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This study concerns the feasibility of a Markov chain model for projecting housing values and racial mixes. Such projections could be used in planning the layout of school districts to achieve desired levels of socioeconomic heterogeneity. Based upon the concepts and assumptions underlying a Markov chain model, it is concluded that such a model is not by itself an adequate tool for forecasting future changes in the distribution of housing values. However, such a technique is found to be a powerful descriptive device which provides insights essential to the construction of a deterministic model not having the Markovian weaknesses. Results obtained in test runs of the model suggest that it is worthwhile to pursue the technique in a larger context. (TT)

FINAL REPORT

Contract No. OEC-0-8-080006-3470(010)

REGIONAL ECONOMIC DEVELOPMENT INSTITUTE, INCORPORATED PITTSBURGH, PENNSYLVANIA

HOUSING VALUE PROJECTION MODEL RELATED TO EDUCATIONAL PLANNING: THE FEASIBILITY OF A NEW METHODOLOGY

July 1968

U.S. DEPARTMENT OF
HEALTH, EDUCATION AND WELFARE

Office of Education
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July 30, 1968

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U.S. DEPARTMENT OF
HEALTH, EDUCATION AND WELFARE

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REGIONAL ECONOMIC DEVELOPMENT INSTITUTE, INCORPORATED

HOUSING VALUE PROJECTION MODEL RELATED TO EDUCATIONAL PLANNING: THE FEASIBILITY OF A NEW METHODOLOGY

SUMMARY STATEMENT

The Markov chain model has been found to be extremely valuable in the context of projecting housing values for educational planning. The value of this technique lies not with its applicability as a projective device, but as an analytical framework within which complex temporal changes in housing value distributions, or other social and economic variables, may be described efficiently, and readied for translation into deterministic relationships.

A basic Markov chain model was implemented in both a Philadelphia study area, population about 90,000, and a Pittsburgh study area, population about 50,000, using average contract rent value data for 1940, 1950 and 1960 census blocks. The results of these tests were extremely interesting in that the structural movements of blocks through rental classes in both areas appeared to be quite similar, and an amazing degree of correspondence between the probabilities of certain transitions was apparent.

An expanded Markov chain model, which accommodates a racial dimension simultaneously with rental changes, was

calibrated with data from the Pittsburgh study area. This three-dimensional model represents the first known application of such a technique. Although the data requirements exceeded the study area in which it was tested, the results were strongly suggestive.

The study concludes by recommending a follow-up research project of larger scale designed to fully implement the three-dimensional Markov chain model, and "couple" it with a simple deterministic projection device.

SECTION I

INTRODUCTION

Part of the revolution presently taking place in American education is reflected in the form and size of the educational unit, most notably in the development of large attendance districts and super schools for major urban areas covered by the rubric "educational park". These parks are being considered in their various forms for future educational systems in Philadelphia, Seattle, and elsewhere; one variant, the Great High School, will be implemented in Pittsburgh, Pennsylvania.

The rationale for these large-scale educational agglomerations is based on two fundamental assumptions. First, it is presumed that large-scale educational facilities are necessary to improve the quality of education. While large units in themselves provide no guarantees of improvement in quality, greater scope of course offerings, more special services, and more flexibility in the use of space and teaching routines are thought to be possible and more economical only in very large schools. With increased size and capacities of new large schools it is possible to employ some of the new educational technologies, particularly those learning and counseling techniques linked to centralized computer facilities. In short, the large school systems beset with urban ills not of their making tend to take the view that unless novel approaches to education of the nation's youth are adopted, educational quality is not likely to improve.

The second assumption surrounding the educational park concept is that large school districts must be drawn to service heterogeneous public school populations in each school unit. Where heterogeneity is a goal, each school ought to tend toward the city's average in its share of ethnic composition and students with different socio-economic backgrounds. Given large networks of feeder schools, appropriate districting can achieve this goal throughout the system.

The maintenance of heterogeneity, however, depends upon the stability within the attendance districts. Probably the most important index of stability can be derived from the study of the housing market. Trends in neighborhood housing values reflecting its demographic and socio-economic character convey the changing structure of an area.

The prime objective of this study has been the evaluation and testing of a method, not hitherto applied to educational planning, which can be used to describe the changing pattern of housing values for areas within a city. Further, the study has been designed to determine the feasibility of using this new methodology as a tool for projecting future patterns of housing values. The methodology investigated, termed Markov chain analysis, has created a great deal of excitement among social science researchers in recent years, and has been employed to a wide variety of research problems.

The format of the present study called for the review of existing Markov chain theory and applications, followed by the design and implementation of a Markov chain model of housing values for sections of Pittsburgh and Philadelphia. Section II presents the highlights of Markov chain theory and indicates the relationship of the Markov chain model to the family of Markovian models and to the broader class of stochastic models. This section also discusses briefly some of the more relevant social science applications of Markov chain models. A basic housing value model is introduced in Section III, and the results of implementing the model in Pittsburgh and Philadelphia are discussed. Section IV presents an expanded version of the basic Markov chain model, which is capable of treating changes in racial composition simultaneously with changes in rent value. This expanded model was calibrated in the Pittsburgh study area, and the results of this calibration are described. The final section presents conclusions and recommendations based upon the findings of the study.

