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The Madison Project's Approach to a Theory of Instruction.

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Reported is the everyday teaching and curriculum planning activities of the Madison Project, operated by teachers and mathematicians. Two kinds of informal explanatory experiences are provided in order to involve students in the discovery of significant mathematical concepts--experiences where children do something and experiences where a "seminar" of children discuss something, under the leadership of a teacher. The Project has developed a set of seven criteria for selecting appropriate classroom "informal exploratory experiences:" (1) The child should have adequate previous "readiness", (2) informal exploratory experiences should be related directly to fundamental ideas, (3) the student must have an active role, (4) concepts must be learned in context, (5) interesting patterns must be developed in every task, (6) the experiences should be appropriate to the age of the child, and (7) the sequence of "informal exploratory experiences" must seem to be worthwhile. A list of some of the objectives of the Madison Project is indicated under two headings--Mathematical objectives and general objectives. (RP)

The Madison Project's Approach to a Theory of Instruction [1]

Robert B. Davis

In many fields today one finds, on the one hand, the "practical" man on the everyday firing line, and, on the other hand, the "theoretical" man back in the laboratory. Presumably it suggests a condition of health for a subject when the two work harmoniously together. In the present remarks, I can speak only as the "practical" man, reporting on the everyday teaching and curriculum planning activities of the Madison Project, which is operated by teachers and mathematicians. We are greatly interested in a "theory of instruction," and we wish to contribute to it as much as possible from our admittedly untheoretical position. The theoretical work of Piaget and Bruner gives me considerable hope that our "practical" decision-making can be related to a broader theoretical perspective. [2]

1. The Madison Project. [3] By way of background, let me explain that the Madison Project is one of the currently-active "curriculum revision" projects, sponsored by the

[1] Financial support for the Madison Project is provided by the National Science Foundation, the Division of Co-operative Research of the U.S. Office of Education, and by other agencies.

[2] Cf. Jerome S. Bruner, "Needed: A Theory of Instruction," Educational Leadership, Vol. 20, No. 8 (1963), May, p. 523 ff.

[3] Cf. i) Robert B. Davis, "The Evolution of School Mathematics," Journal of Research in Science Teaching, Vol. 1 (1963), pp. 260-264; ii) Robert B. Davis, "The Madison Project: A Brief Introduction to Materials and Activities," The Madison Project, 1962; iii) Robert B. Davis, "Report on the Madison Project," Science Education News (1962), December, pp. 15-16; iv) Robert B. Davis, "Report on the Syracuse University-Webster College Madison Project," American Mathematical Monthly, Vol. 71, No. 3 (1964), March, pp. 306-308; v) Robert B. Davis, Experimental Course Report/ Grade Nine (Report #1, June 1964) available from The Madison Project.

National Science Foundation, the U. S. Office of Education, and other agencies. We have basically set ourselves the task of seeking "the best experience with mathematics which can be provided for children at the pre-college level." As one result of this quest we have developed a program of supplementary work in algebra and co-ordinate geometry for grades 2-8 (i.e., chronological ages 7 years through 13 years, approximately), we have developed a complete 9th grade mathematics course, and we are carrying on tentative exploration at the level of nursery school and kindergarten (chronological ages 3 years to 5 years, approximately).

We have two aspects of direct interest to this conference: for one thing, we make films showing actual classroom lessons, and follow the same children (where possible) for many consecutive years.[4] You can watch, on film, one of our classes begin in grade 3, and gradually progress to grade 7; in another case, you can watch a group of students begin the study of Madison Project materials in grade 5, and continue through grade 9. Much is revealed by these films, and we hope that many of you will wish to borrow them, and to analyze them or comment upon them from your various points of view. I can think of nothing better than to accumulate essays or analyses of these films by, say, a cultural anthropologist like Jules Henry, psychologists like Jean Piaget, Jerome Bruner, Jerome Kagan, Henry Murray, Richard Suchman, and Richard de Charms, by appropriate mathematicians, logicians, teachers, psychoanalysts, and so on. I shall return to this later when we discuss the "description problem."

[4] Cf. Robert B. Davis, "Report on Madison Project Activities, September 1962 - November 1963." A report submitted to the National Science Foundation, December 1963.

Secondly, the Project is, as I have mentioned, deeply interested in contributions to a "theory of instruction." A description of our everyday decision making procedures -- to the extent that we know them explicitly -- constitutes the rest of this paper.

2. What Kinds of Mathematical Experiences Shall We Provide for our Students?

In general, there are two kinds of experiences which we provide for the children: experiences where children do something, and experiences where a "seminar" of children discuss something, under the leadership of a teacher. Both kinds of experiences are so different from usual "mathematics lessons" that we have had to give them a distinctive name -- informal exploratory experiences -- in self-protection against unsympathetic observers who have told us "Why, there was no teaching in that lesson!"

It might be well to give one or two examples of each kind of "experience."

Angles and Rotations.[5] In these lessons, which we refer to as "experience" lessons, the children (at the 4th or 5th grade level in most cases) are shown pictures of angles drawn on the blackboard, asked to guess the measure (in degrees) of the angle, and thereafter check their guesses by trying to measure the angle with a protractor, or with pie-shaped "units" (circular sectors of 10° central angle). They do "right-face," "about-face," and other turns with their own bodies (including turns through 30°, $^-30^\circ$, 360°, 720°, and so forth), and rotate wheels through specified angles (of positive or negative measure). This kind of thing we refer to as "experience with angles." In a sense those observers are right

[5] Cf the films Experience with Estimating and Measuring Angles and Experience with Angles and Rotations.

who say "Why, there was no teaching in that lesson!" We believe there was, however, considerable learning. The teacher has tried to bring the children into a direct face-to-face confrontation with the mathematics itself. When such lessons work well, our teachers often have the feeling that they have somehow been able to step backward, out of the way, and the child has been in direct communication with the mathematics.

Weights and Springs.[6] This lesson is mainly an "experience" lesson, but it sometimes turns suddenly into a seminar discussion lesson. Children (at the level of grade 4-9), perhaps working in small groups, hang weights on (A) a metal spring, and (B) a chain of rubber bands, recording at each step the weight on the spring (or rubber bands), and the amount that the spring stretches. We do not structure this by suggesting what weights they use, etc. We do suggest that they represent this data by a graph (but we do not tell them how to make the graph[7]). We also suggest that, if possible, they write an algebraic expression for the functional relationship shown on the graph.

