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This programed booklet is designed for the engineering student who is familiar with the techniques of integral calculus and electrical networks. The booklet teaches how to determine the current and voltages across a resistor, inductor, and capacitor after the switch in a network has been closed. This is a classical problem in engineering, the solution of which is obtained in the classical sense by first determining the current and then obtaining voltages from three well known relations. The classical approach has two significant limitations--(1) it does not emphasize physical principles and (2) it does not readily lend itself to extension to nonlinear cases. For these reasons, this booklet presents a more modern approach where the solution is obtained simultaneously for the current and voltage across the capacitor. (RP)

No. 9

TRANSIENT RESPONSE
of a
SECOND ORDER SYSTEM
(Using state variables)

by

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INTRODUCTION

The circuit shown in Fig. 1 embodies the essential features of a large number of circuits employed in electrical technology, and also represents many other situations which are not electrical, such as a weight suspended on a spring. In this particular example, there is a charge on the capacitor before the switch closes. When the switch

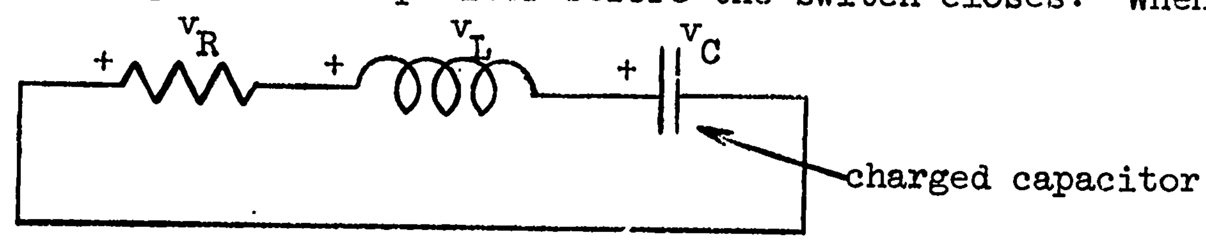


Fig. 1

closes, this charge moves around the circuit, resulting in a current. Many variations of Fig. 1 occur in practice. For example, this is essentially the circuit of a conventional automobile ignition system, except the coil is part of a transformer, there is a battery, and a switch is across the capacitor, as in Fig. 2.

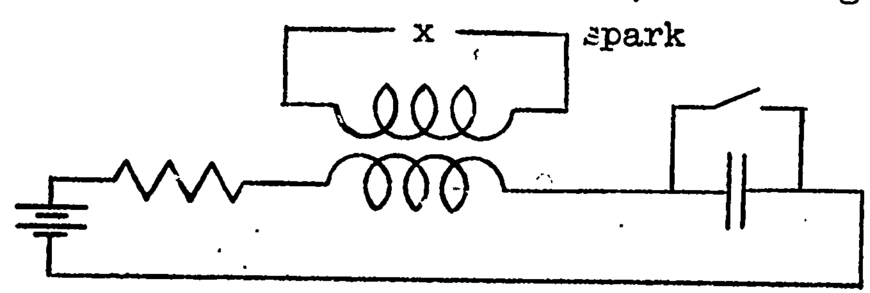


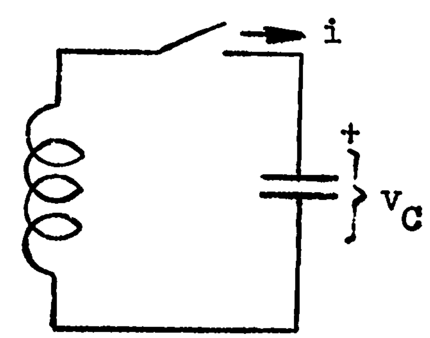
Fig. 2

Once the case shown in Fig. 1 has been analyzed it will be easy to see how other cases, such as Fig. 2, can be treated. Thus, in the beginning we shall concentrate on Fig. 1. Our objective is to determine how current i and voltages v_R, v_L, and v_C vary with time (t) after the switch is closed.

This is one of the classical problems of engineering, the solution of which is obtained in the classical sense by first finding current i, and then obtaining voltages from the three relations

$$v_R = Ri \qquad v_L = L\frac{di}{dt} \qquad v_C = \frac{1}{C}\int_0^t i\,dt$$

This classical approach has many satisfactory features, but it has two significant limitations; first, it does not emphasize physical principles and, second, it does not readily lend itself to extension to the <u>nonlinear</u> cases (where R, L, or C vary with i). For these reasons, we shall use a more modern approach where the solution is obtained simultaneously for two variables; i and v_C. The advantage of doing this arises from the fact that a graphical interpretation becomes possible, an interpretation which makes possible important insights as to why certain things happen, without excessive reliance on the subtleties involved in the interpretation of formulas. You can see it "right there on the graph" as it were.



Initially charged capacitor

Fig. 3

PART I. CIRCUIT WITH ZERO RESISTANCE

In order not to have too many things to consider at once, let us first consider the simplified (but not practically feasible) circuit shown in Fig. 3. We have set $R = 0$.

When the switch is closed, current can begin to flow, from the initially charged capacitor, and voltage v_C and current i will subsequently vary with time. When we want to emphasize that these are functions of time t we shall write $v_C(t)$ and $i(t)$. On other occasions, where no misunderstanding seems possible, we shall merely write v_C or i.

Similarly, when we write

$\frac{di}{dt}$ or $\frac{dv_C}{dt}$, we shall mean ______________ or ______________.

4

$$\frac{dv_C(t)}{dt} \quad \text{or} \quad \frac{di(t)}{dt}$$

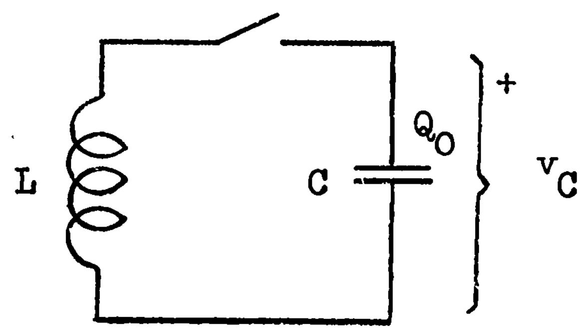


Fig. 3

Q_O = charge on capacitor before switch closes

The problem to be solved in the ensuing pages is to find v_C and i, which are functions of the variable __________. If we arbitrarily take $t = 0$ for the instant when the switch closes, we want to find $v_C(t)$ and $i(t)$ when t is ______________.

From physical principles (which we assume you know), it is known that the voltage across a capacitor, or the current in an inductor, cannot experience a sudden jump.* (A review of these principles will be found on page 7).

From these principles, and the conditions described in Fig. 3, we can say the following about the quantities v_C and i:

$$v_C(0) = \_\_\_\_\_\_\_\_, \quad i(0) = \_\_\_\_\_\_\_\_$$

---

*In mathematical terminology, it is said that v_C and i must be <u>continuous</u> functions of t.

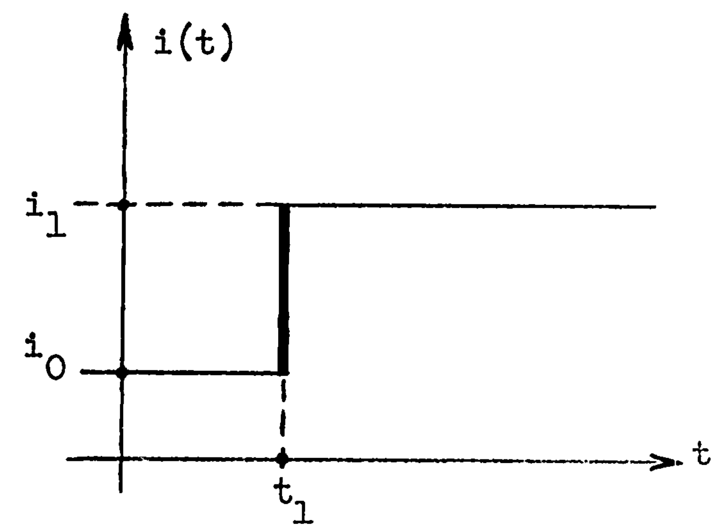
t, or time

t is greater than zero, (or more precisely, when t is greater than or equal to zero).

$v_C(0) = \frac{Q_0}{C}$

$i(0) = 0$

Note that $v_C(0)$ and $i(0)$ are values of $v_C(t)$ and $i(t)$ when $t = 0$. These must be the same as the values just before $t = 0$. The capacitor voltage is Q_0/C, and the current must be zero, because the switch is open prior to $t = 0$.



For a discussion of the reason for continuous functions, GO TO PAGE 7. Otherwise, GO TO PAGE 9.

An Optional Review

The principle that voltage across a capacitor or current in an inductor cannot experience sudden jumps comes from the assumption that voltages and currents remain finite. For example, in an inductor

$$v_L = L \frac{di_L}{dt}$$

Recalling that the derivative is the slope of a curve, and referring to the figure on the left, it is evident that di/dt would be infinite at any point where there was a sudden jump. Thus, if all voltages are finite, including v_L, there can be no sudden jump in i_L.

For a capacitor,

$$i_C = C \frac{dv_C}{dt}$$

and by similar argument, v_C can experience no sudden jump (so dv_C/dt will remain finite) if i_C is finite.

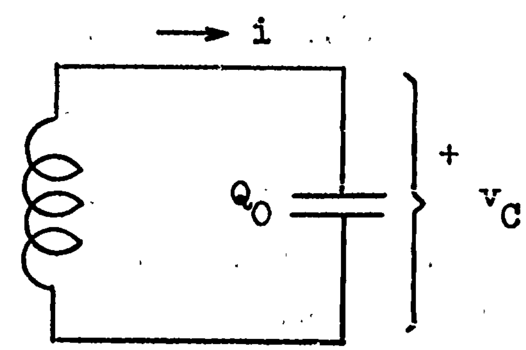


Fig. 3

The value $v_C(0)$ and $i(0)$ are respectively called the <u>initial values</u> of v_C and i, or merely the <u>initial conditions</u> of the circuit.

The ________________ are merely two numbers which are the values of v_C and i at the t = 0 point of the graphs of v_C and i.

We also have two physical laws relating v_C and i, namely:

(for L) $L\frac{di}{dt} = -v_C$; (for C) $C\frac{dv_C}{dt} = i$

Note the minus sign on the left, which arises from the reference direction of i as related to the reference polarity for v_C.

Suppose the current referenc direction had been opposite to that shown. The corresponding equations would be

$L\frac{di}{dt} =$ ________________ ; $C\frac{dv_C}{dt} =$ ________________

10

initial conditions

v_C ; $-i$

(if i reference direction had been reversed.)

<u>Reference</u>

$$L \frac{di}{dt} = -v_C$$

$$C \frac{dv_C}{dt} = i$$

Definition

Let

$$x_1 = \sqrt{\frac{L}{C}}\, i$$

$$x_2 = v_C$$

The equations just given are repeated on the right-hand side of page 10. These are two simultaneous equations in the two variables ________________ and ________________, which at this point in this text must be regarded as unknowns. Very soon we shall solve these equations, but the process will be simpler if we change to the variables x_1 and x_2 defined in the box on page 10. In the space below, change the given equations to equations containing x_1 and x_2 as the cnly variables, and write your answers by completing the equations below.

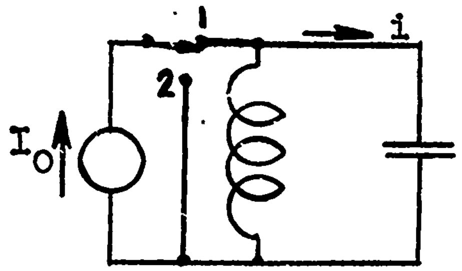
$$\frac{dx_1}{dt} =$$

$$\frac{dx_2}{dt} =$$

v_C and i

$$\frac{dx_1}{dt} = -\frac{1}{\sqrt{LC}} x_2$$

$$\frac{dx_2}{dt} = \frac{1}{\sqrt{LC}} x_1$$



d-c current source

Fig. 4

Comment: It was not absolutely necessary to replace v_C by x_2, and to have done so may appear to be unreasonable, since it amounts merely to a substitution of one letter for another. However, simplification of notation is an aid to thought and, by doing this, we get the two equations on the left which are <u>symmetrical</u>. By virtue of the symmetry one equation can be obtained from the other merely by a <u>change in sign</u> and an <u>interchange</u> of <u>subscripts</u>.

Note: In this text we shall use the symbol $i(0)$ to mean $i(0+)$ as used in the R-L and R-C texts.

Symbols x_1 and x_2 stand for functions of ________ and therefore can also be written ________ and ________. They are called <u>state variables</u> of the circuit, because at any instant of time their values completely determine the state of the system.

From results obtained so far, it is known that the initial values of x_1 and x_2, the ________ ________, are, respectively,

$$x_1(\quad) = \_\_\_\_\_\_\_\_ \text{ and } x_2(\quad) = \_\_\_\_\_\_\_\_$$

Other circuit arrangements yield different initial values. For example, consider Fig. 4 in which a current source is switched out of the circuit by moving the switch from (1) to (2). Using the same notation as before, the respective initial values of x_1 and x_2 are

$$x_1(\quad) = \_\_\_\_\_\_\_\_ \text{ and } x_2(\quad) = \_\_\_\_\_\_\_\_$$

time

$x_1(t)$ and $x_2(t)$

$x_1(0) = 0$

$x_2(0) = \frac{Q_0}{C}$

$x_1(0) = -\sqrt{\frac{L}{C}}\, I_0$

$x_2(0) = 0$

} If you got these, go to page 17.

If you got any part of this last set of answers wrong, go to page 15.

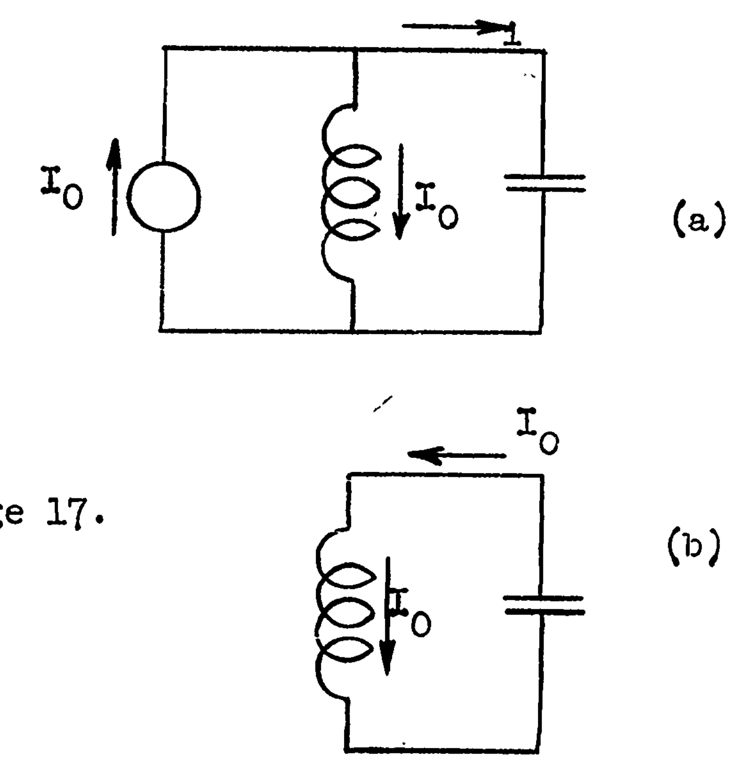


Fig. 4

YOU HAVE BEEN DIRECTED TO THIS FRAME BECAUSE YOU GOT SOMETHING WRONG IN THE LAST ANSWERS.

Your error was probably in x_1. Recall that

$$x_1(t) = \sqrt{\frac{L}{C}}\, i(t)$$

and look at Fig. 4a. When the d-c source is connected, the d-c current cannot flow through C, and hence i will be zero and the inductor current will be I_0, in the direction shown. If the current source is removed, as by the switch in Fig. 4, this inductor current cannot change suddenly, and hence I_0 must flow in the circuit, as in Fig. 4b. Thus, observing that I_0 is opposite to the reference direction of i, it is evident that

$$i(0) = -I_0, \text{ and hence that } x_1(0) = -\sqrt{\frac{L}{C}}\, I_0$$

For $x_2(0) = v_C(0)$ it is recalled that L is an ideal inductance having zero resistance. Therefore, with the <u>constant</u> current I_0 through L, the voltage across L is zero.

<u>Review</u>

For the circuit of Fig. 4 we have defined two variables

$$x_1(t) = \sqrt{\frac{L}{C}}\, i \text{ and } x_2(t) = v_C$$

and shown two examples of how initial conditions $x_1(0)$ and $x_2(0)$ can be determined from physical conditions.

Furthermore, it has been shown that $x_1(t)$ and $x_2(t)$ must be solutions of the simultaneous equations.

$$\frac{dx_1}{dt} = - \frac{1}{\sqrt{LC}} x_2$$

$$\frac{dx_2}{dt} = \frac{1}{\sqrt{LC}} x_1$$

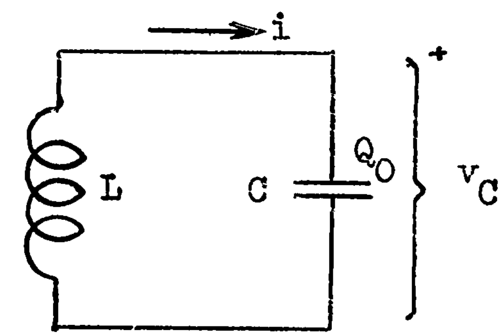


Fig. 4

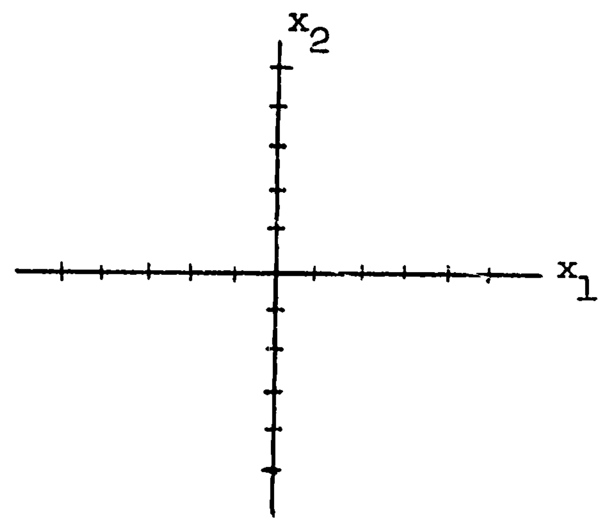


Fig. 5

Referring to the review on the previous page, the factor $1/\sqrt{LC}$ will appear repeatedly, so that it is convenient to have a symbol for it. Accordingly, we define

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and then our equations become

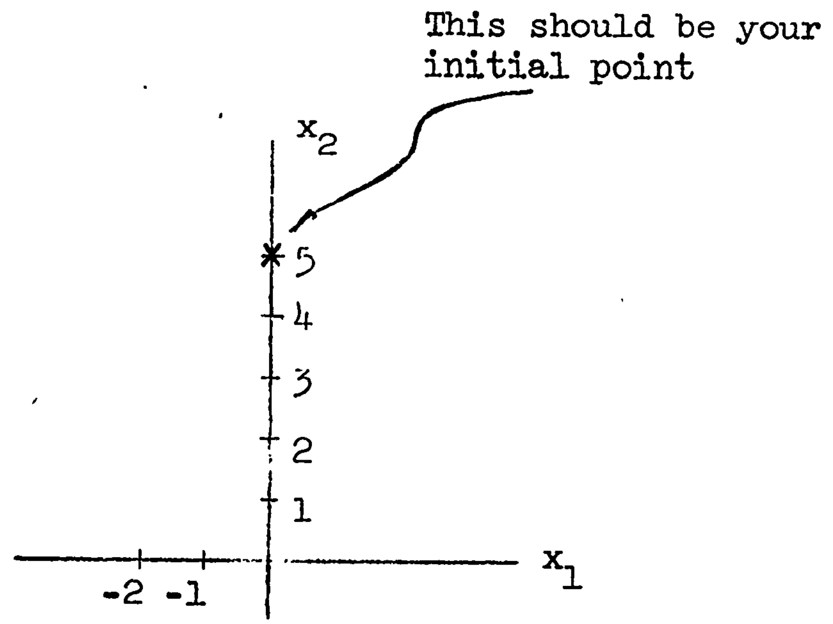
$$\frac{dx_1}{dt} = -\omega_0 x_2 \qquad \frac{dx_2}{dt} = \omega_0 x_1$$

Having two variables suggests plotting them on a set of axes like Fig. 5. Since the axes are state variables, we shall say they define a state plane. As t increases, x_1 and x_2 will change, causing a curve to be traced out in Fig. 5. Let us obtain the starting point of this curve, for the example of Fig. 4. Place a dot on Fig. 5 to represent this initial point, and label it, using the values

$$L = .125h \qquad C = 2 \times 10^{-6} f \qquad Q_0 = 10^{-5} \text{ coulomb}$$

Do it now, and also find ω_0 = ______________.

$\omega_0 = 2000$



Exact Formulas	Approximate Formulas
$\frac{dx_1}{dt} = -\omega_0 x_2$	$\Delta x_1 = -(\omega_0 x_2)\Delta t$
$\frac{dx_2}{dt} = \omega_0 x_1$	$\Delta x_2 = (\omega_0 x_1)\Delta t$

We can solve our equations "analytically", and will do so presently. However, in order not to appear to be indulging only in formula manipulation, let us first look at the approximate formulas shown on the right of page 18. Furthermore, let us apply them to our numerical case. Thus, at $t = 0$, $x_2 =$ __________ and $x_1 =$ ________. Now suppose t increases by a small amount, $\Delta t = .0001$ sec. Our approximate equations give

$$\Delta x_1 = -2000(5)(.0001) = -1 \text{ and } \Delta x_2 = 0.$$

Why is $\Delta x_2 = 0$? __________. Thus, after .0001 sec. the point representing the state of the circuit has moved approximately to values $x_1 =$ ________, $x_2 =$ __________. Plot these, and label the point (1).

