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ABSTRACT

This booklet has been written for elementary school teachers as an introductory survey of the real number system. The topics which are developed include the number line, infinite decimals, density, rational numbers, repeating decimals, irrational numbers, approximation, and operations on the real numbers. (RS)

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THE SYSTEM OF REAL NUMBERS

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TOPICS IN MATHEMATICS

FOR ELEMENTARY SCHOOL TEACHERS

BOOKLET 11

THE SYSTEM OF REAL NUMBERS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
Washington, D.C. 20036

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PREFACE

This booklet is one of nine new units in a series introduced in 1964 by the National Council of Teachers of Mathematics (NCTM). Because the first eight booklets were so well received, being reprinted several times, it was felt that an extension of subject matter would be valuable.

Like the earlier booklets (Nos. 1–8), the new units are written for elementary school teachers rather than their pupils. Each booklet presents an exposition of a basic topic in mathematics. The topics are among those with which elementary school teachers need to be familiar in order to treat with understanding the mathematics usually taught in the elementary school. The booklets present introductions to topics, not exhaustive treatments; the interested reader may study the subjects in greater depth in other publications.

The topics were chosen especially with a view to providing background material for those teachers who believe that the learning experiences provided for children in their early school years should include a simple introduction to some of the *central unifying concepts in mathematics*. Many teachers have found that their professional education did not prepare them to teach arithmetic in a manner consistent with this view. It is the hope of the authors and of the NCTM that this series of booklets may be helpful to these teachers as well as to others, and indeed to all interested in improving mathematics instruction.

The earlier titles are these:

Booklet No. 1: *Sets*
Booklet No. 2: *The Whole Numbers*
Booklet No. 3: *Numeration Systems for the Whole Numbers*
Booklet No. 4: *Algorithms for Operations with Whole Numbers*
Booklet No. 5: *Numbers and Their Factors*
Booklet No. 6: *The Rational Numbers*
Booklet No. 7: *Numeration Systems for the Rational Numbers*
Booklet No. 8: *Number Sentences*

The new titles are as follows:

Booklet No. 9: *The System of Integers*
Booklet No. 10: *The System of Rational Numbers*

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Booklet No. 11: *The System of Real Numbers*
Booklet No. 12: *Logic*
Booklet No. 13: *Graphs, Relations, and Functions*
Booklet No. 14: *Informal Geometry*
Booklet No. 15: *Measurement*
Booklet No. 16: *Collecting, Organizing, and Interpreting Data*
Booklet No. 17: *Hints for Problem Solving*

It is suggested that, ordinarily, the books be read in the order of the numbers assigned them, since the spiral approach was used to some extent in their preparation.

The new booklets were begun in 1966 by the members of a summer writing group. The writers herewith express their deep appreciation to the following persons for reading parts of the manuscripts and consulting with the writers during the preparation of the booklets: Joseph M. Trotter, principal of San Luis Rey School, and Bonita Trotter, teacher at Laurel School, both of the Oceanside Union School District; John M. Hoffman, director of the Community Educational Resources Section of the San Diego County Department of Education; and James E. Inskeep, Jr., professor of education at San Diego State College. The writers are especially indebted to Alice C. Beckenbach for extensive help in organizing and editing the material for several of the booklets. They are most grateful, also, to Elaine Barth and her fine staff of typists for their excellent work in manuscript preparation.

The new project, undertaken to carry forward the work of the earlier one, was initiated and sponsored by the NCTM Supplementary Publications Committee under the chairmanship of William Wooton. Financial support was provided by the NCTM, which now extends its appreciation to members of the writing group that produced the present extension of the "Topics" series. Their names are given below.

George Arbogast
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Marguerite Brydegaard
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Helen L. Curran
Patricia Davidson
Walter Fleming
Joseph Hashisaki
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David Johnson
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William Wooton
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CONTENTS

BOOKLET NUMBER ELEVEN:

THE SYSTEM OF REAL NUMBERS

INTRODUCTION

In Booklet No. 10 of this series, the system of *rational numbers* was discussed. This system includes certain number systems previously investigated: the system of *whole numbers* (Booklet No. 2), the system of *nonnegative rational numbers* (Booklets No. 6 and 7), and the system of *integers* (Booklet No. 9). Now we are going to study a still more extended system, the system of *real numbers*.

We should first give special attention to some characteristics of the rational number system that will be of particular importance to the development of our study. One is the matter of notation. Rational numbers are most commonly denoted by *fractions*, such as 3/2, 5/4, and 6/8. In the fraction 3/2, for example, the number 3 is the *numerator* and the number 2 is the *denominator*. Another common representation is the *decimal* notation, in which the position of a digit has a place value that is used to convey the idea that the denominator is a power of ten. Thus we write 0.25 for 25/100 or 1/4. Remember that 2/8, 1/4, and the decimal numeral 0.25 are all names for the same rational number. (It will often be convenient to use simply the term "decimal" in place of the longer term, "decimal numeral.")

In Booklet No. 7: *Numeration Systems for the Rational Numbers*, an extended type of decimal representation was mentioned. This is the repeating infinite-decimal notation. Here the word "infinite," which we shall use repeatedly in this booklet, reminds us that these decimals, unlike 0.25, do not have a last digit. The whole idea is of such importance that it will be covered in detail later. For the present, let us look at an illustration or two that will remind us of some facts: For example, among the notations for the rational number denoted by 1/3 is 0.333 $\cdots$, where the dots indicate an infinite extension of the pattern of the repeated digit 3. Or again, 1/11 can be written as 0.090909 $\cdots$ and 2/9 as 0.222 $\cdots$.

Another important property of the rational numbers is quite difficult to phrase carefully without further development of our study. This is the property of *betweenness*, which permits us to use rational numbers to express measurement or calculation with greater and greater precision. In fact, all computations, all measurements, all practical uses of numbers, are made with rational numbers alone.

Why, then, extend the system of rational numbers? We must carry out an extension in order to avoid clumsy and inexact use of language and to gain a clear picture of the relationship between numbers and the number line. When we have performed the extension, it will be much easier to talk about the unlimited possibility of conceptual accuracy to which we alluded in the last paragraph. We are going to develop a new property called *completeness*, which the system of rational numbers lacks.

Without attempting at the moment to define this concept of completeness, let us look at an example or two. First, consider a "positive number" that when multiplied by itself, or squared, is 2. We want a number, which we shall designate by $\sqrt{2}$ and call the positive square root of 2, such that $\sqrt{2} \times \sqrt{2} = 2$ and $\sqrt{2} > 0$. Sometimes we see inexact expressions, such as 1.414, for $\sqrt{2}$. But 1.414 is not a square root of 2; it is a square root of 1.999396, as you can determine by multiplying 1.414 by 1.414. We shall see later that *there is no rational number that is the positive square root of 2.*

Shall we give up and say there is no number that is the positive square root of 2? Or shall we invent a new number system such that 2 will have a positive square root among the numbers of the system? The ancient Greeks followed the first course and turned aside from arithmetic because it was incomplete in this way. Modern mathematicians have chosen the second course.

It would be possible to invent a system barely sufficient to provide 2 with a positive square root and to satisfy the basic rules of arithmetic. In such a system, it would turn out that 3 did not have a square root. Then we should have to start all over again to remedy this situation. Actually, it would be possible to formulate a system in which every positive number in the system has a positive square root, but in which there would not exist any number representing the ratio of the circumference of a circle to its diameter.

We shall, however, find it quite possible to do a *complete* job in the sense that all ratios of geometrical quantities—all values that could be the result of conceptual measurements—will be in the system we construct.

The procedure we shall use will be formulated in the next section. It may be thought of as "filling the number line."

As we create the new numbers, we should also create definitions for arithmetical operations on them—addition, subtraction, multiplication, and division—that will preserve the basic properties of these operations as they apply to the rational numbers. We want the new system to include the rational numbers not only as a subset but also as a *subsystem*, just as the system of rational numbers may be considered to include the system of integers. (See Booklet No. 10: *The System of Rational Numbers*.) This goal will be accomplished by defining the operations on the new numbers in terms of operations on rational numbers.

THE NUMBER LINE AND INFINITE DECIMALS

The Rolling Circle

Let us make a number line in the following way. We draw a horizontal line, which we assume to be perfectly straight and entirely immune from all imperfections of the real world. It is not disturbed by lumps in the paper; it has no gaps due to spaces between molecules. This line exists only in our imagination, of course, but drawing parts of it will aid our thought and help us to communicate information.

On the line, we choose an arbitrary point and mark it 0. To the right of this point, we choose a second arbitrary point and mark it 1. (See Fig. 1.) Using the distance between these two points as a basic unit, we mark additional points to the right as 2, 3, $\cdots$. Then we reflect to the left as in a mirror held at 0, marking the reflections $^{-}1$, $^{-}2$, $\cdots$. (See Booklet No. 9: *The System of Integers*.)

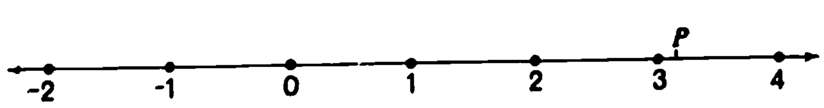

FIGURE 1

One further point, P, has been put on the line between the points marked 3 and 4, in order to illustrate our next concept, the idea of a *decimal expansion*. The point was obtained, at least theoretically, in this way: We made a mark on the circumference of a circle whose diameter measures the same as the distance between the points 0 and 1 marked on our line. Matching that mark to the zero point of the line, we rolled the circle on the line (without slipping) until the mark again met the line. This point of contact was then called P, as indicated in Figures 1 and 2.

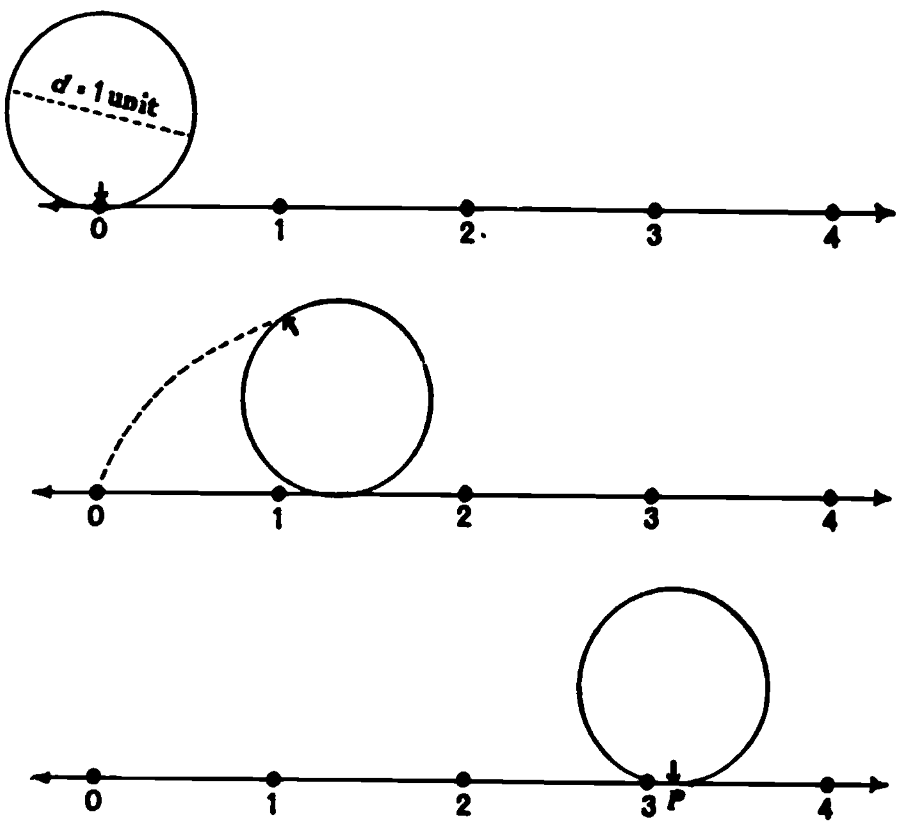

FIGURE 2

Suppose that we subdivide *each* of the intervals between the points marked with integers into *ten congruent smaller intervals.* In order to pursue our example more directly, we shall draw the result only for the segment between the points marked 3 and 4, as shown in Figure 3. As the work progresses we shall continue to concentrate on intervals containing the illustrating point P. For convenience, such phrases as "the point marked 3" will be replaced by "point 3." Although the numeral names both a number and a point, the context will tell us which meaning is intended. This should not prove confusing; the same names are used for different types of things in many other areas of our experience. Consider the sentence "Rose picked a rose," for example.