This lesson becomes a "seminar" or "discussion" lesson if the children need to work together in a total group, under teacher guidance, in order to obtain the algebraic

[6] An earlier, and somewhat different, version of this lesson, with a 6th grade class, can be viewed on the film entitled Weights and Springs. A generally similar kind of lesson can be viewed on the film Average and Variance.

[7] However, the class will have had plenty of prior experience with functions and graphs. Cf. the films Experience with Linear Graphs, Second Lesson, Postman Stories, Circles and Parabolas, and Guessing Functions.

representation of the function, or if they wish to discuss sources of measurement error, the peculiar behavior of the rubber band, whether any apparent linearities are descriptive of the spring or are artifacts of the experimental procedure and of the measurement methods, etc.

Matrices.[8] This lesson is usually a "seminar discussion" type lesson. Students who are already familiar with the structure of the rational numbers, and who know how to add and multiply matrices, are asked to explore the algebraic structure of the system of 2-by-2 matrices. (Grade level: 5 through 9, inclusive.) The point of the lesson might be stated as follows: in their previous work with the structure of the system of rational numbers, the children were getting experience in "exploring an unknown mathematical terrain." We now want to see how sure-footedly they can go about the task of exploring another new mathematical terrain. The hope is that the children will know what kind of questions to ask, and what kind of answers to seek, as well as how to find these answers. (I shall return to this example later, under the discussion of "short-cutting.") Obviously, where the children falter, the teacher tries to step in as unobtrusively as possible.

One way the teacher may do this is by making a suggestion that is, in fact, inappropriate. In the process of explaining to the teacher why the teacher's suggestion is inappropriate, the students are, of course, forced to peer more deeply into the mathematical structure itself. Once again, the teacher has tried to remove himself from the role of middleman: he has tried to step out of the way, and let the child look directly at the mathematical

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[8] Cf. the film Matrices, and the accompanying pamphlet, which is also entitled "Matrices."

structure itself.[9]

3. Criteria for Choosing Experiences. In selecting mathematical experiences to present to children, at the pre-college level, it is obvious that the danger of including inappropriate experiences is about as great as the danger of omitting valuable ones. The Project has developed a set of seven criteria for choosing appropriate experiences:

i) Adequate previous "readiness." We try to make sure that, prior to the lesson in question, the children have had enough previous experience with essential ideas or techniques so that the desired new learning will be able to take place. I think our attention to prerequisites is unusually meticulous, in the sense that we seek a very careful breakdown of concepts into their simple "atomic" constituents.[10]

On the other hand, our pace is much more rapid than is customary in pre-college classes in the United States; we believe that our rapid pace is not the result of a neglect of readiness building, but is the result of a more optimistic expectation of student performance, a greater reliance upon the mathematical structure itself as a source of cognitive simplification ("reduction of cognitive strain")[11] and of motivation, and

[9] Cf. Robert B. Davis, Discovery in Mathematics, A Text for Teachers. Addison-Wesley, Inc., 1964, pp. 8-15.

[10] Cf. the film Sequencing and Elementary Ideas.

[11] Cf. Jerome S. Bruner, J. J. Goodnow, and G. A. Austin, A Study of Thinking, Wiley, 1956, p. 82 ff. These same pages are also suggestive in relation to our notion of "degree of autonomous control," as discussed on page 15 of the present report.

perhaps other similar factors.

ii) Relation to fundamental ideas. We do not wish to squander valuable momentum by a relatively unprofitable exploration of by-ways. Consequently we make up a list of (what appear to us to be) fundamental concepts and techniques. This list includes: variable, function, Cartesian co-ordinates, open sentence, truth set, matrices, vectors, implication, contradiction, axioms and theorems, uniqueness, "mapping" or transformation, linearity, limit of a sequence, monotonicity, etc. We require each "informal exploratory experience" to relate directly to these fundamental ideas.

iii) The student must have an active role. By this we mean to include activities such as problem-solving, arguing, criticizing, etc., as well as activities such as measuring, estimating, or performing an experiment. We believe that many children fail in mathematics because they assume too passive a role. In order to avoid this danger (which, in our view, is very great) we almost never lecture, and we make very little use of required reading of routine material. (Indeed, we probably make too little use of reading, but our fear of letting some students drop into a passive receptivity, with subsequent failure, is very great, and we tread carefully.)

iv) Concepts must be learned in context. It appears to us that this is an important point. All of the paraphernalia of science or mathematics -- concepts, equipment, data, techniques, even attitudes and expectations -- arise out of the act of tackling problems, arise out of inquiry. We want the concepts which the students form to arise in this same way. We believe this gives the ideas a different kind of meaning than they would have if they had sprung full-grown from the head of the teacher.

v) Interesting patterns must lurk under the surface of every task. We want the students to form the habit of questioning even when there are no explicit external cues suggesting that they question. We wish them to be in the habit of asking: Did that really work? Can we extend it? When does it work? When would it fail? Is there a better way to do it?

In order to cause this "looking beyond the immediate problem" to become virtually habitual, we wish to give the students extensive practice in seeking underlying patterns. Our method for doing this is to attempt to arrange our material in such a way that there nearly always are some interesting patterns lurking just beneath the surface. They are usually not requisite for performance of the immediate task -- but if one does observe them, they are interesting in themselves, they can usually be used to shorten greatly the effort required to perform the immediate task, they may suggest a more powerful attack that will solve harder problems, etc. One such example occurs when the immediate task is to practice the use of variables and the arithmetic of signed numbers -- but this practice occurs in a context of quadratic equations, and sharp-eyed students can "discover" the important coefficient rules for polynomial equations lurking just beneath the surface. This discovery confers greatly added power to those who make it.

