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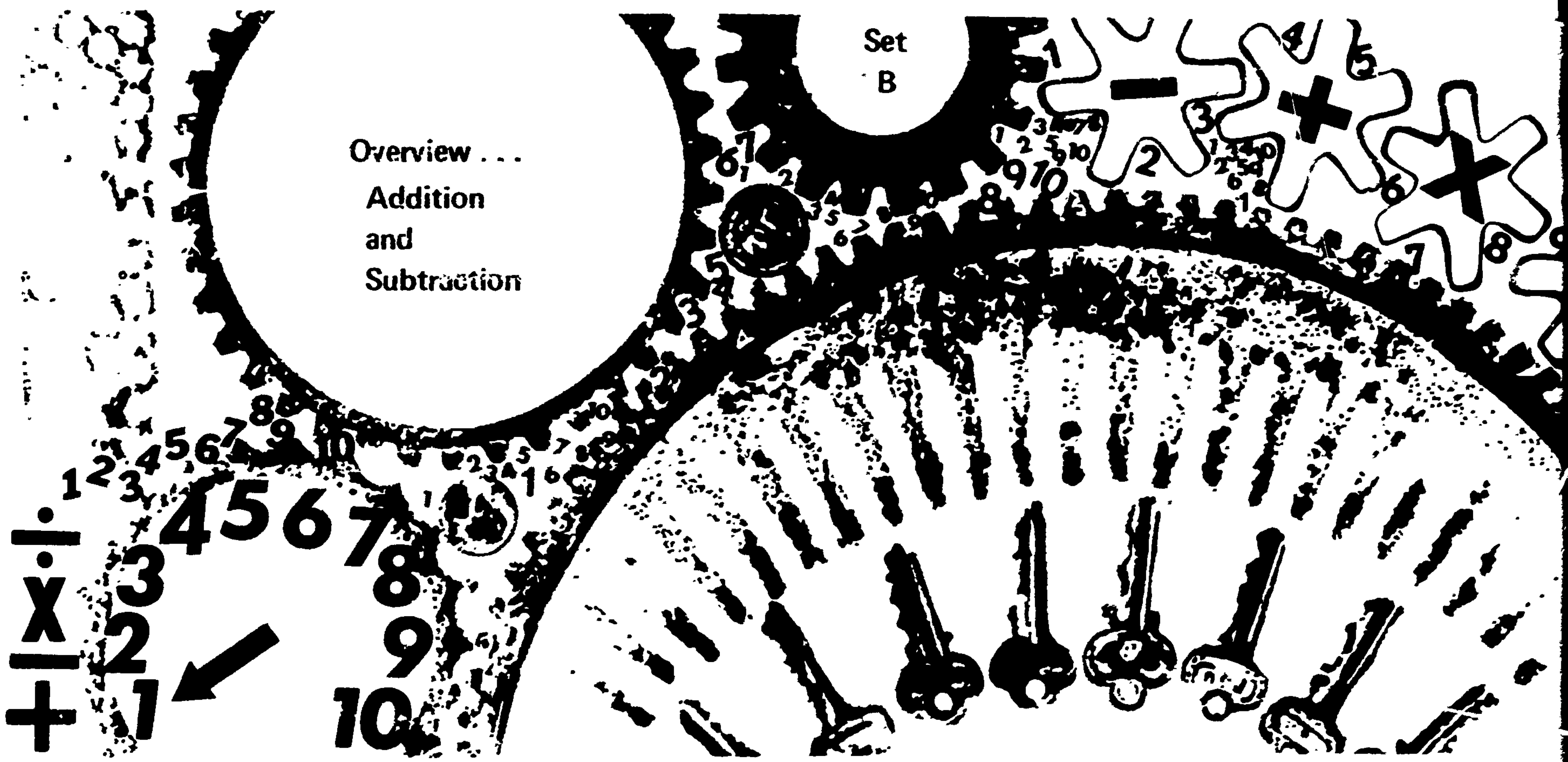
ABSTRACT