Now let t take another jump, to .0002 sec. The value of Δt is ____________ and $\Delta x_1 =$ __________ and $\Delta x_2 =$ ___________. From there, find $x_1 =$ ____________ and $x_2 =$ ___________ for $t = .0002$ sec., and plot the corresponding point, labeling it (2).

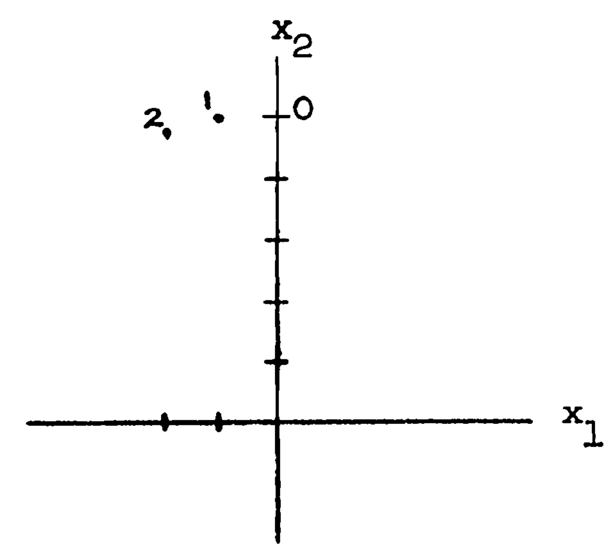
$\Delta x_2 = 0$ because $x_1 = 0$

After Δt, $x_1 = -1$, $x_2 = 5.0$

$\Delta t = .0001$ sec.

$\Delta x_1 = -2000(5)(.0001) = -1$
$\Delta x_2 = 2000(-1)(.0001) = -0.2$

$x_1 = -2$
$x_2 = 4.8$



Let us do one more point by this approximate method. It will help to give a hint as to what kind of curve is being generated. Let's repeat the equation for Δx_1 and Δx_2 here, for this numerical case. The formulas are:

$$\Delta x_1 = \_\_\_\_\_\_\_\_ \quad \text{and} \quad \Delta x_2 = \_\_\_\_\_\_\_\_$$

Furthermore, approximate values at point (2) are

$$x_1(\qquad) = \_\_\_\_\_\_\_\_ \quad \text{and} \quad x_2(\qquad) = \_\_\_\_\_\_\_\_.$$

Insert numbers in the parentheses to represent values of t, and give numbers for x_1 and x_2.

Now let's get point (3), corresponding to t = .0003. We have, in numerical values,

$$\Delta x_1 = \_\_\_\_\_\_\_\_ \quad \text{and} \quad \Delta x_2 = \_\_\_\_\_\_\_\_$$

giving, for point (3), the values

$$x_1(\qquad) = \_\_\_\_\_\_\_\_ \quad \text{and} \quad x_2(\qquad) = \_\_\_\_\_\_\_\_.$$

$\Delta x_1 = -2000\ x_2 \Delta t,\ \Delta x_2 = 2000\ x_1 \Delta t$

At point (2)

$x_1(.0002) = -2$

$x_2(.0002) = 4.8$

$\Delta x_1 = -(2000)(4.8)(.0001) = -0.96$

$\Delta x_2 = 2000(-2)(.0001) = -0.4$

$x_1(.0003) = -2-0.96 = -2.96$

$x_2(.0003) = 4.8-0.4 = 4.4$

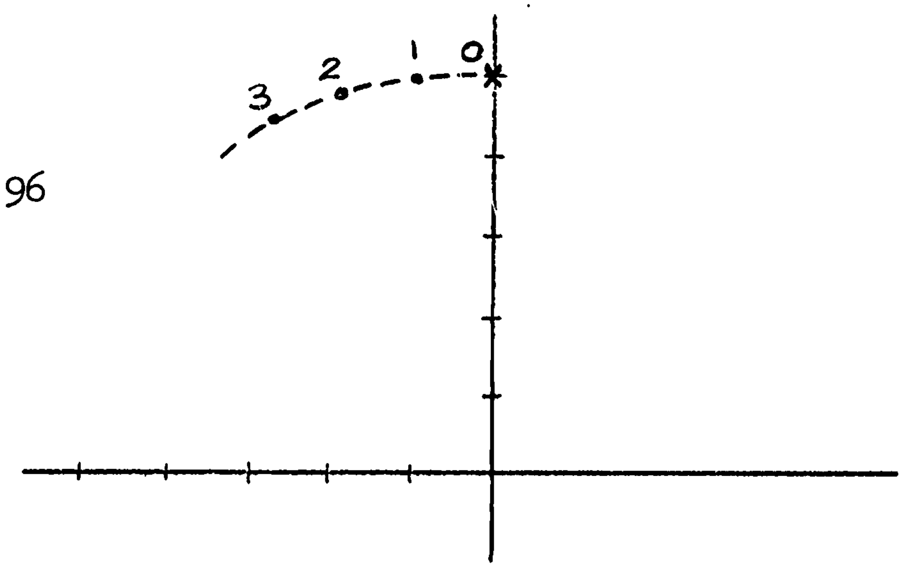


Fig. 6

What we have done should help you visualize that a moving point will start at the point representing the _______ _________ and will trace out a curve as time progresses. This moving point is called the <u>state point</u>, and the curve is called a <u>trajectory</u>. On Fig. 6 label point (3) "sample state point", and label the curve "trajectory".

So far you have done part of a step-by-step calculation of an approximate solution, a process that would easily be done by a computer.

By way of this approximate solution you have learned that the circuit behavior can be described by the motion of a state point which traces out a trajectory.

Would you hazard a guess as to the shape of this trajectory for the L-C circuit being considered?

initial conditions

The shape of this curve will be determined by the analytical solution which follows. We shall let your guess "ride" until you get to the answer.

ANALYTICAL SOLUTION

For certain complicated cases (where L and/or C are not constants) the step-by-step solution is necessary. However, an analytical solution is possible for the cases we consider in this book.

Let us now get the exact solution, determining the nature of the curve which we began to develop in Fig. 6. Beginning with our equations

$$\frac{dx_1}{dt} = -\omega_0 x_2$$

$$\frac{dx_2}{dt} = \omega_0 x_1$$

multiply the first by x_1 and the second by x_2, and then add the equations. Write your result in the space below.

$$x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = 0$$

A note about derivatives:

$$x \frac{dx}{dt} = \frac{1}{2} \frac{dx^2}{dt}$$

Therefore

$$x_1 \frac{dx_1}{dt} = \underline{\hspace{4cm}}$$

$$x_2 \frac{dx_2}{dt} = \underline{\hspace{4cm}}$$

and so

$$x_1 \frac{dx_1}{dt} + x_x \frac{dx_2}{dt} = 0$$

becomes

$$\frac{dx_1^2}{dt} + \frac{dx_2^2}{dt} = 0$$

Observe the note on the previous page, and fill in the blanks below the note on page 26. You arrived at the conclusion that x_1 and x_2 must satisfy the equation

$$\frac{dx_1^2}{dt} + \frac{dx_2^2}{dt} = 0$$

or

$$\frac{d(x_1^2 + x_2^2)}{dt} = 0$$

Thus, $x_1^2 + x_2^2$ is a function whose derivative is zero, and so this function must be a constant which we shall call K^2. Therefore, the solution is

$$x_1^2 + x_2^2 = K^2 \quad .$$

Recall what you know about algebraic equation. You conclude that this curve that you began to <u>trace</u> out (what are traced curves called? __________), is in reality a __________ and K is the ____________.

For the numerical case shown in Fig. 6, K has the value ____________. Thus it is seen that K is determined by the ____________ ____________ of the circuit.

trajectory

circle, and K is its radius

$K = 5$

K is determined by the initial conditions of the circuit

Note: This simple result (a circle) is the main reason for changing from variables i and v_C to x_1 and x_2. If this had not been done, the resulting curve would have been an ellipse.

Refresher

Energy storage:

in an inductor: $\frac{1}{2}Li^2$

in a capacitor: $\frac{1}{2}Cv_C^2$

ENERGY

In the equation $x_1^2 + x_2^2 = K^2$, (arrived at in the last frame), let us go back to the original variables i and v_C. You should remember that they are related to x_1 and x_2 by

$$x_1 = \underline{\qquad\qquad}; \quad x_2 = \underline{\qquad\qquad}$$

Substituting these in the above equation gives

$$Li^2 + Cv_C^2 = CK^2$$

Now note the equations on page 28 for energy storage in an inductor and a capacitor. This makes it appropriate to change the above to

$$\frac{Li^2}{2} + \frac{Cv_C^2}{2} = \frac{CK^2}{2}$$

This shows that the total stored energy in the circuit is ____________________.
(a property, not a value)

Let W be this total energy. We see, then, that the radius of the trajectory is related to the total energy by

$$K = \underline{\qquad\qquad\qquad}$$

30

$x_1 = \sqrt{\frac{L}{C}}\, i$

$x_2 = v_C$

Total energy is constant.

$K = \sqrt{\frac{2W}{C}}$

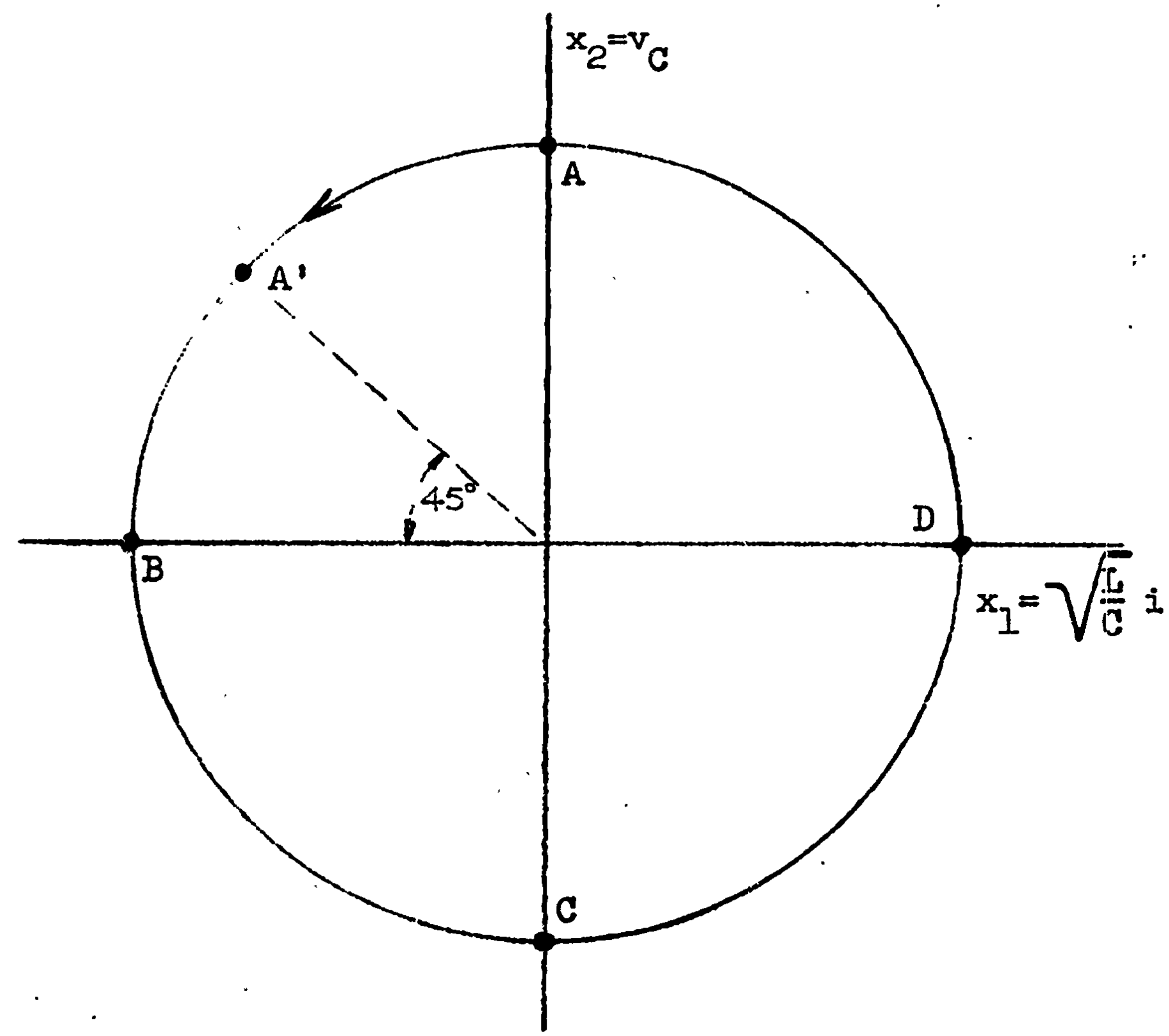


Fig. 7

Let us think about energy a bit more, referring to the circular trajectory shown in Fig. 7. Again using W for the total energy, fill in the following table, showing the energy in each element at the instants of time corresponding to points A,B,C,D.

Points	Energy in L	Energy in C
A		
B		
C		
D		

Furthermore, at point A', in terms of W, the energy in L is ____________ and the energy in C is ________________.

Thus it is seen that the energy W flows back and forth between the capacitor and the inductor, while the total remains __________.

From physical principles, explain why it is reasonable that the total energy should remain constant: ____.

	Energy in L	Energy in C
A	0	W
B	W	0
C	0	W
D	W	0

At A' energy in L = energy in C = $\frac{W}{2}$.

Constant.

W is constart because there is no resistance, and hency no i^2R loss.

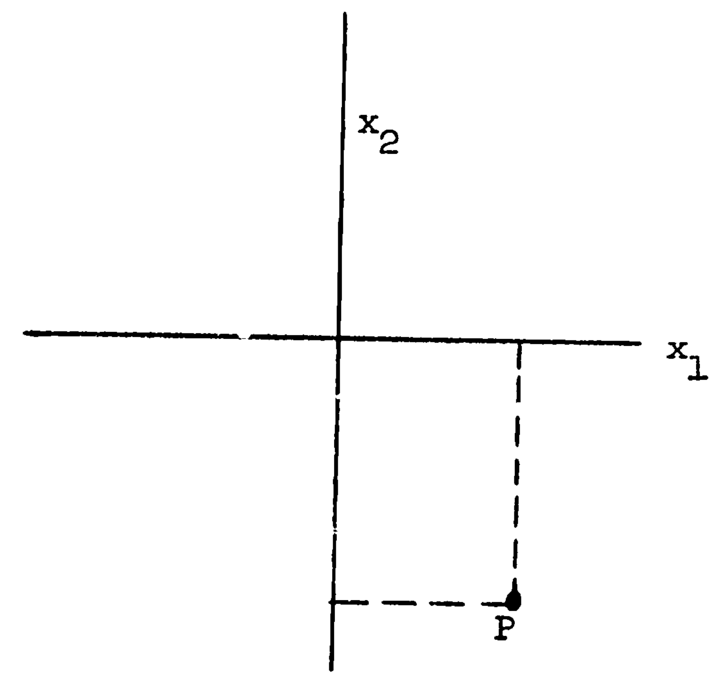


Fig. 8

$$\frac{dx_1}{dt} = -\omega_0 x_2$$

$$\frac{dx_2}{dt} = \omega_0 x_1$$

VELOCITY ALONG THE TRAJECTORY

Our next consideration is to find the velocity of the moving point. What is this point called? ____________.

Point P in Fig. 8 is an example. The point is moving parallel to the x_1 axis with velocity dx_1/dt and parallel to the x_2 axis with velocity dx_2/dt. For the point shown: x_1 is ________ (+ or -) and x_2 is ________ (+ or -). Hence the signs of dx_1/dt and dx_2/dt are respectively ________ and ________.

Draw velocity arrows at point P in Fig. 8 to represent these two components of velocity. Note that these magnitudes are respectively proportional to the magnitude of x_2 and x_1, and draw their lengths in approximately correct proportion.

state point

x_1 is positive

x_2 is negative

$\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ are both positive.

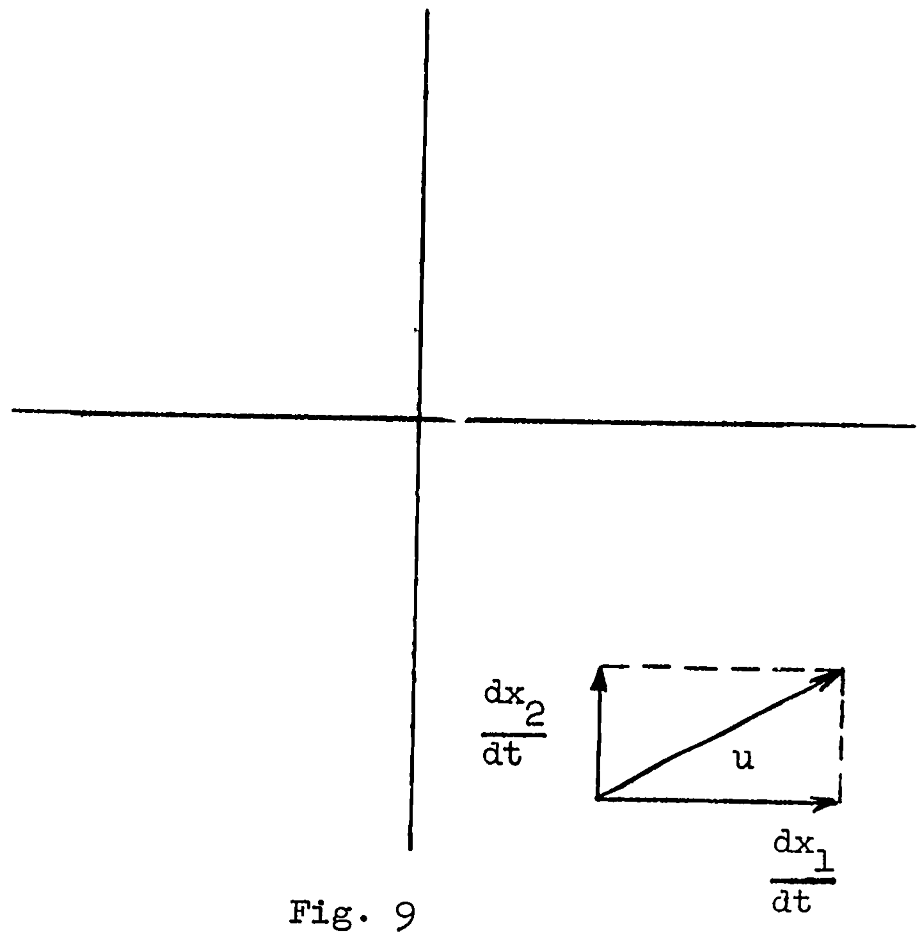


Fig. 9

These velocity components can be added vectorially to give a velocity of magnitude u, as in Fig. 9. This magnitude is

$$u = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2} .$$

But $\frac{dx_1}{dt}$ = __________ and $\frac{dx_2}{dt}$ = __________, and so

$$u = \omega_0 \sqrt{x_1^2 + x_2^2} .$$

Furthermore, $\sqrt{x_1^2 + x_2^2}$ = __________, which is __________ of the circle. Thus,

u = __________

$$\frac{dx_1}{dt} = -\omega_0 x_2$$

$$\frac{dx_2}{dt} = \omega_0 x_1$$

$$\sqrt{x_1^2 + x_2^2} = K, \text{ the radius}$$

$$u = \omega_0 K$$

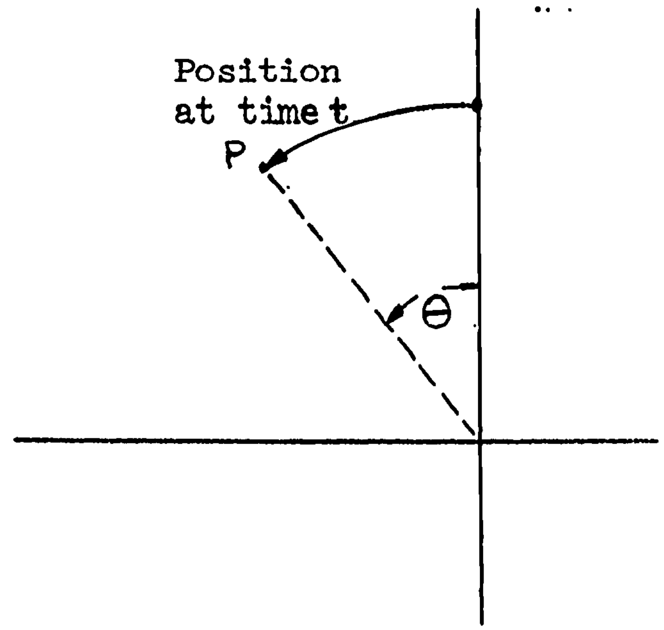


Fig. 10

Parameter ω_0 is more convenient to interpret than u, because it does not depend on the radius. To interpret ω_0, refer to Fig. 10. In a time interval t the state point will move through an arc length equal to the magnitude of the velocity multiplied by time. This arc length can also be written as the product of subtended angle multiplied by radius. Equate the two, using symbols ω_0, K, θ, and t appropriately.

$$\overline{\text{arc length in terms of velocity}} = \overline{\text{arc length in terms of angle}}$$

This gives

$$\omega_0 = \frac{\theta}{t}$$

In other words, ω_0 is the rate of change of angle θ. It is called the <u>angular velocity</u> of the rotating line drawn from the origin to moving point P. ω_0 is usually specified in <u>radians per second</u>.