A second example occurs when we try to graph the truth set for linear equations. The immediate task can be handled by merely "plotting points," but a hugely extended power will be his who discovers the patterns of "slope" and "intercept."

vi) The experiences should be appropriate to the age of the child. This may seem to go without saying, but in ordinary education this precept seems honored mainly by non-compliance. We are finding, in our admittedly limited experience, that fifth graders (age about 10 years, chronologically) are "natural intellectuals," and can enjoy choosing up a set of algebraic axioms and proving, from them, a variety of algebraic theorems. (This topic was formerly encountered in the latter years of college, or in graduate school.) By contrast, seventh and eighth graders are not "intellectuals"; it might come closer to say they are "engineers" at heart. For 7th and 8th graders, the usual school regime of sitting at desks, reading, writing, and reciting seems to ignore the basic nature of the child at this age; he wants to move around physically, to do things, to explore, to take chances, to build things, and so on.[12] At this age we prefer to get the children out of their seats and, where possible, to get them out of the classroom, even to get them out of the school. We do vector problems by

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[12] The importance of this question should not be overlooked. None of my reading in psychology, nor most of my contacts with psychologists, have attached much special importance to the 5th-6th-7th-8th-grade developmental pattern. Yet for virtually every one of the "new curriculum" projects in mathematics and science which deal experimentally with this age range, this is the single decisive, elusive, and discouraging phenomenon. The 5th grader is very good at mathematics and science. This same child, at grade 6, begins to perform less well. In grades 7 and 8 he is usually a total loss -- he will perform routine tasks to a mediocre standard, but in situations calling for great creativity he usually creates chaos. Since encountering this catastrophe, we have accumulated about 20 alternative explanations, from psychoanalysts, teachers, physiologists, and parents. They include: i) the sex theory: a sexual revolution and awakening is occurring, and all else is secondary; ii) the energy theory: the child's energy is tied up in physical growth (which occurs rapidly at this age); iii) the metabolic theory: the child's metabolic rate shifts, and it takes him several years to adjust his behavior to the new metabolic rate; iv) the "noise" theory: everything we teach is wrong; by grade 6 the child has been in school long enough to accumulate so much

(Footnote 12 continued on page 10)

hanging twenty pound weights on yarn, and predicting whether the yarn will hold or will break; we do graphical differentiation and rate-of-change problems in the context of velocity and acceleration, using actual automobiles in the school driveway; we determine the height of the school flagpole by similar triangles; and so on.[13]

We do not know whether we are on the right track or not, but to our amazement we find no established and well-accepted theory to help to decide this problem: what kind of school experience should the 7th and 8th grader get? In our own clinical interviews with children of this age, we find they greatly prefer "moving around" subjects (like gym, dancing, shop, art, music, home economics, science laboratory, etc.) over sedentary subjects (such as Latin, English, mathematics, and social studies). We need to understand this far more than we do; in the meantime, we are asking ourselves if mathematics, social studies, etc., need to be sedentary subjects at this grade level. [Notice that traditional dogma emphasizes physical activity for younger

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12 (Continued from page 9)
misinformation that he is lost; v) the "nobody loves junior high school" theory: junior highs get the almost-discarded buildings, the almost-discarded teachers, and the almost-discarded objectives and methods; vi) the peer-group theory: the 5th grader loves adults; the 7th grader knows better, and believes his contemporaries, unpromising though they appear, are in the long run a better bet; vii) the neo-Pareto theory: every generation has to take over from its elders, and grade 7 is the place to start; viii) the "finding yourself" theory; ix) the "finding reality" theory: 5th graders are remote from reality and allow themselves an interest in abstract things; a 7th grader is becoming sensitive to power, and so he demands more "practical" employment; x) the "poor self-concept of the junior high teacher" theory: high school teachers really want to be college teachers, and commit the folly of frustrated emulation; junior high teachers imitate the imitators, and that's even worse. Many other theories have been proposed.

13 Cf., for example, Alfred North Whitehead, Aims of Education. (Mentor Books) Macmillan Company, 1929.

children -- some of whom, in our experience (e.g., fifth graders) do not especially require it -- but places less emphasis upon physical movement for the older 7th and 8th graders. Obviously the over-all school program must be considered as a causal factor, also; the younger children, in most schools, may "get enough" chance for physical movement, whereas in junior high school they may not.]

vii) The sequence of "informal exploratory experiences" must seem to "add up" to something worthwhile. By this we mean that the teacher (or other observer) must feel that the lesson, the day, the week, the year have each made their proper contribution to the child's growth toward mathematical maturity and sophistication.

In selecting "informal exploratory experiences," we keep in mind all seven criteria listed above.

4. The Flexibly-Programmed Discussion Sequence. In our "experience" lessons we structure the situation as little as possible. (In practice, we sometimes structure it too little, and sometimes too much.[14]) In our "seminar discussion" lessons there is at once an appearance of a relatively highly structured situation, and at the same time an appearance of great flexibility. After wondering about this seeming paradox for some time, we have come to believe that the "good" Madison Project teacher possesses, in his head, the ability to construct suitably designed "branching programs" at a moment's notice.

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[14] See, for example, the film Creative Learning Experiences.

Let me give an example. The film A Lesson with Second Graders shows a sequence where the teacher poses the problem

$$^{+}5 + \square = {}^{+}3,$$

and asks the students what number can be substituted to produce a true statement. A girl named Charlotte responds immediately with "positive two." At this point the teacher's internal program-constructing facility devises a suitable "error-correction loop." He asks: "How much is

$$^{+}5 + {}^{+}2 = \ ? \text{"}$$

Charlotte thoughtfully (or hesitantly) volunteers the answer "positive seven ..."

The teacher now asks: what can we write in the $\square$ to make a true statement from the open sentence

$$^{+}5 + \square = {}^{+}3,$$

and Charlotte correctly answers "negative two."

I do not mean that every Madison Project teacher would describe his method of operation in the terms I have used here, but I believe that good Madison Project teachers do behave this way.

Indeed, the study of the properties of the "program sequence" which teachers use, perhaps together with a study of the cues which determine teacher decisions, may represent one of the most powerful points of attack in developing a theoretical understanding of "Madison Project teaching."

5. Reinforcement Schedules. Psychologists[15] observing Madison Project lessons have repeatedly emphasized the quite unusual use (or non-use) of reinforcement schedules in Project classes. We should admit at the outset that we use the ordinary "rewards" such as praise and affectionate warmth, etc., in securing reasonable social behavior. We try never, however, to use a teacher-imposed external reinforcement schedule to determine what a child thinks, how he answers a question, or how he attacks a problem. (This last may be a mild over-statement; perhaps the proper description would be: well, hardly ever.)

We try to use two forms of reinforcement only: first, intrinsic rewards derived from solving a problem, from the reduction of cognitive strain which follows upon the discovery of an important concept or relationship, from the gratification of experimental verification of a prior theoretical prediction, etc.;[16] and, second, the reward that comes from being able to tell your classmates, or your teacher, about what you have just discovered or have just accomplished.

As a consequence of our inclusion of this second motivating factor, there is a good deal of communication going on in Project classrooms. In particular, we have never really developed the highly individualized instruction represented by "each student working alone,"

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[15] For example, Dr. Richard Anderson of the University of Illinois, and Dr. Carl Pitts of Webster College.