Material included in this bulletin provides an overview of research studies which pertain to teaching addition and subtraction of whole numbers in the primary grades. The studies mentioned relate directly to the following questions: (1) What foundation for addition and subtraction do children have upon entering school? (2) What is the relative difficulty of addition and subtraction facts? (3) Should addition and subtraction be introduced at the same time? (4) What type of problem situation should be used for introductory work with subtraction? (5) How can number facts be taught effectively? (6) How should subtraction with renaming be taught? (7) What is the role of drill in teaching addition and subtraction? A list of selected references is supplied at the end of the paper. (FL)



U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
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Using Research: A Key to Elementary School Mathematics

ADDITION AND SUBTRACTION WITH WHOLE NUMBERS

What foundation for addition and subtraction do children have upon entering school?

The ability to count is, of course, of particular importance as a foundation for developing addition and subtraction concepts and skills. Ability to recognize the number of a set without counting and to "conserve numerousness" is also helpful. Surveys have shown that most children can count to at least 19 by the time they enter school, and many can solve addition and subtraction examples which are presented orally.

What is the relative difficulty of addition and subtraction facts?

It has been found in many studies done under a drill method of teaching that:

(1) An addition combination and its "reverse" form tend to be of equal difficulty.

(2) Size of addend is the principal indicator of difficulty.

(3) Combinations with a common addend appear to be of similar but equal difficulty.

(4) The doubles in addition and those in which 1 is added with a greater number appear to be easiest in addition, while those with differences of 1 or 2 are easiest in subtraction.

<u>However</u>, the order of difficulty seems to be a function of teaching method -- thus research is presently being done to reconsider difficulty level for the meaningful methods in use today.

Should addition and subtraction be introduced at the same time?	In the few studies reported, stress on the relationship between addition and subtraction is found to facilitate understanding, and some increase in achievement has been noted when they are taught together.
What type of problem situation should be used for introductory work with subtraction?	"Take-away" problems are easiest, then "additive" problems, and finally "comparative" problems. Recent research has shown that an approach in which sets are separated into subsets is effective for developing understanding of subtraction situations.
How can number facts be taught effectively?	Experiences with concrete materials have been found to be essential for developing understanding of addition and subtraction concepts. Materials should be appropriate to the child's achievement level and rate of learning. It has been found that children use various ways of obtaining answers to combinations -- guessing, counting, solving using known combinations, and meaningful recall -- and apparently attain mastery only after meaning becomes clear to them.
How should subtraction with renaming be taught?	Decomposition is the renaming procedure used almost exclusively in the United States today. When it is taught meaningfully, understanding and accuracy are better than when it is taught mechanically. Use of the equal additions procedure may lead to even greater accuracy, but possibly at the expense of understanding.
What is the role of drill in teaching addition and subtraction?	Drill must be preceded by meaningful instruction. Accuracy has been and is accepted as a goal in mathematics, but the type of thinking which is developed and the child's facility with the process of thinking is of greater importance than mere recall. Drill and practice should be included at appropriate points; they should be planned to meet the needs of the child.

