DOCUMENT RESUME

ED 025 446 24 SE 006 273

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The Teaching of Two Concepts in Beginning Calculus by Combinations of Inductive and Deductive Approaches. Final Report.

Illinois Univ., Urbana.
Spons Agency- Office of Education (DHEW), Washington, D.C. Bureau of Research.
Bureau No- BR-6-8651
Pub Date Jun 68
Grant- OEG-2048-99-7
Note- 363p.
EDRS Price MF-$1.50 HC-$18.25
Descriptors- *Calculus, Comparative Analysis, Grade 11, Grade 12, *Instruction, *Mathematics, *Programed Instruction, *Secondary School Mathematics, Teaching Procedures

The purpose of this study was to determine if different teaching approaches produced differences in learning the limit and the derivative concepts in beginning calculus as measured by a common criterion test. An ordered combination of two teaching approaches--the concrete inductive and the abstract deductive--was used for the two sequential topics involving the limit and the derivative concepts. All four possible pairings for the two units were considered and constituted the four experimental treatments. Specifically, the problem in this study was to determine if there were a statistically significant difference in the four treatments. Programed units were used to control the teacher variable. The units were read by advanced high school mathematics students, who were divided into high and low achievers on the basis of pretest scores. Contrary to the pilot study and related research results, when a difference in teaching methods existed in the derivative and total treatment studies, the deductive method was favored. Further correlation and regression analyses revealed that a student's prior mathematical knowledge, a" measured by the pretest and limit test, was the determining factor in predicting the limit and derivative test scores, even though the deductive treatment was found to be superior. (RP)

BR-6-8651
PA-24
OEBR

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June, 1968

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

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Lois M. Lackner

University of Illinois

Urbana, Illinois

June 28, 1968

The research reported herein was performed pursuant to a grant with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

U.S. DEPARTMENT OF
HEALTH, EDUCATION AND WELFARE

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ACKNOWLEDGEMENTS

The investigator wishes to thank Professor Robert E. Pingry, who served as major adviser and chairman of the thesis committee. He has given generously of his time, effort, and guidance throughout this study. Professors Josephine Chanler, Kenneth B. Henderson, Stewart Jones and Wilson M. Zaring, committee members, were helpful in the writing of the text material for this experiment. They also gave useful suggestions for the statistical design and analyses in the three studies in the experiment.

Mr. Robert Rundus, mathematics teacher at Urbana Senior High School, Urbana, Illinois, Mr. Allen Holmes, mathematics teacher at University High School, Urbana, Illinois, Mr. Dale Haywood, mathematics teacher at Bremen High School, Midlothian, Illinois, and Dr. Ronald Ady, education professor at Illinois Teachers College-South, Chicago, Illinois provided classes for the pilot studies leading to the final programmed text materials used in the experiment.

Dr. Ronald M. Shelton wrote the inductive and deductive programmed limit texts used in this study. He was helpful in giving guidance for the writing of the inductive and deductive derivative units used in this experiment. His experiences with pilot studies and his experiment comparing inductive and deductive teaching of the limit concept in beginning calculus were heavily relied upon in the investigator's study.

Mrs. Ronald M. Shelton gave generously of her time to meet typing deadlines of the text material and the thesis.

Superintendents and administrators of Chicago suburban high schools helped secure the appropriate number of students for the experiment. So helpful were they that some high schools had to be refused participation. For those schools that did participate, thanks go to the mathematics department chairmen and teachers who cooperated so willingly in the preliminaries of testing, the administration of the programmed materials, and the reporting of test scores. These chairmen and teachers are:

Mrs. Helen N. Antich, Mr. Brendan J. Flynn, Miss Joyce Haerther, and Mr. James Ulrich, Arlington Heights High School, Arlington Heights, Illinois,

Mr. Edward Paine, Downers Grove North High School, Downers Grove, Illinois,

Mr. Russell Lundstrom and Mr. Richard Wyllie, Downers Grove South High School, Downers Grove, Illinois,

Mr. Alfred Umberger, Forest View High School, Arlington Heights, Illinois,

Mr. Reino Takala, Hinsdale Township High School, Hinsdale, Illinois,

Mr. Ray Janata, Rich Township High School, Park Forest, Illinois,

Mr. John Flannigan, West Leyden High School, Northlake, Illinois,

Mr. Leon Benson, Miss Janice Flake, Mr. Ted Nelson, Mr. Donald Peterson, and Mr. Anthony Portincaso, York Community High School, Elmhurst, Illinois.

Samuel A. Mueller and Donald E. Betts performed the computer analysis of data for the pilot studies. John Keck developed the analysis of covariance program used in the three studies in the experiment.

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CHAPTER I

INTRODUCTION

Orientation to the Problem

Since mathematics, and the sciences in general, are becoming increasingly important subject areas in the school curriculum on all educational levels, much revision and restructuring in content has been and is being done. Such reorganization is evidenced by the many experimental programs in the "new math." Such programs are the University of Illinois Committee on School Mathematics (UICSM), the Madison project at Webster College in Missouri, the School Mathematics Study Group (SMSG) at Stanford University, the Ball State University experimental mathematics program in Muncie, Indiana, and the Minnimath Program at the University of Minnesota. New approaches to content presentation in physics (Physical Science Study Committee - PSSC), Biology (Biological Sciences Curriculum Study - BSCS), and chemistry (CHEM Study Program) are also being tried.

In the field of mathematics, Marks, Purdy and Kinney discuss this activity toward reorganization of content:

> The school mathematics curriculum at every level from the primary grades through the high school and college presents a scene of reexamination and innovation. Different emphases, different grade placement, new topics, changed methods for promoting learning, and new materials - both textbooks and multisensory aids - are characteristic at all levels. (37, p. 1)

Spencer and Brydegaard say:

> Today's school program emphasizes problem-solving in which the learner is challenged to question, to experiment,

> and to explore, in order to find basic ways of attacking problems. Learning is conceived to be a personal achievement that can be accomplished only through problem-solving behavior on the part of the learner. (65, p. v)

If one glances through a mathematics textbook with a current publication date, he finds very different content and format than was used in a text in the same area ten years ago. Mathematics textbook authors admit the influence of the experimental work cited above. Robison states in the preface of a college text in modern algebra and trigonometry:

> In selecting the content of this book, I have been guided by the recommendations of various curriculum study groups, such as the Committee on the Undergraduate Program in Mathematics (CUPM). The textbooks of the School Mathematics Study Group (SMSG) influenced much of the writing. (51, p. 7)

In the preface of a beginning college text that uses a somewhat rigorous approach, Britton, Kriegh and Rutland state:

> This is the first of two volumes that are intended to provide college and university students with a sensible continuation of the modern approach to mathematics that is being introduced in most elementary and secondary schools, with more emphasis than in the past placed on an understanding of fundamental concepts. Certain advanced topics in algebra and trigonometry, along with analytic geometry and calculus, are unified into a sequential exposition that eliminates much unnecessary duplication and is conducive to an efficient development and use of ideas and techniques. Fundamental concepts are discussed in a reasonably rigorous fashion, with adequate emphasis on important skills, and without an excess of sophistication. Many applications of mathematics have been included, and they have frequently been made the motivation for the introduction of mathematical concepts. An intuitive discussion often precedes the formal treatment of a new idea. (8, preface)

Although content reorganization is important for effective teaching, so is teaching method. We often hear Bruner's thesis debated: a child can learn any topic if it is presented in an "intellectually honest" manner. (9) In mathematics, in particular, Kaplan believes that not only is teaching method important, but that it has changed as a result of change in content. He writes:

> By now, most people are aware that a "revolution" has taken place in the teaching of arithmetic - mathematics - in our schools. This revolution is not only of content but of approach. (26, preface)

Yet, educators would not dispute the fact that content and method are intimately related. Swenson concludes:

> True, some teachers do overemphasize subject matter; they do act as if the arithmetic in and of itself were more important than the people who are to learn arithmetic. The critics of this attitude, however, are just as naive when they declare that we should teach children instead of arithmetic. Both these points of view lead us nowhere; they advocate purpose impossible of achievement. (68, p. 12)

Others feel that personality and rapport of the teacher are as important as content and teaching method. This is true because the latter creates positive or negative attitudes toward the subject area. In a doctoral study dealing specifically with attitudes of prospective elementary school teachers, Purcell concludes:

> With students entering (college) with better preparation in mathematics and the resulting more favorable attitudes (as shown by the study), it will be possible to increase these favorable attitudes even more as a result of planned college instruction directed at changing attitudes toward elementary mathematics. The teacher shortage will continue, but the experimental study results give indications that more favorable attitudes are being achieved. Thus, the teachers entering on their careers have more favorable attitudes toward elementary mathematics. (48, p. 89)

In summary, it can be said that a combination of effective teaching method, wholesome teacher personality, and appropriate content is to be sought in any satisfactory pupil-teacher relationship. Johnson supports this statement.

> Mathematics instruction must do more than build an understanding of the logical structure of mathematics, even while acknowledging that this is the basic foundation for understanding mathematics.
>
> The mathematics program must in addition, strive for broader objectives, such as creativeness, habits, attitudes and values - objectives which are increasingly difficult to attain and to measure. (24, p. 185)

Let us now look more closely into the area of teaching methods, considering the teaching of advanced mathematics topics in particular. We can ask, "If subject matter is taught by two different teaching approaches, by which approach will a student learn more effectively?" Such a question is empirically verifiable and researchable.

For the teaching of advanced topics of a discipline, teaching methods have rarely been investigated. Such an investigation, if conducted at all, has usually been secondary to some other main area of investigation. (11) Perhaps little research exists in this area because it is felt that those studying an advanced topic are above-average in intelligence (however measured), highly motivated, and will learn under almost any reasonable teaching conditions. In a study by Ady at the University of Wisconsin, an experimental group and a control group of student teachers in an advanced education course were given two review methods over lecture material. He concludes:

> Although there were no measurable differences in learning results between the visually programmed self-evaluation item group and the verbal leaderless discussion group, the subjects did express different opinions about these instructional methods. The visual programmed item group believed their method was "organized," "helpful," and contributed to "high learning" and "high retention." On the other hand, the verbal leaderless discussion group believed their method was "lively," "interesting," "good," and "intelligent." (1, p. 127)

Also, those teaching advanced topics may not have teaching as their primary concern. As Sharp states:

> . . . I can think of at least three reasons for university teachers neglecting undergraduate students; . . . 2) Despite pious declarations in faculty handbooks and administrative speeches good teaching is not rewarded. If one wants to get ahead, one publishes; dull or disorganized teaching will not hurt one's chances much, and successful teaching will be noticed only by the students. (59, p. 75)

Yet, this investigator believes that even in an exact discipline such as mathematics, a highly motivated student may experience more permanent and meaningful learning of an advanced topic by one teaching approach than by another. The investigator also believes that when superior teaching approaches of advanced topics are used, good teaching will be recognized and rewarded.

Review of the Literature

There is a decided absence of empirical studies relating to the teaching of advanced topics in mathematics including the teaching of calculus.

In the field of science, however, two studies are relevant. A study by William Schefler investigated the teaching of college freshman biology by an inductive laboratory approach and a traditional lecture approach. Four groups of students were taught by two instructors. Each instructor had both an experimental and a control group. A pre-test on genetics was given. The original hypothesis favored the inductive method. An analysis of covariance was performed at the .05 level of significance, investigating the following hypothesis:

> When an inductive laboratory approach to teaching a unit on genetics is compared with a traditional lecture-illustrative

> laboratory approach, students taught by the inductive laboratory approach will show a significantly greater achievement in terms of the following criteria:
>
> (1) Knowledge of facts and principles, and their applications, as measured by a genetics test.
>
> (2) An understanding of the nature and methodology of science, as measured by the "Test on Understanding Science."
>
> (3) An interest in science, as measured by the scientific subscale of Kuder Preference Test, Form C.
>
> (4) A positive attitude toward scientists and science, as measured by an application of the semantic differential technique. (56, p. 7)

A significant difference was found in group test scores only on the basis of instructors. No support for the original hypothesis was found. Schefler concludes:

> Referring back to the main hypothesis of this study, it is apparent that within the limitations imposed on the data, the hypothesis concerning a significant difference in achievement between students taught by an inductive laboratory method and those taught by a traditional lecture-illustrative laboratory approach is not supported by the evidence. Specifically, this study has provided no evidence that the experimental method was superior to the control method in terms of achievement as measured by the specified criterion instruments. It should also be reiterated that this study has at the same time produced no evidence that the inductive laboratory approach is inferior to the traditional method used.
>
> The evidence provided by the data suggests that the effects of teacher difference may be of greater significance than the effects of method difference, and suggests a possible need for further research in the area of teacher effects on achievement. (56, p. 54)

In the area of high school chemistry, O'Connell compared an inductive and deductive teaching approach. She concluded that inductively taught students had a more thorough knowledge of chemistry than those taught deductively. (44, p. 1679)

Wallace studied the effects of two self-instructional methods of improving spelling in high school and college. A traditional deductive text and a programmed inductive text, of 103 frames, were used with 606 high school and college students in 26 paired experimental classrooms. One member of the pair used the traditional text and the other the programmed text. The following conclusions were formulated as a result of the study:

> 1. An analysis of covariance on raw scores on the Traxler High School Spelling Test, Form I, before instruction, combined with the mean score on 13 tests during instruction showed no significant difference for method alone.
>
> 2. An analysis of variance on Terminal Traxler, Form 2 test scores after instruction showed the boys with the programmed text made higher scores, significant at the .05 level. Such a finding was absent for the girls in the study.
>
> 3. An analysis of variance showed girls were better spellers before, during and after instruction than boys.
>
> 4. General improvement in spelling was found, due to students' self-instructional efforts, irrespective of method. Effort seemed more decisive than method. (72, p. 5801)

For studies investigating different teaching approaches in advanced topics in mathematics, we can cite Shelton's findings for his hypotheses on teaching the limit concept in beginning calculus by an inductive and deductive approach. He concludes:

> Any generalizations based on results of this study must be made with caution, but for the particular population, treatments and criterion test used in this study the following conclusions were drawn:
>
> 1. No advantage in achievement of either treatment program was apparent.
>
> 2. No difference in achievement between the two levels was found.

3. No advantage in achievement of either treatment program for a particular level was apparent. (60, p. 61)

Kenneth Cummins replicated a study in the teaching of selected topics in calculus at the secondary level for a one-quarter course.

> The one-quarter experimental sections were conducted for two quarters in an atmosphere rich in encouraging discovery, whereas the control groups were taught more or less traditionally by capable men of long university teaching experience. The same text was used in all sections. (14, p. 163)

Using a regression analysis involving previous grades, a pre-test, the American Council of Education Psychological Test, a test designated Test A for the traditional section, and a test designated Test One for the experimental section, the following results were obtained:

a. The students in the experimental group scored on the average 27.10 points higher on Test One than would be expected on the basis of their preliminary test scores (significant at the 1% level).

b. The students in the traditional group scored 51.59 points lower on Test One than would be expected (significant at the 1% level).

c. The difference was not significant on Test A. (14, p. 168)

Many articles by mathematics teachers, educators, and research personnel at all academic levels state views on inductive methods which are sometimes called discovery or heuristic methods and deductive methods of presentation. Let us consider some of the views of these authors. In questioning and supporting the value of both procedures, Clark writes:

> . . . Should both procedures be used? Most teachers today are searching for workable transitions from the informal-intuitive to the formal-deductive. This search is a

> significant aspect of the current "reform movement" in teaching elementary mathematics. (12, p. 99)

Courant writes:

> The interplay between generality and individuality, deduction and construction, logic and imagination - this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far-reaching development, all of them will be involved. Generally speaking such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy: after this flight comes specific goals in newly surveyed low plains of individual "reality." In brief, the flight into abstract generality must start from and return again to the concrete and specific. (13, p. 43)

To support an inductive method, Schlinsog states:

> Contemporary conceptions of teaching place less emphasis on the familiar "telling and showing" approach and more emphasis on student discovery. While psychologists have not devised an adequate theory of instruction and while they hold many conflicting ideas, there are some basic principles upon which they tend to agree. Readiness, motivation, exploration and discovery, feedback and reinforcement have been widely discussed elsewhere. (57, p. 293)

In a dialogue between a student and teacher involving the introduction of the commutative property of addition in the elementary school, Rupkey favors the inductive method. He wishes one to draw this conclusion when he questions:

> Both teachers taught topics from modern mathematics, but were both teaching modern mathematics? Can a teacher use a modern text and yet fail to accomplish the most important objectives of modern mathematics? Which method of teaching - inductive or deductive - is more useful in teaching modern mathematics? (52, p. 220)

In teaching the specific process of differentiation in calculus, Saxelby says:

> . . . it too often happens that a student . . . acquires a merely fatal facility in differentiation, regarding it as a mechanical juggling with symbols but having no conception of its relation to experience. (54, p. v)

He then adds, in support of an inductive approach,

> . . . this intuitional direct vision method is intended, not to take the place of, but to prepare the way for, a more rigorous analytical study of the subject. . . . The most natural method of advance is by a series of successive approximations to logical rigor, and, in fact, this is the way in which the subject has actually grown up. . . . The process by which the science itself was formed is also the most natural for the mind of the student. (54, pp. v-vi)

The number of empirical studies of inductive and deductive teaching methods is small, particularly in advanced mathematics topics. Thus, it might be helpful to consider the presentation of the topic of this study, the derivative, as given in calculus textbooks.

In these texts, the teaching of the derivative concept in calculus eventually presents the definition of the derivative as the limit of the difference quotient $\frac{\Delta y}{\Delta x}$, the derivative interpreted as the slope of the tangent line to a curve, and instantaneous velocity and general rate of change. The ancillary notions of limit and continuity are not considered. The deductive approach is considered as proceeding through the above topics in the order of the definition of the derivative, followed by one or all of its interpretations, in any order. The inductive approach begins with some or all of the applications of the derivative, followed by the definition of the derivative, again possibly followed by one or more of the remaining applications. Thus, we see that there can be a variety of both the inductive and deductive approaches.

Most textbooks in beginning calculus present the topic of the derivative in a deductive manner. This is probably a typical approach to the writing of a text book. Such an expository presentation is given for both ease of writing and conservation of space. The reader may be confronted with the phrase "the reader can easily prove" or a similar statement. In this case, a deduction is implied, for which the reader is to supply his own proof. Appropriate examples follow such a statement.

Or the reader may be confronted with the phrase "it is intuitively obvious" or a similar statement. Here a form of induction is sometimes implied. In these cases, the reader is asked to make the inductive leap to the desired generalization by a series of reasonable and "obvious" examples.

The investigator reviewed 33 calculus books dating from 1911 to the present. These books are listed in the bibliography. The topic of the derivative is deductively by 23, inductively by nine, and by a combined approach in one. This last approach, in Menger, Calculus, A Modern Approach, is on the abstract level, so one might say it is deductive. Yet, for the derivative in particular, the slope of the tangent line to a curve is used throughout the text as a way of introducing other more formal ideas, theorems and corollaries. For one familiar with Menger's approach to calculus, a further analysis is difficult, since his presentation is somewhat unconventional and cannot be categorized as clearly inductive or deductive as defined above.

Most revisions (second or third editions) are inductive or deductive according to the original printing. This is probably the case because revisions frequently involve only changes of language, increased precision of wording, small inserts of supplementary material, and correction of errors. The total structure remains essentially the same as the original edition. This we see in the deductive approach of Love, Differential and Integral Calculus (1943), Love and Rainville, Differential and Integral Calculus (1954), and Rainville, Unified Calculus and Analytic Geometry (1961), each of which follows the sequence of the definition of the derivative, slope of the tangent line to a curve, instantaneous velocity and acceleration, and related rates.

In Thomas, Calculus (1953) and Thomas, Calculus and Analytic Geometry (1962), inductive approaches follow the sequence of the slope of the tangent line to a curve, the definition of the derivative, instantaneous velocity, and related rates. Leighton, Calculus and Analytic Geometry (1960) reorders the chapters of Leighton, Calculus (1958), inserting a chapter on curve discussion before the derivative discussion. This is understandable since the added topic of analytic geometry is presented in the latter edition. In both editions, the derivative presentation is deductive; i.e., definition of the derivative, instantaneous velocity, related rate, and slope of the tangent line of a curve are presented deductively.

The editions Wade, Calculus (1953) and Taylor and Wade, University Calculus (1962) are exceptions to the inductive or deductive approach being used in both editions. The former presents an inductive approach for the derivative (slope of the tangent line to a curve, definition of the derivative, instantaneous velocity, general rate of change). The latter gives a deductive approach (definition of the derivative, slope of the tangent line to a curve, instantaneous velocity, related rates).

Morrey, University Calculus and Analytic Geometry (1962) and Protter and Morrey, College Calculus with Analytic Geometry (1964) follow essentially the same deductive approach (definition of the derivative, slope of the tangent line to a curve, instantaneous velocity, and related rates). One is not surprised at no change in this case since a lapse of only two years prompted no more than a superficial revision.

Granville, Elements of the Differential and Integral Calculus (1952), a reprint of the 1911 version, Granville, Smith and Longley, Elements of the Differential and Integral Calculus (1941) and (1946), and Longley, Smith and Wilson, Analytic Geometry and Calculus (1952) all have almost identical deductive approaches (definition of the derivative, slope of the tangent line to a curve, instantaneous velocity, related rates), with the interchange, addition and expansion of certain selected chapters in the last text.

Peterson, Elements of Calculus (1950) and Peterson, Calculus with Analytic Geometry (1960) both exhibit inductive approaches (general rate of change, definition of the derivative, slope of the tangent line to a curve, instantaneous velocity and acceleration). There is an insertion of two chapters on analytic geometry in the latter edition.

For the inductive approach in Calculus in the First Three Dimensions, Stein writes:

> The introduction of many concepts, such as the definite integral, the derivative and the limit of a sequence begin with numerical examples and exercises. This is done not only to

make the abstract concrete, but also to compensate for a lack of down-to-earth mathematical experience in high school. In particular, both the definite integral and the derivative are preceded by four of their applications. (66, p. vi)

Nature of the Study

This study is concerned with the problem discussed in the last paragraph of the first section, that of investigating two teaching methods in an advanced subject matter area. The specific area of teaching method investigated is mathematics. In this area, the limit and derivative concepts in calculus are the topics. Allendoerfer has stated that the limit is an important concept in calculus.

> The essential idea in calculus is that of limit, and without a clear exposition of limits any calculus course is a failure. . . . There are those, however, who begin the course with a brief, but full dress, treatment of limits, using the epsilon-delta technique. This almost universally is wasted on the class, for they are confronted with a difficult new idea without an intuitive preparation. (2, p. 484)

The present study is an outgrowth of a similar study by Ronald M. Shelton in the teaching of the limit concept in beginning calculus by two different methods. Shelton presented this concept by a concrete inductive approach defined as:

> Concrete inductive approach: a presentation of a sequence of items leading from specific numerical examples in which students will calculate the limits of particular functions at a definite point by appeal to intuition to the general case of a general function at any point. After the general case is reached rigorous proofs will be presented. (60, p. 8)

He also presented the material on the limit by an abstract deductive approach defined as:

> Abstract deductive approach: a presentation of a sequence of items leading from the abstract δ-ε -definition of a limit of a function to calculating the limits of particular functions at a definite point. (60, pp. 8-9)

The samples Shelton used in two independent studies were small, one of 24 subjects and the other of 28. The two groups were divided into high and low achievers (levels) on the basis of pre-test scores. From a 2 x 2 (levels X treatments) analysis of covariance design, tests of significance made at the .05 level showed no statistically significant difference in treatments, between levels, or in interaction, as measured by a criterion test constructed by Shelton.

The investigator believes that perhaps there might have been a significant difference in treatments, had the samples been larger and had other variables been controlled. Yet, results of similar studies to be cited later are not encouraging. In the present study the investigator replicates Shelton's study and, in addition, develops two programmed units that teach the derivative concept in calculus by an inductive and a deductive approach. The present study then attempts to determine the effectiveness of learning, as measured by an achievement test, that results from using various combinations of inductive and deductive methods of presentation of the limit and derivative topics.

The purpose of using programmed materials in the experiment is to remove the "teacher variable" and thus control the method of presentation. Scandura discusses the problems with the "teacher variable" in educational research. He states:

> As you well know, the traditional methods paradigm for research on teaching and learning has been designed to assess

> the relative effectiveness of two or more instructional methods. A major difficulty with this sort of research is that too often both method and content are varied simultaneously but not independently. Such an approach allows one to say nothing about either separately. (55, p. 131)

It should be noted that a study of teaching methods may yield different results for "live" teaching compared to "canned" or programmed teaching. Students may tend to become bored with the latter and lose interest in the material. Thus, a real difference in teaching methods presented by programmed materials may be masked. Experimental and control groups may both receive low scores on a criterion measure. Their dislike of the method of presentation may cause hostility and low scores in all students to a greater degree than teaching method might cause differences in scores in the experimental and control groups.

During the course of this study, classroom teachers were to do no more than answer individual student questions over the written text material. This procedure was stressed to insure that learning took place from the programmed units entirely. This way, any change in criterion test scores could be attributed to the teaching methods programmed into the units. If some students were reading far less than the minimum number of frames per day, these students were allowed to take their units home to finish the reading in approximately a week's time. In such a short time period it seemed reasonable to assume that a student would not have time to read the alternate treatment.

The investigator wrote two programmed units of comparabl^ length to teach the derivative concept in beginning c lculus. One of these units was written by an inductive approach and the other by a deductive approach. (See Shelton's definitions of these terms cited above.) These units were paired with the limit units to insure that all students participating in the study had comparable preparation for the derivative unit. The derivative unit assumed familiarity with the limit concept as prerequisite knowledge. Thus, there were four treatments: inductive limit-inductive derivative, inductive limit-deductive derivative, deductive limit-inductive derivative, deductive limit-deductive derivative.

Each participant was given the pre-test for the limit unit. On the basis of the score he received on this pre-test, he was assigned to a high or low achievement group (level). On either level, the four treatments were randomly assigned. Scores on the common derivative criterion test are used in an analysis of covariance to determine if a statistically significant difference existed in treatments, between levels, or in interaction.

A pilot study on the derivative units showed that the text material did indeed teach the derivative concept somewhat effectively. The units were revised from suggestions obtained by students participating in the pilot studies.

It is to be noted that this study is not intended to evaluate programmed instruction. This method of text format is only the vehicle of instruction. Yet, some ancillary effects of the programmed texts may be derived.

Statement of the Problem

This study is to determine if there is a difference in learning the limit and derivative concepts in beginning calculus as measured by a common criterion test. An ordered combination of two teaching approaches for the two sequential topics of the limit and derivative is used. The two approaches are concrete inductive and abstract deductive and the two sequential topics are the limit and derivative, in that order. All four possible pairings for the two units are considered and constitute the four treatments: inductive limit-inductive derivative, inductive limit-deductive derivative, deductive limit-inductive derivative, and deductive limit-deductive derivative. The problem is to determine if there is a statistically significant difference in the four treatments among the two levels (high and low) used in the study, employing an analysis of covariance on the common criterion measure.

The study attempts to determine if ability and knowledge of mathematics before beginning the study of calculus, and the method of presentation of a beginning topic in calculus, would have an effect on learning this topic of the derivative by its respective method of presentation. Shelton's study is also replicated.

Statement of Hypotheses

A replication of Shelton's original study is felt appropriate, to test his non-significant findings for small groups of 24 and 28 students in each of his two independent studies of an inductive and deductive treatment of the limit concept. In the present study, Shelton's three null hypotheses are investigated:

L 1. There is no difference in the results on the achievement test on limits after adjustment for the scores on the pre-test between the two treatments.

L 2. There is no difference in achievement as measured by the test on limits between the two levels used in the experiment.

L 3. There is no interaction between treatments and levels -- the treatments will produce similar results at both levels. (60, p. 11)

In the present study, two areas of investigation are of interest. These are the total treatment used for both the limit and derivative units, controlling for the pre-test score, and the treatment used only in the derivative unit, controlling for the pre-test score. For the total treatment, three null hypotheses are investigated:

T 1. There is no difference in results on the achievement test on the derivative among the four total treatments after adjustment for the scores on the pre-test.

T 2. There is no difference in achievement as measured by the test on the derivative between the two levels used in the study, controlling for the pre-test score.

T 3. There is no interaction between the four total treatments and two levels.

For the derivative treatment alone, since results from the pilot studies seemed to favor the inductive approach, we might state three one-tailed hypotheses. However, evidence from other studies give little support for the superiority of an inductive approach over a deductive approach. Hence, the three hypotheses under investigation for the derivative unit are also two-tailed hypotheses:

D 1. There is no difference in results on the achievement test on the derivative, controlling for pre-test, between the two derivative treatments.

D 2. There is no difference in achievement as measured by the test on the derivative, controlling for the pre-test, between the two levels used in the experiment.

D 3. There is no interaction between the derivative treatments and levels.

CHAPTER II

DEVELOPMENT OF TREATMENT AND EVALUATION INSTRUMENTS

Selection of Material to be Learned

It is felt that a sequence of concrete applications of the derivative (slope of the tangent line to a curve, instantaneous velocity, general rate of change), leading to the formal limit definition of the derivative, is more effective than the presentation of the formal definition, followed by applications. If such is the case, it might be possible to teach the important and basic topic of the derivative more meaningfully and effectively in the future.

The derivative units, both inductive and deductive approaches, are written assuming a knowledge of the limit concept which normally precedes that of the derivative in a beginning calculus course. Both units are written to contain the same content: the definition of the derivative; the application of the derivative as the slope of a tangent line to a curve; theorems for the derivative of sums, differences, products and quotients of algebraic functions, the constant function, the independent variable, a real power of the independent variable, and a composite function. A section on composite functions is included before the presentation of the theorem on composite function differentiation. Corollaries for the derivative of a constant multiplied by a function and the derivative of a real power of a polynomial function are presented. Numerous examples and exercises are identical in both units, or otherwise the same in content and difficulty. The section on the composite function review and the first 17 frames of each unit, which involved a change of notation for the limit, from $\lim_{x \to x_1} f(x)$ to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$, are identical. From either unit the student is to be able to differentiate simple algebraic, rational, and polynomial functions and to apply this knowledge to the derivative interpreted as an instantaneous velocity and general rate of change.

The units are approximately the same in length, 346 frames in the inductive unit and 359 frames in the deductive unit. Each unit was prepared to take a high school student approximately five 55-minute periods to read, a week's time in most secondary school schedules.

Development of the Abstract Deductive Derivative Unit

The deductive unit proceeds, without exception, from abstract statements of theorems and definitions to numerous examples and exercises. This approach parallels a rul-eg (rule to examples) programming sequence. (36) The strategy of exposition of a deductive approach is used. Under the strategy of exposition, logical deduction is also used. (22)

Exposition follows the teaching model of: 1) stating the item of subject matter to be taught - a generalization, theorem, definition, algorithm, etc.; 2) clarifying or paraphrasing the item, giving examples and/or stressing various components; 3) justifying the statement; 4) summarizing the teaching by restating the initial item of subject matter or giving an application; and 5) making a transition to another item of subject matter.

After a brief section on change in notation for the limit of a function from $\lim_{x \to x_1} f(x)$ to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$, this section identical to one appearing at the start of the inductive unit, the student is given the definition of the derivative by exposition, as $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, provided this limit exists. This section is followed by several examples that express the derivative as the limit of this difference quotient for simple polynomial functions. This approach leading to the definition of the derivative is just the reverse of that used in the inductive derivative unit.

The abstract forms of the theorems for the derivatives of the independent variable, the constant function, a real power of the independent variable, and sums, differences, products and quotients of algebraic functions and the corollary for the derivative of a constant multiplied by a function are abstractly stated. Each theorem and the corollary is proved by logical deduction. Following each theorem appear several concrete numerical examples to illustrate the particular theorem. Exercises for the student to complete are given at appropriate places in the text material. These exercises use the definition of the derivative and theorems proved up to that point in the development.

A section on composite function identification, which is introduced by exposition and is of a review nature, then follows. This section is the same as one appearing in the inductive unit, and is presented there for the same purpose. The theorem for the differentiation of a composite function is then stated and proved by logical deduction. Again, concrete numerical examples follow.

Functions not possessing derivatives at all points are introduced by exposition using the expressions defining them and their graphs. There functions are identical to those in the inductive unit, and are defined by the following expressions: $f(x) = 1/x$, $g(x) = |x-1|$, $h(x) = \sqrt{x-2}$.

Finally, again by exposition, the application of the derivative as the slope of the tangent line to the graph of f at the point with x-coordinate x_1 is presented and related to the same notion expressed in the limit unit. Following these applications, the writer next includes several examples of functions exhibited by both their defining expressions and their graphs. Some of these functions possess and some lack tangent lines at certain points on their graphs. The next topic presented in the unit is the writing of equations of tangent lines, both when the derivatives exist at the point of tangency and when the derivatives don't exist but the tangent lines exist. The format used is similar to that in a corresponding section in the inductive unit.

Tracing the development of this deductive program, we see that it is characterized by exposition and logical deduction. Logical deduction is used in proving the various derivative theorems. Throughout, the abstract definitions or theorems are given and concrete numerical examples and exercises follow.

The first written format of the deductive derivative unit was read by seven male high school students in their fourth year of mathematics at Urbana Senior High School, Urbana, Illinois in the spring semester of the school year 1965-1966. It was read also by four more students in the same high school in an advanced third year mathematics course, and by one fourth year male student at University High School, Urbana, Illinois. These students gave the investigator suggestions for revising mathematical content. At the same time, six students in a programmed learning course at Illinois Teachers College, Chicago-South read the unit and offered programming techniques to improve it. None in the last group felt competent to criticize the unit regarding mathematical content.

The 359 frames making up the deductive unit were duplicated by the multilith process. Each 8 1/2 by 11 inch page contains two or three frames, depending on the length of the frame. Each frame has its underlined answer on the page following, with the letter "A" followed by the frame number. Most of the frames require a one-word response of a fill-in-the-blank nature, or a short answer. A few require more than one answer and some are simple expository or introductory frames, requiring no answer. This format is the same as that in the limit unit. It is used to provide continuity in reading the two programs. More important is the elimination of a second possible variable, that of type of format in the limit and derivative units, by programming the units in the same way.

The student read through the frames at the top of the pages, to the last page, and then returned to page one for the answer to the frame on the last page. He proceeded reading through the pages again, following the frames across the middle of the pages on this reading.

He then returned to page one for the answer to the last frame on the middle of the page. He proceeded through the pages one last time, reading across the bottom of the pages. The deductive derivative unit was covered in white cover stock, labeled Treatment 4, and fastened with two large staples.

The inductive and deductive limit units that were to precede the derivative units were bound in the same manner. The inductive limit unit was labeled Treatment 1 and the deductive limit unit Treatment 2. A page of instructions was provided in both units, giving directions for the reading of the two sequential units. A copy of the abstract deductive derivative unit is included as Appendix B.

Development of the Concrete Inductive Derivative Unit

In general structure, the inductive unit proceeds according to the strategies of simple enumeration and difference and agreement of an inductive presentation. (23)

The strategy of simple enumeration of an inductive teaching approach gives only confirming instances of the item of subject matter to be taught. No counterinstances are exhibited. The inductive strategy of difference and agreement is obtained by combining the two inductive strategies of "the method of agreement" and "the method of difference." (6, p. 296) The basic logic of the strategies of the joint method of agreement and difference and of simple enumeration is the same; the joint method simply provides a more plausible argument.

The method of agreement provides that every generalization or other item of subject matter to be taught has the property that every case of p is also a case of q. Each instance confirming the generalization says that in addition to p and q, other factors, r, s, t, ..., are present or absent. Only p and q occur in all cases, and no case is found wherein p occurs and q doesn't; i.e., no contrary evidence is present. Thus, probably every case of p is also a case of q and the agreement of all confirmatory instances has been demonstrated in the presence of only p and q.

The method of difference takes the same form as that of the method of agreement - every case of p is also a case of q. The initial instance confirming the generalization states that in addition to p and q, other factors, r, s, t, ..., are present, also as in the method of agreement. The next confirmatory instance, however, says that when factors r, s, t, ... are present and q is absent, p is also absent. (23)

Exceptions to an inductive approach in the unit are the proofs of the theorems for the derivatives of the sum and product of functions, the discussion of the non-existence of $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, the writing of the equations of the tangent lines, the review section on composite functions, and the introductory section in the first 17 frames of the program, presenting the change of notation for the limit from $\lim_{x \to x_1} f(x)$ to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$. These sections are presented in an expository manner to effect economy of time and text space, thus keeping the two formats of the derivative units somewhat the same length. The inductive unit is programmed by an eg-rul (example to rule) technique. (36)

Based on his knowledge of the limit from the preceding unit, by simple enumeration the student is lead to the reasonable conclusion that if $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ exists, this limit can be interpreted as the slope of a tangent line to the graph of f, at a point with x-coordinate x_1. This is achieved by having the student calculate this limit for several elementary polynomial functions accompanied by a graphical representation. In fact, the slope of a tangent line to the graph of f is then defined in this manner, if the limit exists.

The notion that this limit may not exist for selected values of x_1 is introduced by several functions. These functions are defined by $f(x) = 1/x$, $g(x) = |x-1|$, $h(x) = \sqrt{x-2}$. They are the same as those appearing in the deductive unit. The slope of the tangent line, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, may not exist at a point if the slope of the tangent line is undefined at the point or if the function itself is undefined at the point. By difference and agreement, it is then emphasized that <u>if</u> this limit exists, the derivative of the function f, evaluated at x_1, is defined as this limit. This is the first time the term "derivative" is used. Throughout the remainder of the unit, "derivative" and "slope of the tangent line to the graph of f, if it exists," are used interchangeably.

The slope of the tangent line to a curve is induced by simple enumeration. This is developed from a sequence of slopes of secant lines to a curve through the point of tangency and a nearby point. Tables of values showing slopes of secant lines approaching in values the slope of the tangent line at the fixed point are to be completed by the student. It is then an easy step to proceed to the writing of the equation of the tangent line, if the latter exists, at a fixed point. For the sake of economy of time and available text space, as mentioned above, presentation of the writing of the equation of the tangent line is expository in nature.

The next section is developed using examples of simple polynomial functions. Some of these examples were previously discussed in introducing the definition of the derivative. Theorems for the derivative of the constant function, the independent variable, a real power of the independent variable, sums, differences, products, and quotients of functions, and a composite function are induced by simple enumeration. The proofs for the theorems involving the sum and product of two functions and a real power of the independent variable are given deductively for reasons cited above. Before the theorem for the differentiation of a composite function is stated, a review section enabling the student to identify composite functions is given. This review is the same as that appearing in a corresponding section in the deductive unit. A suitable number of exercises enforce and confirm student learning at various points throughout the text. These exercises are identical to or similar in content and difficulty to those in the deductive unit. Figure 1 exhibits the logical development of the units in the form of a flow chart. Figures 2 and 3 exhibit the content development of the units in the form of flow charts.

A first draft of the inductive unit was read by a class of 19 fourth year high school students at Bremen High School, Midlothian, Illinois in the fall semester of the 1966-1967 academic year. From the suggestions of these students, mathematical content and programming features were revised. The text material, as revised, appears as Appendix A.

FIGURE 1

FLOW CHART OF DEVELOPMENT OF DERIVATIVE UNITS

TREATMENT 3
(Concrete Inductive Unit)

$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ exists $\xrightarrow[\text{Enumeration}]{\text{Simple}}$ Definition of slope of a tangent line to a graph at x_1.

$\xrightarrow[\text{Enumeration}]{\text{Simple}}$ Abstract common properties. $\xrightarrow[\text{Agreement}]{\text{Difference and}}$ $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

may or may not exist.

Definition of the derivative.

$\xrightarrow[\text{Enumeration}]{\text{Simple}}$ Theorems and corollaries for derivatives of sums, differences, products, quotients of functions and other selected functions.

TREATMENT 4
(Abstract Deductive Unit)

$\lim_{x \to x_1} f(x) = \lim_{\Delta x \to 0} f(x_1 + \Delta x)$ $\xrightarrow{\text{Exposition}}$ Definition of the derivative. $\xrightarrow[\text{Deduction}]{\text{Logical}}$

Theorems and corollaries for derivatives of sums, differences, products, quotients of functions and other selected functions. $\xrightarrow{\text{Exposition}}$

The derivative may not always exist. $\xrightarrow{\text{Exposition}}$ Application of the derivative as slope of a tangent line to a curve.

FIGURE 2

FLOW CHART OF CONTENT OF INDUCTIVE DERIVATIVE UNIT

Change notation from $\lim_{x \to x_1} f(x)$ to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ ⟶

$f(x)=x^2$
$f(x)=x^2 + x-6$
$f(x)=\sqrt{25-x^2}$
$f(x)=x$
$f(x)=k$

} $\lim_{\Delta x \to 0^-} f(x_1 + \Delta x) =$
$\lim_{\Delta x \to 0^+} f(x_1 + \Delta x) =$
$\lim_{\Delta x \to 0} f(x_1 + \Delta x)$

⟶ There exist functions such that $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ exists at x_1. ⟶ Definition of slope of the tangent line to a curve at x_1.
Examples
Exercises

$f(x)=x^2$
$f(x)=x^2 + x-6$
$f(x)=x$
$f(x)=k$

} Equations of tangent lines. ⟶ Generalize $(y-y_1)= f'(x_1)(x-x_1)$

$f(x)=\sqrt{x-2}$
$f(x)=1/x$
$f(x)=|x-1|$

} $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ doesn't exist at x_1. ⟶ There exist functions such that $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ doesn't exist at x_1.

Definition of the derivative. (Same as the definition of the slope of the tangent line.) ⟶

<u>Theorem 1</u>. Derivative of a constant function.
<u>Theorem 2</u>. Derivative of the independent variable.
(Previously proved as problems.)

$f(x)=x^3$
$f(x)=x$
$f(x)=x^2$

} Previous examples or exercises. ⟶ $f(x)=x^4$

<u>Theorem 3</u>. Derivative of a real power of the independent variable.
Proof
Exercises

$f(x)=x^2 + x-6$
$f(x)=x+2$
$f(x)=x^3 + x^2 + x+11$

} Previous examples or exercises. ⟶

<u>Theorem 4</u>. Derivative of a sum.
Proof
Exercises

FIGURE 2 (Continued)

$f(x)=x-2$
$f(x)=x^2-x+6$
$f(x)=x^3-x^2-x-11$

} Previous examples. → Theorem 5. Derivative of a difference. No proof

$f(x)=x^3 \cdot x^3$
$f(x)=x^4 \cdot x^2$ →
$f(x)=x^5 \cdot x$

Theorem 6. Derivative of a product. Proof Exercises →

$f(x)=7x^5$
$f(x)=-6(x+1)$
$f(x)=\pi(x^5+x^3+x^2+1)$

→ Theorem 7. Derivative of a quotient. No proof Examples Exercises →

$f(x)=\sqrt{x-2}$
$f(x)=\sqrt{x^2-2x}$
$f(x)=\sqrt{x-2x^2}$

→ Composite function review. → Theorem 8. Derivative of a composite function. No proof Examples Exercises

→ Corollary. Derivative of a power of a function of x.

→
$f(x)=\sqrt{25-x^2}$
$f(x)=(x^2-2x-3)$
$f(x)=(x^4-1)^{-2}$

Exercises

FIGURE 3

FLOW CHART OF CONTENT OF DEDUCTIVE DERIVATIVE UNIT

Change notation from $\lim_{x \to x_1} f(x)$ to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ → Definition of the derivative. → Examples
$f(x)=3x$
$f(x)=x$
$f(x)=x^2$
$f(x)=2x^2$
$f(x)=x-x^2$

Theorem 1. Derivative of the independent variable.
Theorem 2. Derivative of a constant function.
(Previously proved as problems.)
→ Theorem 3. Derivative of a real power of the independent variable.
Proof

Examples
$f(x)=x^3$
$f(x)=x^7$
$f(x)=x^2$
Exercises
→ Theorem 4. Derivative of a sum.
Proof
→ Examples
$f(x)=x+7$
$f(x)=x^3-6$
$f(x)=x^4+x$
$f(x)=x^{8/7}-x^2$

→ Theorem 5. Derivative of a difference.
No proof
→ Examples
$f(x)=\pi-x$
$f(x)=x-x^2$
$f(x)=x^{9/7}-(\sqrt{2})^7$
→

Theorem 6. Derivative of a product.
Proof
→ Examples
$f(x)=x^2 \cdot x^3$
$f(x)=x(1-x)$
$f(x)=(x^2+1)(x^3+1)$
→ Corollary. Derivative of a constant multiplied by a function of x.

Examples
$f(x)=7x^5$
$f(x)=3(x^3+x^4)$
$f(x)= -6(x+1)$
→ Theorem 7. Derivative of a quotient.
Proof
→ Examples
$f(x)=x^7/x^2$
$f(x)=(x+1)/(x-3)$
$f(x)=\frac{x^2+3x}{x^3+1}$

FIGURE 3 (Continued)

Exercises
Composite function
review. →

Theorem 8. Derivative of a composite function. →
Examples
Exercises

Corollary. Derivative of a power of a function of x. →
Proof

Examples
$f(x)=\sqrt{25-x^2}$
$f(x)=(x^2-2x-3)^{7/2}$ →
$f(x)=(x^4-1)^{-2}$
$f(x)=\sqrt{x-2x^2}$
Exercises

There exist functions such that $\lim_{\Delta x \to 0} f(x_1+\Delta x)$ doesn't exist at x_1. →
Example: $f(x)=|x-1|$

Definition of the slope of the tangent line to a curve at x_1. →
(Same as the definition of the derivative.)

Examples
$f(x)=x^2$
$f(x)=\sqrt{25-x^2}$
$f(x)=x^2+x-6$ →
$f(x)=x^3-2x^2+5x-1$

Equation of a tangent line.
$y-y_1=f'(x_1)(x-x_1)$ →
Exercises

There exist functions such that the slopes of the tangent lines at x_1 don't exist.

Examples
$f(x)=|x-1|$
$f(x)=\sqrt{25-x^2}$ →
$f(x)=x^{1/4}$
$f(x)=\sqrt{x-2}$
$f(x)=1/x$

Graphical interpretation of theorems 1 and 2.
Exercises

Development of the Evaluation Instrument

A common 28-item criterion test on the derivative units was designed to measure student knowledge of differentiation of simple algebraic and rational expressions, and application of this knowledge to the derivative as the slope of the tangent line to a curve, instantaneous velocity and general rate of change. Care was taken to avoid wording and item construction favoring one treatment over the other. The correct choices for each of the 24 multiple-choice items are randomly distributed among the four possible alternatives. Each multiple-choice answer is worth one point. The last four short answer questions involve one proof, one computation and two discussions; each of these answers is worth three points. The total number of possible points on the test is 36.

The test was revised from results of the pilot study on the deductive derivative unit. The primary revision was the addition of a brief explanation for the questions involving instantaneous velocity and general rate of change. The revised text material did not contain these last two topics. However, they were felt important in an understanding of the derivative and are included in the evaluation.