[16] Köhler, for example, speaks of certain situations -- such as an expensive cup and saucer teetering precariously on the very edge of a table -- that possess "demand quality." We feel that virtually every good lesson should be based upon tasks, situations, problems, questions, seeming contradictions, etc., that possess "demand quality." In this sense we believe that we can entirely eliminate "drill" -- that is to say, we hope to eliminate all of those tasks for which the child sees no reason other than to please the teacher. (Wolfgang Köhler, The Place of Value in a World of Fact, Liveright Publishing Company, 386 Park Avenue South, New York 16, New York.

as this is seen in graduate chemistry laboratories, or in the use of some of Z. P. Dienes' arithmetical materials at the elementary school level (as in Leicestershire, England, or at Shady Hill School in Cambridge, Massachusetts).

Personally, I suspect the Project may have developed too deep an attachment to "seminar-type discussion" lessons, and has, in consequence, failed to develop enough "informal exploratory experiences" in a form where each child works alone, or where the children co-operate in quite small groups (with perhaps three children per group). We are now trying to remedy this, but progress is slow. Is it possible that this slow progress results from the fact that "each child working alone" is not an entirely natural and satisfactory situation? Perhaps the reinforcement of showing the whole class how clever you are really is virtually essential. The tendency of mature scientists to announce their results, often at considerable length, should not be overlooked in this connection.

6. Autonomous Decision Procedures. We believe that the child should, everywhere possible, have a method for telling whether an answer is right or wrong that is independent of the teacher and independent of the textbook. For physical scientists the laboratory ostensibly fills this need. For mature mathematicians, logic ostensibly fills this need.[17] We have tried to fill this need in the earlier grades by using counting to verify work in arithmetic, and in later grades we try to provide multiple methods for solving problems as one way to

[17] Cf., however, Eves and Newsom, The Foundations and Fundamental Concepts of Mathematics, Holt, Rinehart, Winston, 1964, and also J.R. Newman and E. Nagel, Gödel's Proof, New York University Press, 1960.

decide correctness (that is, by the agreement or disagreement of results obtained by different methods). As a second "autonomous decision procedure" we try to provide models, as in the case of "postman stories" for the arithmetic of signed numbers.[18]

It appears to us that removing the "correctness" of mathematics from the authority of the teacher greatly increases student motivation. In the words of Jerrold Zacharias, "Science is a game played against nature; it is not a game played against the teacher."

7. Degree of Autonomous Control. We also believe that the more freedom we can give the children, the more easily can we maintain a really high level of motivation.[19] Since we follow the same children for as long as five years, this is a matter of some importance to us.

The nature of "freedom" has puzzled mankind for a long time, and I do not claim that we understand it. We can, however, say that when a child feels that a task is artificial, capriciously (or thoughtlessly) imposed by the teacher, he does not usually regard it as a serious challenge. Where (as usually, in our work) the task is determined by the teacher, the greater the extent to which a child is free to define his own method of attack, to define

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[18] Cf. W. J. Sanders, "The use of models in mathematics instruction," The Arithmetic Teacher, Vol. 11, No. 3 (1964), March, pp. 157-165. For the specific use of "postman stories," cf. the film entitled Postman Stories.

[19] Cf. W. A. Graham, "Individualized teaching of fifth and sixth-grade arithmetic," The Arithmetic Teacher, Vol. 11, No. 4 (1964), April, pp. 233-234. Cf. also, Hughes Mearns, Creative Power, Dover Publications, 1958; A. S. Neill, Summerhill, Hart Publishing Co., 1960; Ruth Brecher and Edward Brecher, "Gifted Children Need Freedom to Learn," Parents' Magazine (1962), June, p. 44 ff.

the "boundary conditions," and to define for himself what shall constitute an acceptable answer, the more he is inclined to take the whole matter seriously.

I do not claim that we understand "freedom," but we can recognize situations where the child enjoys but a very low degree of autonomous control, and in such situations we do not believe that very much good learning takes place. Even more apparent, in such situations the level of continuing motivation is not usually very high for most students.

8. Assimilation and Accommodation. As a gesture toward what the engineers would call "good impedance matching," it might be well to describe our understanding of part of Professor Piaget's remarks: specifically, those dealing with the gradual modification of existing cognitive structure.

In the first place, an orientation based upon the notion of the gradual modification of the individual's internal cognitive structure appears to us as highly appropriate for studying the learning of mathematics. This is the kind of task with which the math teacher and the math learner is confronted. Indeed, at one time I thought that the two basic tasks of the mathematics teacher were to show important differences between matters which the student erroneously considered to be the same, and to point out similarities between situations that the student had failed to relate to one another: that is to say, to sever notions incorrectly related, and to relate notions incorrectly separated.[20]

[20] Cf. Jerome S. Bruner, On Knowing. Essays for the Left Hand, Harvard University Press, 1963, pp. 12-15; pp. 18-20. Professor Bruner's remarks also call to mind Leonard Bernstein's delightful essay on the greatness of Beethoven ["Why Beethoven?," an essay included in the volume Seven Arts, No. 2, edited by Puma. Permabooks (Doubleday), Garden City, New York, 1954, pp. 33-40].

If I now believe that the teacher's task is more complex than this, I do not for a moment doubt that both teacher and learner begin with the learner's "initial" cognitive structure, and seek to re-shape this into a "more sophisticated" cognitive structure. This, again, is a powerful point of attack for studying what is involved in teaching and learning mathematics.

We, as teachers, often think of assimilation and accommodation in terms of the task of learning to find your way around a strange city. At first there is so little cognitive structure that you cannot make sense out of directions, observations, etc. Presently one builds up such basic concepts as a knowledge of the principle streets and main landmarks. One can either extend the picture by introducing additional detail, or -- when "paradoxes" are encountered -- modify the picture by removing major errors that, previously unnoticed, have suddenly become important.

This view of the learning of mathematics has many direct implications for the classroom: for one thing, it encourages the use of "readiness building" and preliminary "unstructured exploration," in order to allow the child to build some basic relevant cognitive structure which more systematic instruction can then seek to modify. Again, this picture is consonant with the fact that everything that any of us knows is wrong. You cannot -- from memory -- even sketch a floorplan of a home you have lived in for years, without committing many "errors" and leaving plenty of room for further improvement. Hence it is foolish to say that "we shall see that the child learns everything correctly from the very beginning." It is equally foolish to say that "we shall never allow a child to leave a class with a misconception in his mind." Every idea of every one of us on every subject is wrong -- partly wrong, that is. We learn by successive approximations, and there is no final and absolutely perfect "ultimate version" in any of our minds. We are wrong, but we can learn; having learned, we shall still be wrong, but less so; and, after that, we can still develop a yet more accurate cognitive representation, within our minds, for the various structures that exist independently of our minds.