SECTION II

MARKOV-CHAINS AND OTHER STOCHASTIC PROCESS MODELS

Theoretical Background

Stochastic process models, of which Markov-type models represent one class, are descriptive. They do not require that casual factors or inter-relations among variables be specified, or input into the model, in any way. This characteristic underlies both the strengths and the weaknesses of a stochastic process model, and should be borne in mind when deciding whether or not this family of models is appropriate to a particular research problem.

Many questions to which the social scientist addresses himself involve processes which are the observable manifestation of an exceedingly complex interplay of forces and counter-forces. Often the researcher is unable to identify all of the relevant determinants acting to shape the process which he is trying to explain. Of those determinants which are successfully identified, knowledge is frequently limited, and the accurate quantification, or measurement, of relevant determinants usually presents a major source of headaches. At this point, the social science researcher is faced with the question of whether to employ some type of deterministic model, or turn to a stochastic or probabilistic model. If he elects the former, he must possess some reasonably sound ideas about the determinants at work in the process under investigation. Further, he must be able to measure these determinants, or at least be able to substitute and measure reliable proxy variables. Finally, he must be willing to write off any variation in the observed process, which he does not explain with the determinants and proxies he has selected, as unimportant, noise, or an insignificant randomized component.

On the other hand, if the researcher selects some form of a stochastic model, he explicitly allows for uncertainty of outcome, and thus compensates for both his limited knowledge of the relevant process determinants and any variability inherent in the process itself.

This argument should not be interpreted as a blanket endorsement of stochastic models over deterministic models for all kinds of social science problems. On the contrary, the fact that a stochastic model requires no understanding of causality, or variable inter-relationships, means that such a model will have nothing specific to say about such things once it has done its job. That is, a stochastic model may be used to project the results of a process through time, but the model will not provide any details of the factors causing changes in the process. For certain questions and certain types of processes, this may be enough, but for other questions, where an understanding of the causes behind the process is required, a stochastic model may be inadequate.

The Markov-chain model, the central focus of this study, is a member of the broader Markovian class of models, which in turn is but one class of the probabilistic or stochastic model. Somewhat formally, the Markov chain process is described as follows:

> A Markov chain process is determined by specifying the following information: There is a given set of states (S1, S2............S_n). The process can be in one and only one of these states (or classes) at a given time, and it moves successfully from one state to another. Each move is called a step. The probability that the process moves from S_i to S_j depends only on the state S_i that it occupied before the step. The transition probability P_{ij} which gives the probability that the process will move from S_i to S_j is given for every ordered pair of states. Also an initial starting state is specified at which the process is assumed to begin.[1]

By analogy, this process may be thought of in terms of a frog on a lily-pad in a pond.[2] As time passes, the frog jumps from one lily-pad to another depending upon

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1 J. G. Kemy, et.al., Finite Mathematical Structures (New York: Prentice Hall, 1959), p. 148.

2 Credit for this analogy belongs to Ronald A. Howard, Dynamic Programming and Markov Processes. (Cambridge, Mass., The MIT Press, 1960).

his comforts and appetities. The lily-pads are states, and the state of the system is defined by the pad currently occupied by the frog. Each time the frog leaps to a new pad the system undergoes a state transition, or step, and the probability that he will leap to any given pad in the pond is only a function of his present resting place.

If the number of lily pads is finite, and the frog rests for some period of time between leaps, the Markov chain described by this analogy is termed a discrete-time finite chain. This particular kind of Markov chain has received the majority of attention by researchers interested in applying a Markov technique, and has been the central focus of the current study.

A discrete-time finite Markov chain is characterized by two basic and critical assumptions:

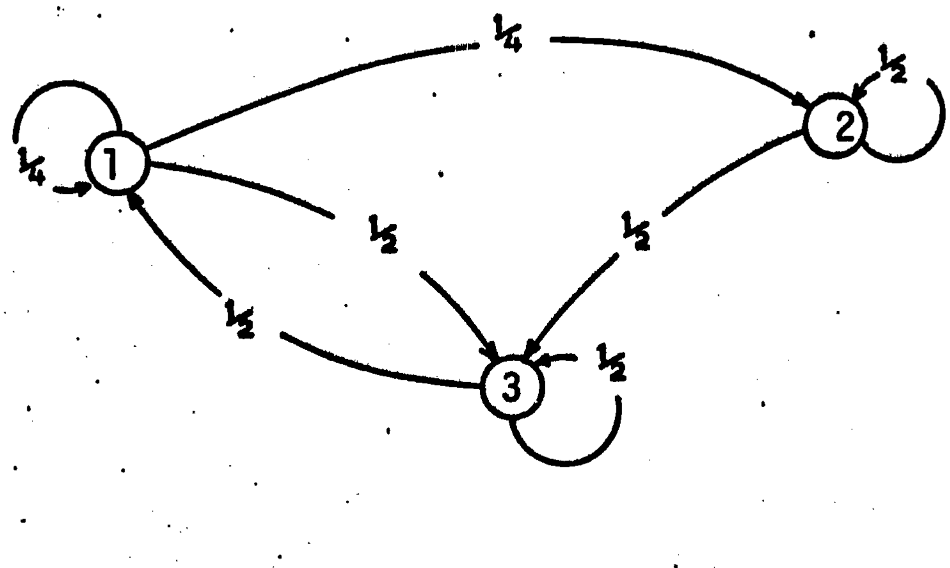
1) knowledge of the past sequence of events beyond the last state of the process is not relevant, and