The multiple-choice items are arranged in order of difficulty from easiest to hardest, in clusters of related topics. A split-half reliability coefficient of .69 was calculated from the pilot study for the inductive derivative text material. The split-half reliability coefficient in the present study is .78. The odd item-total item score correlation is .87 and the even item-total item score correlation .93.

The correlations for the derivative study are significant ($p < .001$). Inter-coder reliability was obtained for the last four short answer questions, using the instructors of the classes in the pilot studies and present study, as well as university calculus teachers.

The test was to be given during two 55-minute class periods, splitting the instrument after item 18. The pre-test and criterion test for the limit units, developed by Shelton, are used in the statistical evaluation of the present study. This procedure was incorporated into the study for a more complete analysis. The derivative test was constructed according to the format of the limit test: 24 multiple-choice items worth one point each and four short answer items worth three points each, for a total of 36 points. A copy of the criterion test for the derivative units appears as Appendix C.

A table of specifications for the derivative test, constructed according to Bloom's Taxonomy of Educational Objectives (7), was helpful in constructing the derivative test and appears as Table 1.

Preliminary Studies

Pilot Study I. The deductive unit was written while the investigator was in residence at the University of Illinois, Urbana, Illinois, for the doctoral degree, during the spring of 1966.

As mentioned in a preceding section, seven boys at Urbana Senior High School, Urbana, Illinois in a fourth year mathematics course participated in the pilot study. They read the programmed material as part of their course requirement near the end of the 1965-1966 academic year. None of the boys had had a former exposure to calculus. Four advanced

TABLE I

CONTENT ANALYSIS OF DERIVATIVE CRITERION TEST

Item of knowledge needed to complete questions.	Question Number													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1. Definition of the derivative.												x	x	x
2. Definition of the slope of a tangent line.												x	x	x
3. The derivative of the independent variable.	x		x	x	x		x	x	x	x	x	x	x	x
4. The derivative of a constant function.	x	x	x	x	x	x	x	x	x	x	x	x	x	x
5. The derivative of a real power of the independent variable.	x	x	x	x	x	x	x	x	x		x	x	x	
6. Derivative of a sum.	x	x	x	x		x		x	x		x	x		
7. Derivative of a difference.	x	x	x		x	x	x		x	x	x	x	x	x
8. Derivative of a product.		x	x	x	x	x	x	x	x	x	x	x	x	x
9. Derivative of a quotient.				x	x					x				
10. Derivative of a constant multiplied by a function of x.	x	x	x			x	x	x	x		x	x	x	
11. Derivative of a composite function.						x	x	x	x	x	x		x	x
12. Derivative may not exist if function undefined at x_1.											x		x	
13. Derivative doesn't exist.														
14. Equation of the tangent line.														
15. Instantaneous velocity.														
16. General rate of change.														

TABLE I (Continued)

Item of knowledge needed to complete questions.	15	16	17	18	19	20	21	22	23	24	25	26	27	28
	Question Number													
1. Definition of the derivative.	x	x	x	x	x	x	x	x	x	x	x	x	x	x
2. Definition of the slope of a tangent line.	x	x	x	x							x	x		x
3. The derivative of the independent variable.		x		x			x	x					x	
4. The derivative of a constant function.	x	x	x	x	x	x	x	x	x		x		x	
5. The derivative of a real power of the independent variable.	x		x	x	x	x	x		x	x			x	
6. Derivative of a sum.			x		x	x	x						x	
7. Derivative of a difference.	x	x	x	x	x		x						x	
8. Derivative of a product.			x		x	x	x	x	x	x			x	
9. Derivative of a quotient.													x	
10. Derivative of a constant multiplied by a function of x.			x		x		x	x	x				x	
11. Derivative of a composite function.	x	x	x		x	x							x	
12. Derivative may not exist if function undefined at x_1.		x												
13. Derivative doesn't exist.												x		x
14. Equation of the tangent line.	x	x	x											
15. Instantaneous velocity.					x	x	x							
16. General rate of change.								x	x	x				

students (three boys and one girl) in a third year mathematics course at the same high school expressed a desire to read the material. Their backgrounds represented a good foundation in college algebra. A fourth year male student in mathematics at University High School, Urbana, Illinois also participated. The last student's background was probably the best of the high school students. He was then enrolled in a course in vector geometry. Six students in a programmed learning course at Illinois Teachers College, Chicago-South read the program at the same time. The high school students' comments were used for revising the mathematical content of the program. The college students' comments were used for revising the programming techniques of the program.

Before the derivative unit was read, each of the high school students was given the programmed unit on the limit concept, the inductive and deductive approaches being randomly distributed. The reading of these units was to provide the students with the necessary background to read the derivative unit. Shelton's pre-test was administered to each student before either unit was distributed. His test for the limit unit was also given. The criterion test on the derivative unit was given after the completion of the investigator's unit. The administration of the three tests served as student motivation and teacher evaluation.

Each participant was told to read his limit and derivative units. He was to write the answer for each frame on his own paper. The seven boys at Urbana High School were allowed to ask questions of their instructor over the material they could not understand. When the investigator was present during the class sessions, she answered such questions. The other participants in Urbana were allowed to call the investigator by telephone or meet with her personally if questions arose. Very few questions were asked during the course of the pilot study. It took the high school students approximately three weeks to complete both units and the three tests. They spent a 50-minute class period, or its equivalent for those working out of class, five days each week.

In general, the comments from the high school students on the deductive derivative units were favorable. In informal conversations with the writer the participants said they believed they had learned from the material. The ample number of examples, graphs and exercises were cited. It was suggested by some that the section on composite functions be reworked. Other criticism concerned the format of presentation of frame answers. These answers appeared next to their corresponding frames. The students preferred the format of the limit unit, in which an answer to a frame appears on the following page.

The college students were given only the derivative unit. They criticized the unit from the standpoint of programming techniques. They felt too much material was covered and the "steps" between material were too large.

Pilot Study II. The inductive unit was written during the summer of 1966, while the author was in residence at the University of Illinois, Urbana, Illinois.

The mathematical content of the inductive unit was "cut down" slightly from that of the deductive unit. This was due to the suggestions of the students in the programmed learning course at the Teachers College. The sections on instantaneous velocity and general rate of change that appeared in the deductive unit were omitted in the inductive unit. They were finally omitted in both units in their final form. The criterion test content was the same as that administered to the deductive group. The only difference was that an introductory sentence or two were necessary to explain what was wanted of the student in questions involving instantaneous velocity and general rate of change.

The unit was read by 19 students in their first semester of fourth year mathematics at Bremen High School, Midlothian, Illinois in the fall of 1966. The class contained ten girls and nine boys. Each student had had a course through modern algebra in the Dolciani series.

Only the deductive limit units were used in this study, but the other features of this pilot study were conducted in a manner similar to those of the first study. Both the teacher of the class and the investigator, when present, answered questions of the students during the 40-minute class period. Most of the questions were asked on the deductive limit unit. The students found this unit difficult and became somewhat bored with it. The reading of both the limit and derivative units and the administration of the three tests took approximately one month. The students did all their work during the mathematics class period.

The inductive derivative unit was more to the students' likings. By individual questioning by the investigator, the class said it found the derivative unit "easier" than the preceding unit. No mention was made by the investigator or the class instructor of the different approaches in the two units. The students seemed to find the material interesting and were able to answer all but about ten of the original total of 381 frames. This was determined by an analysis of written responses. The difficulty centered on the composite functions section primarily, which was later revised.

The students commented on their liking the "discovery approach." In many cases they formed the generalization before reading it. A few thought some of the "discovery" could have been accomplished in less time. No mention was made of proofs being too rigorous or frequent. The class seemed to think the criterion test on the derivative unit was "fair" and covered the material presented in the unit.

CHAPTER III

THE EXPERIMENT

Experimental Design

The subjects in this study were divided into two levels, high and low, on the basis of points on a pre-test. For both levels, the four total treatments were randomly distributed. These treatments are inductive limit-inductive derivative designated as treatment A, inductive limit-deductive derivative designated as treatment B, deductive limit-inductive derivative designated as treatment C, and deductive limit-deductive derivative designated as treatment D.

The difference between the means of the four total treatments and between the two levels are to be tested for significance on the basis of criterion test scores. The table below exhibits the treatments by levels design of this experiment.

TABLE II

Treatments X Levels Experimental Design for Total Treatment Study

	Treatments			
Levels	1. A	2. B	3. C	4. D
1. High	A-H	B-H	C-H	D-H
2. Low	A-L	B-L	C-L	D-L

The final analysis of the criterion measures employs a 4 x 2 treatments X levels analysis of covariance to test for significance between the adjusted criterion total treatment means, the adjusted level means, and interaction.

For the total treatment, the predictor variable is the score on the pre-test. This test includes material on absolute-value inequalities and graphing primarily. Items from algebra, trigonometry, and analytic geometry considered necessary for the learning of the limit and derivative concepts in beginning calculus also appear. The final criterion variable is the score on the common derivative test. This test was given to all students upon completion of both units.

An analysis of covariance was chosen to statistically control the pre-test score and to refine further the results. The primary variables controlled are (1) length of the two derivative treatments and the derivative criterion test, (2) content of the derivative treatments and the derivative criterion test, and (3) administration of the derivative treatments and the derivative criterion test by randomization within levels and between treatment groups. These variables were similarly controlled for the two limit treatments and the limit criterion test in Shelton's study. The .05 level of significance was deemed appropriate in this exploratory study.

The hypotheses, in null form, to be tested for the total treatment are:

T1. There is no difference in results on the achievement test on the derivative, controlling for the pre-test score, among the four total treatments.

T2. There is no difference in achievement as measured by the test on the derivative between the two levels used in the study, controlling for the pre-test score.

T3. There is no interaction between the four treatments and the two levels.

Shelton's study is replicated using only the inductive and deductive limit units. A test of significance of the difference between the means of his two treatments and between the means of his two levels (high and low), on the basis of limit criterion test score only, is to be performed. The table below shows Shelton's treatments by levels design.

TABLE III

Treatments X Levels Experimental Design for Limit Study

	Treatments	
Levels	1. Inductive	2. Deductive
1. High	I-H	D-H
2. Low	I-L	D-L

An analysis of covariance is employed on the criterion measures to test for significance between the adjusted criterion limit treatment means, the adjusted level means, and interaction.

For Shelton's replicated study, procedures similar to those for the total treatment are followed. The predictor variable is the score on the pre-test. The criterion variable is the score on the common limit test. This test was taken by all students in the study upon completion of the limit unit only.

The hypotheses, in null form, to be tested for Shelton's limit treatments only, are:

L1. There is no difference in the results on the achievement test on limits after adjustment for the scores on the pretest between the two limit treatments.

L2. There is no difference in achievement as measured by the test on limits between the two levels used in the experiment.

L3. There is no interaction between limit treatments and levels - the limit treatments will produce similar results at both levels.

An analysis of covariance is also performed to test for significance of the difference between the means of the two derivative treatments and between the means of the two levels (high and low), on the basis of derivative criterion test scores only. The table below shows the treatments by levels design in this experiment.

TABLE IV

Treatments X Levels Experimental Design for Derivative Study

	Treatments	
Levels	1. Inductive	2. Deductive
1. High	I-H	D-H
2. Low	I-L	D-L

For only the derivative treatments the predictor variable is again the score on the pre-test. The final criterion variable is the score on the common derivative criterion test. Other procedures are similar to those of the total and limit treatments procedures.

The hypotheses, in null form, to be tested for the derivative treatment only are:

D1. There is no difference in results on the achievement test on the derivative, controlling for the pre-test score, between the two derivative treatments.

D2. There is no difference in achievement as measured by the test on the derivative, controlling for the pre-test score, between the two levels used in the experiment.

D3. There is no interaction between the derivative treatments and levels.

Population and Sampling

Eight suburban Chicago high schools participated in this study. They supplied a total of 449 students. These 449 students were enrolled in 22 third and fourth year mathematics courses in the fall term of the 1967-1968 year. There were 338 males and 111 females in the total study. Table V gives a more detailed description of the population in the experiment.

All the students in the fourth year classes had had courses in algebra, geometry, and trigonometry. Some had had analytic geometry. The students in the third year classes were generally taking trigonometry at the time of the study and had had courses in algebra and geometry.

Procedures

A pre-test, written by Shelton, was given to each of the 449 students in the study the first full class day of the 1967 fall school term by their respective teachers. The pre-test consisted of 36 multiple-choice questions. Each teacher graded his pre-tests by a pre-determined

TABLE V

Population and Sampling of Chicago Suburban Schools in Experiment

School	Mathematics Classes	Number of Students
Arlington Heights High School Arlington Heights	Seven fourth year classes taught by three instructors	147
Downers Grove North High School Downers Grove	One fourth year class	29
Downers Grove South High School Downers Grove	Two fourth year classes, each taught by a different instructor	38
Forest View High School Arlington Heights	One fourth year class	14
Hinsdale Township High School Hinsdale	One fourth year class	26
Rich East Community High School Park Forest	Three fourth year classes, each taught by a different instructor	22
West Leyden High School Northlake	Two fourth year classes taught by the same instructor	33
York Township Community High School Elmhurst	Two third year classes taught by one instructor	34
	Four fourth year classes taught by three instructors	106
	TOTAL	449

key and sent the scores to the investigator. The scores were arranged from a high of 35 out of a total of 36 raw score points, to a low of three raw score points. In the cases of numerous duplicate scores the schools were arranged alphabetically and the students within each school were also arranged alphabetically. The 222 students in the high groups received scores of 21 points or higher on the pre-test; the remaining 227 students constituted the low group.

The four total treatments, A, B, C, and D, were randomly assigned in the high group from the highest to the lowest scores, proceeding through duplicate scores and beginning with treatment A. A similar procedure was followed for the low group. Several students were (randomly) removed from cells to give equal numbers in each cell. With equal cell numbers in the high group, there were 50 students reading treatment A, 50 treatment B, 50 treatment C, and 50 treatment D. For the low group, the cell numbers are the same for the four treatments. Thus, no one school in the study had an equal number of students for each treatment at each level. Table VI (c) displays the mean pre-test scores of the students in each cell.

For the four cells in the experimental design of Shelton's replicated limit study, 100 subjects in each cell were obtained by randomly removing the excess subjects. Table VI (a) displays the mean pre-test scores of the students in each cell of this experimental design.

One hundred subjects were similarly assigned to the four cells in the experimental design for the derivative unit. Table VI (b) displays the mean pre-test scores of the students in each cell of this experimental design.

Each school participating in the study had class periods 55 minutes in length. The programs for both units were passed out at the beginning of each class period to the proper students and collected at the end of the period. This procedure was followed to attempt to eliminate exposure of a student to both formats of the limit and derivative units. The teachers were present in the classes during reading of the units, and were allowed and encouraged to answer only individual questions of the students. They were not to conduct a general discussion. Such a procedure attempts to control the teacher variable and insure that the students learned from the programmed material only.

Most students asked questions over both deductive units, the deductive limit and the deductive derivative units. Students found the deductive units more difficult than the inductive ones and tended to become somewhat bored. Since a number of students were initially confused by the immediate presentation of the definition of a limit in the deductive limit unit, some teachers conducted a limited discussion in a confined section of the classroom for these students only. None of the students reading the inductive limit unit were involved in this restricted discussion. Such a procedure is considered acceptable within the framework of the procedures of the study.

Because some students were reading far less than the minimum number of 50 frames per day, they were allowed to take their units home. In such a short period of time it seemed reasonable to assume the students would not have time to read the alternate treatment. To insure against this type of contamination even further, answer sheets for the responses

TABLE VI (a)

Mean Pre-test Scores by Cells in the Three Studies

Limit Treatment

Level	Inductive	Deductive
High	25.87	25.56
Low	15.72	15.58

Derivative Treatment (b)

Level	Inductive	Deductive
High	25.72	25.71
Low	15.62	15.68

Total Treatment (c)

Level	A	B	C	D
High	25.92	25.82	25.52	25.60
Low	15.68	14.76	15.56	15.60

to the programmed text material were required of each student before he was allowed to take the criterion test over the particular unit.

The average reading time for the 309 frames in the limit units was six days. The administration of the criterion test on the limit unit, given upon completion of the unit, took two class periods. The average reading time for the 346 frame inductive derivative unit and the 359 frame deductive derivative unit was seven class periods. The administration of the criterion test on this unit took two class periods. This test was given upon the completion of the derivative unit.

The investigator visited each school at least once during the study. During these visits she observed the classes reading the units and answered student questions. She met with the class instructors to answer their questions and give them further instructions if needed. Numerous telephone calls were made to the schools during the course of the study to judge student progress. Correspondence was sent when schools requested further directions.

The teachers in the respective schools were more than willing to cooperate in this study. This was evidenced by their constant communication by telephone and letter to check procedural policy and report progress and results during the study. Their attitudes remained favorable throughout the duration of the study. However, as the study progressed students became bored with the procedure of reading programmed material for 55 minutes, five days a week for at least two weeks. An attitude measure of mostly negative responses to programmed instruction was obtained and should be considered in interpreting any statistical findings.

Statistical Treatment of Data

All subjects in this study are to learn elementary concepts of the limit and derivative in beginning calculus by programmed texts. Thus, any positive or negative influence of attitude toward programmed material is equated for all subjects, levels, and treatments. The novelty of participation in an experimental study should also be equated for all subjects, levels and treatments.

The criterion test over the limit unit was given in two class periods immediately after completion of this unit. The investigator's pilot studies indicated the test was too long to be completed in one 55-minute period. Each of the first 24 items is of the four-alternative, multiple-choice type, worth one point each. Questions 25-28 call for two definitions and two proofs of theorems, each worth three points. The total number of possible points is 36, the same total as the pre-test.

The teachers of the classes involved in the study administered and graded the tests by a predetermined key. Each teacher was provided with sample answers to questions 25-28. This procedure was followed to provide for a uniform grading of these test items, particularly in the case of partial credit. For all classes the investigator requested the students' written answers for the limit criterion test to check the grading on the last four items. Eighty-three scores on the limit criterion were lowered and 16 raised as a result of this checking.

The criterion test on the derivative unit (Appendix C) was given immediately after the completion of the total study. Two class periods were allowed, splitting the test after item 18. Pilot study results indicated the test was too long to be completed in one 55-minute period. The questions are constructed and weighted similarly to those in the limit criterion test; the first 24 questions are four-alternative multiple-choice items, worth one point each, followed by four short answer questions worth three points each. Each teacher was again provided with sample answers to the last four questions. The investigator requested student answer sheets for regrading of these last four questions. Fifty-seven scores on the derivative criterion test were lowered and 13 raised as a result of this regrading.

The scores on the limit criterion test range from 32 to 3 in the high group, with a mean of 15.93. The scores on the same test range from 26 to 0 in the low group, with a mean of 9.20.

The scores on the derivative criterion test range from 35 to 3 in the high group, with a mean of 17.27. The scores on the same test range from 31 to 2 in the low group, with a mean of 8.99.

A more thorough discussion of both the limit and derivative criterion test scores appears in the next chapter.

CHAPTER IV

RESULTS AND DISCUSSION

Summary of Procedures

The statistical design used in the three studies in this experiment is an analysis of covariance. The limit study tests for a difference in inductive and deductive teaching of this concept. The derivative study tests for the same difference in teaching methods for the derivative concept. A total treatment study tests for differences in teaching the ordered combination of the limit and derivative concepts by the four possible pairings of inductive and deductive approaches.

The pre-test score is the covariate in all three studies. The limit criterion test scores are used in the analysis of covariance for the limit study. The derivative criterion test scores are used in the analyses of covariance for the derivative and total treatment studies.

Criterion Test Scores

The scores on the limit criterion test have a range of 0 to 32. The scores on the derivative criterion test have a range of 2 to 35. Tables VII, VIII and IX give a summary of pre-test and criterion test scores and standard deviations by cell, treatment and level for the three studies.

Figure 4 shows the total regression line for the limit study with the dependent variable being the limit criterion test score and the

TABLE VII

Summary of Pre-test Scores, Criterion Test Scores, and Standard Deviations by Cell for the Limit Study

	Inductive	Deductive	Total
	$\bar{X}$=25.87	$\bar{X}$=25.56	$\bar{X}$=25.71
	σ_x= 3.57	σ_x= 3.42	
High			
	$\bar{Y}$=15.94	$\bar{Y}$=15.92	$\bar{Y}$=15.93
	σ_y= 5.32	σ_y= 8.73	
	n= 100	n= 100	
	$\bar{X}$=15.72	$\bar{X}$=15.58	$\bar{X}$=15.65
	σ_x= 3.33	σ_x= 3.24	
Low			
	$\bar{Y}$= 9.11	$\bar{Y}$= 9.29	$\bar{Y}$= 9.20
	σ_y= 5.02	σ_y= 4.35	
	n= 100	n= 100	
	$\bar{X}$=20.79	$\bar{X}$=20.57	$\bar{X}$=20.68
Total	$\bar{Y}$=12.53	$\bar{Y}$=12.61	$\bar{Y}$=12.57
			$_x$= 6.07
			$_y$= 6.96
			N= 400

KEY TO SCORES
X=pre-test score
Y=criterion test score

TABLE VIII

Summary of Pre-test Scores, Criterion Test Scores, and Standard Deviations by Cells for Derivative Study

	Inductive	Deductive	Total
	$\bar{X}$=25.72	$\bar{X}$=25.71	$\bar{X}$=25.71
	σ_x= 3.48	σ_x= 3.52	
High			
	$\bar{Y}$=16.16	$\bar{Y}$=18.38	$\bar{Y}$=17.27
	σ_y= 7.95	σ_y= 8.19	
	n= 100	n= 100	
	$\bar{X}$=15.62	$\bar{X}$=15.68	$\bar{X}$=15.65
	σ_x= 3.38	σ_x= 3.20	
Low			
	$\bar{Y}$= 7.97	$\bar{Y}$= 9.91	$\bar{Y}$= 8.99
	σ_y= 4.71	σ_y= 6.22	
	n= 100	n= 100	
Total	$\bar{X}$=20.68	$\bar{X}$=20.69	$\bar{X}$=20.68
	$\bar{Y}$=12.07	$\bar{Y}$=14.15	$\bar{Y}$=13.12
			$_x$= 6.07
			$_y$= 8.14
			N= 100

KEY TO SCORES
X=pre-test score
Y=criterion test score

TABLE IX

Summary of Pre-test Scores, Criterion Test Scores, and Standard Deviations by Cells for Total Treatment Study

	A	B	C	D	Total
High	$\bar{X}$=25.92 σ_x= 3.56 $\bar{Y}$=15.44 σ_y= 7.95 n= 50	$\bar{X}$=25.82 σ_x= 3.58 $\bar{Y}$=18.84 σ_y= 6.84 n= 50	$\bar{X}$=25.52 σ_x= 3.39 $\bar{Y}$=16.88 σ_y= 7.89 n= 50	$\bar{X}$=25.60 σ_x= 3.45 $\bar{Y}$=18.32 σ_y= 9.35 n= 50	$\bar{X}$=25.71 $\bar{Y}$=17.37
Low	$\bar{X}$=15.68 σ_x= 3.57 $\bar{Y}$= 7.78 σ_y= 4.95 n= 50	$\bar{X}$=14.76 σ_x= 3.07 $\bar{Y}$= 9.56 σ_y= 5.54 n= 50	$\bar{X}$=15.56 σ_x= 3.17 $\bar{Y}$= 8.16 σ_y= 4.46 n= 50	$\bar{X}$=15.60 σ_x= 3.31 $\bar{Y}$=10.26 σ_y= 6.82 n= 50	$\bar{X}$=15.40 $\bar{Y}$= 8.94
Total	$\bar{X}$=20.70 $\bar{Y}$=11.61	$\bar{X}$=20.29 $\bar{Y}$=14.20	$\bar{X}$=20.54 $\bar{Y}$=12.52	$\bar{X}$=20.60 $\bar{Y}$=14.29	$\bar{X}$=20.68 $\bar{Y}$=13.10 $_x$= 6.07 $_y$= 8.14 N= 400

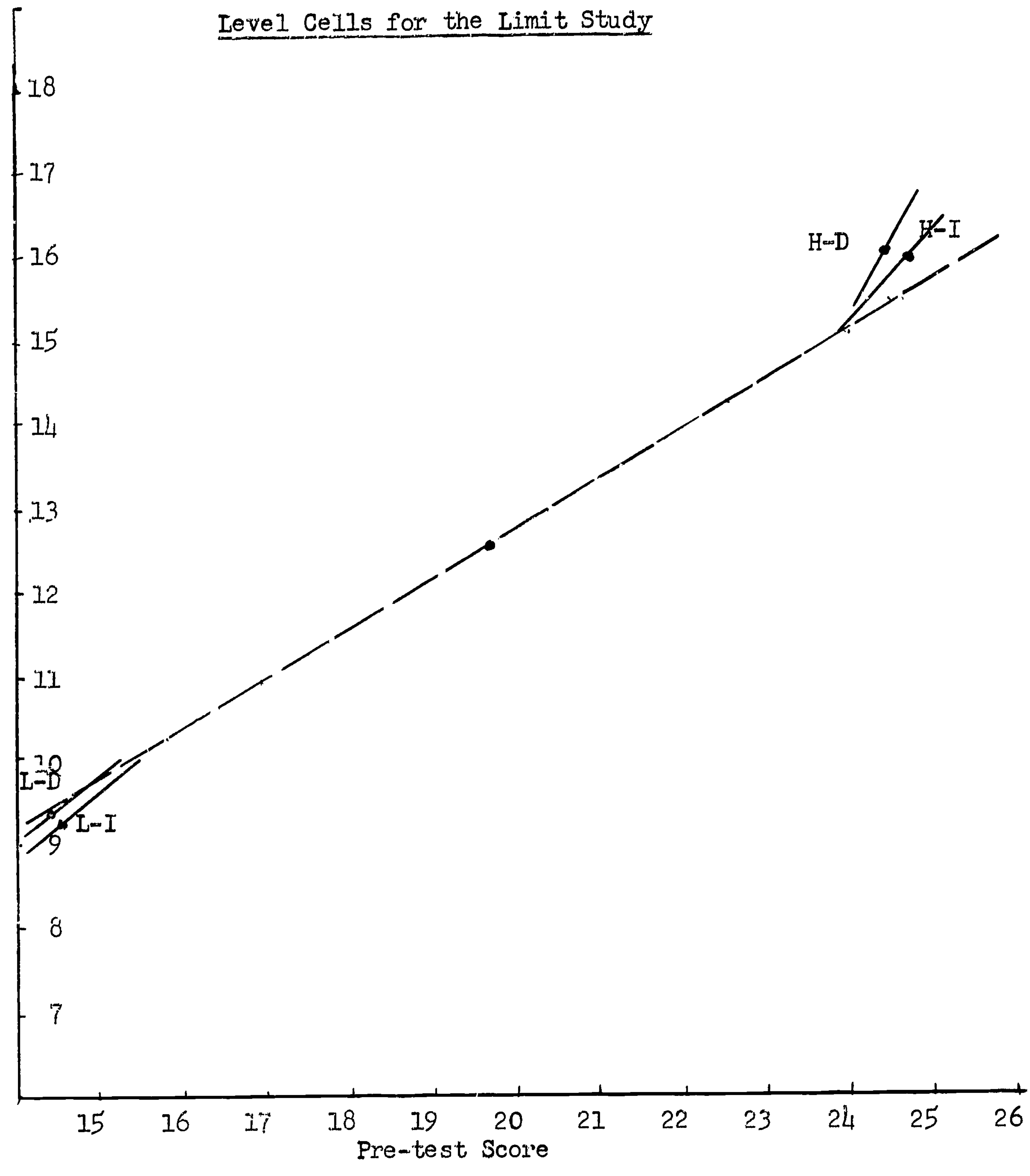
Key to Treatments
- A=inductive limit-inductive derivative
- B=inductive limit-deductive derivative
- C=deductive limit-inductive derivative
- D=deductive limit-deductive derivative

Key to Scores
- X=pre-test score
- Y=criterion test score

FIGURE 4

Regression Lines and Means for the Two High and Two Low

Level Cells for the Limit Study



KEY TO TREATMENT GROUPS
H-I= High-Inductive
H-D= High-Deductive
L-I= Low-Inductive
L-D= Low-Deductive

KEY TO GRAPHS
Within Cell Regression ____
Total Regression Line - - -

independent variable being the pre-test score. Also on the graph are the points corresponding to the mean values of these two variables and the accompanying regression line for each cell. On each level for the two treatments there is very little difference in criterion test score means. In raw score points the difference is .02 for the high groups and .18 for the low groups, with the deductive treatment means higher at each level.

Figure 5 shows a corresponding graph for the derivative study. For both levels the deductive mean scores are higher by approximately two raw score points than the inductive mean scores. The actual differences are 1.94 for the low groups and 2.22 for the high groups. In the analysis of covariance we will see that such a difference in raw score points is significant.

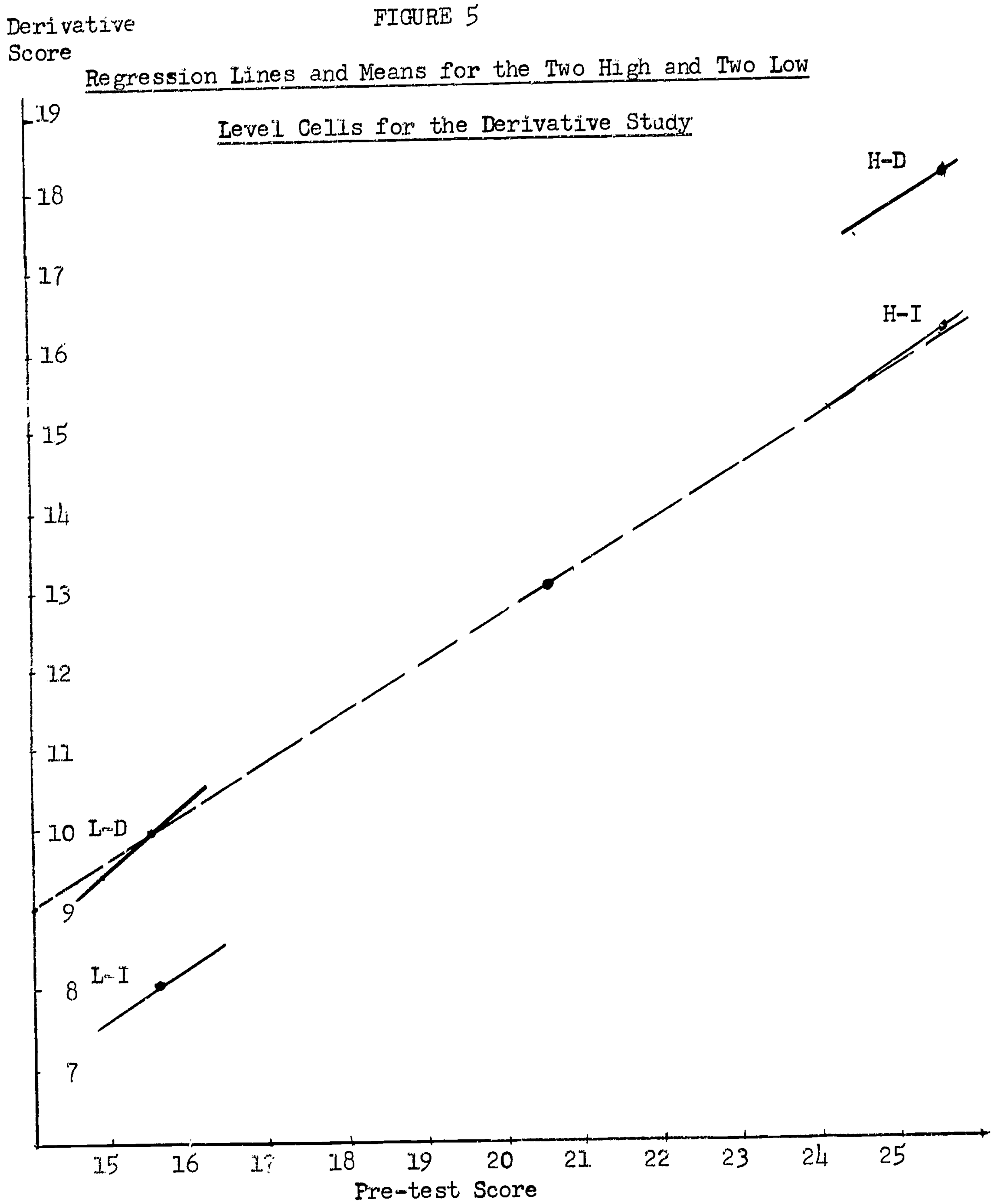
Figure 6 shows the total regression line for the study involving the four total treatments of Y (the derivative criterion test score) on X (the pre-test score). The greatest differences in raw score points between mean scores exist for inductive limit-inductive derivative (A) and deductive limit-deductive derivative (D) treatments at the low level and the inductive limit-inductive derivative (A) and inductive limit-deductive derivative (B) treatments at the high level. We might thus expect significant differences between treatments A and D and treatments A and B in the statistical analyses in the next section.

Results of the Data Analyses for the Limit Study

All the statistical analyses of data in the three studies were performed on the University of Illinois, Chicago Circle Campus 360-50 IBM

FIGURE 5

Regression Lines and Means for the Two High and Two Low Level Cells for the Derivative Study



KEY TO TREATMENT GROUPS

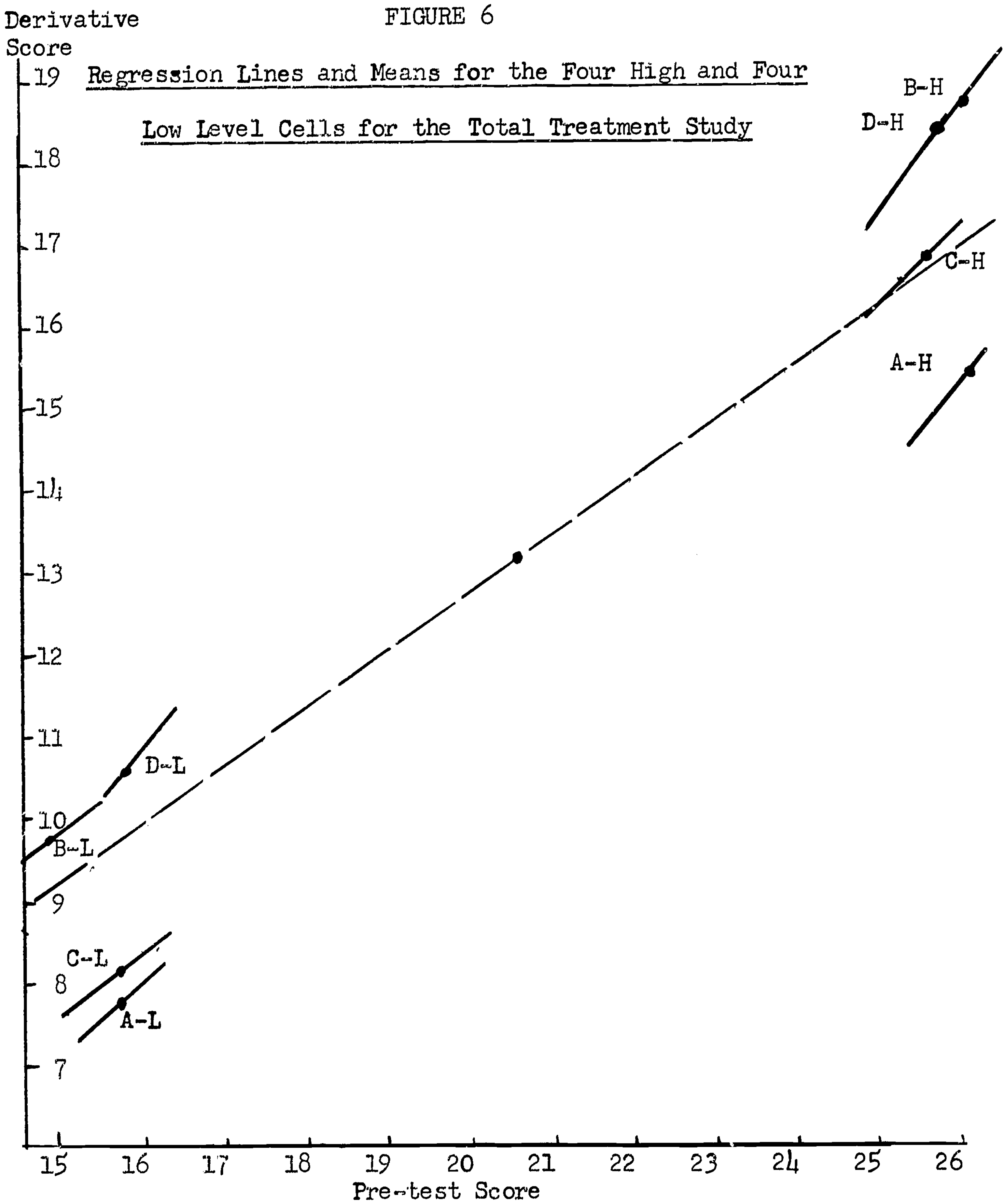
H-I= High-Inductive
H-D= High-Deductive
L-I= Low-Inductive
L-D= Low-Deductive

KEY TO GRAPHS

Within Cell Regression ______
Total Regression Line - - - -

FIGURE 6

Regression Lines and Means for the Four High and Four Low Level Cells for the Total Treatment Study



KEY TO TREATMENT GROUPS

A= inductive limit-inductive derivative
B= inductive limit-deductive derivative
C= deductive limit-inductive derivative
D= deductive limit-deductive derivative

H= high
L= low

KEY TO GRAPHS

Within Cell Regression _______
Total Regression Line - - - -

computer. Sixteen significant digits (double precision) were used in all computations.

Hypothesis L 1. There is no difference in the mean achievement test scores on limits, after adjustment for the scores on the pre-test, for the deductive and inductive treatments.

A 2 x 2, treatments by levels analysis of covariance was performed on the limit criterion test scores, replicating Shelton's study. An analysis of variance was also performed as a check for interaction in the analysis of covariance.

The results of these two analyses are exhibited in Table X. For the analysis of variance there is no significant difference in treatments. For the analysis of covariance, the F value for the two treatments (.12) is not significant at the .05 level for 1 and 395 df. Since the F value is not significant at the .05 level, we do not reject the null hypothesis. We have no evidence that there is a difference in achievement between the two treatments.

Hypothesis L 2. There is no difference in the adjusted mean achievement scores as measured by the test on limits between the two levels in the experiment.

For the analysis of variance there is a significant difference ($p<.001$) between the high and low achievement levels used in the study, since the experimental design was constructed using a high and low achievement group on pre-test scores. The F of 1.13 between levels in the analysis of covariance is not significant at the .05 level. It should be noted that there is an increase in the probability levels for the F between levels, proceeding from the analysis of variance to the

TABLE X

Inductive Limit Treatment versus Deductive

Limit Treatment for Limit Study

Analysis of Variance Summary Table

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	.64	1	.64	.02	-
Levels	4529.29	1	4529.29	120.72	.001
Treatments X Levels	1.00	1	1.00	.03	-
Error (Within)	14857.38	396	37.52		

Analysis of Covariance Summary Table

The pre-test score is the control variable.

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	4.23	1	4.23	.12	-
Levels	38.44	1	38.44	1.13	.77
Treatments X Levels	.28	1	.28	.01	-
Error (Within)	13419.26	395	33.97		

KEY TO TREATMENTS
Inductive limit
Deductive limit

KEY TO LEVELS
High
Low

analysis of covariance. This is due to the covariate determining the levels for the experimental design.

The null hypothesis concerning achievement and the two levels used in the experiment is not rejected.

The bar graph of limit criterion test score means for the four groups in the limit study in Figure 7 may be helpful in understanding the results of the tests of the preceding two hypotheses.

Hypothesis L 3. There is no interaction between the inductive and deductive limit treatments and the high and low levels - - the treatments produce similar results at both levels.

The analysis of covariance of Table X shows that the treatments by levels interaction for the limit criterion test is not significant at the .05 level. The analysis of variance table displaying no interaction effects supports this finding. Thus, the null hypothesis concerning interaction achievement is not rejected.

Lack of evidence to reject the three foregoing hypotheses is exactly the conclusion Shelton reached in his limit study.

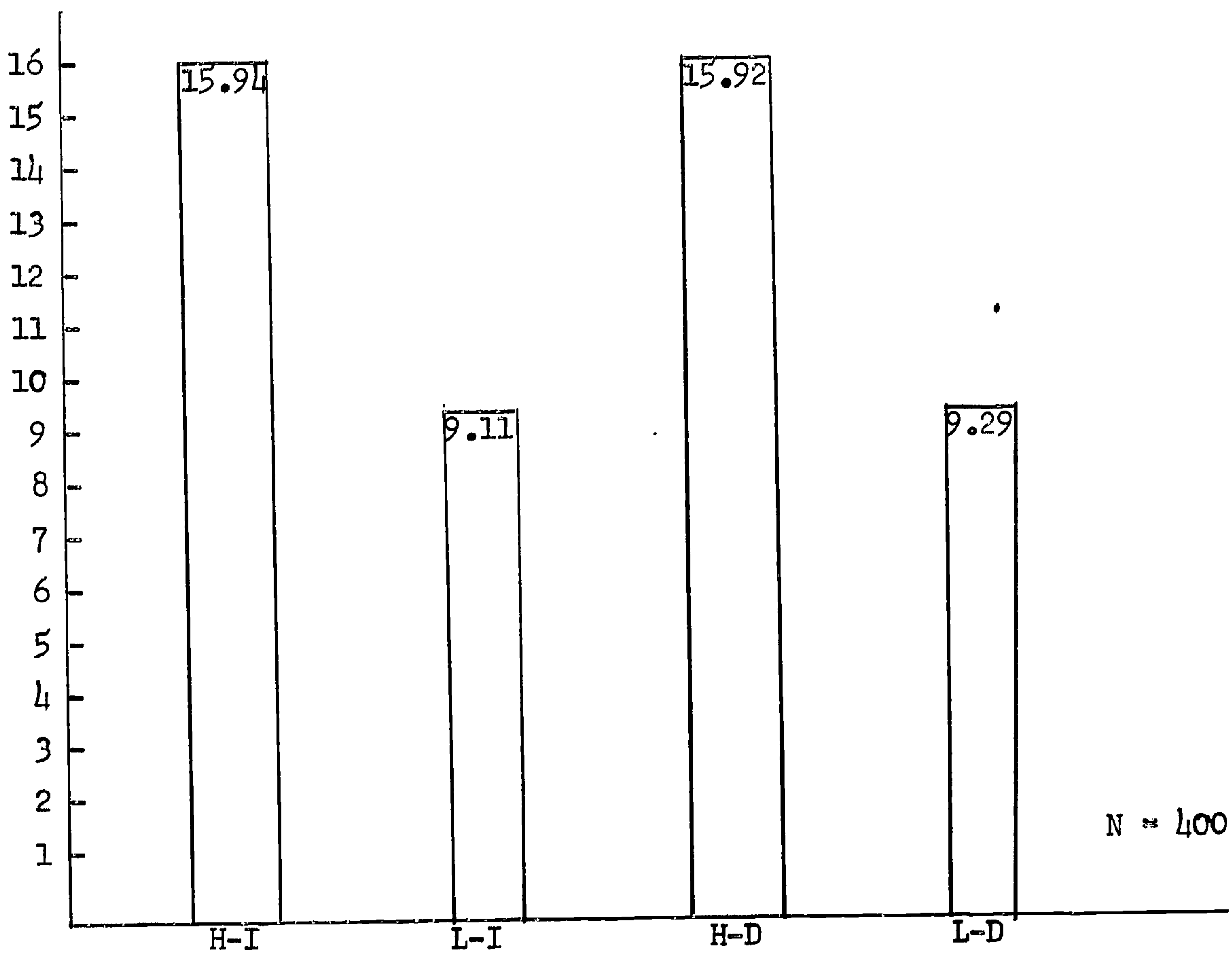
Results of the Data Analyses for the Derivative Study

Hypothesis D 1. There is no difference in mean achievement on the derivative criterion test scores, controlling for the pre-test, between the inductive and deductive derivative treatments.

A 2 x 2, treatments by levels analysis of covariance was performed on the derivative criterion test scores with the pre-test score as the independent variable. An analysis of variance was also performed as a check for interaction effects. The results of these two analyses are exhibited in Table XI.

FIGURE 7

Mean Limit Criterion Test Scores of High and Low Achieving Students for the Inductive and Deductive Limit Units



KEY TO TREATMENTS
H-I=High-Inductive
L-I=Low-Inductive
H-D=High-Deductive
L-D=Low-Deductive

TABLE XI

Inductive Derivative Treatment versus Deductive Derivative Treatment for Derivative Study

Analysis of Variance Summary Table

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	432.64	1	432.64	8.95	.01
Levels	6938.89	1	6938.89	143.62	.001
Treatments X Levels	1.96	1	1.96	.040	-
Error (Within)	19132.10	396	48.31		

Analysis of Covariance Summary Table

The pre-test score is the control variable.

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	424.50	1	424.50	10.30	.01
Levels	5.59	1	5.59	.14	-
Treatments X Levels	2.81	1	2.81	.07	-
Error (Within)	16283.06	395	41.22		

KEY TO TREATMENTS
Inductive derivative
Deductive derivative

KEY TO LEVELS
High
Low

For the analysis of variance there is a significant difference between treatments ($p<.01$) for 1 and 396 df. For the analysis of covariance, there is a significant difference between treatments ($p<.01$) for 1 and 395 df.

Thus, the null hypothesis that there is no difference in results on the derivative criterion test, controlling for pre-test scores, between the two derivative treatments can be rejected from the above evidence. There is a significant difference in derivative achievement scores between the two derivative treatments. The graphs in Figures 5 and 8 show that it is the deductive derivative treatment that has a higher mean score on the derivative criterion test than the inductive derivative treatment on both levels.

Hypothesis D 2. There is no difference in mean achievement as measured by the test on the derivative, controlling for the pre-test, between the two levels used in the experiment.