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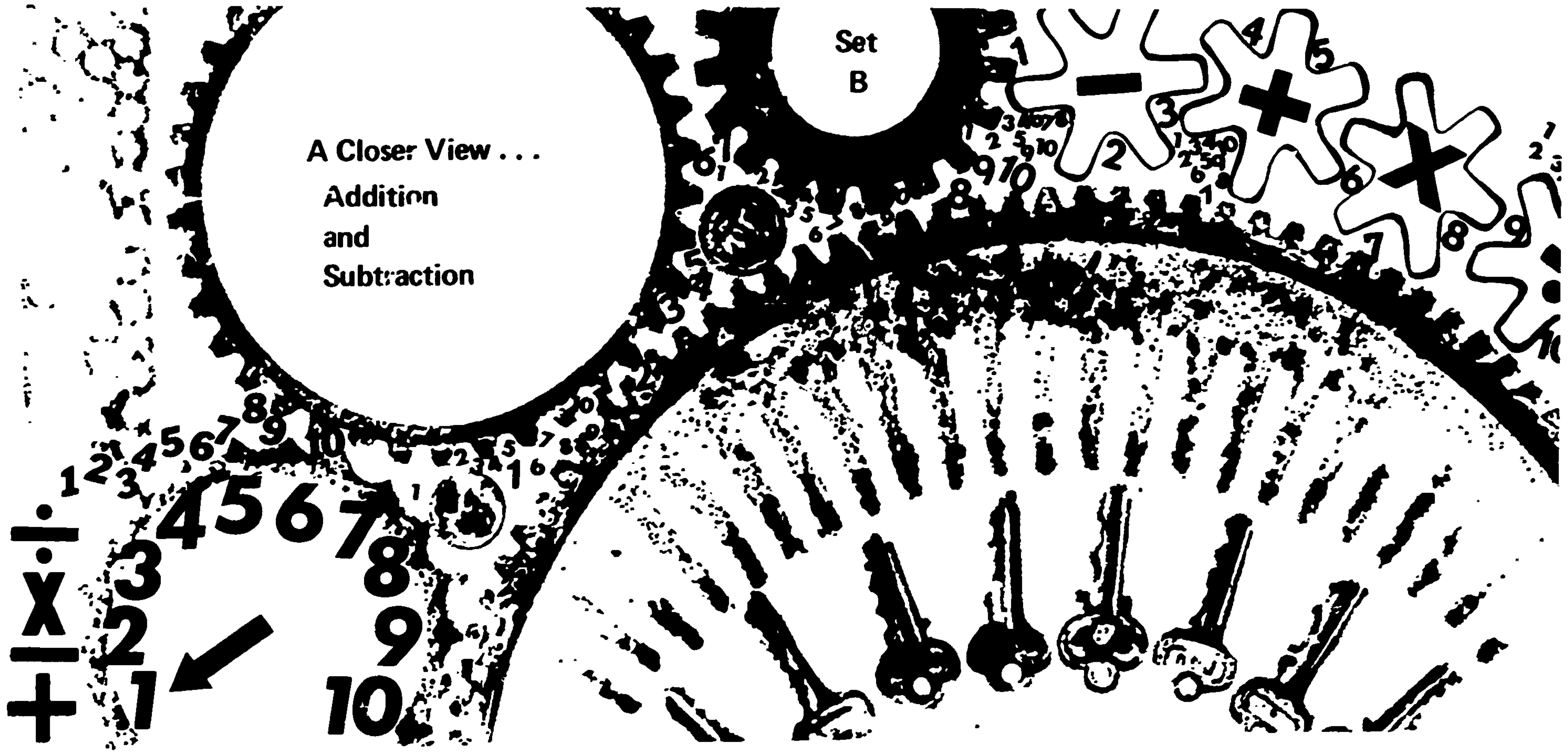
If you would like more information about the research whose findings are cited above, contact MARILYN N. SUYDAM, Project Director, at The Pennsylvania State University, University Park, Pennsylvania, 16802.



Using Research: A Key to Elementary School Mathematics

ADDITION AND SUBTRACTION WITH WHOLE NUMBERS

What foundation for addition and subtraction do children have upon entering school?

As teachers are well aware, a foundation for the development of skills in addition and subtraction is formed long before the first grade. The ability to count is of particular importance: children use counting as a primary means of ascertaining and verifying addition and subtraction facts. The ability to recognize the number of a set without counting is also helpful.

While few experimental studies have been done to determine what <u>can</u> be taught, many surveys have been conducted to ascertain the mathematical ideas and abilities possessed by the pre-school child. The surveys indicate that almost all kindergarten children could count by ones, with most children counting both rotely and rationally to at least 19 (e.g., Bjonerud, 1960; Brace and Nelson, 1965). Less than

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The bulletin was prepared by MARILYN N. SUYDAM, The Pennsylvania State University, Project Director, and J. FRED WEAVER, The University of Wisconsin-Madison, Project Consultant. Art by Ed Saffell.

