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AUTHOR Suydam, Marilyn N.; Weaver, J. Fred
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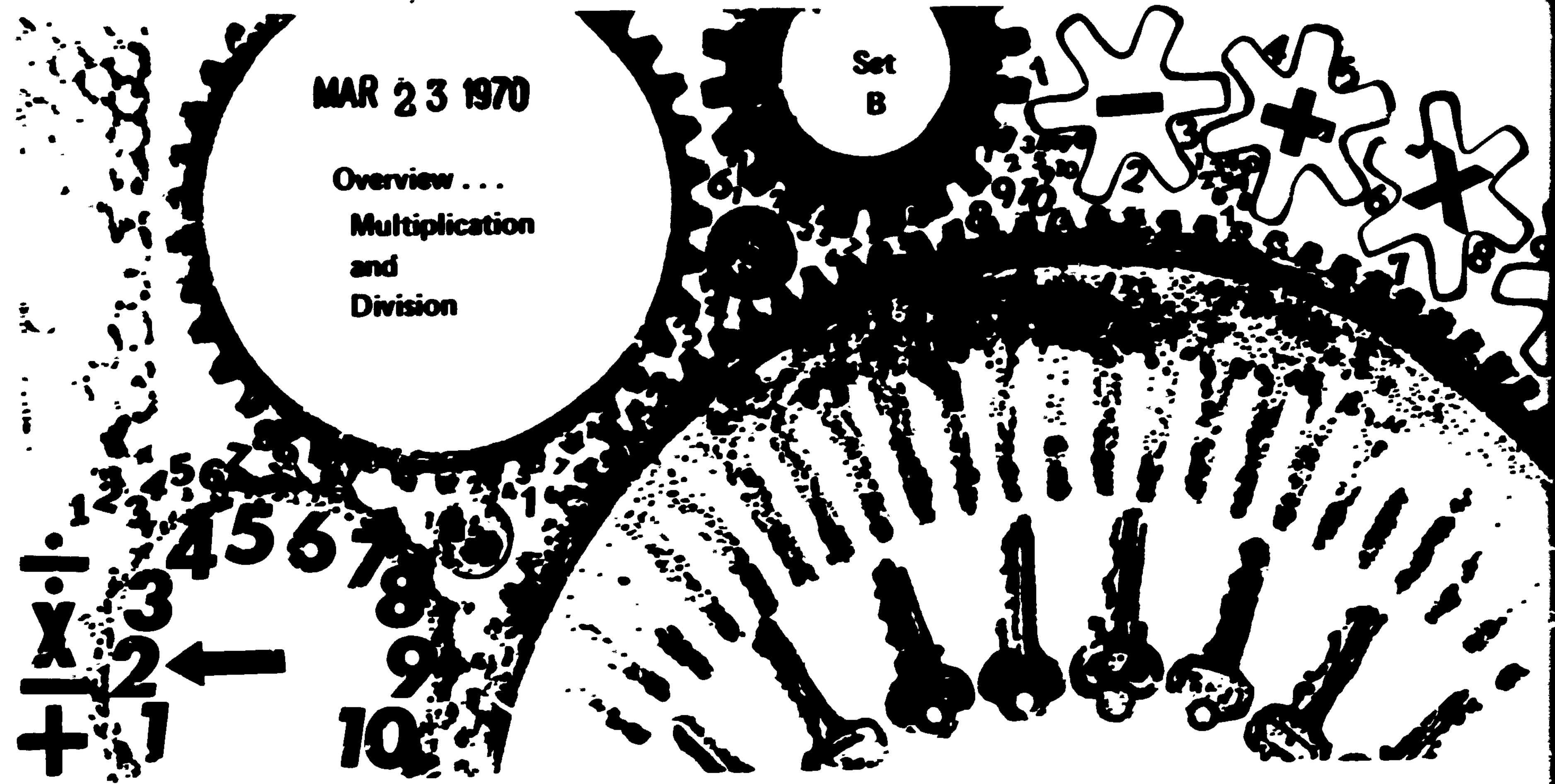
ABSTRACT