$\omega_0 Kt = K\theta$

T = <u>period</u>
(time for one revolution)

f = <u>frequency</u>
(number of revolutions per second)

Two other quantities are often associated with circular motion. These are defined on page 38. Let us see how they are related to ω_0, the ____________. The value of θ corresponding to $t = T$ is θ = __________. Thus, putting these in the equation for ω_0 gives

$$\omega_0 = \frac{(\quad)}{(\quad)}$$

Furthermore, if the time required for one revolution is T, the number of revolutions per second will be 1/T, which is also ____________ (use words) or ____________ (use a symbol).

Thus,

$$\frac{1}{T} =$$

and so we also have

$$\omega_0 = (\quad)(\quad)$$

angular velocity ω_0

$\theta = 2\pi$

$\omega_0 = \frac{2\pi}{T}$

frequency or f

$\frac{1}{T} = f$

$\omega_0 = 2\pi f$

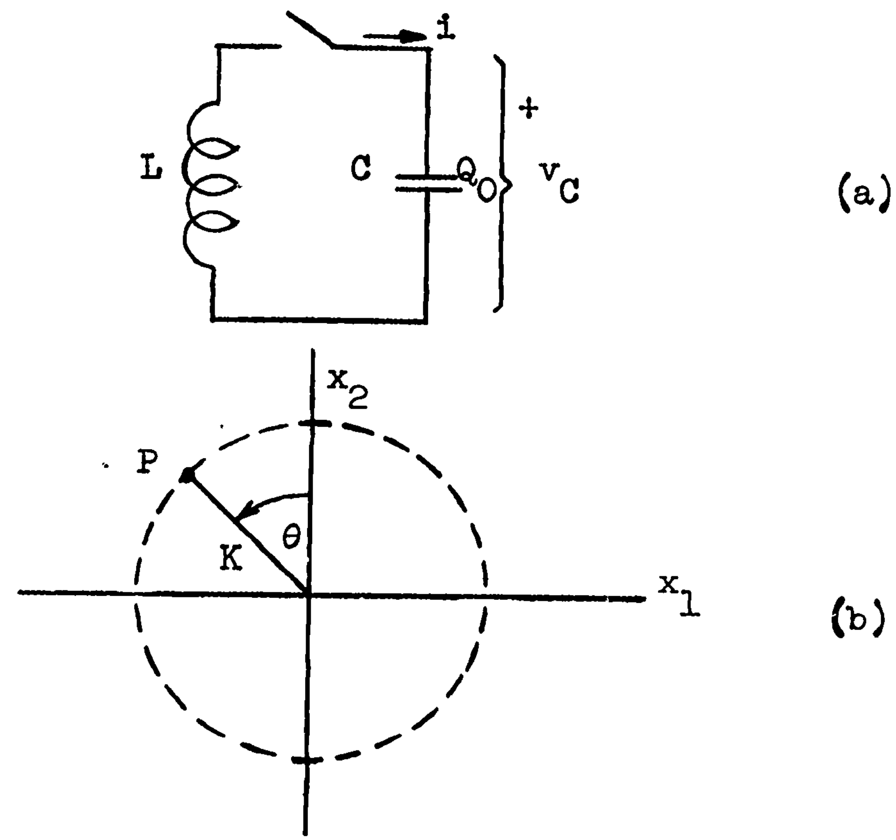


Fig. 11

REVIEW

1. When a charge Q_0 is released in the circuit of Fig. 11a, the energy W in the circuit remains ____________ at the value ____________.
2. The variation of i and v_C can be described by a simple geometrical picture by using the variables

   x_1 = ____________ x_2 = ____________
3. This picture is the circle shown in Fig. 11. The values of v_C and i at any instant of time are represented by the position of point P, called the ____________. Point P moves in such a way that a line drawn to it from the origin rotates at constant ____________ ____________ ω_0.
4. ω_0 is related to the circuit parameters, L and C, by the equation ω_0 = ____________.
5. The radius of the circle K can be expressed in terms of the energy in the system, as K = ____________.
6. In terms of W, L, and C, the maximum values of i and v_C are, respectively,

   max i = max v_C =

1. constant $= \frac{1}{2}\frac{Q_0^2}{C}$ or $\frac{1}{2} CV_C^2$
2. $x_1 = \sqrt{\frac{L}{C}}\, i$ $\quad x_2 = v_C$
3. state point $\quad$ constant angular velocity
4. $\omega_0 = \frac{1}{\sqrt{LC}}$
5. $K = \sqrt{\frac{2W}{C}}$
6. max. $i = \sqrt{\frac{2W}{L}}$; max $v_C = \sqrt{\frac{2W}{C}}$

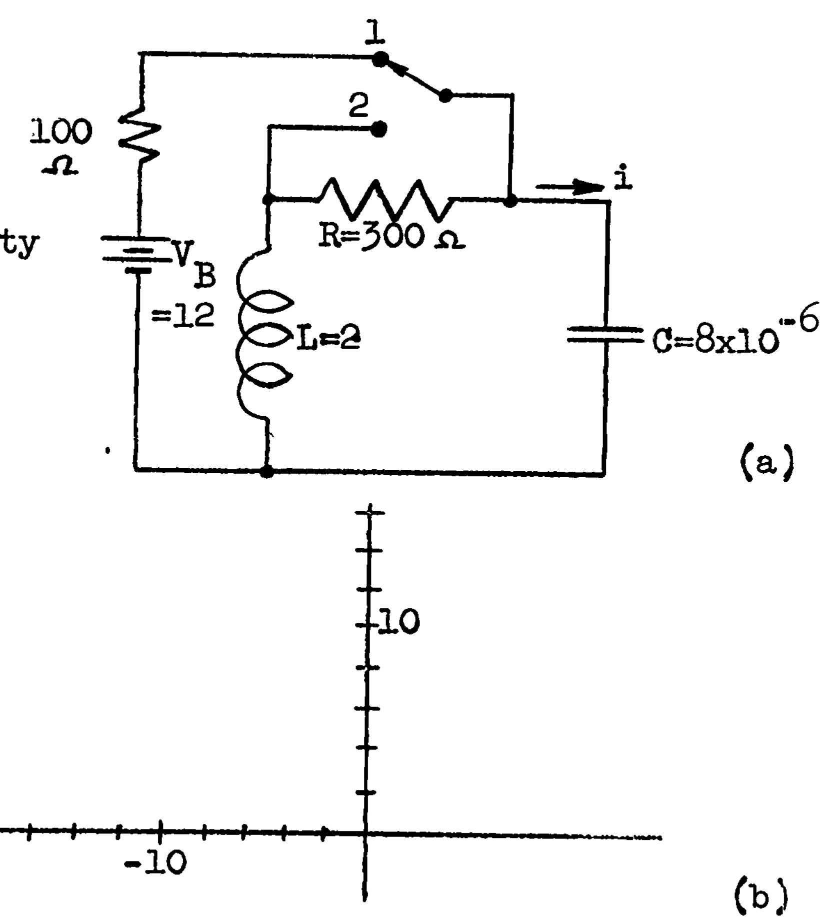


Fig. 12

OTHER INITIAL CONDITIONS

More about the circle! All we need to know to specify a circle is its center and its radius. The center will always be at the origin, and another way to know the radius is to know the starting point. This idea is particularly useful when there is an initial current in the inductor as well as an initial current in the capacitor. Look at Fig. 12a as an example. The switch is assumed to have been in position (1) for a long time. The initial current $i(0)$ and initial $v_C(0)$, to be used for the instant $(t=0)$ when the switch is thrown to position (2), are

Initial i = __________ Initial v_C = __________

Below, calculate the initial coordinates (x_1 and x_2), and plot them on Fig. 12b. Also, determine the radius of the trajectory, and the angular velocity.

-.03 amp. (Note the - sign.)

9 volts

Initial $x_1 = 9$

Initial $x_2 = \sqrt{\frac{2}{8}} \times 10^3(.03) = 15$

Radius = $\sqrt{81 + 225} = \sqrt{306} = 17.5$

$\omega_0 = 250$ c.p.s.

Answer these also:

What is the maximum v_C? ______________

What is the maximum i? ______________

$$K = \sqrt{[x_1(0)]^2 + [x_2(0)]^2}$$

$$W = \frac{L[i(0)]^2}{2} + \frac{C[v_C(0)]^2}{2}$$

You have just completed a numerical example in which there was an initial inductor current and initial capacitor voltage. We now consider the question of whether, in this general case, the radius is still given by

$$K = \sqrt{\frac{2W}{C}}$$

if W is now the total energy at the initial instant (sum of inductance and capacitance energies).

The answer is easily obtained when we recall that for all points on the circle W is ______________. Thus, the answer as to whether the above formulas is always correct is ____________.

Let's also confirm this by formula. For initial current $i(0)$ and capacitor voltage $v_C(0)$, the radius of the circle and energy are given on the opposite page. Show that the above relationship between W and K is satisfied.

46

constant

yes

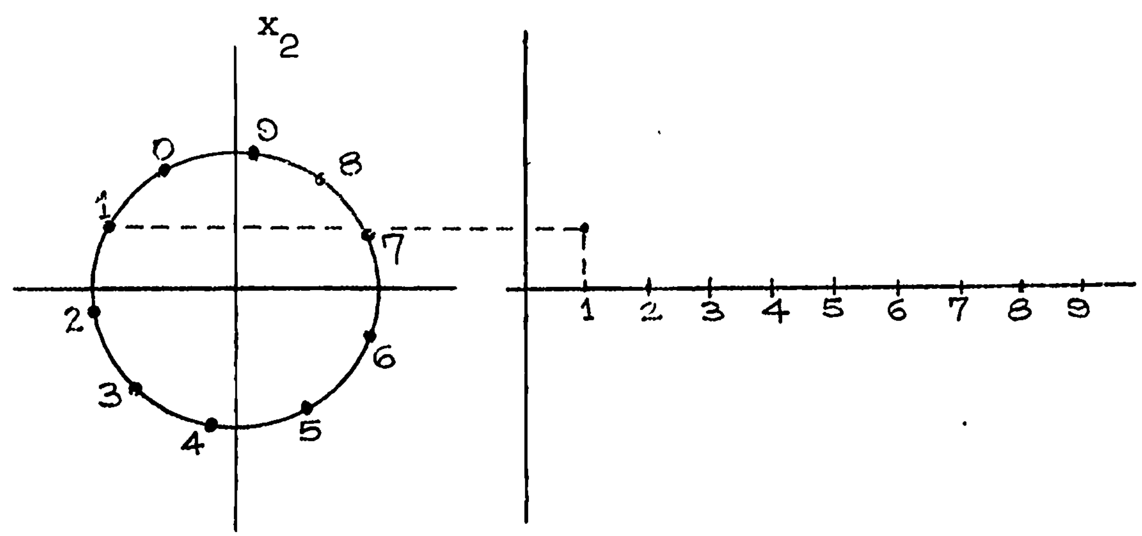


Fig. 13

You have now seen how to determine the trajectory locus for any L-C circuit with any initial inductor current and any initial capacitor charge.

Next let us consider how to get the variables i and v_C as functions of time. Consider v_C first: it is equal to ____________. Thus, referring to Fig. 13 suppose points 0,1,2,3... are positions of the state point at successive uniformly spaced instants of time. These time intervals are marked on the horizontal (t) axis to the right.

Variable $x_2 = v_C$ is the vertical coordinate of each point, and so the corresponding points on the time graph can be obtained by drawing horizontal lines across until they intersect vertical lines at corresponding values of t, like the one shown.

Do this for each dot, and sketch in the wave. Observe that it will repeat itself after one revolution. The wave is periodic, and therefore has a frequency.

The frequency is ________________ (in terms of L and C).

$$v_C = x_2$$

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

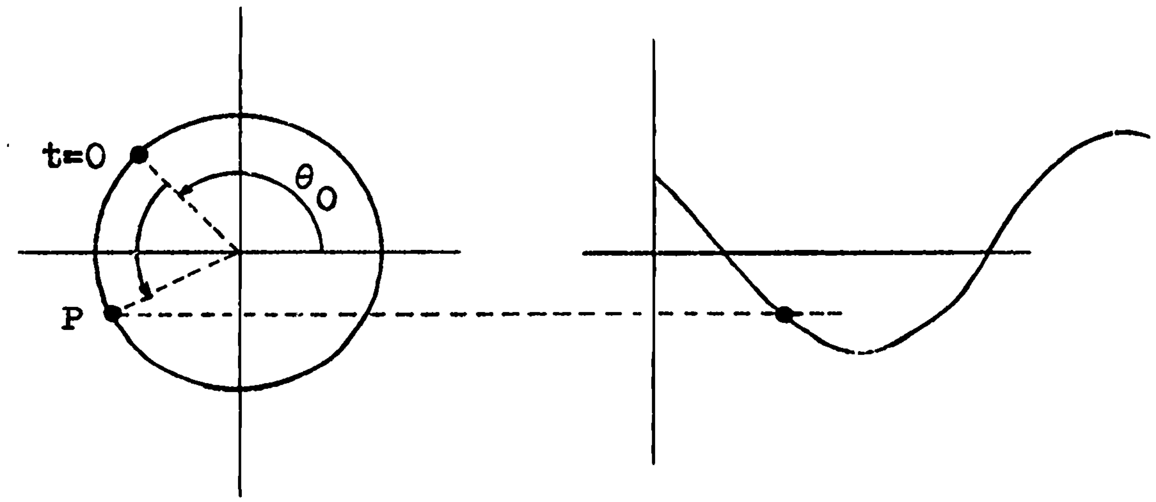


Fig. 14

It is fine to be able to draw a graph like this, and a great deal of insight can be obtained from it. However, the discussion would not be complete without also obtaining the formula for this wave.

Figure 14 is like the previous one, except that a general state point P is shown, rather than a sequence of specific ones. The unlabeled curved arrow indicates an angle to the general point measured from the starting point. At the general point the time is t, and since the angular velocity is ω_0, we see that this angle is ____________. Label it on the figure. Also, let the starting point be located θ_0 radians from the positive horizontal axis.

At general point P, the vertical projection will be the radius times a trigonometric function of the angles shown. Write this function

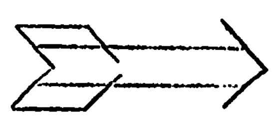
$$x_2 = v_C = (\underset{\text{radius}}{\_\_\_\_\_\_})\ \underset{\text{trig function}}{\_\_\_\_\_\_\_\_\_\_}\ (\underset{\text{angles}}{\_\_\_\_\_\_\_\_})$$

50

$\omega_0 t$

$v_C = K \sin(\omega_0 t + \theta_0)$ or

$v_C = \sqrt{\frac{2W}{C}} \sin(\omega_0 t + \theta_0)$



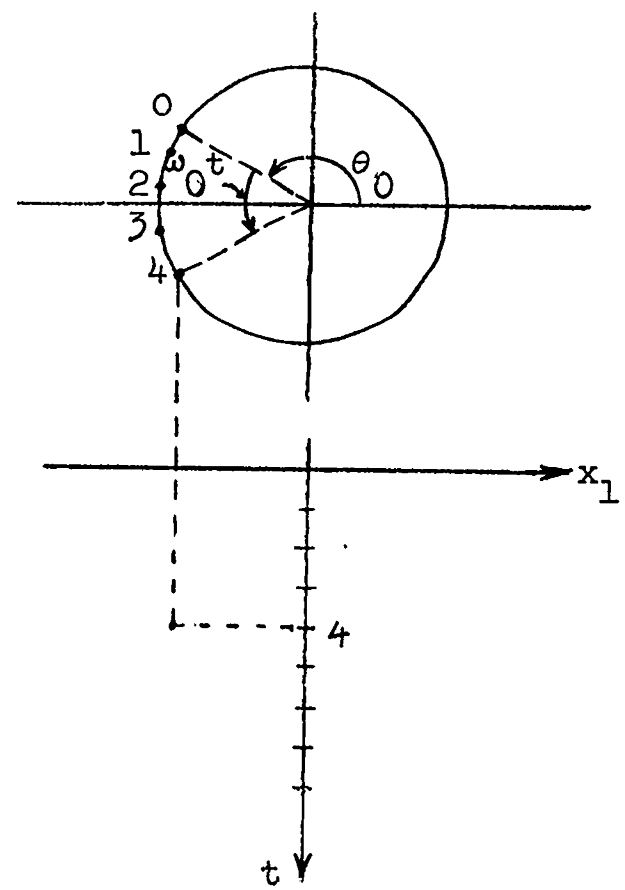


Fig. 15

It is desirable to have a construction similar to Fig. 13 for finding how x_1 (or i) varies with time. One way to do this is shown in Fig. 15. The x_1 axis of the time plot is drawn parallel to the original x_1 axis. But this forces the time axis to be drawn vertically. To see the result as a normal plot, view it from the direction of the arrow.

A sample construction is shown here. Construct the wave in the same way you did in Fig. 13.

Finally, the formula for this wave, constructed from trigonometry, is

$$x_1 =$$

and the current is

$$i =$$

$$x_1 = K \cos(\omega_0 t + \theta_0)$$

$$x_1 = \sqrt{\frac{2W}{C}} \cos(\omega_c t + \theta_0)$$

$$i = \sqrt{\frac{2W}{L}} \cos(\omega_0 t + \theta_0)$$

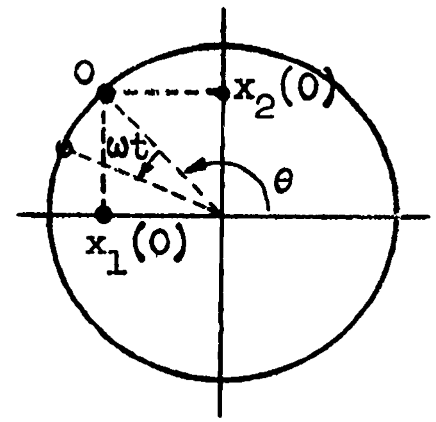


Fig. 16

$$\left.\begin{aligned} x_1 &= K \cos(\omega_0 t + \theta) \quad &(a)\\ x_2 &= K \sin(\omega_0 t + \theta) \quad &(b)\end{aligned}\right\} (1)$$

A summary of results is given in Fig. 16 and the two equations below. Let us now obtain equivalent expressions, in which the initial conditions $x_1(0)$ and $x_2(0)$ appear explicitly. Recall the trigonometric identities

$$\cos(\omega_0 t+\theta) = \cos\theta \cos\omega_0 t - \sin\theta \sin\omega_0 t$$

$$\sin(\omega_0 t+\theta) = \sin\theta \cos\omega_0 t + \cos\theta \sin\omega_0 t$$

Substitute these in the expressions for x_1 and x_2, and complete the following equations

$$x_1 = K(\qquad) \cos\omega_0 t + K(\qquad) \sin\omega_0 t$$

$$x_2 = K(\qquad) \cos\omega_0 t + K(\qquad) \sin\omega_0 t$$

Now observe that the initial values are related to K and θ as

$$x_1(0) = K\cos\theta \quad \text{and} \quad x_2(0) = K\sin\theta$$

Thus, in terms of $x_1(0)$ and $x_2(0)$, the expressions are

$$x_1 =$$

$$x_2 =$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\left.\begin{aligned} x_1 &= x_1(0)\cos\omega_0 t - x_2(0)\sin\omega_0 t \quad &(a) \\ x_2 &= x_2(0)\cos\omega_0 t + x_1(0)\sin\omega_0 t \quad &(b) \end{aligned}\right\} \quad (2)$$

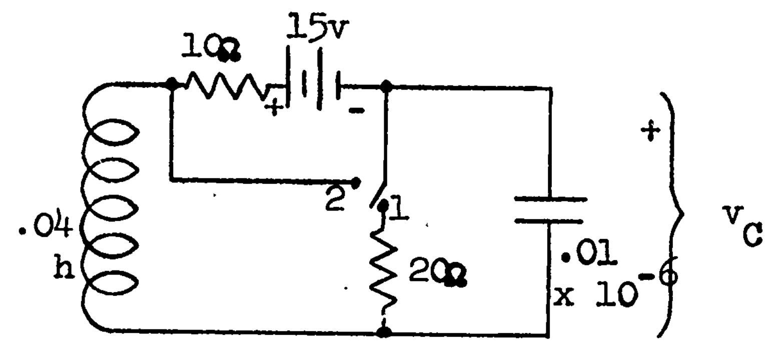


Fig. 17

REVIEW PROBLEM

This is a good point at which to have a checkup. In Fig. 17 assume the switch has been in position 1 for a long time and then, at $t = 0$, is moved to position 2. Do the following, in the space below, and on the other side, if necessary.

Find: 1. The dimensions of the trajectory circle, including the starting point.

2. The equations for i and v_C in the forms of both Eqs. (1) on page 50 and Eqs. (2) on page 52, and state the frequency of oscillation.

Note: If you need help and cannot find it in the previous pages, see your teacher.