It should be noted that research is variable with respect to its quality; hence, the same degree of confidence cannot be placed in all findings. An attempt has been made to take this fact into consideration in preparing this bulletin.

one-fourth of the children could also count by twos, fives, and tens. Many children could solve addition and subtraction examples in an oral context.

Whether rote counting or rational counting should be taught first is a recurrent question, but has not been explicitly answered by research. Generally, the pre-school child learns to say the number names and then begins to say them in order before he associates the names with sets of objects.

The relationship of the work Piaget has done with "conservation" seems to have applicability to the classroom. Steffe (1968) pointed out that one type of ability possessed by children who do better in first grade mathematics is the ability to "conserve numerousness" -- that is, to be able to specify that "if two sets are matched, one-to-one, the number of objects in each is the same, regardless of the arrangement or rearrangement of the two sets."

At the end of first grade, he administered tests of addition problems and facts to children at four levels of ability to conserve numerousness. Children at the lowest level performed significantly less well on both tests than did children in the upper three levels. At all levels of conservation of numerousness, problems with accompanying physical and pictorial aids seemed to be of about equal difficulty; however, problems with no aids were significantly more difficult. Problems in which one of two sets is described as being moved to the other were also significantly easier than problems in which the two sets are static.

Steffe concluded that ability to conserve numerousness thus seems to be related to achievement on addition problems.

LeBlanc (1968) reported on a parallel study with subtraction problems and facts. Children who were in the highest level of conservation of numerousness performed better than children in the lowest two levels. Problems accompanied by aids and those with a description of movement were significantly easier than other types of problems. LeBlanc suggested that a test of conservation of numerousness would provide a basis for a readiness test for first graders.

What is the relative difficulty of addition and subtraction facts?

At one time, especially when stimulus-response theories of learning were prevalent, there was great interest in ascertaining the relative difficulty of the basic number facts or combinations -- e.g., 5 + 2 = 7, 9 + 6 = 15, 8 - 3 = 5, 17 - 9 = 8. Textbook writers as well as classroom teachers used the results of such research to determine the order in which facts would be presented. The assumption was that if the combinations were sequenced appropriately, the time needed to memorize them could be reduced.

The relative difficulty of the combinations generally was derived from a study of either (1) the number of errors made on each combination, (2) reaction time, (3) retention after a period of non-use, (4) the number of repetitions needed for immediate recall during initial learning, or (5) familiarity with combinations among children entering school. The varying procedures are, in part, the reason for lack of agreement among the studies.

Nevertheless, some common findings were evident which, despite the age of the studies, may in part still be applicable (e.g., MacLatchy, 1933; Washburne and Vogel, 1928; Wheeler, 1939):

(1) An addition combination and its "reverse" form tend to be of equal difficulty.

(2) Size of addend is the principal indicator of difficulty, rather than size of sum.

(3) Combinations with a common addend appeared to be of similar but not equal difficulty.

(4) The "doubles" in addition and those in which 1 is added with a greater number appear to be easiest in addition, while those with differences of 1 or 2 are easiest in subtraction.

Swenson (1944) questioned whether results on relative difficulty obtained under repetitive drill-oriented methods of learning are valid when applied in learning situations not so definitely drill-centered. When second graders were taught by drill, by generalization, and by a combined method, it was found that the order of difficulty seemed to be, at least in part, a function of teaching method. Thus research which aims at establishing the difficulty of arithmetic skills and processes should probably do so in terms of a clearly defined teaching and learning method.

Recently, Suppes (1967) has been interested in using the data-gathering potential of the computer to explore the relative difficulty of mathematical examples, including the basic facts. A drill-and-practice program which presents addition and subtraction combinations has been used as the vehicle to determine a suggested order of presentation and amount of practice.

Should addition and subtraction be introduced at the same time?

It is somewhat surprising, considering how frequently this question is asked, to find that there has been little research on the topic. Early studies (such as Brownell, 1928) found that higher achievement resulted when addition and subtraction facts were taught together. Spencer (1968) recently reported that there may be some intertask interference, but emphasis on the relationship facilitates understanding.

Research has generally found that the subtraction combinations are harder for children to learn than those in addition, even when addition and subtraction are taught together.