This paper reports on the answers indicated by research studies to nine questions concerning the teaching of multiplication and division. (1) Should multiplication facts be memorized? Answer: Yes, at an appropriate time. (2) How should multiplication be conceptualized? Answer: Normally addition of equal addends; sometimes arrays or Cartesian-product. (3) Is distributivity important? Answer: Yes, for comprehension, transfer and retention. (4) What other approaches work? Answer: Inductive. (5) What aids advanced work? Answer: Knowledge of multiplication properties. (6) How does the difficulty level change in division? Answer: No conclusion. (7) Which division algorithm should be used? Answer: No conclusion. (8) How should quotient digits be estimated? Answer: No conclusion. (9) How are measurement and partition situations associated with division? Answer: Subtractive algorithm with measurement, distributive with partition; partition problems generally more difficult. (RS)






Using Research: A Key to Elementary School Mathematics

# MULTIPLICATION AND DIVISION WITH WHOLE NUMBERS

**Should children be encouraged to memorize basic multiplication facts?**

Of course children should achieve immediate recall of the basic facts -- <u>at an appropriate time in the learning sequence</u>. Understanding of the nature of multiplication should <u>precede</u> work which focuses on such memorization, however. Use of properties of multiplication will help pupils in this learning.

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**How should multiplication be conceptualized for children?**

Multiplication usually has been conceptualized in terms of the addition of equal addends. Arrays are also suggested as a way of representing multiplication, though little research has been done using them. Cartesian-product problems appear to be more difficult for young children to conceptualize.

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**Is attention to distributivity helpful in early work with multiplication?**

Emphasis on distributivity is especially effective in promoting transfer and retention. Research on this adds further support to a growing body of evidence on advantages to be expected from instruction which emphasizes understanding. The "pay-off" may not always be evident in immediate achievement of skills, but rather in relation to factors such as comprehension, transfer, and retention.

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**What has been found about other approaches to early work with multiplication?**

Do you usually introduce multiplication with verbal problems? If you do this, and then guide pupils in developing the multiplication fact from each problem (by counting, using pictures and diagrams, adding, and using the number line), recall and retention of the facts should be facilitated. Such an inductive approach, where each pupil can work at his own level of maturity, has been shown to be better than one in which the teacher presents the facts to the pupil through examples.

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**What things contribute to pupils' success with more advanced work in multiplication?**

If you <u>only</u> want pupils to achieve speed and accuracy, then readiness for work with two-place factors should consist of practice on the 100 multiplication facts. If, however, you want pupils to achieve the objectives of increased understanding of the process, increased problem solving ability, <u>and</u> increased computation skills, then readiness work should emphasize the properties of multiplication. Use of the algorithm in which partial products are shown appears to aid these same objectives.

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**Which division algorithm should be used?**

Pupils using a subtractive algorithm may achieve greater understanding of division and increased ability to transfer than do pupils using the distributive algorithm which has been common for some years. Use of the distributive algorithm may aid in some problem solving situations, and seems equally effective on retention measures.

```
3)52
  30    10 x 3
  etc.

   1
3)52
  3
  etc.
```

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**What is the most effective method of teaching pupils to estimate quotient digits?**

If success on first trial is the criterion, then "round-both-ways" (42 → 40, 47 → 50) would be recommended. However, corrections must be made by <u>either</u> increasing <u>or</u> decreasing the estimate. With the "round-down" method (42 → 40, 47 → 40) the estimate is corrected by decreasing it, while with the "round-up" method (42 → 50, 47 → 50) the estimate is corrected by increasing it. This last method parallels the procedure used in the subtractive algorithm.

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**What is the role of measurement and partition situations in teaching division?**

Partition problems appear to be more difficult than measurement problems. Use of the subtractive algorithm for measurement situations and the distributive algorithm for partition situations has been suggested.

