) = {}^{-}(a \odot b)$;

(b) $(^{-}a) \odot (^{-}b) = a \odot b$.

Some of the following may be found easier by using ordered pair notation.

(3) Prove that the sum of two positive integers is a positive

integer; and that the sum of two negative integers is a negative integer.

(4) Prove that the product of two positive integers is positive; and that the product of two negative integers is a positive integer.

(5) Prove that the product of a positive and a negative integer is a negative integer.

8.3 Subtraction and Division of Integers

Definition: The binary operation, " $\ominus$ " , is defined as follows: For integers, a and b, if there exists an integer c such that $a = b \oplus c$, then $a \ominus b = c$.

Definition: The binary operation, " $\oplus$ ", is defined as follows: For integers, a and b, if there exists an integer c such that $a = b \odot c$, then $a \oplus b = c$, $(b \neq 0)$.

Clearly, the domain of $\ominus$ is $\mathbb{Z} \times \mathbb{Z}$, while the domain of $\oplus$ is a proper subset of $\mathbb{Z} \times \mathbb{Z}$. This latter idea means that division is a restricted operation for $\mathbb{Z}$.

8.4 Cancellation Laws for Integers

These two cancellation laws hold for the integers:

(1) $a \oplus b = a \oplus c \Rightarrow b = c$, and

(2) $a \odot b = a \odot c \Rightarrow b = c$, $a \neq 0$.

Both of these appear as problems in EXERCISE 19.

To assist in proving the second cancellation law, we first prove this well-know fact:

$$a \odot b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

The student should give this proof very careful study, as it may seem tricky if you don't check every reference and answer each "why"?

If $a \odot b = 0$, then $(^-a) \odot (^-b) = 0$, by a problem of EXERCISE 18. Also, we know that: $(^-a) \odot b = a \odot (^-b) = {}^-(a \odot b)$, also from EXERCISE 18. Since $a \odot b = 0$, then $^-(a \odot b) = 0$, why? and $(^-a) \odot b = 0$, $a \odot (^-b) = 0$. We now have that all four products, $^-(a \odot b)$, $(^-a) \odot b$, $a \odot (^-b)$, and $(^-a) \odot (^-b) = 0$. One of these four products must then be made up of positive integers since all combinations of positive and negative integers are represented in the products. But, all the products equal zero, so one of the integers, a ^-a, b, ^-b, must be zero. (Why?) This means that either a or b is zero.

From the statement, the cancellation law for multiplication of integers can now be proved. In fact, for the set of integers, the cancellation law and the above theorem about the product of two factors equaling zero are equivalent statements.

If we had deferred a proof that the additive inverse of an integer x is unique until we had prove the cancellation law, it can be written as follows. To prove this, i.e., the additive inverse, ^-x, of x is unique, we use an indirect proof: by assuming that x has two additive inverses we shall be led to a contradiction.

Assume that ^-x and $^-x'$ are two different additive inverses of x. These equations are then true:

$$x \oplus {}^-x = 0 \quad \text{and} \quad x \oplus {}^-x' = 0.$$

Therefore, $x \oplus {}^-x = x \oplus {}^-x'$,

which, by the cancellation law, becomes:

$$^-x = {}^-x'.$$

This last statement gives the contradiction. We assumed that ^-x and $^-x'$ were <u>different</u> and found that $^-x = {}^-x'$. Our conclusion is that ^-x is unique.

For natural numbers x, y, and z, the following result has already been proved:

$$(1) \qquad x = y \Longrightarrow z + x = z + y.$$

This result is the <u>converse</u> of the cancellation property for natural numbers. This conclusion is now applied to prove a similar statement about integers:

$$(2) \qquad b = c \Longrightarrow a \oplus b = a \oplus c, \text{ for all integers, } a, b, c.$$

To prove (2) we separate the hypothesis into three cases: $b = c = 0$, b and c are positive; and b and c are negative. For $b = c = 0$, it is clear that (2) holds by virtue of the fact that zero is the identity element for addition of integers.

If b and c are positive, then $b = [b', 0]$ and $c = [c', 0]$, where b' and c' are natural numbers. Furthermore, $a = [a', 0]$, where a' is a natural number. Since $b = c$, then $[b', 0] = [c', 0]$, or $b' = c'$. The sum $a \oplus b = [a' + b', 0]$ and $a \oplus c = [a' + c', 0]$. But $a' + b' = a' + c'$ because of (1), above, for natural numbers. Therefore, $[a' + b', 0] = [a' + c', 0]$ and $a \oplus b = a \oplus c$.

For b and c negative, $b = [0, b']$ and $c = [0, c']$. Since $b = c$, $b' = c'$. The sum $a \oplus b = [a', b']$ and $a \oplus c = [a', c']$. But, $[a', b'] = [a', c']$ and, therefore, $a \oplus b = a \oplus c$. This concludes the proof of (2), the converse of the cancellation law for the integers.

The definition of subtraction is equivalent to the fact that all equations of the form $x \oplus a = b$ can be solved <u>uniquely</u> in the set of integers. The solution of this equation, according to the definition of subtraction, is $x = b \ominus a$. It should be apparent that $x \oplus a = b$ can also be solved by adding the negative of a to the value of both sides of the equation:

$$[x \oplus a] \oplus (^{-}a) = b \oplus (^{-}a).$$

By the associative property for $\mathbb{Z}$:

$$x \oplus (a \oplus (^{-}a)) = b \oplus (^{-}a).$$

Using the property of additive inverses:

$$x \oplus 0 = b \oplus (^{-}a).$$

Finally, the identity element 0 yields:

$$x = b \oplus (^{-}a)$$

which is a statement that subtraction is the same as adding opposites, or inverses. This statement is not uncommon to junior high school students, now.

8.5 Properties of Order

We have already defined the positive integers. Formally, these are the integers of the type $[a,0]$ where a is a non-zero natural number. This integer was denoted by ^{+}a. In EXERCISE 18, we proved that the sum and product of two positive integers is again a positive integer - the positive integers are closed under $\oplus$ and $\odot$. Also, we know from our construction of the integers that for a given integer, x, x is either positive, negative, or $x = 0$. This latter property is called the Law of Trichotomy.

Definition: The integer a is greater than the integer b (written $a \circledcirc b$) if and only if $a \ominus b$ is positive. If $a \circledcirc b$, then we can also write $b \circledcirc a$, which is read "b is less than a."

Immediately, from this definition, the reader should observe that if a is positive, then $a \circledcirc 0$ since $a \ominus 0$ is positive; and that if b is negative, then $0 \circledcirc b$ or $b \circledcirc 0$ since $0 \ominus b$ is positive.

8.6 Trichotomy

With this definition of "greater than" we can state the trichotomy property in another form:

Law of Trichotomy: For any $a, b \in \mathbb{Z}$, one and only one of the following holds:

$$a \circledgt b, \quad b \circledgt a, \quad \text{or} \quad a = b.$$

We now give a proof of this form of the trichotomy law:

Because of the definition of integers as ordered pairs of naturals and the equivalence relation, $\circleddash$, defined on these ordered pairs, we see that at least one of the above statements must hold. If one desires a detailed proof of this he shall work with a and b in the form of ordered pairs of naturals (This is a good exercise.).

With this in mind, the problem reduces to one of showing that only one of the above statements holds true at a time:

I. Assume $a \circledgt b$ and $b \circledgt a$. We shall show that these two statements cannot hold simultaneously.

   If $a \circledgt b$ then $(b \ominus a) \in \mathbb{Z}^+$(set of positive integers) by definition on page 75. Now that $(b - a) \in \mathbb{Z}^+$, then $^-(b - a) = (a - b) \in \mathbb{Z}^-$(set of negative integers). As the second part of our assumption states that $b \circledgt a \Rightarrow (a - b) \in \mathbb{Z}^+$. Clearly, $(a - b) \in \mathbb{Z}^+$ and $(a - b) \in \mathbb{Z}^-$ cannot hold simultaneously, since this would indicate that the integer $(a - b)$ is a member of two distinct equivalence classes. One would be of the form $[x,0]$, with x naturals, and the other would be of the form $[0,y]$, with $y \in$ naturals. This is the required contradiction in this part of the proof.

II. Assume $a \circledgt b$ and $b = a$. If these two statements hold true, then:

1) $a \ominus\!\!\!\!<\, b \Longrightarrow (b \ominus a) \in \mathbb{Z}^+$

2) $b = a \Longrightarrow (b \ominus a) = 0$

These two statements are dissonant since (1) implies that (b - a) is a member of an equivalence class of the form $[x,0]$, while (2) implies that (b - a) is a member of an equivalence class of the form $[0,0]$. Again, recalling our equivalence relation, ⊜, (b - a) cannot be a member of two distinct equivalence classes.

III. Assume b ⧀ a <u>and</u> a = b. The student can supply a proof for this case.

Thus we see that <u>no</u> two of these can hold simultaneously; so, clearly, all three of these cannot hold simultaneously. Therefore, it must be the case that exactly one holds in any given case.

8.7 Absolute Value

For every integer x, its absolute value, denoted by $|x|$ is the non-negative number of the pair x and ^-x. Notice the absolute value of an integer is never a negative number. One can think of "absolute value" as of function defined from the set of integers onto the set of non-negative integer.

This concept can be defined formally as,

<u>Definition</u>: For any integer x,
i) $|x| = x$ when x ⧁= 0
and
ii) $|x| = {}^-x$ when x ⧀ 0.

It should be observed that the absolute value of zero is zero. When you represent integers as points on a number line,

$|x|$ is the distance of the graph of x from the origin or graph of 0.

According to the definition:

(a) $|{}^{+}2| = {}^{+}2$

(b) $|{}^{-}2| = {}^{+}2$

(c) $|{}^{-}354| = {}^{+}354$

(d) $|{}^{+}6 \oplus {}^{-}5| = {}^{+}1$

(e) $|{}^{+}5 \oplus {}^{-}6| = {}^{+}1$

(f) $|{}^{+}2 \odot {}^{-}3| = {}^{+}6$

(g) $|{}^{-}2 \odot {}^{-}3| = {}^{+}6$

EXERCISE 19

(1) Prove the cancellation law for addition of integers.

(2) Prove the cancellation law for multiplication of integers.

(3) Show that $\oplus$ is a restricted operation in $\mathbb{Z}$.

(4) Prove the transitive property for $\textcircled{>}$.

(5) Prove, for integers a, b, c, that: $a \textcircled{<} b \Rightarrow a \oplus c \textcircled{<} b \oplus c$.

(6) Prove, for integers a, b, c, that: $a \textcircled{<} b$ and $c \textcircled{>} 0 \Longrightarrow$ $a \odot c \textcircled{<} b \odot c$.

(7) Prove, for integers a, x, y, that: If $a \textcircled{<} 0$, then $x \textcircled{<} y$ $a \odot x \textcircled{>} a \odot y$.

(8) From the definition of "greater than" "given in this chapter, prove that the standard definition:

$$b \textcircled{>} a \Longleftrightarrow \exists c \in \mathbb{Z}^{+} \ni a \oplus c = b, \quad a, b, c \in \mathbb{Z},$$

is a true statement.

(9) Prove the converses of (5) and (6) above.

(10) Prove that the equation: $x^2 + 1 = 0$, has <u>no solution</u> for $x \in \mathbb{Z}$.

8.8 Isomorphism Between Naturals and Non-negative Integers

We have discussed two number systems, the set of naturals and the set of integers. You have seen how the elements of each of these sets are defined, how operations on these sets are defined, and what some of the fundamental properties of these operations are. We are now going to examine a relation which exists between the non-negative integers and the naturals.

You should first recall that the non-negative integers are closed with respect to $\oplus$ and $\odot$. In fact, to return to our formal notation for this discussion:

(1) $[a,0] \oplus [b,0] = [a + b,0]$, and

(2) $[a,0] \odot [b,0] = [a \cdot b, 0]$,

where $a, b \in \mathbb{N}$.

We are now going to show that the non-negative integers "behave" the same way as the naturals, a statement which should be somewhat apparent from (1) and (2) above. To clarify this we start with the following obvious one-to-one correspondence between naturals and non-negative integers:

Naturals		Non-negative Integers
0	⟷	[0,0]
1	⟷	[1,0]
2	⟷	[2,0]
3	⟷	[3,0]
4	⟷	[4,0]
.		.
.		.
.		.