2) the transition probability, once determined, of moving from S_i to S_j will remain constant for each succeeding generation of the model.

The full impact of these assumptions may become clearer when they are translated into terms of the frog and lily pad analogy. The first assumption states that no matter what is known about the path followed by the frog to reach the pad on which he is currently resting, it is not important, and the only thing which will influence the destination of his next leap is his current pad.

Interpretation of the second assumption calls for the assignment of numbers to the frog analogy. For simplicity, let us consider the case where there are a total of three lily pads in the pond, and the frog makes a decision to leap, or not to leap, every three minutes. Based upon our observation of this frog, of his likes and dislikes in the way of pads, of his jumping range, of the spatial arrangement of the pads, and so forth, it is possible to assign a set of probabilities for his leaps, or non-leaps, from every pad to every other pad as illustrated below in Figure 1.

FIGURE I



This diagram indicates, for example, that when the frog is on lily pad 1, there is a 25% chance he will either remain there, or jump to pad 2, and a 50% chance he will jump to pad 3 when the next time for a decision arrives. Assuming he jumps to pad 3, which is his most probable move, he will be faced with a new set of possible decisions in the next decision period, i.e., t + 6 minutes. Under the first Markov chain assumption, the frog's decision, once on pad 2, will be in no way influenced by the fact that he jumped there from pad 1, and under the second assumption, the initial probabilities shown in Figure 1 do not change for his second, or any subsequent leaps.

It is customary to represent the frog's decision probabilities in the form of a matrix:

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left[\begin{array}{ccc} 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array}\right] \end{array} \begin{array}{c} \Sigma \\ = 1 \\ = 1 \\ = 1 \end{array}$$

The matrix is termed a transition probability matrix (P), and it will be noted that each row sum is equal to unity. This is the case in any matrix of transition probabilities, since the elements of the ith row represent the probabilities for all possible moves when the process is in state a_1. A zero entry indicates that the transition is impossible.

Since P represents the matrix of probabilities of moving from a_i to a_j in exactly one step, the P^2 (the square of matrix P)[3] can be interpreted as the matrix of probabilities of moving from a_i to a_j in exactly two steps. In this case,

$$P^2 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} .31 & .19 & .50 \\ .25 & .25 & .50 \\ .38 & .12 & .50 \end{bmatrix} \end{array}$$

Similarly, P^n is the matrix of probabilities of moving from a_i to a_j in exactly n-steps.

Eventually, after a large number of steps, the probability that the process is in state a_j approaches some constant value regardless of what the initial probabilities are.[4] In the frog problem, it is only necessary to take the transition matrix to the 4th power to see the matrix coverage to an equilibrium vector;

---

[3] Given a matrix P with elements P_{ij}, the square of P is given by the following equation:

$$P^2_{ij} = P_{i1} = P_{i2} + \ldots . P_{in} P_{nj}.$$

[4] This is true only if the transition probability matrix is regular, that is, if some power of the matrix has only positive non-zero components.

$$P^4 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} .33 & .17 & .50 \\ .33 & .17 & .50 \\ .33 & .17 & .50 \end{bmatrix} \end{array}$$

Thus, under the conditions outlined in Figure 1 above, it would be found that after some long period of time the frog would have spent half of his time on lily pad 3, one-third of his time on lily pad 1 and one sixth of his time on lily pad 2, no matter on which pad he initially started.

Social Science Applications of Markov Chains

Of the broad class of Markov process models, only Markov chains have so far found much application in the study of social processes. There are probably many reasons for this exclusive concentration on the Markov chain model, not the least of which are its relative mathematical simplicity and an undefined property which might be described as "seductive".

This study has been primarily concerned with reviewing applications which show a concern for a spatial dimension.

Brown,[5] for example, defined transition states spatially in his liquid propane tank diffusion study. Rogers,[6] demonstrated a similar concern for spatial process in his analysis of inter-regional migration flows in California. The range of applications is exceedingly wide, however, and a glance through the bibliographical portion of this study will reveal that the Markov chain model has appeared seductive to social scientists with broadly varied interests.