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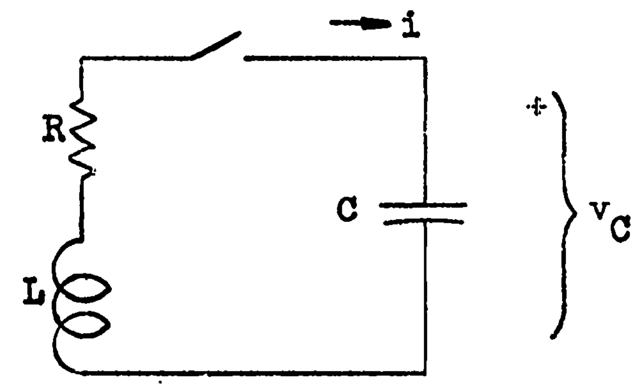


Fig. 18

2

PART II. CIRCUIT WITH NON-ZERO RESISTANCE

We shall now consider the solutions of the circuit of Fig. 18, following the same pattern as before, but taking R into account. Previously, with R = 0, we had

$$L \frac{di}{dt} = -v_C \quad \text{and} \quad C \frac{dv_C}{dt} = i$$

Now complete the following equations relating v_C and i, when R is not zero.

$$L \frac{di}{dt} =$$

$$C \frac{dv_C}{dt} =$$

Also, we will want to continue with the same variables x_1 and x_2 as before, which are related to v_C and i by

x_1 = ____________, x_2 = ________________

$$L\frac{di}{dt} = -Ri - v_C \longrightarrow L\sqrt{\frac{C}{L}}\frac{dx_1}{dt} = -R\sqrt{\frac{C}{L}}\,x_1 - x_2$$

$$C\frac{dv_C}{dt} = i \longrightarrow C\frac{dx_2}{dt} = \sqrt{\frac{C}{L}}\,x_1$$

$$x_1 = \sqrt{\frac{L}{C}}\,i$$

or

$$\frac{dx_1}{dt} = -\frac{R}{L}x_1 - \omega_0 x_2$$

$$x_2 = v_C \qquad \frac{dx_2}{dt} = \omega_0 x_1$$

Please check through the change of variables shown on the right of page 58. The quantity ω_0 appears in the final form. This is the same quantity we used before; it is related to circuit parameters by

$$\omega_0 = \_\_\_\_\_\_\_\_\_\_ .$$

We shall introduce a second parameter

$$\alpha = \frac{R}{L}$$

Thus, when the x variables are used, it is found that the equations depend on only two quantities, namely __________ and __________. (Answer in terms of R, L, and C.)

Suppose you have two circuits, R_1, L_1, C_1 and R_2, L_2, C_2. If $C_2 = 2C_1$, the equations in x_1 and x_2 will be the same if $L_2 =$ __________ and $R_2 =$ __________.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$\frac{R}{L}$ and LC (or $\frac{1}{\sqrt{LC}}$)

$$L_2 = \frac{1}{2} L_1$$

$$R_2 = \frac{1}{2} R_1$$

The equations to be solved are:

$$\frac{dx_1}{dt} = -\alpha x_1 - \omega_0 x_2$$

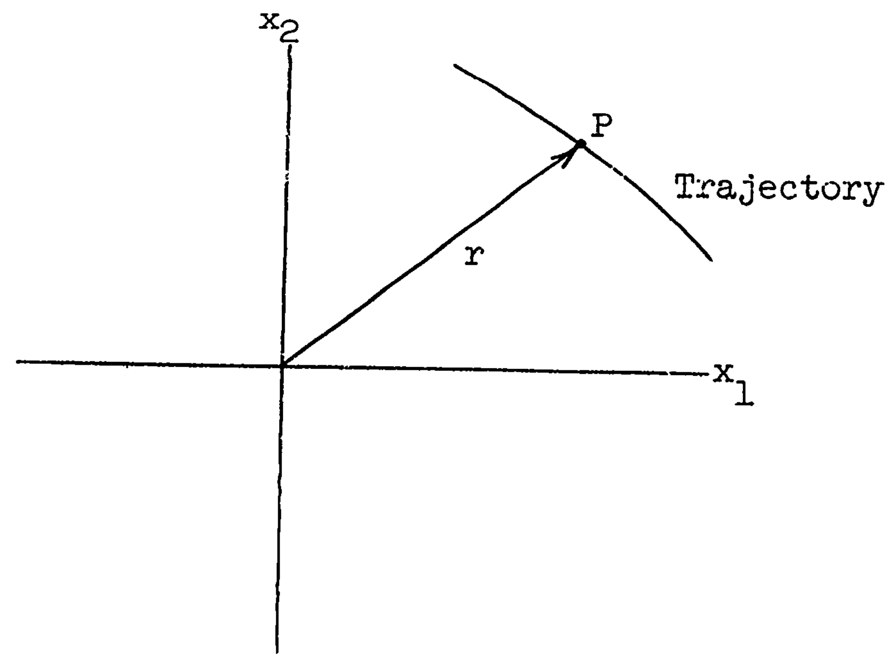
$$\frac{dx_2}{dt} = \omega_0 x_1$$

The first step in Part I, where $\alpha = 0$, was to multiply the first equation by x_1 and the second by x_2, and to add them. Do this again for the present case, in the space below, recalling that

$$x \frac{dx}{dt} = \frac{1}{2} \left(\frac{dx^2}{dt}\right)$$

The result is:

$$\frac{d(x_1^2 + x_2^2)}{dt} = -2\alpha x_1^2$$



$$r^2 = x_1^2 + x_2^2$$

Fig. 19

Observe that $x_1^2 + x_2^2$ can be interpreted as r^2, where r is the radius to the state point P, as in Fig. 19. Thus, the last equation obtained can be written

$$\frac{1}{2}\frac{d(r^2)}{dt} = \alpha x_1^2 \qquad (3)$$

or

$$\frac{dr}{dt} = -\alpha \frac{x_1^2}{r} \qquad (4)$$

In view of this result, how does r vary with time? ______________________.
(increase, constant, decrease)

State any exceptions: ______________________. Also, since $x_1^2 + x_2^2 = \frac{L}{C} i^2 + v_C^2$, we know that $\frac{r^2}{2} = \frac{1}{C}$ (______________________) and $\alpha x_1^2 = \frac{R}{L}(\frac{L}{C})i^2 = \frac{1}{C} Ri^2$, which is $\frac{1}{C} \times$
(name of quantity)

______________________.
(physical description of the quantity)

Thus, Eq. (3) is a statement concerning energy in the circuit. Translate this statement into words concerning energy: ______________________

______________________.

r decreases, since $\frac{dr}{dt}$ is negative, when x_1 is not zero.

When $x_1 = 0$, $i = 0$, and $\frac{dr}{dt} = 0$.

$\frac{1}{2} r^2 = \frac{1}{C}$ (stored energy).

rate of energy dissipation in the resistor.

The rate of decrease of stored energy equals the rate of energy dissipation

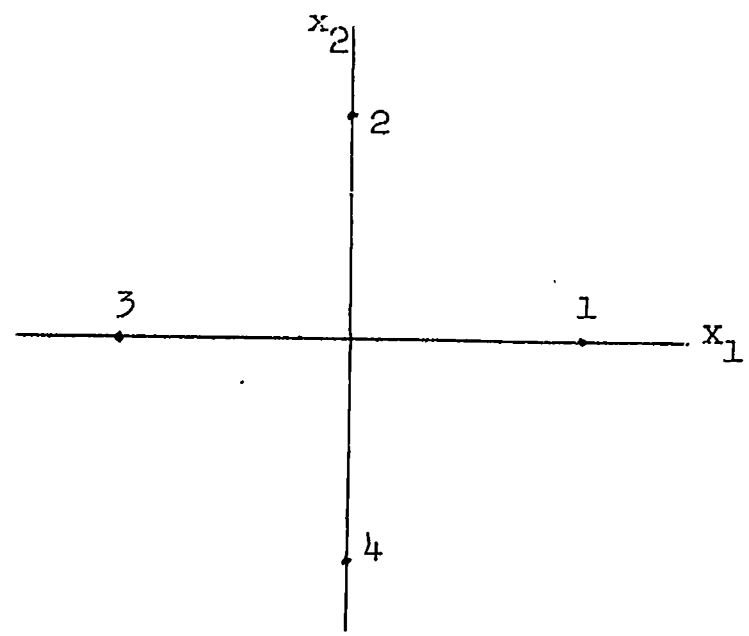


Fig. 20

We have the general result that the distance of the state point from the origin is decreasing with time, except when ______________.

In view of this property, sketch arcs of trajectory at the four sample state points shown in Fig. 20.

Do you know the exact slopes at points (1) and (3)? ____.

Do you know the exact slopes at points (2) and (4)? ____.

66

$x_1 = 0$

No (we only know that the slope is negative, since r is decreasing).

Yes (the slope is zero, because $x_1 = 0$ at points 2 and 4, and hence the radius is not changing).

In the previous frame you found that the slope of the trajectory is known where it crosses the x_2 axis, but not where it crosses the x_1 axis. The latter can be found, however, by using the relation

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}}$$

and the known equations for dx_2/dt and dx_1/dt. The general result is

$$\frac{dx_2}{dx_1} =$$

and when the trajectory crosses the x_1 axis,

$$\frac{dx_2}{dx_1} =$$

$$\frac{dx_2}{dx_1} = \frac{\omega_0 x_1}{-\alpha x_1 - \omega_0 x_2}$$

(in general)

$$\frac{dx_2}{dx_1} = -\frac{\omega_0}{\alpha}$$

when $x_2 = 0$ (crossing the x_1 axis)

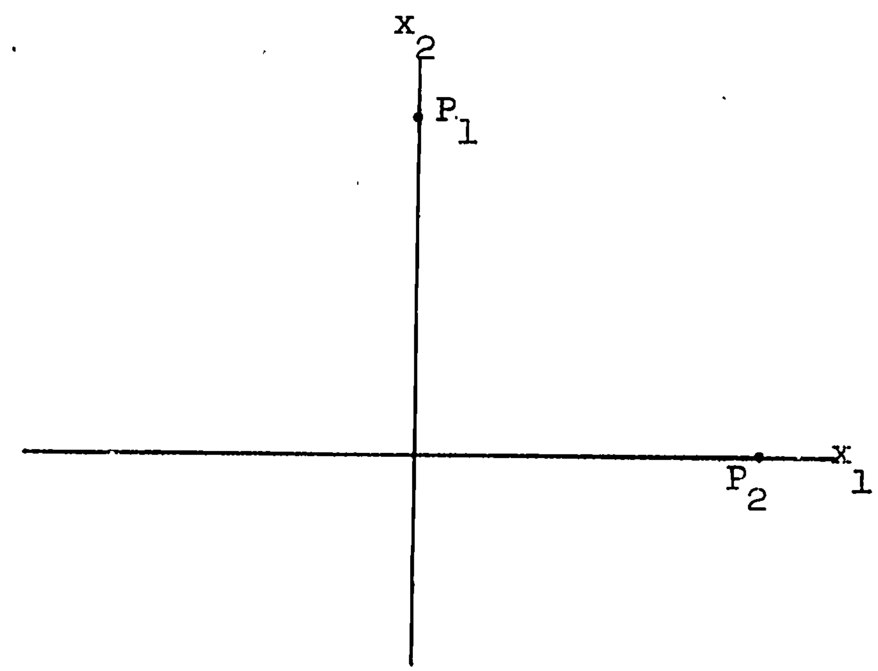
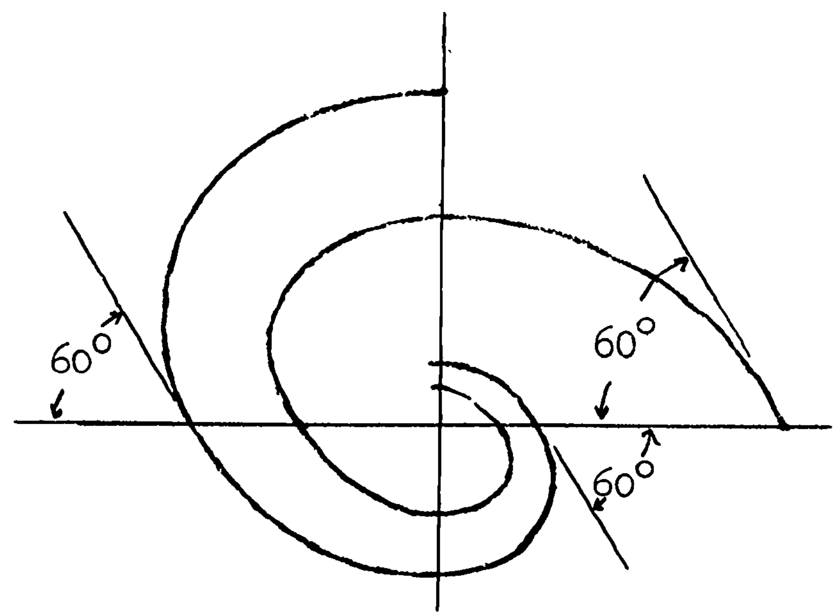


Fig. 21

In Fig. 21, P_1 and P_2 are two sample initial points, equidistant from the origin. Take the case where $\omega_0 = \sqrt{3}\,\alpha$ and sketch in what you consider to be reasonable approximate trajectories. Do this twice, once for each of the given initial points. Carry each trajectory far enough to have two intersections with each axis.



Observe that this is far from a complete solution. However, merely by knowing the few properties so far determined, the <u>general nature</u> of the trajectory is known.

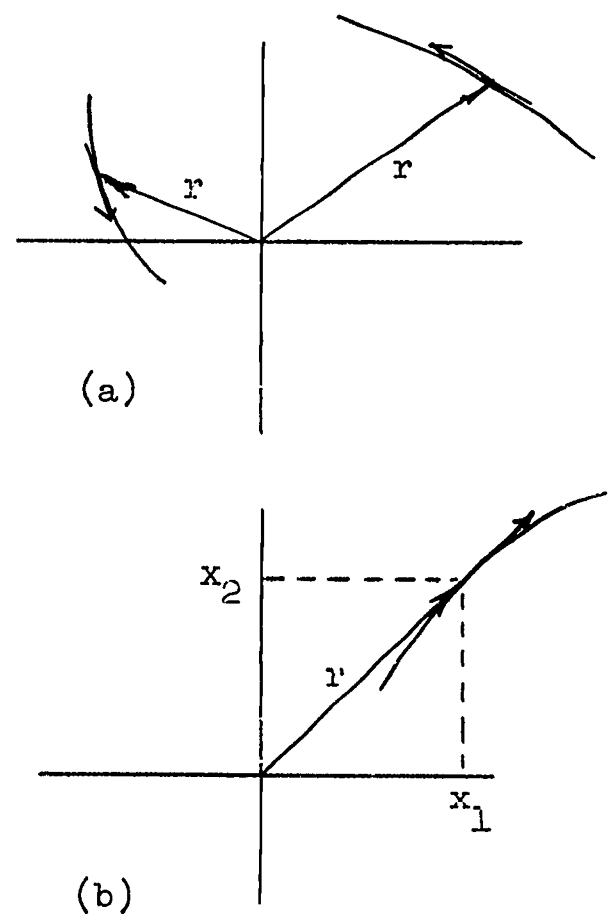


Fig. 22

One might surmise that the trajectories shown are typical of all cases. The distinctive feature in the ones shown is a continued spiraling around while approaching the origin. As in Fig. 22(a), at each point this trajectory makes a non-zero angle with radial line **r**.

Let us see if it is ever possible that the trajectory shall not have this property. In other words, let us see whether the condition shown at Fig. 22(b) can occur, where

$$\frac{dx_2}{dx_1} = \frac{x_2}{x_1} \quad \text{or} \quad \frac{\omega_0 x_1}{-\alpha x_1 - \omega_0 x_2} = \frac{x_2}{x_1}$$

Describe in words how radial line **r** and the trajectory are related when this equation is satisfied: ____

____.

We can consider the ratio (x_2/x_1) as a variable, and write the above

$$\frac{1}{-\frac{\alpha}{\omega_0} - \left(\frac{x_2}{x_1}\right)} = \left(\frac{x_2}{x_1}\right)$$

Can you write a solution for (x_2/x_1)?

The condition portrayed is that the radial line r shall be <u>tangent</u> to the trajectory (or an equivalent statement).

This yields the quadratic equation

$$\left(\frac{x_2}{x_1}\right)^2 + \frac{\alpha}{\omega_0}\left(\frac{x_2}{x_1}\right) + 1 = 0$$

$$\frac{x_2}{x_1} = -\frac{\alpha}{2\omega_0} \pm \sqrt{\left(\frac{\alpha}{2\omega_0}\right)^2 - 1}\ .$$

You obtained a formula, namely

$$\frac{x_2}{x_1} = -\frac{\alpha}{2\omega_0} \pm \sqrt{\left(\frac{\alpha}{2\omega_0}\right)^2 - 1}$$

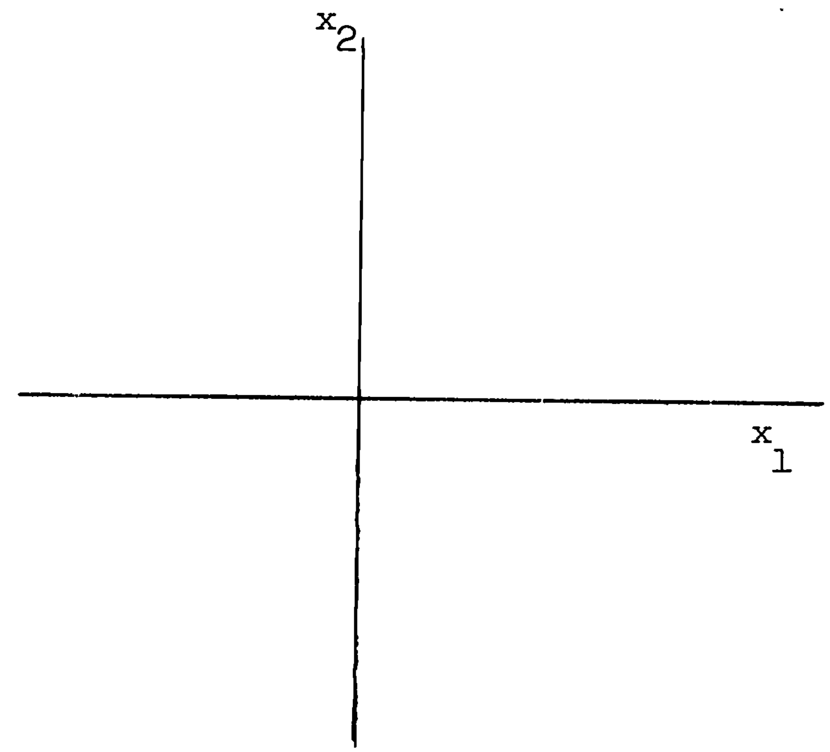
for x_2/x_1 such that r will be tangent to the trajectory. What can you say about different relationships between α and ω_0, and whether or not r <u>can</u> <u>ever</u> be tangent to the trajectory.

Real values of x_2/x_1 exist only if $\alpha \geq 2\omega_0$.

If $\alpha < 2\omega_0$, there are no values and r cannot be tangent to the trajectory. Thus, our previously sketched trajectories were for this condition (indeed, we took $\omega_0 = \sqrt{3}\ \alpha$).

If $\alpha = 2\omega_0$, there is one set of values of x_2/x_1.

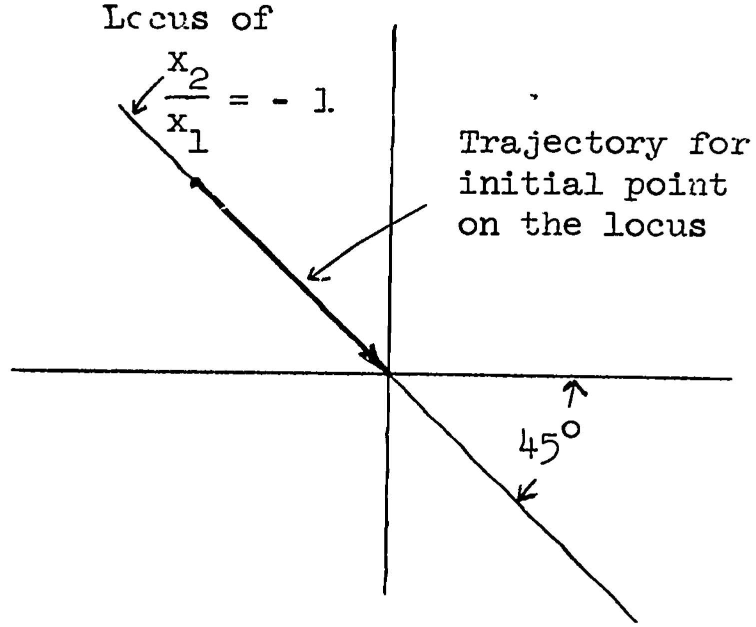
If $\alpha > 2\omega_0$, there are two sets of values of x_2/x_1, or two conditions in which r will be tangent to the trajectory.



Consider the case $\alpha = 2\omega_0$. On the opposite page, sketch the locus of values of x_2/x_1 such that r is tangent to the trajectory. Answer the following questions:

Can the trajectory cross the line you drew? Explain your answer. ____

Under what conditions can the state point be <u>on</u> this locus? ______________. What is the trajectory in that case? ______________.



Trajectory <u>cannot</u> cross the locus, because in so doing it would make a non-zero angle with r.