Notice that under this correspondence the sum of two naturals corresponds to the sum of the integers. For instance:

$$\begin{array}{ccccccl} 2 & + & 3 & = & 5 & & \text{(Naturals)} \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ [2,0] & \oplus & [3,0] & = & [5,0] & & \text{(Integers)} . \end{array}$$

Or, in general:

$$\begin{array}{ccccccl} a & + & b & = & a+b & & \text{(Naturals)} \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ [a,0] & \oplus & [b,0] & = & [a+b,\ 0] & & \text{(Integers)} . \end{array}$$

The same is true for products:

$$\begin{array}{ccccccl} 2 & \cdot & 3 & = & 6 & & \text{(Naturals)} \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ [2,0] & \odot & [3,0] & = & [6,0] & & \text{(Integers)} . \end{array}$$

In general, we have:

$$\begin{array}{ccccccl} a & \cdot & b & = & a \cdot b & & \text{(Naturals)} \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ [a,0] & \odot & [b,0] & = & [a \cdot b,\ 0] & & \text{(Integers)} . \end{array}$$

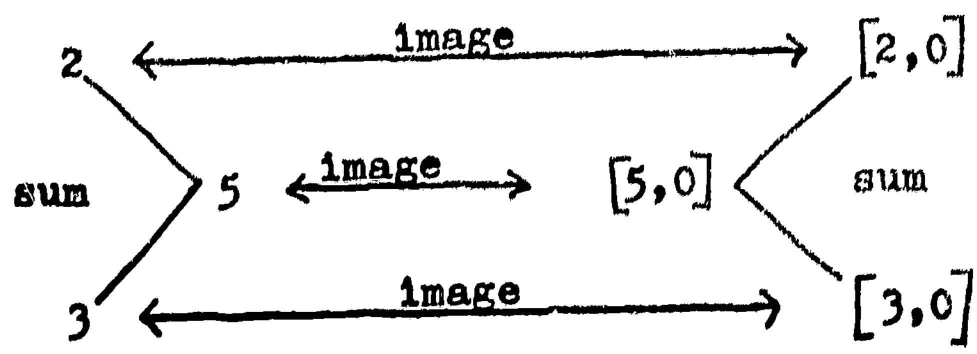
It is possible to say that under the one-to-one correspondence (which means the existence of a one-to-one function and its inverse) from the naturals onto the set of non-negative integers:

(a) the image of the sum of two naturals is the same as the sum of the images of the two naturals, and

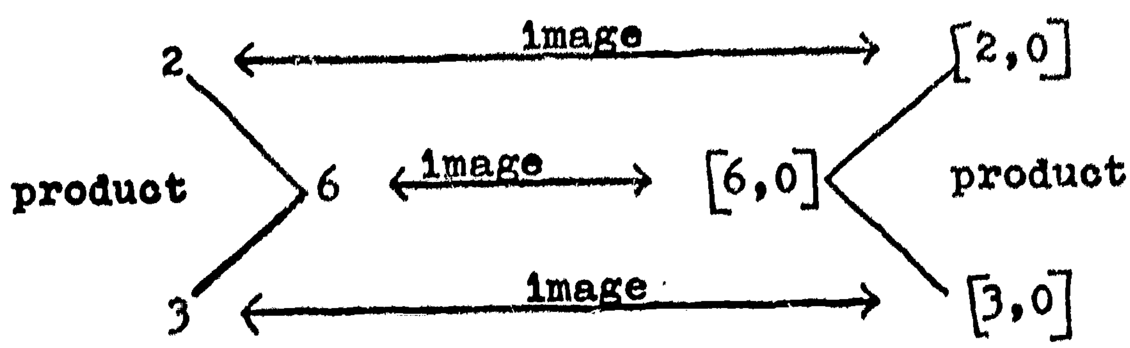
(b) the image of the product of two naturals is the same as the product of the "images of the two naturals."

Pictorially, this can be seen as the following:

ADDITION



MULTIPLICATION



When two sets are in a one-to-one correspondence and possess properties (a) and (b) above, we say the correspondence is an <u>isomorphism</u>, and the two sets are <u>isomorphic</u>. From the mathematical point of view, since these two systems (the naturals and the non-negative integers) "behave" the same way, they can be considered as indistinguishable for all practical purposes. Therefore, it is not necessary to specify whether we are dealing with naturals of non-negative integers in the following statement:

$$2 + 7 = 9.$$

Indeed, in practice we drop the positive signs if we mean integers. It is because of the isomorphism that the distinctions between the naturals and the non-negative integers fade. Because these distinctions fade, the naturals are considered to be a subset of the integers, or the negative integers are an extension of the naturals.

EXERCISE 20

(1)

+	E	0
E	E	0
0	0	E

This table shows the sums obtained when adding even numbers (E) and odd numbers (0). For example, the sum of an even number and an odd number is an odd number (E + 0 = 0).

+	0	1
0	0	1
1	1	0

This table shows the addition facts for the modules system $\{0, 1\}$. For example, 1 + 1 = 0. For the operation "+", are the two systems isomorphic? Explain your answer.

(2) The multiplication tables for both systems of problem 1, above, are:

x	E	0
E	E	E
0	E	0

x	0	1
0	0	0
1	0	1

Are the systems isomorphic for the operation of multiplication ("x")? Explain your answer.

What can you say about a general isomorphism between these two systems?

(3) Prove: $a = {}^{-}a \Rightarrow a = 0$.

(4) Show that the correspondence:

$$\begin{array}{ccc} 0 & \longleftrightarrow & 0 \\ 1 & \longleftrightarrow & {}^{-}1 \\ 2 & \longleftrightarrow & {}^{-}2 \\ 3 & \longleftrightarrow & {}^{-}3 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

is not an isomorphism.

(5) Prove: The quotient of two negative integers is positive; the quotient of a negative integer by a positive integer is a positive integer; the quotient of a positive integer by a negative integer is a negative integer.

(Assume that the quotient exists.)

(6) Given $T = \{2, 4, 6, \ldots, 2n, \ldots\}$ and $T' = \{3, 6, 9, \ldots, 3n, \ldots\}$; show that T is isomorphic to T with respect to addition.

(7) Prove that T and T′ are isomorphic with respect to multiplication.

CHAPTER IX: Rational Numbers

9.1 Introduction to Rational Numbers

The isomorphism between the non-negative integers and the naturals (see pp. 79-81) allowed us to think of these integers and naturals as being indistinguishable from each other. Because of the structural bond between these two sets, for all practical purposes the statements:

(a) $2 + 3 = 5$

(b) $(^{+}2) \oplus (^{+}3) = {}^{+}5$

are one and the same. Defined in quite different ways, the set of naturals and the set of non-negative integers, and their respective operations of addition and multiplication, turn out to be duplications of one another. For this reason, we, too, shall drop the distinguishing characteristics which we have preserved up to this point. We shall use the simpler "+" and "." to denote addition and multiplication for any two naturals or integers, and we shall omit the distinction between a positive integer and a natural. When we write, 7 we shall mean the natural or the positive integer.

The reader has observed that the naturals were limited since we could not subtract <u>any</u> two natural numbers or divide <u>any</u> two natural numbers. The set of integers was restricted by the fact that while subtraction was possible, division was not possible for <u>any</u> two integers. Because of this restriction, we shall now define and examine the set of rational numbers, a number system in which division will be possible for any two non-zero numbers. By means of an isomorphism between the integers and a subset of the rationals we shall see that the latter set can be thought of as an "extension" of the integers.

Another way to state the limitation of the integers is to say that we cannot always find an integer x which solves the equation:

$$x \cdot b = a,$$

for any two integers a and b. The set of rationals will permit solution to this equation as long as $b \neq 0$.

9.2 Definition of Rationals

We begin with the integers from which we define the rationals. Consider this subset of $\mathbb{Z} \times \mathbb{Z}$: $\mathbb{Z} \times T = \{(a,b)\ a, b \in \mathbb{Z}$, and $b \neq 0\}$. Notice that T is the set of non-zero integers.

We choose this subset for reasons which will be apparent later. Again, we define a new relation $\boxminus$:

$$(a,b) \boxminus (c,d) \Longleftrightarrow a \cdot d = b \cdot c,$$

where $\boxminus$ is the definition of equality for any two ordered pairs of $\mathbb{Z} \times T$. Note that the definition of $\boxminus$ is equivalent to $a \cdot d = b \cdot c$, where $a,b,c,d \in \mathbb{Z}$.

Examples

(1) $(2,3) \boxminus (4,6) \Longleftrightarrow 2 \cdot 6 = 3 \cdot 4$,

(2) $(^{-}2,5) \boxminus (6,^{-}15) \Longleftrightarrow (^{-}2) \cdot (^{-}15) = (5) \cdot (6)$.

The reader should think of (2,3) as $\frac{2}{3}$ and (4,6) as $\frac{4}{6}$ so that he can see the motivation behind the ordered pair development. The ordered pairs (2,3) and (4,6) will belong to the same equivalence class and, thus, we shall be able to say that (2,3) is equivalent to (4,6); i.e., that $\frac{2}{3}$ is equivalent to $\frac{4}{6}$ - a phrase common to some third grade children.

Having given you the definition of when two ordered pairs of integers are equivalent, the temptation is again great to ask you (see page 59) to develop the system of rational numbers from this definition. Again, we resist the temptation but perhaps you can give a thought to the way you would define the rationals and operations on them. We shall offer the details with your help.

First we shall ask you to do what is the next obvious step - prove that $\boxed{\equiv}$ is an equivalence relation (see EXERCISE 21-A). As you have seen earlier with the naturals and integers, the equivalence relation generates the numbers in question. You should be aware that the equivalence theorem is an important cornerstone in our development. Application of this theorem to $\mathbb{Z} \times T$, once we have the equivalence relation $\boxed{\equiv}$, partitions $\mathbb{Z} \times T$ into non-overlapping subsets called equivalence classes. The elements of each class are equivalent to each other, and are not equivalent to any elements of any other class.

The <u>standard elements</u> of each class are those ordered pairs which one might expect - those which represent the fractional form in lowest terms. Thus, (2,3) is the standard element of the class containing (2,3), (4,6), (8,12), etc. In short, the <u>standard element</u> of each equivalence class is that ordered pair whose two elements have the integer $^{+}1$ and $^{-}1$ as their only common divisors. Since two ordered pairs qualify for standard pairs, according to this criterion (such as (2,3) and ($^{-}2$, $^{-}3$), or ($^{-}2$,3) and (2,$^{-}3$),) this rule shall be adopted:

(1) Both elements will be positive, or

(2) The first element will be negative and the second element will be positive.

In the case of the class containing (0,1), (0,$^{-}1$), (0,2), (0,$^{-}2$), (0,3), (0,$^{-}3$), etc., the standard element will be (0,1).

Now we can present a table of <u>some</u> equivalence classes:

...	$(^-1,2)$	$(^-2,5)$	$(^-1,3)$	$(^-3,10)$	$(0,1)$	$(3,10)$	$(1,3)$	...
...	$(1,^-2)$	$(2,^-5)$	$(1,^-3)$	$(3,^-10)$	$(0,^-1)$	$(^-3,^-10)$	$(^-1,^-3)$	...
...	$(^-2,4)$	$(^-4,10)$	$(^-2,6)$	$(^-6,20)$	$(0,2)$	$(6,20)$	$(2,6)$	...
...	$(2,^-4)$	$(4,^-10)$	$(2,^-6)$	$(6,^-20)$	$(0,^-2)$	$(^-6,^-20)$	$(^-2,^-6)$	...
...	$(^-3,6)$	$(^-6,15)$	$(^-3,9)$	$(^-9,30)$	$(0,3)$	$(9,30)$	$(3,9)$	...
...	$(3,^-6)$	$(6,^-15)$	$(3,^-9)$	$(9,^-30)$	$(0,^-3)$	$(^-9,^-30)$	$(^-3,^-9)$	...
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

Each of the columns or classes generated by $\boxed{\equiv}$ over $\mathbb{Z} \times T$ is a rational number. We define each rational number to be a different class. The set of rationals is the set of all these classes. The rational number two-thirds is: $\{(a,b) \mid (a,b) \boxed{\equiv} (2,3)\}$; the rational number one-half is: $\{(m,n) \mid (m,n) \boxed{\equiv} (1,2)\}$. We shall shorten this set notation again and use $[(2,3)]$ to stand for the class of all ordered pairs of integers equivalent to (2,3). Remember, although we use the brackets, [...] , to denote an equivalence class as when we defined the integers, we are now speaking of equivalence classes of ordered pairs of integers, not ordered pairs of naturals. Although we chose to denote positive two-thirds by $[(2,3)]$, we can also use, among others, $[(^-2,^-3)]$ or $[(4,6)]$ to represent the same rational number. Thus, we can write these statements:

$$[(2,3)] = [(^-2,^-3)] = [(4,6)] \text{ , and}$$
$$[(1,2)] = [(2,4)] = [(^-1,^-2)] \text{ ,}$$

where the equality sign is used to denote equality of <u>sets of</u>

ordered pairs.