The term seductive is carefully chosen, for although the Markov chain would appear superficially to be a highly flexible and exceedingly versatile tool applicable to any number of social processes, it must, in fact, be used with extreme discretion. It will be recalled from the preceeding section, that a Markov chain is characterized by two basic assumptions, one of which is a condition of stationarity, that is, the transition matrix is invariant over time. A social process which is to be modeled by a Markov chain must, therefore, also obey the condition of stationarity. In a migration context, for example, the probability of a person moving from Pittsburgh to Philadelphia must always remain the same,

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5 Lawrence A. Brown, "The Diffusion of Innovation: A Markov Chain-Type Approach," Department of Geography, Northwestern University, Discussion Paper, No. 8, 1963.

6 Andrei Rogers, "A Markovian Policy Model of Interregional Migration," Papers and Proceedings of the Regional Science Association, Vol. 17, 1966, pp. 205-224.

once it is initially established, no matter how the relative attractiveness between the two places may vary. Obviously, not all the social processes which have been modeled by Markov chains pay strict allegiance to the assumption of stationarity.

A study by Clark[7] tested the applicability of Markov chain models to the analysis of rental housing value movements. In this study average rental values for urban census tracts in several cities are recorded for 1940, 1950 and 1960. A transition probability matrix is then built for each city based upon the 1940 through 1960 data. At first glance, this method of constructing a transition matrix might seem to violate the basic Markov assumption that knowledge of past movements beyond that movement just passed is not relevant. In other words, it would seem more proper to use just the 1940 to 1950 or 1950 to 1960 movements in calculating probabilities. However, by using two separate moves and treating them as if they were a single move, Clark has accomplished two useful purposes. First, he has in effect doubled his number of observations, and thus minimized the problem of having enough data points to build a reliable Markov chain model. Second, and even more importantly, Clark

[7] W.A.V. Clark, "Markov Chain Analysis in Geography: An Application to the Movement of Rental Housing Areas", Annals of the Assn. of American Geographers, Vol. 55, 1965, pp. 351-359.

has averaged out the characteristic differences which set the 40s apart from the 50s. The implications of this averaging action in terms of the condition of stationarity are rather interesting. If Clark had simply used the 1950 to 1960 movements of census tracts through average rent classes (states), and had then gone on to square the resulting transition matrix to determine a 1970 distribution, he would have been implying that those conditions which influenced the behavior of rent during the 50s would act in exactly the same way during the 60s. By basing a 1970 projected distribution on both the 1940 to 1950 and 1950 to 1960 moves, it would be possible to filter out some of the unique characteristics of each decade by projecting on a broader base. In all fairness to Clark, it should be pointed out that he did neither, and in fact did not use his model as a projective device. As will be expanded later in the paper, it does not appear desirable that the type of Markov chain model investigated by this study should be used as a means of projecting rental values.

SECTION III

DESIGN AND IMPLEMENTATION OF A MARKOV CHAIN MODEL

Model Design

Accomplishment of the stated basic objective of this study required the implementation of a Markov chain model to test its usefulness in projecting housing values for educational planning purposes. After some initial experimentation and inquiries into the availability of reliable housing value data in Pittsburgh, it was decided to employ census block data from the 1940, 1950 and 1960 Censuses of Housing. This is not to imply that it is impossible to obtain housing value data at a greater level of detail from sources other than the census, but due to the needs and operation of the model itself, it is unnecessary to expend the sizeable additional effort required to obtain data at a grain finer than the census block.

One of the first minor problems encountered in compiling housing value data from the Census of Housing, was the complete lack of that particular statistic in the 1940 census. It was therefore decided to use average contract rent per census block as the basic unit of measure. Rent may be considered a reasonable proxy for housing value, and, as it turned out, the vast majority of residents in the areas examined were renters rather than home owners.

Study areas were selected in both Philadelphia (Map I) and Pittsburgh (Map II). The Philadelphia area consists of a sector of the central city extending from the Schuykill River out to the Cobbs Creek Parkway, which effectively coincides with the Philadelphia city limits. This sector is bounded by Market Street on the north and Baltimore and South avenues on the south. There are 546 census blocks in the Philadelphia study area, and in 1960 a total of 92,192 people resided in the area.

The Pittsburgh study are consists of 640 census blocks which housed some 50,866 persons in 1960. This area contains most of the Manchester, North Side and Spring Hill sections of the city, and is bounded on the south, east and west by the Allegheny and Ohio Rivers and the north by an extension of Island Avenue.