This may seem obvious. Nonetheless, we encounter time and again the teacher (not a Madison Project teacher!) who says that her students must "get things exactly right from the very beginning, so as not to learn any wrong ideas," and the question (asked, for example, of Richard Suchman, after a demonstration class in Los Angeles recently) "Would you allow that student to leave school that day with a wrong idea?"

The anguish of living with a complex reality may leave us sorely tempted to seek absolute and over-simplified revisions, which are not to be subjected to further modification -- but this is not the way to learn a creative approach to modern science and mathematics, nor is it the way to run a democracy. Growth by successive approximations is the most we can hope to achieve.[21]

There is an important point, however.

When we say we cannot protect the child from "wrong" ideas, have we removed all possible values from a theory of instruction?

The answer, of course, is that we have not. We can think of a sequence of cognitive structures,

$$\ldots \quad C_m, \quad C_{m+1}, \quad C_{m+2}, \quad \ldots$$

where each is "more suitable" or "more sophisticated" than any that came before. Each

[21] "Oh, Lord," Hermann Melville once wrote, "shall we never be done with growing?"

picture is an imperfect model of "reality," but it has other attributes upon which we can pass judgment. If we cannot protect the child from "error," we can (and must) pass judgments on such matters as:

Is this particular cognitive structure, C_p, suitable for the assimilation of the ideas with which we are presently working?

Is this particular cognitive structure, C_p, a suitable one from which we can ultimately get to a more sophisticated structure, C_{p+1}?

Are the emotional, social, and cognitive aspects of our classroom such that the child will easily move from one cognitive structure, C_p, to various more suitable ones, C_{p+1}, C_{p+2}, ... ?

At what point is it desirable for the child to become aware of some of the limitations of a given structure, C_p? When should he develop the new structure C_{p+1}? What should we, as the teacher, do in order to play midwife to the birth of structure C_{p+1}?

The ex-student who graduates from our program of educational experiences should, among other desirable attributes, be adept in discarding one cognitive structure and replacing it with a more adequate new one, and he should have the wisdom to know when this is advisable.[22]

[22] In the words of John W. Gardner, "The ultimate goal of the educational system is to shift to the individual the burden of pursuing his own education." (Science, February 14, 1964).

Incidentally, this notion of a sequence of cognitive structures helps illuminate an important remark of David Page, that every child always gives the right answer to every question, as seen from his point of view. One can re-word this to say that the child must map the question into his cognitive structure as best he can, he must seek within his cognitive structure for answers, and -- within these constraints -- his answer is probably the most appropriate that can be given.

9. The Danger of "Short-Cutting." There is a peculiar phenomenon which we do not understand. Put briefly, it is this: if one states a specific set of really explicit objectives for an educational experience, this list seems always to be significantly incomplete: it is always possible to meet all of the stated requirements, without actually achieving what was really desired.[23]

Why this should be so, or to what extent it *is* so, we do not feel we understand. However, it causes us to be wary of any approach to education which presumes an explicit *a priori* listing of objectives. At present we feel that any approach which depends upon a specific listing of objectives -- however rational this may *seem* -- is in fact an open invitation to somehow losing sight of the subtle, but unstated, values which are the real point of it all.[24]

[23] The situation suggests Mark Twain's comment on his wife's use of profanity: "She knows all the words, but she can't quite get the tune."

[24] Cf. Pierre Boulle, The Test, Popular Library, 1960.

Here are some examples: a large vocabulary may mark the well-educated man; but we can train people to use a large vocabulary while somehow omitting many essentials of a good education.

Again, the enthusiasm of a man with an idea may often be a measure of the worth of the idea, or of the nature of the man. Yet, a certain popular program for "making a good appearance" advises students to show enthusiasm even when they don't feel it, and even when they don't really have very much of an idea in mind at all.

We regard this kind of "short-cutting" -- which achieves the appearance without regard for the reality -- as pernicious in the extreme. But one invites precisely this whenever one lists, in advance, those appearances which will constitute "success." Indeed, as we shall see in the next section, traditional ninth-grade algebra put great emphasis on getting students to write formulas that appear correct, even though the students did not know what the formulas meant.[25]

10. Mathematics Teaching, U.S.A., 1964. I have made some remarks on the kind of school experiences that the Madison Project seeks to achieve, and on some that it seeks to avoid. Let me now describe what we have seen in our role as observers, sitting in the back of non-Madison Project classes. This is what happens in "traditional" teaching in the U.S.A. in 1964:[26]

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[25] A major automobile manufacturer is said to have a department concerned only with the sound when a door is slammed, presumably on the theory that the sound of the door slamming is used by many people to judge the "solidness" of the car.

[26] Cf also Jules Henry, "American Schoolrooms: Learning the Nightmare," Columbia University Forum, Vol. 6, No.2, Spring 1963, pp. 24-30.

In the first place, the student encounters a slow-moving sequence of pedestrian tasks that, motivationally, require him merely to do what he is told, when he is told to, in the way that he is told to do it. From an intellectual or cognitive point of view he may encounter difficulties, but they are not the difficulties endemic to the subject matter; they are the difficulties engendered by obscure communications between teacher and student, and by uncertainties or vagueness within the teacher's own mind. I could give many anecdotes to show that the pace is really incredibly slow; here is one: a 9th grade class of rather bright children spent two consecutive periods -- a total of nearly two hours -- responding to questions such as

$$x^2 \cdot x^3 = \quad ?$$

$$p^{10} \cdot p^7 = \quad ?$$

by answering, respectively, "x^5" and "p^{17}."

An hour and a half later we find these children still working at this same task:

Teacher: $x^5 \cdot x^3 = \quad ?$

Class: x^8.

This is apparently the neo-Pavlovian method for teaching algebraic notation to dogs. It should be noticed that no algebraic concepts are involved, merely typesetters' symbols that seemingly denote nothing. Moreover, if these students show up in college, they will be so conditioned that the stimulus

$$x^2 + x^3$$

will probably evoke the response

$$x^5.$$

Whose fault is this? Is this really the human use of human children? Is this really what psychologists teach teachers to believe? The teacher in question -- though surely misguided -- was conscientious, and legally certified.