We shall call the equivalence class $[(1,2)]$ positive one-half; $[(^-1,2)]$ negative one-half; $[(2,5)]$ positive two-fifths; and $[(^-2,5)]$ negative two-fifths; $[(3,10)]$ positive three-tenths; and $[(^-3,10)]$ negative three-tenths. Note, also, that there will be equivalence classes such as $[(1,1)]$, $[(2,1)]$, $[(3,1)]$, ... ; and $[(^-1,1)]$, $[(^-2,1)]$, $[(^-3,1)]$, ... We shall call those rational numbers positive one, positive two, positive three, etc.; and negative one, negative two, negative three, etc. The class $[(0,1)]$ will be called the rational number 0.

If we let $\mathbb{Q}$ stand for the set of rational numbers, then:

$$\mathbb{Q} = \{[(a,b)] \mid (a,b) \text{ is a standard element}\}.$$

9.3 Addition of Rationals

Definition: Let $x = [(a,b)]$ and $y = [(c,d)]$ be two rational numbers. The sum of x and y, denoted by $x \boxplus y$, is defined by:

$$x \boxplus y = [(a\cdot d + b\cdot c,\ b\cdot d)].$$

In short, $[(a,b)] \boxplus [(c,d)] = [(a\cdot d + b\cdot c,\ b\cdot d)]$.

According to the definition, we are "adding" equivalence classes - that is, the operation $\boxplus$ is a binary operation on $\mathbb{Q} \times \mathbb{Q}$. The operation $\boxplus$ is defined quite differently from + for naturals and $\oplus$ for integers. Note that $(a\cdot d + b\cdot c,\ b\cdot d)$ is a result of multiplying and adding integers.

Recall that a binary operation is a function from a set A into a set B. To insure that the operation $\boxplus$ is a bona fide binary operation the following property must be proven for arbitrary rational numbers:

Let r, s, and t be rational numbers such that if $r = s$, then $r \boxplus t = s \boxplus t$.

In order to prove this property consider r, s, and t as the rational numbers defined by the equivalence classes $[\![(a,b)]\!]$, $[\![(c,d)]\!]$, and $[\![(e,f)]\!]$, respectively.

Consider the following sums:

$$r \boxplus t = [\![(a,b)]\!] \boxplus [\![(e,f)]\!] = [\![(a \cdot f + b \cdot e,\ b \cdot f)]\!],$$

$$\text{and} \quad s \boxplus t = [\![(c,d)]\!] \boxplus [\![(e,f)]\!] = [\![(c \cdot f + d \cdot e,\ d \cdot f)]\!].$$

In order to prove the desired property these two sums must be equal. The proof will start with the desired conclusion and work to a point where equality of known facts is reached. This procedure is acceptable since each step in the proof can be reversed, i.e., if one so desires he can copy the steps of the proof in the reversed order and be able to supply a legitimate reason for each step.

<u>Proof</u>:

$$\text{(i)} \quad [\![(a \cdot f + b \cdot e,\ b \cdot f)]\!] = [\![(c \cdot f + d \cdot e,\ d \cdot f)]\!]$$
$$(a \cdot f + b \cdot e,\ b \cdot f) \boxminus (c \cdot f + d \cdot e,\ d \cdot f)$$

Now by definition (page 85),

$$\text{(ii)} \quad (a \cdot f + b \cdot e)\, d \cdot f = (c \cdot f + d \cdot e)\, b \cdot f .$$

It should be recognized that (ii) is an equality in $\mathbb{Z}$. Hence, all the known properties of elements and operations pertaining to integers can be used. Also, observe that b, d, and f are non-zero integers; thus, (ii) can be written in the following form:

$$\text{(iii)} \quad a \cdot d \cdot f \cdot f + b \cdot d \cdot e \cdot f = b \cdot c \cdot f \cdot f + b \cdot d \cdot e \cdot f .$$

By the cancellation properties of the integers (iii) becomes:

$$\text{(iv)} \quad a \cdot d = b \cdot c .$$

Statement (iv) is known to be true by hypothesis, since:

$$\text{(v)} \quad r = s \Rightarrow [\![(a,b)]\!] = [\![(c,d)]\!] \Rightarrow (a,b) \boxminus (c,d) \Rightarrow a \cdot d = b \cdot c.$$

This proves the property in question, so ⊞ is a bona fide binary operation. But, by the remarks made earlier if so possessed you could begin with statement (v) and work backward to statement (i).

Examples

(1) $[\!(2,3)\!] \boxplus [\!(1,3)\!] = [\!(2\cdot 3 + 3\cdot 1,\ 3\cdot 3)\!]$ or

$[\!(2,3)\!] \boxplus [\!(1,3)\!] = [\!(9,9)\!]$ or

$[\!(2,3)\!] \boxplus [\!(1,3)\!] = [\!(1,1)\!]$.

(In everyday language we would write: $\left({}^{+}\frac{2}{3}\right) + \left({}^{+}\frac{1}{3}\right) = {}^{+}1$).

(2) $[\!({}^{-}1,2)\!] \boxplus [\!({}^{-}3,5)\!] = [\!({}^{-}1\cdot 5 + 2\cdot {}^{-}3,\ 2\cdot 5)\!]$ or

$[\!({}^{-}1,2)\!] \boxplus [\!({}^{-}3,5)\!] = [\!({}^{-}11,\ 10)\!]$.

(In everyday language we would write: $\left({}^{-}\frac{1}{2}\right) + \left({}^{-}\frac{3}{5}\right) = \left({}^{-}\frac{11}{10}\right)$.)

As an exercise (see EXERCISE 21-A) the reader is asked to show that $\mathbb{Q}$ is closed with respect to addition and that ⊞ is a commutative and associative operation. Also, we ask the reader to show that $[\!(0,1)\!]$ is <u>an additive identity</u>. Furthermore, he will show that for each rational number, $[\!(a,b)\!]$, there is a rational number $[\!(c,d)\!]$ such that :

$$[\!(a,b)\!] \boxplus [\!(c,d)\!] = [\!(0,1)\!] ,$$

that is, that each rational number has an <u>additive inverse</u>.

<u>EXERCISE 21-A</u>

(1) Prove that ⊜ is an <u>equivalence relation</u>.

(2) Show that $\mathbb{Q}$ is <u>closed</u> with respect to ⊞ .

(3) Prove that ⊞ is <u>commutative and associative</u>.

(4) Prove that $[\!(0,1)\!]$ is the <u>additive identity</u> for $\mathbb{Q}$.

(5) Find the <u>additive inverse</u> for an arbitrary rational number, $[\![(a,b)]\!]$.

(6) Show how the additive inverse is needed and used to solve:

$$\frac{1}{7} + x = 35.$$

9.4 Additive Inverses for Rationals

In EXERCISE 21-A it was shown that $[\![(0,1)]\!]$ is the additive identity for the set of rational numbers, i.e., for each $[\![(a,b)]\!] \in \mathbb{Q}$:

$$[\![(a,b)]\!] \boxplus [\![(0,1)]\!] = [\![(a,b)]\!].$$

Does each rational number have an additive inverse, or a negative? We found the additive inverse for each rational number in EXERCISE 21-A and thus we have this theorem:

<u>Theorem</u>: An additive inverse of $[\![(a,b)]\!]$ is $[\![(^{-}a,b)]\!]$.

The proof is accomplished by means of actual computation, recalling that ^{-}a stands for the negative of the integer a.

$$[\![(a,b)]\!] \boxplus [\![(^{-}a,b)]\!] = [\![(a \cdot b + b \cdot {}^{-}a,\ b \cdot b)]\!],$$

which is equivalent to $[\![(0,1)]\!]$. This can be recognized by using properties of $\mathbb{Z}$ to simplify the first element of the ordered pair, i.e., $b(a + {}^{-}a)$ which is $b \cdot 0 = 0$. Thus,

$$[\![(a,b)]\!] \boxplus [\![(^{-}a,b)]\!] = [\![(0,1)]\!].$$

The uniqueness of the additive inverse will be proven on page 100.

9.5 Multiplication of Rational Numbers

<u>Definition</u>: Let $x = [\![(a,b)]\!]$ and $y = [\![(c,d)]\!]$ be two rational numbers. The product of x and y, denoted by $x \boxdot y$, is defined by:

$$x \boxdot y = (a \cdot c,\ b \cdot d).$$

In short, $[\![(a,b)]\!] \boxdot [\![(c,d)]\!] = [\![(a \cdot c,\ b \cdot d)]\!]$.

(Again, we remind the reader to think of (a,b) as $\frac{a}{b}$ and (c,d) as $\frac{c}{d}$.)

Example:

$$[\![(^{-}3,4)]\!] \boxdot [\![(2,3)]\!] = [\![(^{-}3\cdot 2,\ 4\cdot 3)]\!] \text{ or}$$
$$[\![(^{-}3,4)]\!] \boxdot [\![(2,3)]\!] = [\![(^{-}6,12)]\!] = [\![(^{-}1,\ 2)]\!].$$

As was the case with the operation of $\boxplus$, it is necessary to examine the following property for $\boxdot$:

> If r, s, and t are rational numbers such that $r = s$, then $r \boxdot t = s \boxdot t$.

The reasoning behind the proof will parallel that described for the similar property of $\boxplus$ given on page 89.

Proof:

Let r, s, and t be the rational numbers defined by the equivalence classes $[\![(a,b)]\!]$, $[\![(c,d)]\!]$, and $[\![(e,f)]\!]$, respectively. Consider the following products:

$$r \boxdot t = [\![(a,b)]\!] \boxdot [\![(e,f)]\!] = [\![(a\cdot e,\ b\cdot f)]\!] \text{, and}$$
$$s \boxdot t = [\![(c,d)]\!] \boxdot [\![(e,f)]\!] = [\![(c\cdot e,\ d\cdot f)]\!].$$

We claim that

$$[\![(a\cdot e,\ b\cdot f)]\!] = [\![(c\cdot e,\ d\cdot f)]\!] \text{, that is,}$$
$$(a\cdot e,\ b\cdot f) \boxed{\equiv} (c\cdot e,\ d\cdot f) \Longrightarrow a\cdot e\cdot d\cdot f = c\cdot e\cdot b\cdot f.$$

Now, this is a statement about integers, with b, d, e, and f being non-zero integers. Thus,

$$a\cdot e\cdot d\cdot f = c\cdot e\cdot b\cdot f$$

by the cancellation property of the product of integers yields

$$(1)\quad a\cdot d = c\cdot b.$$

Statement (1) is known to be true from the hypothesis, since $r = s \Rightarrow [\![(a,b)]\!] = [\![(c,d)]\!] \Rightarrow$
$(a,b) \boxed{\equiv} (c,d) \Rightarrow a \cdot d = b \cdot c$.

Since this process can be reversed see page 89 to construct a rigorous line of reasoning, we have proven the property in question. Hence, $\boxdot$ is a binary operation.

The reader again will inspect the properties of this new operation, $\boxdot$. He will show (see EXERCISE 21-B) that $\mathcal{Q}$ is <u>closed</u> with respect to $\boxdot$, that $\boxdot$ is <u>commutative and associative</u>, and that $[\![(1,1)]\!]$ is a <u>multiplicative identity</u>:

$$[\![(a,b)]\!] \boxdot [\![(1,1)]\!] = [\![(a \cdot 1,\ b \cdot 1)]\!] = [\![(a,b)]\!] .$$

The distributive property of multiplication over addition, which the reader may wish to verify (see EXERCISE 21-B), also holds for the rational numbers:

$$[\![(a,b)]\!] \boxdot \left([\![(c,d)]\!] \boxplus [\![(e,f)]\!] \right) = \left([\![(a,b)]\!] \boxdot [\![(c,d)]\!] \right) \boxplus \left([\![(a,b)]\!] \boxdot [\![(e,f)]\!] \right) .$$

<u>EXERCISE 21-B</u>

(1) Show that $\mathcal{Q}$ is <u>closed</u> with respect to $\boxdot$.