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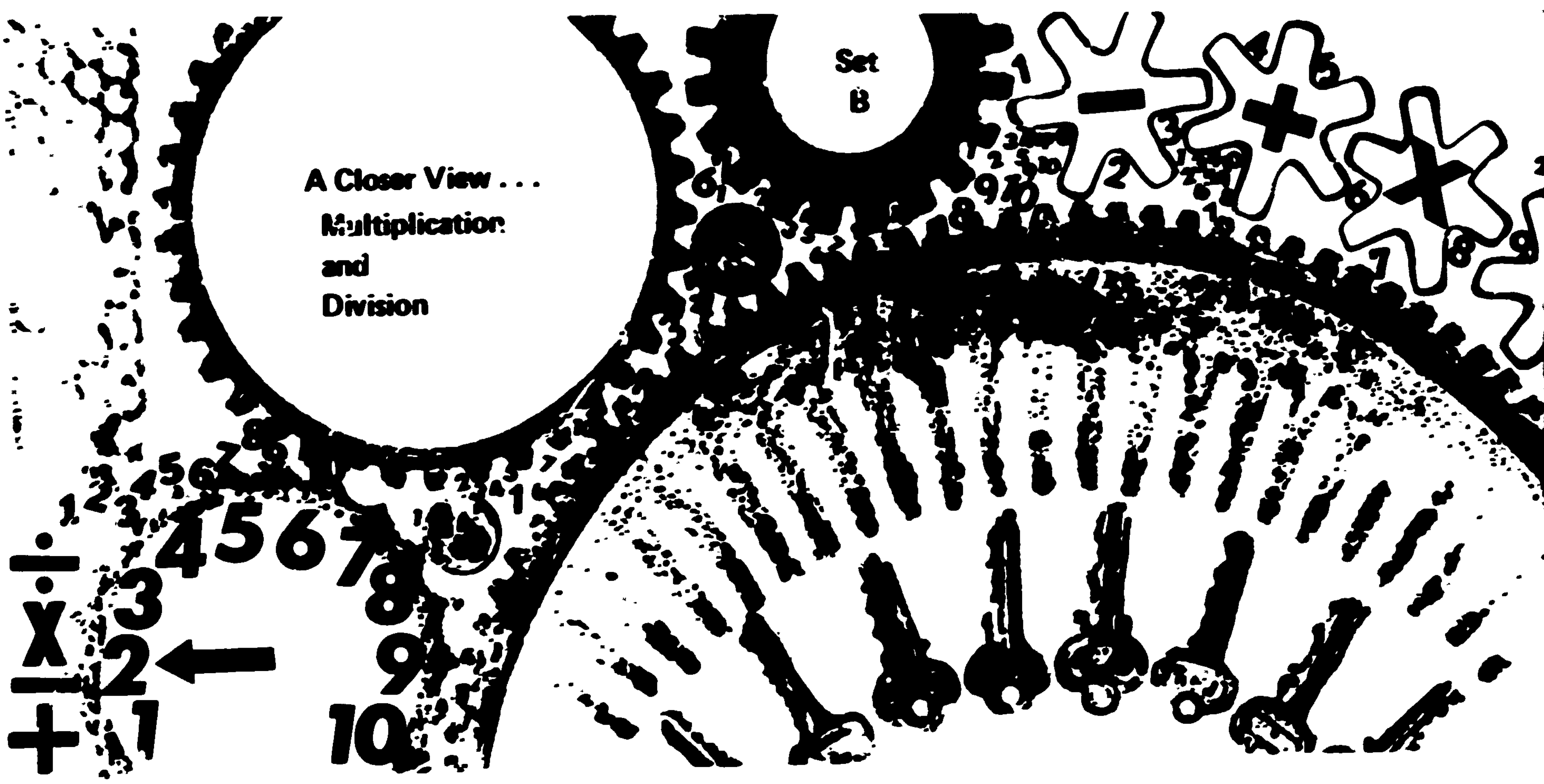
If you would like more information about the research whose findings are cited above, contact MARILYN N. SUYDAM, Project Director, at The Pennsylvania State University, University Park, Pennsylvania, 16802.




Using Research: A Key to Elementary School Mathematics

## MULTIPLICATION AND DIVISION WITH WHOLE NUMBERS

**Should children be encouraged to memorize basic multiplication facts?**

At an appropriate time in the learning sequence it is desirable that children strive to achieve immediate recall of basic multiplication facts (3 x 5 = 15, 6 x 4 = 24, 7 x 8 = 56, 9 x 9 = 81, etc.).

Findings from a comprehensive investigation with children in grades three to five by Brownell and Carper (1943) suggest that activities and experiences which contribute to pupils' understanding of the mathematical nature of multiplication should precede work which focuses on memorization of facts.

Teachers know that the number of specific basic facts to be memorized is reduced substantially if pupils are able to apply the properties of multiplication illustrated by the following examples:

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The bulletin was prepared by MARILYN N. SUYDAM, The Pennsylvania State University, Project Director, and J. FRED WEAVER, The University of Wisconsin-Madison, Project Consultant. Art by Ed Saffell.

It should be noted that research is variable with respect to its quality; hence, the same degree of confidence cannot be placed in all findings. An attempt has been made to take this fact into consideration in preparing this bulletin.

(a) 3 x 5 = 15 and 5 x 3 = 15. (Commutative property of multiplication)

(b) 8 x 1 = 8 and 1 x 8 = 8. (Identity property for multiplication)

(c) 7 x 0 = 0 and 0 x 7 = 0. (Zero property for multiplication)

Hall's (1967) research on teaching selected multiplication facts to third-grade pupils appears to support an emphasis upon the commutative property.