Average contract rent value data was recorded for each block in both of these study areas for 1940, 1950 and 1960. Rent data for 1950 and 1960 was then deflated by the Consumer Price index for rents to offset a general rise in prices. Census maps were compared in each year to insure areal

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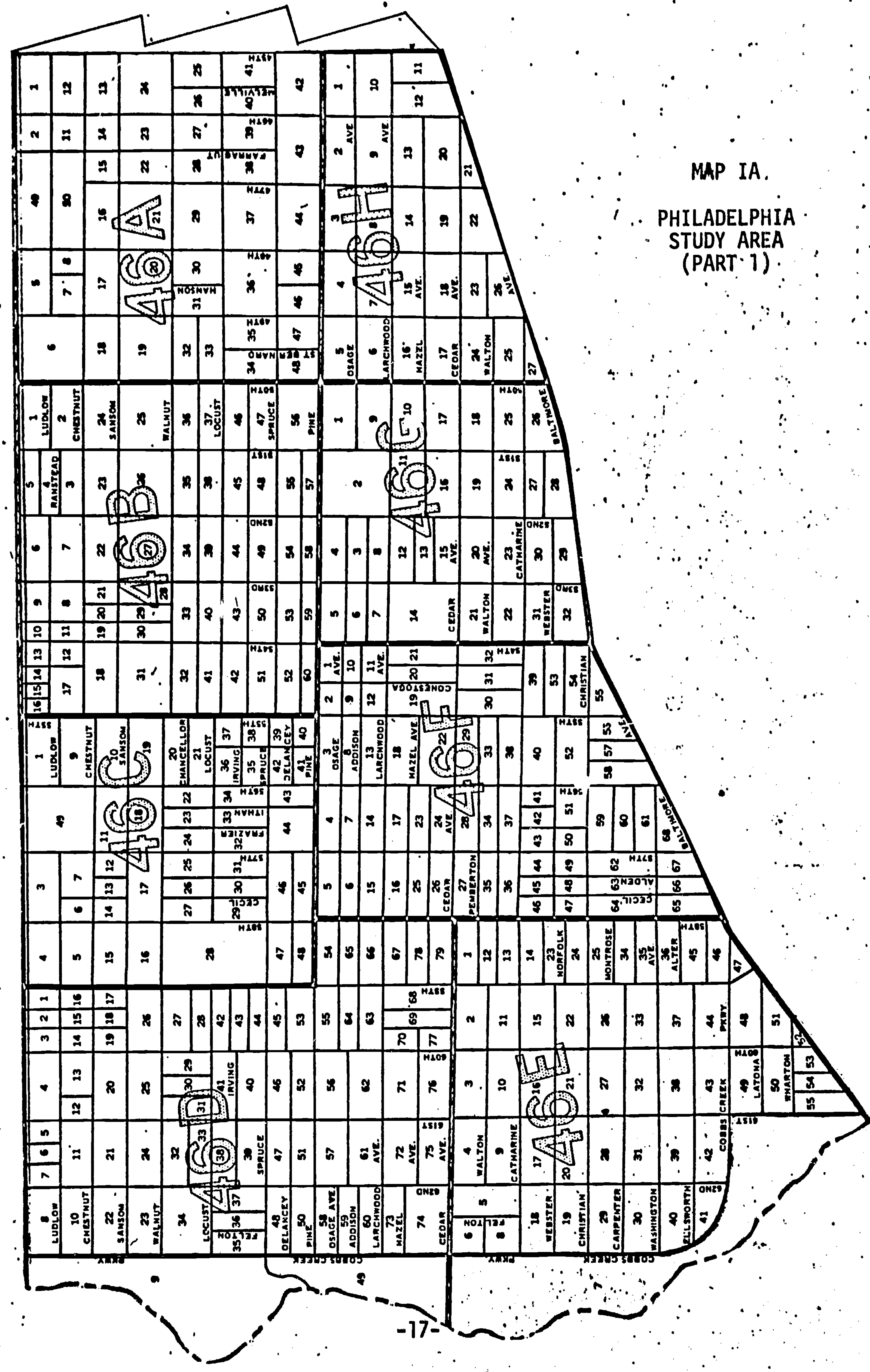
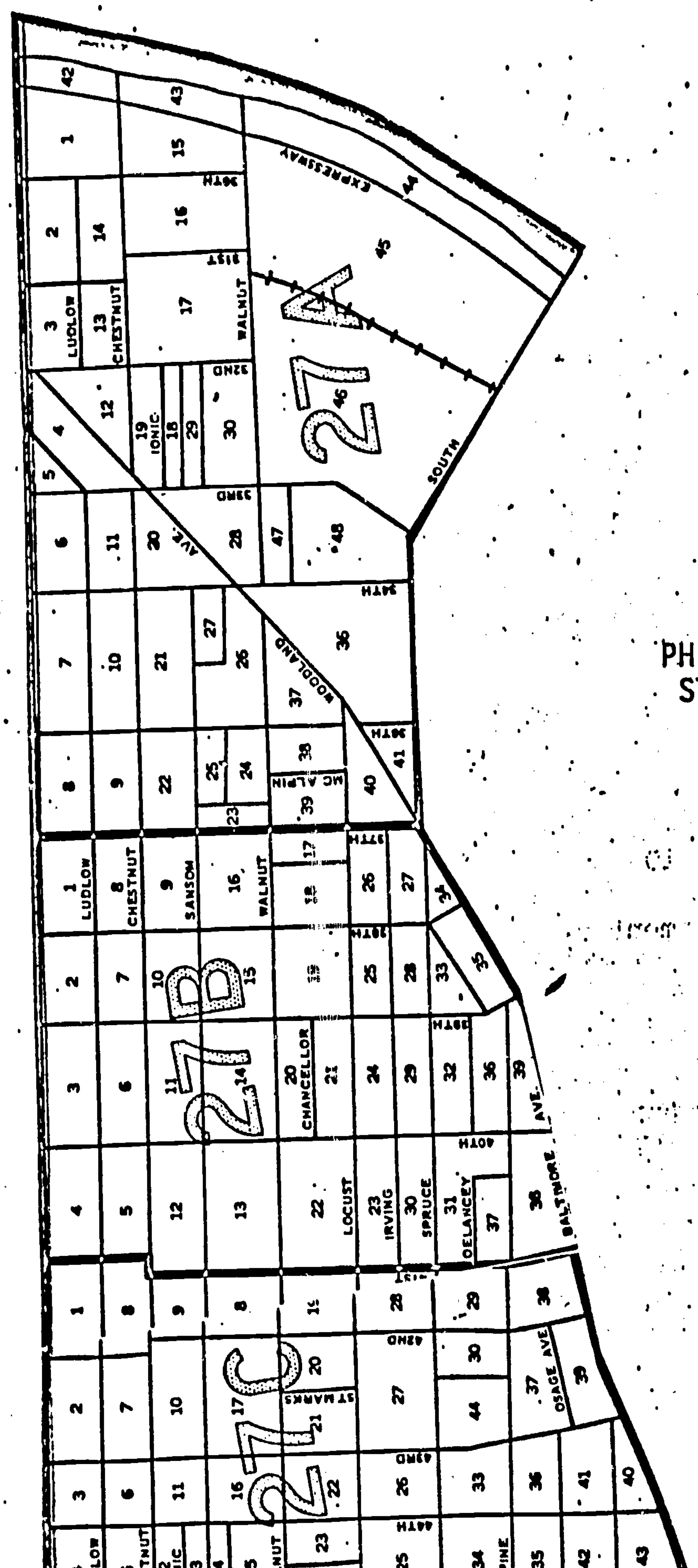
MAP IA.