(2) Prove that $\boxdot$ is <u>commutative</u> and <u>associative</u>.

(3) Show that $[\![(1,1)]\!]$ is the <u>multiplicative identity</u> for $\mathcal{Q}$.

(4) Verify the distributive property for $\mathcal{Q}$:

$$[\![(a,b)]\!] \boxdot \left([\![(c,d)]\!] \boxplus [\![(e,f)]\!] \right) = \left([\![(a,b)]\!] \boxdot [\![(c,d)]\!] \right) \boxplus \left([\![(a,b)]\!] \boxdot [\![(e,f)]\!] \right) .$$

9.6 <u>Multiplicative Inverses for Rational Numbers</u>

The number $[\![(1,1)]\!]$ is the multiplicative identity for $\mathcal{Q}$ because $[\![(a,b)]\!] \boxdot [\![(1,1)]\!] = [\![(a,b)]\!]$ for any rational number $[\![(a,b)]\!]$. Although the only integers for which there were multiplicative inverses are $^{+}1$ and $^{-}1$, all rational numbers except $[\![(0,1)]\!]$ have multiplicative inverses. To show this consider the definition of multiplicative inverse for any rational number, $[\![(a,b)]\!]$:

$$[\![(a,b)]\!] \boxdot [\![(c,d)]\!] = [\![(1,1)]\!].$$

The rational number $[\![(c,d)]\!]$ will be a multiplicative inverse for $[\![(a,b)]\!]$ if this equation holds. This equation is equivalent to:

$$[\![(a\cdot c,\ b\cdot d)]\!] = [\![(1,1)]\!].$$

These two sets are equal if and only if the ordered pairs $(a\cdot c,\ b\cdot d)$ and $(1,1)$ are equivalent.

$$(a\cdot c,\ b\cdot d) \boxminus (1,1) \Longleftrightarrow (a\cdot c)\cdot 1 = (b\cdot d)\cdot 1, \text{ or } a\cdot c = b\cdot d .$$

Clearly, an instance when these pairs will be equivalent is when $c = b$ and $d = a$. Thus <u>a</u> multiplicative inverse of $[\![(a,b)]\!]$ is $[\![(b,a)]\!]$.

<u>Examples:</u>

(1) The multiplicative inverse of $[\![(^{-}2,3)]\!]$ is $[\![(3,^{-}2)]\!]$ or $[\![(^{-}3,2)]\!]$, in standard form.

(2) Each rational number except $[\![(0,1)]\!]$ is a multiplicative inverse and has a multiplicative inverse.

(3) The multiplicative inverse of $[\![(a,1)]\!]$ is $[\![(1,a)]\!]$.

We shall call the multiplicative inverse by a special name - <u>the reciprocal</u> - just as we gave a special name - the negative - to the additive inverse. The fact that $[\![(0,1)]\!]$ does not have a reciprocal is left for the reader to show (EXERCISE 21-C).

The uniqueness of the multiplicative inverse will be proved on page 100-101.

Thus, the rationals have one property which the integers do not possess - each rational number, except zero, has a multiplicative inverse. This additional characteristic allows division to be a non-restrictive operation since division is just multiplication in reverse. It is this property which permits the solution of such equations as:

$$7 \cdot x = 84,$$

since the multiplicative inverse allows us to "isolate" x to yield the equivalent equation:

$$\frac{1}{7} \cdot 7 \cdot x = \frac{1}{7} \cdot 84\text{ , or}$$

$$1 \cdot x = \frac{1}{7} \cdot 84\text{ , or}$$

$$x = \frac{84}{7}$$

The use of the multiplicative inverse with rational numbers to solve equations is known to many 7th and 8th graders.

EXERCISE 21-C

(1) Prove that every rational number, except $[(0,1)]$, has a multiplicative inverse.

(2) Show how the multiplicative inverse is used to solve: $23 \cdot x = 72$.

(3) Show how the additive and multiplicative inverses are used to solve: $3x + 8 = 73$.

(4) Why doesn't $[(0,1)]$ have a multiplicative inverse?

9.7 Isomorphism Between Integers and Subset of Rationals

Consider all rationals in the standard form $[(a,1)]$. We can set up an obvious one-to-one correspondence between this subset of $\mathcal{Q}$ and the set of integers:

Integers		Subset of Rationals
p	$\longleftrightarrow$	$[(p,1)]$

Because of our definition of addition and multiplication of rationals:

$$[\![(p,1)]\!] \boxplus [\![(q,1)]\!] = [\![(p+q,\ 1)]\!] \text{ , and}$$

$$[\![(p,\ 1)]\!] \boxdot [\![(q,1)]\!] = [\![(p \cdot q,\ 1)]\!] .$$

Thus we see that under the above correspondence the sum of two integers p + q corresponds to the sum of the respective rational numbers:

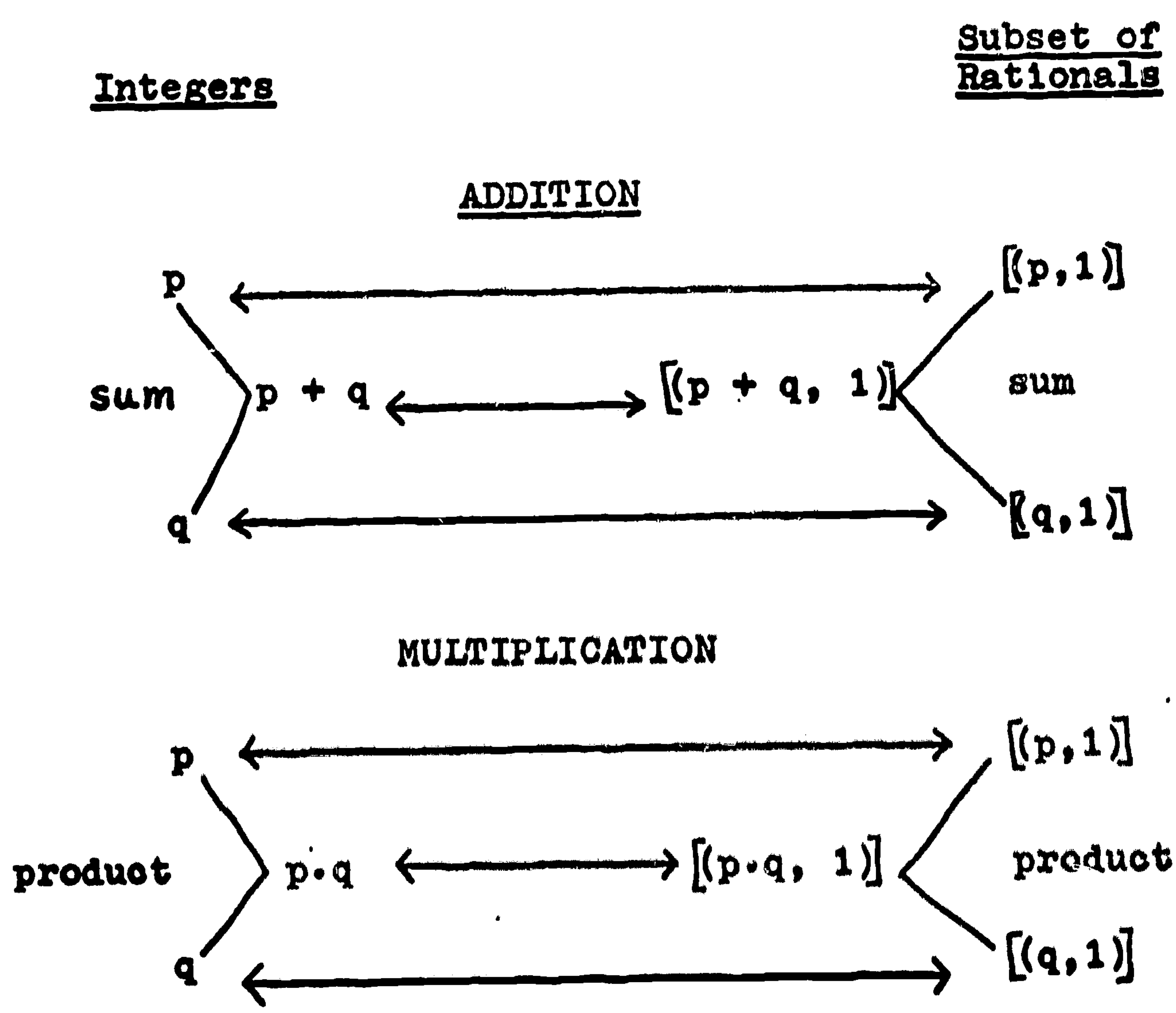
$$p + q \longleftrightarrow [\![(p+q,\ 1)]\!] .$$

Similarly, for multiplication:

$$p \cdot q \longleftrightarrow [\![(p \cdot q\ ,\ 1)]\!] .$$

We can now conclude that under the one-to-one correspondence above, the sum of the images of two integers is equal to the image of the sum of two integers; the product of the images of two integers is equal to the image of the product of two integers.

Here are the results in a diagram:



What these diagrams show is that the correspondence: $p \longleftrightarrow [(p,1)]$ from the set of integers into $\mathbb{Q}$ is an isomorphism. Because of the isomorphism, the integers and a subset of $\mathbb{Q}$, $\{[(p,1)] \mid p$ is an integer$\}$, behave in the same manner. For all practical purposes the integer $^{-}3$ and the rational $[(^{-}3,1)]$ are indistinguishable. For this reason, in practice, the symbol "$^{-}3$" is sufficient for both the integer and the rational number. In fact, as the reader will detect, because of the previous isomorphism he studies between the naturals and the non-negative integers, the symbol "5" can stand for a natural, an integer, or a rational number. We can't tell what "5" stands for, but the two isomorphisms say that it doesn't matter.

CHAPTER X: Properties of Rationals

10.1 Notation for Rationals

We shall again drop the cumbersome notations which was useful in our theoretical work, but will only get in the way in our practical work. We shall denote the rational number $[(a,b)]$ by $\frac{a}{b}$. Furthermore, we shall again drop the special symbols for the operations ⊞ and ⊡ and use simply, + and ·.

The isomorphisms allow us to think of the integers as a subset of the rationals, and the naturals as a subset of the integers. Thus, + and · will serve all numbers.

The operations on two rationals now become:

$$\frac{x}{y} + \frac{a}{b} = \frac{xb + ya}{yb}$$

$$\frac{x}{y} \cdot \frac{a}{b} = \frac{xa}{yb}$$

The reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$. In particular, the reciprocal of x is $\frac{1}{x}$ and vice-versa. We have this equation:

$$\frac{x}{1} \cdot \frac{1}{y} = \frac{x}{y}$$

This equation tells us that every rational number is the product of an integer and the reciprocal of an integer.

We call $\frac{a}{b}$ a fraction, which means that a fraction stands for a rational number or an equivalence class. A fraction is a symbol, not a number. We call a the numerator of this symbol and b the denominator; that is, numerators and denominators are symbols, too.

The symbol ^{-}a means the negative of the rational number a ; $^{-}\left(\frac{x}{y}\right)$ means the negative of the rational number represented by $\frac{x}{y}$. Both of these equations will be true:

$$a + (^{-}a) = 0$$

$$\frac{x}{y} + {}^{-}\left(\frac{x}{y}\right) = 0$$

10.2 Uniqueness of Inverses

We now prove the uniqueness of the additive inverse of each rational number, r. Suppose r' and r'' are rational numbers, and are two different additive inverses of r; that is:

$$r + r' = 0$$

$$r + r'' = 0$$

Therefore,

$$r + r' = r + r''.$$

If we add r' (r'' would do, also) to both sides of this last equation, we obtain:

$$(r' + r) + r' = (r' + r) + r'', \text{ or}$$

$$0 + r' = 0 + r'', \text{ which is equivalent to}$$

$$r' = r''$$

This proves our assertion that r' (or r'') is the unique additive inverse.