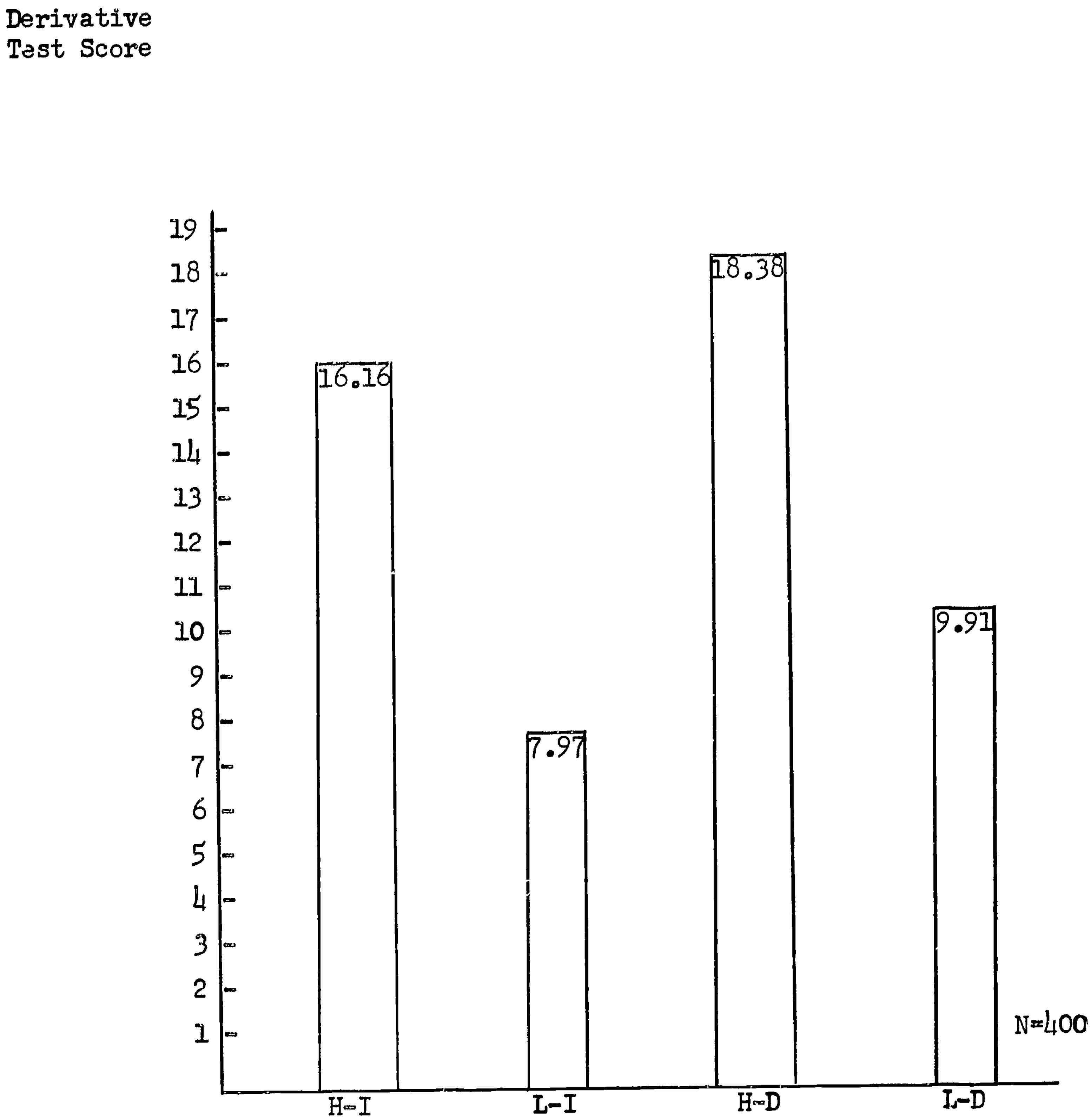
For the analysis of variance, there is a significant difference ($p<.001$) for the levels of the experiment. As in the limit study this difference is due to the experimental design. By use of analysis of covariance it was found that there is no significant difference in levels. Thus, the null hypothesis concerning achievement and the two levels used in the experiment is not rejected. The bar graph in Figure 8 is helpful in interpreting the result of the test of this hypothesis.

Hypothesis D 3. There is no interaction between the inductive and deductive derivative treatments and the high and low levels.

The analysis of covariance of Table XI shows that the treatments by levels interaction for the derivative criterion test is not

FIGURE 8

Mean Derivative Criterion Test Score of High and Low Achieving Students for the Inductive and Deductive Derivative Units



KEY TO TREATMENTS
H-I=High-Inductive
L-I=Low-Inductive
H-D=High-Deductive
L-D=Low-Deductive

significant at the .05 level. Again, no interaction effects in the analysis of variance design confirm this finding. Thus, the null hypothesis concerning interaction achievement is not rejected.

Results of the Data Analyses for the Total Treatment Study

Hypothesis T 1. There is no difference in results on the achievement test for the derivative, controlling for the pre-test scores, among the four total treatments, inductive limit-inductive derivative (A), inductive limit-deductive derivative (B), deductive limit-inductive derivative (C), and deductive limit-deductive derivative (D).

A 4 x 2, treatments by levels analysis of covariance was performed on the derivative criterion test scores for the total treatment study. An analysis of variance was also performed to check for interaction effects in the analysis of covariance. The results of the two analyses are exhibited in Table XII.

Both the analysis of variance and the analysis of covariance summary tables show significant differences in the four treatments, the former with probability less than .05 and the latter with probability less than .01. Both of these probabilities satisfy the .05 level of significance of this study.

Scheffé's method of post-hoc comparisons shows a significant difference at the .05 level between the inductive limit-inductive derivative (A) and deductive limit-deductive derivative (D) treatments. A glance at Figures 6 and 9 exhibit these differences in raw score points between treatments A and D at both levels and bear out the above finding. In the graphs we see that for the high and low groups, a difference in

TABLE XII

Combinations of Inductive Treatments versus Deductive Treatments for Teaching Both the Limit and Derivative Concepts in the Total Treatment Study

Analysis of Variance Summary Table

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	478.25	3	159.42	3.28	.05
Levels	6938.89	1	6938.89	142.68	.001
Treatments X Levels	24.41	3	8.14	.17	-
Error (Within)	19064.04	392	48.63		

Analysis of Covariance Summary Table

The pre-test score is the control variable.

Source	Sum of Squares	df	Mean Square	F	Probability Level
Treatments	496.33	3	165.44	4.00	.01
Levels	4.36	1	4.36	.11	-
Treatments X Levels	30.79	3	10.26	.25	-
Error (Within)	16183.28	391	41.39		

KEY TO TREATMENTS
- A=inductive limit-inductive derivative
- B=inductive limit-deductive derivative
- C=deductive limit-inductive derivative
- D=deductive limit-deductive derivative

KEY TO LEVELS
- High
- Low

total raw score points of almost three exists between treatments A and D. Since the means of treatment D at both levels are higher than those of treatment A, the superiority of the deductive treatment over the inductive treatment is indicated. These findings correspond to those of the derivative study.

Using Scheffé's method the difference between the inductive limit-inductive derivative (A) and inductive limit-deductive derivative (B) treatments approaches significance at the .05 level. This is also seen graphically in Figures 6 and 9 and might again indicate the superiority of the deductive derivative treatment. For the low group the difference between treatments A and B is almost two raw score points, and for the high group the difference is over three raw score points.

There is a raw score point difference between treatments B and C (deductive limit-inductive derivative) of less than two points on both levels. Treatment B is superior in either case. This may indicate that the inductive limit-deductive derivative (B) treatment is slightly better than the deductive limit-inductive derivative (C) treatment. The differences using Scheffé's method for these treatments are not significant. However, in the derivative study we did find the superiority of the deductive derivative treatment.

The other differences in treatments for the total study are not significant by Scheffé's method.

Hypothesis T 2. There is no difference in mean achievement as measured by the test on the derivative between the two levels in the total treatment study, controlling for pre-test score.

FIGURE 9 (a)

Mean Derivative Criterion Test Score of High Achieving Students for Total Treatment Study

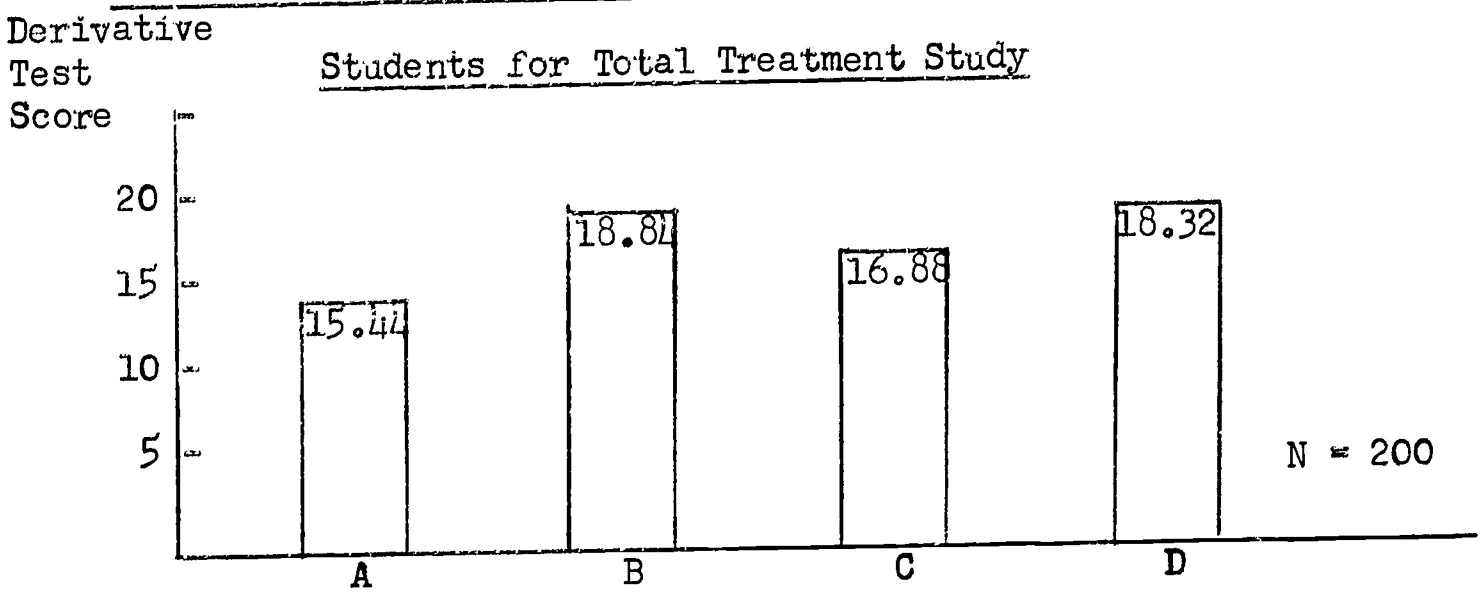
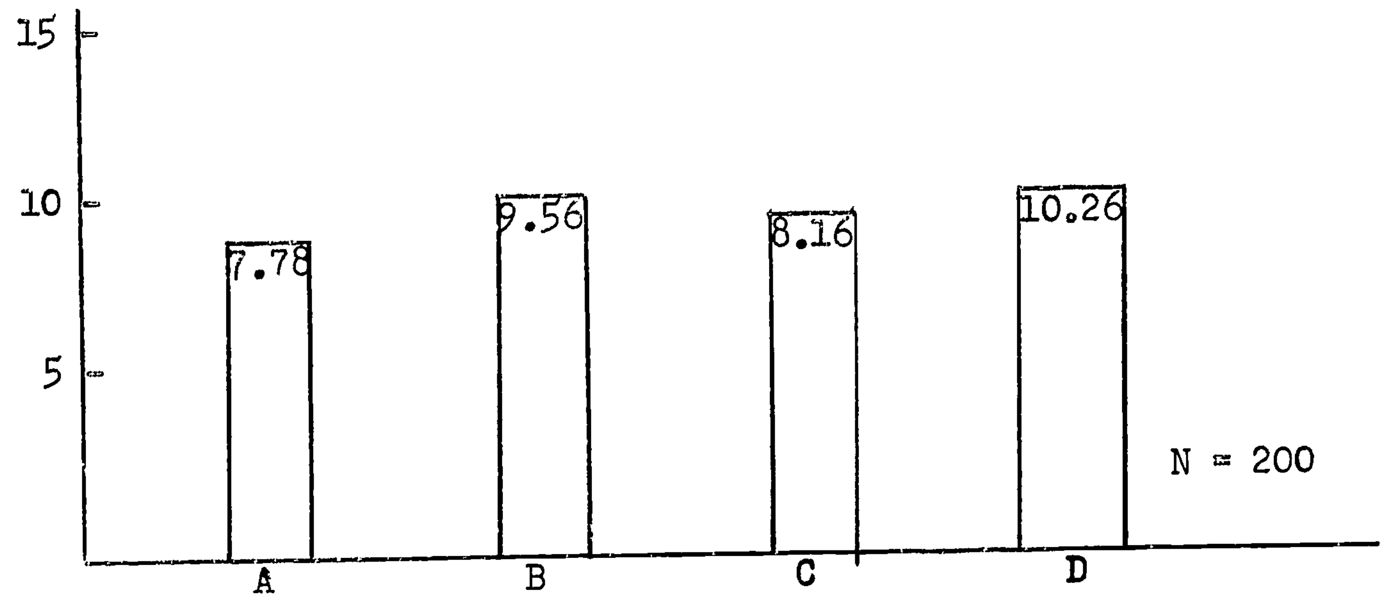


FIGURE 9 (b)

Mean Derivative Criterion Test Score of Low Achieving Students for Total Treatment Study



KEY TO TREATMENTS
A=inductive limit-inductive derivative
B=inductive limit-deductive derivative
C=deductive limit-inductive derivative
D=deductive limit-deductive derivative

The significant difference in levels for the analysis of variance was expected. However, in Table XII we see there is no significant difference for the analysis of covariance concerning achievement of the derivative topic and the two levels in this study. The null hypothesis that there is no difference in achievement as measured by the test on the derivative between the two levels used in the study is not rejected.

Hypothesis T 3. There is no interaction between the four total treatments and the two levels.

The analysis of covariance of Table XII shows no treatments by levels interaction for the total treatment study at the .05 level of significance. The analysis of variance table displaying no interaction effects supports this finding. Thus, the null hypothesis concerning interaction achievement is not rejected.

Further Statistical Analyses

It was felt that if any significant difference in treatments were to exist in the three studies of this experiment, the difference would be in favor of the inductive treatments. Such a belief was supported by the pilot studies and related research studies. Before the analyses of variance and covariance were performed, correlation and multiple regression coefficients were computed. In these preliminary analyses the inductive treatments were coded +1 and the deductive treatments -1.

Table XIII displays the correlation matrix with the five variates (pre-test score, limit treatment, limit score, derivative treatment, derivative score) in the experiment. This analysis includes all 449 subjects in the experiment. We see that there are moderate correlations between pre-test score and limit score (.621) and between pre-test score and derivative score (.607). Former analyses indicated both these correlations significant ($p < .001$). Thus, the pre-test score may be a somewhat good predictor of the limit and derivative scores.

There is a correlation of .745 between the limit score and the derivative score, significant for t ($p < .001$) by a former analysis. Thus, the limit score is a fairly good predictor of the derivative score. In fact, it may be an even better predictor of the derivative score than the pre-test score.

The negative correlation of -.137 between the derivative treatment and derivative score is significant ($p < .005$) by a former analysis. The negative value of this correlation coefficient shows that the deductive derivative treatment, coded -1, produced a higher derivative score than the inductive treatment. It is worth noting that there is a very low, non-significant correlation (.016) between limit treatment and limit score. This indicates a negligible effect of limit treatment on limit score. The other correlations in the table are very low and non-significant.

To further support the high correlations among pre-test score, limit score and derivative score, and the experimental superiority of the deductive derivative treatment, prediction equations were derived using standardized beta weights. Table XIV (a) displays the standardized beta weights and their probability levels of t for predicting the limit score from the pre-test score and limit treatment. Table XIV (b) shows the standardized beta weights and their probability levels of t

TABLE XIII

Correlation Matrix for the Five Variates in the Three Studies

Number	Name	1	2	3	4	5
1	Pre-test Score	1.000	.621	.010	.607	-.003
2	Limit Score	.621	1.000	.016	.745	-.002
3	Limit Treatment	.010	.016	1.000	-.032	-.009
4	Derivative Score	.607	.745	-.032	1.000	-.137
5	Derivative Treatment	-.003	-.002	-.009	-.137	1.000

for predicting the derivative score from the other four variates in the experiment. The variates are numbered as in Table XIII.

Below each table is the linear multiple regression prediction equation for the respective score. A multiple regression model is appropriate since none of the analyses of variance showed interaction effects. The multiple R, the correlation between the actual score and the predicted score, is also displayed. Again an inductive treatment is coded +1 and a deductive treatment -1.

In Table XIV (a) we again see that it is the pre-test's beta weight of .621 ($p < .001$) that seems to contribute considerably more to the predicted limit score than does the limit treatment beta weight. Table XV (a) shows the per cent of the total and explained variance in the limit criterion test score contributed by the pre-test score and limit treatment. We see that 38.56 per cent of the total variance and 99^{+} per cent of the explained variance is due to the pre-test score in predicting the limit score. The per cent of the total variance and the explained variance contributed by the limit treatment is negligible.

Table XIV (b) shows that the pre-test score, limit score, and derivative treatment seem to contribute more to the predicted derivative score than the limit treatment. The beta weights for the pre-test score, limit score and derivative treatment are all significant ($p < .001$). Since the beta weight for the derivative treatment is negative, this favors the deductive derivative treatment. A higher predicted derivative score for the deductive treatment will result since the last term in the derivative score prediction equation will be positive.

TABLE XIV (a)

Standardized Beta Weights for Predicting Limit Score

Number	Predictor Variate	Standardized Beta (β_i)	Student's t for Beta	Probability Level for t
1	Pre-test Score	.621	16.692	.001
3	Limit Treatment	.010	.270	.078

Standardized predicted limit score = β_1(standardized pre-test score)
+β_3(limit treatment)

Multiple R = .622

TABLE XIV (b)

Standardized Beta Weights for Predicting Derivative Score

Number	Predictor Variate	Standardized Beta (β_i)	Student's t for Beta	Probability Level for t
1	Pre-test Score	.233	6.139	.001
2	Limit Score	.601	15.802	.001
3	Limit Treatment	-.045	-1.496	.135
5	Derivative Treatment	-.135	-4.544	.001

Standardized predicted derivative score = β_1(standardized pre-test score)
+β_2(standardized limit score)
+β_3(limit treatment)
+β_5(derivative treatment)

Multiple R = .780

TABLE XV (a)

Per Cent of the Total Variance and Explained Variance in Limit Criterion Test Score Contributed by the Pre-Test Score and Limit Treatment

Number	Predictor Variate	r	β_i	$r\beta_i \times 100$	$\frac{r}{R^2}\beta_i \times 100$
1	Pre-test Score	.621	.621	38.56	99.97
3	Limit Treatment	.010	.010	.01	.03
Total				38.57	100.00

TABLE XV (b)

Per Cent of the Total Variance and Explained Variance in Derivative Criterion Test Score Contributed by the Other Four Variates

Number	Predictor Variate	r	β_i	$r\beta_i \times 100$	$\frac{r}{R^2}\beta_i \times 100$
1	Pre-test Score	.607	.233	14.14	23.22
2	Limit Score	.745	.601	44.77	73.51
3	Limit Treatment	-.032	-.045	.14	.23
5	Derivative Treatment	-.137	-.135	1.85	3.04
Total				60.90	100.00

KEY TO COLUMN HEADINGS

β_i = standardized beta weight

r = raw correlation coefficient

$r\beta_i x100$ = per cent of total variance accounted for by the variable

$\frac{r\beta_i}{R^2} x100$ = per cent of explained variance accounted for by the variable

Table XV (b) shows the per cent of the total and explained variance in the derivative criterion test score contributed by the pre-test, limit score, limit treatment, and derivative treatment. The pre-test score contributes 14.14 per cent of the total variance and the limit score contributes 44.77 per cent of this variance. The per cent of the total variance contributed by the limit treatment and derivative treatment is negligible. The pre-test score contributes 23.22 per cent of the explained variance and the limit score contributes 73.51 of this variance. Again the per cent of the explained variance contributed by the limit treatment and the derivative treatment is inconsequential.

In this study we can see that a student's past mathematical knowledge accounts for much more of the total variance than the teaching method does. In learning the limit concept the pre-test score was the important determinant of achievement. For the derivative concept knowledge of the immediately preceding topic of the limit was most important, followed in importance by the prerequisite knowledge measured by the pre-test.

All the foregoing results substantiate those discussed for the correlation matrix of Table XIII.

General Discussion

The high, significant correlations among pre-test score, limit and derivative scores, and derivative treatment indicate a definite relation between these variables in this experiment. This relationship is presented in the graphs in Figures 4, 5, and 6. Thus, some precision might be gained by using an analysis of covariance design rather than an analysis of variance. However, using the covariate to determine the levels of the experimental design in the analysis of covariance is statistically questionable.

Shelton's study for the inductive and deductive limit units was replicated. Failure to reject any of his three hypotheses in the present study supports Shelton's conclusions. Figure 4 indicates no appreciable difference in the mean scores for the inductive and deductive limit treatments at either level. To explain his results, Shelton writes:

> . . . Care was taken to insure that the two programs had the same mathematical content. The same mathematical theory was covered in both programs, and most of the numerical examples were the same. The main difference was in the order of development of the ideas. It may be that the students rearranged the order of development in their minds after completing the programs. (60, pp. 53-54)

For the present study using the inductive and deductive derivative units, a significant difference was found between treatments ($p<.01$). The hypothesis of no difference between derivative treatments is rejected. From tables of correlation coefficients and multiple regression analyses, the deductive derivative treatment is found to be superior to the inductive derivative treatment. Figure 5 shows the deductive derivative groups at both levels with higher derivative criterion test score means than the inductive derivative groups. This is a surprising result since pilot studies and related research pointed to the superiority of an inductive approach if any difference in treatments existed. Further computation of a multiple regression equation to predict the derivative score shows the pre-test score and limit test score to be the determining factors in predicting the derivative score. The derivative treatment carries very little weight in predicting the derivative score.

Considering the four total treatments at the two levels, a significant difference ($p<.05$) was found among treatments, as in the derivative study. Again, the difference favors the deductive treatments, since the difference between the inductive limit-inductive derivative treatment (A) and the deductive limit-deductive derivative treatment (D) is significant, with the means of treatment D higher than those of treatment A on both levels. Figure 6 presents this finding graphically.

In none of the three studies are interaction effects evident. We thus have no evidence that effectiveness of these teaching units is dependent upon mathematical level as measured by the pre-test.

The results on both the limit and derivative criterion tests indicate that the programs do indeed teach their respective topics. On the basis of chance alone a mean of six would be expected on the first 24 four-alternate multiple-choice questions on each test. However, means of 12.57 and 13.12 for the 449 subjects in the experiment were obtained for the limit and derivative tests, respectively.

The pilot studies for the derivative units also showed evidence of the units' teaching. We can only hope that the content of the units and the criterion tests contain the types of achievement necessary for the testing of the nine hypotheses in this experiment.

CHAPTER V

SUMMARY AND CONCLUSIONS

Re-statement of the Problem

The original purpose of this study was to study the merits of an inductive and deductive teaching approach for the derivative concept in beginning calculus. The "teacher variable" was controlled by using programmed text material for each teaching approach. To insure that a student participating in the study had the necessary mathematical background to learn the derivative, a unit on the limit concept was first read by each student. This unit was also an inductive or deductive programmed text.

It was then decided to expand the study. A former study to evaluate the effectiveness of an inductive or deductive approach to teaching the limit concept in beginning calculus by the programs used in this study was replicated. A total treatment study was also conducted, using both the limit and derivative programs. The four paired teaching treatments (inductive limit-inductive derivative, inductive limit-deductive derivative, deductive limit-inductive derivative, deductive limit-deductive derivative) were compared.

Students were divided into a high and low level on the basis of pretest scores to check for interaction between treatments and levels. This was done to see if those students at the high level might learn better from one treatment, while those at the low level might learn better from an alternate treatment.

The effectiveness of the treatments in each study was determined by a limit criterion test and a derivative criterion test. The limit criterion test was used in the statistical analyses of the limit study and the derivative criterion test in the statistical analyses of both the derivative and total treatment studies.

Instructional Programs

The treatments constituted the reading of two linearly programmed sequential texts on the limit and derivative concepts over a two week period. Four programs were used, two to teach the limit concept and two to teach the derivative concept. The main difference in the two programs for each topic was the method of presentation of the material. One program was written by an inductive format, proceeding from concrete, numerical examples to a general abstract case. The other program was written by a deductive approach, proceeding from an abstract generalization to concrete numerical examples. Both programs for each topic contained essentially the same content of basic theorems, corollaries, and numerical examples. The time exposure allowed for each treatment was controlled by the number of frames in the units.

Experimental Design

A treatments by levels analysis of covariance was used in each of the three studies in this experiment. All subjects were assigned to a high or low level on the basis of scores, on a pre-test designed to measure prerequisite mathematical knowledge for the study of the limit concept in beginning calculus. For the limit and derivative studies each level was divided into two treatment groups, inductive and deductive, giving rise to a 2 x 2, treatments by levels design. For the total treatment study each level was divided into four groups, thus establishing a 4 x 2, treatments by levels design.

Each subject in the experiment received one limit treatment and one derivative treatment. Criterion scores for the limit and derivative tests were compared for treatments, levels and interaction by the analyses of covariance. Analyses of variance were performed for each study to check for interaction effects in the analyses of covariance and to interpret the analyses of covariance results more meaningfully. Tests of significance were made at the .05 level.

Preliminary computations of correlation coefficients and multiple regression equations were also made.

Population and Sampling

The experiment was conducted in eight Chicago suburban high schools, using eleventh and twelfth grade mathematics students. Of the total of 463 students who began the study, 449 completed it. The scores of these 449 students were used in the statistical analyses.

The students were enrolled in 22 mathematics classes in the eight high schools in the fall of 1967. There were 338 males and 111 females in the total study. Four hundred scores were randomly selected for the statistical analyses in the three studies. Thus, there were 100 subjects in each of the four cells of the limit and derivative experimental designs and 50 subjects in each of the eight cells of the total treatment experimental design.

Administrative Procedures

The pre-test, limit and derivative units, and limit and derivative criterion tests were administered to each class as it met in its respective high school classroom. The class teacher proctored all the classroom reading time, answered all the individual student questions except when the investigator was present, and graded all the multiple-choice questions on the criterion tests. The treatments were distributed and collected each class session. For students needing additional reading time, extra class sessions during or after the school day were arranged. A few very slow reading students were allowed to take their units home for extra work.

Criterion Tests

Two paper and pencil criterion tests were used in the three studies as a measure of achievement. The limit criterion test was developed in a former research study. It consisted of 24 four-alternate multiple-choice questions and four short answer questions. Two of the short answer questions required proofs and two required definitions.

The derivative criterion test was developed for this study. It had the same format as the limit criterion test - 24 four-alternate multiple-choice questions and four short answer questions. One short answer question required writing a proof, one giving an explanation, one doing a computation, and the fourth exhibiting an example of a function satisfying certain conditions. Care was taken to avoid favoring either treatment in the item construction of the test.

Both tests were given in two parts on successive days, immediately after the completion of the respective program.

Results

The three research hypotheses in each of the three studies were tested by an analysis of covariance. For the limit study, the investigator found:

1. There were no statistically significant differences in achievement between the two limit treatment groups shown by the adjusted limit criterion test means.

2. There were no statistically significant differences in achievement shown by the adjusted limit criterion test means between the two levels used in the study.

3. There was no statistically significant interaction between the limit treatments and levels as measured by the limit criterion test.

For the derivative study, it was found that:

1. There was a statistically significant difference in achievement between the two derivative treatment groups shown by the adjusted derivative criterion test means. The deductive treatment was favored.

2. There were no statistically significant differences in achievement between the two levels used in the study shown by the adjusted derivative criterion test means.

3. There was no statistically significant interaction between the derivative treatments and levels as measured by the derivative criterion test.

For the total treatment study, the investigator found:

1. There was a statistically significant difference in achievement between the four total treatment groups shown by the adjusted derivative criterion test means. This difference was between the inductive limit-inductive derivative and deductive limit-deductive derivative treatments, the latter superior.

2. There were no statistically significant differences in achievement between the two levels used in the total treatment study shown by the adjusted derivative criterion test means.

3. There was no statistically significant interaction between the total treatments and levels as measured by the derivative criterion test.

Conclusions

In drawing any generalizations in this experiment, we must keep in mind the particular sample, treatments and their method of presentation, as well as the evaluation instruments used. Aware of these restrictions we can conclude:

1. No advantage in achievement of either limit treatment was apparent, but advantages were noted in the derivative treatment and the limit-derivative (total) treatment. The deductive treatments were favored.

2. No difference in achievement between the two levels was found in the limit, derivative or total treatment study.

3. No advantages in achievement of the treatments for a particular level were apparent in the limit, derivative, or total treatment studies.

The novelty of the programmed texts used in this study was definitely a negative motivating factor. After two weeks of learning the calculus material only by reading, with no class discussion and little teacher interaction, most students expressed a negative attitude toward programmed texts. This was indicated by a response to an attitude question asked of each student at the end of the experiment. Yet, since all students used materials of the same format and since these materials were randomly distributed among all the subjects in the experiment, it is felt that any negative or positive effects of the programmed texts were present for all students.

In summary, the results of Shelton's replicated limit study indicate that it is not the teaching method but the student's prior knowledge of mathematics, as measured by a pre-test, that enables him to learn the limit concept in beginning calculus. The results of the derivative study show that the student's prior knowledge of mathematics, indicated by a pre-test and limit test score, has important weight in learning the derivative concept. The teaching method is secondary in such learning. This is true even though the deductive teaching method for the derivative study was shown to be significantly better than the inductive teaching method.

The results of the total treatment study show that for the teaching of the combination of the limit and derivative concepts in beginning calculus to eleventh and twelfth grade students, as given in these programs, the deductive approach in teaching both concepts together is superior to the inductive approach in teaching both concepts together. If each of the concepts is taught by a different method, no difference in student learning, as measured by an achievement test on the final derivative unit, is evident. There is also no difference between a mixed treatment and a strictly inductive or deductive treatment.

Cautions of Interpretations of Methods Studies

Any interpretations of the conclusions of this study must be made with caution, particularly when attempting to apply the findings to an actual classroom teaching situation. The following limitations should be considered:

1. In attempting to control for the teacher variable in this study, programmed text material was used. The high school students learned entirely from a written format for two weeks, with little teacher interaction. The students may have had discussions over the material after class, although these discussions would have been of a limited nature since the materials were not to be taken from the classroom. The teacher could not guide or stimulate student discussion. He could not let his personality or skill enter into his preference for an inductive or deductive teaching approach. If human interaction and after school work had been allowed, the results of this experiment might have been different.

2. The time exposure to the material in this study was limited to only two weeks. Perhaps this was too short a period of time to test the effectiveness of a teaching method. If the study had been extended over a whole semester or whole school year, a teaching method found effective for a short period of time might not be so for a longer time. Or a teaching method found effective for an isolated topic might not be so for a total unit of study involving a number of different concepts.

3. The students in this study were rather mathematically sophisticated eleventh and twelfth graders. Such students may be accustomed to getting much of their teaching from text books written by a deductive approach. They may learn from just one example leading to or following a generalization. For these students economy of learning time may be important, and such economy may best be effected by a deductive teaching approach. If junior high school students had been subjects in this experiment, they might have shown a preference for a less formal, more heuristic, inductive teaching method.

4. The students' mathematical backgrounds in this study were controlled only to the extent of pre-test scores. Since this pre-test is not a single, perfect measuring instrument, we might better have considered a student's total past academic performance. Results of this study favored the deductive derivative treatment. We might find that students reading the deductive treatments had higher total grade point averages in mathematics, or in all academic subjects, than did those reading the inductive treatment.

5. The psychological constructs operating in the testing-inference design of this experiment are also to be questioned. Is a test score a true indication of student learning? Will the material taught in the respective limit and derivative units be available for later recall and transfer when it is really needed?

Implications and Questions for Further Research

In terms of the limitations of these studies just cited, we can see a number of implications for further research in the area of mathematics, particularly mathematics taught at an advanced subject matter level.

As indicated in the first limitation, programmed text material may not be the best vehicle to use in testing a teaching method. A study might be devised to use actual classroom teachers in testing for differences in teaching methods. Each teacher might be "programmed" to teach both inductively and deductively. If some classes of a particular teacher taught by one method show higher achievement than those taught by the same teacher by the alternate method, certain personality constructs might be inferred to be operating in both the students and teacher.

The permanence of the effectiveness of a teaching method in mathematics might be investigated. If a particular teaching method is superior for the teaching of an isolated mathematical concept over a short period of time, will this method still be effective over a longer time period, with a change in concepts to be taught?

The factor of student year in school may determine the effectiveness of teaching method. One might investigate if an inductive or deductive teaching approach is as effective with elementary school children as the same teaching method is with high school students. Is a teaching method effective with high school students as effective with college students or is type of teaching method no longer a factor in student learning at this advanced learning level?

Up to this time very little research has been directed toward the teaching of advanced topics in a subject area. Perhaps teaching method is not really important. Maybe prerequisite knowledge is the determining factor. Or perhaps students learning an advanced topic may be so highly motivated, they will learn by any acceptable teaching method. After being introduced to an idea these mature students have their own methods of learning.

Finally, perhaps one should be more concerned with level of achievement related to a teaching method than with differences in achievement. Perhaps the scores produced in the present study of the teaching of the limit and derivative concepts were too low to be really acceptable as a minimum level of achievement for either teaching method. Maybe an acceptable score could only be obtained by a combination of several teaching approaches.

Certain questions also come to mind. Perhaps the results of this study might have been different had the length of instruction been extended or abbreviated. The limit and derivative units required approximate reading time of two weeks in the average high school program. What if a whole semester or whole school year had been used to develop a particular teaching method?

A deductive method of teaching may be more successful for a short period of time and an inductive method for an extended period of time. This might be particularly true when teaching isolated concepts such as the limit and derivative in the larger discipline of the calculus. Or perhaps certain concepts in calculus are learned better by a deductive approach rather than an inductive one, namely the derivative concept in beginning calculus. Other topics in calculus may be taught more effectively by an inductive approach. For yet other concepts, the teaching method may be unimportant, but the prerequisite knowledge in mathematics may be the important variable. We showed this in the teaching of the limit concept in this experiment.

The quality of the material in the two programs was intended to be similar. Care was taken to make content and length of exposure to topics the same. Most of the numerical examples were the same. Comparable programming techniques were used to write both inductive and deductive programs. If a student reordered the material after its presentation, the experimental design had no control over this. Thus, the individual student could have used his own combination of inductive and deductive teaching methods to learn the material in the studies.

The removal of the human element from the teaching situation in this experiment should cause concern. If teachers, rather than text material, had been "programmed" to teach only inductively or deductively, results might have been different. As mentioned before, many students in this study were bored by learning only by reading for an extended period of time. Even if there were again no differences in teaching approaches in an actual classroom situation, we might find higher scores on the criterion measure than were obtained by programmed texts.

One can also argue a long-range effect of teaching -- later appropriate recall of the material learned. Might it be that a teaching approach could produce higher immediate criterion test scores but an alternate approach produce higher retention and transfer test scores?

It can also be said that the level of the student may have an effect on his reception of a certain teaching method. Perhaps the rather mathematically mature eleventh and twelfth grade students in this study were more conditioned to a deductive type of teaching. These students were obviously accustomed to learning much of their mathematics from text books, which are primarily deductive in exposition as stated in Chapter I. Even more important, these same students probably had the knowledge to abstract from given instances and, conversely, to apply given abstractions.

Perhaps junior high school students (seventh and eighth grade students) would learn mathematics better from a more numerical, illustrative, non-verbal, inductive approach. They may still find the reading of text material difficult. For them discovery may be an important learning method for building up prior empirical knowledge. They may not yet have the experience essential for making concepts and generalizations meaningful.

Implications for Education

If we again take caution to avoid overgeneralizing, some interesting educational implications emerge from this study.

Although programmed materials were used as the vehicle of instruction in this experiment, several interesting results of their use are evident. Programmed texts can and do teach, but probably not as effectively as a classroom teacher. As this experiment also shows, programmed texts can be used in a meaningful way in research studies if conservative generalizations are made.

The results of this experiment show several interesting findings. From pilot study results and related research, it was believed that a difference in teaching approaches might favor an inductive method. When differences did exist in this study, the deductive teaching method was found superior. Perhaps for the advanced level of students in this experiment, a formal, abstract, deductive teaching method was more effective. If a less mathematically mature group of students had been chosen as subjects, a concrete, numerical, inductive approach might have produced higher achievement scores. If a combination of inductive and deductive teaching had been used, mean criterion test scores might have been higher than for either method used alone.

As this experiment clearly indicated, perhaps it is the student's prior mathematical knowledge that determines his proficiency in mathematics. This may be what Ausubel meant when he wrote:

> The cognitive structure of the particular learner must include the requisite intellectual capacities, ideational content, and experiential background. It is on this basis that the potential meaningfulness of learning material varies with such factors as age, intelligence, occupation, cultural membership, etc. (5, p. 20)

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APPENDIX A

CONCRETE INDUCTIVE DERIVATIVE PROGRAMMED UNIT

1. The derivative is a very important topic in mathematics and related sciences. It is the basis of a beginning course in calculus and provides a foundation for more advanced courses.

A118. -2

119. The equation of this tangent line passing through the point (-1, 1) is ________________.

A233. $x^4 < x^2$

234. We know by Theorem 3 that if $w(x) = x^4$, $w'(x_1) =$ ________.

2. Since we will need the limit in studying the derivative of a function, let us review the notion of a limit of a function. We know that if $\lim_{x \to x_1} f(x) = L$, we can make $f(x)$ as "close" to L as we please for x in a suitably chosen deleted neighborhood of x_1. (The student should review the mathematically precise δ-ϵ definition of a limit of a function.) Which of the numbers in the set $\{2.9, 2.99, 2.999, 3.001, 3.1\}$ belongs to the deleted neighborhood defined by the inequality $|x-3| < .01$?

A119. $y-1 = -2(x + 1)$

120. For the second function f defined by $f(x) = x^2 + x - 6$, for which we expressed the slope $m_t = 2x_1 + 1$ of the tangent line to the graph of f at any point, at the point (3,6) the slope of the tangent line is ___.

A234. $4x_1^3$

235. We know by Theorem 3 that if $v(x) = x^2$, $v'(x_1) =$ _________.

A2. 2.999, 3.001

3. We will now introduce an alternate notation for $\lim_{x \to x_1} f(x) = L$. We know that $\lim_{\Delta x \to 0}(x_1+\Delta x) = x_1$ where Δx is a variable, its value usually small in magnitude. Thus, if Δp is a variable, $\lim_{\Delta p \to 0}(p_1+\Delta p) =$ ______.

A120. 7

121. The equation of the tangent line passing through the point (3,6) is ______________.

A235. $2x_1$

236. Consider the following table:

Rule Defining the Function, Evaluated at x_1	Derivative Evaluated at x_1
$w(x_1) = x_1^4$	$4x_1^3$
$v(x_1) = x_1^2$	$2x_1$

Do you see that it is possible to obtain the derivative of $f(x) = x^6$, i.e., $(6x_1^5)$, by taking the sum of the products of the terms on the opposite ends of the arrows? That is, $x_1^4 \cdot 2x_1 + x_1^2 \cdot 4x_1^3 =$ ________.

A3. p_1

4. If Δx approaches 0, $(x_1 + \Delta x)$ approaches x_1. Therefore, $\lim_{x \to x_1} f(x) = \lim_{\Delta x \to 0} f(x_1 + \Delta x)$. If Δx approaches 0, $(x_1 + \Delta x)$ approaches x_1, so $\lim_{x \to x_1} g(x) = \lim_{\Delta x \to 0} g(\_\_\_\_)$.

A121. $y-6 = 7(x-3)$

122. For the point with coordinates (-2,-4) on the graph of f defined by $f(x) = x^2 + x - 6$, the equation of the tangent line at this point is ____________________.

A236. $6x_1^5$

237. Let $f(x) = x^6$ be rewritten $x^6 = x^5 \cdot x$. In this case $f(x) = r(x) \cdot s(x)$, where $r(x) = x^5$ and $s(x) =$ ____.

A4. $x_1 + \Delta x$

5. If $x_1 = 5$, $x_1 + \Delta x = 5 + \Delta x$. As Δx approaches 0, $(x_1 + \Delta x)$ approaches ______.

A122. $y + 4 = -3(x + 2)$

123. For the fourth function f defined by $f(x) = x$, which we discussed in frame 72, we found that the slope of the tangent line at any point on the graph of f had numerical value _____.

A237. x

238. So $D_x(x^6) = D_x($______$) = 6x_1^5$.

A5. $\underline{5}$

6. Therefore, $\lim_{x \to 5} f(x) = \lim_{\blacksquare} f(5 + \Delta x)$.

(Your answer should correctly complete the shaded box.)

A123. $\underline{1}$

124. We noted that the graph of the function and the tangent line at any point A with x-coordinate x_1 coincided, so the equation of the tangent line at any point on the graph of f is ______________.

A238. $\underline{x^5 \cdot x}$

239. We know by Theorem 3 that if $r(x) = x^5$, $r'(x_1) =$ ________.

A6. $\Delta x \to 0$

7. If $x_1 = -3$, $x_1 + \Delta x = -3 + \Delta x$. As Δx approaches 0, $x_1 + \Delta x$ approaches ____.

A124. $y = x$ (or $f(x) = x$)

125. We can check the above statements by writing the equation of the tangent line to the graph of f defined by $f(x) = x$. For the point with coordinates (2,2), since $m_t = 1$, the equation of the tangent line at this point is $y - 2 = 1(x - 2)$ or (in simplified form).

A239. $5x_1^4$

240. We also know by Theorem 2 that if $s(x) = x$, $s'(x_1) =$ ____.

A7. -3

8. Therefore, $\lim_{x \to -3} g(x) = \lim_{\Delta x \to 0} g(\_\_\_\_)$.

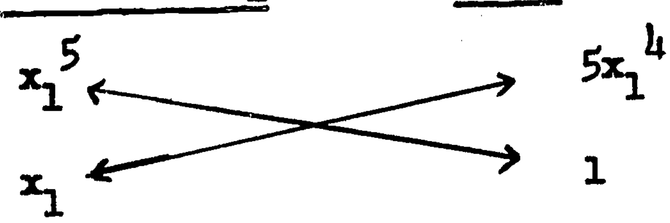
A125. $y = x$ (or $f(x) = x$)

126. For the point with coordinates (0,0), the equation of the tangent line at this point is ____________________.

A240. 1

241. Consider the following table:

Rule Defining The Function, Evaluated at x_1	Derivative Evaluated at x_1



Do you see that it is possible to obtain the derivative of $f(x) = x^6$ by taking the sum of the products of the terms on the opposite ends of the arrows? Thus, $x_1^5 \cdot \_\_\_ + x_1 \cdot \_\_\_ = 6x_1^5$.

A8. $-3 + \Delta x$

9. It may be the case that x is to the right of x_1 and sufficiently close to x_1, or that x is to the left of x_1 and sufficiently close to x_1. We recall from the section on limits that $\lim_{x \to x_1} f(x)$, now shown to be equivalent to $\lim_{\Delta x \to 0} f(x_1+\Delta x)$, exists if f(x) approaches the same value L when we consider numbers which may be either greater or ______ than x.

A126. $y - 0 = 1(x - 0)$ (or $y = x$)

127. For the last function f defined by $f(x) = k$ which we discussed, we found that the slope of the tangent line at any point A with x-coordinate x_1 on the graph of f had a value of ____.

A241. 1, $5x_1^4$

242. Let $f(x) = x^6$ be rewritten $x^6 = x^3 \cdot x^3$. In this case, $f(x) = p(x) \cdot q(x)$, where $p(x) =$ _____ and $q(x) =$ _____.

A9. less (smaller)

10. If we consider numbers only greater than x_1, then we denote the limit by $\lim_{x \to x_1^+} f(x) = L$. If we restrict our consideration to numbers less than x_1, then we denote the limit by $\lim_{x \to x_1^-} f(x) = L'$. L is called the right hand limit. Hence, the left hand limit of f(x) would be ____.

A127. 0

128. We noted that the graph of this function and the tangent line at any point A coincided, so the equation of the tangent line at any point on the graph of f is ______________.

A242. x^3, x^3

243. Thus, $D_x(x^6) = D_x(x^3 \cdot x^3) =$ ______.

A10. L'

11. Let us now translate the notation for right and left hand limits into Δx notation. If Δx approaches 0 from the right, then $x_1 + \Delta x$ approaches x_1 and $(x_1 + \Delta x)$ is (greater than, less than) x_1. This means x approaches x_1 from the right and we write $x \rightarrow$ ____.

A128. $y = k$ (or $f(x) = k$)

129. Let us verify the above reasoning for the function f defined by $f(x) = k$. For the point (2,k) on the graph of f, since $m_t = 0$, the equation of the tangent line at this point is $y - k = 0(x-2)$ or __________.

A243. $6x_1^5$

244. We know by Theorem 3 that if $p(x) = q(x) = x^3$, $p'(x_1) = q'(x_1) =$ ____.

A11. greater than, x_1^+

12. If Δx approaches 0 from the left, then $(x_1 + \Delta x)$ approaches x_1, $(x_1 + \Delta x) < x_1$, and we write $x \to$ ____.

A129. $y = k$ (or $f(x) = k$)

130. For the point (7,k), the equation of the tangent line at this point is (in simplified form).

A244. $3x_1^2$

245. Consider the following table, filling in the missing entries.

Rule Defining the Function, Evaluated at x_1	Derivative Evaluated at x_1
x_1^3	____
____	$3x_1^2$

A12. x_1^-

13. Then $\lim_{x \to x_1^+} f(x) = \lim_{\Delta x \to 0^+} f(x_1 + \Delta x)$ and $\lim_{x \to x_1^-} f(x) = \lim_{\Delta x \to 0^-} f(x_1 + \Delta x)$.

The right hand limit is thus expressed as $\lim_{\Delta x \to 0^+} f(x_1 + \Delta x)$ and the left hand limit is expressed as ________.

A130. $y = k$

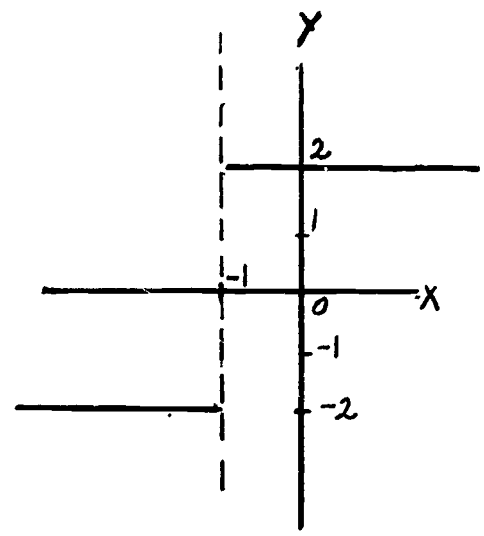
131. Let us now extract some of the common properties from the above discussion. For any function f, we can express the slope of the tangent line at the point with x-coordinate x_1 as m_t = (limit form).