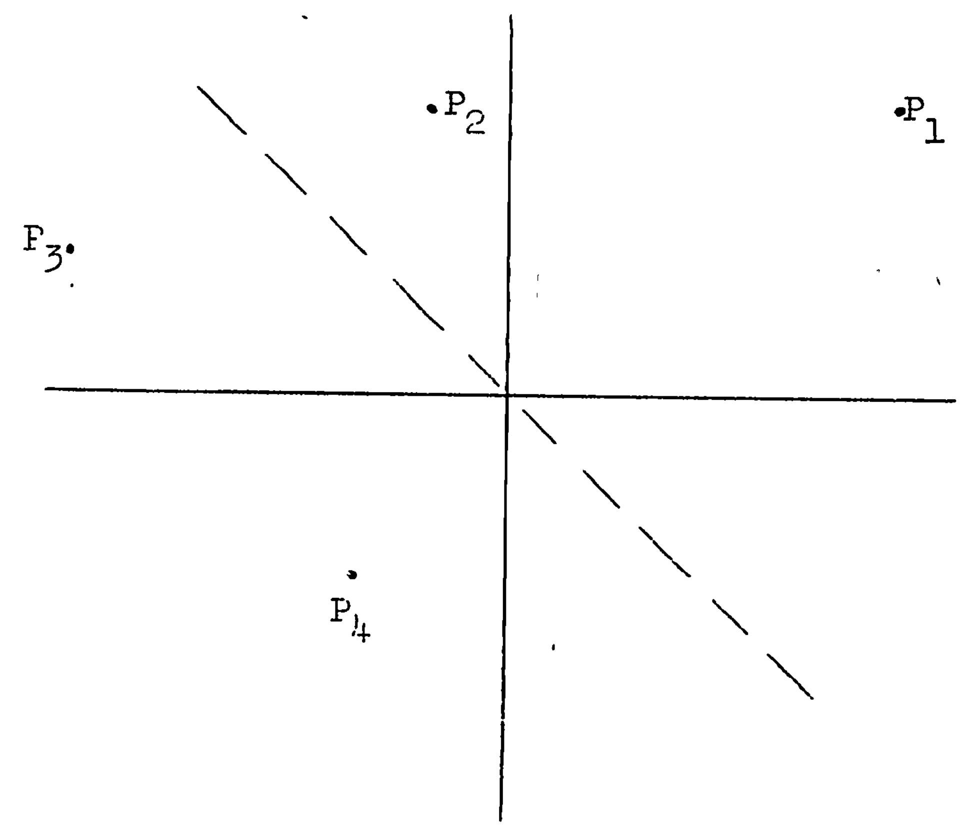
A state point can be <u>on</u> the locus if the initial point is <u>on</u> the locus. In that case the trajectory will be a straight line coincident with the locus.

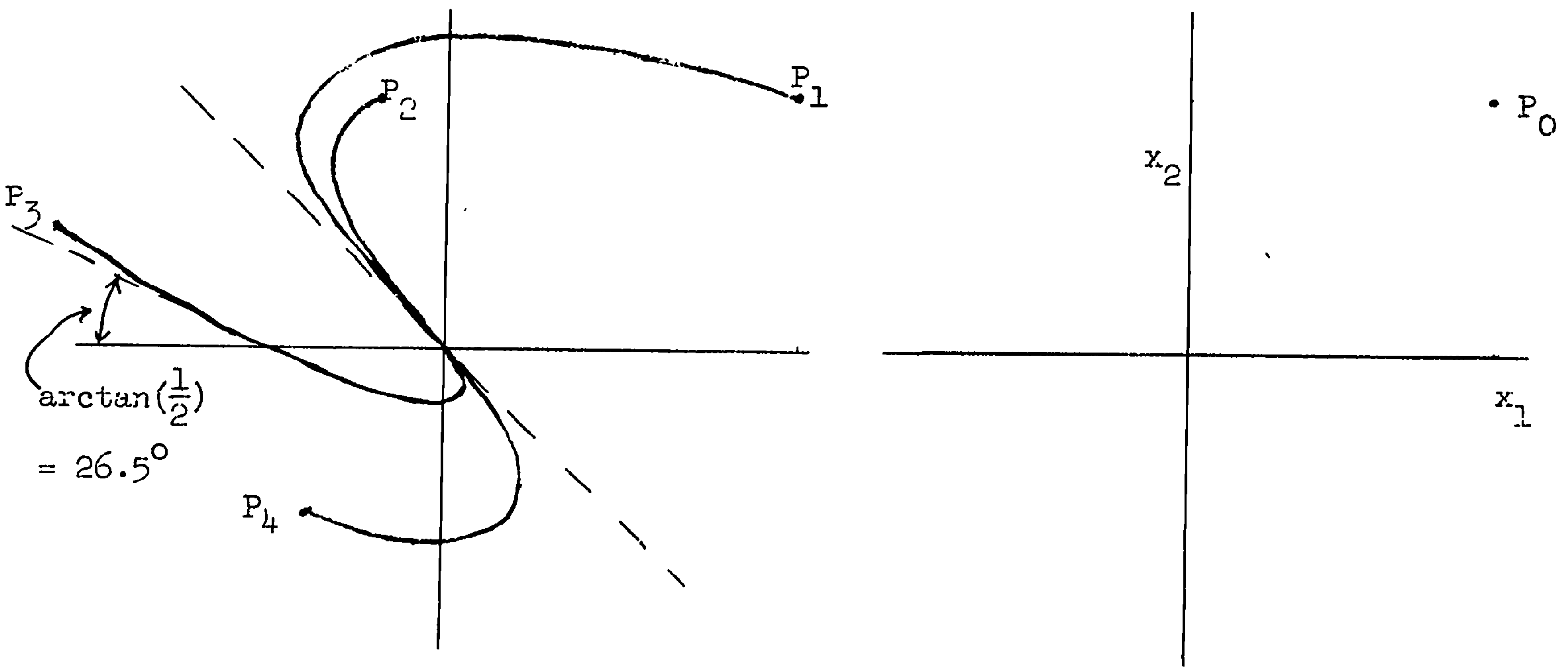
The general equation for the slope is

$$\frac{dx_2}{dx_1} = \frac{\omega_0 x_1}{-\alpha x_1 - \omega_0 x_2}$$

$$= \frac{1}{-\frac{\alpha}{\omega_0} - \left(\frac{x_2}{x_1}\right)}$$

For each of the four starting points shown below, sketch reasonable trajectories, consistent with the properties you have learned. This is for the case $\alpha = 2\omega_0$.

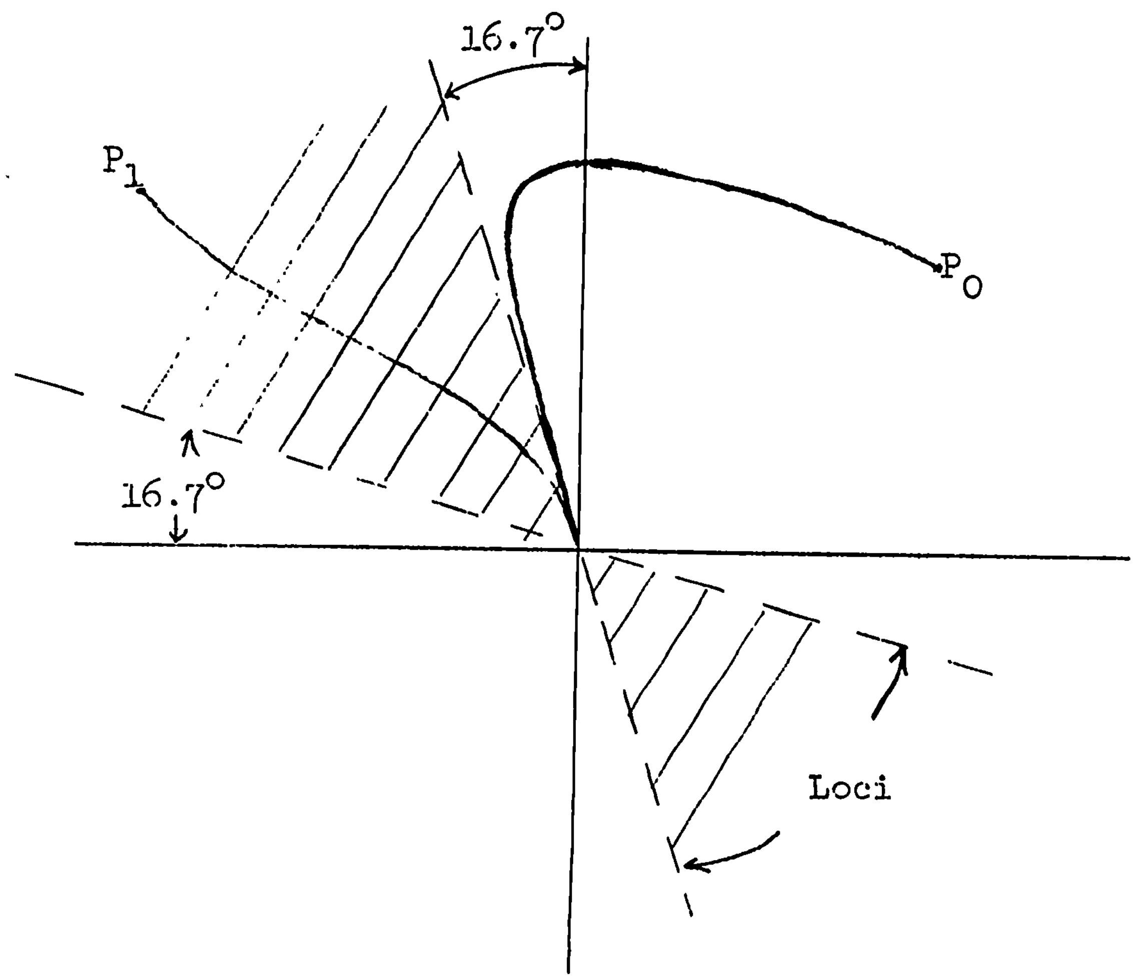



P_0
x_1
x_2
P_1
P_2
P_3
P_4
$\arctan(\frac{1}{2})$
$= 26.5^{\circ}$

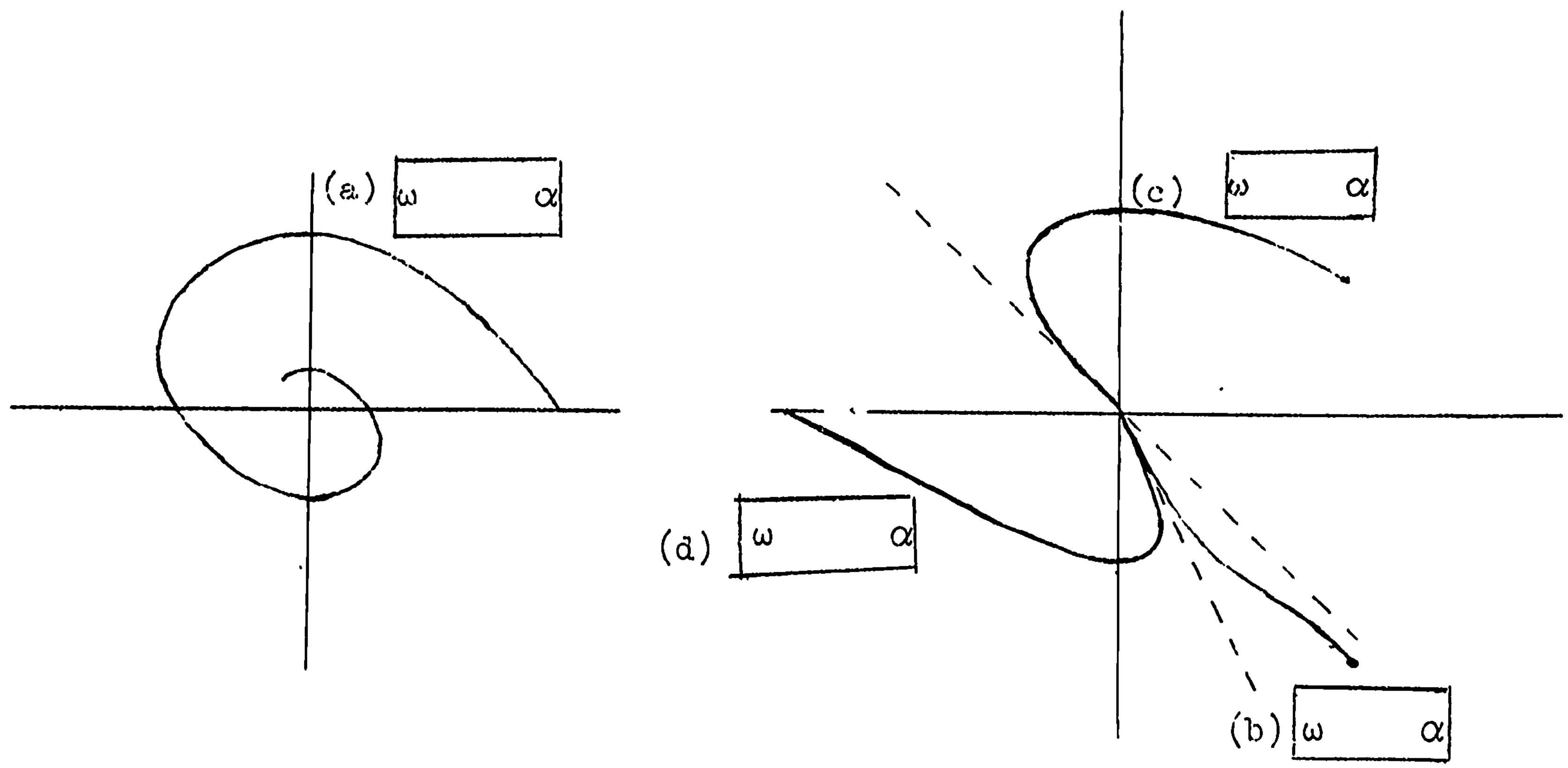
Return to the equation for the locus for which r is tangent to the trajectory, namely

$$\frac{x_2}{x_1} = -\frac{\alpha}{2\omega_0} \pm \sqrt{\left(\frac{\alpha}{2\omega_0}\right)^2 - 1}$$

for the case where $\alpha > 2\omega_0$. For example, let $\alpha = 4\omega_0$. Sketch the loci for this case, and a trajectory starting from point P_0. Do this on page 78.


16.7°
P1
P0
16.7°
Loci

A trajectory is also shown for initial point P_1, on the previous page. How is it known that the trajectory from P_1 must remain within the shaded wedge-shaped area (the shaded <u>sector</u>)?



Because the trajectory cannot
cross either locus.

Review:

1. For each of the four cases shown on page 82, within each box complete the relationship between ω and α.

2. What happens to the shaded sector on page 80 as α becomes infinite?

3. For the case

$$L = .2 \text{ h}$$
$$C = .8 \times 10^{-6} \text{ f}$$

what is the range of R such that the trajectory will encircle the origin?

1. (a) $\alpha < 2\omega_0$

   (b) $\alpha > 2\omega_0$

   (c) $\alpha = 2\omega_0$

   (d) $\alpha > 2\omega_0$

2. The sides of the shaded sector approach the coordinate axes.

3. $\omega_0 = \dfrac{10^3}{\sqrt{.2 \times .8}} = \dfrac{1000}{.4} = 2500$

   $\alpha = \dfrac{R}{L} < 2\omega_0 = 5000$

   $R < .2 \times 5000 = 1000$ ohms

   $0 < R < 1000$ ohms

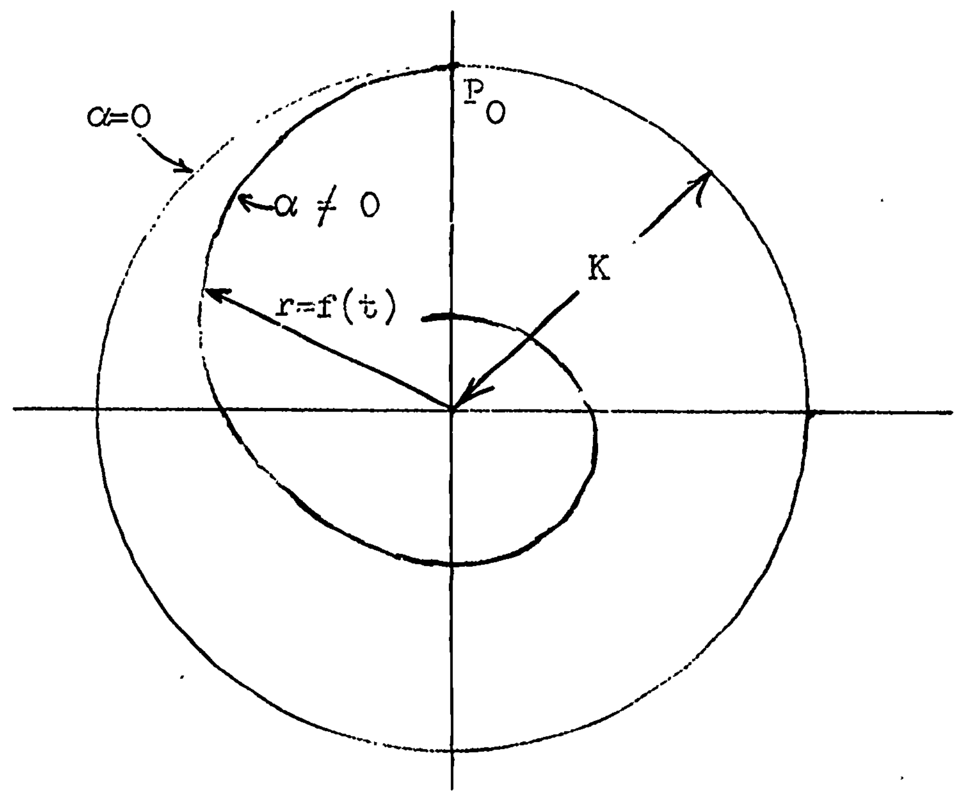


Fig. 23

PART III. ANALYTICAL SOLUTION FOR $\alpha < 2\omega_0$

So far, we have established sufficient properties of the trajectories to enable approximate sketches to be made. However, exact shapes are not known, and the time scale is completely missing.

We shall now obtain a quantitative solution for the case where the initial point is on the x_2 axis. Refer to Fig. 23, and recall the $\alpha = 0$ solution which, for P_0 as an initial point, would be

$$x_1 = -K \sin \omega_0 t \quad , \quad x_2 = K \cos \omega_0 t$$

Now, for $\alpha \neq 0$, since P_0 is on the x_2 axis, x_1 will be zero when $t = 0$, and hence a sine function would be appropriate for x_1. However, when $\alpha \neq 0$, r is gradually decreasing. Let the instantaneous value of r be an unknown function f(t), and let an unknown number ω (without subscript 0) be the angular frequency. With these assumptions, a trial solution for x_1 will be

x_1 = ______________________

$x_1 = -f(t) \sin \omega t$

This is a trial solution.

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -\alpha x_1 - \omega_0 x_2 \qquad (a) \\ \frac{dx_2}{dt} &= \omega_0 x_1 \qquad (b) \end{aligned}\right\} (3)$$

From Eq. (3a)

$$x_2 = -\frac{1}{\omega_0}\left[\alpha x_1 + \frac{dx_1}{dt}\right]$$

Using the trial x_1, we have

$$\frac{dx_1}{dt} = -f'(t)\sin\omega t - \omega f(t)\cos\omega t$$

$$\alpha x_1 = -\alpha f(t)\sin\omega t$$

$$x_2 = \frac{[f'(t)+\alpha f(t)]\sin\omega t + \omega f(t)\cos\omega t}{\omega_0}$$

Check this in Eq. (3b)

$$\frac{dx_2}{dt} = \frac{1}{\omega_0}\left\{\begin{array}{l} [f''(t)+\alpha f'(t)]\sin\omega t \\ \quad +\omega f'(t)\cos\omega t \\ +[\omega f'(t)+\alpha\omega f(t)]\cos\omega t \\ \quad -\omega^2 f(t)\sin\omega t \end{array}\right\}$$

or

$$\frac{dx_2}{dt} = \frac{1}{\omega_0}\left\{\begin{array}{l} [f''(t)+\alpha f'(t)-\omega^2 f(t)]\sin\omega t \\ +\omega[2f'(t)+\alpha f(t)]\cos\omega t \end{array}\right\}$$

Some routine algebra is given on page 86, beginning with a trial solution and the original equations. Answer the following questions concerning these steps:

1. Why is it not sufficient to stop with the bottom formula on the left, which seems to be a solution for x_2? ______________________________

2. The last formula obtained, for dx_2/dt, must be identical to what? ______________________________.

3. In view of your answer to (2), what can be said about the quantity $[2\omega f'(t) + \alpha\omega f(t)]$? ____________ Why? ____________

4. What is f(t) in terms of a geometrical property of the trajectory? ______________________________

5. What are the mathematical meanings of the symbols f'(t) and f''(t)? ______________________________

1. This formula includes unknowns $f(t)$ and ω. We can't stop yet!
2. The formula for $\frac{dx_2}{dt}$ must be identical to the formula for $\omega_0 x_1$, namely $-\omega_0 f(t)\sin\omega t$.
3. Identity means "for all t", hence the coefficient $[2\omega f'(t) + \alpha\omega f(t)]$ of $\cos \omega t$ must be zero for all values of t.
4. r is the instantaneous distance from the origin to the state point.
5. $f'(t)$ is the derivative which could also be written $\frac{df(t)}{dt}$, and $f''(t)$ is the second derivative $\frac{d^2f(t)}{dt^2}$.

We established that it must be true that

$$2f'(t) + \alpha f(t) = 0$$

if the trial solution is to be satisfactory. This is also

$$2\frac{df(t)}{dt} + \alpha f(t) = 0$$

Its solution is

$$f(t) = Ae^{-(\alpha/2)t}$$

Try it, to convince yourself that this is the solution!

Have you seen it before?

Now refer to the right-hand side of page 88. It is shown there that the unwanted cosine term in the equation for dx_2/dt will drop out if

$$f(t) = Ae^{-(\alpha/2)t}$$

What can you say about the quantity A? ______________________________

When f(t) is given by the above, referring to page 86, we see that

$$\frac{dx_2}{dt} = \frac{1}{\omega_0}\,[f''(t) + \alpha f'(t) - \omega^2 f(t)]\,\sin \omega t$$

Substitute the known formula for f(t), and write dx_2/dt in what you consider to be its simplest form:

$$\frac{dx_2}{dt} =$$

This must be identical to $\omega_0 x_1$, which is

$$\omega_0 x_1 =$$

A is <u>any</u> constant. (It is the value of $f(t)$ at $t = 0$, which can be any constant).

$$\frac{dx_2}{dt} = \frac{A}{\omega_0} e^{-(\alpha/2)t} [\frac{\alpha^2}{4} - \frac{\alpha^2}{2} - \omega^2] \sin \omega t$$

$$= - \frac{A}{\omega_0} e^{-(\alpha/2)t} [\frac{\alpha^2}{4} + \omega^2] \sin \omega t$$

$$\omega_0 x_1 = - \omega_0 A e^{-(\alpha/2)t} \sin \omega t$$

Don't forget, we are attempting to make dx_2/dt and $\omega_0 x_1$ ________ (a word). This will be accomplished if

________ = ________

which can be solved for ω to give

$\omega =$

For what range of α will ω be a real number? ________

If L and C are constant, and recalling that $\alpha = R/L$, write a brief statement about the value of ω as $R = 0$ and as R gradually increases in value. ________

____

Do you know anything about A yet? ________

identical.

$$\frac{\alpha^2}{4} + \omega^2 = \omega_0^2$$

$$\omega = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$$

ω is real for $\alpha < 2\omega_0$

At $R = 0$, $\omega = \omega_0$. As R increases, ω decreases.