Likewise, we prove the uniqueness of the multiplicative inverse. Suppose that s' and s'' are two rational numbers and they are different multiplicative inverses for the (non-zero) rational number r. Therefore,

$$r \cdot s' = 1$$

$$r \cdot s'' = 1, \text{ and}$$

$$r \cdot s' = r \cdot s''.$$

Multiply each side of this last equation by s' (s'' would do, also) to obtain:

$$(s' \cdot r) \cdot s' = (s' \cdot r) \cdot s'' \text{, or}$$

$$1 \cdot s' = 1 \cdot s'' \text{, which is equiva-}$$

lent to

$$s' = s'' .$$

Hence, the multiplicative inverse is unique.

10.3 Subtraction and Division

We shall define these operations a little differently from before.

Definition: For any two rational numbers, a and b, $a \boxminus b$ is defined by:

$$a \boxminus b = a \boxplus (^{-}b) .$$

Thus, we define subtraction by the addition of a negative.

Definition: For any two rational numbers, a and b, $a \boxed{\div} b$ is defined by:

$$a \boxed{\div} b = a \boxdot \frac{1}{b} .$$

We define division by the multiplication of a reciprocal.

It is clear that by the definition of division:

$$a \boxed{\div} b = \frac{a}{b} .$$

NOTE: Again, we shall drop the particular symbols $\boxminus$ and $\boxed{\div}$ for subtraction and division of two rationals, respectively. From now on, we shall use $-$ and $\div$ to stand for subtraction and division of rationals.

Examples:

(1) $\frac{2}{3} - \frac{5}{8} = \frac{2}{3} + \frac{^{-}5}{8} = \frac{16 + (^{-}15)}{24} = \frac{1}{24}$.

(2) $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{12}{15} = \frac{4}{5}$

(3) $\dfrac{\frac{7}{8}}{\frac{2}{5}} = \frac{7}{8} \div \frac{2}{5} = \frac{7}{8} \cdot \frac{5}{2} = \frac{35}{16} = 2\frac{3}{16}$.

The only restriction imposed upon division is that division by 0 is impossible (see Exercise 22.) As mentioned earlier we can now solve these equations:

$$ax = b,$$

where a and b are rational numbers, and $a \neq 0$.

10.4 Ordering the Rationals

We define the set of positive rationals as all those rationals $[\![(a,b)]\!]$, in standard form, such that $a > 0$ means the integer a is greater than 0. Recall that in standard form b is always greater than 0 .

Definition: The rational number x is greater than y (in symbols, $x \boxed{>} y$) if and only if $x - y$ is positive. The expression $y \boxed{<} x$ is equivalent to $x \boxed{>} y$, and is read "x is greater than y."

Note that if the rational number r is positive, then $r \boxed{>} 0$. The negative rationals are those in standard form $[\![(a,b)]\!]$ where $a < 0$ (as an integer.)
Note that if r, a rational, is negative then $0 \boxed{>} r$ or $r \boxed{<} 0$ (r is less than 0).

It should be clear from the formation of rational numbers and their standard forms that a rational is either positive, negative, or zero (Law of Trichotomy.)

Again, we shall drop the new notation $\boxed{<}$ and $\boxed{>}$ and use the symbols $<$, $>$ again because of the isomorphisms.

EXERCISE 22

(1) Use the definition of division to show why division by 0 is impossible.

(2) Prove the cancellation laws for addition and multiplication of rationals:

(a) For $a, b, c \in Q$, $a \boxplus b = a \boxplus c \Longrightarrow b = c$;

(b) For $a, b, c \in Q$, $a \neq 0$, $a \boxdot b = a \boxdot c \Longrightarrow b = c$.

(3) Show that the definitions of subtraction and division of rationals are equivalent to previous definitions used with integers and naturals.

(4) Convince yourself that Numbers 4, 5, 6, 7, 8, and 9 of Exercise 19 also hold for the rationals.

(5) Show that there is no smallest positive rational number.

(6) Show that if $\frac{a}{b} > 0$ then there exists a natural number, n, such that:

$$0 < \frac{1}{n} < \frac{a}{b}.$$

*(7) Define the absolute value of r as follows:

$$|r| = r, \text{ if } r \geq 0,$$

$$|r| = {}^{-}r, \text{ if } r < 0.$$

(a) Show that $|0| = 0$.

(b) If $|r| = 0 \Longrightarrow r = 0$.

(c) $|{}^{-}r| = |r|$.

(d) $|rs| = |r| \cdot |s|$.

(e) $|r + s| \leq |r| + |s|$. (Triangle Inequality)

*(8) Why is this equation not solvable for rationals:

$x^2 = 2$? (Look up a proof in a textbook)

(9) (a) Prove that the sum of two positive rationals is positive.

(b) Prove that the sum of two negative rationals is negative.

(10) (a) Prove that the product of two positive rationals is positive.

(b) Prove that the product of two negative rationals is positive.

(c) Prove that the product of a positive rational and a negative rational is negative.

10.5 Converse of Canellation

If the proofs that binary operations $\boxplus$ and $\boxdot$ are well defined (pp. 89 and 92), are examined, it will be discovered that the properties under consideration are precisely the converses of the cancellation laws.

The converses of the cancellation laws are:

For r, s, and t rational numbers such that:

(a) If $r = s$, then $r \boxplus t = s \boxplus t$, and

(b) If $r = s$, then $r \boxdot t = s \boxdot t$.

Hence, there is no need to give proofs of these converses since they follow from the definitions of the binary operations of $\boxplus$ and $\boxdot$.

10.6 Further Properties of the Rational Numbers

Theorem: Let a and b be rational numbers. If $a < b \Rightarrow$ there exists a rational number c such that $a < c < b$.

Proof: We shall show that $\frac{a+b}{2}$ is a suitable choice for c. First, show that $\frac{a+b}{2} < b$. We must prove that $b - \frac{a+b}{2}$ is positive OR that $\frac{b-a}{2}$ is positive. But, $b - a$ is positive, by hypothesis. The rational number $\frac{1}{2}$ is also positive. Thus, the product $\frac{1}{2} \cdot (b - a) = \frac{b-a}{2}$ is positive (see Exercise 22, Number 10.) Now, $a < \frac{a+b}{2} \Leftrightarrow \frac{a+b}{2} - a$ is positive. The number $\frac{a+b}{2} - a$ is equal to $\frac{b-a}{2}$ which we have just shown to be positive. Therefore, $\frac{a+b}{2} - a$ is also positive and $\frac{a+b}{2} > a$. Thus, we have shown $a < \frac{a+b}{2}$ and $\frac{a+b}{2} < b$. So let $c = \frac{a+b}{2}$ and we have shown the existence of a (rational) c such that $a < c < b$.

It follows from the above theorem that there are an infinite number of rational numbers between any two rationals (Why?)

10.7 Decimal Equivalence of Fractions

The reader is familiar with the decimal equivalent of fractions. For example, to determine the decimal equivalent for $\frac{16}{7}$, we interpret this fraction as

$16 \div 7$:

```
      2.285714
   7)16.000000
     14
      20
      14
       60
       56
        40
        35
         50
         49
          10
           7
          30
          28
           2
```

It is clear from this process of long division that $\frac{16}{7}$ has a non-terminating decimal equivalent, i.e., this decimal equivalent does not have a zero remainder. But, while $\frac{16}{7}$ does have a non-terminating decimal equivalent, the decimal repeats after every six decimal places. The digits 285714 repeat indefinitely. The reader is already familiar with decimal equivalents of fractions which terminate and with those which repeat. For example, $\frac{1}{3}$ is a non-terminating repeating decimal, while $\frac{1}{4}$ is a terminating decimal.

A little inspection of the long division process will reveal that all rational numbers can be represented by either terminating decimals or by non-terminating repeating decimals (Do you see why?)

On the other hand, <u>all</u> repeating non-terminating decimals and terminating decimals represent rational numbers. Here are two examples which will indicate how this statement can be proved generally.

Consider the decimal 28.143 . If we multiply this decimal by $\frac{1,000}{1,000}$ then $28.143 = 28.143 \cdot \frac{1,000}{1,000} = \frac{28,143}{1,000}$, a rational number.

Next, examine the decimal: 24.1234 1234 1234 ...., a repeating non-terminating decimal. Let N denote this number, i.e.:

$$\text{(a)} \qquad N = 28.1234\ 1234\ 1234\ \ldots.$$

If we multiply this decimal by 10,000, then:

$$\text{(b)}\ 10,000\ N = 241,234.1234\ 1234\ 1234\ \ldots.$$

Subtract (a) from (b) : $9,999\ N = 241,210$,

$$\text{(c)} \qquad N = \frac{241,210}{9,999}.$$

N is now expressed as a fraction which represents a rational number. We call the student's attention to the link between the number of decimal places which repeat (4 in the number, N, above) and the power of 10 used to multiply this decimal (4 to get 10^4 or 10,000). To show that the number N can be expressed in fractional form, it is necessary to multiply using this rule.

The conclusion that we reach upon inspection of decimal equivalents for rational numbers is that every rational can be represented by a decimal equivalent which is periodic (non-terminating) or is terminating, and conversely.

10.8 Geometric Construction

Given a straight line L, it is possible to represent the rational number 0 (at any point called the origin of the line,) the positive rationals (to the right of the origin), and the negative rationals (to the left of the origin) on L. To accomplish this job of identifying points for rational numbers, one needs a unit of length and an orientation (choose one of the half-lines to be positive, the other to be negative). By this process of measurement, each rational number $\frac{a}{b}$ can be associated with a point P of L. We shall soon see that each point P of L is not (generally) associated with a rational number. What we have is a one-to-one correspondence from $\mathbb{Q}$ into L. The range of this correspondence from $\mathbb{Q}$ into L is the set of points whose distances from the origin are measurable by ruler methods.

10.9 Field Properties of the Rationals

The reader may be familiar with the properties of a field. He should realize that $\mathbb{Q}$ satisfies all these properties for + and · :

(1) Closure, uniqueness for + and ·.

(2) Commutative Properties for + and ·.

(3) Associative Properties for + and ·.

(4) Identity Elements (Additive and Multiplicative)

(5) Distributive Property for · over + .

(6) Inverse Elements (Additive and Multiplicative)

The set of integers $\mathbb{Z}$ satisfies all these except the possession of a multiplicative inverse for each of its non-zero elements. The presence of these elements among the rationals makes division possible.

The set of rational numbers ($\mathbb{Q}$) satisfies these properties:

(1) Trichotomy: for any given rational $\frac{a}{b}$, $^{-}\left(\frac{a}{b}\right)$ is positive, or $\frac{a}{b}$ is positive, or $\frac{a}{b}$ is 0.

(2) The sum of two positive rationals is positive.

(3) The product of two positive rationals is positive.

The student may wish to verify (1) as an exercise; the student has verified (2) and (3) in Exercise 22, Numbers 9 and 10.

10.10 Well Ordered

Any set containing these properties is called an ordered set. Thus, $\mathbb{Z}$ is an ordered set. $\mathbb{Q}$ is an ordered set. Since $\mathbb{Q}$ is a field,

$\mathbb{Q}$ is called an ordered field.

Definition: If every non-empty subset S_1 of set S contains a smallest element, then S is well-ordered.

From this definition, one can see that the set of positive integers is well-ordered. The set of negative integers is not well-ordered. Nor is the set of positive rationals or negative rationals.

Note that the well-ordering property is a property which describes the nature of (infinite) sets. It is a property which is "outside" of the elements and hence is non-algebraic, as opposed to, say, the commutative property which deals with the structure of the set and is an algebraic property.

CHAPTER XI: Real Numbers

11.1 Introduction to the Real Numbers

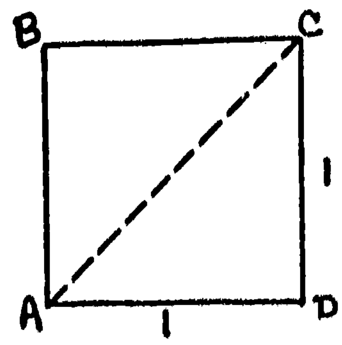
By now the student is familiar with the method which was employed to generate the different number systems. First, an equivalence relation is defined using only those elements which have been previously defined. The equivalence theorem applied to the elements of the equivalence relation partitions these elements into equivalence classes. These classes are defined to be the numbers.

We shall again use this procedure to arrive at the set of real numbers, although there will be some modifications enroute. The set of rational numbers satisfies many properties. Subtraction and division are possible as long as we do not divide by zero. Another way of saying this last statement is to state that additive and multiplicative inverses exist for each rational number, except that 0 does not have a multiplicative inverse.