What type of problem situation should be used for introductory work with subtraction?

Gibb (1956) explored ways in which pupils think as they attempt to solve subtraction problems. In interviews with 36 second graders, she found that pupils did best on "take-away" problems and poorest on "comparative" problems. For instance, when the question was, "How many are left?", the problem was easier than when it was, "How many more does Tom have than Jeff?". "Additive" problems, in which the question might be, "How many more does he need?", were of medium difficulty and took more time. She reported that the children solved the problems in terms of the situation, rather than conceiving that one basic idea appeared in all applications.

Schell and Burns (1962) found no difference in performance on the three types of problems. However, "take-away" situations were considered by pupils to be easiest -- thus they are generally considered first in introductory work with subtraction.

Coxford (1966) and Osborne (1967) found that an approach using set-partitioning, with emphasis on the relationship between addition and subtraction, resulted in greater understanding than the "take-away" approach. Consideration of this finding is important to those who want to develop set-subset concepts as a strand in the curriculum.

How can number facts be taught effectively?

Brownell (1928, 1941) and McConnell (1934) found that pupils use various ways of obtaining answers to combinations -- guessing, counting, and solving from known combinations, as well as immediate recall. Brownell stated, "Children appear to attain 'mastery' only after a period during which they deal with procedures less advanced (but to them more meaningful) than automatic responses."

In general, experiences with concrete materials provide an essential base for developing understanding of addition and subtraction concepts. Encouraging pupils to use drawings as well as objects may help those having difficulty learning combinations (Brownell, 1928).

Generally, researchers have concluded that understanding is best facilitated by the use of concrete materials; followed by semi-concrete materials such as pictures, and finally by the abstract presentation with words and/or numerals.

Gibb (1956) also found that abstract contexts were poorest. She reported, however, that pupil performance was better on subtraction examples presented in a semi-concrete context, rather than with concrete materials. Nevertheless, she noted, "Children have less difficulty solving problems if they can manipulate objects or at least think in [the] presence of objects with which the problems are directly associated than when solving problems wholly on a verbal basis."

Klausmeier and Feldhusen (1959) are among those who have found that curriculum materials should be appropriate to the learner's achievement level and rate of learning. Then both initial achievement and retention are not significantly different across intelligence levels.

Transfer was studied by Olander (1931). Pupils who had studied only 55 addition and 55 subtraction combinations (omitting the "reverse" forms) were also able to answer most of the 90 which they had not studied, doing almost as well as those who studied all 200 combinations.

How should subtraction with renaming be taught?

Over the years, researchers have been very concerned with procedures for teaching subtraction involving renaming (once commonly called "borrowing"). The question of most concern has been whether to teach subtraction by equal additions or by decomposition.

How do you do this example?

$$\begin{array}{r} 91 \\ -\ 24 \\ \hline 67 \end{array}$$

You're using decomposition if you do it this way:

11 - 4 = 7 (ones); 8 - 2 = 6 (tens)

If you do it this way, you're using equal additions:

11 - 4 = 7 (ones); 9 - 3 = 6 (tens)

In a classic study, Brownell (1947; Brownell and Moser, 1949) investigated the comparative merits of two algorithms (decomposition and equal additions), in combination with two methods of instruction (rational or meaningful, and mechanically):

	rational (meaningful)	mechanical
decomposition	a	b
equal additions	c	d

He found that, at the time of initial instruction:

(1) Rational decomposition [a] was better than mechanical decomposition [b] on measures of understanding and accuracy.

(2) Rational equal additions [c] was significantly better than mechanical equal additions [d] on measures of understanding.

(3) Mechanical decomposition [b] was not as effective as either equal additions procedure [c or d].

(4) Rational decomposition [a] was superior to each equal additions procedure [c, d] on measures of understanding and accuracy.

It was concluded that whether to teach the equal additions or the decomposition algorithm depends on the desired outcome.