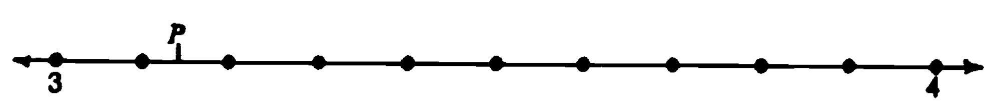

FIGURE 3

Denote now the end points of the intervals from left to right by 3.1, 3.2, and so on, to 3.9. To keep our notations consistent, replace 3 and 4 by 3.0 and 4.0, respectively. These points named by decimals with *one* digit after the decimal point will be called *marking points of the first stage.*

The corresponding numbers, of course, are rational numbers. Now P is between 3.1 and 3.2. (See Fig. 4.)

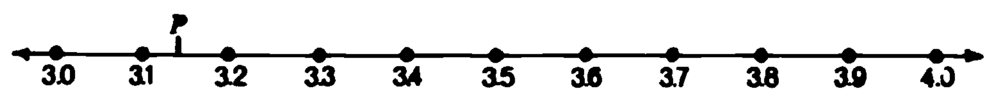

FIGURE 4

Let us continue the process of subdivision. For each successive step a drawing will show the relevant part of the previous line segment magnified by a factor of ten. It will be as if we looked at the part of the line containing P under a succession of microscopic lenses, each lens ten times as powerful as the one before.

In the next stage we see a magnification of the interval between 3.1 and 3.2. It has, in turn, been subdivided into ten congruent smaller intervals, as shown in Figure 5. We adjoin decimal digits as before, to

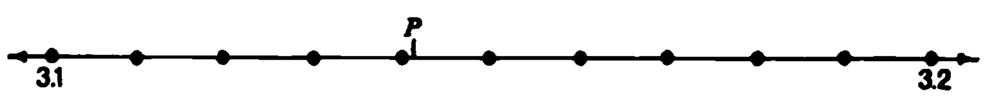

FIGURE 5

name the subdividing points. (See Fig. 6.) Now, however, there are *two* digits after the decimal point, and the points so marked are called *marking points of the second stage*. Again, they correspond to rational numbers. Note that marking points of the first stage are also marking points of the second stage. They get a new name by the adjunction of a 0 at the right; but it also is not unusual for an object, whether mathematical or other, to have several different names. For example, Dad is Mr. Brown or Gregory or Uncle Greg, depending on who is speaking about him. P is now to be seen (Fig. 6) between 3.14 and 3.15.

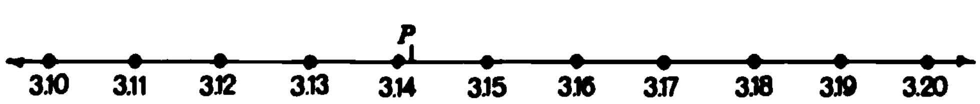

FIGURE 6

Let us magnify this second-stage segment. Now, however, we shall show it only after we have finished naming the appropriate *marking points*

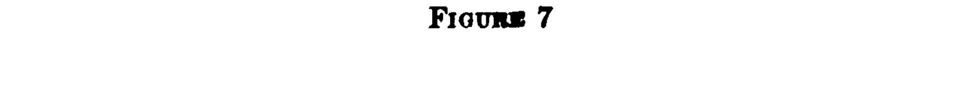

FIGURE 7

of the third stage. (See Fig. 7.) This time the higher magnification shows us that P is between 3.141 and 3.142.

Try another magnification; again we illustrate only the segment containing P. In Figure 8 it appears after the *marking points of the fourth*

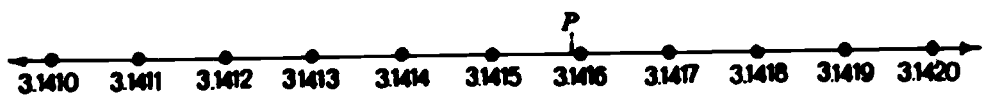

FIGURE 8

stage have been indicated and named. We see that P is most of the way from 3.1415 to 3.1416. Still another magnification would indicate P between 3.14159 and 3.14160.

To obtain an overview of the whole process thus far, we combine four magnifications in the single representation of Figure 9.

We can simplify the descriptions of the location of P by a simple convention. At any given stage, instead of describing the location of a point as being in the segment between two marking points of that stage, we shall simply name the marking point immediately to the left of the point. (Of course, if a point is a marking point in its own right, we name that point according to its appropriate stage.) Thus, P would be indicated by 3.1 at the first stage, 3.14 at the second, and then successively by 3.141, 3.1415, and 3.14159. The standard way of saying this is to indicate the stage by the number of decimal digits and to use constructions such as "The decimal expansion of P to two decimal digits is 3.14," or simply "P is given to two decimals by 3.14."

Remember that this is an imaginary experiment. We are rolling a "perfect" circle on a "perfect" line. If we actually tried the experiment, our accuracy would be limited by imperfections in the circle and in the line, by the circle's slipping as it rolled, and by our inability to mark and measure with exact precision. Already we are at or beyond the limits of careful machining for sizes of things that we customarily handle. Industrial standards, for example, require that the pistons and piston rings of precision engines be accurate to better than one ten-thousandth of an inch (0.0001 in.), whereas we are already talking about intervals whose length is a hundred-thousandth (0.00001) of our original unit. Fortunately, mathematical calculation is not limited by such mundane considerations. By the use of electronic calculators, others have carried our theoretical experiment through ten thousand stages, working with trigonometrical formulas, which do not require any real circle or real line.

But our imagination is not limited by the cost or the time required for computation on a modern computing machine. We can imagine the process continuing literally forever, as decimal digit follows decimal digit

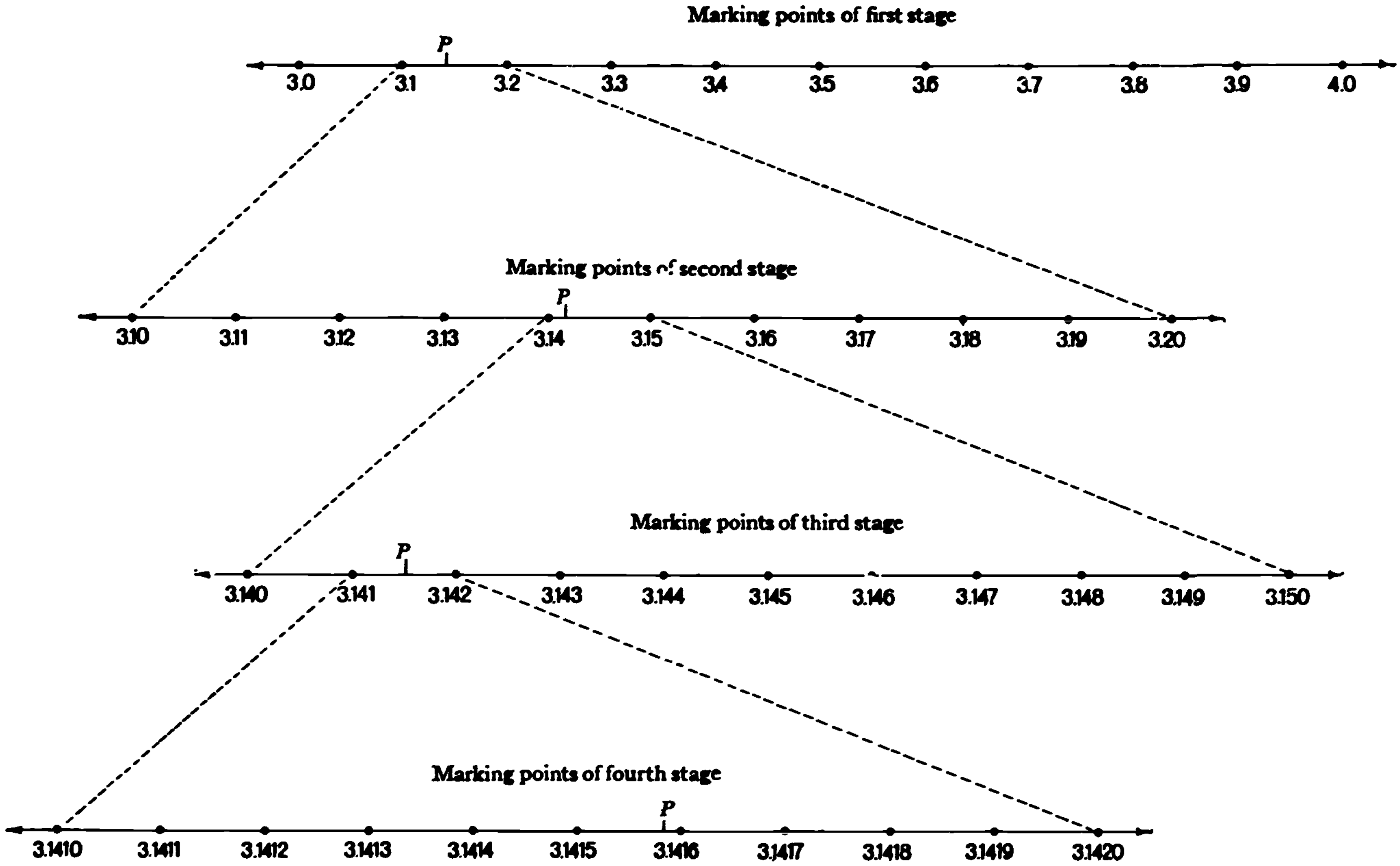

FIGURE 9

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in the designation of P. It is this limitless, or infinite, decimal numeral that we think of as the name for P. All the finite preliminary numerals are merely initial stages.

The Real Numbers

We want to consider two other ideas that are related to the process we have been discussing.

The first idea is this: Each marking point has been given a decimal numeral. This decimal, as we have said, also denotes a *number*. Thus, *every marking point has an associated rational number*. This number specifies the length, in terms of our unit, of the segment bounded by zero and the marking point. We now invent a "number," designated by the numeral for P, that describes the distance from 0 to P. Note that this is perfectly in accord with our previous use of infinite decimals. The point one-third of the distance from 0 to 1 is denoted by $0.333 \cdots$, and the numeral $0.333 \cdots$ is associated with the rational number 1/3.

The second idea we have already suggested by implication. While our attention was fixed on the specific part of the line that contained P, we were aware of the possibility of performing the subdivisions at each stage on the *whole line*. At the first stage, *every* integer interval is divided into ten parts. At the second, there are one hundred subdivisions between *any two successive* integers, and so on. For example, at the second stage the marking point just to the right of 1.00 is 1.01; the point just to the left of 2.00 is 1.99. What comes just to the right of 0.50? Just to the left? The points 0.51 and 0.49, of course.

We should pay particular attention to that part of the line to the left of 0, because it may not be sufficiently clear merely to say that we reflect the right-hand side and use negative signs. At the first stage, the segment from ⁻1.0 to 0.0 would have the appearance shown in Figure 10.

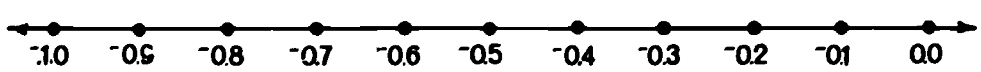

FIGURE 10

If our circle had rolled to the left, where would we have marked the resulting point? Between ⁻3.1 and ⁻3.2. Call the point Q instead of P, to distinguish between the experiments. (See Fig. 11.)

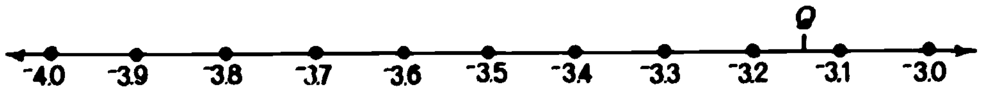

FIGURE 11

What decimal names Q at this stage? Remember the reflection idea. Think of a mirror mounted at 0. The "negative" part of the line is the image in the mirror of the "positive" part. When we chose 3.1 to represent P at the first stage, we chose the marking point between P and 0. Q is the reflection of P; and ¯3.1, between Q and 0, is the reflection of 3.1. So we associate Q and ¯3.1.

Now let us reconsider the number line in Figure 10. At the second stage, what are the marking points just left and right of ¯1.00? They are ¯1.01 and ¯0.99, respectively, as shown in Figure 12.

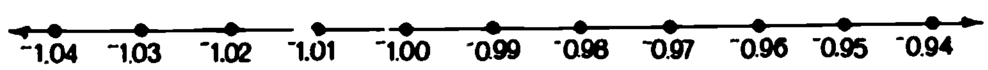

FIGURE 12

Now think of the whole line as we go from one stage to the next. The subdivisions increase in number; the intervals become shorter and shorter. Our intuition suggests that the intervals ultimately narrow to a point.

This concept turns out to be very difficult to examine rigorously in full logical detail, and almost as difficult to think about clearly in an intuitive manner. Some sources of our difficulties are the following:

1. *The problem of imagining the infinite.*—Infinite subdivisions and infinite decimals are far beyond our day-to-day experience.