Rational numbers are usually represented by fractions and quite often by decimal numerals, or decimals. Every rational number can be expressed either as a terminating decimal (such as .75) or as a repeating non-terminating decimal (such as .743743743...). And vice-versa, each terminating decimal and each repeating, non-terminating decimal stands for a rational number. As we shall soon see, non-repeating non-terminating decimals do not represent rational numbers. They represent numbers which are called irrational numbers, some of which are familiar to school children such as π , $\sqrt{2}$ and $\sqrt{3}$. Each of these numbers cannot be represented by repeating decimals. The irrational numbers, in a sense, serve to "enlarge" the rational number system, just as the non-integral rational numbers enlarged the set of integers. Together, the rationals and the irrational numbers comprise the real number system. Our goal in this chapter is to define the real numbers and show (through an ismorphism) how the rational numbers can be thought of as a subset of the real numbers.

At this point, we would like to leave the reader with an additional thought concerning the decimal equivalents for rational and irrational numbers. The set of irrational numbers consists of all numbers which can be represented by non-terminating, non-repeating decimals. It can be shown that this set has more elements in it than the set of rational numbers $\mathbb{Q}$. (see Appendix B for a proof.) By "more elements" we mean that it is impossible to find a one-to-one correspondence between the set of irrational numbers and $\mathbb{Q}$. From the results in Appendix B, it will be noticed that neither set is finite.

A further property of $\mathbb{Q}$ is that each rational number can be represented on a straight line, or number line. But not all points of the line represent rational numbers. For example the length of $\overline{AC}$ in the diagram below is not a rational number. When this segment is marked off on the number line with one endpoint on the origin, the other endpoint does not coincide with a rational point on the line.



The distance from A to C (by the Pythagorean Theorem) is equal to a number whose square is 2, or $\sqrt{2}$. In an earlier exercise (Exercise 22, No. 8) the student showed that $\sqrt{2}$ is not a rational number. It is an irrational number. The proof that $\sqrt{2}$ is not a rational number is worth repeating here because it is derived from a similar proof known to ancient Greek mathematicians.

Theorem: The equation $x^2 = 2$ is not solvable by a rational number.

Proof: Assume that the equation were solvable by a rational number represented by $\frac{a}{b}$, where $\frac{a}{b}$ is a standard element (see pp. 86-87). Therefore, $x = \frac{a}{b}$ or:

$$(1) \qquad \frac{a^2}{b^2} = 2.$$

From equation (1):

$$(2) \qquad a^2 = 2b^2$$

which implies that a^2 is even. Thus, a is also an even number. If a is even, $a = 2m$, where m is an integer. Thus, from equation (2):

$$a^2 = (2m)^2 = 4m^2 = 2b^2.$$

So,

$$b^2 = 2m^2.$$

Therefore, b is even. We deduce that since a and b are even, $\frac{a}{b}$ is not in lowest terms (i.e., not a standard element), which is a contradiction. Thus, we conclude that $\sqrt{2}$ is not rational.

This theorem also asserts that there is at least one point on the number line that does not represent a rational number. A proof equivalent to this one can be used to show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc., are also irrational numbers. The theorem also implies that the decimal equivalent of $\sqrt{2}$ must be non-terminating and non-repeating. Recall that a previous examination of the decimal equivalents of rational numbers allowed this generalization: each rational number can be represented by a terminating decimal or by a repeating non-terminating decimal, and conversely.

Exercise 23

(1) Prove this statement: The square of an integer is divisible by 3 if and only if the integer itself is divisible by 3.

(2) Prove that $\sqrt{3}$ is irrational.

(3) Prove that $\sqrt{6}$ is irrational.

(4) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

(5) Prove that if γ is irrational, then $-\gamma$ is irrational.

(6) Prove or disprove: the sum of two irrational numbers is irrational (closure under addition.)

(7) Prove or disprove: the product of two irrational numbers is irrational (closure under multiplication.)

(8) Suppose s is an irrational number. Prove that $\frac{1}{s}$ is also irrational.

(9) Suppose γ is an irrational number and r is a non-zero rational number. Prove that $\gamma + r$ is irrational, and that $\gamma - r$ is irrational.

(10) See problem (9). Prove that $r\gamma$ and $\frac{r}{\gamma}$ are irrational.

(11) Prove that if γ is irrational, then $\sqrt{\gamma}$ is also irrational.

(12) Prove: If α and β are irrational numbers and $\alpha + \beta$ is rational then $\alpha - \beta$ is irrational.

11.2 Limitations of the Rationals

Previously, we attempted to cite the limitations of the natural numbers and the integers. These limitations served as motivation to extend these systems. The limitation of the rational numbers is somewhat more obscure, but no less important than those indicated earlier for other number systems.

Definition: A number, x, is an upper bound of set S if and only if for each $s \in S$, then $x \geq s$.

Example 1: For $S = \{x \mid 0 < x < 1, \text{ and } x \text{ is rational}\}$, an upper bound is 4. Some other upper bounds of S are 3, 2, and 1. In short, all numbers greater than, or equal to 1 are upper bounds.

Example 2: The set $W = \{0, 1, 2, 3, \ldots\}$ has no upper bound.

Definition: The least of all upper bounds is called the least upper bound (l.u.b.).

Example 3: The least upper bound of S (see Example 1, above) is 1, which is not a member of S.

Example 4: The set W (see Example 2, above) has no l.u.b.

Example 5: Let $T = \{x \mid 0 \leq x \leq 1 \text{ and } x \text{ is rational}\}$.
The l.u.b. is 1, a member of the set, T.

Example 6: For $Z^- = \{{}^-1, {}^-2, {}^-3, {}^-4, \ldots\}$, the least upper bound is $^-1$, an element of the set.

Example 7: The least upper bound of $F = \{{}^-\frac{1}{2}, {}^-\frac{1}{4}, {}^-\frac{1}{8}, \ldots\}$ is 0, which is not an element of the set.

Consider next the set $P = \{x \mid x^2 < 2$, and x is rational$\}$. This set consists of all rational numbers whose squares are less than 2, such as 1, 1/2, 1.4, 1.41, and 1.414. It has many upper bounds: 1.5, 2, $2\frac{1}{2}$, 3, 4, and 7. In fact, all rational numbers greater than 1.5 will serve as upper bounds of P, although there are others. But P does not have a least upper bound which is a rational number. For the least upper bound of P is not a rational number; it is the irrational number $\sqrt{2}$.

The set P serves to point out that the set Q of rational numbers does not possess the least upper bound property.

Definition: A non-empty set, H, satisfies the least upper bound property if and only if each subset G of H which has an upper bound, has a least upper bound which is an element of H.

Although Q does not possess the least upper bound property, the set of real numbers does. Our work in the next section begins a development which leads to this conclusion.

11.3 Definition of Real Numbers

For a definition of a real number, we can use only previously defined numbers. Specifically, we shall use the rational numbers to arrive at the real numbers.

Consider the sequence of rational numbers:

$$a_1 = 1$$
$$a_2 = 1.4$$
$$a_3 = 1.41$$
$$a_4 = 1.414$$
$$a_5 = 1.4141$$
$$a_6 = 1.41413$$
$$\vdots \qquad \vdots$$

When we square each of these numbers, we find that the square gets closer and closer to 2. By continuing this sequence of numbers in this manner, we can arrange for the squares of each of these numbers to get as close to 2 as we like. Although, in practice, the actual computation of each decimal place is laborious, the reader should appreciate that theoretically, at least, the job can be done. By means of a sequence $\{a_1, a_2, a_3, a_4, \ldots\}$ of rational numbers, the irrational number $\sqrt{2}$ will be defined. <u>All real numbers will be defined through sequences of rational numbers.</u>

Let us begin by defining what is meant by a sequence of rational numbers. Although a sequence is rarely indicated as such, it is a set of ordered pairs; in other words, a relation. The first elements of the ordered pairs of any sequence are the positive integers.

Definition: A sequence of elements of set Y is a function from $\mathbb{Z}^+$, the set of positive integers, into Y, such that the ordered pairs of the function retain the same order as that of the positive integers, i.e., there is a first element, a second element, etc.

Note 1: All sequences are functions whose domains are $\mathbb{Z}^+$.

Note 2: A sequence of rational numbers, which is the type of sequence which interests us most at present, is defined by replacing Y by $\mathbb{Q}$ in the definition.

Note 3: We use the symbol $\langle \ldots \rangle$ to enclose a sequence.

Example 1: The sequence $\langle (1,1), (2,\frac{1}{2}), (3,\frac{1}{3}), (4, \frac{1}{4}), \ldots \rangle$ is a sequence whose range is a subset of the rational numbers. The range of the sequence, then, is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

Note 4: Since the domain of all sequences is $\mathbb{Z}^+$, we denote a sequence writing only its range. Thus, the sequence of Example 1 is written:

$$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle.$$

Example 2: The sequence $\langle \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \ldots \rangle$ consists only of one element in the range.

Example 3: The sequence $\langle 0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots \rangle$ consists of three elements in the range.

Note 5: The elements of the range of a sequence are called the terms of the sequence. The definition of a sequence requires that a sequence be an infinite set.

Example 4: The terms of the sequence $\langle 1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}$ $\ldots \rangle$ consist of the reciprocals of the odd positive integers. The general or n^{th} term of the sequence is expressed by $\frac{1}{2n-1}$. Thus, if $n = 7$, the 7^{th} term is $\frac{1}{13}$. The number n represents the corresponding positive integer of the domain.

Consider the sequence:

$$\left\langle \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots, \frac{1}{10^n}, \ldots \right\rangle .$$

It is apparent that if we "go out far enough" in this sequence, the difference (in absolute value) between any two consecutive terms of the sequence can be made as small as we like. For instance, the absolute difference can be made smaller than $\frac{1}{10^6}$, or one-millionth. If we check the absolute difference between the sixth and seventh terms, it is $\left| \frac{1}{10^6} - \frac{1}{10^7} \right| = \frac{9}{10^7}$ which is less than $\frac{1}{10^6}$. After the fifth term of the sequence, the absolute difference between <u>any</u> two consecutive members is also less than $\frac{1}{10^6}$. In fact, after the fifth term, the absolute difference between <u>any</u> two members is less than $\frac{1}{10^6}$.

For example, the absolute difference between the eighth and 20^{th} terms is:

$$\left| \frac{1}{10^8} - \frac{1}{10^{20}} \right| = \frac{10^{12} - 1}{10^{20}} = \frac{99,999,999,999}{10^{20}}$$

which is less than $\frac{1}{10^6}$.

Instead of choosing the eighth and 20^{th} terms, this can be generalized for two arbitrary terms, occurring beyond the fifth term. Suppose we choose the m^{th} and n^{th} terms, i.e., $6 \leq m$ and $6 \leq n$. Since we are choosing natural numbers m and n, we know from earlier work, $m < n$, or $m = n$, or $n < m$. We take $m \neq n$, since two arbitrary terms beyond the fifth term must be considered. It is unimportant whether we have $n < m$ or $m < n$, so choose $m < n$. We would like to show:

$$\left| \frac{1}{10^m} - \frac{1}{10^n} \right| < \frac{1}{10^6} \text{ where } 6 \leq m < n .$$

Clearly, $\left| \frac{1}{10^m} - \frac{1}{10^n} \right| = \left| \frac{10^{n-m} -1}{10^n} \right|$. Since $n > m$, it follows that $\left| \frac{10^{n-m} -1}{10^n} \right| = \frac{10^{n-m} -1}{10^n}$. Hence, the problem becomes one of showing $\frac{10^{n-m} -1}{10^n} < \frac{1}{10^6}$.

Now $\frac{10^{n-m} -1}{10^n} < \frac{1}{10^6}$, if $\frac{10^{n-m} -1}{10^n} < \frac{10^{n-6}}{10^n}$; but this is true provided $10^{n-m} -1 < 10^{n-6}$; this will follow if $-1 < 10^{n-6} - 10^{n-m}$.

We observe this last inequality is true since it can be written as $-1 < 10^n \cdot 10^{-6} - 10^n \cdot 10^{-m} = 10^n \left(\frac{1}{10^6} - \frac{1}{10^m} \right)$. Now observe that $\frac{1}{10^6} - \frac{1}{10^m} > 0$, since $\frac{1}{10^6} \geq \frac{1}{10^m}$ for $m \geq 6$.