Brownell and Carper also suggested that development of the facts may lead to the organization of a "table":

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 8 | 10 | |
| 3 | 0 | 3 | 6 | 9 | 12 | 15 | |

This can aid in the identification of patterns and relationships; pupils can find answers to such questions as:

-- If 1 is a factor, what pattern is true?

-- If 5 is a factor, what digit will be in the units place in the product?

-- If one factor is even, will the product be odd or even?

Ascertaining the relative difficulty of the multiplication facts was once a matter of great concern, based on the assumption that there is a fixed rank for each. Little commonality of levels of difficulty was evident among the studies, however, since this is apparently a function of (1) whether pupils are studied at the time of initial learning, or later; (2) the order and organization of the facts; and (3) the method of teaching, whether meaningful, with emphasis on relationships, or drill-oriented. Thus we need to ask, "Difficulty level for whom? at what age? under what method of instruction?"

Two findings that were frequently cited in the early studies (conducted under a drill approach) were that combinations involving zero presented difficulty, and that the size of the product was positively correlated to difficulty. Whether these remain true today, where a more meaningful teaching approach is used, has not been ascertained by research, but nevertheless should be considered by the teacher.

**How should multiplication be conceptualized for children?**

Traditionally multiplication of whole numbers has been conceptualized for children in terms of the addition of equal addends. For instance, "4 x 7" has been interpreted to mean "7 + 7 + 7 + 7." But there are logical difficulties inherent in this interpretation when the first factor in a multiplication example is 0 or 1.

Some recent research has investigated the feasibility of using other conceptualizations of multiplication. One of these interpretations, which is independent of addition, is based upon the following relationship: if set A has a members and set B has b members, the Cartesian product of sets A and B has a x b members. Hervey (1966) reported that second-grade pupils had significantly greater success in solving, conceptualizing, and visually representing equal-addends problems than Cartesian-product problems. Cartesian-product problems were conceptualized and solved more often by high achievers than by low achievers, more often by boys than by girls, and more often by pupils with above-average intelligence. Hervey was not able to determine the extent to which her findings may be influenced by the nature of prior instruction or by differences inherent in the mathematical nature of the two conceptualizations.

Another conceptualization of multiplication may be associated with rectangular arrays -- either independent of or in conjunction with Cartesian products. At the third-grade level Schell (1964) investigated achievement of pupils who used array representations exclusively for their introductory work with multiplication, as compared with pupils who used a variety of representations. He found no conclusive evidence of a difference in achievement levels.

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**Is attention to distributivity helpful in early work with multiplication?**

We know, for example, that 3 x (4 + 7) = (3 x 4) + (3 x 7). This is an instance of the distributive property of multiplication over addition which (in one form or another) is used to some extent in contemporary programs of mathematics instruction. Specific instances of this property often are illustrated with arrays.

Although Schell (1964) reported some findings regarding third-graders' ability to use distributivity, his observations were based upon a very limited amount of instruction: two introductory lessons. Such findings are tenuous at best.

From a more comprehensive investigation with third-grade pupils and their beginning work with multiplication, Gray (1965) found that an emphasis upon distributivity led to "superior" results when compared with an approach that did not include work with this property. The superiority was statistically significant on three of four measures: posttest of transfer ability, retention test of multiplication achievement, and retention test of transfer. On the remaining measure -- posttest of multiplication achievement -- children who had worked with distributivity scored higher than those who had not, but the difference was not statistically significant.

Gray's findings add further support to a growing body of evidence on advantages to be expected from instruction which emphasizes mathematical meaning and understanding. The "pay-off" may not always be particularly evident in terms of skills-achievement immediately following instruction. Rather, the pay-off is much more clearly evident in relation to factors such as comprehension, transfer, and retention.

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What has been found about other approaches to early work with multiplication?

Fullerton (1955) compared two methods of teaching the "easy" multiplication facts to third-graders: (1) an inductive method by which pupils developed multiplication facts from word problems, using a variety of procedures; and (2) a "conventional" method which presented multiplication facts to pupils without involving them in the development of such facts. In this instance a significant difference in favor of the inductive method was found on a measure of immediate recall of taught facts as well as on measures of transfer and retention.