PHILADELPHIA STUDY AREA (PART 1)





MAP IB

PHILADELPHIA
STUDY AREA
(PART 2)

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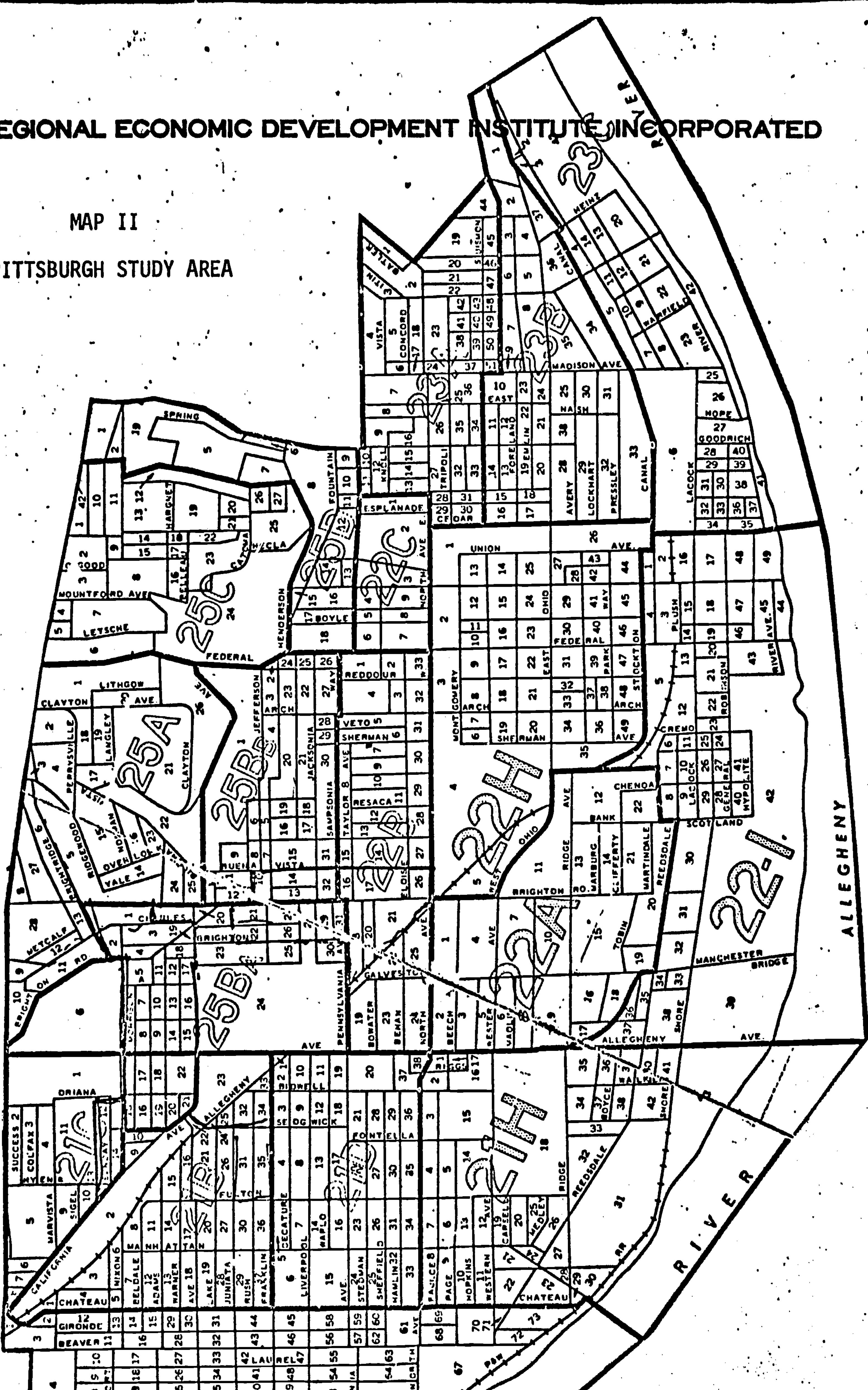
MAP II