No. A remains an unknown constant.

<u>Summary</u>

Original equations:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -\alpha x_1 - \omega_0 x_2 \quad &(a) \\ \frac{dx_2}{dt} &= \omega_0 x_1 \quad &(b) \end{aligned}\right\}(5)$$

Solution for x_1:

$$x_1 = -Ae^{-(\alpha/2)t} \sin \omega t$$

where

$$\omega = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$$

We had a formula for x_2 on page 86, in terms of $f(t)$ and its derivative. We could substitute into that, but in order to keep the original equations in mind (see page 92), let us obtain x_2 directly from Eq. (5a), which we again write as

$$x_2 = - \frac{1}{\omega_0} [\alpha x_1 + \frac{dx_1}{dt}]$$

Now we have

$$\alpha x_1 =$$

$$\frac{dx_1}{dt} =$$

and so

$$x_2 = Ae^{-(\alpha/2)t} [(\quad) \cos \omega t + (\quad) \sin \omega t]$$

If the initial point, which is on the x_2 axis, is at coordinate $x_2(0)$, what is A?

$$A =$$

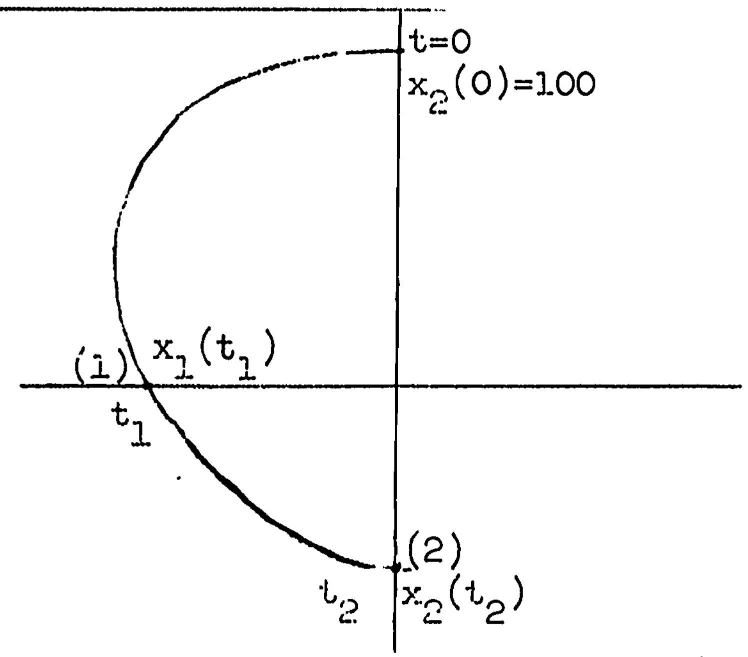
$$\alpha x_1 = -Ae^{-(\alpha/2)t}\,\alpha \sin \omega t$$

$$\frac{dx_1}{dt} = -Ae^{-(\alpha/2)t}\left[-\frac{\alpha}{2}\sin \omega t + \omega \cos \omega t\right]$$

$$x_2 = Ae^{-(\alpha/2)t}\left[\frac{\omega}{\omega_0}\cos \omega t + \frac{\alpha}{2\omega_0}\sin \omega t\right]$$

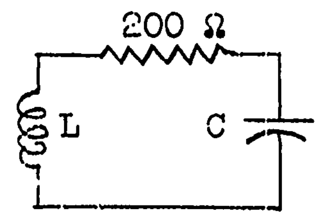
$$x_2(0) = \frac{A\omega}{\omega_0}$$

Therefore $A = x_2(0)\,\frac{\omega_0}{\omega}$



$$\omega_0 = 250$$
$$\alpha = 100$$
$$x_2(0) = 100$$

$$\omega = \sqrt{(250)^2 - (50)^2}$$
$$= \sqrt{60000} = 245$$



Let us obtain some practice with these results, for the numerical values given on page 94.

What values of L and C, for the circuit shown, will correspond to the given values of α and ω_0? L = ________________, C = ________________.

Rather than to compute the entire trajectory, let us obtain the intercepts and values of t at points (1) and (2). Point (2) is easier than point (1). Therefore we will do it first.

At point (2) the value of $t = t_2$ is

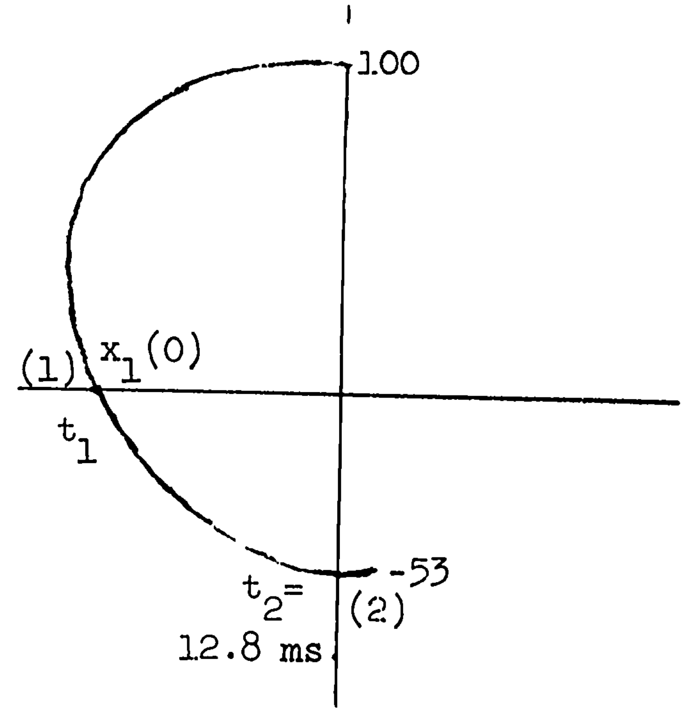
and the value of $x_2 = x_2(t_2)$ is

$L = 2$ henries
$C = 8 \times 10^{-6}$ farad

At t_2, we know that $\cos \omega t$ must have gone through one half-cycle:

$$t_2 = \frac{\pi}{\omega} = 12.8 \text{ milliseconds}$$

$$x_2(t_2) = 100\, e^{-50(.0128)}(\cos \pi)$$
$$= -53$$



$$x_1 = -Ae^{-(\alpha/2)t} \sin \omega t$$

$$x_2 = Ae^{-(\alpha/2)t} \left[\frac{\omega}{\omega_0} \cos \omega t + \frac{\alpha}{2\omega_0} \sin \omega t\right]$$

Now consider point (1). What is the distinctive feature of this point, from which the value of t_1 can be obtained?

As a result of the above,

$$\tan \omega t_1 =$$

and finally

$$t_1 =$$

The value of $x_1(t_1)$ is

$x_2 = 0$ at point (1)

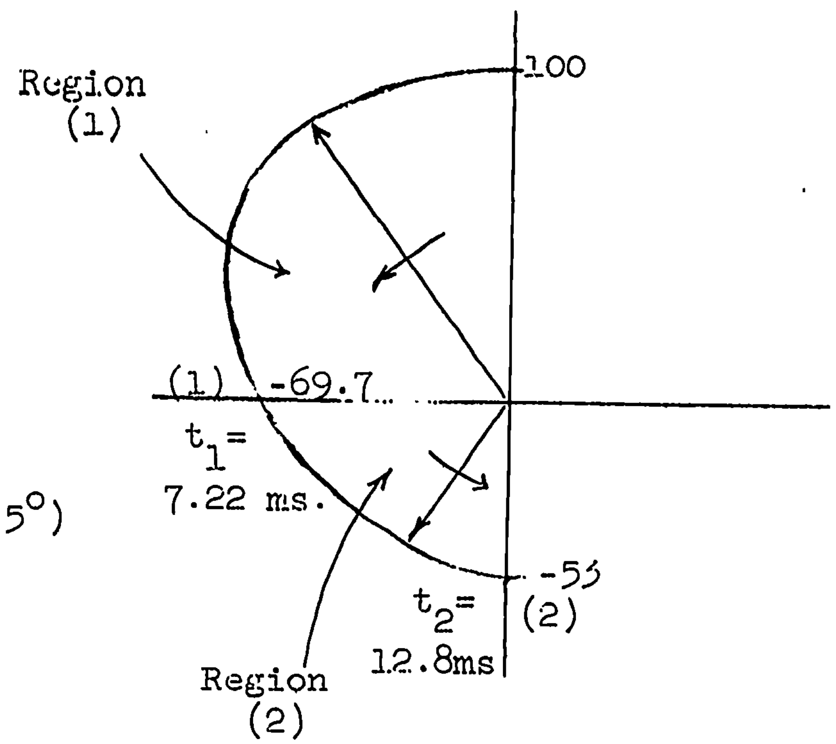
$$\tan \omega t_1 = -\frac{2\omega}{\alpha} = -4.9$$

$$\omega t_1 = 1.77 \text{ radians } (101.5^\circ)$$

$$t_1 = 7.22 \text{ millisec.}$$

$$x_1(t_1) \doteq -100\left(\frac{250}{245}\right) e^{-(50)(.00722)} \sin(101.5^\circ)$$

$$= -69.7$$



In the R = 0 (circular trajectory) case, we found that the radial line to a state point rotates with constant angular velocity, which is ________.
(symbol)

Let us use the numerical values just calculated to learn something about the angular velocity for the $R \neq 0$ case.

First, what is the <u>average</u> angular velocity between points (0) and (2)? ________ ________________ (give a symbol, and a numerical value).

Also, for regions between points (0) and (1), and between points (1) and (2), respectively labeled regions 1 and 2 in the opposite figure, the numerical values for the <u>average</u> angular velocity are:

For region (1), average angular velocity =

For region (2), average angular velocity =

In general, it may be said that the angular velocity ____________ with time.

100

ω_0

$$\text{Average} = \frac{\pi \text{ rad.}}{.0128 \text{ sec.}} = 245 \frac{r}{s} = \omega$$

$$\text{region (1)}: \quad \frac{\pi}{2(.00722)} = 218 \text{ rad/sec.}$$

$$\text{region (2)}: \quad \frac{\pi}{2(.0128-.00722)} = 310 \text{ rad/sec.}$$

varies with time

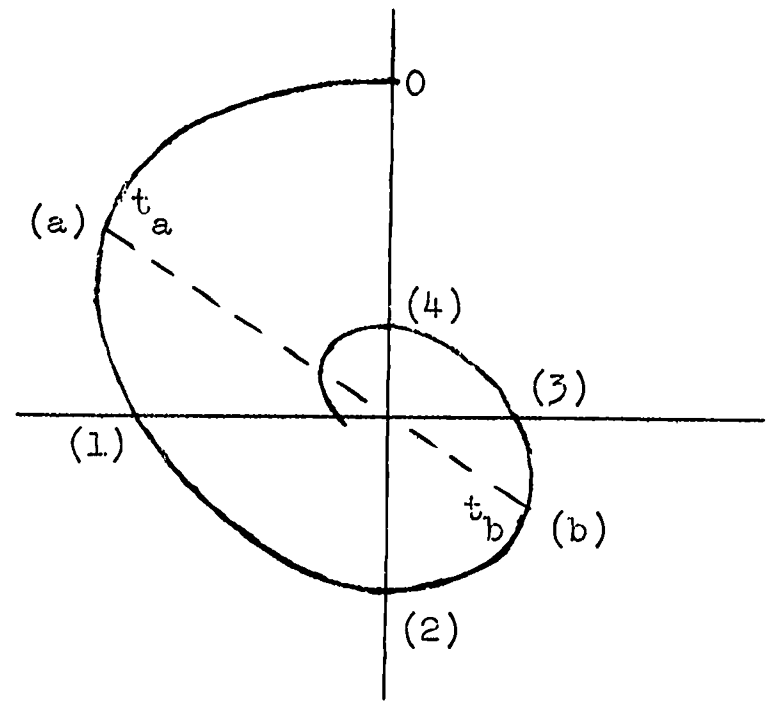


Fig. 24

Similar conclusions regarding the angular velocity can be drawn from the first and fourth quadrants, as indicated in Fig. 24.

What will be the average angular velocity between points (1) and (3)? __________, between points (2) and (4)? ________________.

What will be the time required to go between any two diametrically opposite points, like (a) and (b), and what will be the average angular velocity between these points? (This is for the same numerical example.) This answer is not obvious, you must do some work to get it.

time =

average angular velocity =

$\omega = 245$ in all cases.

For points (a) and (b), x_2/x_1 must be the same, and so

$$\frac{\frac{\omega}{\omega_0}\cos\omega t_a + \frac{\alpha}{2\omega_0}\sin\omega t_a}{-\sin\omega t_a} = \frac{\frac{\omega}{\omega_0}\cos\omega t_b + \frac{\alpha}{2\omega_0}\sin\omega t_b}{-\sin\omega t_b}$$

or

$$\frac{\omega}{\omega_0}\cot\omega t_a = \frac{\omega}{\omega_0}\cot\omega t_b$$

cot x repeats when x increases by π.

Therefore, $\omega t_b = \omega t_a + \pi$ or $t_b - t_a = \frac{\pi}{\omega} = \frac{\pi}{245}$

$= 12.8$ ms.

Average angular velocity $= \frac{\pi}{.0128} = 245$ (which is ω).

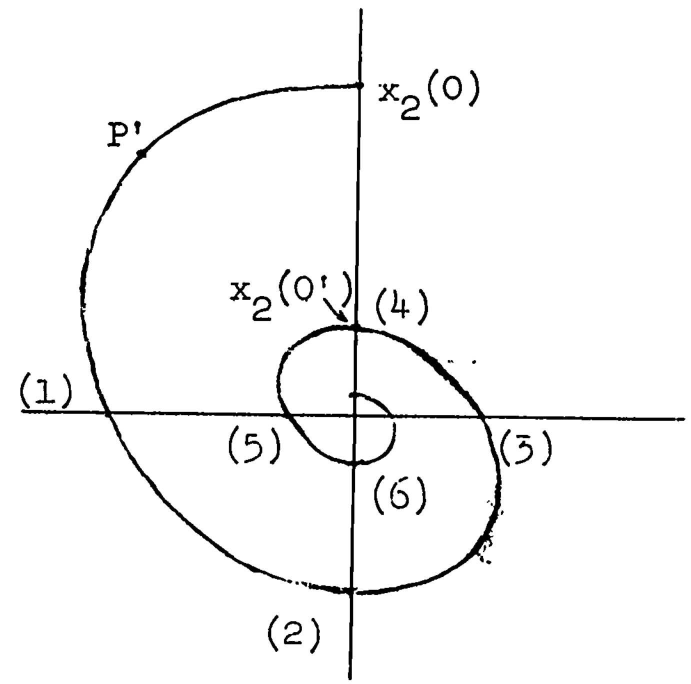


Fig. 25

So far, we have considered the time elapsed and average angular velocity for portions of the first complete encirclement of the origin. What can we say for the second, and third, etc.? A second encirclement is portrayed in Fig. 25, to which your attention is now directed. Consider the following:

When we calculated the time interval from (0) to (1), or from (1) to (2), did the intercept $x_2(0)$ have any influence on the answer? ____________

Any particular state point on the trajectory can be regarded as a possible initial point. For example, subsequent to point P' the trajectory is the same if either $x_2(0)$ or P' is the initial point. Likewise, point (4) could be regarded as an initial point. Recall that we found that t_1 and (t_2-t_1) are respectively 12.8 and 7.22 millisec.

What conclusion do you reach concerning t_5-t_4 and t_6-t_5?

$t_5-t_4 =$

$t_6-t_5 =$

104

No.

Since the initial value has no influence on the time intervals, the time required to pass through the quadrant after point (4) is the same as after point (0). Thus,

$$t_5 - t_4 = 12.8$$

$$t_6 - t_5 = 7.22$$

Slow	Fast
Fast	Slow

Review

$$x_1 = -Ae^{-(\alpha/2)t} \sin \omega t$$

$$x_2 = Ae^{-(\alpha/2)t}[\frac{\omega}{\omega_0} \cos\omega t + \frac{\alpha}{2\omega_0} \sin\omega t]$$

These results have been obtained for a specific numerical case. However, the elapsed time required for the state point to pass through a quadrant, can be determined for the general case. When we refer to "passing through a quadrant", we mean one of the quadrants of the coordinate axes, not any 90° angle.

Either x_1 or x_2 is zero at the boundary of each of these quadrants. These boundaries are:

"Slow" quadrant from ___ = 0 to ___ = 0 } Insert x_1 or x_2
"Fast" quadrant from ___ = 0 to ___ = 0 } appropriately

Let t_s and t_f be the intervals of time required to pass through "slow" and "fast" quadrants, respectively. What are t_s and t_f, in terms of the general parameters α and ω?

t_s = ________________ t_f = ________________

What is the average angular velocity for any half revolution of the state point? ________________. For any complete revolution? ________________

Slow: from x_1 - 0 to $x_2 = 0$

Fast: from $x_2 = 0$ to $x_1 = 0$

To get t_s, let $x_1 = 0$ be the initial point.

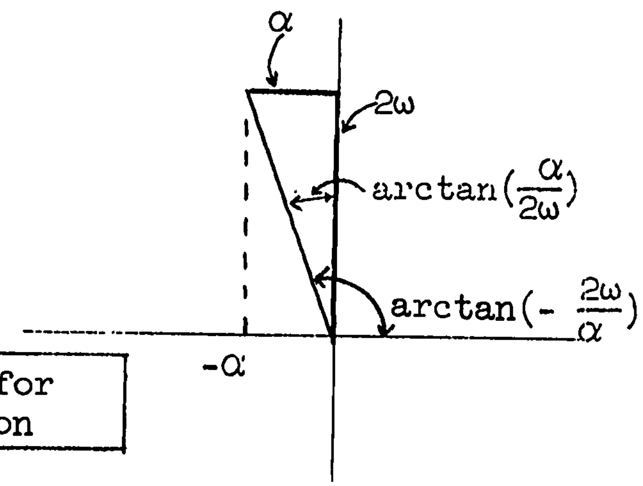
Then,

$$\frac{\omega}{\omega_0} \cos \omega t_s + \frac{\alpha}{2\omega_0} \sin \omega t_s = 0$$

or

$$t_s = \frac{1}{\omega} \arctan(-\frac{2\omega}{\alpha}) = \frac{1}{\omega}[\frac{\pi}{2} + \arctan \frac{\alpha}{2\omega}]$$

See figure for explanation



and, since $t_s + t_f = \frac{\pi}{\omega}$,

$$t_f = \frac{1}{\omega}[\frac{\pi}{2} - \arctan \frac{\alpha}{2\omega}]$$

The average angular velocity is ω for any half or full revolution. This is worked out on page 102.

Corresponding to these results, what are the average angular velocities (ω_s and ω_f) for fast and slow quadrants?

$\omega_s =$

$\omega_f =$

What are the values of ω_s and ω_f when $\alpha = 0$? $\omega_s =$ ____________ $\omega_f =$ ____________. Do these agree with what you should expect from the R = 0 case? ____________. Is the average <u>velocity</u> in a "fast" quadrant always higher than in a "slow" quadrant? ____________

Suppose $\alpha = \sqrt{2}\,\omega_0$. What are the values of ω_s and ω_f?

$\omega_s =$

$\omega_f =$

$$\omega_s = \frac{\pi}{2t_s} = \frac{\omega}{1 + \frac{2}{\pi}\arctan\frac{\alpha}{2\omega}}$$

$$\omega_f = \frac{\pi}{2t_f} = \frac{\omega}{1 - \frac{2}{\pi}\arctan\frac{\alpha}{2\omega}}$$

$$\omega_s = \omega_f = \omega$$

Yes.

No ("slow" and "fast" refer to <u>angular</u> velocities).

When $\alpha = \sqrt{2}\ \omega_0$, $\omega = \frac{\omega_0}{\sqrt{2}}$, $\frac{\alpha}{2\omega} = 1$

Thus, since $\arctan(1) = \frac{\pi}{4}$

$$\omega_s = \frac{2}{3}\omega$$

$$\omega_f = 2\omega$$

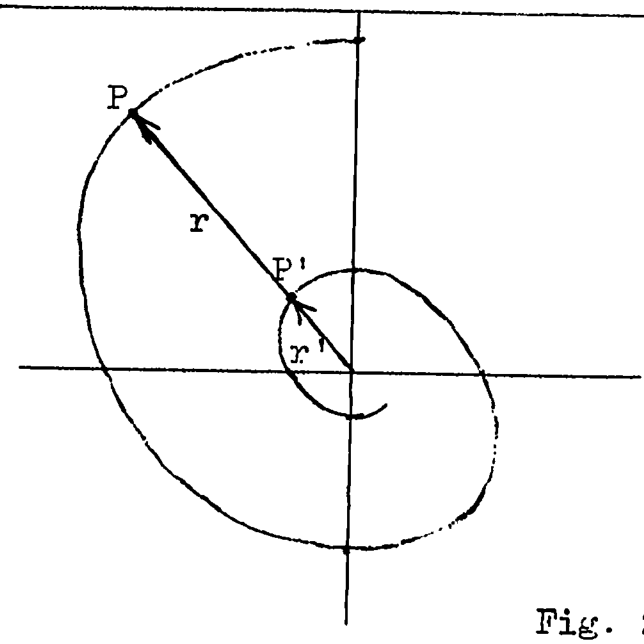


Fig. 26

Review

$$x_1 = -Ae^{-(\alpha/2)t}\sin\omega t$$

$$x_2 = Ae^{-(\alpha/2)t}\left[\frac{\omega}{\omega_0}\cos\omega t + \frac{\alpha}{2\omega_0}\sin\omega t\right]$$

Review

The solutions, which are repeated on page 108, include terms like $\sin \omega t$ and $\cos \omega t$. The parameter ω is related to the motion of a radial line to the state point. Choose the correct description of that relationship:

1) ω is the angular velocity at any instant;
2) ω is the angular velocity averaged over any quadrant;
3) ω is the angular velocity averaged over any half revolution.