Second, as the preceding example shows, teachers never think to use the internal structure of the mathematics itself as a source of motivation, a safeguard against confusion, a protection against forgetting, an alternative to unmindful drill. As Warwick Sawyer has put it, "the poor teacher asks the child: 'don't you remember the rule?,' whereas the good teacher asks 'don't you see the pattern?.'"

Having opted against organizing learning in terms of the internal structure of the subject to be learned, what can the teacher put in its place? Typically, she finds four things: imitation (in the "rule-example-drill" trilogy), the use of external reinforcement schedules ("There's no reason! It's company policy!"), neo-Pavlovian drill (as in the case above), and cultural determinism. This last point is of some interest. We can take any present-day citizen of the United States, and get him to answer certain questions "correctly," even though he has no idea what he's talking about, simply by using the fact that he is a present-day citizen of the United States. This is commonly done, and is evil.

Here is an example. Within many number systems, every element has an additive inverse. The additive inverse is itself an element of the system, and so it has an additive inverse. Can you discuss this last-mentioned element?

Perhaps not -- indeed, on the bare facts given, several possible answers might be correct, and none is obvious.

The problem can be re-stated. In place of the mathematically accurate phrase "additive inverse," substitute the suggestive word "opposite." It is now obvious to the student that the opposite of the opposite is the element you started with! Obvious, but unfortunately not universally true! This kind of culturally-determined answer creates a spurious structure which is not the genuine structure of the mathematics itself, but an ersatz structure "borrowed" from the culture. The student who learns this comes to see things with an out-of-focus fuzziness that makes it very hard for him to see the real structure of the mathematics, which is what he was supposed to have been learning.

As a second example, teachers sometimes say "negative one times negative one makes positive one, because two negatives make a positive." Some weaknesses of this argument are at once apparent: Why negative one, why not negative two? Does the analogy with English imply that, when Professor Piaget is in the United States, he should write

$$^{-}1 \times {}^{-}1 = {}^{+}1,$$

but when he returns to his French-speaking home, where two negatives work quite differently, he should write

$$^{-}1 \times {}^{-}1 = {}^{-}1 \text{ ?}$$

The worst part of this kind of argument, though, is that, by substituting a spurious "culturally-determined" structure, it badly obscures the student's view of the genuine structure of mathematics.

A consequence of "teaching" mathematics so that it is the moral equivalent of memorizing the telephone directory is that students know not what their words could mean and should mean, and they know not that they know not. We have called this the "superficial verbal problem." (Professor Raphael Salem, of M.I.T., once remarked that any M.I.T. freshman could recite "the derivative of log x is $\frac{1}{x}$," but the student didn't know what a "log" is, and didn't know what a "derivative" is.)[27]

[27] One can distinguish "culturally-determined" responses from "mathematically determined" responses by seeking examples where the two responses would be different. For example, cf. the question:

A set which is not open is ________.

The "culturally-determined" response is surely "closed." This, however, is mathematically incorrect, since mathematical sets may be closed, open, neither, or both!

11. The Teacher Needs To Listen to the Child. It may seem superfluous to mention that the teacher needs to listen to the child as carefully as she can. Nonetheless -- and this may be symptomatic of the teacher's view of the process of learning -- many teachers do not, and many fine and creative responses of students are rejected as "wrong" because they do not conform to the *a priori* expectations of the teacher.

Indeed, listening to the child's suggestions appears to us to be one of the cornerstones upon which "new mathematics" is built. Don't require the child to read your mind; don't require him to "do it your way" -- show a simple respect for the child's intellectual, analytical, and problem-solving autonomy and you will discover that he is a much cleverer child than you had ever imagined!

How can the child improve his own personal internal cognitive structure when it is only the teacher's cognitive structure that is ever discussed? [28]

12. The "Description" Problem. In working with children, using our "informal exploratory experiences," we sometimes get excellent results. We have had fifth graders conjecture special cases of the binomial theorem, and prove them from a set of axioms selected by the children themselves. We have had sixth graders use an isomorphism between rational

[28] Cf. Robert B. Davis, "Math Takes a New Path," *The PTA Magazine*, Vol. LVII, No. 6, February 1963, pp. 8-11, and the film entitled *Education Report: The New Math*.

numbers and a subset of the set of 2-by-2 matrices in order to solve the equation

$$x^2 = {}^-4$$

and to introduce complex numbers. We have had 8th graders work out a theory of infinite sequences, and use this as a foundation for the introduction of irrational numbers. In some classes interest of virtually all students runs high, and continues to run high for five consecutive years. In some situations, students voluntarily attend classes before school in the morning, or on Saturdays.

But in the case of other classes, this fails to happen. Student interest does not become great, and what there is does not maintain itself at a very high level. Student participation may be poor, and "brilliant student achievements" may not occur.

What makes the difference?

The response does not seem to be adequately explained in terms of I.Q. differences. We have, at various times, wondered about many possible explanations, including these:

i) Recalling our extensive (perhaps excessive) use of "seminar-type" informal discussion, do we get better results with homogeneous classes, where a wise guidance counselor hovers in the wing to maintain a "compatible" group of children in the seminar?

ii) Is it a question of which children in the class set the dominant tone for the class?

iii) Is it a question of the kind of status fights that are going on among the children?

iv) Since much of the Madison Project material is taught by a "visiting specialist teacher," is it a question of the effectiveness of the classroom teacher in welding the children into a cohesive, co-operative team that can work together, much like a good basketball team, where "setting up" intellectual shots contributes to "making baskets"?

v) Since Madison Project teaching lays considerable stress upon freedom, autonomy, and responsibility, is it a question of how consistent this orientation is with that of the regular classroom teacher, of the school, and of the community?

vi) Recalling again that the "informal exploratory experiences" are usually supervised by a "visiting specialist teacher," is the success or failure determined by the degree of "moral support" provided by the regular classroom teacher (or its opposite -- by the deliberate attrition effected by the regular classroom teacher)?

vii) Does success or failure depend upon parental attitudes? (This, and item vi, include also the status or prestige assigned to the visiting specialist teacher, which is sometimes very high and sometimes is not.)

In answering questions of this sort, we encounter at least three obstacles:

i) We are not interested in individual variables operating *in vacuo*; the question, for example, of "teacher attitude" must be operative in an actual Madison Project situation, where the other parameters fall within a "typical" range of values;

ii) We cannot alter school situations at our will; in the words of Bruner, it is not true that "the universe is spread before one, and one has freedom of choice as to what one will take as an instance for testing." Instead, the plight of the experimenter is "that he must make sense of what happens to come along, to find the significant grouping in the flow of events to which he is exposed, and over which he has only partial control."