PITTSBURGH STUDY AREA



continuity in the definition of each census block. In cases where the boundaries of a block were re-defined, and continuity could not be insured, the observation was omitted. Similarly, in cases where a block was listed with no average contract rent value in a census year, no transition was recorded for that decade. That is, contract rents were not allowed to proceed to or from zero.

Once the average contract rent values for each block in the Philadelphia and Pittsburgh study areas were tabulated, a transition matrix was constructed for each city. In each case the states of the matrix were defined as rent classes with dollar-value limits. The Philadelphia matrix consisted of 12 classes, or states, which were defined by the following values: 0 to \$25, state a_1; \$75 to ∞, state a_{12}; and \$5 intervals from \$25 to \$75, states a_2-a_{11}. The definition of states in the Pittsburgh matrix differed slightly due to the overall lower value of rents in the Pittsburgh study area. Here, only 10 states were used. State a_1 was bounded by 0 and \$15, state a_{10} by \$55 and ∞ , and states a_2 through a_7, were defined by \$5 intervals from \$15 to \$55.

Observation matrices were developed by aggregating the movements of each block through the rent states. The starting state of a block was determined by its 1940 value for a 1940 to 1950 transition and its 1950 value for a 1950 to 1960 transition. For example, census block 21B-3 with an average rent of $16 in 1940 starts in state a_2 ($15-20). This block then enters the matrix at row a_2 (Figure 2). In 1950, block 21B-3 has an average rent of $24, which means that the block has moved from state a_2 to a_3. An entry is therefore made in the cell defined by row a_2, column a_3. The 1960 rent of block 21B-3 is noted at $44, and in order to record the 1950 to 1960 transition, the matrix is entered at row a_3 (a new starting state for the 1950 value) and an entry is made in column a_7. The movements of each block in the study areas were recorded in this manner until the observation matrices reflected the decade movement of all blocks during the 1940-1960 period.

The next step involved conversion of the observation matrices into transition probability matrices. If the number of entries in each cell is labeled b_{ij}, then the probability of a move from i to j (P_{ij}) can be determined by summing all b_{ij}s in a row, and dividing b_{ij} by the row sum. Symbolically,

$$P_{ij} = \frac{b_{ij}}{\sum_{j=1}^{n} b_{ij}}, \text{ where } n = \text{no. of states.}$$

DATA

Census Block	Average Contract Rent 1940	1950	1960
21B - 3	$16	$24	$44
4	18	25	30
5	20	22	48
6	21	32	27
.	.	.	.
.	.	.	.
.	.	.	.