In recent years, the decomposition procedure has been used almost exclusively in the United States, since it was considered easier to explain in a meaningful way. However, some question has recently been raised about this: with increased emphasis in many programs on properties and on compensation in particular, the equal additions method can also be presented with meaning. For instance, pupils are learning that:

(a) $9 - 3 = \square$ means that $\square + 3 = 9$

or $3 + \square = 9$

They are learning that:

(b) $7 - 4 = 3$ is equivalent to

$$(7 - 4) + 2 = 3 + 2$$

Development of such ideas should facilitate the teaching of the equal additions procedure. Whether there will again be a shift toward wider use of this procedure remains to be seen. Evidence from other studies indicates that use of it leads to greater accuracy.

What facilitates the learning of more complex skills?

Brownell (1947) studied the use of a crutch such as

$$\begin{array}{r} \overset{4}{\cancel{5}}{}^{1}6 \\ -\ 39 \\ \hline 17 \end{array}$$

This seemed to facilitate understanding, but attempts to have pupils stop using the crutch were not wholly successful. Some persons suggest that this crutch should *only* be taught when it is *needed*.

Overman (1930) found that if pupils were taught to generalize about the renaming procedures in two-place addition and subtraction, they were able to do three-place examples. This was less time-consuming than having the teacher present two-place and then three-place examples separately.

Ekman (1967) reported that when third graders manipulated materials before presentation of an addition algorithm, both understanding and ability to transfer increased. Use of materials was better than use of only pictures before introduction to the algorithm, or development of the algorithm without either aid.

What is the role of drill in teaching addition and subtraction?

Discussions on the teaching of mathematics in the primary grades once centered on whether programs should consist of isolated, repetitive drill or of an integrated approach involving the presentation of interrelated ideas. Prior to the 1930's, much research was done on the effectiveness of various types of drill. For instance, Knight (1927) reported on a successful program of drill in which the distribution of practice on basic facts was carefully planned -- no facts were neglected, but more difficult combinations were emphasized.

Accuracy has been and is accepted as a goal in mathematics, and it is in an attempt to meet this goal that drill is stressed. In a series of articles, Wilson advocated no less than 100% mastery. He showed that, with a carefully planned set of materials, the goal was not as unattainable as some persons believed it to be.

Many other studies have shown that drill per se is not effective in developing mathematical concepts. Programs stressing relationships and generalizations among the addition and subtraction combinations were found to be preferable for developing understanding and the ability to transfer (McConnell, 1934; Thiele, 1938). This has been supported by many studies since that time.

Brownell and Chazal (1935) summarized their research work with third graders by stating that drill must be preceded by meaningful instruction. The type of thinking which is developed and the child's facility with the process of thinking is of greater importance than mere recall. Drill in itself makes little contribution to growth in quantitative thinking, since it fails to supply more mature ways of dealing with numbers.

Pincus (1956) also found that whether drill did or did not incorporate an emphasis upon relationships was not significant, when drill followed meaningful instruction.

Should non-paper-and-pencil practice be provided?

Many mathematical problems which arise in everyday life must be solved without pencil and paper. Providing a planned program of non-paper-and-pencil practice on both examples and problems has been found to be effective in increasing achievement in addition and subtraction, as for other topics in the curriculum (Flournoy, 1954). Other researchers have suggested that certain "thought processes" which are especially suited to such practice should be taught. For instance, a left-to-right approach to finding the sum or difference is useful, rather than the right-to-left approach used in the written algorithm. "Rounding," using the principle of compensation, and renaming are also helpful. Increased understanding of the process may result.

Should children "check" their answers?

The answers which research has provided to this question are not in total agreement. We encourage children to check their work, since we believe that checking contributes to greater accuracy. There is some research evidence to support this belief.

However, Grossnickle (1938) reported data which should be considered as we teach. He analyzed the work of 174 third graders who used addition to check subtraction answers. He found that pupils frequently "forced the check," that is, made the sums agree without actually adding; in many cases, checking was perfunctory. Generally, there was only a chance difference between the mean accuracy of the group of pupils when they checked and their mean accuracy when they did not check.

What does this indicate to teachers? Obviously, children must understand the purpose of checking -- and what they must do if the solution in the check does not agree with the original solution.

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