Methodology for such situations does, of course, exist, but it requires the creation of a large situation for study, that in turn requires very extensive teacher training, the development of a large repertoire of "informal exploratory experiences," the recruiting of a large number of co-operating schools, and so on.

iii) Especially fascinating is the difficulty that few behavioral scientists are willing to look at <u>what goes on in the school and in the classroom</u>. Each has his own personal professional specialty, and, usually disdaining an over-all view of what is going on, he prefers to get to work at once in terms of his own specialty: motivation, or anxiety levels, or peer-group status structure, or clarity of cognitive aspects of discussion, or problems in the measurement of divergent thinking, or whatever. Just as it has been difficult to find mathematicians with a deep interest in education, it has also been difficult to find behavioral scientists who try to locate the forest first, before they try to focus on individual trees. This is admittedly a common problem in academic life today, and has parallels everywhere -- for example, in medicine, where most physicians will focus on certain aspects, without seeing the patient as a whole.

What we seek, then, is a general description of what goes on in the classroom and in the school, from which we can begin to identify those variables which appear to be most decisive in determining success or failure, in the long run, for our program of "informal exploratory experiences."

13. The Measurement of Dependent Variables. What happens to children as a result of their participation in an extended program of "informal exploratory experiences" in mathematics? In trying to measure the outcome in terms of explicit dependent variables, we have encountered many obstacles, including these:

i) In current parlance, we are concerned with "divergent thinking." Actually, I believe the "divergent-convergent" dichotomy is a mis-statement of the real problem, but we are, in any event, concerned with the way the child explores on his own, his original "creative" ideas, and other responses which he will not necessarily produce in response to external cues.

ii) The more noteworthy student achievements are unlikely events anyhow, and are easily masked by statistical "noise" in testing procedures.

iii) The child's attitude is surely a most important aspect of the outcome.

iv) Especially difficult is the fact that in education, as in quantum physics and brain surgery, the act of measuring requires a significant alteration of the situation. In an informal situation where teacher and students are colleagues and co-workers, with great mutual respect and autonomy, can the teacher afford to seem to pass judgment on the student? Can the teacher poke and probe without finding that the student is cognitively ticklish?

14. What Is "Discovery"? It should be clear that one of the key questions before us today is the question of what we mean by "learning by discovery," and what good is it?

I feel sure that disagreement over the nature and value of discovery is rooted mainly in disagreement over values. If one thinks of arithmetic as a routine skill which the student should master -- if this view is uppermost in your mind -- then you will probably find no advantage in teaching by "discovery." [29]

The pre-school child (as witness my own two-year old daughter) lives a waking day that is an unending orgy of exploration and discovery. When the child enters school we try to discipline this exploratory propensity. There is abundant evidence that the desire to explore -- at least, as Alan Waterman says, in academic situations -- withers and just about dies. The decline of inquisitiveness, in academic situations, is perhaps not too pronounced by grade 5; by grade 6 it will usually make an unmistakable appearance, and by grades 7, 8, or 9 you can no longer feel any pulse.

[29] Cf. D.P. Ausubel, "Some psychological and educational limitations of learning by discovery," The Arithmetic Teacher, Vol. 11, No. 5 (1964), May, pp. 290-302; cf. also the very fine discussion by Bert Y. Kersh, "Learning by discovery: what is learned?" The Arithmetic Teacher, Vol. 11, No. 4 (1964), April, pp. 226-232.

Here is an actual example: ask 5th graders what they can write in the " $\square$ " to produce a true statement from the open sentence

$$\square \times \square = 2.$$

They try 1; it is too small: $1 \times 1 = 1 < 2$.

They try 2; it is too large: $2 \times 2 = 4 > 2$.

They try 1.5; it is too large: $1.5 \times 1.5 = 2.25 > 2$.

They try 1.4; it is too small: $1.4 \times 1.4 = 1.96 < 2$.

Now they're off to the races, and they produce a great spate of suggestions, approximations, and relevant questions.

Propose the same question to older children; the older the child, the less creative and enthusiastic the response, until finally, by the college freshman year, students very frequently respond by saying: "I don't think we had that in our high school." That is to say: <u>if nobody ever told me, then I cannot possibly know!</u>

Is this a triumph or a failure of education?

To clarify, if possible, our interest in "discovery," let me list some of the objectives of Madison Project teaching: [30]

I. "Cognitive" or "mathematical" objectives:

i) the ability to discover pattern in abstract situations;

[30] Cf. R. B. Davis <u>Experimental Course Report/Grade Nine</u>. Experimental Course Report No. 1, June, 1964. Available from The Madison Project. See especially Appendix E.

ii) the ability (or propensity) to use independent creative explorations to extend "open-ended" mathematical situations;

iii) the possession of a suitable set of mental symbols that serve to picture mathematical situations in a pseudo-geometrical pseudo-isomorphic fashion, somewhat as described by the psychologist Tolman [31] and the mathematician George Polya;

iv) a good understanding of basic mathematical concepts (such as variable, function, isomorphism, linearity, etc.) and of their inter-relations;

v) reasonable mastery of important techniques;

vi) knowledge of mathematical facts.

II. More general objectives:

i) a <u>belief</u> that <u>mathematics</u> <u>is</u> <u>discoverable</u>;

ii) a realistic assessment of one's own ability to discover mathematics;

iii) an "emotional" recognition (or "acceptance") of the open-endedness of mathematics;

iv) honest personal self-critical ability;

v) a personal commitment to the value of abstract rational analysis;

vi) recognition of the valuable role of "educated intuition";

vii) a feeling that mathematics is "fun" or "exciting" or "challenging" or "rewarding" or "worthwhile."

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[31] Cf. E.C. Tolman, Behavior and Psychological Man, University of California Press, 1958, Chapter Nineteen. See also W.J. Sanders, "The use of models in mathematics instruction," The Arithematic Teacher, Vol. 11, No. 3 (1964), March, pp. 157-165.

Actually, there is another important objective. We want the child to know who he is, in relation to the human cultural past. By developing mathematics through discovery and through student initiative, we have brought history right into the classroom! The students have struggled with

$$x^2 = 2,$$

have been stymied, have tried various tangents and flank-attacks, and have finally witnessed a major historical breakthrough when some student proposed the adoption of an axiom that every bounded monotonic sequence converges.

These students really know what a "historical breakthrough" means; they have lived through many, right in their own classroom.