A245. x_1^3 $3x_1^2$

24. From the preceding table, $x_1^3 \cdot 3x_1^2 + x_1^3 \cdot 3x_1^2 =$ ________.

A13. $\lim_{\Delta x \to 0^-} f(x_1 + \Delta x)$

14. Consider the function f discussed in the section on limits, defined by $f(x) = \frac{2(x+1)}{|x+1|}$.



For this function, $\lim_{x \to -1^+} f(x) = \lim_{\Delta x \to 0^+} f(-1 + \Delta x)$ $= 2$ and $\lim_{x \to -1^-} f(x) = \lim_{\Delta x \to 0^-} f(-1 + \Delta x) =$ ______.

A131. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

132. Since the point A has coordinates $(x_1, f(x_1))$ or (x_1, y_1) we can write the general equation of the tangent line at point A on the graph of f, with slope m_t, as ________________.

A246. $6x_1^5$

247. Let us now collect the information from the preceding frames and determine if a pattern exists in evaluating the derivative of the product of two functions. We have:

If $f(x) = x^6 = x^4 \cdot x^2$, $f'(x_1) = x_1^4 \cdot 2x_1 + x_1^2 \cdot 4x_1^3$ $(= 6x_1^5)$

If $f(x) = x^6 = x^5 \cdot x$, $f'(x_1) = x_1^5 \cdot 1 + x_1 \cdot 5x_1^4$ $(= 6x_1^5)$

If $f(x) = x^6 = x^3 \cdot x^3$, $f'(x_1) = x_1^3 \cdot 3x_1^2 + x_1^3 \cdot 3x_1^2$ $(= 6x_1^5)$

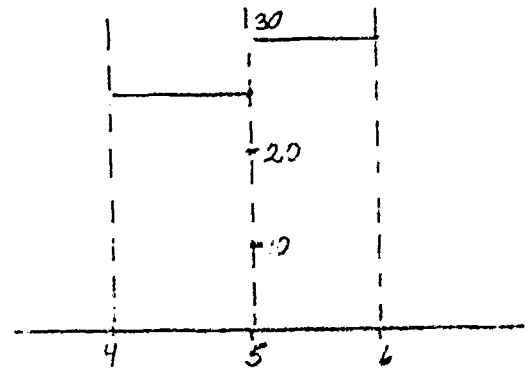
If $f(x) = w(x) \cdot v(x)$, $f'(x_1) = w(x_1) \cdot$ ______ $+$ ______ $\cdot w'(x_1)$.

A14. -2 [Note: If you do not recall this, take time now to convince yourself that $\lim_{x \to -1^+} f(x) = 2$ and $\lim_{x \to -1^-} f(x) = -2$.]

15. For the postage stamp function discussed in the section on limits, $P = f(w)$, where P is the postage and w is the weight, we have

$\lim_{\Delta w \to 0^+} f(5 + \Delta w) = 30$, where Δw is a variable.

Also, $\lim_{\Delta w \to 0^-} f(5 + \Delta w) =$ ______.



A132. $y - y_1 = m_t(x - x_1)$

133. For the exercises in Frame 115, write the equations of the tangent lines at the indicated points.

A247. $v'(x_1)$, $v(x_1)$

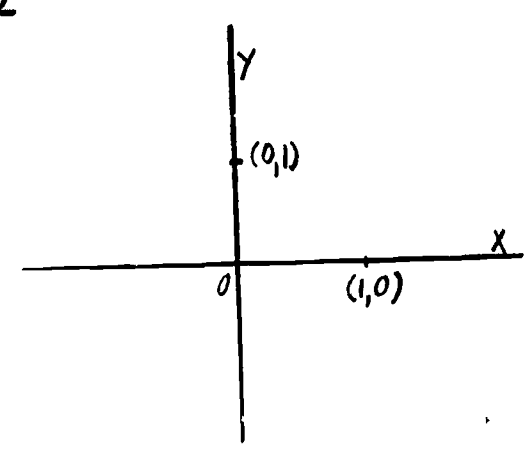
248. In words the above pattern, [if $f(x) = w(x) \cdot v(x)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1)$] states that the derivative of the product of two functions, evaluated at x_1, is the sum of the first function multiplied by the derivative of the ______ function and the second function multiplied by the derivative of the ______ function, assuming the two derivatives exist. This generalization is true, as you will prove later. It will be our sixth theorem. Enter this theorem on your list as Theorem 6.

A15. 25

16. Recall that the limit of a function exists as $x \to x_1$ if the limit of the function as $x \to x_1$ through values of x greater than x_1 is the same as the limit of the function as $x \to x_1$ through values of x less than x_1. Thus, $\lim_{x \to x_1} f(x) = L$ if $\lim_{x \to x_1^-} f(x) = L$ and $\lim_{x \to x_1^+} f(x) =$ ____.

A133. (a) $y = -2$ (or $f(x) = -2$); (b) $y = x$; (c) $y = x-1$ (or $x-y-1 = 0$); (d) $y = 0$ (or x-axis); (e) $y-3 = x+2$ (or $x-y+5 = 0$)

134. Thus far, we have considered functions whose graphs possess tangents at each point with x-coordinate x_1. Consider the function f defined by $f(x) = |x-1|$. Graph this function on your own paper on axes similar to those at right.



A248. second, first

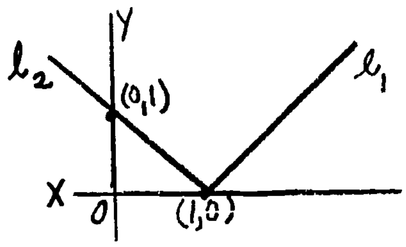
249. It should be noted that it is not necessarily the case that if the derivative of one of the functions in the product of 2 functions doesn't exist at some point, the derivative of the product won't exist at this point. Consider $w(x) = x + 2$ and $v(x) = \frac{1}{x + 2}$, where v(x) does not possess a derivative at the point with x-coordinate $x_1 =$ ____.

A16. L

17. In our notation, $\lim_{\Delta x \to 0} f(x_1 + \Delta x) = L$ if $\lim_{\Delta x \to 0^-} f(x_1 + \Delta x) = L$ and

________.

A134.



135. Referring to the graph of this function, note that the 2 lines l_1 and l_2 making up the graph and intersecting at point P with coordinates (1,0) make angles of 45 and ____ degrees, respectively, with the positive x-axis.

A249. -2

250. However, for the product of the above functions, $f(x) = w(x) \cdot v(x) = (x + 2) \cdot \frac{1}{(x + 2)} = 1$, $f'(x_1) =$ ____.

A17. $\lim_{\Delta x \to 0^+} f(x_1 + \Delta x) = L$

18. We will begin the discussion leading to the derivative of a function f evaluated at x_1. Consider the slope of a tangent to the graph of f evaluated at a point on the graph whose x-coordinate is ______.

A135. 135

136. Choose a point B on the graph with an x-coordinate slightly greater than $x_1 = 1$; i.e., $x_1 + \Delta x = 1 + \Delta x = 1.1$ where $\Delta x =$ ______. Indicate this point on your graph.

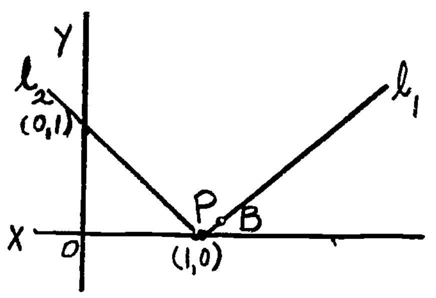
A250. 0

251. Now consider $f(x) = x^2 \cdot x^3$ where $w(x) = x^2$ and $v(x) = x^3$. Here $w(x_1) =$ ______ and $v(x_1) =$ ______.

A18. x_1

19. For the function f defined by $f(x) = x^2$, we showed in the section on limits that the slope of the tangent line at point P with x-coordinate 1 is 2. Graph this function and sketch the tangent line at the point with x-coordinate 1 on a separate paper for reference.

A136. .1

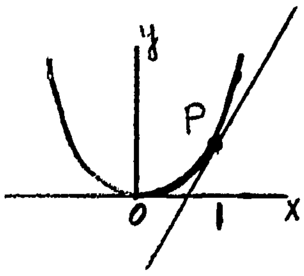


137. For point B, a secant line PB is the same as the line ______.

A251. x_1^2, x_1^3

252. Then $w'(x_1)$ = ______ and $v'(x_1)$ = ______.

A19.



20. Let us recompute this slope. Indicate the coordinates (x_p, y_p) of point P on the graph.

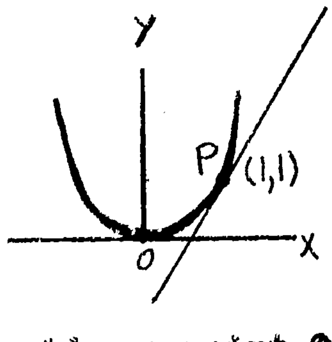
A137. 1_1

138. The limiting position of such a secant line PB as B approaches P is line ______.

A252. $2x_1$, $3x_1^2$

253. Thus if $f(x) = x^2 \cdot x^3$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ ______ + ______ = ______.

A20.



21. Consider a point Q: (x_q, y_q) on the graph of f defined by $f(x) = x^2$, whose x-coordinate is slightly greater than 1; i.e. $x_q = 1 + \Delta x$. For this choice of point Q, $\Delta x = x_q - 1$ and is (greater than, less than) 0. Indicate such a point Q on your graph.

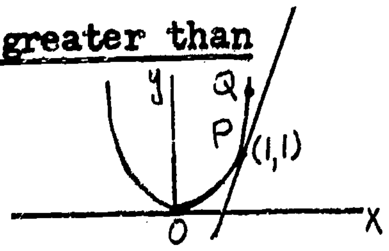
A138. l_1

139. Since l_1 makes a 45 degree angle of inclination with the positive x-axis, the slope of this line is _____.

A253. $3x_1^4$, $2x_1^4$, $5x_1^4$

254. If $f(x) = x^2 \cdot x^3$, $f(x) = x^5$, so by Theorem 3, $f'(x_1) =$ _____. Do the two answers for $f'(x_1)$, obtained in different ways, check?

A21. greater than



22. Points P and Q determine a line which intersects the graph of f in 2 points in a deleted neighborhood of P. Line PQ is called a secant line because, recalling our knowledge of a circle, a secant line intersects a curve in at least ____ distinct point(s).

A139. 1

140. Now consider a point A with x-coordinate slightly less than $x_1 = 1$; i.e. $x_1 + \Delta x = 1 + \Delta x = .5$. Here $\Delta x =$ ____. Indicate this point on your graph.

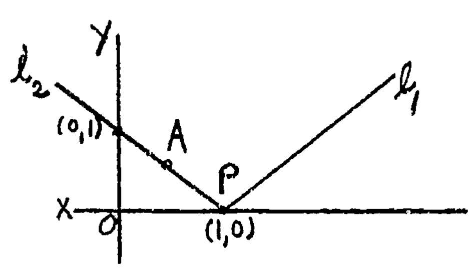
A254. $5x_1^4$, Yes

255. If $f(x) = x(1-x)$, $w(x) = x$ and $v(x) =$ ____, so $w(x_1) =$ ____ and $v(x_1) =$ ____.

A22. 2

23. Let us now compute the slopes of secant lines PQ and see how these slopes are related to the slope of the tangent line at P. For any 2 points P and Q with coordinates (x_p, y_p) and (x_q, y_q) respectively, the slope m of the line PQ, if the slope exists, is m = ______________.

A140. -.5



141. For point A, a secant line AP is the same as line _____.

A255. $1-x$, x_1, $1-x_1$

256. Then for this function, $w'(x_1)$ = ____ and $v'(x_1)$ = _____.

A23. $\frac{y_q - y_p}{x_q - x_p}$ or $\frac{y_p - y_q}{x_p - x_q}$

24. The slope of PQ will not exist if x_p = ____ in which case PQ is parallel to the ____ axis.

A141. l_2

142. Since l_2 makes a 135 degree angle of inclination with the positive x-axis, the slope of this line (l_2) is _____.

Skip a page for the answer to Frame 142.

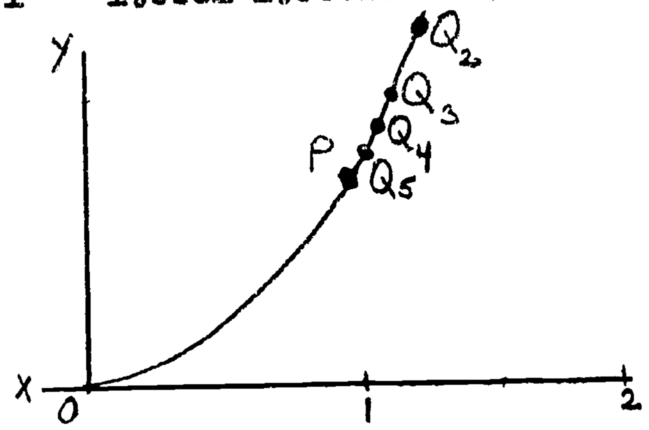
A256. 1, -1

257. Thus, if $f(x) = x(1-x)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ _______ + _______ = _______.

A24. x_q, y

25. We will now investigate the existence of the slope of the tangent line for the function f defined by $f(x) = x^2$, at point P: (1,1). To do this, we will compute the slopes $m_s = \frac{y_q - y_p}{x_q - x_p}$ of secant lines PQ for several values of Δx greater than 0, and surmise the limit of the slopes as Δx approaches 0. Complete the table below, noting the accompanying graph.

	x_p	$y_p = f(x_p)$	x_q	$y_q = f(x_q)$	$\Delta x = x_q - x_p$	$y_q - y_p = f(x_q) - y_p$	$m_s = \frac{y_q - y_p}{x_q - x_p} = \frac{f(x_q) - f(x_p)}{x_q - x_p}$
(PQ_1)	1	1	2	4	1	3	3
(PQ_2)	1	1	1.5	2.25			2.5
(PQ_3)	1	1	1.1	1.21	.1		
(PQ_4)	1	1	1.01	1.0201	.01	.0201	
(PQ_5)	1	1	1.0001	1.00020001	.001	.00020001	



A257. $x_1(-1)$, $(1-x_1) \cdot 1$, $1 - 2x_1$

258. We could have found the derivative of the above function f defined by $f(x) = x(1-x)$ in another manner. Do you see how? If $f(x) = x(1-x)$, $f(x) = x - x^2$, so by Theorem 5, $f'(x_1) =$ ________. Do your answers check?

A25. 0.5 1.25

.21 2.1

2.01

2.0001

26. Note the pattern of values of m_s in your table. Here $\Delta x = x_q - x_p$. As Δx gets smaller, reading down the table, the slopes m_s of the secant lines get close to the value ____.

A142. -1

143. For the function f defined by $f(x) = |x-1|$ at $x_1 = 1$, recalling from the limit section the function f defined by $f(x) = |x|$, you should show that for the present function $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} =$ $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} =$ ____.

A258. $1 - 2x_1$, Yes

259. If $f(x) = 7x^5$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ ________ + ________ $= 7 \cdot$ ________.

A26. 2

27. We surmise that the limit of this sequence of slopes of secant lines is probably _____.

A143. 1

144. For the function f defined by $f(x) = |x-1|$ at $x_1 = 1$, reasoning as above, you should show that $\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} =$ _____.

A259. $7 \cdot 5x_1^4$ or $35x_1^4$, $x_1^5 \cdot 0$ or 0, $5x_1^4$

260. If $f(x) = -6(x+1)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ _______ + _______ $= -6 \cdot$ _____.

A27. 2

28. We expect the slope of the tangent line to the graph of f defined by $f(x) = x^2$ at the point (1,1) to have the value ____.

A144. -1

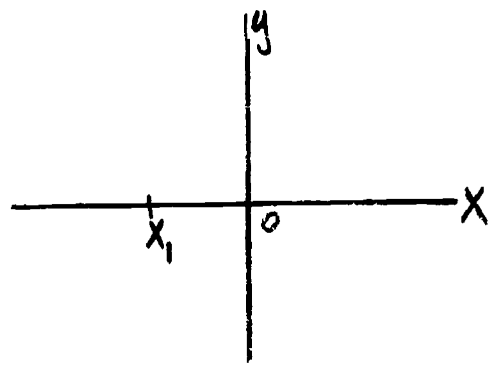
145. Thus, for f defined by $f(x) = |x-1|$, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ evaluated at x_1 (exists, doesn't exist).

A260. $-6 \cdot 1$, $(x_1 + 1) \cdot 0$, 1

261. If $f(x) = \pi(x^5 + x^3 + x^2 + 1)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ ____ $\cdot$ ________ + ________ $\cdot$ ____ $= \pi \cdot$ ________.

A28. 2 (That the slope actually is 2 will now be shown.)

29. We will now compute the slope of the tangent line at any point A: (x_1, y_1) on the graph of the above function f defined by $f(x) = x^2$. For this function, another way of expressing the coordinates (x_1, y_1) of A, since $y = x^2$ or $f(x) = x^2$, is $(x_1, f(x_1))$ or $(x_1,$ ____). Indicate such a point on the graph of f on axes as below.



A145. doesn't exist (since $\lim_{\Delta x \to 0^+} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$

146. Let us consider the function f defined by $f(x) = \sqrt{x-2}$. Graph this function on your own paper.

A261. π, $5x_1^4 + 3x_1^2 + 2x_1$, $x_1^5 + x_1^3 + x_1^2 + 1$, 0, $5x_1^4 + 3x_1^2 + 2x_1$

262. Let us summarize our results from the three previous frames and see if a general rule emerges.

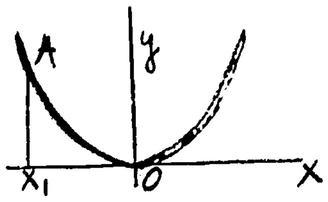
If $f(x) = 7x^5$, $f'(x_1) = 7 \cdot 5x_1^4$

If $f(x) = -6(x+1)$, $f'(x_1) = -6 \cdot 1$

If $f(x) = \pi(x^5+x^3+x^2+1)$, $f'(x_1) = \pi \cdot (5x_1^4 + 3x_1^2 + 2x_1)$

If $f(x) = k \cdot g(x)$ where k is a constant, $f'(x_1) = k \cdot$ ________.

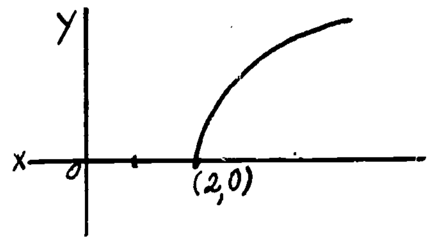
A29. x_1^2



Note: Your point A may have been placed slightly differently.

30. Consider a point B on the graph of the above function whose x-coordinate, $x_1 + \Delta x$, is greater than that of point A. For this choice of point B, Δx is (greater than, less than) 0. Indicate such a point B on your graph.

A146.

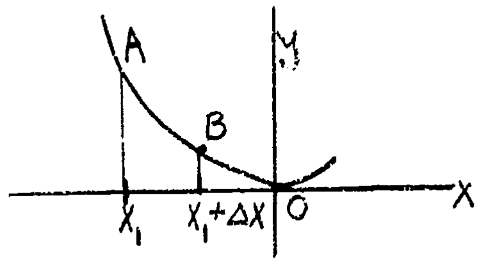


147. For this function, the domain is the set of all real numbers greater than or equal to 2 and the range is ______________________________.

A262. $g'(x_1)$

263. The above generalization [if $f(x) = k \cdot g(x)$ where k is a constant, then $f'(x_1) = k \cdot g'(x_1)$] provided $g'(x_1)$ exists is a theorem which you can easily prove from Theorem 6. A theorem which is easily derived from an immediately preceding theorem is called a ____________.

A30. greater than



31. Let us compute the slope of a secant line AB. The coordinates of point B for the function f defined by $f(x) = x^2$ are $(x_1 + \Delta x, f(x_1 + \Delta x))$ or, since $f(x) = x^2$, $(x_1 + \Delta x,$ ____________________$)$.

A147. the set of non-negative real numbers

148. Referring to the graph of this function, at what point(s) would the tangent line(s) be parallel to the y-axis?

A263. corollary (Enter this corollary on your list.)

264. The corollary, stated in words, says that the derivative of a constant multiplied by a function of x is the constant multiplied by

____.

A31. $(x_1 + \Delta x)^2$ (or $x_1^2 + 2x_1\Delta x + \Delta x^2$)

32. For the points A and B whose coordinates are given above, the slope $m_s = \frac{y_b - y_a}{x_b - x_a}$ of a secant line AB is $\frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ or if $\Delta x \neq 0$ then we simplify $\frac{(x_1 + \Delta x)^2 - x_1^2}{(x_1 + \Delta x) - x_1}$ and get ______________.

A148. (2,0)

149. At what point(s) would the slope(s) of the tangent line(s) not exist?

A264. the derivative of the function evaluated at x_1, if this derivative exists

265. Let us now prove the theorem for the derivative of the product of two functions. Theorem 6. If $f(x) = w(x) \cdot v(x)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1)$, if $w'(x_1)$ and $v'(x_1)$ exist. Before proving this theorem, consider an alternate notation for $w(x_1)$, $w(x_1 + \Delta x)$, $v(x_1)$, $v(x_1 + \Delta x)$, which will simplify the notation in our proof. Let $w(x_1) = w_1$. If x_1 changes by an amount Δx, i.e., $x = x_1 + \Delta x$, then, since w is a function of x, w_1 will change by a corresponding amount, which we will call Δw. Thus, $w(x_1 + \Delta x)$ will be denoted by $w_1 + \Delta w$. Reasoning in a similar manner, let $v(x_1)$ be denoted by v_1 and let $v(x_1 + \Delta x)$ be denoted by ____________________.

A32. $2x_1 + \Delta x$

33. Point B can be made close to point A by choosing Δx __________.

A149. (2,0)

150. Let us now proceed to evaluate the slope of the tangent to the graph of the function f defined by $f(x) = \sqrt{x-2}$ at any point with x-coordinate x_1, and then consider the point where the tangent to the graph is parallel to the y-axis, the point having x-coordinate x_1 = ____.

A265. $v_1 + \Delta v$

266. Returning to the proof of Theorem 6, we will follow the four steps in the "delta process" for finding the derivative of f(x) evaluated at x_1. In step (1), if $f(x) = w(x) \cdot v(x)$, then $f(x_1 + \Delta x) = w(x_1 + \Delta x) \cdot v(x_1 + \Delta x) = (w_1 + \Delta w) \cdot (v_1 + \Delta v)$ and $f(x_1) = w(x_1) \cdot v(x_1) =$ __________.

A33. small ("close to 0" or similar wording)

34. When Δx takes on values close to 0, $m_s = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$= 2x_1 + \Delta x$ takes on values close to ________.

A150. 2

151. If we consider both right and left limits again, we can evaluate

$\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ at $x_1 = 2$. It is not possible to evaluate

$\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ at $x_1 = 2$ because ____________________.

A266. $w_1 \cdot v_1$

267. In step (2) $f(x_1 + \Delta x) - f(x_1) = (w_1 + \Delta w)(v_1 + \Delta v) - w_1 \cdot v_1 =$

(simplified form).

A34. $2x_1$

35. From the section on limits, we know we can rewrite the above condition as $\lim_{\boxed{}} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$. Your answer should correctly complete the shaded box.

A151. the domain of f doesn't include real numbers less than 2

152. In step (1) for finding $\lim_{\Delta x \to 0^+} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ for f defined by $f(x)=\sqrt{x-2}$, $f(x_1+\Delta x)=\sqrt{(x_1+\Delta x)-2}$ and $f(x_1)=$ ________.

A267. $w_1 \cdot \Delta v + v_1 \cdot \Delta w + \Delta w \cdot \Delta v$

268. In step (3), $\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = \frac{w_1 \cdot \Delta v + v_1 \cdot \Delta w + \Delta w \cdot \Delta v}{\Delta x}$

(expressing each term in the numerator over the denominator Δx)

$= w_1 \frac{\Delta v}{\Delta x} +$ ______ $+ \Delta w \cdot \frac{\Delta v}{\Delta x}$.

A35. $\Delta x \to 0$

36. Thus, the slope of the tangent line at point A for the function f defined by $f(x) = x^2$ may be derived from a line AB, where AB is a ________ line to the graph of f.

A152. $\sqrt{x_1 - 2}$

153. In step (2), $f(x_1 + \Delta x) - f(x_1) =$ ________________________.

A268. $v_1 \cdot \frac{\Delta w}{\Delta x}$

269. In step (4),

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} (w_1 \cdot \frac{\Delta v}{\Delta x} + v_1 \cdot \frac{\Delta w}{\Delta x} + \Delta w \cdot \frac{\Delta v}{\Delta x})$$

$$(1) = \lim_{\Delta x \to 0} [w_1 \cdot \frac{\Delta v}{\Delta x}] + \lim_{\Delta x \to 0} [v_1 \cdot \frac{\Delta w}{\Delta x}] + \lim_{\Delta x \to 0} [\Delta w \cdot \frac{\Delta v}{\Delta x}]$$

$$(2) = [\lim_{\Delta x \to 0} w_1][\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}] + [\lim_{\Delta x \to 0} v_1][\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x}] + [\lim_{\Delta x \to 0} \Delta w][\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}]$$

$$(3) = w_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} + [\lim_{\Delta x \to 0} \Delta w] \cdot [\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}]$$

Supply reasons for (1), (2), (3).

A36. secant

37. Since point B can be made as close as desired to point A by taking Δx small enough, a secant line AB has a slope close to the value of the slope of the ________ line at point A.

A153. $\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}$

154. In step (3), expressing the difference quotient,

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \underline{\hspace{6cm}}.$$

A269. (1) The limit of a sum is the sum of the limits, if the limits exist. (2) The limit of a product is the product of the limits, if the limits exist. (3) The limit of a constant is that constant (w_1 and v_1 are independent of Δx and hence are constants).

270. Recalling the alternate notation for the derivative of $y = f(x)$ evaluated at x_1 as $f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we have $\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = w'(x_1)$ and

$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} =$ ________, if $w'(x_1)$ and $v'(x_1)$ exist.

A37. tangent

38. The slope of the secant line AB, $m_s = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, thus gets close to the slope of the tangent line at A as Δx gets close to 0. We may express this condition as a limit, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} (2x_1 + \Delta x)$. This limit has the value ______.

A154. $\frac{\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}}{\Delta x}$

155. Finally, in step (4), $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ______________.

A270. $v'(x_1)$

271. The terms in the last line of frame 269 can now be expressed as

$w_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = w_1 \cdot v'(x_1)$, $v_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = v_1 \cdot$ ______, and

$\lim_{\Delta x \to 0} \Delta w \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \Delta w \cdot$ ______.

A38. $2x_1$

39. Thus, if $f(x) = x^2$, we can express the slope of the tangent line at a point on the graph of f, with x-coordinate x_1, as

$$m_t = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \_\_\_\_\_\_.$$

A155. $\lim_{\Delta x \to 0^+} \frac{\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}}{\Delta x}$

156. If we try to evaluate $\lim_{\Delta x \to 0^+} \frac{\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}}{\Delta x}$, we obtain $\frac{0}{0}$, which is __________.

A271. $w'(x_1)$, $v'(x_1)$

272. Thus, we have

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = w_1 \cdot v'(x_1) + v_1 \cdot w'(x_1) + \_\_\_\_\_\_\_\_\_\_.$$

A39. $2x_1$

40. At the point (1,1) on the graph of f, $x_1 = 1$. For slope of the tangent line at this point, $m_t = 2x_1$ or $m_t =$ ____. Does this answer check with that surmised from our table in frame 25?

A156. undefined (or indeterminate)

157. However, the above limit does exist. To find its value, we can express the difference quotient $\frac{\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}}{\Delta x}$ in a different form. You remember from the limit section that we rationalize the numerator by multiplying both numerator and denominator of the expression by $\sqrt{(x_1 + \Delta x) - 2} + \sqrt{x_1 - 2}$. Thus $\frac{\sqrt{(x_1 + \Delta x) - 2} - \sqrt{x_1 - 2}}{\Delta x}$ becomes

________________________________.

A272. $(\lim_{\Delta x \to 0} \Delta w) \cdot v'(x_1)$

273. The limit of the difference quotient on the left hand side of the expression in frame 269 is $f'(x_1)$. We have $f'(x_1) = w_1 \cdot v'(x_1) + v_1 \cdot w'(x_1) + \lim_{\Delta x \to 0} \Delta w \cdot v'(x_1)$. Reviewing the statement of the theorem on your list [if $f(x) = w(x) \cdot v(x)$, ...] and comparing this statement to our expression for $f'(x_1)$ in the preceding sentence, we must show that the term $(\lim_{\Delta x \to 0} \Delta w) \cdot v'(x_1)$ is ____.

A40. 2, Yes

41. For the point with coordinates (2,4), x_1= _____ so the slope of the tangent line at this point is obtained from $m_t = 2x_1$ and is numerically _____.

A157. $\dfrac{\Delta x}{\Delta x(\sqrt{(x_1+\Delta x)-2}+\sqrt{x_1-2})}$

158. Let us now evaluate this limit; $\lim_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} =$

$$\lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x(\sqrt{(x_1+\Delta x)-2}+\sqrt{x_1-2})} = \lim_{\Delta x \to 0^+} \frac{1}{\sqrt{(x_1+\Delta x)-2}+\sqrt{x_1-2}} =$$

_______________.

A273. zero

274. Since $v'(x_1)$ is a constant, i.e., the value of the derivative of v evaluated at a fixed or constant value x_1, we must show _______ = 0.

A41. 2, 4

42. If we consider the points with coordinates $(-5, 25)$, $(0, 0)$, $(\frac{7}{2}, \frac{49}{4})$ on the graph of f, $x_1 =$ ____, ____, ____ respectively and the slopes of the tangent lines are ____, ____, ____ respectively.

A158. $\frac{1}{2\sqrt{x_1 - 2}}$

159. Why can we write $\lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x(\sqrt{(x_1 + \Delta x) - 2} + \sqrt{x_1 - 2})}$ as

$\lim_{\Delta x \to 0^+} \frac{1}{\sqrt{(x_1 + \Delta x) - 2} + \sqrt{x_1 - 2}}$?

A274. $\lim_{\Delta x \to 0} \Delta w$

275. To do this, we must show that as Δx approaches 0, Δw approaches 0. Consider the expression $w(x_1 + \Delta x) = w_1 + \Delta w = w(x_1) + \Delta w$, in terms of our alternate notation. For $w(x_1 + \Delta x)$, we see that as Δx approaches 0, $\lim_{\Delta x \to 0} w(x_1 + \Delta x) =$ ____________.

A42. $-5, 0, \frac{7}{2}, \quad -10, 0, 7$

43. Consider the function f defined by $y = x^2 + x - 6$ or $f(x) = x^2 + x - 6$. Graph this function and sketch the tangent line at the point P with x-coordinate 2 on a separate paper for reference.

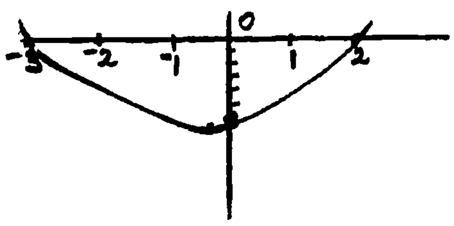
A159. Since $\Delta x \neq 0$, but approaches 0, we can divide both numerator and denominator of the difference quotient by Δx.

160. Let us now consider the point on the graph of f defined by $f(x) = \sqrt{x - 2}$ where the tangent line is parallel to the y-axis. This point has an x-coordinate $x_1 =$ ____.

A275. $w(x_1)$ (or w_1) (Note that we must here assume the continuity of the function w, a concept we will not discuss in this unit.)

276. If this is the case, then the right side of the above equality, $w(x_1 + \Delta x) = w(x_1) + \Delta w$, must approach the same value $w(x_1)$ as Δx approaches 0. This is the same as saying that as Δx approaches 0, Δw approaches ___.

A43.



44. Let us now compute the slope of the tangent line at point P: (2, 0). Consider a point Q: (x_q, y_q) on the graph of f defined by $f(x) = x^2 + x - 6$, whose x-coordinate is slightly greater than 2; i.e., $x_q = 2 + \Delta x$. For this choice of Q, $\Delta x = x_q - x_p$, which is (positive, negative). Indicate such a point Q on your graph.

A160. 2

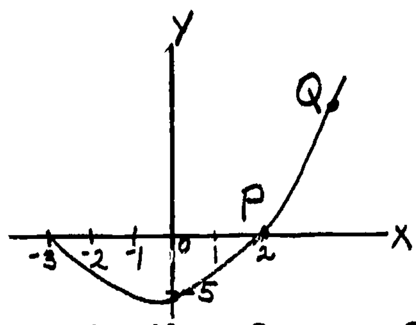
161. The slope of the tangent line at any point (x_1, y_1) such that $x_1 \geq 2$ is $m_t = \dfrac{1}{2\sqrt{x_1 - 2}}$. If $x_1 = 2$, $m_t =$ __________.

Skip a page for the answer to Frame 161.

A276. 0

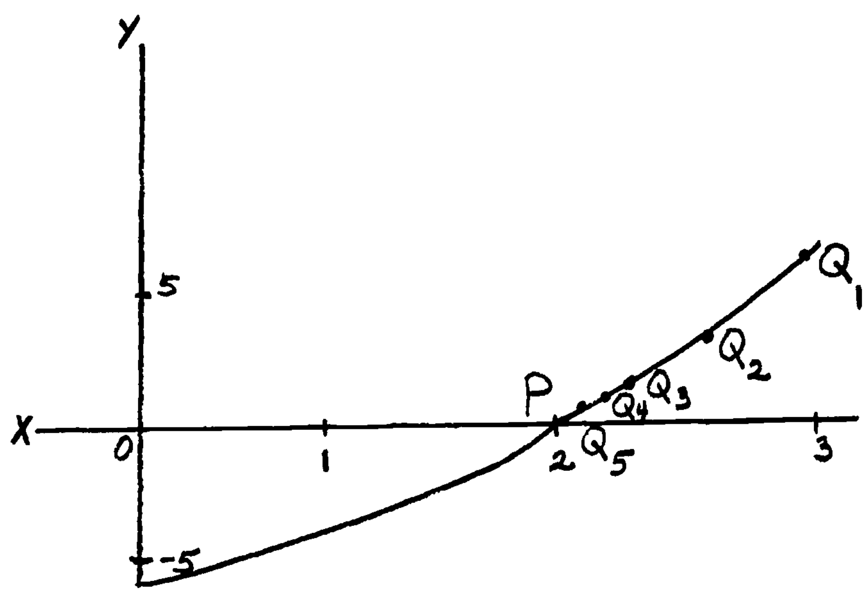
277. Thus, if Δx approaches 0, then Δw approaches 0, so $\lim_{\Delta x \to 0} \Delta w$ can be expressed as $\lim_{\Delta x \to 0} \Delta w = \lim_{\Delta w \to 0} \Delta w$ and this limit has the value _____.

A44. positive



45. We will now compute the slopes of secant lines PQ and surmise the limit of the slopes as Δx approaches 0. Complete the table below, noting the accompanying graph.

	x_p	$y_p = f(x_p)$	x_q	$y_q = f(x_q)$	$\Delta x = x_q - x_p$	$y_q - y_p = f(x_q) - f(x_p)$	$m_s = \frac{y_q - y_p}{x_q - x_p}$
(PQ_1)	2	0	3	6	1	6	6
(PQ_2)	2	0	2.5	2.75			5.5
(PQ_3)	2	0	2.1	.51	.1		
(PQ_4)	2	0	2.01	.0501	.01	.0501	
(PQ_5)	2	0	2.001	.005001	.001	.005001	



A277. 0

278. We have now proved the result required in frame 273, since

$\lim_{\Delta x \to 0} \Delta w \cdot v'(x_1) = 0 \cdot v'(x_1) =$ ____.

A45. .5 2.75

.51 5.1

5.01

5.001

46. Note the pattern of values of m_s in your table. As $\Delta x = x_q - x_p$ gets smaller, the slopes m_s of secant lines get close to ____.

A161. undefined

162. We see that $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ evaluated at $x_1 = 2$ is not defined, so the slope of the tangent line for $x_1 = 2$ (exists, doesn't exist).

A278. 0

279. Reviewing, we have proved the theorem that if $f(x) = w(x) \cdot v(x)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1)$, if $w'(x_1)$ and $v'(x_1)$ exist. We can answer the question posed in frame 230 that sought to determine if the derivative of the product of two functions evaluated at x_1 was the same as the product of the derivatives of the two functions evaluated at x_1. We know from previous examples and Theorem 6 that the answer is ___.

A46. 5

47. We would expect the slope of the tangent line to the graph of f defined by $f(x) = x^2 + x - 6$ at the point (2, 0) to be ______.

A162. doesn't exist

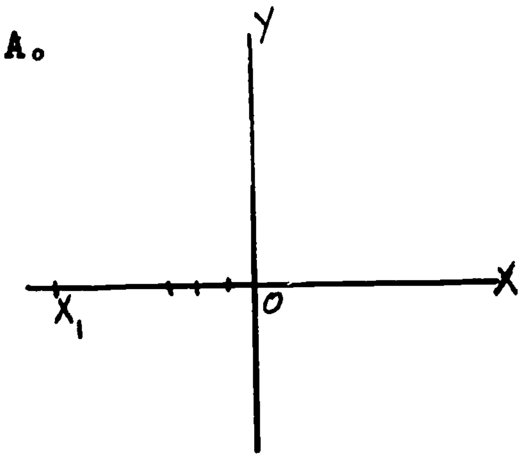
163. Since $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ is the slope of the tangent to the graph of f at the point with x-coordinate x_1, we know that this limit will not exist (will not be defined) at points on the graph for which the slope of the tangent line assumes a (horizontal, vertical) position.

A279. no

280. We will next consider a function f defined as a quotient of two functions; i.e., $f(x) = \frac{w(x)}{v(x)}$ for $v(x) \neq 0$. We know that the limit of the quotient of two functions is the quotient of the limits of the two functions, if these limits exist, so we shall want to investigate if a corresponding relationship holds for the derivative. Since the derivative of the product of two functions was not the product of the derivatives of the functions, would you expect the derivative of the quotient of two functions to be the quotient of the derivatives of the two functions?

A47. 5 (That the slope actually is 5 will now be shown.)

48. Now let us compute the slope of the tangent line at any point A: (x_1, y_1) on the graph of the above function f defined by $f(x) = x^2 + x - 6$. Graph this function on your own paper on axes similar to those below and indicate such a point A.

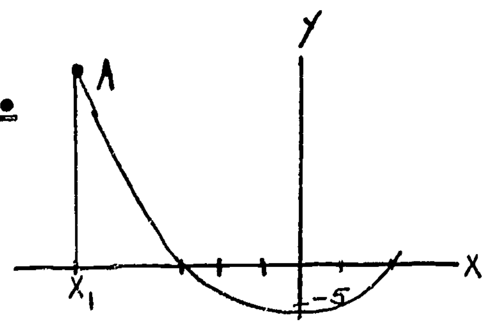


A163. vertical

164. Yet, we can write the equation of the tangent line to the graph of the function f defined by $f(x) = \sqrt{x - 2}$. Recalling that this line is parallel to the y-axis and passes through the point (2,0), its equation is ______.

A280. You would not expect this to be the case. Read the following proof to convince yourself.

281. The theorem reads: Theorem 7. If $f(x) = \frac{w(x)}{v(x)}$, $v(x) \neq 0$ and $w'(x_1)$ and $v'(x_1)$ exist, then $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$ provided ______ $\neq 0$. (Add this theorem to your list.)

A48. 

Note: Your point A may have been placed slightly differently.

49. Consider a point B on the graph of f, as defined above, whose x-coordinate $(x_1 + \Delta x)$ is less than that of A. For this choice of point B, Δx is (greater than, less than) 0. Indicate such a point B on the graph of f.

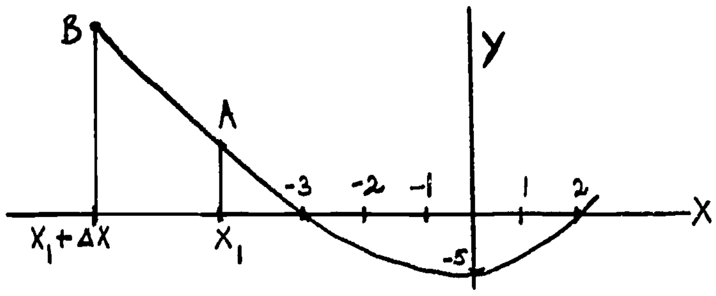
A164. $x = 2$

165. Consider the function f defined by $f(x) = \frac{1}{x}$. Graph this function on your own paper.

A281. $v(x_1)$

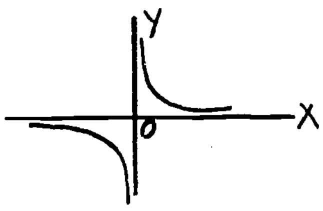
282. Stated in words, the theorem says that the derivative of the quotient of two functions evaluated at x_1, assuming the existence of the derivatives of the two functions, is the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, this quantity divided by ____________________.

A49. less than



50. Points A and B determine a secant line of the graph of f, the slope of which will enable us to find the slope of the tangent line at A. The coordinates of point B are $(x_1 + \Delta x, f(x_1 + \Delta x))$, or since $f(x) = x^2 + x - 6$, $(x_1 + \Delta x,$ ________________$)$. Indicate these coordinates on your graph.

A165.



166. The domain of f defined by $f(x) = \frac{1}{x}$ is the set of all real numbers except zero and the range of the function is ________________________.

A282. the square of the denominator, provided the denominator is nonzero

283. We will omit the proof of this theorem, which can be found in most beginning calculus books. Let us illustrate its use, however. If $f(x) = \frac{x+1}{x-3}$, $x \neq 3$, $w(x) =$ ____, $v(x) =$ ____, $w(x_1) =$ ____, $v(x_1) =$ ____, $w'(x_1) =$ ____, $v'(x_1) =$ ____.

A50. $(x_1 + \Delta x)^2 + (x_1 + \Delta x) - 6$

51. The slope of the secant line AB, $m_s = \frac{y_b - y_a}{x_b - x_a}$ is $\frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1}$ =

$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ or $\frac{[(x_1 + \Delta x)^2 + (x_1 + \Delta x) - 6] - [x_1^2 + x_1 - 6]}{\Delta x}$ =

(in simplest form ____) if $\Delta x \neq 0$.

A166. the same (the set of all real numbers except zero).

167. From the graph of this function, for what value(s) of x_1 would you expect the slope(s) of the tangent line(s) not to exist?

A283. $x + 1$, $x - 3$, $x_1 + 1$, $x_1 - 3$, 1, 1

284. Thus, $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

$= \frac{(x_1 - 3) \cdot 1 - (x_1 + 1) \cdot 1}{(x_1 - 3)^2}$

= (simplifying the numerator), if $x_1 \neq 3$.

A51. $2x_1 + \Delta x + 1$

52. Point B can be made as close to point A as desired, by choosing ____ sufficiently close to 0.

A167. 0

166. Let us proceed to show that the conjecture in the previous frame is indeed true. You should verify that for $f(x) = \frac{1}{x}$,

$$\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = -\frac{1}{x_1^2} \text{ and } \lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = -\frac{1}{x_1^2}$$

if $x_1 \neq$ ____.

A284. $\frac{-4}{(x_1 - 3)^2}$

285. If $f(x) = \frac{x^2 + 3x}{x^3 + 1}$, $x^3 + 1 \neq 0$, $w(x) =$ ______, $v(x) =$ ______, $w(x_1) =$ ______, $v(x_1) =$ ______, $w'(x_1) =$ ______, $v'(x_1) =$ ______.

A52. Δx

53. When Δx is close to 0, $m_s = \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = 2x_1 + \Delta x + 1$, so m_s is close to ______.

A168. 0

169. Thus, if $x_1 \neq 0$, $m_t = \lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = -\dfrac{1}{x_1^2}$ (exists, doesn't exist).

A285. $x^2 + 3x$, $x^3 + 1$, $x_1^2 + 3x_1$, $x_1^3 + 1$, $2x_1 + 3$, $3x_1^2$

286. Thus, $f'(x_1) = \dfrac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

$$= \frac{(x_1^3 + 1)(2x_1 + 3) - (x_1^2 + 3x_1)(3x_1^2)}{(x_1^3 + 1)^2}$$

= (simplifying the numerator) if $x_1^3 + 1 \neq 0$.

A53. $2x_1 + 1$

54. Thus, the slope of the tangent line to the graph of f defined by $f(x) = x^2 + x - 6$ can be derived from the slope of the secant line AB. This slope can be expressed as $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} (2x_1 + \Delta x + 1)$

= ________.

A169. exists

170. For $x_1 = 0$, evaluating $m_t = -\frac{1}{x_1^2}$, we conclude m_t is ________.

A286. $\frac{-x_1^4 - 6x_1^3 + 2x_1 + 3}{(x_1^3 + 1)^2}$

287. If $f(x) = \frac{x^2 - 7}{x + 4}$, $x \neq -4$, $w(x)$ = ______, $v(x)$ = ______, $w(x_1)$ = ______, $v(x_1)$ = ______, $w'(x_1)$ = ______, $v'(x_1)$ = ______.

A54. $2x_1 + 1$

55. At the point (2,0) on the graph of f, $x_1 = 2$. The slope of the tangent line at this point is $m_t = 2x_1 + 1$ or $m_t =$ ____. Does this answer check with that surmised in frame 47?

A170. undefined

171. For the function f defined by $f(x) = \sqrt{x - 2}$, we were able to write the equation of the tangent line at (2,0) even though the slope of this line didn't exist. This was true because the tangent line existed and $f(x) = \sqrt{x - 2}$ was defined at this point and had the value ____.

A287. $x^2 - 7$, $x + 4$, $x_1^2 - 7$, $x_1 + 4$, $2x_1$, 1

288. Thus, $$f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$$

$$= \frac{(x_1 + 4) \cdot (2x_1) - (x_1^2 - 7) \cdot \boxed{}}{[\boxed{}]^2}$$

= (simplifying the numerator) if $x_1 \neq -4$.

Your answer should correctly complete the shaded box.

A55. 5, Yes

56. Considering the points with coordinates $(0, -6)$, $(1, -4)$, $(\frac{1}{2}, \frac{-21}{4})$ on the graph of f, $x_1 =$ ___, ___, ___ respectively and the slopes of the tangent lines at these points are ___, ___, ___ respectively.

A171. 0

172. For the function f defined by $f(x) = \frac{1}{x}$, this function not only lacks a defined slope for the tangent line at a point with x-coordinate $x_1 = 0$, but $f(x) = \frac{1}{x}$ is not defined if $x_1 =$ ___.