In another investigation Haynes (1964) concluded that the Cuisenaire approach to multiplication (based upon Gattegno's texts) was no more effective with third-graders than was a "conventional" method exemplified by a well-known and often-used arithmetic textbook published in 1959. [Research on the Cuisenaire approach within other contexts is reported in Bulletin A-5.]

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What things contribute to pupils' success with more advanced work in multiplication?

On the basis of multiple criteria, Schrankler (1967) evaluated the relative effectiveness of two algorithms for teaching multiplication with whole numbers to fourth grade pupils. As interacting factors, he considered (1) three intelligence levels and (2) two readiness backgrounds. From a variety of findings Schrankler concluded that methods using general ideas based on the structure of the number system are more successful than other methods investigated in achieving the objectives of increased computational skills, understanding of processes, and problem solving abilities associated with the multiplication of whole numbers between 9 and 100.

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What is the difficulty level of division combinations?

Little research has been done on the difficulty level of the basic division facts, but great attention has been given to the difficulties inherent in the algorithm. Osburn (1946) noted 41 levels of difficulty for division examples with two-digit divisors and one-digit quotients. Pupils' ability to divide with two-figure divisors has been found to involve a considerable variety of skills varying widely in difficulty (Brownell, 1953; Brueckner and Melbye, 1940). Examples in which the apparent quotient is the true quotient (as in 43)$\overline{92}$ are (of course) much easier than those requiring correcting (such as 43)$\overline{81}$ ), with difficulty increasing as the number of digits in the quotient increases.

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Which is it better to teach: the subtractive or the distributive form of the division algorithm?

During the 1940's and 1950's, the division algorithm typically taught in elementary school mathematics was:

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      2
  23)552        First think
     46         '2's in 5?'
      92
      etc.
```

(Some people refer to this as the distributive algorithm.)

Today, a multiplicative and subtractive approach to the division algorithm has come back into use:

```
  23)552
     230    10 x 23
     322
     230    10 x 23
      92
      etc.
```

In one investigation comparing use of the conventional (or distributive) and the subtractive forms, Van Engen and Gibb (1956) reported that there were some advantages for each. They evaluated pupil achievement in terms of understanding the process of division, transfer of learning, retention, and problem solving achievement. Among their conclusions were:

(1) Children taught the subtractive method had a better understanding of the process or idea of division in comparison with the conventional method used. Use of this algorithm was especially effective for children with low ability. Those with high ability used the two methods with equivalent effectiveness.

(2) Children taught the conventional (distributive) method achieved higher problem solving scores (for the type of problem in the study).

(3) Use of the subtractive method was more effective in enabling children to transfer to unfamiliar but similar situations.

(4) The two procedures appeared to be equally effective on measures of retention of skill and understanding. This seems to be more related to teaching procedures, regardless of the method of division.

In another study of the division algorithm with twelve fourth grade classes, Dawson and Ruddell (1955) compared the effectiveness of (1) "common textbook practices" and (2) a procedure in which division was presented as "a special case of subtraction." The second procedure also stressed "meaningful" instruction through much use of discussion and manipulative materials. The investigators concluded that this latter approach resulted in significantly higher achievement (immediately following instruction as well as after a retention period of seven weeks), and increased ability to solve examples in a new situation. It also helped pupils to develop greater understanding of division and its interrelationships with subtraction, multiplication and addition than did the "common textbook practices" approach. Whether these findings were related primarily to the emphasis on (1) subtractive concepts or (2) method of instruction or (3) use of materials cannot, however, be ascertained from the design of the study.

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What is the most effective method of teaching pupils to estimate quotient digits?

Meaningful algorithms ultimately may need to be shortened to gain efficiency in division. Then pupils must be able to estimate quotient digits systematically. Several methods have been advocated: (1) the "apparent" or "round-down" method, in which the divisor is rounded to the next lower multiple of 10; and (2) the "increase-by-one" methods, in which the divisor is rounded to the next higher multiple of 10, (a) either "round-both-ways," depending on whether the digit in units' place is less or greater than 5, or (b) "round-up," no matter what. Which method do you use?

| | apparent or round-down | increase-by-one | |
|---|---|---|---|
| | | round-up | round-both ways |
| $42\overline{)216}$ | $\begin{array}{r}5\\4\overline{)21}\end{array}$ | $\begin{array}{r}4\\5\overline{)21}\end{array}$ | $\begin{array}{r}5\\4\overline{)21}\end{array}$ |
| $47\overline{)216}$ | $\begin{array}{r}5\\4\overline{)21}\end{array}$ | $\begin{array}{r}4\\5\overline{)21}\end{array}$ | $\begin{array}{r}4\\5\overline{)21}\end{array}$ |

Efforts to resolve the issue of which method is best have focused on analysis and comparison of the success of each method on a specified population of division examples. Morton (1947), for instance, analyzed 40,014 examples and found that an "increase-by-one" method was "correct" 61% of the time and the "apparent" method was "correct" in 53% of the cases. Grossnickle conducted a series of such studies, as did Osburn (1950), carefully comparing the cases where each method resulted in the "correct" quotient digit.

If success on first trial were the criterion, then "round-both-ways" would be recommended. However, not only must the child learn two rules, but the true quotient digit may then be either greater or less than an estimated quotient digit. Grossnickle (1932) and Osburn (1946) advocated the "apparent" method, since the estimated digit is always corrected (if necessary) by decreasing it. With the "round-up" method, the estimated digit is corrected (if necessary) by increasing it -- a procedure very much like that used in the subtractive algorithm.

Hartung (1957) critically reviewed these and other analytic studies. He concluded that "round-up" was the most useful method, because of the advantages of obtaining an estimate that is less than the true quotient (which decreased the need for erasing), and because of the relative simplicity of a "one-rule" method.

In one of the few experimental investigations on this topic, Grossnickle (1937) studied the achievement of groups taught by "round-down" and "round-both-ways." He concluded that there were no significant differences between the scores of the two groups.

How children apply the method was studied by Flournoy (1959), who found that "round-both-ways" was used as effectively as the "round-down" method. She stressed that perhaps not all children should be taught the "round-both-ways" method. Carter (1960) reported that pupils taught this method were not as accurate as those taught a one-rule method -- nor did pupils always use the method taught.

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**What is the role of measurement and partition situations in teaching division?**

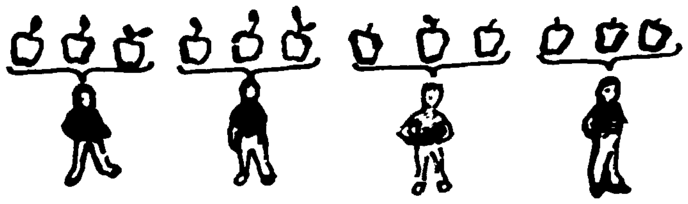
Measurement problems involve situations such as:

If each boy is to receive 3 apples, how many boys can share 12 apples? (Find the number of equivalent subsets.)

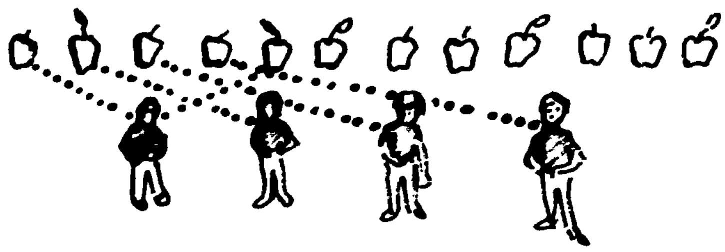
Partition problems involve situations like this:

If there are 4 boys to share 12 apples equally, how many will each boy receive? (Find the number of elements in each equivalent subset.)

In a study with second graders (chosen since commonly children at this level have had little experience with division which would interact with the teaching in the research study), Gunderson (1953) reported that problems involving partition situations were more difficult than problems involving measurement situations. The ease of visualizing the measurement situation probably contributes to this. For instance, for the illustration above, a picture like this could be formed:



For the partition situation, the drawing might be:



and so on!

Zweng (1964) also found that partition problems were significantly more difficult for second graders than measurement problems. She further reported that problems in which two sets of tangible objects were specified, were easier than those in which only one set of tangible objects was specified. In an earlier study, Hill (1952) found that pupils in the intermediate grades indicated a preference for measurement situations, but performance was similar on both types.

In the study in which they compared two division algorithms, Van Engen and Gibb (1956) found that children who used the distributive algorithm had greater success with partition situations, while those who used the subtractive algorithm had greater success with measurement situations.

Scott (1963) used the subtractive algorithm for measurement situations and the distributive algorithm for partition situations. He suggested that: (1) use of the two algorithms was not too difficult for third grade children; (2) two algorithms demanded no more teaching time than only one algorithm; and (3) children taught both algorithms had a greater understanding of division.

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Project on Interpreting Mathematics Education Research
Center for Cooperative Research with Schools
302 Education Building
The Pennsylvania State University
University Park, Pennsylvania 16802

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