2. *Our use of inexact images.*—We draw blobs of ink, pencil, or chalk and call those blobs points and lines. If we say, for example, "No matter how close distinct points are, eventually some marking point will come between them," we may get an answer like this: "But I'll mark two points here so close that they touch. You'll never get another point between them." The answer really implies, "If I make two *blobs* that touch, you can't put a *blob* between them." But blobs are not the same as points, because points have no dimension and cannot touch without being identical.

3. *Logical difficulties with infinite sets and infinite processes.*—These very real difficulties have occupied mathematicians for thousands of years. Only within the last hundred years has there been developed a method of handling infinite sets and infinite processes that is acceptable to the vast majority of modern mathematicians.

To return to our construction of the real numbers: At each step, every point is assigned to a nearby marking point named by a decimal numeral. At each successive step, the decimal numerals contain one more digit. As we subdivide the intervals without limit, we assign to each point its infinite decimal. In turn, each of the finite decimal numerals for the marking points names a particular rational number that indicates more and more closely the distance and direction of the point from the origin, or zero point. The infinite decimal, then, names this latter distance, or "real number."

For every point on the number line, there is an associated infinite decimal that names a real number. The converse of this statement, that for every infinite decimal there is an associated point, is a hypothesis of our development. In a way this is just what was meant when we required that the number line be free from gaps or imperfections.

What we want are the following one-to-one correspondences:

Real Numbers ↔ Infinite Decimals ↔ Points on the Number Line.
(Read "↔" as "correspond[s] to.")

We have given intuitive reasons why there is at least one infinite decimal for each real number and each point. We have also, by definition, made a one-to-one correspondence between real numbers and points by regarding a real number as expressing the length of a segment, with an appropriate sign adjoined to indicate the direction from the origin. We have not discussed the possibility that a point might have more than one infinite decimal associated with it. As a matter of fact, this happens in some special cases, and we must discuss these before going on.

When we were assigning marking points to points in the intervals, we said that if a point *is* a marking point, then it is assigned to itself. Otherwise, at each stage we take the nearest marking point to the left for points on the positive side of the origin and the nearest marking point to the right for points on the negative side of the origin.

What would happen if we always used just one rule, say the following? "At every stage, assign to any point on the positive side of the origin the nearest marking point to its left; and to any point on the negative side assign the nearest marking point to its right." This would of course mean that a marking point could no longer be assigned to itself.

For one thing, we should have to make a special definition for 0, which is on *neither* the positive *nor* the negative side. But other undesirable things would happen. Take, for example, the point originally marked 1. At the first stage, what is the nearest marking point on its left? Clearly, 0.9. What at the second stage? It is 0.99. Continue the process—at the tenth stage it is 0.9999999999. It becomes obvious that the resulting infinite decimal for the point is just "0." followed by an infinite string of 9's. Hence $1.000\cdots$ and $0.999\cdots$ name the same point.

We can see this duplication of names in another way. We know that the point 1/3 of the way from 0 to 1 has the infinite decimal $0.333\cdots$, an infinite string of 3's. This implies that the point 2/3 of the way from 0 to 1 has the decimal $0.666\cdots$, which can be expressed as "$2 \times 0.333\cdots$." Now, how about the point 3/3 of the way? We have

$$3 \times 0.333 \cdots = 0.999 \cdots = \frac{3}{3} = 1.000 \cdots .$$

A similar duplication happens at *every* nonzero marking point of *every* stage. Thus $0.249999 \cdots$ names the same point as $0.250000 \cdots$, and $^{-}1.73219999 \cdots$ names the same point as $^{-}1.7322000 \cdots$.

We are going to rule out of our consideration all infinite decimals that trail off in an infinite string of 9's. It will turn out that this takes care of all duplication difficulties. Our rule of assigning marking points to themselves takes care of this problem.

Order

In our discussion of the rational number system in Booklet No. 10, it was noted that the rational numbers are ordered; that is, for any two different rational numbers, one is less than the other. An ordering of the real numbers should correspond to the relationship between decimals and points on the number line. We try to make this precise with the following rules.

A. Grouping the infinite decimals into three classes (trichotomy)
   1. Zero: $0.000 \cdots$. If desired, it may be prefaced with either a negative or a positive sign; the number named is unchanged.
   2. Positive decimals: At least one digit is not 0 (for example, $0.0001000 \cdots$), and the decimal is unsigned or prefaced with a positive sign.
   3. Negative decimals: At least one digit is a nonzero digit (for example, $^{-}5.000 \cdots$), and the decimal is prefaced with a negative sign.

B. Comparing the numbers
   1. If one of the decimals to be compared is zero—
      a) Zero is *less* than any real number represented by a positive decimal.
      b) Zero is *greater* than any real number represented by a negative decimal.
   2. If the decimals have opposite signs, any real number represented by a negative decimal is less than any real number represented by a positive decimal.
   3. If both decimals are positive, find the first digit at which the decimal representations differ. The number whose representation has the *lesser* digit at this point is the lesser:

$$0.457\underline{3}9876 \cdots < 0.457\underline{4}1245 \cdots .$$

$$\underline{0}7.352 \cdots < \underline{2}1.352 \cdots .$$

If the difference is to the left of the decimal sign, we can still apply the rule strictly if we remember that zeros can fill in the empty places of the numeral. Thus, in the second example above, we can replace $7.352 \cdots$ with 07.352, to balance the beginning points of the two numerals.[1]

4. If both decimals are negative, find the first digit at which the representations differ. The number whose representation has the *greater* digit at this point is the lesser number:

$$^{-}4\underline{4}.372 \cdots < {}^{-}4\underline{3}.372 \cdots .$$

Density

Between any two points with different decimal numerals there is a marking point.

We shall not try to give a formal proof but shall just indicate ways that a marking point can be found. There are in fact an infinite number of choices.

We use the following steps:

1. If the numbers have opposite signs, we know that zero is between them.
2. If the numbers have the same sign, we try "rounding off" the one representing the point further from the origin. Rounding off is the process of replacing with zeros all digits to the right of some particular chosen digit. This can be done at any digit after the first difference between the numerals appears. The process of rounding off is also called "cutting short."
   a) $R \leftrightarrow 0.45739876 \cdots$.
      $S \leftrightarrow 0.45741245 \cdots$.
      Marking points between R and S: $0.45740000 \cdots$,
      $0.45741000 \cdots$, $0.457412000 \cdots$, etc.
   b) $R \leftrightarrow {}^{-}0.0321981 \cdots$.
      $S \leftrightarrow {}^{-}0.0322012 \cdots$.
      Marking points between R and S: ${}^{-}0.0322000 \cdots$,
      ${}^{-}0.32201000 \cdots$, etc.
3. If this does not seem to work, we try the other point and another process:
   a) $R \leftrightarrow 0.4730000 \cdots$.
      $S \leftrightarrow 0.4712000 \cdots$.

[1] Notice that we originally reserved the dot notation at the end of a numeral to imply the projection of an established and readily visible pattern. Now we are dealing with general real numbers that may not have any pattern, but that we cannot write out in full. So we need to use the dot notation just to indicate continuation of real numbers: if no pattern is apparent before the dots begin, assume that none exists. Thus, for the examples in rules 3 and 4 no pattern is implied.

Cutting short 0.4730000··· will not work because it *is* cut short. Instead, we try "rounding up" 0.4712000··· . The process used here is to increase by 1 a digit in the numeral representing the point nearer the origin and then to cut short. Rounding up can be done as soon as a different digit occurs that *can* be increased, that is, a digit that is not 9.

Marking points between R and S: 0.4720000··· , 0.471300··· , etc.

Note how all these suggestions would fail to put a point between those represented by 0.249999··· and 0.25000··· . The former cannot be increased without exceeding the latter. In turn, we cannot decrease 0.25000··· and stay above 0.24999··· . This is another way of seeing that these two decimals denote the same real number. You should recall that we have ruled out the one with the infinite string of nines.

We can put a number associated with a marking point between any two different real numbers. We can approximate any real number as closely as we wish by a number associated with a marking point. That is, given any distance, no matter how small, from a particular point on the number line, we can find a marking point within that distance.

For example, let us find a marking point within 0.00001 unit of the point P in the example of the rolling circle. Recall that P was located between the points denoted by 3.14159 and 3.14160. These are only 0.00001 units apart, and P is *between* them. Consequently, P must be nearer to each of them than 0.00001. Hence, both 3.14159 and 3.14160 are marking points within 0.00001 unit of P.

In the "Introduction" we mentioned two properties of the rational number system, "betweenness" and "completeness." These properties enable us to find a marking point between any two points denoted by different decimals and to approximate any real number as closely as we wish by using a number associated with a marking point. Thus we can use rational numbers in general, and numbers represented by decimal fractions ending in zeros (called terminating decimals) in particular, to approximate all real numbers and measurements.

In Booklet No. 6: *The Rational Numbers*, a similar property of the rational numbers was discussed: the rational numbers are *dense*. That is, between any two rational numbers there is another rational number. Here we are dealing with a generalization. Between every two *real* numbers there is a *rational* number—in fact, one that is denoted by a terminating decimal.

The mathematical usage is as follows. Suppose we have one set S of numbers and another set T that is a subset of S; then $T \subset S$, with the property that, given any two numbers of S, there is a number of T between them; *then T is said to be dense in S.*[2]

For example, the rational numbers are dense in themselves (between every two rational numbers there is a rational number). The rational numbers are dense in the real numbers. (Actually, the irrational numbers also are dense in the real numbers.) The integers are *not* dense in the rational numbers, since, for example, there is no integer between 1/2 and 2/3. Are the integers dense in themselves? Is there an integer between 2 and 3? Certainly not; so the integers are not dense in themselves. Are the numbers with terminating decimals dense in the rational numbers? Yes; it is easy, for example, to find a number with a terminating decimal between $0.72000\cdots$ and $0.73000\cdots$. One such terminating decimal is $0.725000\cdots$.

It is indeed fortunate that this possibility of approximation exists, since direct calculations using real numbers expressed in their infinite-decimal form are generally impossible. If the decimals have a special form, however, then the operation may often be carried out. We have already written

$$3 \times 0.3333\cdots = 0.999\cdots,$$

but the right-hand side is not allowed; so, in permissible notation, we have

$$3 \times 0.3333\cdots = 1.000\cdots.$$

Again, we can write equations like

$$0.333\cdots - 0.111\cdots = 0.222\cdots;$$

but we recognize this as just a cumbersome way of writing $1/3 - 1/9 = 2/9$, as we can check through expanding the fractions into decimals by the division algorithm.

Exercise Set 1

1. At the third stage as defined on page 6 in the text, name the marking point that is—
   a. Just to the left of 1.000.
   b. Just to the right of $^{-}1.000$.
2. Find a marking point between the points named by $^{-}0.20134\cdots$ and $^{-}0.20245\cdots$.

---

[2] This definition is usually replaced in modern advanced mathematics by

T is dense in S if, for each member a of S, there is a member of T as close as we please to a.

The two definitions are not strictly equivalent, but the definition given in the text is better adapted to our discussion of approximation, since we wish to emphasize the choice involved.

3. Arrange the following real numbers in increasing order: $^-0.01234\cdots$, $^-0.1234\cdots$, $1.234\cdots$, $^-4.2310\cdots$, $12.345\cdots$, $^-100.00\cdots$.

4. Replace the following decimal expressions by the correct decimal form (that is, without the infinite string of 9's) designating the same value:

a. $0.20999\cdots$

b. $^-1.12999\cdots$

c. $2.134999\cdots$

RATIONAL NUMBERS AND REPEATING DECIMALS

Repeating Decimals

In Booklet No. 7: *Numeration Systems for the Rational Numbers*, a connection was established between rational numbers and repeating decimals. We are now going to treat this material again from a slightly different viewpoint. We wish to show that the set of rational numbers is identified with a very special subset of the set of real numbers—namely, the subset of real numbers for which the decimals have an infinite repeating pattern. Here are some repeating decimals:

$$\begin{aligned}&0.333\cdots\\&20.202020\cdots\\&^-5.73012012012\cdots\\&13.25000\cdots\end{aligned}$$

In all cases, after perhaps a certain amount of initial hesitation, which may extend to the right of the decimal point, there occurs a digital pattern that continues indefinitely and consists of a fixed number of consecutive digits that repeat in the same order.

In the first example the pattern begins at the decimal point and consists of a single digit, 3.

In the second, the pattern begins two digits to the left of the decimal point and consists of two consecutive digits, 20.

In the third, the pattern does not start until the third digit to the right of the decimal point. It consists of three digits, 012.

The fourth pattern also begins three digits to the right of the decimal point but consists of a single digit, 0.