A288. 1, $x_1 + 4$, $\dfrac{x_1^2 + 8x_1 + 7}{(x_1 + 4)^2}$

289. If $f(x) = \dfrac{3x^3 - 7x^2 + 9x - 4}{6x^4 - 2x^3 + 7x - 6}$, $6x^4 - 2x^3 + 7x - 6 \neq 0$, $w(x) =$ ________, $v(x) =$ ________, $w(x_1) =$ ________, $v(x_1) =$ ________, $w'(x_1) =$ ________, $v'(x_1) =$ ________.

A56. 0, 1, $\frac{1}{2}$, 1, 3, 2

57. Consider the function f defined by $y = \sqrt{25 - x^2}$ or $f(x) = \sqrt{25 - x^2}$. We showed in the section on limits that the slope of the tangent line at point P with x-coordinate 4 was $\frac{-4}{3}$. Graph this function and sketch the tangent line at the point with x-coordinate 4 on a separate paper and indicate the coordinates of point P.

A172. 0

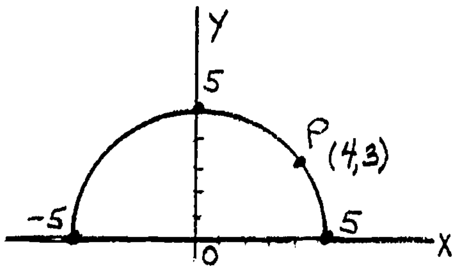
173. It is not possible to write the equation of the tangent line to the graph of the function f defined by $f(x) = \frac{1}{x}$ at a point whose x-coordinate is $x_1 = 0$; i.e., the tangent does not exist at this point, because there is no corresponding second element belonging to this function whose first element is ______.

A289. $3x^3 - 7x^2 + 9x - 4$, $6x^4 - 2x^3 + 7x - 6$, $3x_1^3 - 7x_1^2 + 9x_1 - 4$, $6x_1^4 - 2x_1^3 + 7x_1 - 6$, $9x_1^2 - 14x_1 + 9$, $24x_1^3 - 6x_1^2 + 7$

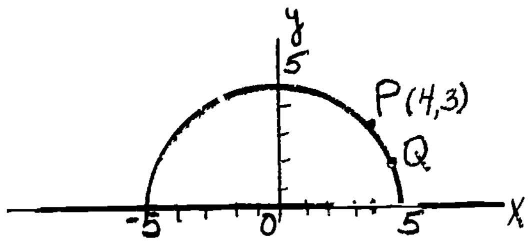
290. Thus, $f(x) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$ = ______________________________ if $v(x_1) \neq 0$.

Skip a page for the answer to Frame 290.

A57.



58. As for the previous functions, we will now attempt to arrive at a reasonable value for the slope of this tangent line at (4,3) and then investigate if it is indeed the correct value for the slope. Consider a point Q: (x_q, y_q) on the graph of f defined by $f(x) = \sqrt{25 - x^2}$, whose x-coordinate is slightly greater than 4; i.e., $x_q = 4 + \Delta x$, where $\Delta x = x_q -$ ______ is greater than 0.



A173. 0

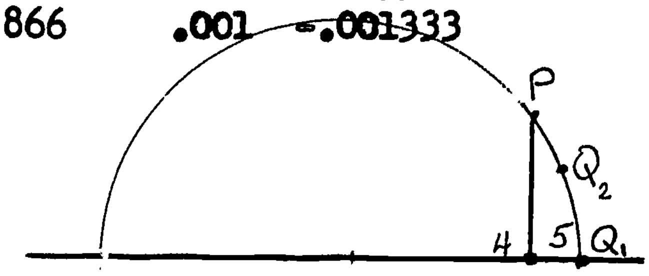
174. Thus, in the case that the slope of the tangent line to the graph of a function is undefined at a point with x-coordinate x_1, it may not be possible to write the equation of the tangent line at this point if ______________________.

Skip a page for the answer to Frame 174.

A58. x_p (or 4)

59. We will determine if the slopes of secant lines PQ approach a limit as Δx approaches 0. Complete the table below, noting the accompanying graphical representation.

	x_p	$y_p = f(x_p)$	x_q	$y_q = f(x_q)$	$\Delta x = x_q - x_p$	$y_q - y_p = f(x_q) - f(x_p)$	$m_s = \frac{y_q - y_p}{x_q - x_p} = \frac{f(x_q) - f(x_p)}{x_q - x_p}$
PQ_1	4	3	5	0	1	-3	-3
PQ_2	4	3	4.5	2.179	0.5		
PQ_3	4	3	4.1	2.862	0.1		-1.38
PQ_4	4	3	4.01	2.98662	.01	-.01338	
PQ_5	4	3	4.001	2.99866	.001	-.001333	



A290. $$\frac{(6x_1^4 - 2x_1^3 + 7x_1 - 6)(9x_1^2 - 14x_1 + 9) - (3x_1^3 - 7x_1^2 + 9x_1 - 4)(24x_1^3 - 6x_1^2 + 7)}{(6x_1^4 - 2x_1^3 + 7x_1 - 6)^2}$$

291. Let us summarize our work. We have introduced the derivative of a function f as the slope of a tangent to the graph of f at the point with x-coordinate x_1, and have presented seven differentiation theorems and one corollary. You should now review your list of these theorems and the corollary. Following is a group of exercises you should be able to complete using these theorems and the corollary.

(a) If $f(x) = (9x - 7)(x^3 + x^2 - 4x - 17)$, $f'(x_1) =$ ____________.

(b) If $f(x) = \frac{7x - 5}{x^2 - 1}$, $x \neq \pm 1$, $f'(x_1) =$ ________ if $x_1 \neq \pm 1$.

(c) If $f(x) = (x^2 - 3)^2$, $f'(x_1) =$ ____________.

(d) If $f(x) = \frac{x(2x^2 - 3)}{x - 5}$, if $x_1 \neq 5$, $f'(x_1) =$ ________ if $x_1 \neq 5$.

(e) If $f(x) = \frac{(4x^3 - 7x^2 + 9)}{(x-2)(x+1)}$ if $x \neq 2, -1$, $f'(x_1) =$ ________ if $x_1 \neq 2, -1$.

A59. -0.821 -1.642
-0.138
-1.338
-1.333

60. As $\Delta x = x_q - x_p$ gets smaller, reading down the table, the slopes m_s of the secant lines get close to the value ________________.

A174. f(x) does not exist at this point

175. An example of such a function f just discussed is defined by $f(x) =$ _____.

A291. (a) $36x_1^3+6x_1^2-86x_1-125$ [or $(9x_1-7)(3x_1^2+2x_1-4)+(x_1^3+x_1^2-4x_1-17)(9)$], (b) $\frac{-7x_1^2+10x_1-7}{(x_1^2-1)^2}$ or $\frac{(x_1^2-1)(7)-(7x_1-5)(2x_1)}{(x_1^2-1)^2}$, (c) $4x_1^3-12x_1$ (or $4x_1(x_1^2-3)$), (d) $\frac{4x_1^3-30x_1^2+15}{(x_1-5)^2}$ or $\frac{(x_1-5)(6x_1^2-3)-x_1(2x_1^2-3)}{(x_1-5)^2}$,

(e) $\frac{(x_1-2)(x_1+1)(12x_1^2-14x_1)-(4x_1^3-7x_1^2+9)(2x_1-1)}{(x_1-2)^2(x_1+1)^2}$

292. We will introduce one more basic theorem to allow us to differentiate functions of functions, or composites of one function by another. Let us first discuss the composite of one function by another, or more simply a composite function. By definition, a function is a set of ordered pairs such that no two distinct ordered pairs have the same __________ element.

A60. $-1.33\ldots$ or $\frac{-4}{3}$

61. We would expect the slope of the tangent line to the graph of f defined by $f(x)=\sqrt{25-x^2}$ at the point (4,3) to be ________________.

A175. $\frac{1}{x}$

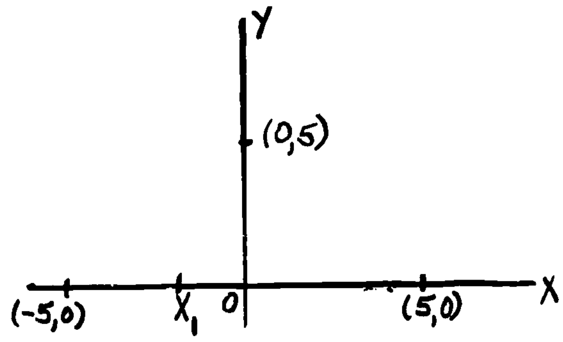
176. It is also true that the slope of a tangent line to the graph of a function may not exist at a point with x-coordinate x_1, but it may be possible to write the equation of the tangent line at this point because ____.

A292. first (Review the definition of a function in the section on limits if you had difficulty answering this item.)

293. A composite function (or a composite of one function by another) is defined by f[g(x)] or (f o g)(x). An example of such a function f o g is defined by $f[g(x)]=(f \circ g)(x)=[(x^2+1)]^2$, $z = g(x) =$ ________, $f(z) = f[g(x)] = z^2$.

A61. -1.33... or $\frac{-4}{3}$ (That the slope actually is $\frac{-4}{3}$ will now be shown.)

62. Now we will compute the slope of the tangent line at any point A: (x_1, y_1) on the graph of the above function f defined by $f(x) = \sqrt{25 - x^2}$. Graph this function on your own paper on axes similar to those below and indicate the coordinates of point A.



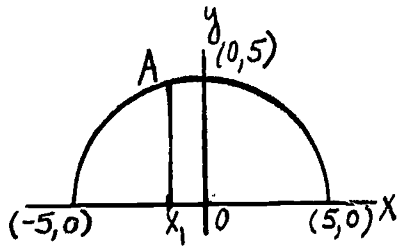
A176. f(x) is defined at this point and the tangent line exists.

177. An example of such a function f previously discussed is defined by f(x) = __________.

A293. $x^2 + 1$

294. For this function, an ordered pair belonging to f o g is (1, ___).

A62.



63. Consider a point B on the graph of the semi-circle whose x-coordinate, $x_1 + \Delta x$, is slightly greater than that of A. In this case, Δx is greater than 0. Indicate the coordinates of such a point B on your graph.

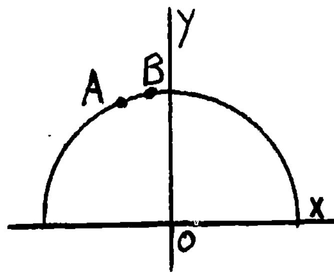
A177. $\sqrt{x - 2}$

178. Thus far, we have considered the slope of the tangent line to the graph of a function f, at a point on the graph of f with x-coordinate x_1, as a limit: $\lim_{\Delta x \to 0}$ ________________.

A294. 4

295. Other ordered pairs belonging to the function are (0,1), (2,25), (-1,____), (3,____). (Note here that we are emphasizing the fact that a composite of one function by another is a function in the "ordered pair" sense discussed in the section on limits.)

A63. $(x_1, f(x_1))$ or $(x_1, \sqrt{25 - x_1^2})$



$(x_1 + \Delta x, f(x_1 + \Delta x))$ or $(x_1 + \Delta x, \sqrt{25 - (x_1 + \Delta x)^2})$

64. The slope of a corresponding secant line AB is $m_s = \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$

or $m_s =$ ___________.

A178. $\dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$

179. We then proceeded to show that this limit does not always exist if $\lim_{\Delta x \to 0} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ is not defined (becomes infinite) or if

$\lim_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} \neq$ ____________________.

A295. 4, 100

296. Another example of a composite function f o g is defined by

$(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$

$z = g(x) = x^3$

$f(z) = f[g(x)] =$ _____.

A64. $\dfrac{\sqrt{25-(x_1+\Delta x)^2}-\sqrt{25-x_1^2}}{\Delta x}$

65. When Δx is close to 0, both numerator and denominator of

$m_s = \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ or $m_s = \dfrac{\sqrt{25-(x_1+\Delta x)^2}-\sqrt{25-x_1^2}}{\Delta x}$ approach ____.

A179. $\lim\limits_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$

180. If $\lim\limits_{\Delta x \to 0} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ is undefined (becomes infinite) or if

$\lim\limits_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} \neq \lim\limits_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$, we know that the slope of the tangent line to the graph of the function, at the point with x-coordinate x_1, (exists, doesn't exist).

A296. $\sqrt{z}$

297. Ordered pairs belonging to the composite function f o g defined above are (1,1), $(2,\sqrt{8})$, (3,______), (0,______).

A65. 0

66. We will thus seek an equivalent expression, if such exists, for the above algebraic expression, in order to evaluate its limit as Δx approaches 0. From the limit section, we know that we can rationalize the numerator of the expression by multiplying both numerator and denominator by ____________________.

A180. doesn't exist

181. Following is a group of exercises for which you are to use the information in the preceding discussion. (a) If $f(x) = |x + 2|$, does $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ for $x_1 = 2$ exist? A graph may be helpful in responding. (b) If $f(x) = \frac{1}{x-3}$, find the equation of the tangent line at the point whose x-coordinate is $x_1 = 3$. (c) If $f(x) = \sqrt{x + 5}$, find the equation of the tangent line at (-5,0). (d) If $f(x) = (x + 2)$, find the equation of the tangent line at (-2,0). (e) If $f(x) = \sqrt{x^3}$, find the equation of the tangent line at (0,0). (f) If $f(x) = x^3 + x^2 + x + 11$, find the equation of the tangent line at (-1,10).

A297. $\sqrt{27}$ (or $3\sqrt{3}$), 0

298. Let us now concentrate on recognizing the form of certain composite functions (or composites of one function by another). This ability will be needed for the next differentiation theorem. Note that $f[g(x)] = (f \circ g)(x)$ denotes that f is a function of g defined by $g(x) = z$ and g is in turn a function defined by the variable ____.

Skip a page for the answer to Frame 298.

A66. $\sqrt{25-(x_1+\Delta x)^2}+\sqrt{25-x_1^2}$

67. Thus, if $\Delta x \neq 0$, $\dfrac{\sqrt{25-(x_1+\Delta x)^2}-\sqrt{25-x_1^2}}{\Delta x} = \dfrac{\sqrt{25-(x_1+\Delta x)^2}-\sqrt{25-x_1^2}}{\Delta x}$

$\cdot \dfrac{\sqrt{25-(x_1+\Delta x)^2}+\sqrt{25-x_1^2}}{\sqrt{25-(x_1+\Delta x)^2}+\sqrt{25-x_1^2}}$ = ________________.

A181. (a) No. For $x_1 = 2$, $\lim_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = 1$ and $\lim_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = -1$, (b) $f(x)$ is not defined for $x_1 = 3$, so the tangent line does not exist at this point., (c) $x = -5$, (d) $y = x + 2$, (e) $y = 0$ (or x-axis), (f) $y - 10 = 2(x + 1)$.

182. To summarize the discussion in the preceding section, you see that there are functions for which the right and left limits exist but are not equal; i.e. $\lim_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$.

Some of these functions, defined as follows, we have discussed: $f(x) = |x-1|$, $g(x) = \sqrt{x-2}$. When it is the case for the function f that $\lim_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = \lim_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$, we know that

$\lim_{\Delta x \to 0} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ (exists, doesn't exist).

A67. $\dfrac{-2x_1+\Delta x}{\sqrt{25-(x_1+\Delta x)^2}+\sqrt{25-x_1^2}}$

68. Since the slope of a secant line AB can now be expressed as

$m_s = \dfrac{-2x_1+\Delta x}{\sqrt{25-(x_1+\Delta x)^2}+\sqrt{25-x_1^2}}$, when Δx is close to 0, m_s is close

to ________________________.

A182. exists

183. If $\lim_{\Delta x \to 0} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ exists for the function f, we define the derivative of f evaluated at x_1 to be this limit. The derivative may also be considered a function.

Definition 2: The derivative of f, evaluated at x_1, is

$\lim_{\Delta x \to 0} \dfrac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ if this limit exists. Add this definition to your list. Referring to the "List of Definitions and Theorems," we see that the derivative of a function f evaluated at x_1 is the same as ____.

A298. x

299. The composite function f o g defined by $(f \circ g)(x) = f[g(x)] = (x^2+1)^2$, considered in frame 293, can be defined in another way as the composite of one function by another. Since $(x^2+1)^2 = \sqrt{(x^2+1)^4}$, we may write $(f \circ g)(x) = (p \circ q)(x)$ or $p[q(x)] = \sqrt{(x^2+1)^4}$, $z = q(x) = (x^2+1)^4$, $p(z) = p[q(x)] =$ _____.

A68. $\dfrac{-x_1}{\sqrt{25 - x_1^2}}$

69. Thus, the slope of the tangent line to the graph of f defined by $f(x) = \sqrt{25 - x^2}$ can be derived from the slope of the secant line AB. This slope can be expressed as $\lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$

$\lim_{\Delta x \to 0} \dfrac{-2x_1 + \Delta x}{\sqrt{25 - (x_1 + \Delta x)^2} + \sqrt{25 - x_1^2}} =$ ________.

A183. the slope of the tangent line to the graph of f at x_1 (or an equivalent expression)

184. The derivative of f, evaluated at x_1, is sometimes denoted by $f'(x_1)$; i.e., $\lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = f'(x_1)$. Other notations for the derivative are y', $D_x y$, $\frac{dy}{dx}$. Differentiation is the name given to the process of finding the derivative of f evaluated at x_1. Thus, we differentiate f(x) to find ________________________________.

A299. $\sqrt{z}$

300. The composite function f o g defined by $(f \circ g)(x) = (x^2 + 1)^2$ may also be defined by $(m \circ n)(x) = m[n(x)] = (\sqrt[3]{x^2 + 1})^6$ where $n(x) =$ ______ and $m(z) = z^6$.

A69. $\dfrac{-x_1}{\sqrt{25 - x_1^2}}$

70. At the point on the graph of f with coordinates (4, 3), $x_1 = 4$. The slope of the tangent line at this point is $m_t = \dfrac{-x_1}{\sqrt{25 - x_1^2}} =$ ______.

Does this answer check with that surmised in frame 61?

A184. $f'(x_1)$ (or $\lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, y', $D_x y$, $\dfrac{dy}{dx}$)

185. Theorems exist that considerably simplify our work in computing this limit. We will now proceed to prove some of these theorems, called derivative theorems. $f'(x_1) = \lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, if this limit exists and is the ________ of the function f evaluated at x_1.

A300. $\sqrt[3]{x^2 + 1}$

301. Thus, you see that a function has more than one representation as a composite function -- in fact, an infinite number of such representations. Can you think of other ways in which $(f \circ g)(x) = f[g(x)] = (x^2 + 1)^2$ is a composite of one function by another, different from the above representations? Some other representations follow, which may be the same as or different from yours. Check your results with your teacher if you have questions.

A70. $\frac{-4}{3}$, Yes

71. Considering the points with coordinates (0, 5), (-4, 3), (1, $\sqrt{24}$) on the graph of f, the slopes of the tangent lines at these points are ____, ____, ____ respectively.

A185. derivative

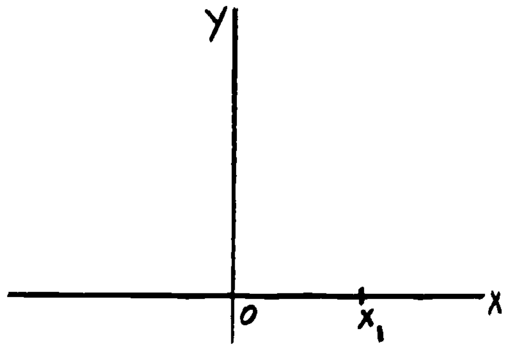
186. We have already proved two derivative theorems in the first section of this program that enable us to find $f'(x_1)$ for the function f evaluated at x_1. The first of these theorems is suggested in frame 97. Theorem 1: If $f(x) = k$, $f'(x_1) = 0$. (Add this theorem to your list.) In words, this theorem states that the derivative of a constant function is ______.

A301. $\left[(x^2+1)^{\frac{1}{4}}\right]^8$, $(\sqrt{x^2+1})^4$, $\sqrt[3]{(x^2+1)^6}$

302. Consider the second composite function f o g given above, defined by $(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$, $x \geq 0$ which may also be defined in a different way as the composite of one function by another. Since $\sqrt{x^3} = (\sqrt{x})^3$, we may write $(f \circ g)(x) = (p \circ q)(x)$ or $p[q(x)] = (\sqrt{x})^3$, $z = q(x) = \sqrt{x}$, $p(z) = p[q(x)] =$ ____.

A71. 0, $\frac{4}{3}$, $\frac{-\sqrt{6}}{12}$

72. Let us now consider the identity function f defined by $f(x) = x$. We wish to compute the slope of the tangent line at any point A, with x-coordinate x_1, on the graph of f. Graph this function on your paper and indicate point A, using axes similar to those below.



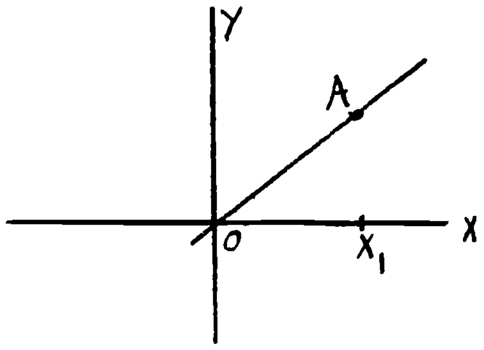
A186. zero

187. This theorem is equivalent to the statement that the slope of the tangent line at a point with x-coordinate x_1 to the graph of f defined by $f(x) = k$ is ______.

A302. z^3

303. The composite function defined by $(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$ may also be defined by $(m \circ n)(x) = m[n(x)] = \sqrt[4]{x^6}$ where $n(x) =$ _____ and $m(z) = \sqrt[4]{z}$.

A72.



73. The point A has coordinates $(x_1, f(x_1))$ or since $f(x) = x$, $(x_1, \_\_\_\_)$.

A187. zero

188. The second theorem concerns the derivative of the identity function f defined by $f(x) = x$, which we showed in frame 85 was one. Written in mathematical notation, this theorem states:

Theorem 2: If $f(x) = x$, $f'(x_1) =$ ____. (Add this theorem to your list.)

A303. x^6

304. We have now expressed $(f \circ g)(x) = \sqrt{x^3}$ as the composite of one function by another in three ways. Can you think of other ways, different from the given three, for which $(f \circ g)(x) = \sqrt{x^3}$ is a composite of one function by another? Some such representations follow.

A73. x_1

74. Consider a point B on the graph of f whose x-coordinate, $x_1 + \Delta x$, is slightly greater than that of point A. For this choice of B, indicate this point on your graph.

A188. 1

189. This theorem is equivalent to the statement that the slope of the tangent line to the graph of f, defined by $f(x) = x$ evaluated at a point with x-coordinate x_1, is ____.

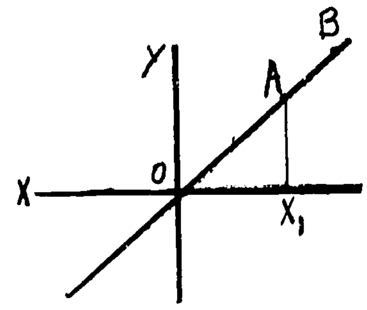
A304. $\sqrt[3]{x^{\frac{9}{2}}}$, $\sqrt[3]{x}^{\frac{9}{2}}$, $\sqrt[8]{x^{12}}$

305. Consider the composite function f o g defined by

$$(f \circ g)(x) = \frac{1}{\sqrt[3]{x+1}} = (x+1)^{-\frac{1}{3}}, \text{ if } x \neq -1,$$

or $f[g(x)] = (x+1)^{-\frac{1}{3}}$ where $g(x) = x + 1$ and $f(z) =$ ______________.

A74.



75. Points A and B determine a secant line of the graph of f. In this case, such a secant line AB coincides with ________________.

A189. one

190. A useful theorem concerns the derivative of a power of x, the function f defined by $f(x) = x^n$, where n is a positive integer. For $f(x) = x^2$, we saw in frame 39 that $f'(x_1) =$ ______.

A305. $z^{-\frac{1}{3}}$ (or $\frac{1}{\sqrt[3]{z}}$) or $\frac{\sqrt[3]{z^2}}{z}$

306. The composite function f o g defined by $(f \circ g)(x) = (x + 1)^{-\frac{1}{3}}$ may also be defined by $(p \circ q)(x) = (\sqrt[3]{x + 1})^{-1}$ where $q(x) =$ ________ and $p(z) =$ ______.

A75. the graph of f (or equivalent wording)

76. We know that the slope of the line which is the graph of f defined by $f(x) = x$ is ____.

A190. $2x_1$

191. For $f(x) = x^3$, (see the first term of the expression in exercise 181(f)) you can conclude $f'(x_1) =$ ______.

A306. $\dfrac{\sqrt[3]{x+1}}{z^{-1} \text{ (or } \frac{1}{z})}$

307. The composite function f o g defined by $(f \circ g)(x) = (x + 1)^{-\frac{1}{3}}$ may also be defined by $(m \circ n)(x) = \sqrt[3]{(x + 1)^{-1}}$ where $n(x) =$ __________ and $m(z) =$ __________.

A76. 1

77. The tangent line at A coincides with ________________.

A191. $3x_1^2$

192. For $f(x) = x$, $f(x)$ is a power of x, namely the ______ power.

A307. $(x + 1)^{-1}$ (or $\frac{1}{x + 1}$), $\sqrt[3]{z}$ (or $z^{\frac{1}{3}}$)

308. Can you define $(f \circ g)(x) = (x + 1)^{-\frac{1}{3}}$ as a composite function by representations different from those cited above? Some such representations follow.

A77. the graph of f (or equivalent wording)

78. Thus, the slope of the tangent line at A is ____.

A192. first

193. For $f(x) = x = x^1$, we showed in Theorem 2 that $f'(x_1) =$ ____.

A308. $\frac{1}{\sqrt[6]{(x+1)^2}}$, $[(x+1)^{-2}]^{\frac{1}{6}}$, $[(x+1)^3]^{\frac{-1}{9}}$

309. Following is a list of functions you are to define as composite functions in two other ways as indicated:

(a1) $F(x) = (f \circ g)(x) = x$ where $g(x) = x^3$, $f(z) = f[g(x)] = z^{\blacksquare}$.

(a2) $F(x) = (f \circ g)(x) = x$ where $g(x) = x^{\blacksquare}$, $f(z) = f[g(x)] = z^7$.

(b1) $H(x) = (f \circ g)(x) = \frac{1}{x} = x^{-1}$, $x \neq 0$, where $g(x) = x^2$, $f(z) = f[g(x)] = z^{\blacksquare}$.

(b2) $H(x) = (f \circ g)(x) = x^{\blacksquare}$, $x \neq 0$, where $g(x) = x$, $f(z) = f[g(x)] = z^{-1}$.

(c1) $Q(x) = (f \circ g)(x) = \left(\frac{1}{\sqrt{x}}\right)^3 = x^{\frac{-3}{2}}$, $x > 0$ where $g(x) = x$, $f(z) = f[g(x)] = z^{\blacksquare}$.

(c2) $Q(x) = (f \circ g)(x) = \left(\frac{1}{\sqrt{x}}\right)^3 = x^{\frac{-3}{2}}$ where $g(x) = x^{\blacksquare}$, $f(z) = f[g(x)] = z^{-1}$.

(d1) $P(x) = (f \circ g)(x) = \frac{1}{(1+x)^{-2}}$ where $g(x) = 1 + x$, $f(z) = f[g(x)] = z^{\blacksquare}$.

(d2) $P(x) = (f \circ g)(x) = \frac{1}{(1+x)^{-2}}$ where $g(x) = (1+x)^{\blacksquare}$, $f(z) = f[g(x)] = \sqrt{z}$.

(e1) $G(x) = (f \circ g)(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$, where $g(x) = (1 - x^2)^2$, $f(z) = f[g(x)] = z^{\blacksquare}$.

(e2) $G(x) = (f \circ g)(x) = \sqrt{(1 - x^2)}$, $-1 \leq x \leq 1$, where $g(x) = (1 - x^2)^{\blacksquare}$, $f(z) = f[g(x)] = z^3$.

A78. 1

79. Let us compute the slope of the tangent line at point A for this function and check it with the above reasoning. The coordinates of point B are $(x_1 + \Delta x, f(x_1 + \Delta x))$ or since $f(x) = x$, $(x_1 + \Delta x,$ ________$)$.

A193. 1

194. Let us now arrange the results of the previous frames. If

$f(x) = x^1$, $f'(x_1) = 1 \cdot x_1^{1-1} = 1 \cdot x_1^0 = 1 \cdot 1 = 1$

$f(x) = x^2$, $f'(x_1) = 2 \cdot x_1^{2-1} = 2x_1^1 = 2x_1$

$f(x) = x^3$, $f'(x_1) = 3x_1^{3-1} = 3x_1^2$

$f(x) = x^4$, $f'(x_1) = 4x_1^{4-1} =$ ______.

A309. (a1) $\frac{1}{3}$, (a2) $\frac{1}{7}$, (b1) $\frac{-1}{2}$, (b2) -1, (c1) $\frac{-3}{2}$, (c2) $\frac{+3}{2}$, (d1) $+2$, (d2) 4, (e1) $\frac{1}{4}$, (e2) $\frac{1}{6}$

310. Consider once again the function G defined by $G(x) = \sqrt{x - 2} = (x - 2)^{\frac{1}{2}}$ discussed in frames 146-164. G is a composite function defined by $G(x) = (f \circ g)(x) = (x - 2)^{\frac{1}{2}}$ where $g(x) = x - 2$ and $f(z) = z$ [redacted].

A79. $x_1 + \Delta x$

80. The slope of a secant line AB is $m_s = \dfrac{y_b - y_a}{x_b - x_a}$ or $m_s = \dfrac{(x_1 + \Delta x) - x_1}{(x_1 + \Delta x) - x_1}$ = _____ if $\Delta x \neq 0$.

A194. $4x_1^{\,3}$

195. Generalizing to the n^{th} power of x, we surmise the following theorem: Theorem 3: If $f(x) = x^n$, where n is a positive integer, $f'(x_1) =$ ________. (Add this theorem to your list.)

A310. $\frac{1}{2}$

311. For this function, $G'(x_1) = \dfrac{1}{2\sqrt{x_1 - 2}}$, as given in frame 158, can also be expressed as written $G'(x_1) = ■(x_1 - 2)^{\frac{1}{2} - 1} \cdot (1) = ■\,(x_1 - 2)^{\frac{1}{2} - 1}$ $\cdot\, D_x(x - 2)$.

A80. 1

81. Point B can be made as close to point A as desired because ________________________________.

A195. $n \cdot x_1^{n-1}$

196. To prove this theorem, we proceed through the four steps for finding $f'(x_1)$ as before. In step (1), if $f(x) = x^n$, $f(x_1) =$ ______ and $f(x_1 + \Delta x) =$ ________.

A311. $\frac{1}{2}$, $\frac{1}{2}$

312. Thus, if $G(x) = (x - 2)^{\frac{1}{2}}$, $G'(x_1) =$ (rewritten form).

A81. Δx can be chosen close to 0 but greater than 0 (or equivalent wording)

82. When Δx is close to 0, the slope of a secant line AB, m_s

$$= \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x} = \frac{(x_1+\Delta x) - x_1}{\Delta x},$$ is close to ____ if $\Delta x \neq 0$.

A196. $x_1^{\,n}$, $(x_1+\Delta x)^n$

197. We note that $f(x_1+\Delta x)=(x_1+\Delta x)^n$ can be expanded by the binomial theorem, since n is a positive integer. Thus,

$(x_1+\Delta x)^n = x_1^{\,n} + \frac{n}{1!} \circ x_1^{\,n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{\,n-2} \cdot (\Delta x)^2 +$ ______________________

______________________ $+ S$, where S is the sum of other terms, all of which contain Δx with exponents greater than 3 if $n > 3$.

A312. $\frac{1}{2}(x_1 - 2)^{\frac{1}{2} - 1} D_x(x-2)$

313. Let us now consider the function H defined by $H(x)=\sqrt{x^2-2x}$.

H is a composite function, i.e., $H(x)=(f \circ g)(x)=(x^2-2x)^{\frac{1}{2}}$ where

$g(x)=(x^2-2x)$, $f(z)=z^{\frac{1}{2}}$.

A82. 1 (Note that m_s is independent of Δx.)

83. We can rewrite the above statement as $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$= \lim_{\Delta x \to 0} 1 =$ ______.

A197. $\frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3$

198. Thus, in step (2), $f(x_1 + \Delta x) - f(x_1) = (x_1 + \Delta x)^n - x_1^n$. If we substitute the expression for $(x_1 + \Delta x)^n$ derived by the binomial expansion, we get $f(x_1 + \Delta x) - f(x_1) =$ ______________________________

$- x_1^n$.

A313. 1

314. Let us compute $H'(x_1)$ by the only method we have at our disposal at this time, that of applying the four steps of the delta process. For $H(x) = \sqrt{x^2 - 2x}$, $H(x_1) =$ ______ and $H(x_1 + \Delta x) =$ ________________.

A83. 1

84. What limit theorem did you use in the above evaluation? ________

A198. $x_1^n + \frac{n}{1!} \cdot x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3 + S$

199. Simplifying $f(x_1 + \Delta x) - f(x_1) = (x_1^n + \frac{n}{1!} x_1^{n-1} \cdot \Delta x + \ldots + \Delta x^n) - x_1^n$ we obtain $f(x_1 + \Delta x) - f(x_1) =$ ________________.

A314. $\sqrt{x_1^2 - 2x_1}$, $\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)}$

315. Combining steps (2) and (3), $\frac{H(x_1 + \Delta x) - H(x_1)}{\Delta x} =$ ________.

A84. $\lim_{x \to a} k = k$

85. Thus, if $f(x) = x$, the slope of the tangent line at any point on the graph of f, with x-coordinate x_1, has a value _____.

A199. $\frac{n}{1!} x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3 + S$

200. In step (3) $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$$= \frac{\frac{n}{1!} x_1^{n-1} \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \Delta x^3 + S}{\Delta x}$$

= ____________________________ (dividing each term by Δx).

A315. $\frac{\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} - \sqrt{x_1^2 - 2x_1}}{\Delta x}$

316. In step (4) $\lim_{\Delta x \to 0} \frac{H(x_1 + \Delta x) - H(x_1)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} - \sqrt{x_1^2 - 2x_1}}{\Delta x}$. Letting Δx approach 0, we obtain ____________________.

A85. 1

86. For the point with coordinates (0,0), on the graph of f, the slope of the tangent line is ____.

A200. $n \cdot x_1^{n-1} + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^2 + \frac{S}{\Delta x}$

201. In step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} (n \cdot x_1^{n-1}$ + the sum of terms, each containing Δx), and since the limit of a sum is the sum of the limits, we have $\lim_{\Delta x \to 0} n \cdot x_1^{n-1} + \lim_{\Delta x \to 0}$ (the sum of terms, each containing Δx). This is equal to ________.

A316. $\frac{0}{0}$, which is undefined (or an indeterminate form.)

317. Recalling the procedure employed in such a case from the section on limits, we rationalize the numerator of

$\frac{\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} - \sqrt{x_1^2 - 2x_1}}{\Delta x}$ to obtain ________________.

A86. 1

87. For the points with coordinates (1,1), (π,π), (10,10), the slopes of the tangent lines at these points are ____, ____, ____ respectively.

A201. $n \cdot x_1^{n-1}$

202. Thus, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = ________. Referring to frame 195, does this result prove the theorem?

A317. $\frac{2x_1\Delta x + \Delta x^2 - 2\Delta x}{\Delta x(\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} + \sqrt{x_1^2 - 2x_1})}$

318. In evaluating the limit now,

$\lim_{\Delta x \to 0} \frac{2x_1\Delta x + \Delta x^2 - 2\Delta x}{\Delta x(\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} + \sqrt{x_1^2 - 2x_1})}$, we can divide both numerator and denominator of the difference quotient by Δx, because ________.

A87. 1, 1, 1

88. For any point P having coordinates (n,n) where n is a real number, the slope of the tangent line of the graph of f defined by $f(x) = x$ is ___.

A202. $n \cdot x_1^{n-1}$, Yes

203. This theorem is sometimes referred to as the power differentiation formula. Using this theorem, if $f(x) = x^5$, $f'(x_1) = 5x_1^4$. If $f(x) = x^7$, $f'(x_1) =$ ______.

A318. Δx is not equal to zero but gets as close to zero as we choose.

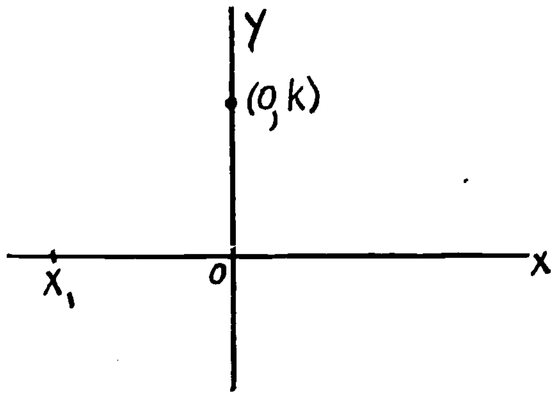
319. Thus, $\lim_{\Delta x \to 0} \frac{2x_1\Delta x + \Delta x^2 - 2\Delta x}{\Delta x(\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} + \sqrt{x_1^2 - 2x_1})}$

$= \lim_{\Delta x \to 0} \frac{2x_1 + \Delta x - 2}{(\sqrt{(x_1 + \Delta x)^2 - 2(x_1 + \Delta x)} + \sqrt{x_1^2 - 2x_1})}$

= ______________________.

A88. 1

89. Let us consider the constant function f defined by $f(x) = k$, where k is any real number. We again want to compute the slope of the tangent line at point A, with x-coordinate x_1, to the graph of f. Graph this function and the point A on axes similar to those below, on your own paper.



A203. $7x_1^6$

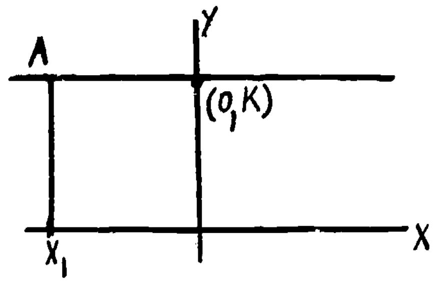
204. It will be proved later in your calculus course that if $f(x) = x^n$, where n is any real number, not simply a positive integer, it is still true that $f'(x_1) = nx_1^{n-1}$. If $f(x) = x^{\frac{4}{3}}$, $f'(x_1) = \frac{4}{3} x_1^{\frac{4}{3} - 1} = \frac{4}{3} x_1^{\frac{1}{3}}$.

If $f(x) = x^{-5}$, $f'(x_1) = -5 \cdot x_1^{-5-1} =$ ________.

A319. $\dfrac{2x_1 - 2}{2\sqrt{x_1^2 - 2x_1}}$ if $x_1^2 - 2x_1 > 0$ (or $x \geq 2$ or $x \leq 0$)

320. The above derivative can be rewritten $\dfrac{2x_1 - 2}{2\sqrt{x_1^2 - 2x_1}} = \square(x_1^2 - 2x_1)^{\frac{1}{2}-1}$ $(2x_1 - 2) = \blacksquare(x_1^2 - 2x_1)^{\frac{1}{2}-1} \cdot D_x(x^2 - 2x)$.

A89.



90. The graph of this function is a line whose slope is ____, so from the preceding discussion, do you think it would seem reasonable to assume that the slope of the tangent line at point A has the same numerical value as the slope of the graph of the function?

A204. $-5 \cdot x_1^{-6}$

205. If $f(x) = x^{-\frac{1}{7}}$, $f'(x_1) =$ ____________.

A320. $\frac{1}{2}$, $\frac{1}{2}$

321. Thus, if $H(x) = \sqrt{x^2 - 2x}$, $H'(x_1) =$ (rewritten form).

A90. 0, We will see.

91. Choose a point B, with x-coordinate $x_1 + \Delta x$, $\Delta x < 0$. Indicate point B on your graph.

A205. $-\frac{1}{7} \cdot x_1^{-\frac{8}{7}}$

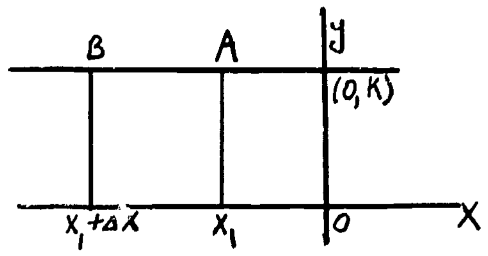
206. If $f(x) = x^{\sqrt{10}}$, $f'(x_1) =$ ______________.

A321. $\frac{1}{2}(x_1^2 - 2x_1)^{\frac{1}{2} - 1} \cdot D_x(x^2 - 2x)$

322. Consider the function P defined by $P(x) = \sqrt{x - 2x^2} = (x - 2x^2)^{\frac{1}{2}}$.

P is a composite function defined by $(f \circ g)(x) = (x - 2x^2)^{\frac{1}{2}}$ where $g(x) = x - 2x^2$, $f(z) = f[g(x)] = z^{\blacksquare}$.

A91.



92. The coordinates of points A and B are (x_1, k), (_____, _____), respectively.

A206. $\sqrt{10} \cdot x_1^{\sqrt{10} - 1}$

207. A useful theorem concerns the derivative of the sum of two or more functions. Recalling the example in frame 54, $f(x) = x^2 + x - 6$, $f'(x_1) = 2x_1 + 1$. From the three theorems we have already proved for the derivative of f evaluated at x_1 (you should have these on your list), we know that the derivative of x^2 which is the first term of the expression $x^2 + x - 6$, evaluated at x_1, is ______.

A322. $\frac{1}{2}$

323. You should verify that $P'(x_1) = \frac{1}{2}(x_1 - 2x_1^2)^{\frac{1}{2} - 1} \cdot (1 - 4x_1)$

$= \frac{1}{2}(x_1 - 2x_1^2)^{\frac{1}{2} - 1} \cdot D_x(x - \_\_\_\_\_)$ if $0 \le x_1 \le \frac{1}{2}$.

A92. $x_1 + \Delta x$, k

93. The slope of a secant line AB, $m_s = \frac{y_b - y_a}{x_b - x_a}$ becomes

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{k - k}{\Delta x} = \_\_\_\_\_.$$

A207. $2x_1$

208. The second term of the expression $x^2 + x - 6$, which is x, by Theorem 2, has a derivative of _____.

A323. $2x^2$

324. Let us now collect the results of the previous frames and determine if a generalization can be formed concerning the derivative of a composite function.

If $G(x) = (x - 2)^{\frac{1}{2}}$, $G'(x_1) = \frac{1}{2}(x_1 - 2)^{\frac{1}{2} - 1} \circ D_x(x - 2)$

If $H(x) = (x^2 - 2x)^{\frac{1}{2}}$, $H'(x_1) = \frac{1}{2}(x_1{}^2 - 2x_1)^{\frac{1}{2} - 1} \circ D_x(x^2 - 2x)$

If $P(x) = (x - 2x^2)^{\frac{1}{2}}$, $P'(x_1) = \frac{1}{2}(x_1 - 2x_1{}^2)^{\frac{1}{2} - 1} \cdot D_x(x - 2x^2)$

⋮

If $F(x) = (f \circ g)(x) = [g(x)]^n$, $F'(x_1) = n[g(x_1)]$ [redacted] $\cdot g'(x_1)$.

A93. 0

94. The slope of the tangent line at point A can be expressed as a limit as ______________________________.

A208. 1

209. The third term which is -6 by Theorem 1, has a derivative of ____.

A324. n-1

325. It should be noted that all composite functions are not powers of a function of x. An example of such a composite function is the trigonometric function f o g defined by (f o g)(x) = $\sin(x^2 + 7)$ where g(x) = ______, f(z) = sin z.

A94. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ or $\lim_{\Delta x \to 0} \frac{k - k}{\Delta x}$ or $\lim_{\Delta x \to 0} 0$

95. The numerical value of the slope of the tangent line at point A is thus ____.

A209. 0

210. Combining the results of the last three frames term by term, if $f(x) = x^2 + x - 6$, $f'(x_1) =$ ____. Does this answer agree with that obtained in frame 54 for the slope of the tangent line to the graph of f at a point with x-coordinate x_1?

A325. $x^2 + 7$

326. Thus, the generalization in frame 324 (If $F(x) = (f \circ g)(x) = [g(x)]^n$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$) is a special case of our next theorem:

Theorem 8. If $F(x) = (f \circ g)(x) = f[g(x)]$, and $f'(g(x_1))$ and $g'(x_1)$ exist, $F'(x_1) = f'[g(x_1)] \cdot g'(x_1)$. (Add this theorem to your list.) The generalization in frame 324 states, for a composite function considered as a power of a function of x, that $f'[g(x_1)]$ as given in the theorem is the same as ______ as given in the generalization.

A95. 0

96. What limit theorem did you use in evaluating the above limit?

A210. $2x_1 + 1$, Yes

211. In exercise (d) of frame 181, if $f(x) = x + 2$, $f'(x_1) = 1$. The first term (x) of the expression $x + 2$ by Theorem 2 has a derivative of _____.

A326. $n[g(x_1)]^{n-1}$

327. We shall be primarily concerned with composite functions considered as powers of functions of x in this unit. Hence, we will state the generalization in frame 324 as a corollary to Theorem 8 and use this corollary in the remainder of the discussion.

Corollary: If $F(x) = [g(x)]^n$, and $g'(x_1)$ exists, $F'(x_1) = n[g(x_1)]^{n-1}$ · _______. (Add this corollary to your list.)

A96. $\lim_{\Delta x \to a} k = k$

97. Thus, if $f(x) = k$, the slope of the tangent line at the point on the graph of f with x-coordinate x_1 is $m_t = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ____.

A211. 1

212. The second term (2) of the expression x + 2, by Theorem 1, has a derivative of ____.

A327. $g'(x_1)$

328. We will not prove this theorem or its corollary, the proof of which may be found in most beginning calculus books. Theorem 8 is sometimes referred to as the chain rule differentiation formula because $F'(x_1)$ is expressed as a chain of derivatives, namely ____ in number as we have stated the theorem. More derivatives may be involved, depending on the complexity of the composite function.

A97. 0

98. To answer the question raised in frame 90, "Is the slope of the tangent line at any point A on the graph of f defined by $f(x) = k$ the same as the slope of the line representing the graph of f?", we can now answer _____.