Thus, the right-side of $^{-}1 < 10^{n}\left(\frac{1}{10^{6}} - \frac{1}{10^{m}}\right)$ is non-negative because it is the product of positive rational and a non-negative rational. Hence, the right-side is greater than $^{-}1$.

$$\therefore \forall\ m \geq 6 \text{ and } \forall\ n > m,\ \left|\frac{1}{10^{m}} - \frac{1}{10^{n}}\right| < \frac{1}{10^{6}}.$$

We could have proposed a number smaller than $\frac{1}{10^{6}}$ with which to test the sequence, but we would find that it is possible to find a term of the series such that from that term on, the difference between <u>any</u> two terms is less than any test number.

We are going to use sequences which satisfy the property shown above to define real numbers. We shall always be using sequences of rational numbers. We now generalize this property of these special sequences.

<u>Definition</u>: Given the sequence $\langle a_1, a_2, a_3, \ldots, a_n, \ldots \rangle$ and any positive rational number, r, (however small), if it possible to find a term of the sequence, for instance a_N, such that $|a_p - a_q| < r$, for all $p > N$ and $q > N$, then $\langle a_1, a_2, \ldots, a_N, \ldots \rangle$ is a <u>Cauchy sequence of rational numbers</u>.

Note: These sequences are special cases of Cauchy sequences of real numbers.

This property means that regardless of how small a number one starts with, it is possible to find a term in the sequence such that the difference between any two terms of the sequence after that term is less than the given number. In the above example, $a_N = \frac{1}{10^{5}}$, $N = 5$, and $r = \frac{1}{10^{6}}$.

Example 1: Here are examples of Cauchy sequences:

(a) $\langle \frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \ldots \rangle$

(b) $\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots \rangle$

(c) $\langle .9, .99, .999, \ldots \rangle$

(d) $\langle 1.4, 1.41, 1.414, 1.4145, \ldots \rangle$

Example 2: Here are examples of sequences which are <u>not</u> Cauchy sequences:

(a) $\langle 1, 2, 3, 4, 5, \ldots \rangle$

(b) $\langle 0, 1, 0, 1, 0, 1, \ldots \rangle$

(c) $\langle 5, 10, 5, 10, \ldots \rangle$

(d) $\langle 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots \rangle$

Note: In example 1, sequence (a) is made up of terms which approach $\frac{1}{3}$; sequence (b) consists of terms which approach 0; the terms of (c) approach 1, and the terms of (d) approach $\sqrt{2}$. The application of these ideas will become more evident when we discuss equivalent Cauchy sequences.

Observe that if, from a Cauchy sequence, we delete or add a finite number of terms, the sequence remains a Cauchy sequence, i.e., the criterion for a Cauchy sequence is not affected by the deletion or addition of a finite number of terms. What we can do is remove the first 100 terms, for example, and the new sequence would still be a Cauchy sequence.

Consider next the sequence:

$$\langle 1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots \rangle$$

By inspection, it is clear that this sequence approaches 0, that is the difference between each term of the sequence and 0 gets smaller and smaller. To state this idea in another way, the difference between any term of the sequence and 0, from some term on, is smaller than any positive rational number. Choose any positive rational number, such as 1/100; then all terms after the 100^{th} term are smaller than 1/100.

Similarly, the sequence $\langle {}^{-}1, {}^{-}1/2, {}^{-}1/3, {}^{-}1/4, \ldots \rangle$ has the same property if we consider the absolute value of the terms. These examples lead to this definition:

Definition: A sequence $\langle a_1, a_2, a_3, \ldots, a_n, \ldots \rangle$ is a <u>null sequence</u> if and only if given any positive rational number r it is possible to find a term of the sequence, a_N, such that $|a_n - 0| < r$, for all $n > N$.

Note: This definition states that one can find a term, a_N, of the sequence such that the difference $|a_n - 0|$ for all terms after a_N is less than any preassigned number, r.

Next, consider two Cauchy sequences $\langle p_1, p_2, p_3, p_4, \ldots, p_n, \ldots \rangle$ and $\langle q_1, q_2, q_3, q_4, \ldots, q_n, \ldots \rangle$. The sequence formed by the difference of corresponding terms of the sequences can be represented by:

$$\langle p_1 - q_1, p_2, p_3 - q_3, \ldots, p_n - q_n, \ldots \rangle .$$

If this sequence formed by the term-by-term difference of to Cauchy sequences is a null sequence, then it can be expected that the Cauchy sequences are <u>equivalent</u> to each other. In other words, the expectation is that since $a_n - b_n$ approaches 0 , then the two Cauchy sequences can be treated as <u>indistinguishable</u> sequences. Thus, it is quite natural to define the relation $\textcircled{\sim}$ between the Cauchy sequences:

<u>Definition</u>: The sequence $\langle p_1, p_2, p_3, \ldots, p_n, \ldots \rangle \textcircled{\sim} \langle q_1, q_2, q_3, \ldots, q_n, \ldots \rangle$ if and only if $\langle p_1 - q_1, p_2 - q_2, p_3 - q_3, \ldots, p_n - q_n, \ldots \rangle$ is a null sequence.

Before we prove that $\textcircled{\sim}$ is an equivalence relation, we give an example of two Cauchy sequences whose term-by-term differences form a null sequence. Consider the sequence:

(1) $\langle 2.9, 2.99, 2.999, 2.9999, \ldots \rangle$

and the sequence

(2) $\langle 3.1, 3.01, 3.001, 3.0001, \ldots \rangle$.

Recognize, first of all, that each of those sequences is a Cauchy sequence, i.e., for each sequence, the absolute difference of any two terms after some term is less than any preassigned positive rational number. Let's look at the sequence formed by term-by-term differences:

(3) $\langle 3.1 - 2.9, 3.01 - 2.99, 3.001 - 2.999, 3.0001 - 2.999, \ldots \rangle$.

Note that we could have subtracted sequence (2) from sequence (1), but the result, in absolute value, would be the same. The difference sequence is:

(4) $\langle .2, \ .02, \ .002, \ .0002, \ \ldots \rangle$

which is clearly a <u>null sequence</u>. It should be observed that the Cauchy sequences (1) and (2) both approach 3 and in that sense are equivalent.

To show that $\circledsim$ is an equivalence relation, we must show that it is a reflective, symmetric, and transitive relation.

Before we prove these properties, we introduce the notation $\langle a_p \rangle$ to represent the sequence $\langle a_1, a_2, \ldots, a_p, \ldots \rangle$. Thus, we use $\langle \frac{1}{p} \rangle$ to denote the <u>entire</u> sequence $\langle 1, 1/2, 1/3, \ldots, 1/p, \ldots \rangle$, and $\langle 2^p \rangle$ to denote the <u>entire</u> sequence $\langle 2, 2^2, 2^3, \ldots, 2^p, \ldots \rangle$.

Proof:

(1) <u>Reflexivity</u>: $\langle a_p \rangle \circledsim \langle a_p \rangle$.

This result follows trivially from the definition of equivalent sequences; notice that the sequence formed of the difference of corresponding terms is composed entirely of zeroes.

(2) <u>Symmetry</u>: If $\langle a_p \rangle \circledsim \langle b_p \rangle$ then $\langle b_p \rangle \circledsim \langle a_p \rangle$.

If $\langle a_p \rangle \circledsim \langle b_p \rangle$, then by definition of equivalent sequences, $\langle a_p - b_p \rangle$ is a null sequence. Therefore, by definition of a null sequence, for any positive number r , there exists a positive integer N such that for all $n > N$, $|(a_n - b_n) - 0| = |(a_n - b_n)| < r$. Now, recalling the conclusion of Problem 7-c on page 105, $|a_n - b_n| = |b_n - a_n|$. Thus, $\langle b_p - a_p \rangle$ is a null sequence, since $|b_n - a_n| < r$, for r any positive number. Hence, $\langle b_p \rangle \circledsim \langle a_p \rangle$.

(3) <u>Transitivity</u>: If $\langle a_p \rangle \circledsim \langle b_p \rangle$ and $\langle b_p \rangle \circledsim \langle c_p \rangle$, then $\langle a_p \rangle \circledsim \langle c_p \rangle$.

The first thing to decide is exactly what must be established in order to prove the statement. It must be shown that given any rational number, r_1, it is possible to find a positive integer, N, such that for all $n > N$, $|(a_n - c_n) - 0| = |(a_n - c_n)| < r_1$.

If this can be shown, then $\langle a_p - c_p \rangle$ will be a null sequence and $\langle a_p \rangle \circledsim \langle c_p \rangle$.

The triangular inequality (page 105, Problem 7-e) will have a central role in this proof. Before attempting the proof, we demonstrate the form that the triangular inequality will take. Suppose we have $|x - z|$, where $x, z \in \mathbb{Q}$. This can be written as $|x + 0 - z|$ without changing its value. Since $0 = {}^-y + y$, for all $y \in \mathbb{Q}$, then $|x - z| = |x + 0 - z| = |x - y + y - z|$. By closure of $\mathbb{Q}$ under subtraction, the expression $|x - y + y - z| = |(x - y) + (y - z)|$ is actually the absolute value of the sum of two rational numbers. This is exactly the form of the triangular inequality as stated on p. 105. Hence, $|x - z| \leq |x - y| + |y - z|$. This will be used by arriving at the right side of the inequality and replacing it by the left side of the inequality.

We are ready to construct a proof for the transitive property. It is given that $\langle a_p \rangle \circledcirc \langle b_p \rangle$ thus, $\langle a_p - b_p \rangle$ is a null sequence. So, for <u>any</u> given rational number, r, there exists a positive integer N_1 such that for all $n > N_1$, $|a_n - b_n| < r$. Since this statement is true for <u>any</u> given rational number r, it will be true for a particular one, namely $\frac{r_1}{2}$, where r_1 is the same rational referred to above at the beginning of this proof (the reason for wanting to look at $\frac{r_1}{2}$ will become obvious later in the proof.)

Likewise, $\langle b_p \rangle \circledcirc \langle c_n \rangle \Longrightarrow \langle b_p - c_p \rangle$ is a null sequence; so for <u>any</u> given rational number, r', there exists a positive integer, N_2 (usually not equal to N_1, above), such that for $n > N_2$, $|b_n - c_n| < r'$. Again, this will be true for a particular rational number, namely $\frac{r_1}{2}$, the same number mentioned above.

Now, if it is known that $|a_n - b_n| < \frac{r_1}{2}$ for $n > N_1$, and $|b_n - c_n| < \frac{r_1}{2}$ for $n > N_2$, then both these statements will be true if we choose for our value of N the larger of N_1 and N_2. Hence, we have that $|a_n - b_n| < \frac{r_1}{2}$ and $|b_n - c_n| < \frac{r_1}{2}$ for all $n > N$.

Consider next the sum of these two expressions, i.e.,

$$(1) \quad |a_n - b_n| + |b_n - c_n| < \frac{r_1}{2} + \frac{r_1}{2} .$$

If we apply the traingle inequality to the left-side of (1) and perform the suggested addition on the right-side of (1), we can conclude that $|a_n - c_n| < r_1$ for all $n > N$. This is our desired result. Therefore, it has been established that the relation   between Cauchy sequences is an equivalence relation.

We have shown that $\approx$ is an equivalence relation on the set of all Cauchy sequences of rational numbers. By the equivalence theorem, $\approx$ partitions this set of Cauchy sequences of rational numbers into non-overlapping classes whose union is the entire set. Each of these classes contains all sequences of rational numbers which are equivalent to each other. The equivalence class of all sequences equivalent to the Cauchy sequence $\langle a_n \rangle$ will be denoted by $[a_n]$.

<u>Definition:</u> A <u>real number</u> is an equivalence class of Cauchy sequences of rational numbers.

Thus, every Cauchy sequence of rational numbers defines a real number. The sequence $\langle a_n \rangle$ defines the real number $[a_n]$.

<u>Theorem</u>: Every sequence $\langle r, r, r, \ldots \rangle$, where r is a rational number, defines a real number.

Since the sequence $\langle r, r, r, \ldots \rangle$ is a Cauchy sequence, it defines the class of all Cauchy sequences equivalent to it.