It is frequently convenient to use bars rather than dots to denote repetitions. Bars are placed over the digits of a pattern, or "block." Bars are never placed to the left of the decimal point. Thus, the four examples could be rewritten

$$0.\overline{3}$$
$$20.\overline{20}$$
$$^{-}5.73\overline{012}$$
$$13.25\overline{0}$$

Sometimes we may not wish to put bars as far to the left as we can. For example, in the equations $1/11 = 0.\overline{09}$ and $10/11 = 0.\overline{90}$ it might be desirable for computational purposes to use bars to show that the patterns are the same:

$$\frac{1}{11} = 0.\overline{09}, \qquad \frac{10}{11} = 0.9\overline{09};$$

or

$$\frac{1}{11} = 0.0\overline{90}, \qquad \frac{10}{11} = 0.\overline{90}.$$

We have two basic points to make in this section:

1. Every repeating decimal is a decimal expression of a rational number.
2. Every rational number has a repeating decimal expansion.

You should be careful to analyze the difference between the two statements. It is conceivable that one might be true and the other false, as is the case with the statements "Every cat is a mammal" and "Every mammal is a cat."

We shall see, however, that statements 1 and 2 are both true; together they establish the result that repeating decimals and fractions are just different representations of the same rational numbers.

Making a Fraction from a Repeating Decimal

To convince ourselves that statement 1 is true, we need a prescription showing how to find an equivalent fraction, given a repeating decimal. We shall choose as a first illustration of the prescription the decimal $0.272727\cdots$, which may be expressed as $0.\overline{27}$.

The method relies on the effect of multiplying by a power of ten a number that is represented by a decimal. Let us begin with the number $a = 0.272727$, which is like our example except that it terminates after six digits:

$$\begin{aligned} a &= 0.272727. \\ 10 \times a &= 2.72727. \\ 100 \times a &= 27.2727. \\ 1{,}000 \times a &= 272.727. \\ 10{,}000 \times a &= 2{,}727.27. \end{aligned}$$

The effect of each successive multiplication by a higher power of ten is to move the decimal point one place to the right in the decimal representation.

Now consider our nonterminating example:

$$b = 0.272727\cdots = 0.\overline{27}.$$
$$10 \times b = 2.727272\cdots = 2.\overline{72} = 2.7\overline{27}.$$
$$100 \times b = 27.272727\cdots = 27.\overline{27}.$$

If we subtract the original value of b from $100 \times b$, we have

$$\begin{array}{r} 100 \times b \\ -b \\ \hline 99 \times b \end{array} \quad \text{or, in this case,} \quad \begin{array}{r} 27.272727\cdots \\ -0.272727\cdots \\ \hline 27.000000\cdots \end{array};$$

so

$$99 \times b = 27, \quad \text{or} \quad b = \frac{27}{99} = \frac{3 \times 9}{11 \times 9} = \frac{3}{11}.$$

Notice how b differs from a. As we multiply a by successive powers of ten the last digit is moved to the left in forming each resulting decimal. There are fewer digits after the decimal point. But for b there is no *last* digit; the decimals for b, $10 \times b$, and $100 \times b$ repeat their blocks indefinitely.

Why did we pick 100 for a multiple? Because it brought the repeating pattern into line with the original. If we had used

$$10 \times b = 2.\overline{72} = 2.7\overline{27},$$

the pattern would not have lined up with $0.\overline{27}$. Of course, 10,000 would work as well as 100:

$$10{,}000 \times b = 2{,}727.\overline{27}.$$

$$\begin{array}{r} 10{,}000 \times b \\ -b \\ \hline 9{,}999 \times b \end{array} \quad \text{or, in this case,} \quad \begin{array}{r} 2{,}727.\overline{27} \\ -0.\overline{27} \\ \hline 2{,}727.\overline{00} \end{array}$$

Thus,

$$9{,}999 \times b = 2{,}727.$$

So

$$b = \frac{2{,}727}{9{,}999} = \frac{3 \times 909}{11 \times 909} = \frac{3}{11}.$$

But why do more work than we must? We choose the least multiple that will do the job.

Let us try another example: $0.\overline{148} = 0.148148\cdots$. This time we need a shift of three places. Therefore, multiply by 1,000:

$$1{,}000 \times 0.\overline{148} = 148.\overline{148} \qquad \text{(why?)};$$

subtract the original:

$$\begin{array}{r} 148.\overline{148} \\ -0.\overline{148} \\ \hline 148.\overline{0} \end{array}$$

We had 1,000 times the original number and subtracted the original number; 999 times the original is left and is equal to 148. So the original number has the value

$$\frac{148}{999} = \frac{4 \times 37}{27 \times 37} = \frac{4}{27}.$$

Here is a general procedure for carrying out the program:

1. Find the pattern length—that is, the number of digits in the pattern.
2. Multiply by the power of ten that will shift one complete pattern to the left.
3. Subtract the original decimal.
4. Form a fraction from the result.
5. Calculate what multiple of the original value resulted from steps 2 and 3.
6. Divide by the result of step 5. The result of the division is a fraction whose value is the same as that of the original decimal.
7. Simplify the fraction.

As a final example, we shall illustrate the method on a decimal whose pattern is slow to establish itself. Take

$$0.4135135\cdots = 0.4\overline{135}.$$

A little more work will be necessary, beginning at step 4, than was needed for the previous examples.

1. Pattern length: $0.4\overline{135}$ (three digits).
2. Multiply by 1,000: $413.\overline{513} = 413.5\overline{135}$.
   (It may be easier to see this without the bars:
   $1{,}000 \times 0.41351351 35\cdots = 413.513513513\cdots = 413.5\overline{135}$.)
3. Subtract:
   $$\begin{array}{rcr} 413.5\overline{135} & = & 1{,}000 \times n \\ -0.4\overline{135} & = & -1 \times n \\ \hline 413.1\overline{0} & = & 999 \times n \end{array}$$
4. Make a fraction: $413.1 = \dfrac{4{,}131}{10}$.
5. Find the multiple: $1{,}000 - 1 = 999$.
6. Divide: $\dfrac{4{,}131}{10} \div 999 = \dfrac{4{,}131}{9{,}990} = n.$

7. Simplify: $n = \frac{4{,}131}{9{,}990} = \frac{153 \times 27}{390 \times 27} = \frac{153}{370}.$

The process will always work because at step 3 it yields a terminating decimal that can be converted to a fraction at step 4.

Making a Repeating Decimal from a Fraction

To find a decimal expression for a rational number, when its fraction name is known, we use the division algorithm. We want to see why the resulting decimal necessarily repeats. This is essentially due to the repetitive nature of the algorithm itself. In performing the calculations we find ourselves eventually repeating the following three steps over and over:

1. Bring down a zero.
2. Divide by the denominator of the original fraction.
3. Get a new remainder.

But how many *different* new remainders can we get? Let us try an example, making a special point of keeping track of the remainders as we go, as in Example 1.

EXAMPLE 1: $12/7 = 1.\overline{714285}$.

```
     1.71428571
 7 | 12.00000000                      Remainder
     7
    -----
    50                                    5
    49
    ---
     10                                   1
      7
     ---
      30                                  3
      28
      ---
       20                                 2
       14
       ---
        60                                6
        56
        ---
         40                               4
         35
         ---                             ---
          50                              5
          49
          ---
           10                             1
            7
           ---
            3                             3
```

After six steps of the algorithm, the remainder 5 repeats. As soon as it shows up again, the whole calculation repeats because each set of three steps is exactly the same when the remainders are the same. The blocks marked by the dotted lines are identical and will occur over and over. Thus,

$$\frac{12}{7} = 1.\overline{714285}.$$

Now, how do we know that some remainder must repeat? This is so because the number of possible remainders is limited.

If we get a remainder of zero, the decimal terminates; that is, it repeats in one-digit blocks consisting of zeros only.

If a zero remainder does not occur, the possible remainders range from 1 to one less than the divisor, inclusive. (In our example, 1, 2, 3, 4, 5, and 6 were possible remainders.) As soon as all possibilities have occurred, something has to repeat.

Of course, not all possible remainders have to occur; we may get a repeat long before all possibilities are exhausted.

EXAMPLE 2: $23/101 = 0.\overline{2277}$.

```
       0.22772
101 ) 23.00000          Remainder
      20 2
       2 80                28
       2 02
         780               78
         707
          730              73
          707
           230             23
           202
            28             28
```

In Example 2 only four out of the one hundred possible nonzero remainders ever appear. In fact, because the three steps mentioned in the algorithm begin at once, since only 0 is ever brought down in this case, the problem repeats itself as soon as a remainder of 23 is obtained. The original problem was to divide 23 by 101, and when 23 occurs as a remainder, this is the situation again. Thus the digits in the answer start to repeat the established pattern at this point.

Whichever way it is, sooner or later the repetition of remainders must occur; and then we have finished a pattern, and start all over again.

Exercise Set 2

1. Expand as repeating decimals:

a. $\frac{2}{13}$ **b.** $\frac{5}{37}$ **c.** $\frac{17}{35}$ **d.** $\frac{4}{13}$

2. Find a fraction (in simplest form) equivalent to

a. $0.202020\cdots$ **b.** $1.2\overline{36}$ **c.** $0.\overline{384615}$ **d.** $0.\overline{12345}$

3. Write the repeating decimals for 1/7, 2/7, 3/7, 4/7, 5/7, and 6/7. Show that if an appropriate starting point is selected, the same pattern occurs every time.

4. Show, by the method developed in this section, that $0.999\cdots = 1.000$.

IRRATIONAL NUMBERS

The Positive Square Root of 2

Real numbers whose decimal expansions do not have the form of an infinite repeated pattern are called "irrational numbers." It is easy to find the decimals for as many irrational numbers as we want. All we have to do is to avoid successively repeating patterns. Patterns are all right as long as they do not repeat. For example, an irrational number is represented by the decimal $0.101001000100001\cdots$, where there is an additional 0 in each block of 0's between successive 1's. Can you find the pattern in the following decimal?

$$0.123456789101112131415161718192021\cdots$$

(When you get to $\cdots 1011$, you may say "ten, eleven" or "one, zero, one, one.")

Although these numbers are interesting, it is natural to ask whether or not there are any irrational numbers more commonly useful. Here we run into complications. In the "Introduction" we mentioned $\sqrt{2}$. How can we *prove* that it is irrational? A computing machine can quickly give us thousands of decimal digits of its numeral, and we can see that no repeating pattern has emerged. But perhaps $\sqrt{2}$ is a rational number whose simplest fraction has a very great numerator and a very great denominator. It might easily require millions of digits, instead of thousands, before a repeat begins. We can never *prove* the number is irrational by looking at digits, no matter how many of them we produce.

Actually, the history of the problem is quite interesting. The Pythagoreans in ancient Greece wanted an expression for the length of a diagonal of a square in terms of the length of one of its sides. They sought the ratio of the length of a diagonal to the length of a side and expected to compute it as the ratio of two counting numbers. It is this idea of ratio that leads to the use of the term "rational" in the sense of "having a ratio."

The Pythagoreans knew that in a right triangle the square of the length of the long side is equal to the sum of the squares of the lengths of the shorter sides. The square of a number is the product when that number is multiplied by itself. The superscript 2 is used to indicate that a number is squared; for example, 3^2 means 3×3, or 9. For a right triangle with sides measuring 3 units, 4 units, and 5 units, as in Figure 13, we have the equation $3^2 + 4^2 = 5^2$; that is, $9 + 16 = 25$.

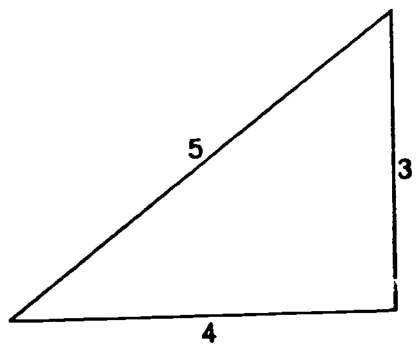

FIGURE 13

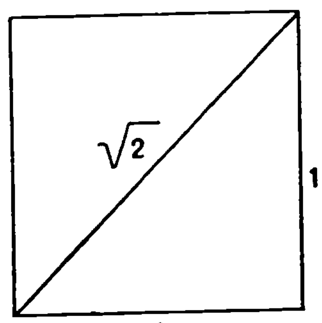

FIGURE 14

Now, if each side of a square figure measures 1 unit, then the square of the length of the diagonal is $1^2 + 1^2 = 1 + 1 = 2$. Therefore, the length of the diagonal is $\sqrt{2}$, as indicated in Figure 14, since $(\sqrt{2})^2 = 2$.

The Pythagoreans wished to write $\sqrt{2}$ as a ratio of integers—that is, as a "fraction." To understand what happened next, we need to review some facts:

A. All whole numbers are either even or odd. No number is both even and odd. (The even whole numbers are 0, 2, 4, 6, $\cdots$; the odd ones are 1, 3, 5, 7, $\cdots$.)

B. Any even whole number can be written as two times another whole number. For example,

$$10 = 2 \times 5,$$
$$12 = 2 \times 6.$$

Any number that is twice a whole number is even.