A212. 0

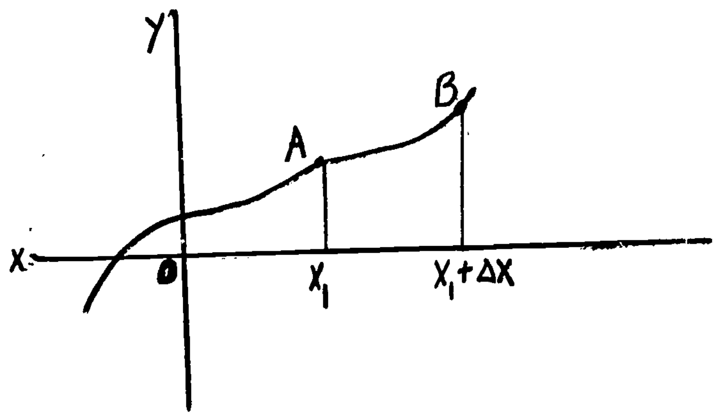
213. Combining the results of the previous two frames term by term, if $f(x) = x + 2$, $f'(x_1) =$ ________. Does this answer agree with the one obtained as the slope of the tangent line in exercise (d) of frame 181?

A328. two

329. Let us consider how the corollary, on which we will focus our attention, is used. If F is defined by $F(x) = \sqrt{25 - x^2}$, for $-5 \leqq x \leqq 5$, F is the composite function defined by $(f \circ g)(x)$ $= f[g(x)] = (25 - x^2)^{\frac{1}{2}}$ where $g(x) = 25 - x^2$, $f(z) =$ ________ for $-5 \leqq x \leqq 5$.

A98. "Yes"

99. Let us now generalize our above discussion. Consider a function f such as the one whose graph appears below. The coordinates of points A and B are (____, ____), (____, ____) respectively.



A213. 1 + 0 = 1, Yes

214. In exercise (f) of frame 181, if $f(x) = x^3 + x^2 + x + 11$, $f'(x_1) = 3x_1^2 + 2x_1 + 1$. The first term of the expression has a derivative of ____, the second term a derivative of ____, the third term a derivative of ____ and the fourth term a derivative of ____, each evaluated at x_1.

A329. $z^{\frac{1}{2}}$ (or $\sqrt{z}$)

330. $F(x) = f[g(x)] = [g(x)]^{\frac{1}{2}}$. The corollary to Theorem 8 states that if $F(x) = [g(x)]^n$, then $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$ where, in this case, n = ____.

A99. x_1, $f(x_1)$, $x_1 + \Delta x$, $f(x_1 + \Delta x)$

100. Referring to the graph of f with points A and B, point B can be made as close to point A as desired by choosing Δx ______________.

A214. $3x_1^2$, $2x_1$, 1, 0

215. Summing the derivatives of each individual term, we see that for this example, as in the previous example, the derivative of a sum of functions is the _____ of the derivatives of the functions.

A330. $\frac{1}{2}$

331. For the composite function $F(x) = f[g(x)] = (25 - x^2)^{\frac{1}{2}}$, $g(x) = 25 - x^2$ so $g(x_1) =$ ________.

A100. close to 0 (or equivalent wording)

101. If a point B′ were chosen to the left of point A, such that $\Delta x < 0$, the coordinates of point B′ would be (______, ______).

A215. sum

216. From this we surmise the following theorem, which reads in mathematical notation:

Theorem 4. If $f(x) = g(x) + h(x) + m(x) + \ldots + z(x)$ and $g'(x_1)$, $h'(x_1)$, $m'(x_1) \ldots z'(x_1)$ exist, then $f'(x_1) =$ ____________________________.

(Add this theorem to your list.)

A331. $25 - x_1^2$

332. For the above composite function, $g(x) = 25 - x^2$, so $g'(x_1) =$ _____.

A101. $x_1 + \Delta x$, $f(x_1 + \Delta x)$

102. The points A and B will determine a secant line having slope

$$m_s = \frac{y_b - y_a}{x_b - x_a} \text{ or } \frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1} = \_\_\_\_\_\_\_\_\_\_\_\_.$$

A216. $g'(x_1) + h'(x_1) + m'(x_1) + \ldots + z'(x_1)$

217. Let us formally prove this theorem for the sum of two functions. We will proceed through the four steps to find $f'(x_1)$ for $f(x) = w(x) + v(x)$. Step (1) for finding the derivative of $f(x)$ evaluated at x_1, gives $f(x_1 + \Delta x) = w(x_1 + \Delta x) + v(x_1 + \Delta x)$ and $f(x_1) =$ ____________.

A332. $-2x_1$

333. Thus, if $F(x) = g(x)^{\frac{1}{2}}$,

$$F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$$

$$F'(x_1) = \frac{1}{2}[25 - x_1^2]^{\frac{1}{2} - 1} \cdot (-2x_1)$$

= (simplify your answer leaving a negative exponent on the quantity $(25 - x_1^2)$).

A102. $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

103. The slope of the tangent line at point A, considering B to the right of point A, or B' to the left of point A on t graph f f, can be expressed in terms of a limit as ________________.

A217. $w(x_1) + v(x_1)$

218. In step (2), $f(x_1 + \Delta x) - f(x_1)$

$$= [w(x_1 + \Delta x) + v(x_1 + \Delta x)] - [w(x_1) + v(x_1)]$$

$$= w(x_1 + \Delta x) + v(x_1 + \Delta x) - w(x_1) - v(x_1)$$

(grouping like terms)

$$= [w(x_1 + \Delta x) - w(x_1)] + [\_\_\_\_\_\_\_\_\_\_]。$$

A333. $-x_1(25 - x^2)^{\frac{-1}{2}}$ (Note that this expression may be rationalized.)

334. If F is defined by $F(x) = (x^2 - 2x - 3)^{\frac{7}{2}}$, $x^2 - 2x - 3 \geqq 0$, F is the composite function $(f \circ g)(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$ where $g(x) = x^2 - 2x - 3$, $f(z) =$ _____.

A103. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

104. From the sketch of the graph of this function f, since the tangent line at a point is unique, we see that this notion can now be visually reinforced. Since the tangent line at A is unique, the value of the slope of the tangent line at A is ______.

A218. $v(x_1 + \Delta x) - v(x_1)$

219. In step (3),

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{[w(x_1 + \Delta x) - w(x_1)] + [v(x_1 + \Delta x) - v(x_1)]}{\Delta x}$$

$$= \frac{[w(x_1 + \Delta x) - w(x_1)]}{\Delta x} + \text{\_\_\_\_\_\_\_\_\_\_}.$$

A334. $z^{\frac{7}{2}}$

335. $F(x) = f[g(x)] = [g(x)]^{\frac{7}{2}}$, so we use the corollary to Theorem 8, which states that if $F(x) = [g(x)]^n$, $F'(x_1) = n[g(x_1)]^{n-1}$. ______.

A104. unique

105. To summarize our discussion up to this point, we see that there are functions for which $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ exists. Some of these functions that we have discussed are defined by $f(x) = x^2$, $g(x) = x^2 + x - 6$, $h(x) = \sqrt{25 - x^2}$, $p(x) = x$, $q(x) = k$. When $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ exists, we define as this limit the slope of the tangent line to the graph of f at the point with x-coordinate x_1.

Definition 1: The slope of the tangent line to the graph of f, at the point with x-coordinate x_1 is $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, if this limit exists. (Write this definition on the enclosed sheet headed "List of Definitions and Theorems.")

A219. $\frac{[v(x_1 + \Delta x) - v(x_1)]}{\Delta x}$

220. Thus, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} + \frac{v(x_1 + \Delta x) - v(x_1)}{\Delta x}\right]$

(1)
$= \lim_{\Delta x \to 0} \frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x_1 + \Delta x) - v(x_1)}{\Delta x}$

(2)
$= w'(x_1) + v'(x_1)$ Supply reasons for (1) and (2).

A335. $g'(x_1)$

336. For $F(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$, n = ____.

106. We thus see that to find the slope of the tangent line,

$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, at a point A with x-coordinate x_1, we

(1) compute $f(x_1 + \Delta x)$ and $f(x_1)$,

(2) find the difference $f(x_1 + \Delta x) - f(x_1)$,

(3) compute the difference quotient $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$,

(4) find $\lim_{\Delta x \to 0}$ ________.

This process is sometimes called the delta process, "delta" referring to the small Greek letter Δ in the notation "Δx."

A220. (1) The limit of the sum of two functions is the sum of the limits of the two functions, if these limits exist. (2) By definition $\lim_{\Delta x \to 0} \frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} = w'(x_1)$ and the same is true for $v'(x_1)$.

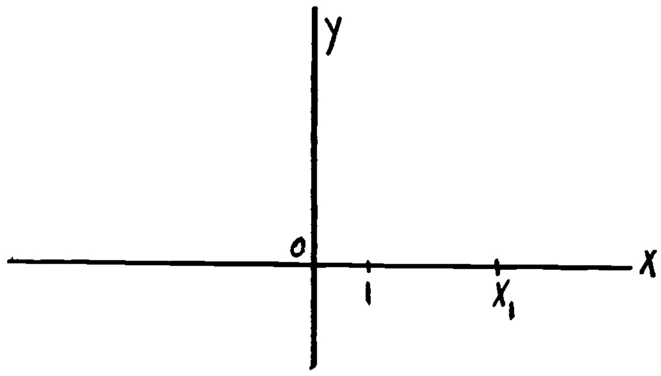
221. If $f(x) = + x - 2$ (rather than $x + 2$) you should expect from term by term differentiation that $f'(x_1) =$ ____.

A336. $\frac{7}{2}$

337. For the above composite function, $F(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$, $g(x) = x^2 - 2x - 3$, so $g(x_1) =$ ________.

A106. $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

107. Let us now apply the four steps to the function f defined by $f(x) = -x^2 + 2x - 1$ to find the slope of the tangent line at any point A with x-coordinate x_1. Graph this function on your own paper and indicate the point A on the graph on axes similar to those below.



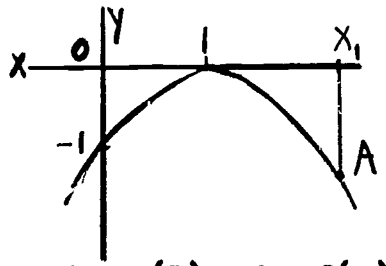
A221. + 1

222. If $f(x) = +x^2 - x + 6$ (rather than $x^2 + x - 6$), you should expect from term by term differentiation $f'(x_1) =$ ________.

A337. $x_1^2 - 2x_1 - 3$

338. For the above composite function, $g(x) = x^2 - 2x - 3$, so $g'(x_1) =$ _____.

A107.



108. In step (1), if $f(x) = -x^2 + 2x - 1$, $f(x_1 + \Delta x) = -(x_1 + \Delta x)^2 + 2(x_1 + \Delta x) - 1$ and $f(x_1) =$ ________.

A222. $+ 2x_1 - 1$

223. If $f(x) = x^3 - x^2 - x - 11$ (rather than $x^3 + x^2 + x + 11$) you should expect from term by term differentiation that $f'(x_1) =$ ____________.

A338. $2x_1 - 2$

339. Now, if $F(x) = [g(x)]^{\frac{7}{2}}$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$ becomes

$F'(x_1) =$ [redacted]$(x_1^2 - 2x_1 - 3)$[redacted] $\cdot (2x_1 - 2)$.

A108. $-x_1^2 + 2x_1 - 1$

109. In step (2), $f(x_1 + \Delta x) - f(x_1) = [-(x_1 + \Delta x)^2 + 2(x_1 + \Delta x) - 1] - [-x_1^2 + 2x_1 - 1] =$ _____________.

A223. $+ 3x_1^2 - 2x_1 - 1$

224. In general, if f(x) can be expressed as the difference of two functions, i.e., $f(x) = w(x) - v(x)$, you should expect that $f'(x_1) =$ ____________________.

A339. $\frac{7}{2}$, $\frac{7}{2} - 1$ (or $\frac{5}{2}$)

340. Thus, if $F(x) = (x^2 - 2x - 3)^{\frac{7}{2}}$, $F'(x_1) =$ (simplify your answer, leaving a fractional exponent on the quantity $(x_1^2 - 2x_1 - 3)$.

A109. $-2x_1 \cdot \Delta x - \Delta x^2 + 2\Delta x$

110. In step (3), $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \dfrac{-2x_1 \cdot \Delta x - \Delta x^2 + 2\Delta x}{\Delta x} =$ __________

if $\Delta x \neq 0$.

A224. $w'(x_1) - v'(x_1)$

225. With the aid of the limit theorem for the difference of two functions, one could prove:

Theorem 5. If $f(x) = w(x) - v(x)$, and $w'(x_1)$ and $v'(x_1)$ exist, $f'(x_1) =$ __________. (Add this theorem to your list of theorems.)

A340. $7(x_1^2 - 2x_1 - 3)^{\frac{5}{2}}(x_1 - 1)$

341. If F is defined by $F(x) = \dfrac{1}{(x^4 - 1)^2}$, $x^4 - 1 \neq 0$, $F(x) = (f \circ g)(x)$ $= f[g(x)] = (x^4 - 1)^{\square}$.

A110. $-2x_1 + 2 - \Delta x$

111. In step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0}(-2x_1 + 2 - \Delta x) =$ ____.

A225. $w'(x_1) - v'(x_1)$

226. Stated in words, this theorem says that the derivative of the difference of two functions is ______________________.

A341. -2

342. So $F(x) = (f \circ g)(x) = f[g(x)] = (x^4 - 1)^{-2}$ where $g(x) = x^4 - 1$, $f(z) =$ ________.

A111. $-2x_1 + 2$

112. Thus, for the function f defined by $f(x) = -x^2 + 2x - 1$, the slope of the tangent line at the point on the graph of f with x-coordinate x_1 is $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = __________.

A226. the difference of the derivatives of the two functions, if these derivatives exist.

227. It should be noted that it is not necessarily the case that if the derivatives of 2 functions don't exist at some point, the derivative of the sum of these functions won't exist at this point. Consider $g(x) = |x|$ and $h(x) = -|x|$, neither of which possess derivatives at the point with x-coordinate $x_1 =$ ______.

A342. z^{-2}

343. $F(x) = f[g(x)] = [g(x)]^2$, so our corollary will be used with $n =$ _____

A112. $-2x_1 + 2$

113. For the point with coordinates (1,0) belonging to f defined by $f(x) = -x^2 + 2x - 1$, the slope of the tangent line is $-2x_1 + 2 = -2(1) + 2 =$ _____.

A227. 0

228. However, for the sum of the above functions, $f(x) = g(x) + h(x) = |x| + [-|x|] = 0$, $f'(x_1) =$ _______.

A343. -2

344. For the above composite function $F(x) = f[g(x)] = (x^4 - 1)^{-2}$, $g(x_1) =$ ________ and $g'(x_1) =$ _______.

A113. 0

114. For the points with coordinates $(0, -1)$, $(-1, -4)$, $(4, -9)$, the slopes of the tangent lines at these points are ____, ____, ____ respectively.

A228. 0 (see Theorem 1)

229. We now have at our disposal five theorems for finding the derivative of f evaluated at x_1. Review these theorems from your compiled list. Following is a set of exercises you should be able to do using these theorems.

(a) If $f(x) = x^3 + 6$, $f'(x_1) =$ ________

(b) If $f(x) = x^4 + x$, $f'(x_1) =$ ________

(c) If $f(x) = x^{\frac{8}{5}} - x^2$, $f'(x_1) =$ ________

(d) If $f(x) = \pi - x$, $f'(x_1) =$ ________

(e) If $f(x) = x^{1.7} - 2^{\sqrt{2}}$, $f'(x_1) =$ ________

(f) If $f(x) = x - x^2$, $f'(x_1) =$ ________

A344. $(x_1^4 - 1)$, $4x_1^3$

345. Now, if $F(x) = [g(x)]^{-2} = (x^4 - 1)^{-2}$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1) =$ (simplify your answer leaving a negative exponent on the quantity $(x_1^4 - 1)$.)

Skip a page for the answer to Frame 345.

A114. 2, 4, -6

115. Following are exercises for which you are to find the slope of the tangent lines at the indicated points.

(a) If $f(x) = -2$, find the slope of the tangent line at $(7,-2)$.

(b) If $f(x) = x$, find the slope of the tangent line at $(-9,-9)$.

(c) If $f(x) = x - 1$, find the slope of the tangent line at $(1,0)$.

(d) If $f(x) = (x - 1)^2$, find the slope of the tangent line at $(1,0)$.

(e) If $f(x) = 1 - 3x - x^2$, find the slope of the tangent line at $(-2,3)$.

A229. (a) $3x_1^2$, (b) $4x_1^3 + 1$, (c) $\frac{8}{5} x_1^{\frac{3}{5}} - 2x_1$, (d) -1,

(e) $1.7x_1^{.7}$, (f) $1 - 2x_1$

230. Let us now consider the derivative of the product of two functions. We know that the limit of the product of two functions is the product of the limits if these limits exist, so we shall want to investigate if the derivative of the product of two functions is the product of the ____________ of the two functions.

A115. 0, 1, 1, 0, 1

116. For the first function f defined by $f(x) = x^2$, for which we expressed the slope $m_t = 2x_1$ of the tangent line to the graph of f at any point with x-coordinate x_1, the slope of the tangent line at the point with coordinates (3,9) is ____.

A230. derivatives

231. For the function f defined by $f(x) = x^6$, by Theorem 3, $f'(x_1)$ = ________.

A345. $-8x_1^3(x_1^4 - 1)^{-3}$

346. Following is a set of functions you should be able to differentiate using the theorems and corollaries developed in this unit. Review your list of these theorems and corollaries at this time, before proceeding to the exercises below.

(a) If $f(x) = (x - 1)^{\frac{5}{2}}$, $f'(x_1)$ = ________.

(b) If $f(x) = (x^2 - 1)^{\frac{5}{2}}$, $f'(x_1)$ = ________.

(c) If $f(x) = \sqrt[3]{x^2}$, $f'(x_1)$ = ________.

(d) If $f(x) = (\sqrt{x - 1})^5$, $f'(x_1)$ = ________.

(e) If $f(x) = (\sqrt[5]{x^2 - 1})^2$, $f'(x_1)$ = ________.

(f) If $f(x) = \sqrt[3]{x^2 - 1}$, $f'(x_1)$ = ________.

A116. 6

117. The equation of a straight line can be written if we know its slope m and a point (x_1, y_1) through which the line passes; i.e., $y - y_1 = m(x - x_1)$. This is the point-slope form for the equation of a straight line. If a tangent line has slope 6 and passes through the point (3, 9), its equation is ____________.

A231. $6x_1^5$

232. Let $f(x) = x^6$ be rewritten as the product $f(x) = x^6 = x^4$. _____.

A346. $\frac{5}{2}(x_1 - 1)^{\frac{3}{2}}$, $5x_1(x_1^2 - 1)^{\frac{3}{2}}$, $\frac{2}{3}x_1^{-\frac{1}{3}}$, $\frac{5}{2}(x_1 - 1)^{\frac{3}{2}}$ (See (a)), $\frac{4}{5}x_1(x_1^2 - 1)^{-\frac{3}{5}}$, $\frac{2}{3}x_1(x_1^2 - 1)^{-\frac{2}{3}}$

A117. $y - 9 = 6(x - 3)$

118. For the point with coordinates $(-1,1)$ belonging to the function f defined by $f(x) = x^2$, the slope of the tangent line at this point is _____.

A232. x^2

233. So, $D_x(x^6)$ (see the alternate notation for a derivative in frame 184) $= D_x(\_\_\_\_\_) = 6x_1^5$.

APPENDIX B

ABSTRACT DEDUCTIVE DERIVATIVE PROGRAMMED UNIT

1. The derivative is a very important topic in mathematics and related sciences. It is the basis of a beginning course in calculus and provides a foundation for more advanced courses.

A125. $5x_1^4 + 3x_1^2 + 2x_1$

126. Do you see that you could have computed the above derivative in a different way? We can express $f(x) = (x^2 + 1)(x^3 + 1)$ as $f(x) = x^5 + x^3 + x^2 + 1$. Generalizing the theorem for the derivative of the sum of two functions to the sum of any finite number of functions, in this case 4 functions, we have $f'(x_1) =$ ____________________. Do your 2 answers check?

A249. zero

250. It follows that the slope of a secant line AB approac' the slope of the tangent line at A, denoted by m_t. Thus, we may write

$m_t = \lim_{\Delta x \to 0}(m_s) = \lim_{\Delta x \to 0}$ ______________.

2. Since we need the limit in studying the derivative of a function, let us review the notation of a limit of a function. We know that if $\lim_{x \to x_1} f(x) = L$, then $f(x)$ is as "close" to L as we please for x in a suitably chosen deleted neighborhood of x_1. (The student should now review the mathematically precise δ-ε definition of a limit of a function.) Which of the numbers in the set 2.9, 2.99, 2.999, 3.001, 3.1 belongs to the deleted neighborhood defined by the inequality $|x - 3| < .01$?

A126. $5x_1^4 + 3x_1^2 + 2x_1$, Yes

127. Do not acquire the mistaken notion that a function which is differentiated by the product rule, as Theorem 6 is sometimes called, can always be differentiated in another manner. We have chosen only very elementary examples to illustrate the theorem. More complicated examples of functions, composed of the product of two functions, which can be differentiated by the product theorem only, will be given later in your calculus course.

A250. $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

251. Note that $m_t = \lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ has the exact form of the derivative of f evaluated at x_1. Thus, the derivative of a function evaluated at x_1 and the slope of the tangent line to the graph of f at a point with x-coordinate x_1, are ______________, when both the derivative and slope exist.

A2. 2.999, 3.001

3. We will now introduce an alternate notation for $\lim_{x \to x_1} f(x) = L$. We know that $\lim_{\Delta x \to 0}(x_1 + \Delta x) = x_1$ where Δx is a variable, usually small in magnitude. Thus, if Δp is a variable, $\lim_{\Delta p \to 0}(p_1 + \Delta p) =$ __________.

126. It is not necessarily the case that if the derivatives of one or more of the functions in the product of two functions don't exist for some value of x_1, the derivative of the product won't exist for this value of x_1. Consider $f(x) = w(x) \cdot v(x) = |x| \cdot [-|x|]$ where $w(x) = |x|$, $v(x) =$ ________.

A251. the same (or equivalent wording)

252. We now have that for f defined by f(x), the slope of the tangent line to the graph of f at a point whose x-coordinate is x_1 is ________ ____.

A3. p_1

4. If Δx approaches 0, $x_1 + \Delta x$ approaches x_1. Therefore, $\lim_{x \to x_1} f(x) = \lim_{\Delta x \to 0} f(x_1 + \Delta x)$. If Δx approaches 0, $x_1 + \Delta x$ approaches x_1, so $\lim_{x \to x_1} g(x) = \lim_{\Delta x \to 0} g(\_\_\_\_\_\_)$.

A128. $-|x|$

129. Neither $|x|$ nor $[-|x|]$ exist for $x_1 =$ ______.

A252. $f'(x_1)$ (or a statement equivalent to the derivative of f evaluated at x_1)

253. Let us now consider the function f defined by $f(x) = x^2$. In the section on limits, we computed the slope of the tangent to the graph of this function at the point (1,1). This slope was 2. The student should now review this section in the limit unit.

A4. $x_1 + \Delta x$

5. If $x_1 = 5$, $x_1 + \Delta x = 5 + \Delta x$. As Δx approaches 0, $x_1 + \Delta x$ approaches ___.

A129. 0

130. However, for the product in the above function,

$f(x) = w(x) \cdot v(x) = |x| \cdot -|x| = -|x|^2 = -x^2$, $f'(x_1) =$ ____________________.

254. We will now obtain the slope of the tangent line for any point on the graph of f defined by $f(x) = x^2$ and then check its value at the point (1,1) with the above result. The derivative of $f(x) = x^2$, evaluated at x_1 is ________.

A5. 5

6. Therefore, $\lim_{x \to 5} f(x) = \lim_{\blacksquare} f(5 + \Delta x)$.

Your answer should correctly complete the shaded box.

A130. 0 for all values of x

131. We have now proved a theorem for finding the derivative of the product of 2 functions. Let us use this theorem to prove a corollary.

Corollary: If $f(x) = k \cdot g(x)$, where k is a constant, and $g'(x_1)$ exists then $f'(x_1) = k \cdot g'(x_1)$. Add this corollary to your list.

A254. $2x_1$

255. The slope of the tangent line at any point with x-coordinate x_1 on the graph of f defined by $f(x) = x^2$ is $m_t = f'(x_1) =$ ______.

A6. $\Delta x \to 0$

7. If $x_1 = -3$, $x_1 + \Delta x = -3 + \Delta x$. As Δx approaches 0, $x_1 + \Delta x$ approaches ______.

132. Stated in words, this corollary says that the derivative of a constant multiplied by a function is the constant multiplied by ______ ______________________________, provided the derivative exists.

A255. $2x_1$

256. For the point with coordinates (1,1) on the graph of f defined by $f(x) = x^2$, $x_1 =$ ____.

A7. -3

8. Therefore, $\lim_{x \to -3} g(x) = \lim_{\Delta x \to 0} g(\_\_\_\_\_)$.

A132. the derivative of the function evaluated at x_1

133. We will apply Theorem 6 to the function f defined by $f(x) = k \cdot g(x)$. Here $w(x) = k$, $v(x) = g(x)$, $w'(x_1) =$ _____, $v'(x_1) =$ ________.

A256. 1

257. Thus, $m_t = f'(x_1) = 2x_1$ for the point with x-coordinate $x_1 = 1$ becomes $f'(1) = 2 \cdot 1 =$ _____. Does this result check with the former result in the limit section?

A8. $-3 + \Delta x$

9. It may be the case that x is to the right of x_1, and sufficiently close to x_1, or that x is to the left of x_1, and sufficiently close to x_1. We recall from the section on limits that $\lim_{x \to x_1} f(x)$, now shown to be equivalent to $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$, exists if f(x) approaches the same value L when we consider numbers which may be either greater or ______ than x_1.

A133. 0, $g'(x_1)$

134. Thus, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) =$ (in simplified form).

A257. 2, Yes

258. We can, of course, evaluate the slope of the tangent line at any point on the graph of f defined by $f(x) = x^2$. If we consider the point (2,4), $x_1 = 2$ so $m_t = f'(x_1) = 2x_1$, or $f'(2) =$ ____.

A9. smaller (less)

10. If we consider numbers only greater than x_1, then we denote the limit by $\lim_{x \to x_1^+} f(x) = L$. If we restrict our consideration to numbers less than x_1, we denote the limit by $\lim_{x \to x_1^-} f(x) = L'$. L is called the right-hand limit. Hence, the left-hand limit of f(x) would be ____.

A134. $k \cdot g'(x_1)$

135. This corollary is very useful, since constants frequently appear with variables in a function. If $f(x) = 7x^5$, k = ____, g(x) = ____, $g'(x_1)$ = ____.

A258. 4

259. If we consider the points (-5,25), (0,0), $(\frac{7}{2}, \frac{49}{4})$ on the graph of f, $x_1 = -5$, 0, $\frac{7}{2}$ respectively for these points, so m_t, the slopes of the tangent lines for these values of x_1, becomes ____, ____, ____ respectively.

A10. L'

11. Let us now translate the notation for right and left hand limits into Δx notation. If Δx approaches 0 from the right, then $x_1 + \Delta x$ approaches x_1 and $x_1 + \Delta x$ is (greater than, less than) x_1. This means x approaches x_1 from the right and we write ______.

A135. 7, x^5, $5x_1^4$

136. Thus, if $f(x) = 7x^5$, $f'(x_1) = k \cdot g'(x_1) = 7 \cdot 5x_1^4 =$ ______.

A259. -10, 0, 7

260. We can write the equation of the tangent line at any point on the graph of f defined by f(x), say the point with coordinates (x_1, y_1). Recall that you can write the equation of a line if you know a point on the line and the ______ of the line.

A11. greater than, $x \to x_1^+$

12. If Δx approaches 0 from the left, then $x_1 + \Delta x$ approaches x_1, $x_1 + \Delta x < x_1$, and we write $x \to$ ____.

A136. $35x_1^4$

137. If $f(x) = 3(x^3 + x^4)$, k = ___, g(x) = _____, $g'(x_1) =$ _________.

A260. slope

261. The slope of the tangent line, m_t, at the point (x_1, y_1) can be expressed as ___________.

A12. x_1^-

13. Then $\lim_{x \to x_1^+} f(x) = \lim_{\Delta x \to 0^+} f(x_1 + \Delta x)$ and $\lim_{x \to x_1^-} f(x) = \lim_{\Delta x \to 0^-} f(x_1 + \Delta x)$.

The right hand limit is thus expressed as $\lim_{\Delta x \to 0^+} f(x_1 + \Delta x)$ and the left hand limit is expressed as __________.

A137. 3, $x^3 + x^4$, $3x_1^2 + 4x_1^3$

138. Thus, if $f(x) = 3(x^3 + x^4)$, $f'(x_1) = k \cdot g'(x_1) =$ ____________.

A261. $f'(x_1)$ (or an equivalent expression for the derivative of f evaluated at x_1)

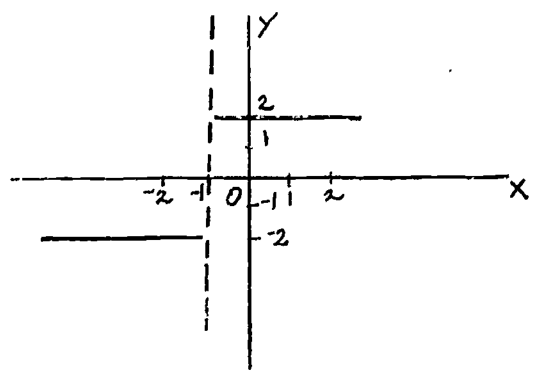
262. Using the point-slope form for the equation of a line $y - y_1 = m(x - x_1)$, we obtain the form for the equation of the tangent line as $y - y_1 = m_t(x - x_1)$ or $y - y_1 =$ _____ $(x - x_1)$.

A13. $\lim_{\Delta x \to 0^-} f(x_1 + \Delta x)$

14. Consider the function f, discussed in the section on limits, defined by $f(x) = \frac{2(x+1)}{|x+1|}$. For this function, $\lim_{x \to -1^+} f(x) = \lim_{\Delta x \to 0^+} f(-1 + \Delta x)$

$= 2$ and $\lim_{x \to -1^-} f(x) = \lim_{\Delta x \to 0^-} f(-1 + \Delta x) =$ ____.

(See the graph below as an aid to responding.)



A138. $3(3x_1^2 + 4x_1^3)$

139. If $f(x) = -6x(x + 1) = -6[x(x + 1)]$, k = ____, $g(x) = x(x + 1)$,

$g'(x_1) =$ (use the product rule - Theorem 6).

A262. $f'(x_1)$

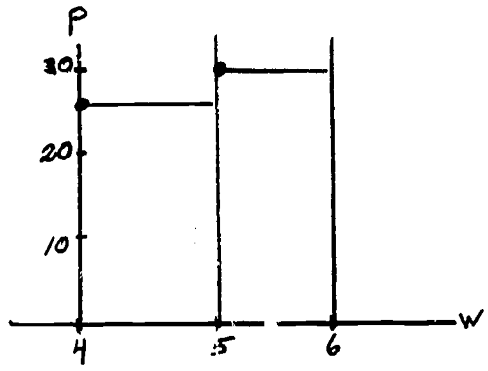
263. For the equation of the tangent line at the point with coordinates (1,1) for the above function f defined by $f(x) = x^2$, $x_1 =$ ____, $y_1 =$ ____,

$m_t = f'(x_1) = 2x_1$ becomes $m_t = f'(1) =$ ____.

A14. -2 (Note: If you do not recall this function, take time now to convince yourself that $\lim_{x \to -1^+} f(x) = 2$ and $\lim_{x \to -1^-} f(x) = -2$.)

15. For the postage stamp function discussed in the section on limits, $p = f(w)$ where p is the postage and w is the weight, we have $\lim_{\Delta w \to 0^+} f(5 + \Delta w) = 30$ where Δw is a variable. Also $\lim_{\Delta w \to 0^-} f(5 + \Delta w) =$

____________. (See the graph below as an aid to responding.)



A139. -6, $2x_1 + 1$

140. Thus, if $f(x) = -6x(x + 1)$, $f'(x_1) = k \cdot g'(x_1) =$ __________.

A263. 1, 1, 2

264. Thus, the equation of the tangent line, $y - y_1 = f'(x_1)(x - x_1)$, in this case is ____________.

A15. 25

16. Recall that the limit of a function exists as $x \to x_1$ if the limit of the function as $x \to x_1$ through values of x greater than x_1 is the same as the limit of the function as $x \to x_1$ through values of x less than x_1. Thus, $\lim_{x \to x_1} f(x) = L$ if $\lim_{x \to x_1^-} f(x) = L$ and $\lim_{x \to x_1^+} f(x) =$ _____.

A140. $-6(2x_1 + 1)$

141. You might expect that there is a theorem for finding the derivative of a function which is the quotient of two functions. From what was surmised concerning the derivative of the product of 2 functions, would you expect the derivative of the quotient of 2 functions to be the quotient of the derivatives of the two functions if these derivatives exist; i.e. if $f(x) = \frac{w(x)}{v(x)}$, $f'(x_1) = \frac{w'(x_1)}{v'(x_1)}$ where $v'(x_1)$ and $w'(x_1)$ exist and $v'(x_1) \neq 0$.

A264. $y - 1 = 2(x - 1)$ (or an equivalent expression)

265. For the above function at the point $(-5, 25)$, $x_1 =$ ____, $y_1 =$ ____, $m_t = f'(x_1) = f'(-5) =$ ____.

A16. L

17. In our notation, $\lim_{\Delta x \to 0} f(x_1 + \Delta x) = L$ if $\lim_{\Delta x \to 0^-} f(x_1 + \Delta x) = L$ and ______.

A141. You shouldn't expect this - such is not the case.

142. As with the derivative of the product of 2 functions, the derivative of the quotient of 2 functions is not what might be expected. Let us state the theorem. Theorem 7. If $f(x) = \frac{w(x)}{v(x)}$ and $w'(x_1)$ and $v'(x_1)$ exist, then $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$ provided $v_1(x_1) \neq 0$.

Add this theorem to your list.

A265. -5, 25, -10

266. The equation of the tangent line at the point (-5, 25) on the graph of f defined by $f(x) = x^2$ is ______________.

A17. $\lim_{\Delta x \to 0^+} f(x_1 + \Delta x) = L$

18. We are now ready to define the derivative of a function.

Definition: The derivative of the function f, evaluated at x_1, is $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, provided this limit exists. The derivative is denoted by $f'(x_1)$ and may be considered a function itself. Other notations for the derivative are y', $D_x y$, $\frac{dy}{dx}$. (Enter this definition on the enclosed sheet headed "List of Theorems and Definitions.")

143. In words, this theorem says that the derivative of the quotient of 2 functions of x, evaluated at x_1 provided these derivatives exist, is the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, this number divided by ______________________.

A266. $y - 25 = -10(x + 5)$ (or an equivalent expression)

267. At the point on the above graph with coordinates $(\frac{7}{2}, \frac{49}{4})$, the equation of the tangent line is ______________.

19. The expression $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ is called a difference quotient. The numerator of this difference quotient can be considered as a small change in $y = f(x)$. A symbol used for $f(x_1 + \Delta x) - f(x)$ is ____.

A143. the square of the denominator, provided this is non-zero

144. We will now proceed with the proof of this theorem in a manner exactly analogous to that used in proving our former theorems. Perhaps you can derive your own proof of this theorem, following the 4 steps for finding the derivative. Try to do so before reading the proof that follows.

A267. $(y - \frac{49}{4}) = 7(x - \frac{7}{2})$ (or an equivalent expression)

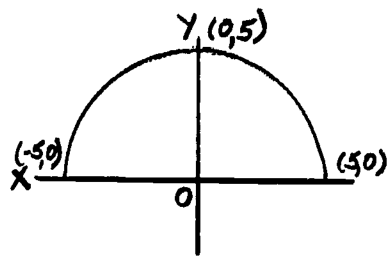
268. Let us now consider another function whose slope you computed in the section on limits, the function F defined by $F(x) = \sqrt{25 - x^2}$. Graph this function on your own paper for future reference.

A19. Δy

20. Thus, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ can be written as $\lim_{\Delta x \to 0}$ ____.

145. Here is our proof. If $f(x) = \frac{w(x)}{v(x)}$, step (1) expresses $f(x_1) = \frac{w(x_1)}{v(x_1)}$ = (alternate notation).

A268.



269. For this function, we will compute the slope of the tangent line at the point with x-coordinate $x_1 = 4$. The corresponding y-coordinate of this point is $y_1 =$ ____.

A20. $\frac{\Delta y}{\Delta x}$ (Note: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is used in many texts as an expression for the derivative.)

21. In the definition of the derivative, it is important to note the condition that the limit of the difference quotient must exist. We say f(x) is differentiable at x_1 if the limit of the difference quotient exists. From the section on limits, we know that this limit (will always exist, may not always exist).

A145. $\frac{w_1}{v_1}$

146. Likewise, $f(x_1 + \Delta x) = \frac{w(x_1 + \Delta x)}{v(x_1 + \Delta x)}$ = (alternate notation).

A269. 3 (Indicate this point on your graph.)

270. In the section on limits, we showed the slope of the tangent line to the graph of F defined by $F(x) = \sqrt{25 - x^2}$ at the point with coordinates (4,3) had the value $\frac{-4}{3}$. The student should now review this discussion in the limit unit.

A21. may not always exist

22. Thus, the differentiability of a function f at x_1 is equivalent to the statement that ______________________.

A146. $\dfrac{w_1 + \Delta w}{v_1 + \Delta v}$

147. In step (2), $f(x_1 + \Delta x) - f(x_1) = \dfrac{(w_1 + \Delta w)}{(v_1 + \Delta v)} - \dfrac{w_1}{v_1}$

$$= \frac{v_1(w_1 + \Delta w) - w_1(v_1 + \Delta v)}{v_1(v_1 + \Delta v)}$$

$=$ (in simplified form).

271. We can also express the slope of the tangent line to the graph of F at the point with x-coordinate x_1 as __________.

(Skip three pages for the answer to frame 271.)

A22. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ exists.

23. Steps for Computing Derivative. There are four steps in computing the derivative of the function f evaluated at x_1:

(1) Express $f(x_1 + \Delta x)$ and $f(x_1)$

(2) Find the difference $f(x_1 + \Delta x) - f(x_1) = \Delta y$

(3) Compute the difference quotient $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{\Delta y}{\Delta x}$

(4) Find $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

A147. $\frac{v_1 \cdot \Delta w - w_1 \cdot \Delta v}{v_1(v_1 + \Delta v)}$

148. In step (3), $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{v_1 \cdot \Delta w - w_1 \cdot \Delta v}{\Delta x \cdot [v_1(v_1 + \Delta v)]}$

$$\stackrel{(1)}{=} \frac{v_1 \cdot \frac{\Delta w}{\Delta x} - w_1 \cdot \frac{\Delta v}{\Delta x}}{v_1(v_1 + \Delta v)}$$

Supply a reason for (1).

24. Let us now find the derivative of f evaluated at x_1, defined by $f(x) = 3x$. Step (1) tells us to express $f(x_1 + \Delta x) = 3(x_1 + \Delta x)$ and $f(x_1) =$ _____.

<u>A148. Both numerator and denominator of a fraction may be divided by the same non-zero quantity without changing its value.</u>

149. Then, in step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{v_1 \cdot \frac{\Delta w}{\Delta x} - w_1 \cdot \frac{\Delta v}{\Delta x}}{v_1(v_1 + \Delta v)}$$

(1)
$$= \lim_{\Delta x \to 0} \left(v_1 \cdot \frac{\Delta w}{\Delta x} - w_1 \cdot \frac{\Delta v}{\Delta x}\right) \Big/ \lim_{\Delta x \to 0}(v_1(v_1 + \Delta v))$$

(2)
$$= \left(\lim_{\Delta x \to 0} v_1 \cdot \frac{\Delta w}{\Delta x} - \lim_{\Delta x \to 0} w_1 \cdot \frac{\Delta v}{\Delta x}\right) \Big/ \lim_{\Delta x \to 0} (v_1(v_1 + \Delta v))$$

(3)
$$= \left(\lim_{\Delta x \to 0} v_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} - \lim_{\Delta x \to 0} w_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) \Big/ \lim_{\Delta x \to 0} v_1 \cdot \lim_{\Delta x \to 0}(v_1 + \Delta v)$$

(4)
$$= \left(v_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} - w_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) \Big/ v_1 \cdot \lim_{\Delta x \to 0} (v_1 + \Delta v)$$

Supply reasons for (1), (2), (3), (4).

A24. $3x_1$

25. Step (2) requires an express[illegible]r $f(x_1 + \Delta x) - f(x_1) = 3(x_1 + \Delta x) - 3x_1 =$ (in simplified form).

A149. (1) The limit of a quotient is the quotient of the limits, provided these limits exist. (2) The limit of a difference is the difference of the limits, provided these limits exist. (3) The limit of a product is the product of the limits, provided these limits exist. (4) $\lim_{x \to a} k = k$ ($w_1 = w(x_1)$ and $v_1 = v(x_1)$ are constant.)

150. Consider the final expression in the previous frame. As in the proof of Theorem 6, $\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = w'(x_1)$ and $\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} =$ ______.

A25. 3Δx

26. The difference quotient in step (3) is $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = ______.

A150. $v'(x_1)$

151. Also, as we reasoned before, assuming v is continuous, Δv approaches 0 as Δx approaches 0, so $\lim_{\Delta x \to 0}(v_1 + \Delta v) = \lim_{\Delta v \to 0}(v_1 + \Delta v) =$ ____.

A271. $F'(x_1)$ (or an equivalent expression)

272. To find the slope of the tangent line at (4,3) on the graph of F defined above, we must evaluate F'(___).

A26. $\frac{3\Delta x}{\Delta x}$

27. Finally, computing the limit of the difference quotient, step (4) gives $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} 3 \frac{\Delta x}{\Delta x} \overset{(1)}{=} \lim_{\Delta x \to 0} 3 = 3$. Supply a reason for (1).

A151. v_1

152. Substituting these results in the right hand side of the expression in the last line of frame 149, we have $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$$= \frac{v_1 \cdot w'(x_1) - w_1 \cdot v'(x_1)}{v_1^{\,2}}$$

= (changing notation for v_1 and w_1) ______________________

A272. 4

273. The function F defined by $F(x) = \sqrt{25 - x^2}$ is a composite function, so $F'(x_1)$ is obtained by using the corollary to the chain rule differentiation formula; i.e. if $F(x) = [g(x)]^n$, $F'(x_1) =$ ______________.

A27. Since $\Delta x \neq 0$, but approaches zero, both numerator and denominator may be divided by Δx.

28. In evaluating $\lim_{\Delta x \to 0} 3 = 3$, what limit theorem did you use?

A152. $\dfrac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

153. We have thus proved the theorem that if $f(x) = \frac{w(x)}{v(x)}$ and $w'(x_1)$ and $v'(x_1)$ exist, then $f'(x_1) =$ ____________________, if $v(x_1) \neq 0$.

A273. $n[g(x_1)]^{n-1} \cdot g'(x_1)$

274. By this corollary, if $F(x) = \sqrt{25 - x^2} = (25 - x^2)^{\frac{1}{2}}$, $F'(x_1) =$ ______.

A28. For a constant k, if $f(x) = k$, $\lim_{x \to a} f(x) = k$

29. Thus, if f is defined by $f(x) = 3x$, the derivative of f, evaluated at x_1, is $f'(x_1) =$ _____.

A153. $\dfrac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

154. Consider $f(x) = \dfrac{w(x)}{v(x)} = \dfrac{x^7}{x^2}$, $x \neq 0$. Here $w(x_1) = x_1^{\,7}$, $v(x_1) = x_1^{\,2}$, $w'(x_1) =$ ______, $v'(x_1) =$ ______.

A274. $-x_1(25 - x_1^{\,2})^{-\frac{1}{2}}$

275. Thus, $F'(x_1) = -x_1(25 - x_1^{\,2})^{-\frac{1}{2}}$ - we will rationalize this expression when we substitute in our value of x_1 - so the slope of the tangent line to the graph of F at the point (4,3) is $F'(x_1)$ or $F'(4) =$ _____.

A29. 3

30. Let us now find the derivative of the function f defined by $f(x) = x$. Step (1) expresses $f(x_1) =$ ____ and $f(x_1 + \Delta x) =$ ______.

A154. $7x_1^6$, $2x_1$

155. Thus, for $f(x) = \frac{x^7}{x^2}$, $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

$$= \frac{x_1^2 \cdot 7x_1^6 - x_1^7 \cdot 2x_1}{[x_1^2]^2}$$

= (in simplified form), if $x_1 \neq 0$.

A275. $\frac{-4}{3}$

276. Does this answer check with the results obtained in the section on limits, as cited in frame 270?

A30. x_1, $x_1 + \Delta x$

31. Then, step (2) requires an expression for $f(x_1 + \Delta x) - f(x_1) =$ (in simplified form).

A155. $5x_1^4$

156. You probably realize you could find the derivative of the above example in a simpler way. How could you rewrite $f(x) = \frac{x^7}{x^2}$, if $x \neq 0$?

A276. Yes

277. Let us now write the equation of the tangent line with slope $\frac{-4}{3}$, passing through the point with coordinates (4,3) for this function. Here $x_1 =$ ____, $y_1 =$ ____, $F'(x_1) = F'(4) =$ ____.

A31. Δx

32. The difference quotient in step (3) is $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = ____.

A156. $f(x) = x^5$

157. What is the derivative of the rewritten expression, i.e. $f(x) = x^5$? Does this answer check with that just obtained?

A277. 4, 3, $\frac{-4}{3}$

278. The equation of the tangent line, $y - y_1 = [F'(x_1)](x - x_1)$ becomes

__________________.

A32. $\frac{\Delta x}{\Delta x}$

33. Finally, computing the limit of the difference quotient, step (4) gives $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \overset{(1)}{=} \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$. Supply a reason for (1).

A157. $5x_1^4$, Yes

158. If $f(x) = \frac{w(x)}{v(x)} = \frac{x + 1}{x - 3}$, $x \neq 3$, $w(x_1) =$ ____, $v(x_1) =$ ____, $w'(x_1) =$ ____, $v'(x_1) =$ ____.

A278. $y - 3 = \frac{-4}{3}(x - 4)$ (or an equivalent expression)

279. Let us consider a function not previously discussed in the section on limits and compute m_t, the slope of the tangent line, for any point on the graph of this function. Graph f defined by $f(x) = x^2 + x - 6$ on your own paper and draw the tangent line at the point (2,0).

A33. Since $\Delta x \neq 0$, but approaches zero, both numerator and denominator may be divided by Δx.

34. In evaluating $\lim_{\Delta x \to 0} 1 = 1$, what limit theorem did you use?

A158. $x_1 + 1$, $x_1 - 3$, 1, 1

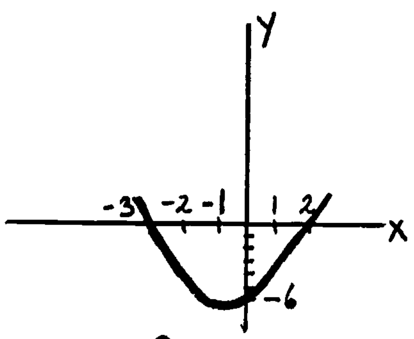
159. Thus, if $f(x) = \frac{w(x)}{v(x)} = \frac{x + 1}{x - 3}$, $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

$$= \frac{(x_1 - 3) \cdot 1 - (x_1 + 1) \cdot 1}{(x_1 - 3)^2}$$

= (combining terms in numerator)

if $x_1 \neq 3$.