Every rational number generates a sequence of the type in the theorem. A terminating decimal, for instance, generates a Cauchy sequence in which all the terms are the same. The rational number represented by 6.25 yields the sequence $\left\langle \frac{625}{100}, \frac{625}{100}, \frac{625}{100}, \ldots \right\rangle$. Non-terminating decimals which represent rational numbers also generate Cauchy sequences.

The decimal 6,24999... represents 6.25. This decimal generates the sequence $\left\langle \frac{624}{100}, \frac{6249}{1000}, \frac{62499}{10000}, \ldots \right\rangle$ which is a Cauchy sequence. This sequence is equivalent to $\left\langle \frac{625}{100}, \frac{625}{100}, \frac{625}{100}, \ldots \right\rangle$. To see that these two sequences are equivalent, we compute the absolute value of the term-by-term differences

$$\left| \frac{625}{100} - \frac{624}{100} \right| = \frac{1}{100}$$

$$\left| \frac{625}{100} - \frac{6249}{1000} \right| = \frac{1}{1000}$$

$$\left| \frac{625}{100} - \frac{62499}{10000} \right| = \frac{1}{10000}$$

The absolute value of the term-by-term differences yield this sequence $\left\langle \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots \right\rangle$, which is clearly a null sequence.

Both of these sequences are in the equivalence class which defines the real number $\frac{625}{100}$ or 6.25.

Similarly, from the representation of $\frac{1}{3}$ by .333···, we have the sequence $\left\langle \frac{3}{10}, \frac{3}{100}, \frac{3}{1000}, \ldots \right\rangle$ which is equivalent to the sequence $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots \right\rangle$. Both of these sequences are in the same equivalence class defining the real number $\frac{1}{3}$. Of course, both 6.25 and $\frac{1}{3}$ are also rational numbers.

A non-repeating, non-terminating decimal also generates a sequence of rational numbers. For example, the non-repeating decimal 2.1347217... produces the sequence:

$$\left\langle \frac{21}{10}, \frac{213}{100}, \frac{2134}{1000}, \ldots \right\rangle$$

which defines a real number. As indicated earlier, this number is irrational.

The decimal equivalent of π is 3.1415925··· . This decimal generates $\left\langle \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \ldots \right\rangle$, a sequence of rational numbers which defines π.

NOTE: In these last two sequences, the three dots, . . ., have been used to indicate that the terms of the sequence follow the digits of the decimal, not that the terms follow a special pattern.

Both numbers shown above, 2.1347217... and π, are irrational numbers. Recall that any non-terminating, non-repeating decimal generates an irrational number. The set of real numbers consists of the set of rational numbers and the set of irrational numbers.

As an additional example, the irrational number $\sqrt{2}$ is defined by means of a Cauchy sequence, each of whose terms, when squared get closer and closer to 2. One such sequence is $\langle 1, 1.4, 1.41, 1.414, 1.4142, \cdots \rangle$ (non-repeating). Cauchy sequences based upon (relatively) familiar decimal equivalents of irrational numbers, such as $\sqrt{3}$ and $\sqrt{5}$, provide the basis for the definition of these numbers. Observe how once again the theoretical foundation in mathematics depends upon experience.

11.4 Operations on Real Numbers

Let p be the real number generated by the Cauchy sequence $\langle p_n \rangle$, i.e., $p = [p_n]$, and let q be the real number generated by the Cauchy sequence $\langle q_n \rangle$, $q = [q_n]$. We now define addition for these two classes $[p_n]$ and $[q_n]$.

Definition: $p + q = [p_n + q_n]$

Note: We should, of course, use another symbol instead of '+' for the sum p + q as we did for the sum of two integers and two rational numbers, but as long as the student recognizes that '+' stands for addition of real numbers p and q when we are using real numbers, we will not need a special symbol.

To show that the definition has validity, it is necessary to prove that:

(a) $\langle p_n + q_n \rangle$ is a Cauchy sequence, and

(b) If $\langle p'_n \rangle$ and $\langle q'_n \rangle$ are two Cauchy sequences difference from $\langle p_n \rangle$ and $\langle q_n \rangle$, respectively, but defining p and q, respectively, then $\langle p'_n + q'_n \rangle$ is equivalent to $\langle p_n + q_n \rangle$.

Proof of (a):

$\langle p_n + q_n \rangle$ is a Cauchy sequence if and only if for any positive rational κ it is possible to find a positive integer N such that

$|(p_m + q_m) - (p_n + q_n)| < \kappa$ for all $m > N$ and all $n > N$.

Further, $\langle p_n \rangle$ and $\langle q_n \rangle$ are both Cauchy sequences, so that it is possible to find terms of the two sequences, p_{N_1} and q_{N_2} such that;

$$|p_m - p_n| < \frac{\kappa}{2} \text{ for } m > N_1 \text{ and } n > N_1 \text{ ; and}$$

$$|q_m - q_n| < \frac{\kappa}{2} \text{ for } m > N_2 \text{ and } n > N_2 .$$

But,

$$|(p_m + q_m) - (p_n + q_n)| = |(p_m - p_n) + (q_m - q_n)|$$
$$\leq |p_m - p_n| + |q_m - q_n| .$$

This inequality is a result of an application of the triangular inequality.

If N equals the maximum of N_1 and N_2, i.e.,

$N = \max(N_1, N_2)$, then

$$|(p_m + q_m) - (p_n + q_n)| \leq |p_m - p_n| + |q_m - q_n| < \frac{r}{2} + \frac{r}{2} = r$$

for all $m > N$ and $n > N$. This is the required statement for $\langle p_n + q_n \rangle$ to be a Cauchy sequence.

Proof of (b);

We must show that the sequence $\langle (p_n + q_n) - p'_n + q'_n) \rangle$ or $\langle p_n + q_n - p'_n - q'_n \rangle$ is a null sequence. We must prove that for any rational number r, one can find a positive integer N such that;

(1) $|p_n + q_n - p'_n - q'_n| < r$ whenever $n > N$.

But $\langle p_n \rangle \circledcirc \langle p'_n \rangle$ and $\langle q_n \rangle \circledcirc \langle q'_n \rangle$, so that $\langle p_n - p'_n \rangle$ and $\langle q_n - q'_n \rangle$ are null sequences. Thus, for each sequence, given an arbitrary rational number $\frac{r}{2}$, one can find positive integers N_1 and N_2 such that;

$|p_n - p'_n| < \frac{r}{2}$ and

$|q_n - q'_n| < \frac{r}{2}$.

From (1), by using the triangle inequality, we have;

$$|p_n + q_n - p'_n - q'_n| = |(p_n - p'_n) + (q_n - q'_n)| \leq |p_n - p'_n| + |q_n - q'_n|.$$

Finally, we are able to conclude that by choosing $N = \max$ (N_1, N_2),

$|p_n + q_n - p'_n - q'_n| \leq |p_n - p'_n| + |q_n - q'_n| < \frac{r}{2} + \frac{r}{2} = r$ for all $n > N$.

Thus, $\langle p_n + q_n - p'_n - q'_n \rangle$ is a null sequence and

$$\langle p_n + q_n \rangle \circledcirc \langle p'_n + q'_n \rangle \ .$$

NOTE: In the definition, $p + q = [p_n + q_n]$, we could describe this sum by writing $p + q = \{\langle z_n \rangle \mid \langle z_n \rangle$ is a Cauchy sequence and $\langle z_n \rangle \circledcirc \langle p_n + q_n \rangle\}$.

We now turn our attention to the definition of multiplication of two real numbers.

<u>Definition</u>: Let p and q be real numbers generated by the Cauchy sequences $\langle p_n \rangle$ and $\langle q_n \rangle$, respectively; then,

$$p \cdot q = [p_n \cdot q_n] \ .$$

NOTE: A comment similar to that on page 134 applies to the symbol '$\cdot$'.

Again, we must shown that $p \cdot q$ is well-defined, i.e.,

c) $\langle p_n \cdot q_n \rangle$ is a Cauchy sequence; and

d) If $\langle p'_n \rangle$ and $\langle q'_n \rangle$ are two Cauchy sequences generating p and q, respectively, then

$$\langle p'_n \cdot q'_n \rangle \circledcirc \langle p_n \cdot q_n \rangle \ .$$

We prove (c) but leave (d) as an exercise for the reader (See Exercise 24). To prove (c) we must show that for any positive rational number r, there exists a positive integer N such that

ii) $|p_m \cdot q_m - p_n \cdot q_n| < r$ for all $m > N$ and all $n > N$.

The absolute value of the inequality (ii) can be expressed as follows;

$$\text{iii)}\quad |p_m \cdot q_m - p_n \cdot q_n| = |p_m \cdot q_m - p_m \cdot q_n + p_m \cdot q_n - p_n \cdot q_n|$$

$$\text{iv)}\quad |p_m \cdot q_m - p_n \cdot q_n| = |p_m (q_m - q_n) + q_n (p_n - p_n)|$$

Now applying the triangle inequality to the right-hand side of statement (iv), we have:

$$\text{v)}\quad |p_m \cdot q_m - p_n \cdot q_n| \leq |p_m (q_m - q_n)| + |q_n (p_m - p_n)| \,.$$

Now applying problem (7-d) from Exercise 22 **to each term of the right side of statement** (v), we have

$$\text{vi)}\quad |p_m \cdot q_m - p_n \cdot q_n| \leq |p_m|\,|q_m - q_n| + |q_n|\,|p_m - p_n| \,.$$

It can be shown (*) that since $\langle p_n \rangle$ and $\langle q_n \rangle$ are Cauchy sequences, then there exists positive rationals K_1 and K_2 such that $|p_n| < K_1$ and $|q_n| < K_2$. (K_1 and K_2 are upper bounds of these sequences) The inequality (vi) becomes

$$\text{vii)}\quad |p_m \cdot q_m - p_n \cdot q_n| < K_1\,|q_m - q_n| + K_2\,|p_m - p_n| \,.$$

But, since $\langle p_n \rangle$ and $\langle q_n \rangle$ are Cauchy sequences, then for rational numbers $\frac{\epsilon}{2K_2}$ and $\frac{\epsilon}{2K_1}$, there exists positive integers N_1 and N_2 such that;

$$|p_m - p_n| < \frac{\epsilon}{2K_2} \quad \text{and} \quad |q_m - q_n| < \frac{\epsilon}{2K_1} \,.$$

(*) See Appendix C

Again choose $N = \max(N_1, N_2)$. Thus for $m > N$ and $n > N$ using inequality (vii) we have

$$\left| p_m \cdot q_m - p_n \cdot q_n \right| < K_1 \cdot \frac{\varepsilon}{2K_1} + k_2 \cdot \frac{\varepsilon}{2K_2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} =$$

which is the desired statement.

Definition: Let a and b be real numbers, then a-b is the real number defined by $\langle a_n - b_n \rangle$, i.e.,

$$a - b = [a_n - b_n] .$$

Definition: Let a and b be real numbers and $b \neq 0$, then $\frac{a}{b}$ is the real number defined by $\left\langle \frac{a_n}{b_n} \right\rangle$ (all $b_n \neq 0$), i.e.,

$$\frac{a}{b} = \left[\frac{a_n}{b_n} \right] .$$

Exercise 24

1. If $\langle a_n \rangle$ and $\langle b_n \rangle$ are Cauchy sequences of rational numbers, show that $\langle a_n + b_n \rangle$ is also a Cauchy sequence.
2. Prove that $\langle a_n - b_n \rangle$ is Cauchy.
3. Prove that $\langle a_n \cdot b_n \rangle$ is Cauchy.
4. Prove that $\left\langle \frac{a_n}{b_n} \right\rangle$ is Cauchy $(b_n \neq 0)$.
5. Let a be defined by $\langle a_n \rangle$ and b be defined by $\langle b_n \rangle$; $\langle a_n \rangle \approx \langle a'_n \rangle$ and $\langle b_n \rangle \approx \langle b'_n \rangle$. Prove $\langle a_n + b_n \rangle \approx \langle a'_n + b'_n \rangle$.
6. Prove: 1) $\langle a_n - b_n \rangle \approx \langle a'_n - b'_n \rangle$ 2) $\langle a_n \cdot b_n \rangle \approx \langle a'_n \cdot b'_n \rangle$

   3) $\left\langle \frac{a_n}{b_n} \right\rangle \approx \left\langle \frac{a'_n}{b'_n} \right\rangle$
7. Prove that the set of real numbers has all the properties of the set of rational numbers.