In Fig. 26 let P be any state point, and P' the corresponding point on the next "branch" of the spiral, and let r and r' be the respective radii to P and P'. In going from P to P', ωt increases by ________________.

Which of the following statements is true?

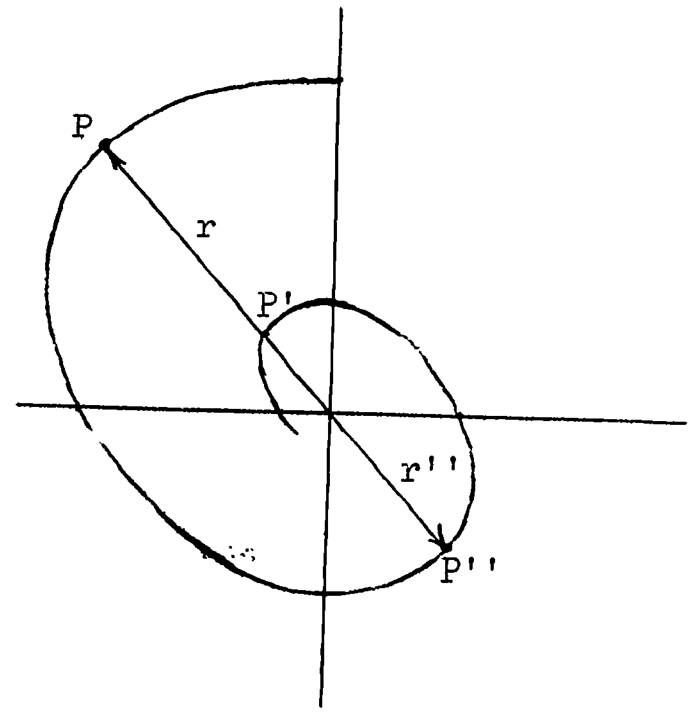
1) r'/r decreases exponentially with t.
2) r'/r is constant (independent of t).
3) r'/r increases exponentially with t.

Statement (3) is correct.

$\omega t = 2\pi$

Statement (2) is correct.

If you got any of these wrong, or do not thoroughly understand why you made correct choices, go to page 111. Otherwise, go to page 113.



You were directed to this frame because you had difficulty with the questions on page 109.

First question: If you got this wrong, go back to page 99. You missed something.

Second question: The average angular velocity from P to P' is ω. Thus, $\omega \times$ (elapsed time from P to P') = 2π, which is also the increase of ωt.

Third question: When ωt increases by 2π, t will increase by $2\pi/\omega$, and x_1 and x_2 will each decrease by $e^{-(\alpha/2)t} = e^{-\alpha\pi/\omega}$. Thus, since they both decrease by the same factor, $r = \sqrt{x_1^2 + x_2^2}$ will also decrease by this factor, which is a constant.

As a check, determine the ratio r''/r, referring to the figure on the opposite page:

$e^{-\alpha\pi/2\omega}$

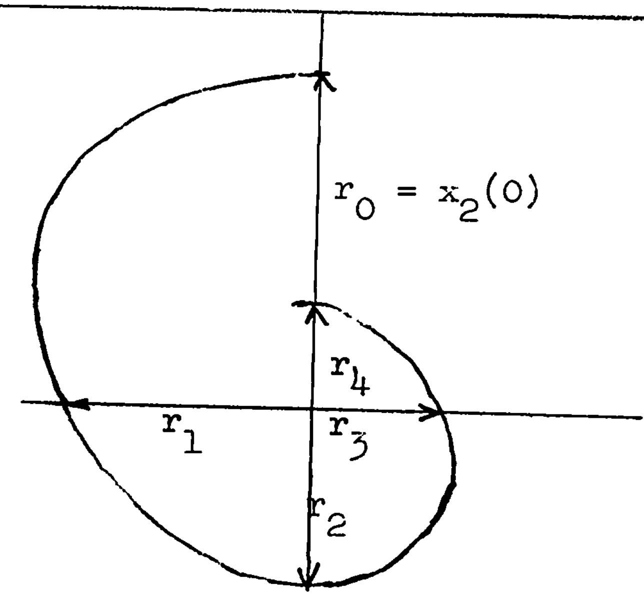


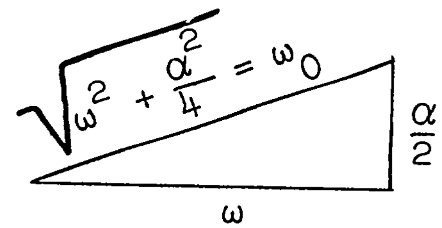
Fig. 27

To find r_1, assuming r_0 is for $t = 0$.

At r_1, $t = t_s = \frac{1}{\omega}(\frac{\pi}{2} + \arctan \frac{\alpha}{2\omega})$ and

$r_1 = -x_1 = Ae^{-(\alpha/2)t_s} \sin \omega t_s$.

We have $\sin \omega t_s = \sin(\frac{\pi}{2} + \arctan \frac{\alpha}{2\omega}) = \cos(\arctan \frac{\alpha}{2\omega})$



Referring to the figure, $\cos(\arctan \frac{\alpha}{2\omega}) = \frac{\omega}{\omega_0}$.

Thus,

$$r_1 = A(\frac{\omega}{\omega_0})\ e^{-(\frac{\alpha}{2\omega})(\frac{\pi}{2} + \arctan \frac{\alpha}{2\omega})}$$

PART IV. PLOTTING THE FUNCTIONS

Now let us consider ratios of radius lines to points where the trajectory intersects the coordinate axes, like r_1, r_2, r_3 and r_4 in Fig. 27. Knowledge of these ratios can be helpful in sketching trajectories without detailed calculations. Let r_0 correspond to the initial point. Thus, $r_0 = x_2(0)$.

Now refer to the derivation of r_1 on page 112, and answer the following:

Why is $r_1 = -x_1$? ______________________________

Explain the steps in arriving at $\sin \omega t_s = \omega/\omega_0$:

Why does $\sin(\frac{\pi}{2} + \arctan \frac{\alpha}{2\omega}) = \cos(\arctan \frac{\alpha}{2\omega})$? ____________________.

Where does $\omega_0^2 = \sqrt{\omega^2 + \alpha^2/4}$ come from? ____________________.

In terms of $x_2(0)$, A = ____________________.

Finally,

$$\frac{r_1}{r_0} = \qquad\qquad \text{(formula)}$$

What is the value of this ratio when $\alpha = \sqrt{2}\,\omega_0$? ____________________.
(number)

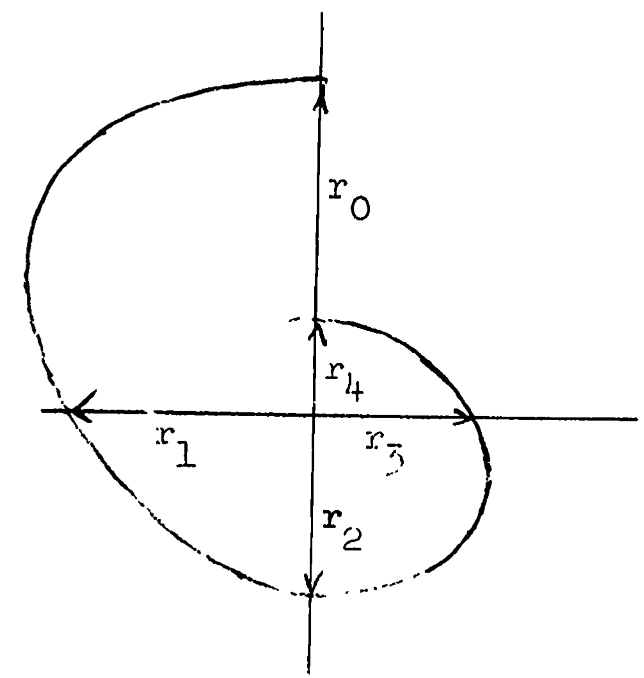
$r_1 = -x_1$ because r_1 is a length (a positive number) and x_1 is a negative intercept.

$$\sin(\frac{\pi}{2} + \arctan\frac{\alpha}{2\omega}) = \sin\frac{\pi}{2}\cos(\arctan\frac{\alpha}{2\omega}) + \cos\frac{\pi}{2}\sin(\arctan\frac{\alpha}{2\omega})$$

ω was defined as $\sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$

$$A = x_2(0)\frac{\omega_0}{\omega}$$

$$\frac{r_1}{r_0} = e^{-(\frac{\alpha}{2\omega})(\frac{\pi}{2} + \arctan\frac{\alpha}{2\omega})}$$



when $\alpha = \sqrt{2}\,\omega_0$, $\omega = \sqrt{2}\,\omega_0$

$$\frac{r_1}{r_0} = e^{-(.5)(\frac{\pi}{2} + \frac{\pi}{4})} = e^{-1.18} = .307$$

To find r_2/r_1, it is recalled that r_2/r_0 has previously been determined to be $e^{-\alpha\pi/2\omega}$ (see pages 111 and 112). Thus,

$$\frac{r_2}{r_1} = \qquad \text{(formula)}$$

Recalling that r_2 can be an initial point, what do you conclude about the ratios r_3/r_2 and r_4/r_3?

$\frac{r_3}{r_2} =$ (this is typical of all ______ (slow/fast) quadrants)

formulas

$\frac{r_4}{r_3} =$ (this is typical of all ______ (slow/fast) quadrants)

Also

$$\frac{r_4}{r_0} = \qquad \text{(formula)}$$

$$\frac{r_2}{r_1} = \left[\frac{e^{-\frac{\alpha\pi}{2\omega}}}{e^{-\left(\frac{\alpha}{2\omega}\right)\left(\frac{\pi}{2} + \arctan\frac{\alpha}{2\omega}\right)}}\right]$$

$$= e^{-\left(\frac{\alpha}{2\omega}\right)\left(\frac{\pi}{2} - \arctan\frac{\alpha}{2\omega}\right)}$$

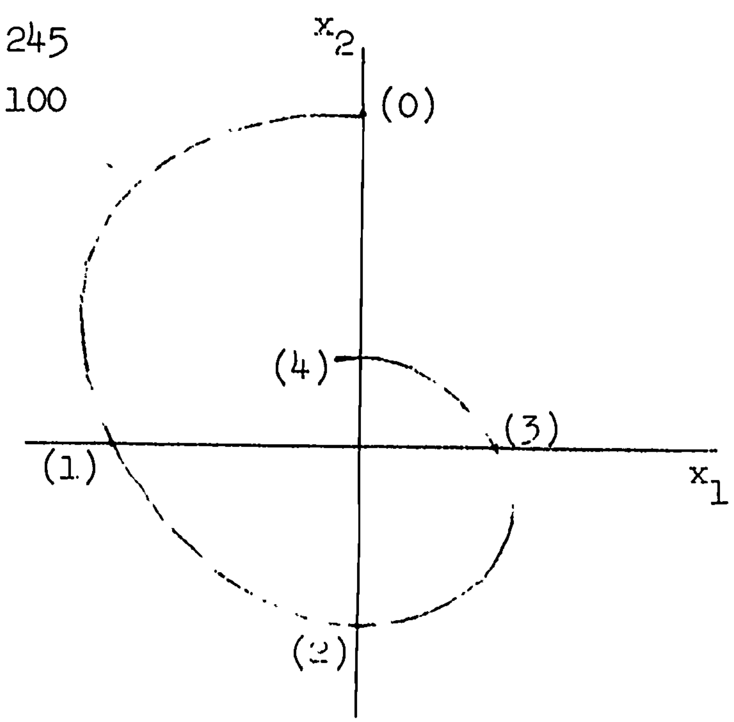
$$\frac{r_3}{r_2} = \frac{r_1}{r_0}$$ (or the formula).

for slow quadrants

$$\frac{r_4}{r_3} = \frac{r_2}{r_1}$$ for fast quadrants

$$\frac{r_4}{r_0} = e^{-\alpha\pi/\omega}$$

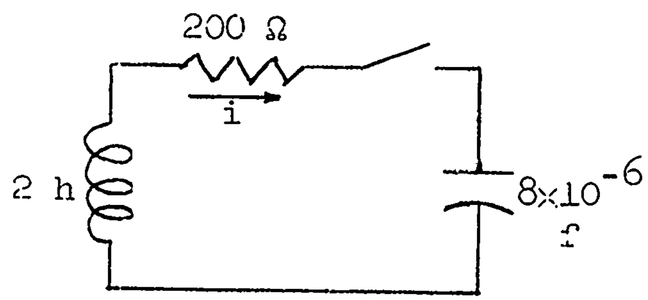
$\alpha = 100$
$\omega_0 = 250$
$\omega = 245$
$x_2(0) = 100$



For purpose of summary, let us return to the numerical case we considered previously, the values for which are repeated on page 116. Calculate the quantities required to complete the table below.

Point	t(millisec.)	Intercept
1		
2		
3		
4		

1.	7.25	.695
2.	12.82	.527
3.	20.07	.366
4.	25.64	.278



Initial capacitor voltage = 100

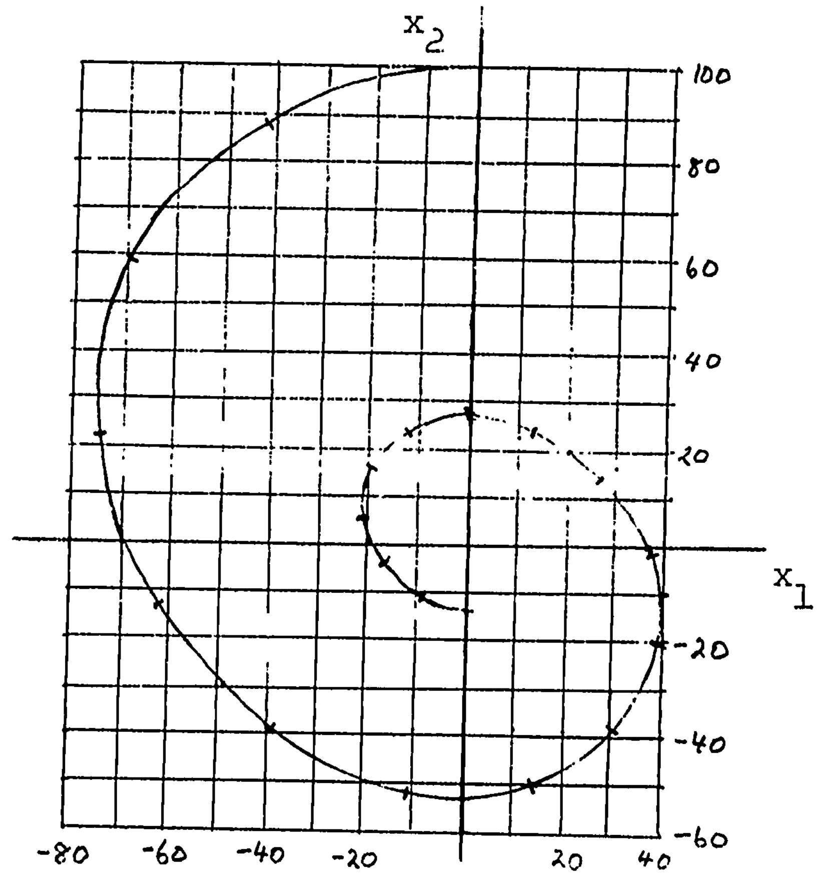


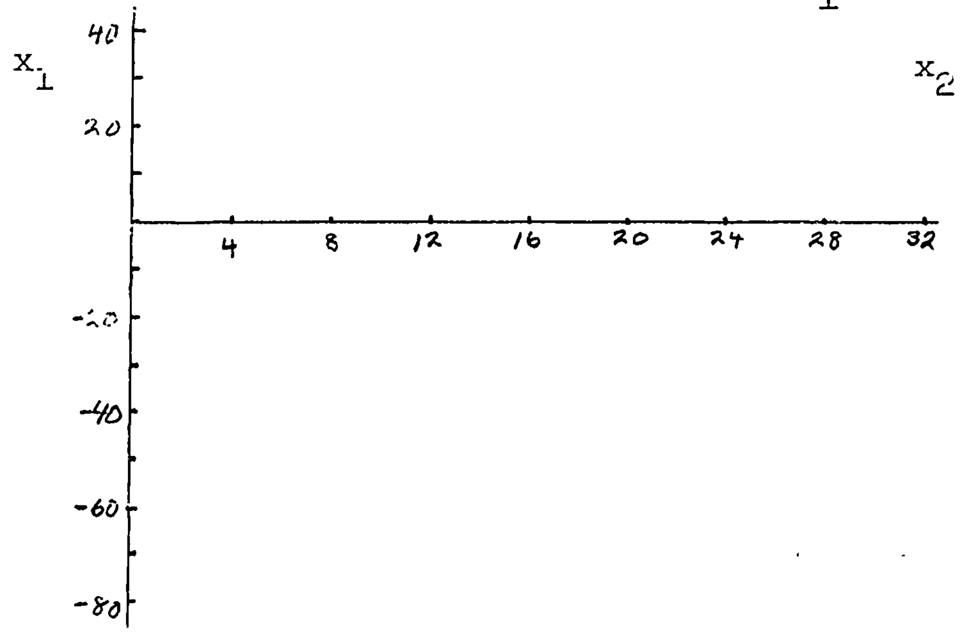
Fig. 28

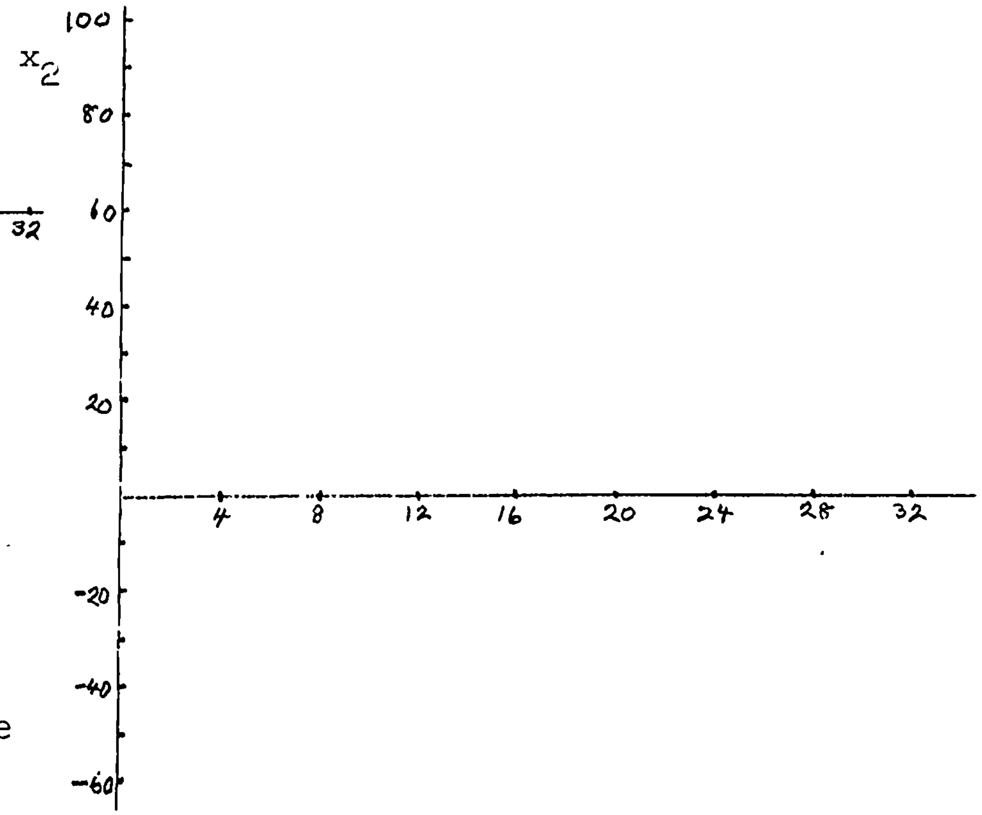
Calculated curve for date of circuit shown

Marks on curve at intervals of .002 sec.

At last! On page 118 you see an accurately drawn trajectory for this numerical case. This curve was produced by a computer.

On the axes below, plot x_1 and x_2 as functions of t.

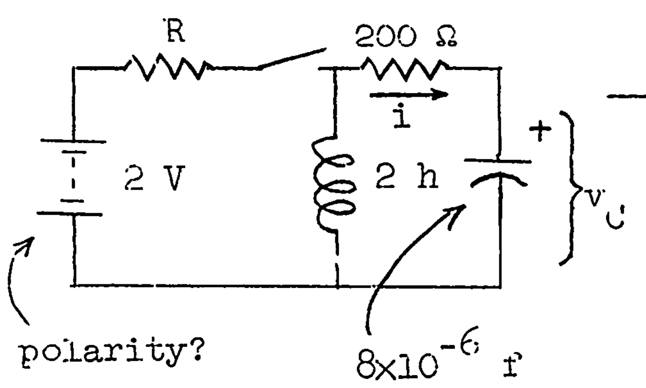




Relabel the x_1 and x_2 axes so these curves will serve for i and v_C of the circuit on page 118, with initial $v_C = 50$.