A279.



280. If $f(x) = x^2 + x - 6$, the slope of the tangent line at any point on the graph of this function whose x-coordinate is x_1 is $f'(x_1) =$ ______.

A34. For a constant k if $f(x) = k$, $\lim_{x \to a} f(x) = k$

35. Thus, if f is defined by $f(x) = x$, then the derivative of f evaluated at x_1, is $f'(x_1) =$ ____.

A159. $\frac{-4}{(x_1 - 3)^2}$

160. If $f(x) = \frac{x^2 + 3x}{x^3 + 1}$, $x^3 + 1 \neq 0$, $w(x_1) =$ ______, $v(x_1) =$ ______, $w'(x_1) =$ ______, $v'(x_1) =$ ______.

A280. $2x_1 + 1$

281. Let us now evaluate $m_t = f'(x_1)$ at the point with coordinates $(2, 0)$. Here $x_1 =$ ____ so $f'(x_1) = 2x_1 + 1$ becomes $f'(2) =$ ____.

A35. 1

36. Let us now find the derivative of f defined by $f(x) = x^2$. Step (1) yields $f(x_1 + \Delta x) = (x_1 + \Delta x)^2$ and $f(x_1) =$ _____.

A160. $x_1^2 + 3x_1$, $x_1^3 + 1$, $2x_1 + 3$, $3x_1^2$

161. Thus, if $f(x) = \frac{x^2 + 3x}{x^3 + 1}$, $f'(x_1) = \frac{v(x_1) \cdot w'(x_1) - w(x_1) \cdot v'(x_1)}{[v(x_1)]^2}$

$$= \frac{(x_1^3 + 1) \cdot (2x_1 + 3) - (x_1^2 + 3x_1)(3x_1^2)}{(x_1^3 + 1)^2}$$

$=$ _______________ if $x_1^3 + 1 \neq 0$.

A281. 2, 5

282. You should now be able to write the equation of the tangent line to the graph of f defined by $f(x) = x^2 + x - 6$ passing through the point (2,0). This equation is _______________.

A36. x_1^2

37. Step (2) in the process of finding the derivative of f defined by $f(x) = x^2$, evaluated at x_1, gives $f(x_1 + \Delta x) - f(x_1) = (x_1 + \Delta x)^2 - (x_1)^2 = (x_1^2 + 2x_1 \Delta x + \Delta x^2) - x_1^2 =$ (in simplified form).

A161. $\dfrac{-x_1^4 - 6x_1^3 + 2x_1 + 3}{(x_1^3 + 1)^2}$

162. Now it's your turn! If $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x^4 + 1}$, $x^4 + 1 \neq 0$,

$f'(x_1) =$ ______________________________.

A282. $y = 5(x - 2)$ (or an equivalent expression)

283. We will now find the equation of the tangent line to the graph of f defined by $f(x) = x^3 - 2x^2 + 5x - 1$ at the point $(-1, -9)$. Perhaps you will want to derive your own solution before reading that which follows. Here m_t, the slope of the tangent line for any general point (x_1, y_1), is $f'(x_1) =$ ____________.

Skip two pages for the answer to frame 283.

A37. $2x_1 \Delta x + \Delta x^2$

38. Expressing the difference quotient, step (3) gives

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{(2x_1 \Delta x + \Delta x^2)}{\Delta x} = \_\_\_\_\_\_\_\_.$$

A162. $\frac{(x_1^4 + 1) \cdot (3x_1^2 + 4x_1 + 1) - (x_1^3 + 2x_1^2 + x_1 - 1) \cdot (4x_1^3)}{(x_1^4 + 1)^2}$ if $(x_1^4 + 1) \neq 0$.

163. Let us summarize our work up to this point. We defined the derivative of a function f as the limit of the difference quotient $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ if this limit exists. We then proved a number of basic theorems to enable us to find derivatives of certain functions. These theorems involved derivatives of sums, differences, products and quotients of functions, the derivative of the constant function, and the derivative of the independent variable. A corollary was proved for the derivative of a constant multiplied by a function of x. Review the theorems, definition and corollary from your list at this time.

A38. $2x_1 + \Delta x$. Since $\Delta x \neq 0$, but approaches zero, both numerator and denominator may be divided by a non-zero quantity.

39. Finally, expressing the limit of the difference quotient, step (4) gives $\lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} 2x_1 + \Delta x =$ ______.

164. Following is a set of exercises you should be able to complete using the theorems, corollary and definitions.

(a) If $f(x) = \pi x^3 - 20x^2 - 9$, $f'(x_1) =$ __________.

(b) If $f(x) = x^{\frac{7}{2}} - x^{2.3} + 7x^{-8}$, $f'(x_1) =$ ____________________.

(c) If $f(x) = (9x - 7)(x^3 + x^2 - 4x - 17)$, $f'(x_1) =$ ____________________.

(d) If $f(x) = \dfrac{7x - 5}{x^2 - 1}$, $x \neq \pm 1$, $f'(x_1) =$ ____________________.

(e) If $f(x) = (x^2 - 3)^2$, $f'(x_1) =$ ____________________.

(f) If $f(x) = \dfrac{x(2x^2 - 3)}{x - 5}$, $x \neq 5$, $f'(x_1) =$ ____________________.

A39. $2x_1$

40. Thus, if $f(x) = x^2$, $f'(x_1) = 2x_1$. If $f(x) = 2x^2$, find the derivative of f evaluated at x_1. Here $f(x_1) =$ _____ and $f(x_1 + \Delta x) =$ __________.

A164. (a) $3\pi x_1^2 - 40x_1$, (b) $\frac{7}{2}x_1^{\frac{5}{2}} - 2.3x_1^{1.3} - 56x^{-9}$,

(c) $(9x_1 - 7)(3x_1^2 + 2x_1 - 4) + 9(x_1^3 + x_1^2 - 4x_1 - 17)$,

(d) $\dfrac{7(x_1^2 - 1) - 2x_1(7x_1 - 5)}{(x_1^2 - 1)^2}$ if $x_1 \neq \pm 1$, (e) $4x_1(x_1^2 - 3)$,

(f) $\dfrac{(x_1 - 5)(6x_1^2 - 3) - (2x_1^3 - 3x_1)}{(x_1 - 5)^2}$ if $x_1 \neq 5$

165. We will introduce one more basic theorem to allow us to differentiate functions of functions, or composites of one function by another. Let us first discuss the composite of one function by another, or more simply a composite function. By definition, a function is a set of ordered pairs such that no two distinct ordered pairs have the same ________ element.

A283. $3x_1^2 - 4x_1 + 5$

284. At the point $(-1, -9)$, the slope of the tangent line to the above graph of f defined by $f(x) = x^3 - 2x^2 + 5x - 1$ is $f'(x_1) = f'(-1) =$ _____.

A40. $2x_1^2$, $2(x_1 + \Delta x)^2$

41. In step (2) in the process of finding the derivative of $f(x) = 2x^2$, evaluated at x_1, $f(x_1 + \Delta x) - f(x_1) =$ (in simplified form).

A165. first. (Review the definition of a function in the limit section if you had difficulty answering this item.)

166. A composite function (or a composite of one function by another) is defined by $f[g(x)]$ or $(f \circ g)(x)$. An example of such a function $f \circ g$ is defined by $f[g(x)] = (f \circ g)(x) = [(x^2 + 1)]^2$

$$z = g(x) = \_\_\_\_\_$$

$$f(z) = f[g(x)] = z^2$$

A284. 12

285. The equation of the tangent line at $(-1, -9)$ with slope 12, for the above function is ________________.

A41. $2(2x_1\Delta x + \Delta x^2)$

42. In step (3), the difference quotient $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = ________.

A166. $x^2 + 1$

167. For this function, an ordered pair belonging to f o g is (1, ____).

A285. $y + 9 = 12(x + 1)$ (or an equivalent expression)

286. In the following set of exercises, find the slope of the tangent line at the indicated point and write the equation of the tangent line at that point.

(a) $f(x) = -2$ at $(7, -2)$

(b) $f(x) = x$ at $(-9, -9)$

(c) $f(x) = x - 1$ at $(1, 0)$

(d) $f(x) = (x - 1)^2$ at $(1, 0)$

(e) $f(x) = 1 - 3x - x^2$ at $(-2, 3)$

A42. $2(2x_1 + \Delta x)$ (Note: We have divided by non-zero Δx.)

43. In step (4), the limit of the difference quotient is expressed as

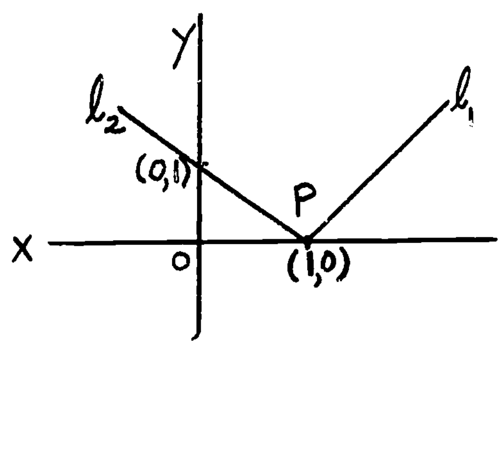
$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$$ (express your answer in final form, after evaluating the limit).

A167. 4

168. Other ordered pairs belonging to the function are (0,1), (2,25), (-1,___), (3,___). (Note that we are here emphasizing the fact that a composite of one function by another is a function in the "ordered pair" sense discussed in the section on limits.)

A286. (a) 0; $y + 2 = 0$ (or $f(x) = -2$), (b) 1; $x = y$, (c) 1; $y = x - 1$, (d) 0; $y = 0$ (or x-axis), (e) 1; $y - 3 = x + 2$

287. Since the derivative of f(x) evaluated at x_1 is the same as the slope of the tangent line to the graph of f at the point with x-coordinate x_1, when this slope exists, we can see why our previous example of the function f defined by $f(x) = |x - 1|$ did not possess a derivative at $x_1 = 1$. Referring to the graph of f below, note that the 2 lines, l_1 and l_2, making up the graph and intersecting at the point P:(1,0), make angles of 45 and _____ degrees, respectively, with the positive x-axis.



A43. $4x_1$

44. Thus, if f is defined by $f(x) = 2x^2$, $f'(x_1) =$ ______.

A168. 4, 100

169. Another example of a composite function f o g is defined by

$(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$, $x \geq 0$

$z = g(x) = x^3$

$f(z) = f[g(x)] = z^{\blacksquare}$ ______.

(Your answer should correctly complete the shaded box.)

A287. 135

288. Choose a point B on the graph of the function on your own paper, with an x-coordinate slightly greater than $x_1 = 1$, say $x_1 + \Delta x = 1 + \Delta x$, where Δx is (greater than, less than) zero. Indicate such a point B on your graph.

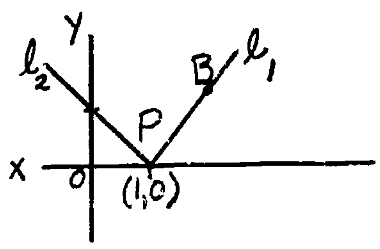
A44. $4x_1$

45. Let us now consider a function f defined by $f(x) = x - x^2$, composed of 2 terms, and compute its derivative. As you might expect, the process will be a bit more complicated in this case. Step (1) gives $f(x_1) =$ ______ and $f(x_1 + \Delta x) =$ ______________.

A169. $\frac{1}{2}$

170. Ordered pairs belonging to the composite function f o g defined above are (1,1), $(2, 2\sqrt{2})$, (3,________), (0,___).

A288. greater than



289. For such a point B, a secant line PB to the graph of f is the same as the line ______.

A45. $x_1 - x_1^2$, $(x_1 + \Delta x) - (x_1 + \Delta x)^2$

46. The difference $f(x_1 + \Delta x) - f(x_1)$ in step (2) is

$[(x_1 + \Delta x) - (x_1 + \Delta x)^2] - [x_1 - x_1^2] =$ ________________.

A170. $\sqrt{27}$ or $3\sqrt{3}$, 0

171. Let us now concentrate on recognizing the form of certain composite functions (or composites of one function by another). This ability will be needed for the next differentiation theorem. Note that $f[g(x)] = (f \circ g)(x)$ denotes that f is a function of g defined by $g(x) = z$, and g is in turn a function defined by the variable _____.

A289. l_1 (or PB)

290. The limiting position of a secant line PB as B approaches P is the line ______.

A46. $\Delta x - 2x_1 \cdot \Delta x - \Delta x^2$

47. The difference quotient $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \dfrac{\Delta x - 2x_1 \cdot \Delta x - \Delta x^2}{\Delta x}$ in step (3) is ____________.

A171. x

172. The composite function f o g defined by $(f \circ g)(x) = f[g(x)] = (x^2 + 1)^2$ considered in frame 166 can be defined in another manner as the composite of one function by another. Since $(x^2 + 1)^2 = \sqrt{(x^2 + 1)^4}$, we may write $(f \circ g)(x) = (p \circ q)(x)$

$$\text{or } p[q(x)] = \sqrt{(x^2 + 1)^4}$$

$$z = q(x) = (x^2 + 1)^4$$

$$p(z) = p[q(x)] = \underline{z\quad}.$$

A290. l_1 (or PB)

291. Since l_1 makes a 45-degree angle of inclination with the positive x-axis, the slope of this line is ____.

A47. $1 - 2x_1 - \Delta x$

48. Finally, the limit of the difference quotient in step (4) is

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} (1 - 2x_1 - \Delta x) = \_\_\_\_\_\_.$$

A172. $\frac{1}{2}$

173. The composite function f o g defined by $(f \circ g)(x) = (x^2 + 1)^2$ may also be defined by $(m \circ n)(x) = m[n(x)] = (\sqrt[3]{x^2 + 1})^6$ where $n(x) =$ ______ and $m(z) = z^6$.

A291. 1

292. Check your answer with our former reasoning for f defined by $f(x) = |x - 1|$ in frame 235. There $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ evaluated at $x_1 = 1$ had numerical value ____.

A48. $1 - 2x_1$

49. Thus, if $f(x) = x - x^2$, $f'(x_1) =$ ______.

A173. $\sqrt[3]{x^2 + 1}$

174. Thus, you see that a function has more than one representation as a composite function - in fact, an infinite number of such representations. Can you think of other ways in which $(f \circ g)(x) = f[g(x)] = (x^2 + 1)^2$ is a composite of one function by another, different from the above representations? Some other representations follow, which may be the same as or different from yours. Check your results with your teacher if you have questions.

A292. 1

293. Now consider a point A on the graph of f defined by $f(x) = |x - 1|$ with x-coordinate slightly less than $x_1 = 1$, say $x_1 + \Delta x = 1 + \Delta x = .5$. Here Δx has numerical value ____. Indicate point A on your graph.

A49. $1 - 2x_1$

50. By now, you realize that the process of differentiating is tedious and could be rather complicated and lengthy. Fortunately, there are a number of theorems that allow us to shorten our work. In fact, we have already proved one of the basic theorems. You recall that in frame 35, if f is defined by $f(x) = x$, then the derivative of f evaluated at x_1 is 1; i.e. $f'(x_1) =$ ____.

A174. $[(x^2 + 1)^{-\frac{1}{2}}]^{-4}$, $(\sqrt{x^2 + 1})^4$, $\sqrt[3]{(x^2 + 1)^6}$

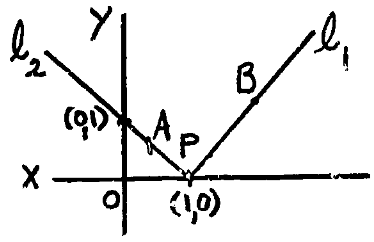
175. Consider the second composite function given above, defined by $(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$, $x \geqq 0$, which may also be defined in a different way as the composite of one function by another. Since $\sqrt{x^3} = (\sqrt{x})^3$, we may write $(f \circ g)(x) = (p \circ q)(x)$ or

$$p[q(x)] = (\sqrt{x})^3$$

$$z = q(x) = \sqrt{x}$$

$p(z) = p[q(x)] = z$ [redacted] ___.

A293. -.5



294. For point A, the secant line AP to the graph of f defined by $f(x) = |x - 1|$ is the same as the line ____.

A50. 1

51. We express this in words, as the following theorem:

Theorem 1. The derivative of the independent variable is one. In mathematical notation, we have: If $f(x) = x$, $f'(x_1) =$ ____. (Enter this theorem, in both forms, on the enclosed sheet headed "List of Definitions and Theorems.")

A175. 3

176. The composite function defined by $(f \circ g)(x) = f[g(x)] = \sqrt{x^3}$ may also be defined by $(m \circ n)(x) = m[n(x)] = \sqrt[4]{x^6}$ where $n(x) =$ ____ and $m(z) = z^{\frac{1}{4}}$.

A294. l_2 (or AP)

295. Since l_2 makes a 135-degree angle of inclination with the positive x-axis, the slope of this line is ____.

A51. 1

52. Let us now state and prove a second theorem in differential calculus that enables us to find the derivative of a function more easily. Theorem 2: The derivative of the constant function, evaluated at x_1, is zero. In mathematical notation, we have: If $f(x) = k$, where k is a constant, then $f'(x_1) =$ ____. (Enter this theorem, in both forms, on your list.)

A176. x^6

177. We have now expressed $(f \circ g)(x) = \sqrt{x^3}$ as the composite of one function by another in three ways. Can you think of other ways, different from the given three, for which $(f \circ g)(x) = \sqrt{x^3}$ is a composite of one function by another? Some such representations follow.

A295. -1

296. Check your answer with our former reasoning for f defined by $f(x) = |x - 1|$ in frame 236. There $\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ____, evaluated at $x_1 = 1$. Do the two results check?

A52. 0

53. We shall proceed to prove this theorem by applying the 4 steps in the process for finding the derivative of the function f evaluated at x_1. In step (1), if $f(x) = k$, $f(x_1) = k$ and $f(x_1 + \Delta x) =$ ____.

A177. $\sqrt[3]{x^{\frac{9}{2}}}$, $(\sqrt[3]{x})^{\frac{9}{2}}$, $\sqrt[8]{x^{12}}$

178. Consider the composite function f o g defined by

$$(f \circ g)(x) = \frac{1}{\sqrt[3]{x+1}} = (x+1)^{-\frac{1}{3}}, \; x \neq -1, \text{ or}$$

$$f[g(x)] = (x+1)^{-\frac{1}{3}} \text{ where}$$

$$z = g(x) = x + 1$$

$f(z) = z$ [redacted] ____.

A296. -1, Yes

297. The derivative of a function f defined by f(x), evaluated at x_1, is the same as the slope of the tangent line to the graph of f at the point with x-coordinate x_1. For the above function, at (1,0) no such tangent line ________.

A53. k (For all x, $f(x) = k$.)

54. In step (2), $f(x_1 + \Delta x) - f(x_1) =$ ____.

A178. $\frac{-1}{3}$

179. The composite function f o g defined by $(f \circ g)(x) = (x + 1)^{\frac{-1}{3}}$ may also be defined by $(p \circ q)(x) = (\sqrt[3]{x + 1})^{-1}$ where $g(x) =$ ________ and $p(z) =$ ________.

A297. exists (or an equivalent expression)

298. The slope of the tangent line at (1,0) doesn't exist because right and left hand limits of the function at this point are unequal; i.e. if $x_1 = 1$ $\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = -1$ and $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ $=$ ____.

A54. 0

55. In step (3), $\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ = ____.

A179. $\sqrt[3]{x+1}$ (or $(x+1)^{\frac{1}{3}}$), z^{-1} (or $\frac{1}{z}$)

180. The composite function f o g defined by $(f \circ g)(x)=(x+1)^{-\frac{1}{3}}$ may also be defined by $(m \circ n)(x)=\sqrt[3]{(x+1)^{-1}}$ where $n(x)=$ __________ and $m(z)=$ __________.

A298. 1

299. Consider the function f defined by $f(x)=x$. We proved as our first theorem, that for this function, $f'(x_1)=$ ____.

A55. 0 (Here again, $\Delta x \neq 0$.)

56. Finally, in step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} 0 =$ ______________

______________.

A180. $(x + 1)^{-1}$ (or $\frac{1}{x + 1}$), $\sqrt[3]{z}$ (or $z^{\frac{1}{3}}$)

181. Can you define $(f \circ g)(x) = (x + 1)^{-\frac{1}{3}}$ as a composite function by representations different from those cited above? Some such representations follow.

A300. 1

301. Let us relate the value of this derivative to the interpretation of the derivative as the slope of the tangent to the graph of f at a point with x-coordinate x_1 if the tangent exists. Graph the function f defined by $f(x) = x$ on your own paper.

Skip a page for the answer to frame 301.

A56. 0, since $\lim_{x \to a} k = k$ for any constant k

57. We have just proved that the derivative of a constant function f defined by $f(x) = k$ is $f'(x_1) =$ ____.

A181. $\frac{1}{\sqrt[6]{(x + 1)^2}}$, $[(x + 1)^{-2}]^{\frac{1}{6}}$, $[(x + 1)^3]^{-\frac{1}{9}}$

182. Following is a list of functions you are to define as composite functions in two other ways as indicated:

(a1) $F(x) = (f \circ g)(x) = x$ where $g(x) = x^3$, $f(z) = f[g(x)] = z$ [redacted].

(a2) $F(x) = (f \circ g)(x) = x$ where $g(x) = x$ [redacted], $f(z) = f[g(x)] = z^7$.

(b1) $H(x) = (f \circ g)(x) = \frac{1}{x} = x^{-1}$, $x \neq 0$, where $g(x) = x^2$, $f(z) = f[g(x)] = z$ [redacted].

(b2) $H(x) = (f \circ g)(x) = x$ [redacted], $x \neq 0$, where $g(x) = x$, $f(z) = f[g(x)] = z^{-1}$.

(c1) $Q(x) = (f \circ g)(x) = (\frac{1}{\sqrt{x}})^3 = x^{-\frac{3}{2}}$, $x > 0$, where $g(x) = x$, $f(z) =$ $f[g(x)] = z$ [redacted].

(c2) $Q(x) = (f \circ g)(x) = (\frac{1}{\sqrt{x}})^3 = x^{-\frac{3}{2}}$, $x > 0$, where $g(x) = x$ [redacted], $f(z) =$ $f[g(x)] = z^{-1}$.

(d1) $P(x) = (f \circ g)(x) = \frac{1}{(1 + x)^{-2}}$ where $g(x) = 1 + x$, $f(z) = f[g(x)] = z$ [redacted].

(d2) $P(x) = (f \circ g)(x) = \frac{1}{(1 + x)^{-2}}$ where $g(x) = (1 + x)$ [redacted], $f(z) = f[g(x)]$ $= \sqrt{z}$.

(e1) $G(x) = (f \circ g)(x) = \sqrt{1 - x^2}$, $-1 \leqq x \leqq 1$, where $g(x) = (1 - x^2)^2$, $f(z) =$ $f[g(x)] = z$ [redacted].

(e2) $G(x) = (f \circ g)(x) = \sqrt{(1 - x^2)}$, $-1 \leqq x \leqq 1$, where $g(x) = (1 - x^2)$ [redacted], $f(z) =$ $f[g(x)] = z^3$.

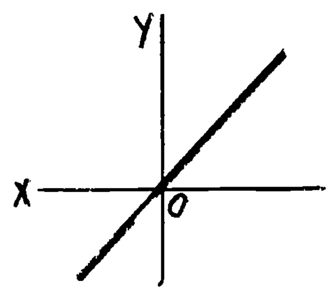
A57. 0

58. Thus, if $f(x) = 7$, $f'(x_1) = 0$. If $f(x) = \frac{-1}{4}$, $f'(x_1) =$ ____.

A182. (a1) $\frac{1}{3}$, (a2) $\frac{1}{7}$, (b1) $\frac{-1}{2}$, (b2) -1, (c1) $\frac{-3}{2}$, (c2) $\frac{3}{2}$, (d1) 2, (d2) 4, (e1) $\frac{1}{4}$, (e2) $\frac{1}{6}$

183. We are now ready to state and prove the theorem for differentiation of a composite function. Theorem 8. If $F(x) = (f \circ g)(x) = f[g(x)]$, and $f'[g(x_1)]$ and $g'(x_1)$ exist, then $F'(x_1) = f'[g(x_1)] \cdot g'(x_1)$. Add this theorem to your list.

A301.



302. Considering any point P on the graph of f with x-coordinate x_1, choose a point B on the graph whose x-coordinate is slightly greater than x_1, say $x_1 + \Delta x$. Indicate such points P and B on your graph.

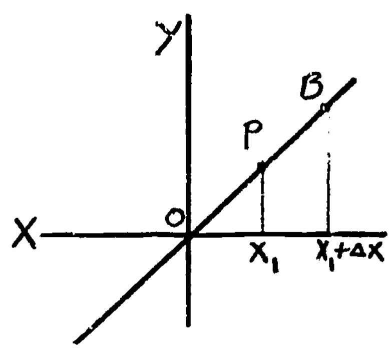
A58. 0

59. A useful and basic theorem of differential calculus involves the derivative of a positive integral power of the independent variable. This theorem reads: Theorem 3. If $f(x) = x^n$, where n is a positive integer, then $f'(x_1) = n \cdot x_1^{n-1}$. Stated in words, this theorem says that the derivative of a positive integral power of the independent variable, evaluated at x_1, is the original exponent, n, multiplied by a power of x_1, and x_1 has ______ as the exponent.

184. This theorem is sometimes called the chain rule of differentiation because $F'(x_1)$ is expressed as a chain of derivatives, namely (fill in numeral) in number as we've stated the theorem.

Skip a page for the answer to frame 184.

A302.



Note: Your points may have been placed differently.

303. A secant line PB to the graph of f is the same as ______________ ______________.

A59. $n = 1$

60. To prove this theorem, we proceed through the four steps in finding the derivative of f evaluated at x_1, as before. In step (1), if $f(x) = x^n$, $f(x_1) =$ ____ and $f(x_1 + \Delta x) =$ ______.

A303. the graph of f defined by $f(x) = x$ (or an equivalent expression)

304. The tangent line to the graph of f defined by $f(x) = x$; i.e. the limiting position of a secant line PB as point B approaches point P, is

__________________.

Skip a page for the answer to frame 304.

A60. x_1^n, $(x_1 + \Delta x)^n$

61. We note that $f(x_1 + \Delta x) = (x_1 + \Delta x)^n$ can be expanded by the binomial theorem, since n is a positive integer. Thus, $(x_1 + \Delta x)^n =$ $x_1^n + \frac{n}{1!} \cdot x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 +$ (next term ?) + S, where S is the sum of other terms, all of which contain Δx with exponents greater than 3 if $n > 3$.

A184. two (More functions than two may be in the chain, depending on the complexity of the composite function.)

185. Proceed to prove this theorem by the same method previously used, that of the four steps in the process for finding the derivative of F evaluated at x_1. In step (1), if $F(x) = f[g(x)]$, $F(x_1) = f[g(x_1)]$ and $F(x_1 + \Delta x) =$ __________.

A61. $\frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3$

62. Thus, in step (2), $f(x_1+\Delta x)-f(x_1)=(x_1+\Delta x)^n-x_1^n$. If we substitute the expression for $(x_1+\Delta x)^n$ derived by the binomial expansion, we have $f(x_1+\Delta x)-f(x_1)=$ ______________________________

$-x_1^n$.

A185. $f[g(x_1+\Delta x)]$

186. In step (2), $F(x_1+\Delta x)-F(x_1)=$ ____________________.

A304. the graph of f

305. The graph of f makes an inclination angle of 45 degrees with the positive x-axis and hence has slope ____.

A62. $x_1^n + \frac{n}{1!} x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3 +$ S

63. Simplifying, $f(x_1 + \Delta x) - f(x_1) = (x_1^n + \frac{n}{1!} x_1^{n-1} \cdot \Delta x + \ldots + \Delta x^n) - x_1^n$, we obtain $f(x_1 + \Delta x) - f(x_1) =$ ______________________________.

A186. $f[g(x_1 + \Delta x)] - f[g(x_1)]$

187. Step (3) gives $\frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} =$ ____________________.

A305. 1

306. Choose a point A on the graph with x-coordinate slightly less than P. Indicate such a point A on your graph.

Skip two pages for the answer to frame 306.

A63. $\frac{n}{1!} x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3 + S$

64. In step (3), $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$

$$\frac{\frac{n}{1!} x_1^{n-1} \cdot \Delta x + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x^2 + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^3 + S}{\Delta x}$$

= ____, dividing each term by Δx.

A187. $\frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{\Delta x}$

188. Let us rewrite the above difference quotient, $\frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{\Delta x}$, before proceeding to step (4). Why can we write

$\frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{\Delta x}$ as $\frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} \cdot \frac{g(x_1 + \Delta x) - g(x_1)}{\Delta x}$?

A64. $n \cdot x_1^{n-1} + \frac{n(n-1)}{2!} x_1^{n-2} \cdot \Delta x + \frac{n(n-1)(n-2)}{3!} x_1^{n-3} \cdot \Delta x^2 + \frac{S}{\Delta x}$

65. In step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$= \lim_{\Delta x \to 0} (n \cdot x_1^{n-1} +$ the sum of terms, each containing $\Delta x)$

$= \lim_{\Delta x \to 0} n \cdot x_1^{n-1} + \lim_{\Delta x \to 0}$ (the sum of terms, each containing Δx)

= ________.

A188. If both numerator and denominator of an expression are multiplied by the same non-zero quantity - in this case $g(x_1 + \Delta x) - g(x_1)$ - the resulting expression is equivalent to the original expression.

189. In step (4), $F'(x_1) = \lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{\Delta x}$

(substituting from above) $= \lim_{\Delta x \to 0} \left(\frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} \cdot \frac{g(x_1 + \Delta x) - g(x_1)}{\Delta x} \right)$

(1) $= \lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} \cdot \lim_{\Delta x \to 0} \frac{g(x_1 + \Delta x) - g(x_1)}{\Delta x}$

(2) $= \lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} \cdot g'(x_1)$

Supply reasons for (1) and (2).

A65. $n \cdot x_1^{n-1}$

66. We have now completed the proof of the theorem which states that if $f(x) = x^n$, where n is a positive integer, $f'(x_1) =$ ________.

A189. (1) The limit of a product is the product of the limits.

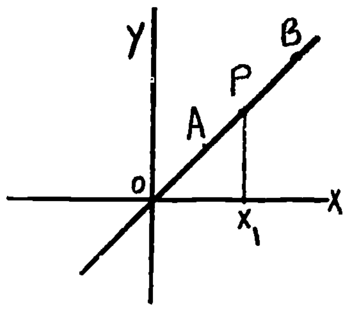
(2) By definition, $\lim_{\Delta x \to 0} \frac{g(x_1 + \Delta x) - g(x_1)}{\Delta x} = g'(x_1)$

190. Referring to the statement of this theorem on your list and considering the final expression for $F'(x_1)$ in the previous frame, i.e.

$$\lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} \cdot g'(x_1),$$ we must show

$$\lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)} = \text{\_\_\_\_\_\_\_\_}.$$

A306.



307. For this point A, let its x-coordinate be $x_1 + \Delta x$, where Δx is (greater than, less than) zero.

A66. $n \circ x_1^{n-1}$

67. This theorem is sometimes referred to as the power differentiation formula. Using this theorem, if $f(x) = x^3$, $f'(x_1) = 3x_1^2$. If $f(x) = x^7$, $f'(x_1) =$ _____.

A190. $f'[g(x_1)]$

191. As we reasoned before for the function w in the proof of the product and quotient rule, where we assumed continuity, as Δx approaches 0, $g(x_1 + \Delta x)$ approaches $g(x_1)$ assuming g is continuous. Thus

$$\lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)}$$

$$= \lim_{\text{[redacted]}} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)}$$

A307. less than

308. The limiting position of a secant line AP, as point A approaches point P, is the tangent line to the graph of f at point P. This is the line ______________________________.

A67. $7x_1^6$

68. Can the derivative of the function f defined by $f(x) = x^2$, whose value is given in frame 40, be evaluated by the theorem just proved? Why?

A191. $g(x_1 + \Delta x) \to g(x_1)$

192. Thus, $\lim_{\Delta x \to 0} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)}$

$$= \lim_{g(x_1 + \Delta x) \to g(x_1)} \frac{f[g(x_1 + \Delta x)] - f[g(x_1)]}{g(x_1 + \Delta x) - g(x_1)}$$

By changing to alternate notation, $g(x_1 + \Delta x) = g_1 + \Delta g$ and $g(x_1) =$ _____.

A308. making up the graph of f

309. This tangent line, obtained from the limiting position of the secant line AP, has slope ____.

A68. Yes, it can because n is the positive integer 2.

69. Thus, if $f(x) = x^2$, $f'(x_1) =$ _____. Does this answer agree with that obtained in frame 40?

A192. g_1

193. Thus,

$$\lim_{g(x_1+\Delta x)\to g(x_1)} \frac{f[g(x_1+\Delta x)] - f[g(x_1)]}{g(x_1+\Delta x) - g(x_1)}$$

$$= \lim_{(g_1+\Delta g)\to g_1} \frac{f[g_1+\Delta g] - f[g_1]}{(g_1+\Delta g) - g_1}$$

$$= \lim_{(g_1+\Delta g)\to g_1} \frac{f[g_1+\Delta g] - f[g_1]}{\text{[redacted]}}.$$

A309. 1

310. Thus, the slopes of the secant lines as we approach P on the graph of f by points to the left and right of P, are numerically (equal, unequal).

A69. $2x_1$, Yes

70. We might also check Theorem 1 (if $f(x) = x$, then $f'(x_1) = 1$) by the power differentiation formula. We see that $f(x) = x$ can be written with a positive integral exponent as $f(x) =$ ____.

A193. Δg

194. But $(g_1 + \Delta g) \to g_1$ as $\Delta g \to 0$, so $\lim\limits_{(g_1 + \Delta g) \to g_1} \dfrac{f(g_1 + \Delta g) - f(g_1)}{\Delta g}$

$= \lim\limits_{\text{[redacted]}} \dfrac{f(g_1 + \Delta g) - f(g_1)}{\Delta g}$.

A310. equal

311. The previous statement is equivalent to the statement that

$\lim\limits_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ________________.

A70. x^1

71. Thus, if $f(x) = x^1$, by the power differentiation formula, $f'(x_1) = 1 \cdot x_1^{1-1} =$ ____________.

A194. $\Delta g \to 0$

195. By definition, $\lim_{\Delta g \to 0} \frac{f(g_1 + \Delta g) - f(g_1)}{\Delta g} =$ ________.

A311. $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

312. Thus, if f is defined by $f(x) = x$, the derivative of f, evaluated at x_1 (the slope of the tangent to the graph of f at $x = x_1$), xists and has numerical value ____. Is this result consistent with Theorem 1?

A71. $1 \cdot x_1^0 = 1 \cdot 1 = 1$

72. Does the answer in the previous frame check with the result of the first theorem?

A195. $f'(g_1)$

196. In alternate notation, $f'(g_1) =$ ________.

A312. 1, Yes

313. Consider the second theorem we proved concerning the derivative of a function f defined by $f(x) = k$, where k is a constant. For this function, $f'(x_1) =$ ____.

A72. Yes

73. It will be proved later in your calculus course that if $f(x) = x^n$, where n is any real number, rather than a positive integer, it is still true that $f'(x_1) =$ ________.

A196. $f'[g(x_1)]$

197. Referring back to frame 190, we have shown the required equality. You should review the proof of this theorem, since it is the most difficult proof demonstrated thus far. Stated again, the theorem reads: If $F(x) = f[g(x)]$, and $f'[g(x_1)]$ and $g'(x_1)$ exist, then $F'(x_1) =$ ________.

A313. 0

314. Graph this function on your own paper, indicating a point P with x-coordinate x_1 on your graph.

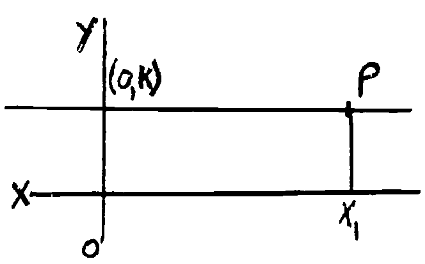
A73. $n \cdot x_1^{n-1}$

74. Thus, if $f(x) = x^{\frac{4}{3}}$, $f'(x_1) = \frac{4}{3} \cdot x_1^{\frac{4}{3} - 1} = \frac{4}{3} \cdot x_1^{\frac{1}{3}}$. If $f(x) = x^{-5}$, $f'(x_1) = -5 \cdot x_1^{-5-1} =$ ________.

A197. $f'[g(x_1)] \cdot g'(x_1)$

198. Since most of the composite functions we want to consider will be powers of a function of x; i.e. $[g(x)]^n$ where n is a real number, we can prove a corollary to Theorem 8. Corollary. If $F(x) = [g(x)]^n$, and $g'(x_1)$ exists, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$. Add this corollary to your list.

A314.



(Note: Your point P may have been placed differently.)

315. Choose a point B on the graph of f defined by $f(x) = k$, with x-coordinate slightly greater than x_1, say $x_1 + \Delta x$. Indicate such a point B on your graph.

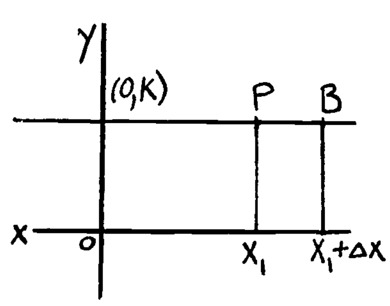
A74. $-5 \cdot x_1^{-6}$

75. We have now proved three basic theorems for finding the derivative of simple functions. These theorems express the derivative of the independent variable, the constant function, and a positive integral power of the independent variable, the last of which we extended to any real exponent. Review these theorems from your list at this time.

199. If $F(x) = f[g(x)] = [g(x)]^n$, where $g(x) = z$

$f[g(x)] = f(z) = z$[redacted].

A315.



316. A secant line PB to the graph of f is the same as the graph of ____.

76. Following are 6 exercises you should be able to do using the theorems.

(a) If $f(x) = x^{\frac{8}{5}}$, $f'(x_1) =$ ______

(b) If $f(x) = 2^{\sqrt{2}}$, $f'(x_1) =$ ____

(c) If $f(x) = x^{1.7}$, $f'(x_1) =$ ______

(d) If $f(x) = 10^0$, $f'(x_1) =$ ____

(e) If $f(x) = 43^x$, $f'(x_1) =$ ______

(f) If $f(x) = x^{\pi}$, $f'(x_1) =$ ______

A199. n

200. If we let $g(x) = z$ in the statement of Theorem 8, we have $F(x) = f[g(x)] = f(z)$, $F'(x_1) = f'[g(x_1)] \cdot g'(x_1) = f'(\_\_\_\_) \cdot z_1'$

A316. f (or an equivalent expression)

317. The tangent line to the graph of f defined by $f(x) = k$; i.e. the limiting position of a secant line PB as point B approaches point P, is ______________.

Skip a page for the answer to frame 317.

A76. (a) $\frac{8}{5}x_1^{\frac{3}{5}}$, (b) 0, (c) $1.7x_1^{.7}$, (d) 0, (e) You do not yet know how to find the derivative of this expression, because the variable is the exponent rather than the base, as is required in the power differentiation formula. (f) $\pi x_1^{\pi-1}$

77. As you might expect, the three theorems we have proved certainly are not sufficient to find the derivatives of all functions one encounters in differential calculus. We will now prove four more theorems that permit differentiation of slightly more complicated functions composed of sums, differences, products and quotients of simpler functions. Because a derivative is a limit, you should expect theorems concerning derivatives of functions composed of sums, differences, products and quotients of functions to depend on limit theorems concerning ______, ________, ________ and ________ of functions.

A200. z_1

201. If we let $g(x) = z$ in the statement of the corollary, we have

$F(x) = [g(x)]^n = z^n$

$F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1) = n[z_1]^{n-1} \cdot$ ____.

A77. sums, differences, products, quotients

78. Our next theorem concerns the derivative of a sum of 2 functions. From the preceding remarks, because the limit of the sum of 2 functions is the sum of the limits of the functions, you should expect that the derivative of the sum of 2 functions is ______________.

A201. z_1'

202. Comparing the final expressions for $F'(x_1)$ in both the theorem and the corollary ($f'(z_1) \cdot z_1'$ and $n[z_1]^{n-1} \cdot z_1'$) we must show $f'(z_1) =$ ________.

A317. the graph of f

318. The graph of f makes an inclination angle of ____ degrees with the positive x-axis and hence has slope ____.

A78. the sum of the derivatives of the functions

79. Our next theorem reads: Theorem 4. If $f(x) = w(x) + v(x)$, where w and v are functions of x as indicated, and $w'(x_1)$ and $v'(x_1)$ exist, then $f'(x_1) = w'(x_1) + v'(x_1)$. Enter this theorem on your list.

A202. $n[z_1]^{n-1}$

203. That's easy! Since $f(z) = z^n$ (see frame 201), by the power differentiation formula, $f'(z_1) = n \cdot z_1^{\square}$.

Your answer should correctly complete the statement when placed in the box.

A318. 0, 0

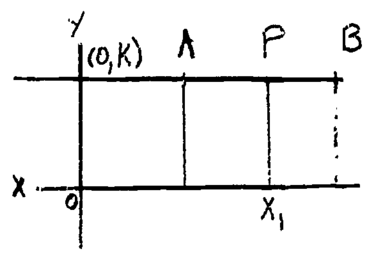
319. Choose a point A on the graph with x-coordinate slightly less than P. Indicate such a point A on your graph.

80. This theorem states that the derivative of the sum of 2 functions is the _____ of the derivatives of the 2 functions, provided the derivatives exist.

A203. n-1

204. Thus, if $F(x) = z^n$, $F'(x_1) = n[z_1]^{n-1} \cdot z'_1$, or if $z = g(x)$, the corollary reads: If $F(x) = [g(x)]^n$, and $[g(x_1)]^{n-1}$ and $g'(x_1)$ exist, $F'(x_1) =$ ________ $\cdot\ g'(x_1)$ which was to be proved.

A319.



320. The limiting position of a secant line AP, as point A approaches point P, is the tangent line to the graph of f at point P. This is

________________________.

A80. sum

81. Let us proceed through the necessary 4 steps to find the derivative, $f'(x_1)$, evaluated at x_1, for $f(x) = w(x) + v(x)$. Step (1) gives $f(x_1) =$ ____________ and $f(x_1 + \Delta x) =$ ____________________.

A204. $n \cdot [g(x_1)]^{n-1}$

205. It should again be emphasized that all composite functions are not powers of a function. An example of such an exception is the trigonometric function f o g defined by $(f \circ g)(x) = \sin(x^2 + 7)$, where $g(x) =$ ______, $f(z) = \sin z$.

A320. the graph of f

321. The graph of f has slope ____.

A81. $w(x_1) + v(x_1)$, $w(x_1 + \Delta x) + v(x_1 + \Delta x)$

82. In step (2), $f(x_1 + \Delta x) - f(x_1) = [w(x_1 + \Delta x) + v(x_1 + \Delta x)] - [w(x_1) + v(x_1)]$

$$= w(x_1 + \Delta x) + v(x_1 + \Delta x) - w(x_1) - v(x_1)$$

(grouping like terms) $= [w(x_1 + \Delta x) - w(x_1)] + [v(x_1 + \Delta x) - \_\_\_\_]$

A205. $(x^2 + 7)$

206. We will not now consider such composite functions as cited in the previous frame. Let us focus on the use of the corollary to Theorem 8. If F is defined by $F(x) = \sqrt{25 - x^2}$, $-5 \leq x \leq 5$, F is the composite function defined by $(f \circ g)(x) = f[g(x)] = (25 - x^2)^{\frac{1}{2}}$, where $g(x) = 25 - x^2$, $f(z) = z$[redacted]___.

A321. 0

322. Thus, the tangent lines at point P exist because as we approach P on the graph of f by points to the left and right of P,

$$\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \_\_\_\_\_\_\_\_.$$

A82. $v(x_1)$

83. Step (3) expresses $\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} = \frac{[w(x_1+\Delta x)-w(x_1)]+[v(x_1+\Delta x)-v(x_1)]}{\Delta x}$

(putting each expression in square brackets over the denominator Δx.) $= \frac{w(x_1+\Delta x)-w(x_1)}{\Delta x} +$ __________

A206. $\frac{1}{2}$

207. $F(x) = f[g(x)] = [g(x)]^{\frac{1}{2}}$, so the corollary states that if $F(x) = [g(x)]^n$, then $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$ where, in this case, $n =$ ____.

A322. $\lim\limits_{\Delta x \to 0^+} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$

323. Thus, if f is defined by $f(x) = k$, the derivative of f evaluated at x_1 (the slope of the tangent line to the graph of f at x_1) exists and has numerical value ____. Is this result consistent with Theorem 2?

A83. $\dfrac{v(x_1 + \Delta x) - v(x_1)}{\Delta x}$

84. In step (4) we take the limit of the above difference quotient:

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} + \frac{v(x_1 + \Delta x) - v(x_1)}{\Delta x}\right)$$

$$\overset{(1)}{=} \lim_{\Delta x \to 0} \frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x_1 + \Delta x) - v(x_1)}{\Delta x}$$

$$\overset{(2)}{=} w'(x_1) + v'(x_1)$$

Supply reasons for statements (1) and (2).

A207. $\frac{1}{2}$

208. For the above composite function, $F(x) = f[g(x)] = (25 - x^2)^{\frac{1}{2}}$, $g(x) = 25 - x^2$, so $g(x_1) =$ ________.

A323. 0, Yes

324. Since the derivative of $f(x)$ evaluated at x_1 can be interpreted as the slope of the tangent to the graph of f at the point with x-coordinate x_1, we know that the derivative would not exist (would not be defined) at points on the graph for which the slope of the tangent line does not exist. At such points, the tangent line may assume a (horizontal, vertical) position.

A84. (1) The limit of a sum of two functions is the sum of the limits of the two functions, if these limits exist. (2) By definition, $\lim_{\Delta x \to 0} \frac{w(x_1 + \Delta x) - w(x_1)}{\Delta x} = w'(x_1)$. The same is true of the second term $v'(x_1)$.

85. We have now proved that if $f(x) = w(x) + v(x)$, and $w'(x_1)$ and $v'(x_1)$ exist, then $f'(x_1) = w'(x_1) + v'(x_1)$. Let us now consider several examples of this theorem. If $f(x) = x + 7$, $w(x) = x$, $v(x) = 7$, $w'(x_1) = 1$, $v'(x_1) = 0$, so $f'(x_1) = w'(x_1) + v'(x_1) =$ ____ + ____.