15. Symbol Systems and Language. Professor Bruner has placed considerable stress on the role of language in cognitive functioning. If one means the English language, this is not terribly useful in mathematics. The symbols don't fit the ideas.[32] If, however, one thinks either in terms of Tolman-esque intuitional symbols of a pseudo-geometrical nature, or else of appropriate mathematical notations, then the greater power conferred by these two (quite different) kinds of symbol systems is dramatic. We are overwhelmed at the greater power our students (in grades 5 - 9) have, even from the simple possession of the notation of matrix algebra. Problems in counter-examples, complex numbers, irrational numbers, vector algebra, simultaneous equations, co-ordinate geometry, and trigonometry are tackled by the

[32] Cf. the science fiction novel The Black Cloud, by the eminent astronomer Fred Hoyle.

students, and solved, using matrix notation -- in many cases where we, their teachers, had not thought to use matrices at all!

16. The Inadequacies of Meta-Languages. There is another reason for using "discovery": in point of fact, you usually cannot "tell" the student what to do. You and he do not share a sufficiently precise meta-language.

The distinction between language and meta-language, emphasized by logicians since the turn of the century, is this:[33] we may use a language in communicating with one another in at least three ways:

i) we can use "everyday" language, fraught with ambiguities, and hope for the best (incidentally, we must start here, for there is nowhere else to start!);

ii) using "everyday" language, we can define a more precise "official" language, just as the vague English language can be used to build the far more precise rules of the game of chess;

iii) we can now seek to discuss our "official" language. To do this, we cannot use the "official" language -- it is the subject of our discussion, not the medium whereby we conduct the discussion. This "medium" is the "meta-language."

Now, meta-language is always troublesome, and never as simple and precise as the "official" language. As a result, when older texts tried to tell the students how (for example)

[33] Actually, I am here making a somewhat metaphorical use of the word "meta-language," but I believe that the metaphor is revealing.

to "add numbers of unlike sign," they usually gave the following rule: "Subtract the smaller from the larger, and use the sign of the larger."

This rule is not merely vague, it is worng! Try it on

$$^{+}10 + {}^{-}3 \ .$$

Now, $^{-}3$ is surely smaller than $^{+}10$ (since it lies to the left on the number line), and so we must subtract $^{-}3$ from $^{+}10$,

$$^{+}10 - {}^{-}3 \ .$$

The result is, of course, $^{+}13$. This already has the "sign of the larger," so further adjustment is unnecessary. It would also be futile; the work is hopelessly in error. We made the mistake of doing what the rule told us to do.

How do students get correct answers? They follow their own correct intuitive ideas, while claiming to be following an incorrect rule which would not yield the right answer! This is both confusion and hypocrisy.

A correct rule would involve such complications as: "determine which number has the smaller absolute value; subtract this absolute value from the absolute value of the number having the larger absolute value; use this result for your answer if the larger absolute value was that of the positive number, or else use the additive inverse of this result in the contrary case."

Unless we wish to write mathematics in legal-document style, or worse, we might better leave it to the students to find the procedure for themselves. That is a very major reason for teaching by discovery.

17. Two Applications. As a test of our primitive theory of learning, let us apply to two other areas, and see if it seems useful:

1) The James Pitman-John Downing Initial Teaching Alphabet.[34] While we cannot, by any means, assess all the causes for difficulty in reading,[35] the intuitive "theory of instruction" used by Madison Project teachers would appear to indicate some considerable virtue in the Pitman-Downing Initial Teaching Alphabet. This is an alphabet of 44 symbols, roughly one symbol per sound, that is designed specifically to bridge the gap between the spoken language with which the child comes to school, and the confused written language of adulthood. Many of its features parallel quite closely various features of Madison Project materials, such as:

i) one symbol per idea; in Madison Project usage, for example, the three different meanings for the traditional symbol " – " are expressed by three different symbols:

$$^{-}5, \qquad 5 - 3, \qquad {}^{o}(^{-}7) = {}^{+}7.$$

ii) seeking a notation that is as nearly self-evident as possible (as in the Madison Project use of "☐").

iii) Avoiding "teaching," and leaving it largely up to the child to "crack the code" -- but giving him a code which is within his ability to crack!

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[34] Cf. M. Gunther, "Cracking the grown-ups' code," Saturday Evening Post, (1964), June 20, pp. 34-35.

[35] Cf., for example, R. Ross, "A description of twenty arithmetic underachievers." The Arithmetic Teacher, Vol. 11, No. 4 (1964), April, pp. 235-241.

iv) Gradual growth, as in the "learning by successive approximations" discussed earlier; the earlier and simpler cognitive structure based upon I.T.A. gradually gives way to the later, and more sophisticated, cognitive structure based upon the usual English alphabet.

v) Conformity to the honest and straightforward approach of the child: for example, I.T.A. reads from left to right, as English fails to do. (In Madison Project notation we have learned, from the children, to re-write

$$y = mx + b$$

instead as

$$(\square \times 3) + 5 = \triangle,$$

and so on.)

vi) Because the code is left to the child to decipher, and because this task is within his reach, intrinsic motivation should be much higher.

vii) Lurking between absolutely intrinsic motivation, and external reinforcement, there is the "almost intrinsic" motivation derived from the fact that, since the alphabet is easier, the child more quickly progresses to the point where reading really is a useful tool.

2) Science in the Elementary School. Although many diverse approaches to elementary school science are presently under discussion, the great importance of having a reasonable cognitive structure, C_n, from which to build "better" versions, C_{n+1}, C_{n+2}...,

seems to argue for a very heavy emphasis on the basic concepts of physics, chemistry, biology, and geology. If this view is correct, it is quite possible that many current elementary science efforts are putting too little emphasis upon the child's acquisition of these basic concepts, upon which future, more sophisticated cognitive structures can most easily be built. (This has almost the appearance of the "critical period" hypothesis!)

18. Conclusion. We live in a world where knowledge -- or, in any event, facts -- are accumulating at an alarming rate. Moreover, obsolescence of facts, theories, attitudes, and even values is more rapid than one can comprehend. How shall we cope with this? It appears that an educational approach based upon a Piaget sequence of successive cognitive structures,

$$\ldots \quad C_k, \quad C_{k+1}, \quad C_{k+2}, \quad \ldots,$$

each growing out of the one which preceded it, may be an especially valuable way to regard both "knowledge" and "learning." Those who think that such a view is basically obvious to the point of banality should try to reconcile it with various other theoretical approaches to the study of learning, which in some cases will appear incompatible. There really is some valuable content here; the theory is not vacuous.

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