120

If you have any difficulty in obtaining these curves, seek help.



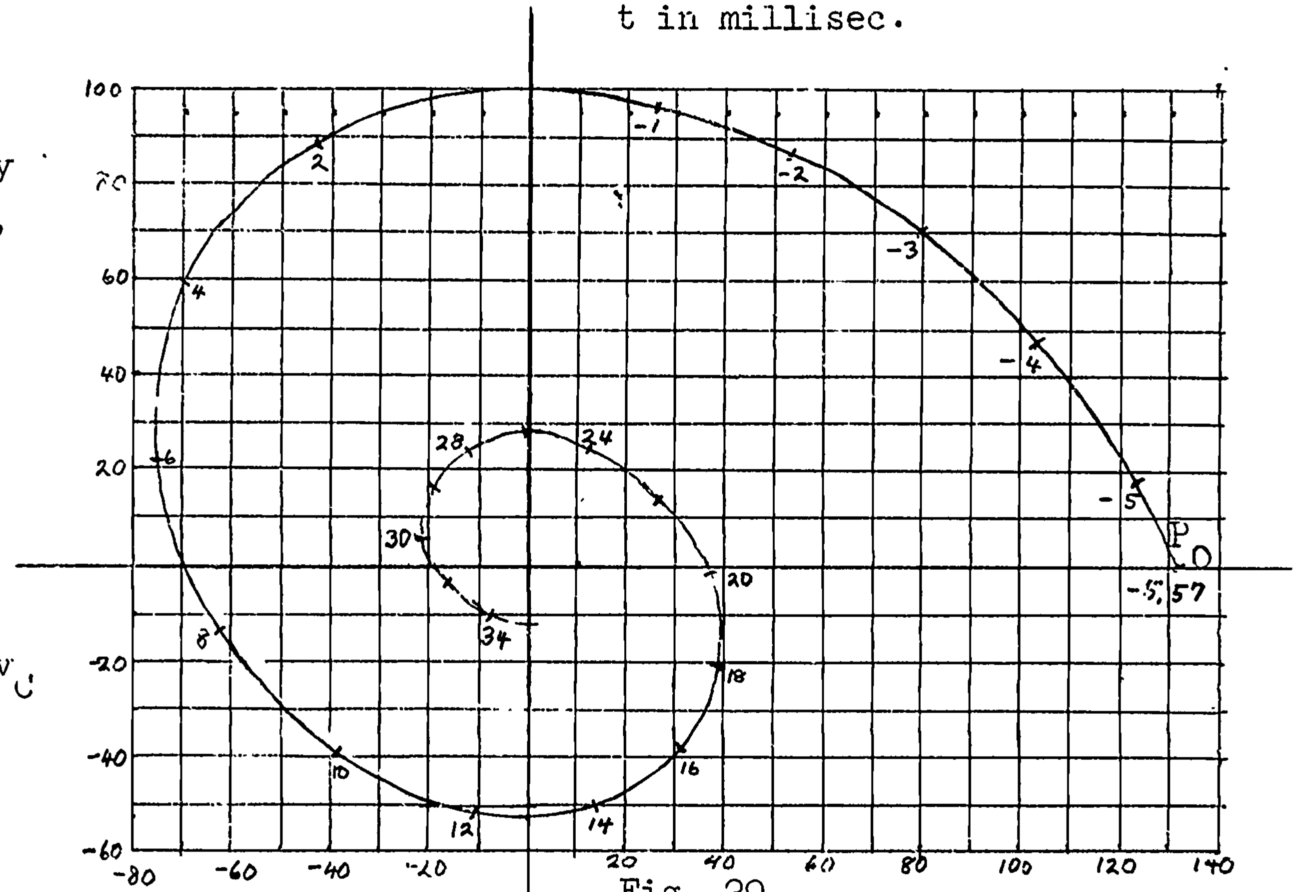


Fig. 29

Calculated curve, carried back to t = -5.57

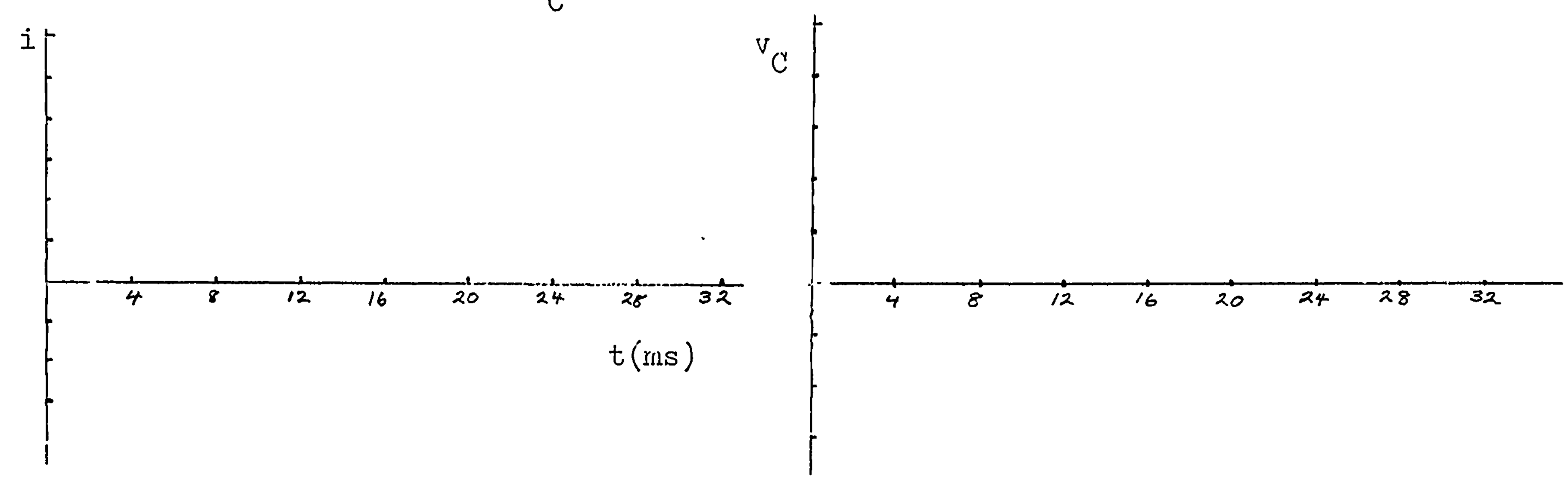
Figure 29 is a repetition of Fig. 28, but including a section calculated for negative t back to t = -5.57 ms. Consider the use of this curve to analyze the circuit on page 120.

Do the following:

1. Mark polarity on the battery, and calculate the value of R to yield point P_0 as the initial condition of this curve.

2. Relabel the time marks such that P_0 corresponds to t = 0.

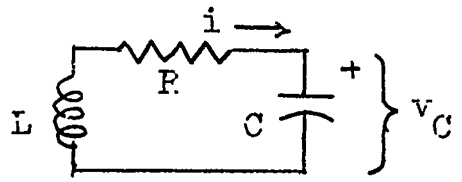
3. Plot curves of v_C and i on the axes below.



122

Refresher

For this circuit



The equations are

$$\frac{dx_1}{dt} = -\alpha x_1 - \omega_0 x_2$$

$$\frac{dx_2}{dt} = \omega_0 x_1$$

where

$$x_1 = \frac{L}{C} i$$

$$x_2 = v_C$$

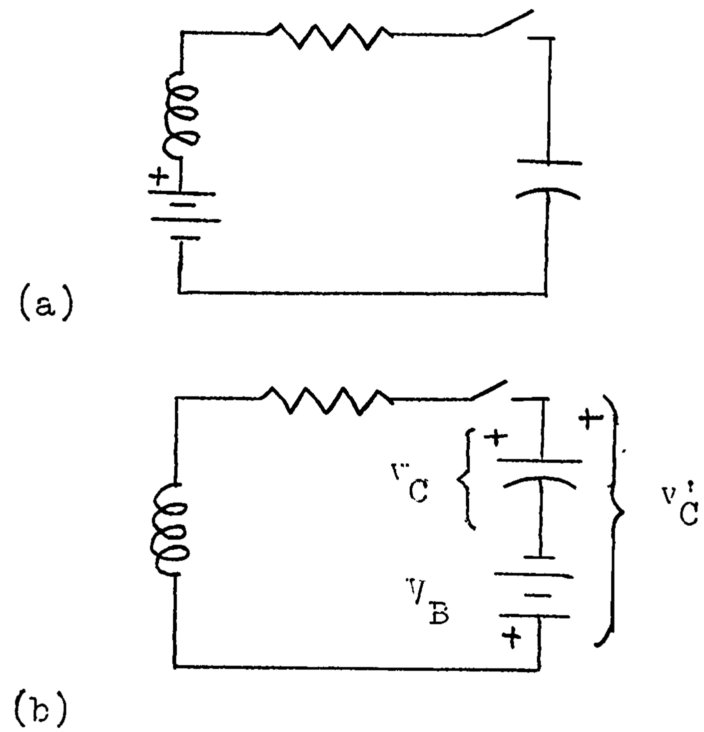


Fig. 30

One circuit arrangement of practical important shown in Fig. 30 has not yet been considered. The battery can appear anywhere in the circuit. Fig. 30(a) is one example. However, wherever it is, for purpose of analysis we shall place it as in Fig. 30(b), and define a voltage v_C'.

Observe the "refresher" on page 122, and decide how to redefine x_2 so that the same equations will apply to Fig. 30(b)

$x_2 =$

Perhaps you will immediately see that

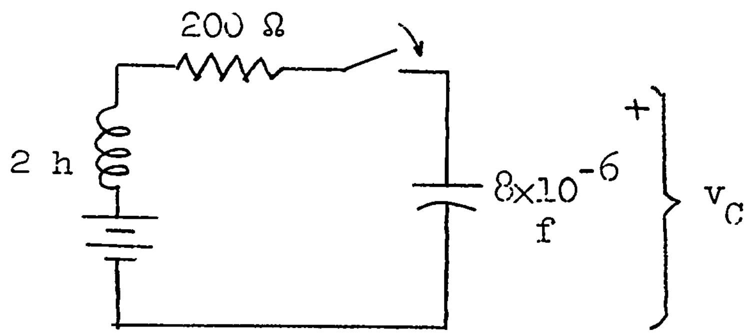
$$x_2 = v_C' = v_C - V_B.$$

If not, go back to the circuit equations

$$L\frac{di}{dt} = -Ri - v_C'$$

$$C\frac{dv_C}{dt} = C\frac{dv_C'}{dt} = i$$

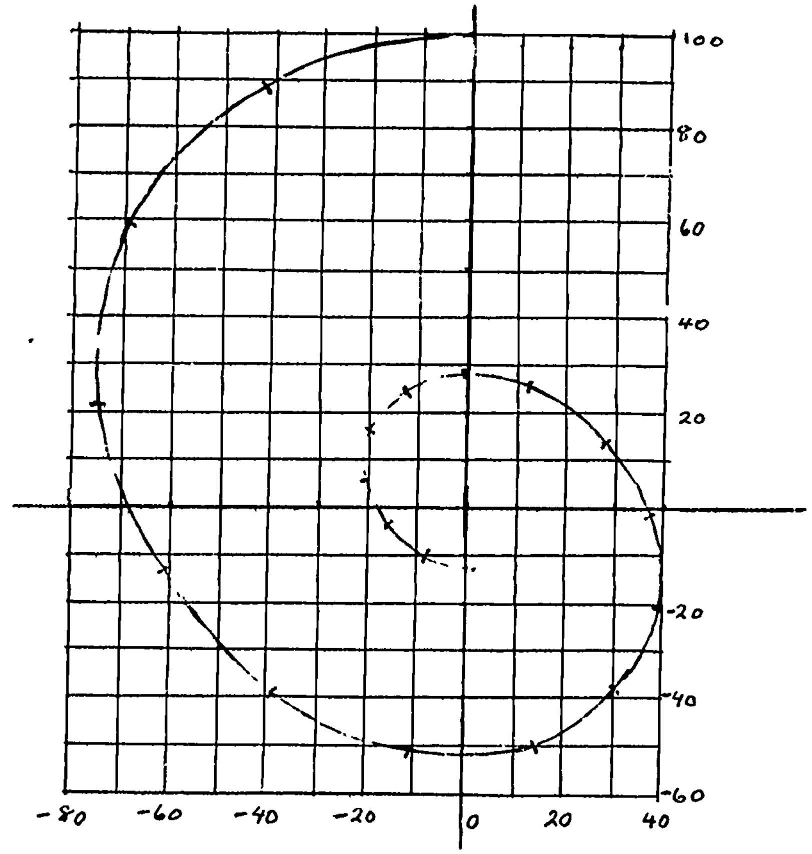
When the variables are changed, the original equations will result.



Capacitor initially uncharged

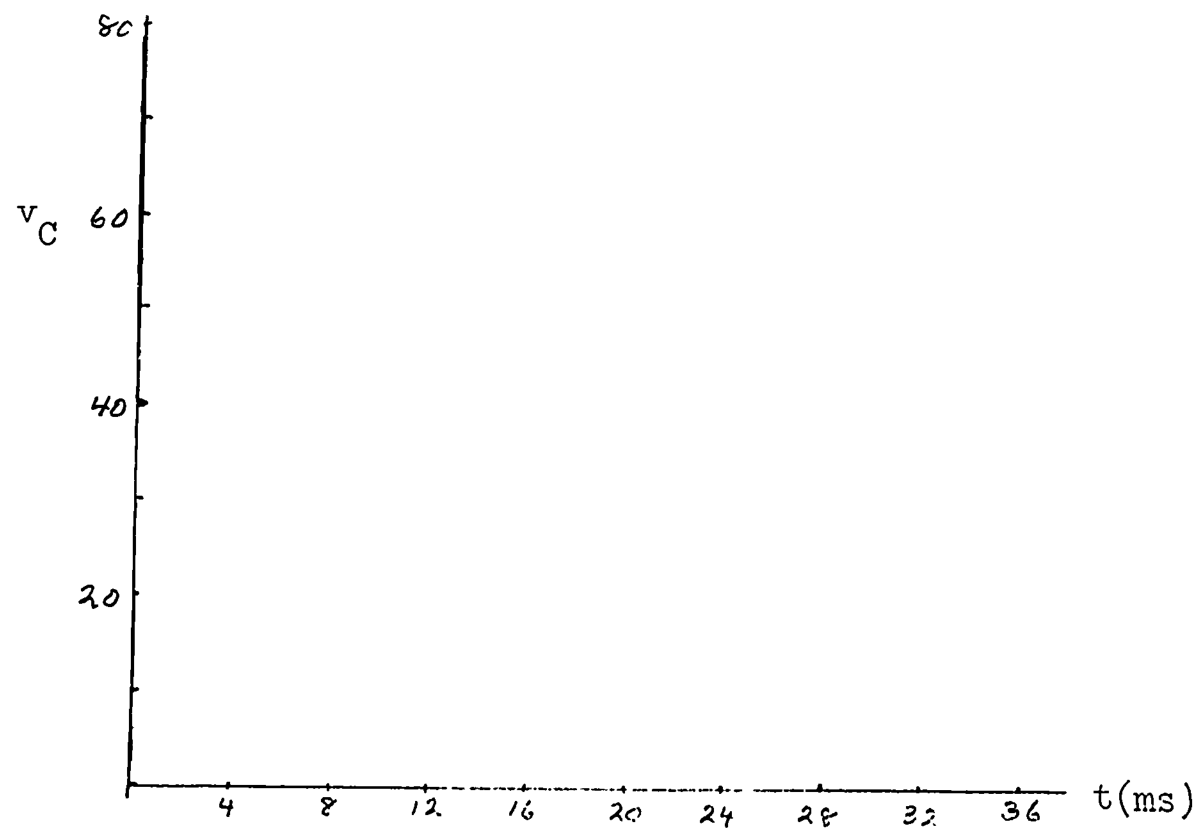
t marked at

.002 sec. intervals

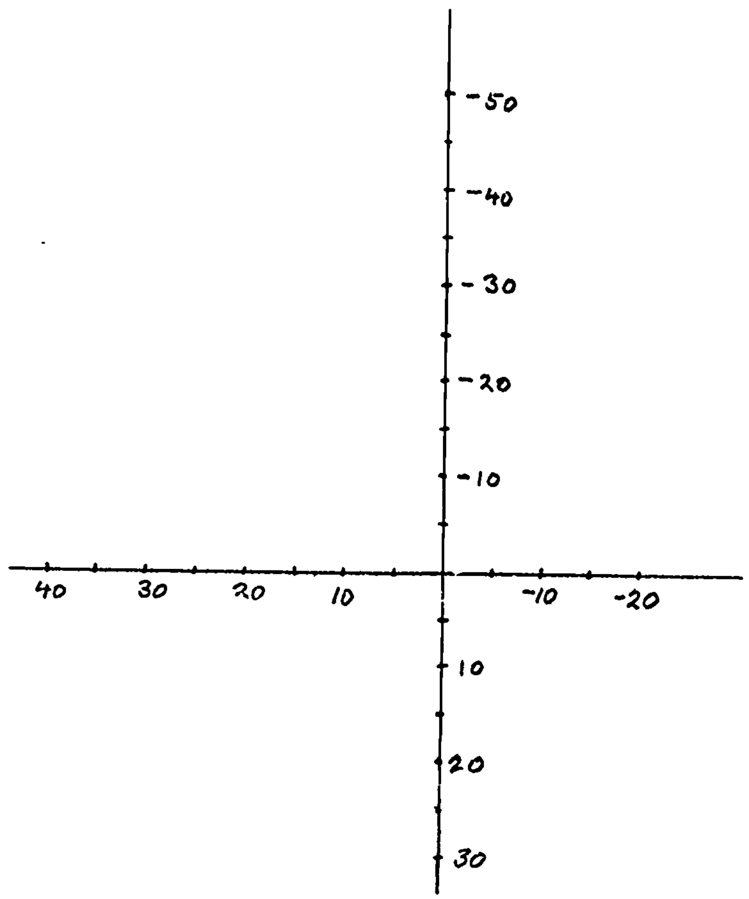


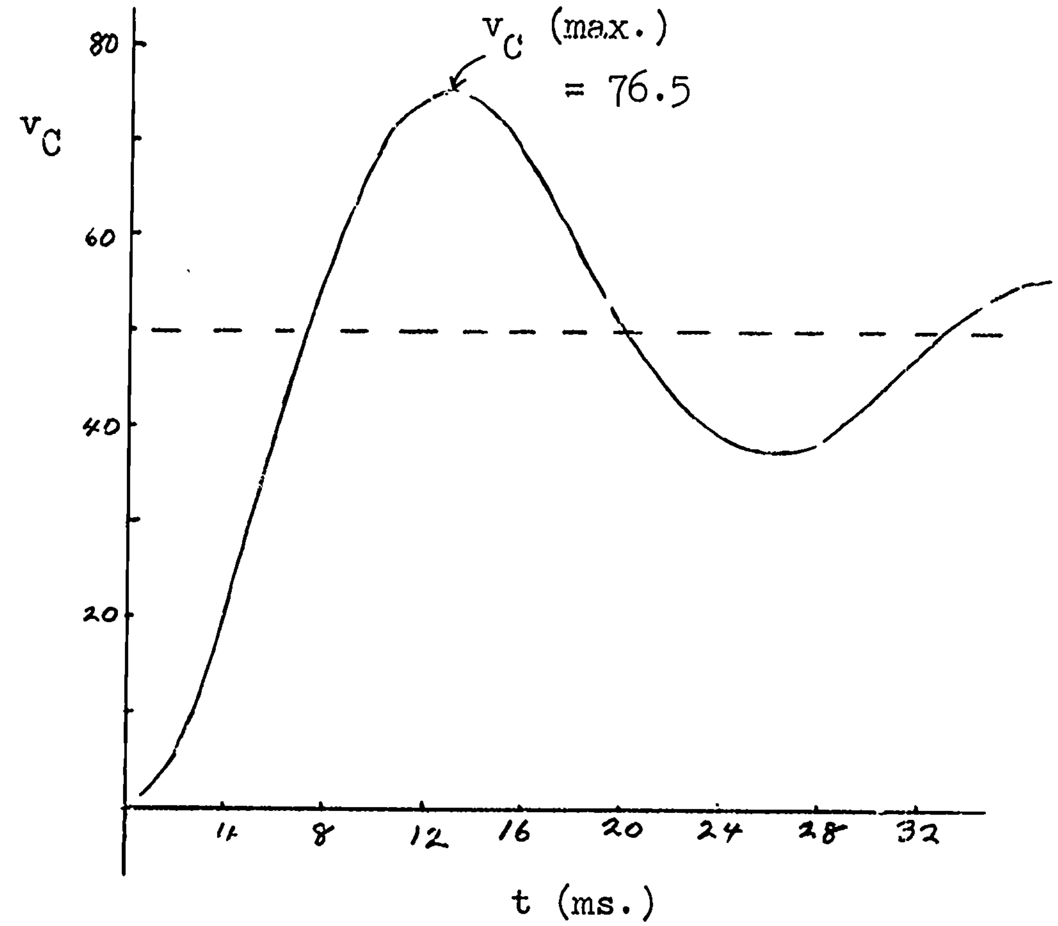
Relabel the axes with numbers and signs

Figure 28 is repeated on page 124. Use it to obtain v_C for the circuit shown. The trajectory curve can be used directly in this case, with $x_2 = v_C - V_B$ (or v_C - ________) if the axes are relabeled. Perform the required relabeling, clearly indicating positive directions. Plot the curve of v_C below.
number



What is the maximum value of v_C?





For this problem, x_2 begins at -50

3