A208. $25 - x_1^2$

209. For the above composite function, $g(x) = 25 - x^2$, so $g'(x_1) =$ ____.

A324. vertical

325. Let us examine this situation in terms of the previously discussed function F defined by $F(x) = \sqrt{25 - x^2}$. The domain of this function is defined by the inequality $-5 \leq x \leq 5$ and the range is defined by the inequality ______________.

A85. 1, 0

86. If $f(x) = x^3 - 6 = x^3 + (-6)$, $w(x) = x^3$, $v(x) = -6$, $w'(x_1) =$ _____,
$v'(x_1) =$ _____, so $f'(x_1) = w'(x_1) + v'(x_1) =$ _____ + _____.

A209. $-2x_1$

210. Thus, if $F(x) = [g(x)]^{\frac{1}{2}}$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$

$$= \frac{1}{2}[25 - x_1^2]^{\frac{1}{2} - 1} \cdot (-2x_1)$$

= (simplify, leaving a negative exponent on the quantity $(25 - x_1^2)$) _____.

A325. $0 \leq y \leq 5$

326. Recalling the graph of this function, at what point(s) would the tangent line(s) be parallel to the y-axis, or assume a vertical position?

A86. $3x_1^2$, 0, $3x_1^2$, 0

87. If $f(x) = x^4 + x$, $w(x) =$ ____, $v(x) =$ ____, $w'(x_1) =$ ____, $v'(x_1) =$ ____, so $f'(x_1) = w'(x_1) + v'(x_1) =$ ______.

A210. $-x_1(25 - x_1)^{-\frac{1}{2}}$ (Note that this expression can be rationalized.)

211. If F is defined by $F(x) = (x^2 - 2x - 3)^{\frac{7}{2}}$, $x^2 - 2x - 3 \geq 0$, F is the composite function f o g defined by $(f \circ g)(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$, where $g(x) = x^2 - 2x - 3$, $f(z) =$ z___.

A326. (-5,0), (5,0)

327. At what point(s) would the slope(s) of the tangent line(s) not exist?

A87. x^4, x, $4x_1^3$, 1, $4x_1^3 + 1$

88. Now it's your turn! If $f(x) = x^{\frac{8}{7}} - x^2$, $f'(x_1) =$ __________.

A211. $\frac{7}{2}$

212. $F(x) = f[g(x)] = [g(x)]^{\frac{7}{2}}$, so we use the corollary to Theorem 8, stating that if $F(x) = [g(x)]^n$, then $F'(x_1) = n[g(x_1)]^{n-1}$. _______.

A327. (-5,0), (5,0)

328. At what point(s) would the derivative(s) of the function not exist (not be defined)?

A88. $\frac{8}{7}x_1^{\frac{1}{7}} - 2x_1$

89. It should be noted that it is not necessarily the case that if the derivatives of two functions fail to exist for some value(s) of x_1, the derivative of the sum of these functions fails to exist at x_1. Consider $f(x) = w(x) + v(x) = |x| + [-|x|]$ where $w(x) = |x|$ and $v(x) =$ ________.

A212. $g'(x_1)$

213. For $F(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$, n = _____.

A328. (-5,0), (5,0)

329. Let us now compute the derivative of $F(x) = \sqrt{25 - x^2}$ to see if the above conjecture is actually the case. For $F(x) = \sqrt{25 - x^2}$, $F'(x_1) =$ __________. (Compute your answer without reference to previous results.)

A89. $-|x|$

90. Neither $w'(x_1)$ nor $v'(x_1)$ exist for $x_1 =$ ______.

A213. $\frac{7}{2}$

214. For the above composite function F defined by $F(x) = f[g(x)] = (x^2 - 2x - 3)^{\frac{7}{2}}$, $g(x) = x^2 - 2x - 3$, $g(x_1) =$ ________________.

A329. $- x_1(25 - x_1^2)^{-\frac{1}{2}}$

330. To find the slopes of the tangent lines at the points (-5,0) and (5,0), we must evaluate F'(___) and F'(___).

A90. 0

91. However, for the sum of the above functions, $f(x) = w(x) + v(x) = |x| + [-|x|] = 0$, $f'(x_1) =$ ______________________.

A214. $x_1^2 - 2x_1 - 3$

215. For the above composite function, $g(x) = x^2 - 2x - 3$, so $g'(x_1) =$

______________.

A330. -5, 5

331. Thus, if $F'(x_1) = -x_1(25 - x_1^2)^{-\frac{1}{2}}$, $F'(-5) =$ ____ and $F'(5) =$ ____.

A91. 0 for all values of x. (See Theorem 2).

92. If an algebraic expression is the difference of 2 functions; i.e., $f(x) = w(x) - v(x)$, you should expect $f'(x_1) =$ ______________.

A215. $2x_1 - 2$

216. Now, if $F(x) = [g(x)]^{\frac{7}{2}}$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g'(x_1)$ becomes

$F'(x_1) =$ [redacted] $(x_1^2 - 2x_1 - 3)$ [redacted] $(2x_1 - 2)$.

A331. Both answers are undefined.

332. What are the equations of the tangent lines to the graph of F defined by $F(x) = \sqrt{25 - x^2}$ at the points $(-5, 0)$ and $(5, 0)$? (Remember that these tangent lines are parallel to the y-axis and pass through the points $(-5, 0)$ and $(5, 0)$.)

A92. $w'(x_1) = v'(x_1)$

93. One could prove the above statement, a theorem, with the help of the limit theorem that states the limit of the difference of two functions is equal to the difference of the ______________________________. (We will omit the proof, but the reader is advised to complete the proof as an exercise.)

A216. $\frac{7}{2}$, $\frac{7}{2} - 1$ (or $\frac{5}{2}$)

217. Thus, if $F(x) = (x^2 - 2x - 3)^{\frac{7}{2}}$, $F'(x_1) = \frac{7}{2}(x_1^2 - 2x_1 - 3)^{\frac{7}{2} - 1}(2x_1 - 2)$

= (simplify your answer, leaving a fractional exponent on the quantity $(x_1^2 - 2x_1 - 3)$).

A332. $x = -5$, $x = 5$

333. In this case, the slopes of the tangent lines don't exist at two points on the graph of F, but F is defined at these points and these tangent lines exist, so it is possible to write their equations. Draw these tangent lines on your graph and label them.

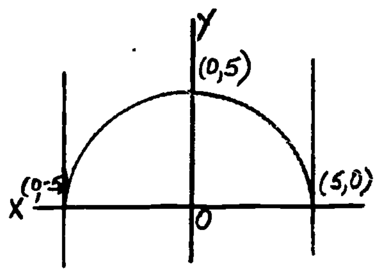
A93. limits of the two functions if these limits exist

94. The theorem reads: Theorem 5. If $f(x) = w(x) - v(x)$, then $f'(x_1) = w'(x_1) - v'(x_1)$ if $w'(x_1)$ and $v'(x_1)$ exist. Add this theorem to your list.

A217. $7(x_1^2 - 2x_1 - 3)^{\frac{5}{2}} (x_1 - 1)$

218. If F is defined by $F(x) = \frac{1}{(x^4 - 1)^2}$, $x^4 - 1 \neq 0$, $F(x) = (f \circ g)(x) = f[g(x)] = (x^4 - 1)$ [redacted] .

A333.



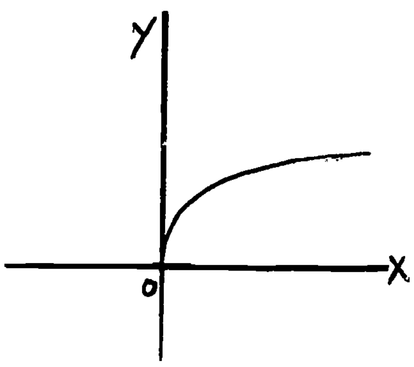
334. Consider the function f defined by $f(x) = x^{\frac{1}{4}}$, $x \geq 0$. Graph this function on your own paper for future reference.

95. If $f(x) = \pi - x$, $w(x) = \pi$, $v(x) = x$, $w'(x_1) =$ ____, $v'(x_1) =$ ____,
$f'(x_1) = w'(x_1) - v'(x_1) =$ ____.

<u>A218. −2</u>

219. So $F(x) = (f \circ g)(x) = f[g(x)] = (x^4 - 1)^{-2}$ where $g(x) = x^4 - 1$,
$f(z) =$ <u>z [redacted]</u>.

<u>A334.</u>



335. The domain of this function is defined by the inequality $x \geq 0$,
as cited above, and the range of f is defined by ________.

A95. 0, 1, -1

96. If $f(x) = x - x^2$, $w(x) =$ ____, $v(x) =$ ____, $w'(x_1) =$ ____, $v'(x_1) =$ ____, $f'(x_1) = w'(x_1) - v'(x_1) =$ ____.

A219. -2

220. $F(x) = f[g(x)] = [g(x)]^{-2}$, so our corollary will apply with n = ____.

A335. the inequality $y \geq 0$

336. From the graph of this function, at what point(s) would the tangent line(s) be parallel to the y-axis?

A96. x, x^2, 1, $2x_1$, $1 - 2x_1$ (Does your answer check with the derivative of this function, obtained by the 4 steps, given in frame 49?)

97. Again it's your turn. If $f(x) = x^{\frac{9}{7}} - \sqrt{2}^{\,7}$, $f'(x_1) =$ __________.

A220. -2

221. For the above composite function, $F(x) = f[g(x)] = (x^4 - 1)^{-2}$, $g(x_1) =$ ______ and $g'(x_1) =$ ______.

A336. $(0,0)$

337. At what point(s) would the slope(s) of the tangent line(s) not exist?

A97. $\frac{9}{7} x_1^{\frac{2}{7}}$ ($\sqrt{2}^7$ is a constant)

98. Since many algebraic expressions are products of simpler functions, we will now prove a theorem concerning the derivative of the product of 2 functions. Consider the function $f(x) = w(x) \cdot v(x)$. Since the limit of a product of 2 functions is the product of the limits of the 2 functions, (i.e. $\lim_{x \to x_1} [w(x) \cdot v(x)] = \lim_{x \to x_1} w(x) \cdot \lim_{x \to x_1} v(x)$) would you expect that the derivative of a product of 2 functions is the product of the derivatives of the 2 functions (i.e. $f'(x_1) = w'(x_1) \cdot v'(x_1)$)?

A221. $x_1^4 - 1$, $4x_1^3$

222. Now, if $F(x) = [g(x)]^{-2} = (x^4 - 1)^{-2}$, $F'(x_1) = n[g(x_1)]^{n-1} \cdot g(x_1) =$ (simplify your answer, leaving a negative exponent on the quantity $(x_1^4 - 1)$).

A337. (0,0)

338. At what point(s) would the derivative(s) of the function not exist?

A98. If you answered "yes," you were tricked. The problem is not that simple.

99. The theorem reads: Theorem 6. If $f(x) = w(x) \cdot v(x)$, then $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1)$ provided $w'(x_1)$ and $v'(x_1)$ exist. Add this theorem to your list.

A222. $-8x_1^3(x_1^4 - 1)^{-3}$

223. For $F(x) = (x - 2x^2)^{\frac{1}{2}}$, $n = \frac{1}{2}$, $g(x_1) =$ ______, $g'(x_1) =$ ______.

A338. (0,0)

339. At the point (0,0), $f'(x_1) = \frac{1}{4} \cdot x_1^{\frac{-3}{4}}$ or $f'(0)$ is ________.

100. Study the statement of this theorem carefully. Stating this theorem in words, we have: The derivative of the product of two functions of x, evaluated at x_1, is the first function multiplied by the derivative of the second function plus the second function multiplied by ________________________, provided these derivatives exist.

A223. $x_1 - 2x_1^2$, $1 - 4x_1$

224. For $F(x) = (x - 2x^2)^{\frac{1}{2}}$, $F'(x_1) =$ (leave a negative exponent on the quantity $(x_1 - 2x_1^2)$.

A339. undefined

340. Can you write the equation of the tangent line for the above function at the point (0,0)? (Remember that this tangent line is parallel to the y-axis and passes through the point (0,0).)

Skip two pages for the answer to frame 340.

A100. the derivative of the first function (You should become familiar with stating all the theorems in words, as an aid to remembering.)

101. Before proving this theorem, consider an alternate notation for $w(x_1)$, $w(x_1+\Delta x)$, $v(x_1)$, $v(x_1+\Delta x)$, which will simplify the notation in our proof. Let $w(x_1)=w_1$. If x_1 changes by an amount Δx, i.e. $x=x_1+\Delta x$, then, since w is a function of x, w_1 will change by an amount, which we will call Δw. Thus, $w(x_1+\Delta x)$ will be denoted by $w_1+\Delta w$. Reasoning in a similar manner, let $v(x_1)=v_1$ and let $v(x_1+\Delta x)$ be denoted by ________.

A224. $\frac{1}{2}(x_1-2x_1^2)^{\frac{-1}{2}}(1-4x_1)$

225. Following is a set of exercises you should be able to differentiate using the theorems and corollaries developed thus far. Review your list of these theorems and corollaries at this time, before proceeding to the exercises below.

(a) If $f(x)=(x-1)^{\frac{5}{2}}$, $f'(x_1)=$ ________.

(b) If $f(x)=(x^2-1)^{\frac{5}{2}}$, $f'(x_1)=$ ________.

(c) If $f(x)=\sqrt[3]{x^2}$, $f'(x_1)=$ ________.

(d) If $f(x)=(\sqrt{x-1})^5$, $f'(x_1)=$ ________.

(e) If $f(x)=(\sqrt[5]{x^2-1})^2$, $f'(x_1)=$ ________.

(f) If $f(x)=\sqrt[3]{x^2-1}$, $f'(x_1)=$ ________.

A101. $v_1 + \Delta v$

102. Returning to the proof of Theorem 6, we will follow the four steps for finding the derivative of f(x) evaluated at x_1. In step (1), if $f(x) = w(x) \cdot v(x)$, then $f(x_1) = w(x_1) \cdot v(x_1) =$ (in alternate notation).

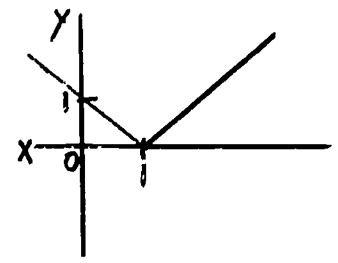
A225. (a) $\frac{5}{2}(x_1 - 1)^{\frac{3}{2}}$, (b) $5x_1(x_1^2 - 1)^{\frac{3}{2}}$, (c) $\frac{2}{3}x_1^{-\frac{1}{3}}$,
(d) $\frac{5}{2}(x_1 - 1)^{\frac{3}{2}}$ (See (a)), (e) $\frac{4}{5}x_1(x_1^2 - 1)^{-\frac{3}{5}}$, (f) $\frac{2}{3}x_1(x_1^2 - 1)^{-\frac{2}{3}}$

226. Let us return to the definition of the derivative of f evaluated at x_1, i.e. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$, when this limit exists. Thus far, we have considered examples of functions such that this limit did exist. However, you know from the section on limits that it might not always be the case that this limit exist. Consider the function f defined by $f(x) = |x - 1|$. Graph this function on a separate paper for future reference.

A102. $w_1 \cdot v_1$

103. If $f(x) = w(x) \cdot v(x)$, $f(x_1 + \Delta x) = w(x_1 + \Delta x) \cdot v(x_1 + \Delta x) =$ $(w_1 + \Delta w) \cdot$ (in alternate notation).

A226.



227. We will proceed to obtain the derivative of $f(x) = |x - 1|$ evaluated at $x_1 = 1$, by proceeding through the four steps in expressing the limit of the difference quotient. If $f(x) = |x - 1|$, $f(x_1) = |x_1 - 1|$ and $f(x_1 + \Delta x) =$ __________.

A340. $x = 0$ (or the y-axis)

341. Again we note that even though the slope of the tangent line doesn't exist at the point (0,0) on the graph of f, since f is defined at (0,0) and the tangent line exists, it (is, is not) possible to write the equation of the tangent line.

A103. $(v_1 + \Delta v)$

104. In step (2), $f(x_1 + \Delta x) - f(x_1) = (w_1 + \Delta w) \cdot (v_1 + \Delta v) - w_1 \cdot v_1 =$

(in simplified form).

A227. $|(x_1 + \Delta x) - 1|$

228. In step (2), $f(x_1 + \Delta x) - f(x_1) =$ ______________.

A341. is

342. If f is defined by $f(x) = \sqrt{x - 2}$, $x \geq 2$, graph this function.

A104. $w_1 \cdot \Delta v + v_1 \cdot \Delta w + \Delta w \cdot \Delta v$

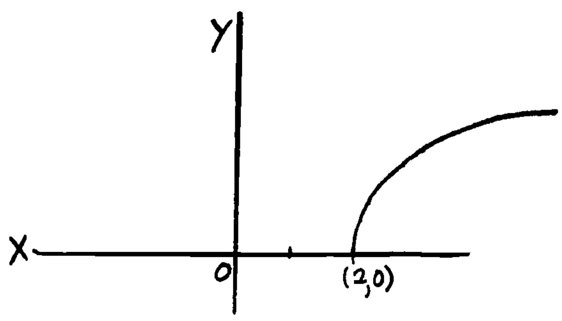
105. In step (3), $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \dfrac{w_1 \cdot \Delta v + v_1 \cdot \Delta w + \Delta w \cdot \Delta v}{\Delta x}$

(dividing each term in the numerator by Δx) $= w_1 \cdot \dfrac{\Delta v}{\Delta x} +$ __________ $+ \Delta w \cdot \dfrac{\Delta v}{\Delta x}$.

A228. $|(x_1 + \Delta x) - 1| - |x_1 - 1|$

229. In step (3), $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ____________________.

A342.



343. The domain of f is defined by the inequality $x \geq 2$, and the range is ______________________________.

Skip a page for the answer to frame 343.

A105. $v_1 \cdot \frac{\Delta w}{\Delta v}$

106. In step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \left(w_1 \cdot \frac{\Delta v}{\Delta x} + v_1 \cdot \frac{\Delta w}{\Delta x} + \Delta w \cdot \frac{\Delta v}{\Delta x}\right)$$

(1)
$$= \lim_{\Delta x \to 0} \left(w_1 \cdot \frac{\Delta v}{\Delta x}\right) + \lim_{\Delta x \to 0} \left(v_1 \cdot \frac{\Delta w}{\Delta x}\right) + \lim_{\Delta x \to 0} \left(\Delta w \cdot \frac{\Delta v}{\Delta x}\right)$$

(2)
$$= \left(\lim_{\Delta x \to 0} w_1\right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} v_1\right)\left(\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \Delta w\right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right)$$

(3)
$$= w_1 \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) + v_1\left(\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \Delta w\right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right)$$

Supply reasons for (1), (2), (3).

A229. $\frac{|(x_1 + \Delta x) - 1| - |x_1 - 1|}{\Delta x}$

230. In step (4), $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ______________________.

A106. (1) The limit of a sum is the sum of the limits, if these limits exist. (2) The limit of a product is the product of the limits, if these limits exist. (3) The limit of a constant is that constant. (w_1 and v_1 are independent of Δx, and hence are constants.)

107. Recalling the alternate notation for the derivative of $y = f(x)$ evaluated at x_1, $f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we have $\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = w'(x_1)$ and

$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} =$ ______.

A230. $\lim_{\Delta x \to 0} \frac{|(x_1 + \Delta x) - 1| - |x_1 - 1|}{\Delta x}$

231. Since we want to evaluate the limit of this difference quotient for $x_1 = 1$, the expression becomes

$$\lim_{\Delta x \to 0} \frac{|(x_1 + \Delta x) - 1| - |x_1 - 1|}{\Delta x} = \lim_{\Delta x \to 0} \frac{|(1 + \Delta x) - 1| - |1 - 1|}{\Delta x} = \underline{\qquad\qquad}.$$

A343. defined by the inequality $y \geqq 0$

344. From the graph of this function, you would expect the derivative(s) (the slope(s) of the tangent line(s)) not to exist at the point(s) with coordinates ______.

A107. $v'(x_1)$

108. The terms in the last line of frame 106 can now be expressed as

$w_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = w_1 \cdot v'(x_1)$, $v_1 \cdot \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = v_1 \cdot w'(x_1)$ and

$\lim_{\Delta x \to 0} \Delta w \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \Delta w \cdot$ ________.

A231. $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$

232. If $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ is to exist, $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} =$ ______________.

A344. (2,0)

345. Let us evaluate the derivative of the function f defined by $f(x) = \sqrt{x - 2}$ and check the above conjecture. Thus $f'(x_1) =$ __________.

A108. $v'(x_1)$

109. Thus, we have $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = w_1 \cdot v'(x_1) + v_1 \cdot w'(x_1) +$

________________.

A232. $\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x}$

233. If $\Delta x \to 0^+$, this means $\Delta x > 0$, so, recalling from the limit section the function f defined by $f(x) = |x|$, $|\Delta x| =$ ____.

A345. $\frac{1}{2}(x_1 - 2)^{-\frac{1}{2}}$ (You should have used the corollary to Theorem 8 for this differentiation.)

346. For the point at which the derivative of the above function should not exist, (2,0), f′(___) = __________.

A109. $\lim_{\Delta x \to 0} \Delta w \cdot v'(x_1)$

110. Since the limit of the difference quotient on the left hand side of the expression in frame 109 is $f'(x_1)$, we have $f'(x_1) = w_1 \cdot v'(x_1) + v_1 \cdot w'(x_1) + \lim_{\Delta x \to 0} \Delta w \cdot v'(x_1)$. Reviewing the statement of Theorem 6 on your list and comparing this statement to our expression for $f'(x_1)$ in the preceding sentence, we must show that the term $\lim_{\Delta x \to 0} \Delta w \cdot v'(x_1)$ has numerical value ________.

A233. Δx

234. If $\Delta x \to 0^-$, this means $\Delta x < 0$, so, reasoning as above, $|\Delta x| =$ ____.

A346. 2, undefined

347. We can write the equation of the tangent line for the function f defined by $f(x) = \sqrt{x - 2}$ at the point (2,0) because f is defined at this point and the tangent line exists. This tangent line has equation

________.

A110. zero

111. Since $v'(x_1)$ is a constant, i.e. the value of the derivative of v evaluated at a fixed or constant value x_1, we must show _______ = 0.

A234. $-\Delta x$

235. Thus, $\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} 1 =$ ____.

A347. $x = 2$

348. In the case that the slope of a tangent line does not exist, the tangent line may not exist either. It is not possible to write the equation of the tangent line to the graph of f if there is no second element, $f(x_1)$, belonging to the function f whose first element is ____.

A111. $\lim_{\Delta x \to 0} \Delta w$

112. Thus, we must show that as Δx approaches 0, Δw approaches 0. Consider the expression $w(x_1 + \Delta x)$ as $w_1 + \Delta w = w(x_1) + \Delta w$, in terms of our alternate notation. For $w(x_1 + \Delta x)$, we see that as Δx approaches 0, $\lim_{\Delta x \to 0} w(x_1 + \Delta x) =$ ___________.

A235. 1 Note that we have used the theorem stating $\lim_{x \to a} k = k$ for a constant k.

236. And, $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \to 0^-} -1 =$ ____.

A348. x_1

349. Consider the function f defined by $f(x) = \frac{1}{x}$. Graph this function on your own paper.

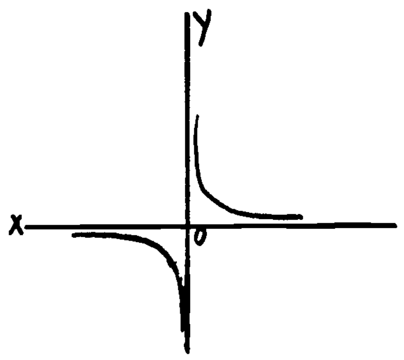
A112. $w(x_1)$ or w_1 (Note that here we must assume the continuity of the function w, a concept we will not discuss in this unit.)

113. If this is the case, then the right side of above equality, $w(x_1 + \Delta x) = w(x_1) + \Delta w$, must approach the same value $w(x_1)$ as Δx approaches 0. This means that if Δx approaches 0, Δw approaches ____.

A236. -1

237. In this case, we see that $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = 1$ and $\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} =$ ____.

A349.



350. The domain of the above function is the set of all real non-zero numbers and the range is the set of ________________.

A113. 0

114. Thus, if Δx approaches 0, then Δw approaches 0, so $\lim_{\Delta x \to 0} \Delta w$ can be rewritten $\lim_{\Delta x \to 0} \Delta w = \lim_{\Delta w \to 0} \Delta w$ and this limit has numerical value ____.

A237. -1

238. So $\lim_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ for $x_1 = 1$ doesn't exist because

____.

A350. all real non-zero numbers

351. From the graph of this function, you would expect the derivative(s) (slope(s) of the tangent line(s)) not to exist for value(s) of $x_1 =$ ____.

A114. 0

115. We have now proved the result required in frame 110, since $\lim_{\Delta x \to 0} \Delta w \cdot v'(x_1) = 0 \cdot v'(x_1) =$ ____.

A238. the left and right hand limits are not equal (or an equivalent expression)

239. Thus, if $f(x) = |x - 1|$, $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ doesn't exist at $x_1 = 1$. This is equivalent to the statement that $f'(x_1) = f'(1)$ ________.

A351. 0 - the tangent lines approach a vertical position for points on the graph of f with x-coordinates near 0.

352. Evaluating the derivative of f defined by $f(x) = \frac{1}{x}$, we have $f'(x_1) =$ ______.

A115. 0

116. Reviewing, we have proved the theorem that if $f(x) = w(x) \cdot v(x)$, and $v'(x_1)$ and $w'(x_1)$ exist, then $f'(x_1) = w_1 \cdot v'(x_1) + v_1 \cdot w'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1)$. Stated in words, the theorem says that the derivative of the product of two functions of x, evaluated at x_1, is the first function multiplied by ________________ plus the second function multiplied by ________________, each term in the sum evaluated at x_1, provided the derivatives exist.

A239. doesn't exist

240. There are other examples of functions that don't possess derivatives at some value(s) of x_1, i.e. $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ (does, does not) exist.

A352. $\frac{-1}{x_1^2}$

353. At the point with x-coordinate $x_1 = 0$, $f'(x_1) = \frac{-1}{x_1^2}$ is ________.

A116. the derivative of the second function, the derivative of the first function

117. Now consider $f(x) = w(x) \circ v(x) = (x^2) \cdot (x^3)$, where $w(x) = x^2$, $v(x) = x^3$, $w'(x_1) = 2x_1$, $v'(x_1) =$ _____.

A240. does not

241. For the moment, we will confine our attention to functions which possess derivatives at all points in the domain of definition of the function. We will return to other exceptional cases when writing equations of tangent lines in the next section. One such exception just cited is $f(x) =$ _____ evaluated at $x_1 =$ ____.

A353. undefined

354. The equation of the tangent line to the graph of the function f defined by $f(x) = \frac{1}{x}$ at a point with x-coordinate $x_1 = 0$ doesn't exist because there is no corresponding second element belonging to this function whose first element is ____.

Skip a page for the answer to frame 354.

A117. $3x_1^2$

118. If $f(x) = w(x) \cdot v(x) = (x^2) \cdot (x^3)$, then $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) = (x_1^2) \cdot (3x_1^2) + (x_1^3) \cdot (2x_1) = 3x_1^4 + 2x_1^4 =$ ______. You should check to see if the theorem is applied as we stated it above; i.e. first times the derivative of the second plus second times the derivative of the first.

A241. $|x - 1|$, 1

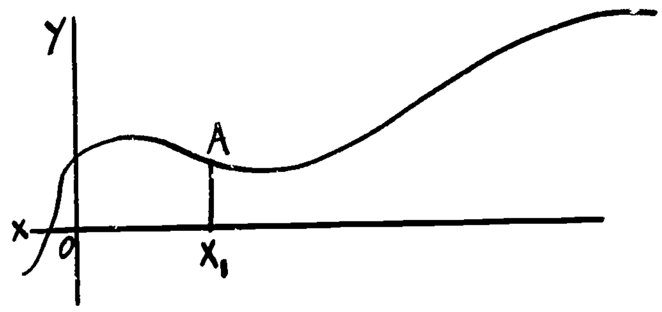
242. Perhaps you have felt that much of our work with the derivative thus far seems to be very much like that in the section on limits. This is not surprising since the derivative is a limit. To extend the analogy even further, we can relate the applications of the limit to applications of the derivative. One such application is that of the derivative considered as the slope of a tangent line to the graph of f evaluated at x_1; i.e. evaluated at a point on the graph whose x-coordinate is ____.

A118. $5x_1^4$

119. Did you realize that you could also find the derivative of $f(x) = x^2 \cdot x^3$, evaluated at x_1, in a different way? Since $f(x) = x^2 \cdot x^3 = x^5$, we can use the power differentiation formula to obtain $f'(x_1) =$ ______.

A242. x_1

243. Consider a general function f such as the one whose graph appears below. The coordinates of point A are (____, ____). Copy this graph on your own paper for future reference.



A354. 0

355. For the functions f defined by $f(x) = \sqrt{25 - x^2}$, $f(x) = x^{\frac{1}{4}}$, $f(x) = \sqrt{x - 2}$, we were able to write the equations of the tangent lines at points on the graphs of these functions where the slopes of these lines didn't exist. This was the case because the tangent lines ______________.

A119. $5x_1^4$ Note that the answers in frames 118 and 119 are the same.

120. Let us now proceed to find the derivative of $f(x) = w(x) \cdot v(x) = x(1-x)$ by the above theorem. Here $w(x) = x$, $v(x) = 1-x$, $w'(x_1) =$ ____, $v'(x_1) =$ ____.

A243. x_1, $f(x_1)$

244. Choose a point B on the graph, in the neighborhood of A, with x-coordinate $x_1 + \Delta x$, where $\Delta x > 0$. Indicate such a point B on your graph.

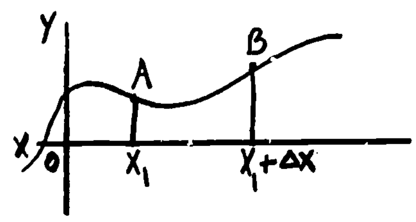
A355. existed (or an equivalent expression)

356. To summarize our discussion in the preceding section, we have presented functions for which the derivative does not exist because $\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \neq$ ____________.

A120. 1, -1

121. Thus, if $f(x) = w(x) \cdot v(x) = x(1-x)$, $f'(x_1) = w(x_1) \cdot v'(x_1) + v(x_1) \cdot w'(x_1) = x_1 \cdot (-1) + (1-x_1) \cdot 1 =$ ______.

A244.



245. Depending on how "near" we want point B to be to point A will depend on how near $(x_1 + \Delta x)$ is to x_1, or how near Δx is to _____.

A356. $\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

357. Also, the derivative of a function will not exist if the expression for $\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ becomes ________, even though $f(x_1)$ exists.

A121. $1 - 2x_1$

122. We could have found the derivative of the above function in another manner. Do you see how? If $f(x) = x(1-x) = x-x^2$, then $f'(x_1) =$ (by Theorem 5 for the derivative of the difference of 2 functions). Does this answer check with that obtained in the previous frame?

A245. zero

246. Regardless of the value of Δx, the point B has coordinates (_____, _____). Indicate these coordinates on your graph.

A357. infinite

358. Both a function and its derivative may fail to exist at x_1. In this case, the tangent line at x_1 (does, does not) exist.

Skip a page for the answer to frame 358.

A122. $1 - 2x_1$, Yes

123. We also computed the derivative of this function before introducing the theorems of differentiation. Check the result here obtained with that obtained for the function $f(x) = x - x^2$ in frame 49. Are all 3 answers the same?

A246. $x_1 + \Delta x$, $f(x_1 + \Delta x)$

247. The two points A and B on the graph will determine a secant line, so defined because this line intersects the curve in at least two points in a neighborhood of A. Such a definition should seem reasonable, because in high school plane geometry, a secant line to a circle is defined as a line which intersects a circle in ____ distinct point(s).

A123. Yes

124. If $f(x) = w(x) \cdot v(x) = (x^2 + 1)(x^3 + 1)$, $w(x) =$ _____, $v(x) =$ _____, $w'(x_1) =$ _____, $v'(x_1) =$ _____.

A247. two

248. We want to show that the derivative of f, evaluated at x_1, is the same as the slope of the tangent line to the graph of f at the point with x-coordinate x_1 when this derivative exists. Let us use a secant line as an aid. Referring to the above graph, the slope of a secant line AB, this slope denoted by m_s, can be expressed in terms of the coordinates of points A and B; i.e., $m_s = \frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1} =$ _____.

A358. does not

359. Following is a set of exercises that will test your understanding of the previous section.

(a) If $f(x) = |x + 2|$, does $f'(-2)$ exist? (A graph may be helpful in responding.)

(b) If $f(x) = \frac{1}{x - 3}$, find the equation of the tangent line at the point with x-coordinate $x_1 = 3$.

(c) If $f(x) = \sqrt{x + 5}$, find the equation of the tangent line at the point $(-5, 0)$.

(d) If $f(x) = \sqrt{x^2 - 9}$, find the equation of the tangent line at the point $(5, 4)$.

(e) If $f(x) = \sqrt{9 - x^2}$, find the equation of the tangent line at the point $(3, 0)$.

(f) If $f(x) = x^{\frac{1}{2}}$, $x \geq 0$, find the equation of the tangent line at the point $(0, 0)$.

A124. x^2+1, x^3+1, $2x_1$, $3x_1^2$

125. Thus, $f'(x_1) = w(x_1)\cdot v'(x_1) + v(x_1)\cdot w'(x_1) = (x_1^2+1)\cdot(3x_1^2) + (x_1^3+1)(2x_1) =$ (combining like terms).

A248. $\dfrac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$

249. A secant line AB assumes a limiting position, that of the tangent line to the graph of f at point A, as Δx approaches ______.

A359. (a) No, for $x_1 = -2$, $\lim\limits_{\Delta x \to 0^+} \dfrac{f(x_1+\Delta x) - f(x_1)}{\Delta x} = 1$ and $\lim\limits_{\Delta x \to 0^-} \dfrac{f(x_1+\Delta x) - f(x_1)}{\Delta x} = -1$.

(b) Neither $f(3)$ nor $f'(3)$ exists, so there is no tangent line at the indicated point.

(c) $x = -5$

(d) $5x - 4y = 9$

(e) $x = 3$ ($f'(3)$ is undefined but $f(3)$ exists)

(f) $x = 0$ (or y-axis)

APPENDIX C

CRITERION TEST ON THE DERIVATIVE

Criterion Test on the Derivative

Mark the correct answer and do your computation on the answer sheet.

Make no marks on the test.

1. If $f(x) = 5x^3 + 6x^2 - 7x + 4$, $f'(x_1) =$

(a) $15x_1^2 + 12x_1 + 4$

(b) $15x_1^2 + 12x_1 - 7x_1$

(c) $15x_1^2 + 12x_1 - 7$

(d) $15x_1^2 + 12x_1 + 4$

2. If $f(x) = (x^2 + 3x - 5)(x^3 + 6x^2 - 5x)$, $f'(x_1) =$

(a) $(2x_1 + 3)(3x_1^2 + 12x_1 - 5)$

(b) $(x_1^2 + 3x_1 - 5)(3x_1^2 + 12x_1 - 5) - (2x_1 + 3)(x_1^3 + 6x_1^2 - 5x_1)$

(c) $(x_1^2 + 3x_1 - 5)(3x_1^2 + 12x_1 - 5) + (x_1^3 + 6x_1^2 - 5x_1)(2x_1 + 3)$

(d) none of the above

3. If $f(x) = x^2(2x - 1)(x^2 + x - 3)$, $f'(x_1) =$

(a) $x_1(10x_1^3 + 4x_1^2 - 21x_1 + 6)$

(b) $4x_1(2x_1 + 1)$

(c) $x_1^2(6x_1^2 + 2x_1 - 7)$

(d) $2x_1^2(2x_1 + 1)$

4. If $f(x) = (x^2 - 4)/(x + 2)$, $x \neq -2$, $f'(x_1) =$

(a) $x_1 + 2$

(b) $2x_1$

(c) 1

(d) $(x_1^2 + 4x_1 + 4)/(x_1 + 2)$

5. If $f(x) = x + 1/(x + 2)(x + 3)$, $x \neq -2, -3$, $f'(x_1) =$

(a) $-(x_1^2 - 2x_1 - 1)/(x_1 + 2)^2(x_1 + 3)^2$

(b) $-x_1^2 - 2x_1 + 1/(x_1 + 2)^2(x_1 + 3)^2$

(c) $x_1^2 - 2x_1 + 1/[(x_1 + 2)(x_1 + 3)]^2$

(d) $1/(2x_1 + 5)$

6. If $f(x) = (2x^3 - 7x^2 + 9)^4$, $f'(x_1) =$

(a) $4(2x_1^3 - 7x_1^2 + 9)^3 \ (6x_1^2 - 14x_1)$

(b) $8x_1(2x_1^3 - 7x_1^2 + 9)^3 \ (3x_1 - 7)$

(c) $(6x_1^2 - 14x_1)^4$

(d) $4(2x_1^3 - 7x_1^2 + 9)^3$

7. If $f(x) = (x^2 - 4)^7(2x - 3)$, $f'(x_1) =$

(a) $14(x_1 - 4)^6$

(b) $2(x_1^2 - 4)^7 + 14x_1(x_1^2 - 4)^6(2x_1 - 3)$

(c) $2(x_1^2 - 4)^7 + 7(2x_1 - 3)(x_1^2 - 4)^6$

(d) none of the above

8. If $f(x) = \sqrt[7]{x^2 + 2x}$, $x^2 + 2x \geq 0$, $f'(x_1) =$

(a) $2/7 \ (x_1 + 1)(x_1^2 + 2x_1)^{-6/7}$

(b) $1/7(x_1^2 + 2x_1)^{-6/7}$

(c) $\sqrt[7]{2(x_1 + 1)}$

(d) $7(x_1^2 + 2x_1)^{5/2}(x_1 + 1)$

9. If $f(x) = (\sqrt[3]{x^3 - 3x + 1})^5$, $f'(x_1) =$

(a) $(\sqrt[3]{3x_1^2 - 3})^5$

(b) $5/3\ (\sqrt[3]{x_1^3 - 3x_1 + 1})^{2/3}$

(c) $5(\sqrt{x_1^3 - 3x_1 + 1})^3(x_1^2 - 1)$

(d) $5(\sqrt[3]{x_1^3 - 3x_1 + 1})^2(x_1^2 - 1)$

10. If $f(x) = \sqrt{x - 7}/\sqrt{x - 6}$, $x \geq 7$, $f'(x_1) =$

(a) $\sqrt{x_1 - 6}/\sqrt{x_1 - 7}$

(b) 1

(c) $1/2\ [(x_1 - 6)(x_1 - 7)]^{-1/2}(x_1 - 6)^{-1}$

(d) $1/2\ [(x_1 - 6)(x_1 - 7)]^{1/2}$

11. If $f(x) = \sqrt{8 - 2x - x^2}$, $8 - 2x - x^2 \geq 0$, $f'(-4) =$

(a) undefined
(b) 6
(c) $\sqrt{6}$
(d) 0

12. The slope of the tangent line to the curve $y = f(x) = 7x^3 - 6x + 1$ at the point $(1, 2)$ is

(a) 2
(b) 15
(c) 1
(d) 0

13. The slope of the tangent line to the curve $y = f(x) = (10 - 3x + x^2 - x^3)^{2/5}$ at the point with x-coordinate $x_1 = 2$ is

(a) 0
(b) $2/5(-11)^{-3/5}$
(c) $\sqrt[5]{121}$
(d) undefined

14. The slope of the tangent line to the curve $y = f(x) = (x - 3)^{1/2}(x - 3)^{1/3}$, $x \geq 3$, at the point (4,1) is

(a) 1
(b) 1/2
(c) 5/6
(d) 1/3

15. The equation of the tangent line to the curve $y = f(x) = \sqrt{16 - x^2}$, $-4 \leq x \leq 4$, at the point (0,4) is

(a) $y - 4 = 0$
(b) $y = 0$
(c) $y = x + 4$
(d) $x + y = 4$

16. The equation of the tangent line to the curve $y = f(x) = \sqrt{5 + x}$ at (-5,0), $x \geq -5$, is

(a) $x = 0$
(b) $y = -5$
(c) $y = 0$
(d) $x = -5$

17. The equation of the tangent line to the curve $x^{1/2} + y^{1/2} = 2^{1/2}$, $0 \leq x \leq 2$, $0 \leq y \leq 2$, at (2,0) is

(a) $y = 0$ (x-axis)
(b) $y = 2$
(c) $x = 0$ (y-axis)
(d) $x = 2$

18. At what point(s) would the tangent line to the curve $y = f(x) = x^2 - x - 12$ be horizontal?

(a) (-3,0), (4,0)
(b) (1/2, -49/4)
(c) (0, -12)
(d) (1, -11)

The velocity at an instant t, the instantaneous velocity, can be expressed as a derivative. If $s = f(t)$, where s is distance traveled by an object in t units of time, the instantaneous velocity at the instant in time t_1, is $f'(t_1) = \lim_{\Delta t \to 0} \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}$.

19. If $s = f(t) = \sqrt[4]{7t^3 - 9t^2 + 3}$, where s is the distance measured in miles and t the time measured in hours, find the instantaneous velocity in miles per hour at time t_1.

(a) $\sqrt[4]{21t_1 - 18t_1}$

(b) $1/4(7t_1^3 - 9t_1^2 + 3)^{-3/4}$

(c) $4(7t_1^3 - 9t_1^2 + 3)^3(21t_1^2 - 18t_1)$

(d) $1/4(7t_1^3 - 9t_1^2 + 3)^{-3/4}(21t_1^2 - 18t_1)$

20. If $s = f(t) = (t^2 + 3)^4$, where s and t are defined as above, find the instantaneous velocity in m.p.h. at time $t_1 = 1$.

(a) 256
(b) 8
(c) 512
(d) 16

21. At what instant in time would a body whose position is given by the distance equation $s = f(t) = -3 + 10t - 3t^2$, where s is the distance measured in feet and the time measured in seconds, be momentarily at rest?

(a) 1/3 sec.
(b) 0 sec.
(c) 3 sec.
(d) 5/3 sec.

The derivative of a function can be interpreted as a rate of change; i.e. the rate of change of the function with respect to the variable defining the function. Thus, if $y = f(x)$, $f'(x_1)$ is the rate of change of y(or f(x)) with respect to x.

22. The perimeter of a square, expressed as a function of x, the side length of the square is $f(x) = 4x$. We know that the perimeter of a square will change as its side length x changes. Using the above formula for the perimeter of a square ($f(x) = 4x$), find the rate of change of the perimeter with respect to the side length x of the square.

(a) 4
(b) 0
(c) $4x_1$
(d) $2x_1$

23. The circumference of a circle can be expressed as a function of x, the radius of the circle. The circumference of a circle will change as its radius x changes. Using the appropriate formula for the circumference of a circle expressed in terms of its radius x, find the rate of change of the circumference with respect to the radius x of the circle.

(a) 2
(b) $2\pi x_1$
(c) 2π
(d) π

24. The area of a square can be expressed as a function of x, the side length of the square. Using the appropriate formula for the area of a square expressed in terms of its side length x, find the rate of change of the area with respect to the side of the square.

(a) x_1^2
(b) $2x_1$
(c) $4x_1$
(d) 2

25. Prove: If $f(x) = k$, where k is a constant, $f'(x_1) = 0$.

26. Does $f'(-7)$ exist for the function $f(x) = |x + 7|$? Why?

You may use a graph to explain your answer.

27. If $f(x) = x^2 - 4x + 7/x^2 + 2x - 3$, $x \neq -3, 1$, find $f'(x_1)$ in two ways and show that your answers are the same in both cases.

28. Give an example of a function that is defined at all points but doesn't have a derivative at all points.

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VITA

Lois Marie Lackner

Lois Marie Lackner was born on February 4, 1936 in Chicago, Illinois. She graduated from William Howard Taft High School, Chicago, Illinois in 1953. The University of Illinois, Urbana, Illinois is her alma mater, where she graduated in 1957 in the upper three per cent of her class. At that time she received a B.S. degree in mathematics teacher training. She was awarded a graduate fellowship at the University, where she received a M.S. degree in the teaching of mathematics in 1958.

She remained in Urbana for the summer of 1958 to teach summer school at Urbana Senior High School. In September, 1958 she began her college teaching career as a mathematics instructor at Ohio University, Athens, Ohio, where she remained for four years teaching freshman and sophomore mathematics. She took a leave of absence for the school year 1960-1961 to teach at the American College for Girls, Istanbul, Turkey. In 1963 she returned to Chicago to join the faculty at Chicago Teachers College-South where she began teaching elementary mathematics methods courses in addition to freshman and sophomore mathematics.

In the summer of 1965 she began work on the doctoral degree in mathematics education at the University of Illinois. After a year's residency in Urbana she returned to Chicago to take charge of the content and methods courses in elementary mathematics at Loyola University.

She has taught in the elementary and secondary schools of Chicago as a substitute teacher. In 1967 she worked with a group of National

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TITLE

The Teaching of Two Concepts in Beginning Calculus by Combinations of Inductive and Deductive Approaches (Thesis)

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University of Illinois, Urbana, Illinois, College of Education

SOURCE CODE

REPORT/SERIES NO.

OTHER SOURCE | SOURCE CODE

OTHER REPORT NO.

OTHER SOURCE | SOURCE CODE

OTHER REPORT NO.

PUB'L. DATE 28-Jun-68 | CONTRACT/GRANT NUMBER 2048-99-7

PAGINATION, ETC.

356 pages

RETRIEVAL TERMS

Inductive, discovery, heuristic teaching
Deductive, expository, tell-and-show teaching
Limit concept in calculus
Derivative concept in calculus

IDENTIFIERS

ABSTRACT

There were three studies in this experiment, a limit study comparing inductive (discovery) teaching and deductive (expository) teaching of this concept in beginning calculus, a derivative study of the same nature, and a total treatment study comparing the four ordered combinations of inductive and deductive teaching of both the concepts. Programmed units were used to control the teacher variable. The units were read by advanced high school mathematics students, who were divided into high and low achievers on the basis of pre-test scores. A limit criterion test was given upon completion of the limit unit and a derivative criterion test upon completion of the derivative unit. The criterion test scores were used in treatments by levels analyses of covariance designs in the three studies. Contrary to pilot study and related research results, when a difference in teaching methods existed in the derivative and total treatment studies, the deductive approach was favored. Further correlation and regression analyses revealed that a student's prior mathematical knowledge, as measured by the pre-test and limit test, was the determining factor in predicting the limit and derivative test scores, even though the deductive derivative treatment was found to be superior.