C. The square of an even number is even. The square of an odd number is odd. (Try some examples to convince yourself.)

D. Any fraction is equivalent to the fraction changed to its simplest form—that is, the form whose numerator and denominator have 1 as their greatest common divisor. For example, 30/42 is equivalent to 5/7, which is in simplest form.

E. If a fraction is in simplest form, then its numerator and denominator are not both even. (For then, according to statement B, we could divide numerator and denominator by 2.) At least one of them is odd.

Now let us try to write $\sqrt{2}$ as a fraction. This fraction (if there is one) can be in simplest form, by statement D. The denominator is some counting number, q, and the numerator is a whole number, p.

Then we have

$$\frac{p}{q} = \sqrt{2},$$

or

$$\frac{p}{q} \times \frac{p}{q} = \sqrt{2} \times \sqrt{2},$$

or

$$\frac{p^2}{q^2} = 2, \quad \text{by the definition of } \sqrt{2}.$$

Now, multiplying both sides of the equation by q^2, we get

$$\frac{p^2}{q^2} \times q^2 = 2 \times q^2,$$

or

$$p^2 = 2 \times q^2.$$

Thus p^2 is even, by statement B; and statement C implies that p is even.

Then q is odd, by statement E. Further, using the result that p is even and applying fact B again, we see that there is some whole number r such that $p = 2 \times r$. Therefore, we can substitute $2 \times r$ for p, in the equation $p^2 = 2 \times q^2$, so that

$$(2 \times r)^2 = 2 \times q^2, \quad \text{or} \quad (2 \times r) \times (2 \times r) = 2 \times q^2.$$

Now the associative and commutative properties of multiplication permit us to regroup the expression on the left as follows:

$$(2 \times 2) \times (r \times r) = 2 \times q^2,$$

or

$$4 \times r^2 = 2 \times q^2,$$

or, when we divide both sides by 2,

$$2 \times r^2 = q^2.$$

Now statements B and C imply that q^2 and q are even.

Hence, q is both even and odd. But, by fact A, this is impossible; q does not exist and there is *no* fraction for $\sqrt{2}$. This completes the proof—but if you have found it difficult, then perhaps you should start again on page 22 to appreciate its subtleties.[3]

This discovery greatly amazed the Greeks. Here was length without an acceptable number to go with it. They stated the fact by saying: "The diagonal of a square and its side are incommensurable." That is, they

[3] See also Edwin F. Beckenbach, "Geometric Proofs of the Irrationality of $\sqrt{2}$," *Arithmetic Teacher*, XV (1968), 244–50.

cannot be measured in terms of each other in the sense that if we repeatedly marked off diagonals and also sides on the same line, starting from the same point, the end points would never match. The length of seven sides is *close* to that of five diagonals, but the two lengths are not *equal*. (If the length of seven sides were equal to the length of five diagonals, then one-fifth the length of a side or one-seventh the length of a diagonal would be a common measure.) From this time on, the Greeks turned their attention almost exclusively to geometry, since arithmetic had proved to be incomplete. The result was that portions of mathematics depending on arithmetical calculations—algebra and trigonometry, for example—were neglected for centuries.

Today we are not so shocked by this proof as was Pythagoras. (Legend says that he swore his associates to secrecy and slaughtered one hundred oxen as a sacrifice!) We know how to approximate $\sqrt{2}$ with numbers represented by finite decimals. One way to do this is to make successive approximations by "bracketing."

To bracket, we begin by getting the value between successive integers; 1 is too small because $1 \times 1 = 1$, which is less than 2; 2 is too large because $2 \times 2 = 4$, which is greater than 2. Now we know that $\sqrt{2}$ is between 1 and 2, and therefore the decimal for $\sqrt{2}$ begins with 1.

Next we try for tenths. We successively multiply 1.1, 1.2, etc., by themselves until we get a product that is greater than 2:

$$1.1 \times 1.1 = 1.21 \text{ (too small)};$$
$$1.2 \times 1.2 = 1.44 \text{ (too small)};$$
$$1.3 \times 1.3 = 1.69 \text{ (too small)};$$
$$1.4 \times 1.4 = 1.96 \text{ (too small, but close)};$$
$$1.5 \times 1.5 = 2.25 \text{ (too large)}.$$

So the decimal for $\sqrt{2}$ begins with 1.4.

Next we go to hundredths and start with 1.41:

$$1.41 \times 1.41 = 1.9881 \text{ (too small)};$$
$$1.42 \times 1.42 = 2.0164 \text{ (too large)}.$$

Now we know that the expansion starts with 1.41.

The next step would be to get the third-stage digit. If we try 1.411, 1.412, 1.413, 1.414, and 1.415, we find that 1.415 is the first to be too large; hence, we know that $\sqrt{2}$ has a decimal approximation, at this stage, of 1.414. The process can obviously be continued as long as our interest or our patience lasts.

Other Irrational Numbers

There are many other interesting irrational numbers—for example, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$. In general, any square root of a counting number is either a counting number or an irrational number.

There is an interesting way of viewing lengths of segments corresponding to these square roots. Start with two perpendicular segments AB and BC, each of length 1. Draw the segment AC (Fig. 15). We have seen that this segment has length $\sqrt{2}$. Starting at C as shown, draw a segment of length 1 perpendicular to the segment of length $\sqrt{2}$ and join its end point D to end point A. What is the length of the new segment, AD? Its square is $1^2 + (\sqrt{2})^2 = 1 + 2 = 3$. Continue in this way. Each new segment from A will have length equal to the square root of the next greater counting number. A spiral will be formed.

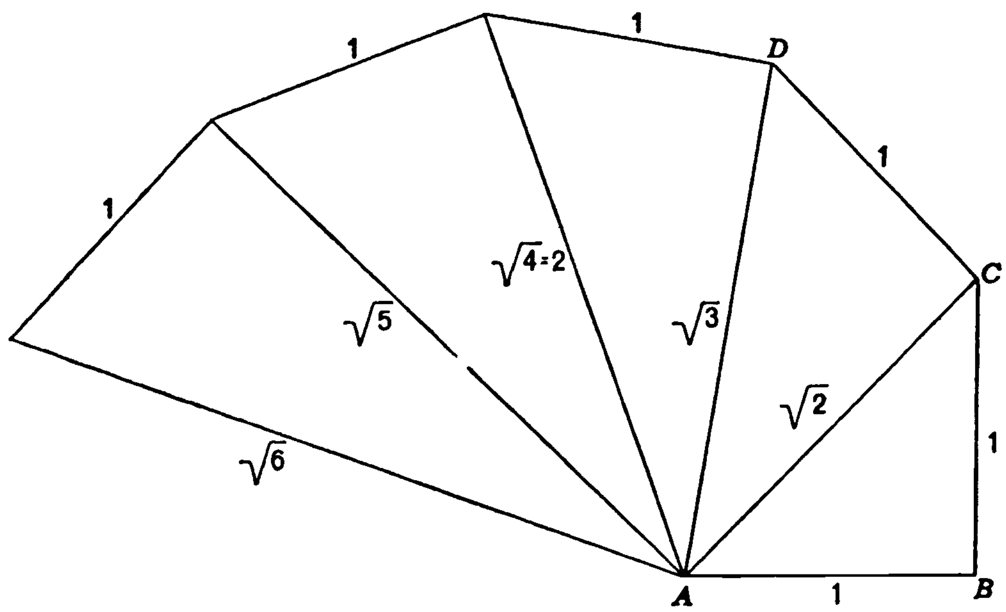

FIGURE 15

Proofs similar to the one given to show that $\sqrt{2}$ is irrational will work for $\sqrt{3}$, $\sqrt{5}$, etc.; but there are other, and more general, methods available to show this. Also, we can define and use higher roots; for example, the positive cube root of 2, written $\sqrt[3]{2}$, is the positive real number having the property that $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$. Again this is irrational, as are most other numbers of this form (except, of course, particular ones like $\sqrt[3]{8}$, which equals 2, since $2 \times 2 \times 2 = 8$).

An especially interesting irrational number is π, the ratio of the circumference of a circle to the length of its diameter. That this ratio is a universal constant, the same for all circles regardless of size, seems to be a matter of very ancient knowledge. The oldest known approximation to π is 3. This appears in the Bible, where it is stated that Solomon, furnishing the temple in Jerusalem, provided a bronze bowl "nine cubits around and three cubits across." This approximation was also used by the Egyptians and the Babylonians.

It does not seem to be very clear when the approximation 22/7 (now so commonly used by school children) first came into use. It was certainly known by the time of the Renaissance. Meanwhile, Archimedes had shown by a geometric proof how to get as close a value as was desired by approximating the circle with polygons.

In 1761 the German mathematician Lambert first proved that π is irrational. No easy proof is known. For practical computation, the values of 3.1416 or 3.14159 are usually sufficiently accurate; but most books of mathematical tables list the value to at least ten digits after the decimal sign (eleven digits in all) for computations that require special precision. Here are the first twenty-five digits in the decimal for π:

$$3.141592653589793238462643.$$

Computation of π to greater and greater accuracy was a favorite mathematical hobby for many years. The coming of the electronic computer put an end to it, and the expansion to ten thousand digits was done merely as a sort of advertising device to show the power of modern computers.

Exercise Set 3

1. Find, by bracketing, to three digits after the decimal sign:
  a. $\sqrt{3}$ **b.** $\sqrt{6}$

2. Find $\sqrt[3]{2}$ to two digits after the decimal sign.

3. Decide which of the following sorts of numbers *cannot* be rational numbers:
  a. Reciprocals of irrational numbers (Hint: If $1/x = a/b$, then you would want to have $x = b/a$.)
  b. Halves of irrational numbers (Hint: If $x/2 = a/b$, then you would want to have $x = 2a/b$.)
  c. Products of irrational numbers by irrational numbers

APPROXIMATIONS AND OPERATIONS

How to Define Operations on the Real Numbers

So far, we have a set of real numbers denoted by infinite decimals and a definition of order on this set. We want to be able to talk about sums, differences, products, and quotients of real numbers. But how? We cannot even imagine the process of multiplying two numbers denoted by infinite decimals.

If the real numbers happen to have some shorter names like $\sqrt{2}$, and if we *assume* the basic laws of arithmetic apply also to the real numbers, then things look a little better. For example, suppose we think about combinations of $\sqrt{2}$ and $\sqrt{3}$. Is $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$? Is $\sqrt{2} + \sqrt{3} = \sqrt{2+3} = \sqrt{5}$?

We might first try similar questions on numbers we know to be rational. Is

$$\sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9} = \sqrt{36}?$$

Yes, since

$$\sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{36} = 6, \quad \text{and} \quad 2 \times 3 = 6.$$

Is

$$\sqrt{4} + \sqrt{9} = \sqrt{4+9} = \sqrt{13}?$$

No!

$$\sqrt{4} + \sqrt{9} = 2 + 3 = 5; \quad \text{and} \quad 5 = \sqrt{25}, \quad \text{not} \quad \sqrt{13}.$$

Again,

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7,$$

but

$$\sqrt{9 + 16} = \sqrt{25} = 5.$$

Therefore, there seems no reason to suspect that $\sqrt{2} + \sqrt{3} = \sqrt{5}$ but good reason to believe that $\sqrt{2} \times \sqrt{3} = \sqrt{6}$.

To decide such questions, we must return to the definition of square root:

> *The square root of a given number is a number whose square is the given number.*

If a real number is the square root of 6, then the result of multiplying it by itself will be 6; thus, we must have $\sqrt{6} \times \sqrt{6} = 6$.

Let us see what happens to $(\sqrt{2} \times \sqrt{3})$ when we multiply it by itself:

$$(\sqrt{2} \times \sqrt{3}) \times (\sqrt{2} \times \sqrt{3}).$$

The associative law provides that we can drop or rearrange parentheses in multiplication, at our convenience. If we can use this law, we may drop the parentheses temporarily and write as follows:

$$\sqrt{2} \times \sqrt{3} \times \sqrt{2} \times \sqrt{3}.$$

The commutative law will let us reorder terms in a product. Interchange the second and third terms:

$$\sqrt{2} \times \sqrt{2} \times \sqrt{3} \times \sqrt{3}.$$

Now put back the parentheses in a convenient way:

$$(\sqrt{2} \times \sqrt{2}) \times (\sqrt{3} \times \sqrt{3}).$$

But the very definition of $\sqrt{2}$ and $\sqrt{3}$ lets us know that $\sqrt{2} \times \sqrt{2} = 2$ and $\sqrt{3} \times \sqrt{3} = 3$, so we have

$$(\sqrt{2} \times \sqrt{2}) \times (\sqrt{3} \times \sqrt{3}) = 2 \times 3$$

and

$$2 \times 3 = 6.$$

Thus $(\sqrt{2} \times \sqrt{3})$ multiplied by itself is 6, and $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, if we assume the associative and commutative laws of multiplication.

If we perform a similar calculation with $(\sqrt{2} + \sqrt{3})$, we obtain

$$\begin{aligned}(\sqrt{2} + \sqrt{3}) \times (\sqrt{2} + \sqrt{3}) &= (\sqrt{2})^2 + 2 \times (\sqrt{2} \times \sqrt{3}) + (\sqrt{3})^2 \\ &= 2 + (2 \times \sqrt{6}) + 3 \\ &= 5 + (2 \times \sqrt{6}),\end{aligned}$$

which is certainly not 5. The expression $5 + (2 \times \sqrt{6})$ is a *name* for a real number, not an order to do something. We cannot proceed any further unless we actually get a decimal representation for $(2 \times \sqrt{6})$. Forms like $(2 + \sqrt{3})$ and $(\sqrt{2} + \sqrt{3})$, which are compound names, are frequently encountered. They are useful; they give a far more precise idea of the quantity named than does any partial decimal representation. Therefore, we use them as they are. A form like $(\sqrt{2} \times \sqrt{3})$ is a name, too, but there is a simpler name for the same number: $\sqrt{6}$. The situation is somewhat analogous to that of the fractions 6/8 and 3/4. In dealing with expressions like $\sqrt{2}$, however, we sometimes do not have any form that is clearly the *simplest*. For example, $\sqrt{1/2}$. $1/\sqrt{2}$, and $\sqrt{2}/2$ all actually name the same real number. Which fo. . is the simplest?

After all this we can still ask: What gives us the right to *assume* all the laws of ordinary arithmetic? The answer is that we *construct* definitions of operations in such a way that we can be certain the laws are valid.

Approximate Calculation

In exercise 1 on page 26, we asked for values of $\sqrt{3}$ and $\sqrt{6}$. Here is a table, in case you did not keep your answers. We also include $\sqrt{2}$ for reference:

$$\begin{aligned}\sqrt{2} &= 1.414 \cdots . \\ \sqrt{3} &= 1.732 \cdots . \\ \sqrt{6} &= 2.449 \cdots .\end{aligned}$$

We are going to try to check the statement that $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. We shall use a method similar to the bracketing scheme used to calculate square roots. Take bracketing approximations to $\sqrt{2}$ and $\sqrt{3}$:

$$\begin{aligned}1.4 < \sqrt{2} < 1.5. \\ 1.7 < \sqrt{3} < 1.8.\end{aligned}$$

Now if we multiply 1.4 by 1.7, are we sure that the answer will be less than $\sqrt{2} \times \sqrt{3}$? Is 1.5×1.8 necessarily greater than $\sqrt{2} \times \sqrt{3}$?

One difficulty in answering these questions is that we still have not carefully defined $\sqrt{2} \times \sqrt{3}$.

A diagram will give a clearer picture. In Figure 16, $\overline{AB}$ is a segment of length $\sqrt{2}$ units, $\overline{AC}$ is of length 1.4, and $\overline{AD}$, 1.5. Similarly, $\overline{AE}$ is $\sqrt{3}$ units long while $\overline{AF}$ and $\overline{AG}$ are 1.7 and 1.8, respectively.

If we multiply 1.4 by 1.7, we calculate the area of the rectangle $ACPF$. If we compute 1.5×1.8, the area of $ADRG$ is obtained. By $\sqrt{2} \times \sqrt{3}$ we mean the area of $ABQE$. Clearly, the area of the rectangle $ABQE$ should be greater than that of the rectangle $ACPF$ and less than that of the rectangle $ADRG$.

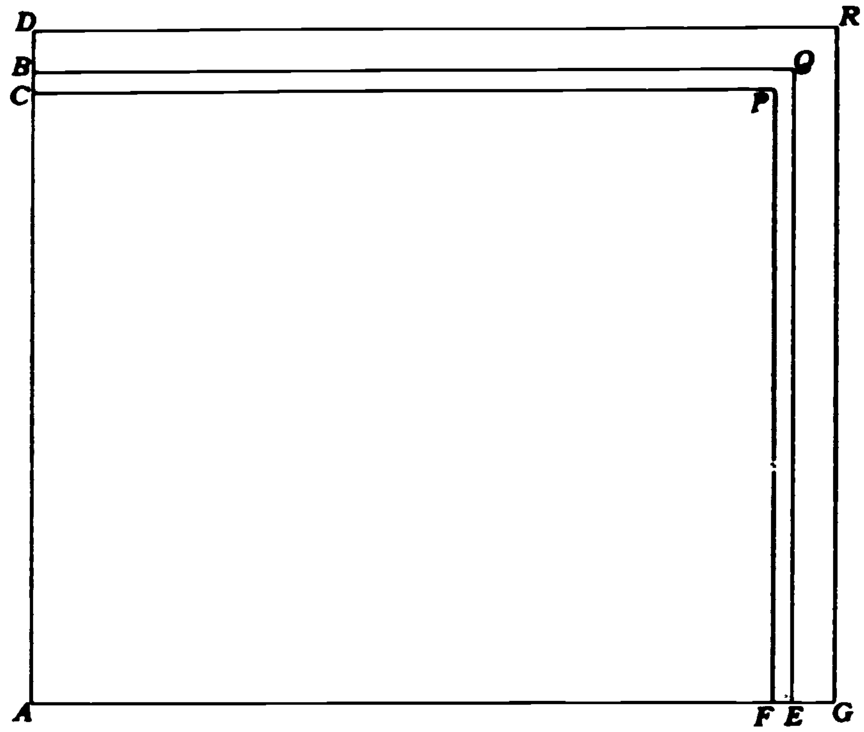

FIGURE 16

The upside-down and backward L-shaped region bounded by $CDRGFPC$ can be thought of as a region of fuzziness whose area represents our uncertainty about the exact answer.

Now, what happens if we make better approximations? Suppose C moves up toward B, and D down toward B. Similarly, let F move outward and G inward toward E. The widths of the bands of uncertainty shrink; the possible errors diminish. Let us look at Table I to see what happens.

What was the answer we expected? Look back at the value of $\sqrt{6} = 2.449\cdots$, which you computed in exercise 1 on page 26. Everything is coming out as we expected.

The final position is that we define the product of two real numbers as the real number that results from the process of using, without limitations,

TABLE I
BRACKETING FOR MULTIPLICATION

Too Small	Too Large	Difference
1.4 × 1.7 = 2.38	1.5 × 1.8 = 2.70	0.32
1.41 × 1.73 = 2.4393	1.42 × 1.74 = 2.4708	0.0315
1.414 × 1.732 = 2.449048	1.415 × 1.733 = 2.452195	0.003147

the products of better and better approximations to the two numbers. That there is essentially one and only one such real number becomes intuitively clear when we think of the L-shaped region shrinking to a pair of perpendicular segments as the uncertainty decreases without limit.

Note that we do not have to approximate by bracketing. Whenever we take better approximations, we get better answers.

Incidentally, we are frequently able to improve our accuracy by taking one factor too large and one too small. For example, suppose we used 1.4 and 1.8 as approximations to $\sqrt{2}$ and $\sqrt{3}$, respectively. The result, $1.4 \times 1.8 = 2.52$, is much closer to the "right" answer, $2.449\cdots$, than 2.38 and 2.70, the products of 1.4×1.7 and 1.5×1.8, respectively.

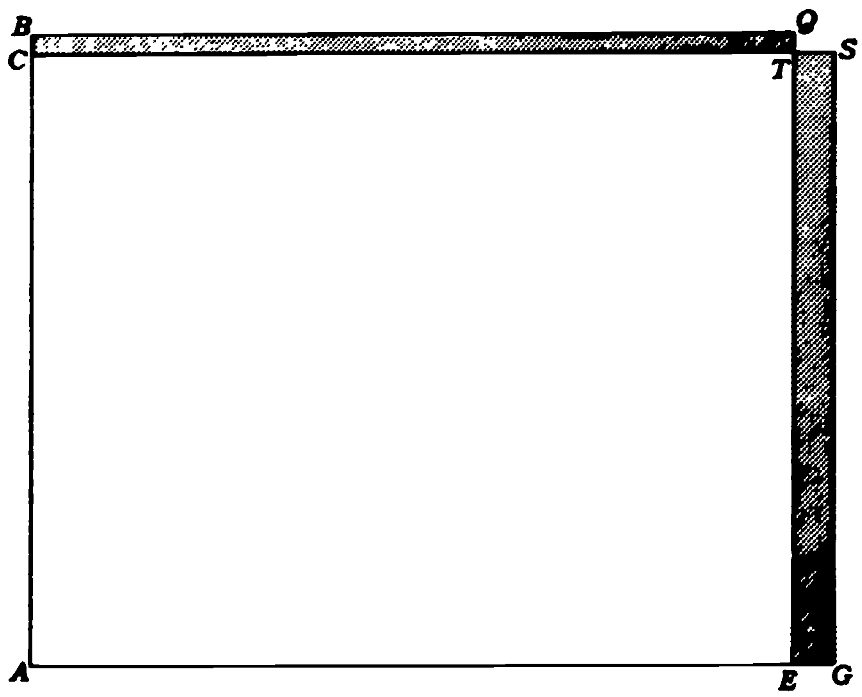

FIGURE 17

We can show the reason for this improvement by redrawing the rectangles just for this example (see Fig. 17). The same letters will be used to designate the segments retained from Figure 16; that is, $\overline{AB}$ is our segment of length $\sqrt{2}$ units, $\overline{AC}$ has length 1.4, $\overline{AE}$ is $\sqrt{3}$ units long, and $\overline{AG}$ is 1.8 units long. The new letters, T and S, designate new points located by extending segment CP to cross segments EQ and GR of the original drawing.

The "correct" rectangle is $ABQE$. Our product gave the area of $ACSG$. This is too small by the area of the upper shaded portion, which lies inside the correct rectangle but was missed, and is too large by the shaded area on the right, which is outside the exact rectangle and was included. Thus the errors partially cancel out.

What about addition? The ideas are very similar. If we want to know part of the decimal numeral for $\sqrt{2} + \sqrt{3}$, we can make a bracketing table. (See Table II.)

TABLE II
BRACKETING FOR ADDITION

Too Small	Too Large	Difference
1.4 + 1.7 = 3.1	1.5 + 1.8 = 3.3	0.2
1.41 + 1.73 = 3.14	1.42 + 1.74 = 3.16	0.02
1.414 + 1.732 = 3.146	1.415 + 1.733 = 3.148	0.002

The same idea works, but you can observe from the tables that for addition the comparative uncertainty (difference) at each step is rather less than for multiplication.

What about subtraction? Watch out! The bracketing table must be made up in a different way. As an example, we shall use $\sqrt{3} - \sqrt{2}$. If we want a resulting approximation that is definitely less than the right answer, we subtract a too-large approximation for $\sqrt{2}$ from a too-small approximation for $\sqrt{3}$. (Why?) To get a result that is definitely greater than the right answer, we subtract a too-small approximation for our larger number, $\sqrt{3}$, from a too-large approximation for $\sqrt{2}$. (See Table III.)

TABLE III
BRACKETING FOR SUBTRACTION

Approximations for Too-small Answer		Result	Approximations for Too-large Answer		Result	Difference
Too Small for $\sqrt{3}$	Too Large for $\sqrt{2}$		Too Large for $\sqrt{3}$	Too Small for $\sqrt{2}$		
1.7	− 1.5	= 0.2	1.8	− 1.4	= 0.4	0.2
1.73	− 1.42	= 0.31	1.74	− 1.41	= 0.33	0.02
1.732	− 1.415	= 0.317	1.733	− 1.414	= 0.319	0.002

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For division, the situation is similar to that of subtraction. To get an answer that we are certain is too small, we must divide an approximation that is too small for the dividend by one that is too large for the divisor. (The greater the divisor, the less the answer.)

So far, all our examples and illustrative calculations have been done with positive real numbers. What about operations involving negative numbers? No new problem arises that was not already present in defining operations with negative rational numbers. (See Booklet No. 10: *The System of Rational Numbers.*) The definitions of operations are given in terms of the same operations on rational numbers, and we simply follow those same rules. For example, if we wish to compute $(\sqrt{2} \times {}^{-}\sqrt{3})$, the value is ${}^{-}\sqrt{6}$ because the rule $a \times {}^{-}b = {}^{-}(a \times b)$ for rational numbers also applies in the extended system.

Whatever operation we are trying to perform, the basic result always is that *if we approximate close enough to the operands (the original real numbers), then we will get a result as close to a fixed real number as we wish. The real number thus approached is designated the result of the operation.*

Since all operations on real numbers are defined in terms of the operations on rational numbers and agree, to as close an approximation as we please, with the corresponding rational operations, the basic algebraic properties of the rational number system are, indeed, carried over to the real number system. For convenience, we shall summarize the whole structure by listing each of its essential properties.

Before doing so, let us remind ourselves of certain basic inclusion properties that are features of the systems constructed in mathematics. The integers are included in the real numbers. In the course of our construction, the integer usually named 2 got the rather unwieldy name $2.0000\cdots$, but we now feel free to simplify this and all other cumbersome decimals so that our terminating decimals really terminate.

Here is a more general statement:

System Inclusion

{Counting Numbers} ⊂ {Whole Numbers} ⊂ {Integers}
⊂ {Rational Numbers} ⊂ {Real Numbers}.

Closure

Addition: The sum of any two real numbers is a unique real number.
Subtraction: The difference of any two real numbers is a unique real number.
Multiplication: The product of any two real numbers is a unique real number.
Division: The quotient of any two real numbers, when the divisor is not zero, is a unique real number.

Commutativity

Addition: If a and b are real numbers, then $a + b = b + a$.
Multiplication: If a and b are real numbers, then $a \times b = b \times a$.

ASSOCIATIVITY

Addition: If a, b, and c are real numbers, then $(a + b) + c = a + (b + c)$.
Multiplication: If a, b, and c are real numbers, then $(a \times b) \times c = a \times (b \times c)$.

IDENTITIES

Addition: There is a real number, 0, such that if a is a real number, then $a + 0 = 0 + a = a$.
Multiplication: There is a real number, 1, such that if a is a real number, then $a \times 1 = 1 \times a = a$.

INVERSES

Addition: If a is a real number, then there is a real number ^-a such that $a + {^-a} = 0$.
Multiplication: If a is a real number not equal to zero, then there is a real number $1/a$ such that $a \times 1/a = 1$.

ORDER

Trichotomy: If a is a real number and b is a real number, then either $a = b$, $a < b$, or $b < a$, and only one of these holds.
Addition: If a and b are real numbers such that $a < b$, and c is any real number, then $a + c < b + c$.
Multiplication: If a and b are real numbers such that $a < b$, and c is a real number such that $c > 0$, then $ac < bc$.

DENSITY

If a and b are real numbers such that $a < b$, then there is a real number c such that $a < c < b$. (In fact, c may always be selected so that it is rational.)

COMPLETENESS

To each point on the number line there corresponds a real number, and to each real number there corresponds a point on the number line.

It is the last-named property—the property of completeness—that *really* distinguishes the real number system from the rational number system.

Exercise Set 4

1. In a city park, paths were arranged as follows: A square 100 yards on each side was bounded by a circular path circumscribed around it.

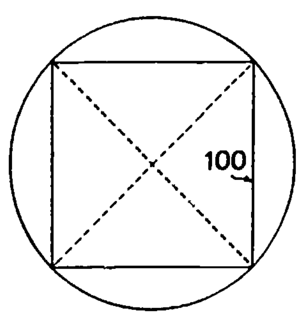

FIGURE 18

Notice that it requires a walk of 400 yards to go around the park, following the sides of the square. The question arose as to how much farther it was to walk around the circle. Is the circular path more or less than a quarter of a mile around? Note that its diameter is $100 \times \sqrt{2}$ yards. A quarter of a mile is 440 yards.

2.a. Take each of the systems mentioned on page 32 and the list of laws on pages 32 and 33 and check each to see how many laws are not satisfied when the words "real number" are replaced throughout by "counting number." For example, the counting numbers do not satisfy the identity property for addition because the statement would read "There is a counting number, 0, such that if a is a counting number, then $a + 0 = 0 + a = a$." But 0 is not a counting number.

Pay particular attention to closure for subtraction and division, to inverses for addition and multiplication, to density, and to completeness.

b. Repeat exercise 2a, replacing the words "real number" by "whole number."

c. Repeat exercise 2a, replacing the words "real number" by "integer."

d. Repeat exercise 2a, replacing the words "real number" by "rational number."

HOW MANY?

We have seen that a rational number is in essence a very special kind of real number. In this section we are going to try to answer, at least partly, the question "How special?" Our trouble lies with a general fuzziness that involves all such questions when we deal with infinite sets.

It is easy to maintain that half of the squares on a checkerboard are black. You can prove it by counting them, and any way of counting that is not obviously wrong will always give 32 black squares and 64 squares in all. Counting, you will recall, is the process in which we put a set into one-to-one correspondence with a set of ordered counting numbers, where the latter set must have the property that if it contains any counting number it must also contain all the previous counting numbers. That is, we are not allowed to count at random (for example, "2, 7, 4, 11"); we must count in order, "1, 2, 3, 4." Then the last counting number used is called the "cardinal number" of the set counted, or simply the number of things in the set.

Now what happens if there is no last counting number used? In this case (if the set has any members at all) we say that it is "infinite." With infinite sets we at once run into problems. Think of this entirely reasonable-seeming assertion: "Half of the counting numbers are even." Suppose someone argues about it. You proceed to "count" all the counting numbers

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and then all the even ones, but it takes *all* the counting numbers to count just the even ones.

$$\begin{array}{rccccc} \text{Even numbers:} & 2 & 4 & 6 & 8 & 10 \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \text{Counting numbers:} & 1 & 2 & 3 & 4 & 5 \cdots \end{array}$$

So there are just as many even counting numbers as counting numbers!

You can, of course, maintain your original assertion in some such manner as this: "Suppose I take the counting numbers up to some large number; then, depending on where I stop, either exactly half or half of just one less than the stopping number are even. In either case, the *proportion* of even ones is very close to one-half and differs from one-half by as little as I want. For example, if I stop at 1,000,000, it is exactly half; if I stop at 1,000,001, it is $\frac{500,000}{1,000,001}$, which equals 0.4999995 $\cdots$."

In the case of the even counting numbers, we saw a subset that could be put into one-to-one correspondence with the entire set of counting numbers. It should not be any further surprise to discover that the counting numbers can be used to count sets that contain the counting numbers as proper subsets.

For example, let us think of the set of all counting numbers and all halves of counting numbers:

$$H = \{\tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, 3, 3\tfrac{1}{2}, 4, \cdots\}.$$

Write the elements all with denominator 2:

$$H = \{\tfrac{1}{2}, \tfrac{2}{2}, \tfrac{3}{2}, \tfrac{4}{2}, \tfrac{5}{2}, \tfrac{6}{2}, \tfrac{7}{2}, \tfrac{8}{2}, \cdots\}.$$

Now make the correspondence with the numerators:

$$\begin{array}{rcccc} H\text{:} & \tfrac{1}{2} & \tfrac{2}{2} & \tfrac{3}{2} & \tfrac{4}{2} \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \text{Counting numbers:} & 1 & 2 & 3 & 4 \cdots \end{array}$$

Every member of H is counted.

A slightly harder problem is to count all the integers:

$$I = \{\cdots, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \cdots\}.$$

You can make a correspondence like that in Figure 19. Here the even

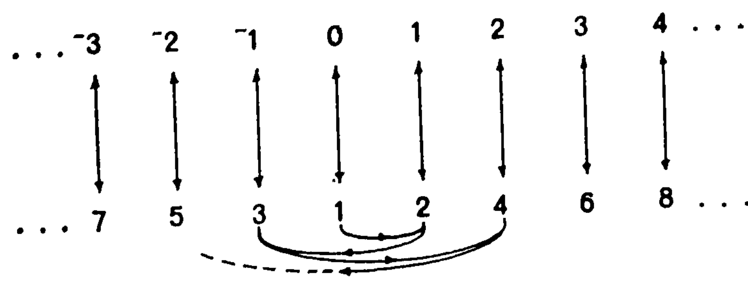

FIGURE 19

counting numbers count the positive integers, and the odd counting numbers count zero and the negative integers.

It is time to stop and remind ourselves of what we mean by "counting" a set. We are merely establishing a one-to-one correspondence between the given set and a set of counting numbers, beginning with 1 and not jumping or skipping any counting number. Then if there is a last counting number used, it is the cardinal number of the set. When, as in the cases we have shown, every member of the set has a counting number but there is no last one used, the cardinality of the set is called aleph-zero ($\aleph_0$), a name given by Georg Cantor about a hundred years ago. In spite of writing and printing difficulties, the name has remained.

So far, we have produced the following sets with cardinality aleph-zero:

{The counting numbers}
{The even counting numbers}
{Halves of counting numbers}
{The integers}

The reader is invited to produce (and to count) other such sets. Here are some for counting practice:

1. The set of counting numbers greater than 5, that is, {6, 7, 8, $\cdots$}
2. The set of counting numbers whose last three digits are 573, that is, {573, 1573, 2573, $\cdots$}
3. The set of integers not divisible by 3
4. The set of fractions, in simplest form, whose denominators are 1, 2, 3, or 6

A question now arises. Can *every* nonempty set be counted? Obviously, by the very definition a finite set can be counted because the property of being finite and nonempty simply consists of having a counting set with a last counting number. So what we really want to know is this: "Can every infinite set be counted?" In our new terminology we ask: "Does every finite set have cardinality aleph-zero; or are there larger sets that just cannot be covered by the counting numbers, no matter how clever we are?"

Note that sometimes we do have to be clever. There are many profligate ways to use all the counting numbers without counting all the integers, for example. If we were careless enough, we could use the entire set just on the positive even integers. It takes some cunning to decide to pick up the positive and the negative ones more or less simultaneously.

What about the set of rational numbers? This seems to be a much larger set than we have tried to count so far. No trick anywhere near so simple as any we have used up to now will wörk this time. Recall that the set of points representing rational numbers is dense on the number line!

Let us write fractions for the positive rational numbers, in a table thought of as being infinite in two directions. In the first row we write those with denominator 1; in the second, those with denominator 2, and so on. In the first column are all fractions with numerator 1; in the second column are those with numerator 2, and so on. (See Fig. 20.)

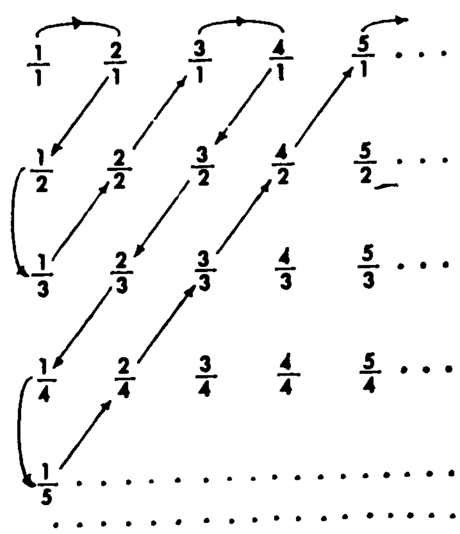

FIGURE 20

The rows are infinite and the columns are infinite, but the right-up, left-down diagonals are *finite*. Follow the arrows and number the fractions as you go. First, number 1/1 with 1; this is the only fraction whose numerator and denominator add up to 2. Now number the fractions (two of them) whose numerator and denominator add up to 3; next, the three fractions that give a sum of 4, and so on.

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1	2	3	4	5	6	7	8	9	10	...

Does every fraction get a counting number? Yes! Just add the numerator and denominator of a fraction, and you know which diagonal it is in. Further, since there are only a finite number of fractions in each diagonal (one less than the sum of the numerator and denominator for that diagonal), we shall have expended only a finite number of counting numbers before we get there. A finite number of finite numbers is finite. Think about this argument until you are sure that you are convinced.

In fact, if we do a bit of algebra, we can make a rough approximation of the counting number that will go with any given fraction.

We mentioned that first we counted one fraction in the first diagonal, then two in the next diagonal, then three in the next, then four, and so on. Each diagonal has one more fraction in it than the one before. Now, how many were there altogether when we finished each diagonal? There was one after the first diagonal; there were three after the second (we added two), six after the third (we added three), ten after the fourth, and so on. (See Table IV.)

TABLE IV

Sum of Numerator and Denominator	Fractions with This Particular Sum	Fractions with This or Smaller Sum
2	1	1
3	2	$3 = 1 + 2$
4	3	$6 = 3 + 3$
5	4	$10 = 6 + 4$
6	5	$15 = 10 + 5$

Now suppose we consider the following formula:[4] Take one-half the sum of numerator and denominator multiplied by one less than this sum—that is,

$$\frac{\text{Sum} \times (\text{Sum} - 1)}{2}.$$

For a sum of 2: $\frac{2 \times 1}{2} = 1.$

For a sum of 3: $\frac{3 \times 2}{2} = 3.$

For a sum of 4: $\frac{4 \times 3}{2} = 6.$

For a sum of 5: $\frac{5 \times 4}{2} = 10.$

For a sum of 6: $\frac{6 \times 5}{2} = 15.$

Working out the formula for each sum gives us the numbers in the third column of Table IV. We shall not prove this formula here, but you can check it until you are reasonably convinced that it works.

What number counts the fraction 1/5? It is the first fraction in the diagonal where the sum is $1 + 5 = 6$. How many fractions had smaller sums? Referring to Table IV, look at the fourth line of the third column, opposite the "5" in the first column; or compute $(5 \times 4)/2 = 10$. So for each of ten fractions the sum of numerator and denominator was

[4] This can be compared with the section titled "Unusual Problem Solving," in Booklet No. 17: *Hints for Problem Solving*.

less than or equal to 5. Then 1/5 is the eleventh fraction counted, or $1/5 \leftrightarrow 11$.

What counting number will go with 7/9? The numerator and denominator add up to 16. It took us $(15 \times 14)/2 = 210/2$, or 105, numbers to finish those that add up to 15. To complete those that add up to 16 will take $(16 \times 15)/2 = 240/2 = 120$. So the number will be somewhere between 106 and 120, inclusive.

We can do even better; our arrows swing up when the totals are even, down when they are odd. (Look at the diagram; notice the arrows for totals 2, 3, 4, 5, and 6.) Our total is 16, and 7/9 is the seventh fraction numbered in this diagonal. (Why?) The first fraction is numbered 106; the second, 107; so the seventh will be numbered 112. If you do not believe this, finish the diagram through fifteen diagonals and count for yourself; then go back through the mathematical argument.

In any case, every fraction gets numbered. You may complain that we started talking about numbering *rational numbers* and ended up counting *fractions*. We have numbered 1/1 differently from 2/2 and 3/3, $\cdots$. Each rational number has been assigned an infinite number of different counting numbers. Can we fix this? Sure! Just leave out of each row any fraction that is equivalent to a fraction in an earlier row, but keep the pattern, as shown in Figure 21.

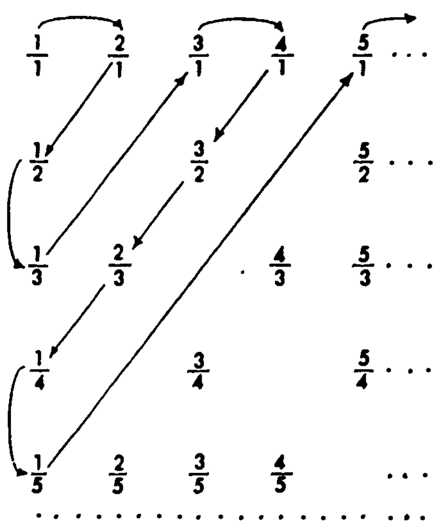

FIGURE 21

Now, number again:

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{5}{1}$	$\cdots$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1	2	3	4	5	6	7	8	9	10	11	$\cdots$

Some of our diagonals may have only a few elements; but this just saves time and counting numbers. Of course, the formulas do not work any more; but they still give a *maximum*. We know that a rational number will be numbered at or before the former number of any of its equivalent fractions.

How about all rationals, or all fractions—positive, negative, and zero? We can make up a pattern, as in Figure 22, that will give a counting method! The arrows go around in spirals, starting at 0/1.

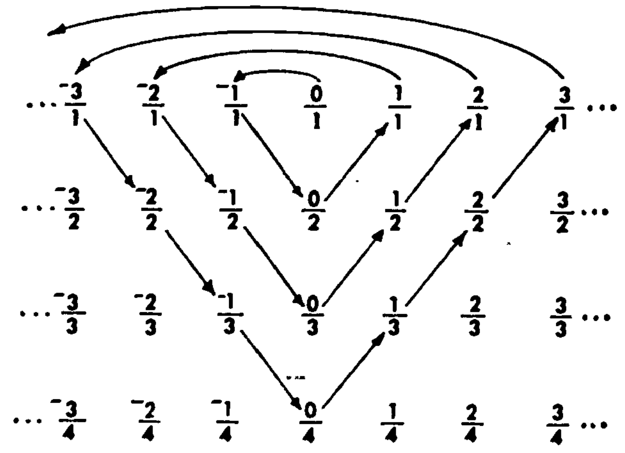

FIGURE 22

Probably you will have already phrased the next question: How about the real numbers? In 1874 Georg Cantor proved that we *cannot* count them. There is no way to stretch the counting numbers this far.

Before we begin the actual proof, it might be well to carry on some discussion of mathematical impossibility. All too often a statement in mathematical terms that something or other is impossible is understood to mean "I can't do it" or "Nobody has done it yet." This misunderstanding has caused tremendous wastes of time, both to the people who did not understand and promptly tried to do the impossible, and to the people who had to find the inevitable errors that resulted from these attempts.

Practically every mathematical statement can be converted to a statement of impossibility. Even "2 + 2 = 4" can be read as, "It is impossible, under the standard rules of arithmetic and numerical notation, that the sum of two and two should differ from four."

The statement "It is impossible to give a straightedge and compass construction for trisecting an arbitrary angle" is essentially no different from the preceding statement. It could be expressed positively: "Tools other than a straightedge and compass are needed to make a construction that will trisect an arbitrary angle."

Our situation is similar. We must show that no one, regardless of his genius or cunning, can ever succeed in putting the set of real numbers into one-to-one correspondence with the counting numbers. In fact, it is a logical contradiction to imply such a correspondence.

The way we are going to demonstrate this is to assume that someone claims to have made the correspondence and presents us with the result. Then we shall look at his list and point out a real number that is not in the list. Indeed, there will be infinitely many real numbers not in the list, but we need only point out *one* to defeat his claim. It is important to note that our method will work no matter how often our friend with the list goes "back to the old drawing board." In theory we can always glance at any list he produces and write down a real number that he has left out.

We shall work only with the real numbers greater than zero and less than one and shall show that not even these can be counted. These numbers will be represented, for convenience of discussion, by infinite decimals without the customary 0 to the left of the decimal point. Some examples are

$$.1000000000 \cdots ,$$
$$.1010010001 \cdots ,$$
$$.1212121212 \cdots .$$

Now imagine the correspondence like this:

Counting Number	Infinite Decimal	
1 ↔	_ _	⋯
2 ↔	_ _	⋯
3 ↔	_ _	⋯
⋯	⋯	⋯

where some digit stands above each underline and the table consists of infinite decimals and extends infinitely far down.

In the following manner we specify an infinite decimal A that is *not* in this list: Look at the first digit of the first numeral listed. If it is not 5, then the first digit of A will be 5; if the first digit of the first numeral is 5, then the first digit of A will be 4. Thus, A starts with either .4 or .5; and we are sure that, whatever A is going to be, it is not going to be the first numeral in the table, because it starts off differently.

Next, look at the second digit of the second numeral (the one corresponding to 2). If this digit is not 5, make the second digit of A equal to 5; if it is 5, use 4 as the second digit of A. So A starts in one of the four ways: .55, .54, .45, or .44. Now we know that A is not going to be equal to either the first or the second numeral in the table.

Let us go on. We look at the third digit of the third numeral and pick either 4 or 5 as a third digit for A, just as before. By so doing we guarantee that A is different from the first, second, and third numerals.

Let us continue. If the list actually began

$$\begin{aligned}
1 &\leftrightarrow .\underline{1}4372956 \cdots \\
2 &\leftrightarrow .3\underline{5}219784 \cdots \\
3 &\leftrightarrow .71\underline{9}85132 \cdots \\
4 &\leftrightarrow .312\underline{9}6540 \cdots \\
5 &\leftrightarrow .1200\underline{0}000 \cdots \\
6 &\leftrightarrow .90763\underline{8}41 \cdots \\
7 &\leftrightarrow .132041\underline{5}2 \cdots \\
8 &\leftrightarrow .5555555\underline{5} \cdots ,
\end{aligned}$$

then A would begin

$$.54555544\ldots$$

because of the successive values of the underscored digits.

Continue this process and then think about the result. We get an infinite decimal for A. It is not *any* of the decimals in the list because it differs from *every* member of the list. So A wasn't in the list at all.

There is nothing special about our rule that picked 4 or 5. Any rule will do as long as it guarantees that A is different from each member of the list and that A neither equals 0 nor trails off in an infinity of 9's, both of which choices are barred. (We use only positive reals in our list, and earlier in this booklet we specified that those decimals ending in an infinite string of 9's were not acceptable.) Another perfectly good system would be to work with numbers 1 and 7, instead of 4 and 5. Try making up your own rules, choosing your own pairs of numbers.

This proof has been named the "diagonal proof" because we go down the diagonal of the table from left to right, making up A, digit by digit, by changing the digits in this diagonal.

What do we know now? We see, by a convincing demonstration, that there are more real numbers than rationals. The counting numbers could be stretched to count the rational numbers; they cannot count the real numbers. The cardinal number of the reals—that is, the number that is associated with each of the sets that *can* be put into one-to-one correspondence with these reals—is called C. Is C the greatest cardinal? No. It can be shown that there is no limit to the cardinality of sets that can be created, but the proof is too long and complicated to give here.

Exercise Set 5

1. Suppose the correspondence began

$$\begin{aligned}
1 &\leftrightarrow .12345678 \cdots \\
2 &\leftrightarrow .11111111 \cdots \\
3 &\leftrightarrow .55544444 \cdots \\
4 &\leftrightarrow .55455555 \cdots \\
5 &\leftrightarrow .10000000 \cdots \\
6 &\leftrightarrow .01010101 \cdots
\end{aligned}$$

ERIC

How would A appear, explicitly to six digits, if you followed the rule given first to construct it?

2. Given: $A = .554544 \cdots$.
Give the first six decimal digits of six different infinite decimals in which the first member differs from A only by its first digit, the second member differs from A only by its second digit, and so on.

SUMMARY

The system of real numbers has been constructed to remedy a defect, an incompleteness, in the system of rational numbers: the fact that there are ratios of lengths of segments that cannot be evaluated in terms of ratios of whole numbers.

We have represented the real numbers by infinite decimal numerals, and we have seen which of these numerals corresponded to the rational numbers as a subset of the real numbers.

Operations corresponding to the arithmetical operations on the rational numbers have been defined for the real numbers; and methods have been given to approximate, as closely as we wish, calculations with real numbers.

Finally, we have seen that the extension of the rational number system to the real number system involves a very great extension indeed. In fact, it requires a completely new order of infinity, a new infinite cardinal number.

For Further Reading

Among the various helpful books dealing with the subject here introduced, one is *Numbers: Rational and Irrational*, by Ivan Morton Niven ("New Mathematical Library," Vol. I [New York and Toronto: Random House, 1961]). This 136-page book is available, in hardback and paperback editions, from Random House, Inc., 501 Madison Avenue, New York, New York 10022.

ANSWERS TO EXERCISES

Exercise Set 1 (p. 14)

1. a. 0.999 **b.** $^-0.999$

2. $^-0.202400\cdots$ (or $^-0.20135\cdots$, $^-0.20200\cdots$, etc.)

3. $^-100.00\cdots$, $^-4.2310\cdots$, $^-0.1234\cdots$, $^-0.01234\cdots$, $1.\bar{2}34\cdots$, $12.345\cdots$

4. a. $0.21000\cdots$ **b.** $^-1.13000\cdots$ **c.** $2.135C00\cdots$

Exercise Set 2 (p. 20)

1. a. $0.\overline{153846}$ **b.** $0.\overline{135}$ **c.** $0.4\overline{857142}$ **d.** $0.\overline{307692}$

2. a. $\frac{20}{99}$ **b.** $\frac{68}{55}$ **c.** $\frac{5}{13}$ **d.** $\frac{4115}{33333}$

3. $0.\overline{142857}$, $0.2857\overline{142857}$, $0.42857\overline{142857}$, $0.57\overline{142857}$, $0.7\overline{142857}$. $0.857\overline{142857}$

4. $x = 0.\bar{9}$, $10x = 9.\bar{9}$. Subtract $9x = 9$, $x = 1$.

Exercise Set 3 (p. 26)

1. a. 1.732 (Try squaring 1.732 and 1.733.) **b.** 2.449

2. 1.25 (Notice that $1.25 \times 1.25 \times 1.25 = 1.953125$;
$1.26 \times 1.26 \times 1.26 = 2.000376$.)

3. The types of numbers named in exercises 3a and 3b cannot be rational numbers. But $\sqrt{2} \times \sqrt{2} = 2$, so products of irrational numbers *can* be rational numbers.

Exercise Set 4 (p. 33)

1. The circular path is more than 444 yards long, since

$$100\sqrt{2} > 141.4 \quad \text{and} \quad \pi > 3.14.$$

2. a. Laws not satisfied by substitution of "counting number":
closure for subtraction
closure for division
additive identity
additive inverse
multiplicative inverse
density
completeness

b. Same as exercise 2a except additive identity

c. Same as exercise 2b except additive inverse

d. Completeness

Exercise Set 5 (p. 42)

1. .554455 ⋯

2. There are many correct answers. Here is one:

$$\begin{aligned} 1 &\leftrightarrow .454544 \cdots \\ 2 &\leftrightarrow .544544 \cdots \\ 3 &\leftrightarrow .555544 \cdots \\ 4 &\leftrightarrow .554444 \cdots \\ 5 &\leftrightarrow .554554 \cdots \\ 6 &\leftrightarrow .554545 \cdots \end{aligned}$$