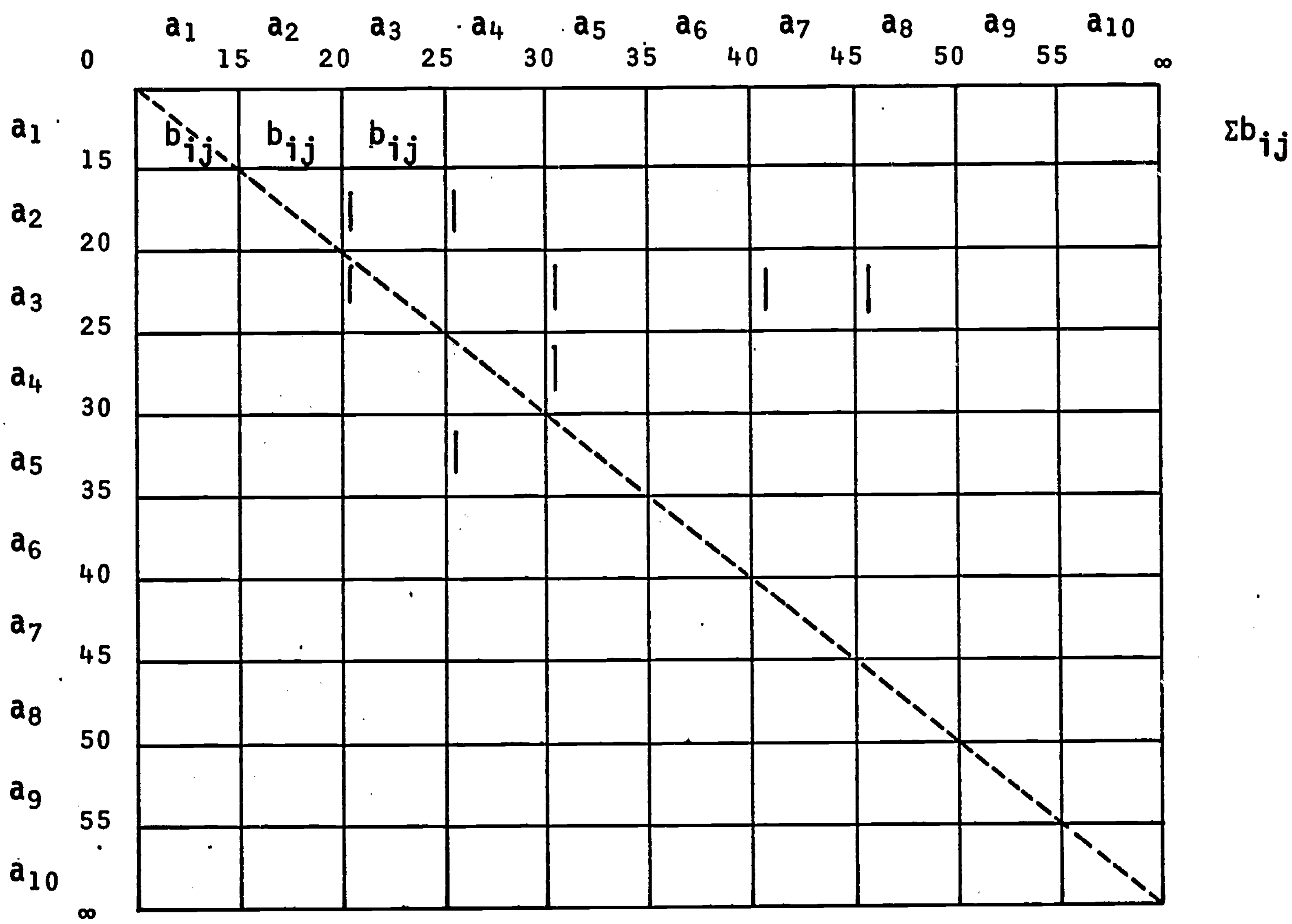


Figure 2

Construction of the Observation Matrix from Census Data

The transition matrix for the Philadelphia and Pittsburgh study areas are presented in Figures 3 and 4.

Results

The transition matrices for the Philadelphia and Pittsburgh study areas are summarized in Table 1. Only 7 states are compared in this table due to small number of observations recorded in some of the rent classes originally designated. Bearing in mind that all dollar values have been deflated to 1940 constant dollars by the Consumer Price Index for rent, the results of this table may be summarized in the following significant points:

1. A census block which starts a decade with average rents two lowest value states excluding state a_1, because of the small number of observations, will hold the same value or increase in value by no more than $10 at the end of the decade with a probability varying between 0.82 and 0.88.

a. The probability of a block in state a_2 or a_3 remaining in its starting state at the end of the decade is 0.28 for the Philadelphia study area and 0.35 for the Pittsburgh area. (i.e., blocks with average rents in the class intervals indicated experienced an average rent change of less than $5 during a decade with the probabilities stated).

0	25	30	35	40	45	50	55	60	65	70	75	∞
25	.25	.25	.31	.12	.06							
30	.05	.28	.40	.14	.08	.05	.01					
35	.01	.07	.28	.39	.14	.06	.02	.01	.01			
40		.01	.18	.44	.24	.08	.03	.01				.01
45			.05	.38	.34	.18	.03	.01				.01
50			.06	.21	.38	.24	.06	.02	.01	.01		.01
55				.14	.28	.28	.14	.10	.02	.02	.02	
60			.08		.08	.42	.33					.08
65					.21	.14	.36	.07	.14	.07		
70						.67		.33				
75					.33		.33	.33				
∞							.20					.80

Figure 3

Philadelphia Rent Value Transition Probability
Matrix: 1940-1950-1960

0	15	20	25	30	35	40	45	50	55	∞
15	.20	.40	.30	.10						
20	.06	.34	.40	.14	.04			.01		.01
25		.10	.35	.39	.13	.02	.01	.01		
30		.03	.17	.42	.28	.07	.02	.01		.01
35		.02	.09	.31	.38	.13	.01	.04		.01
40		.06	.06	.19	.28	.25	.14			
45			.08	.08	.23	.46	.15			
50										
55			1.0							
∞										

Figure 4

Pittsburgh Rental Value Transition Probability
Matrix: 1940-1950-1960

TABLE 1

COMPARISON OF PHILADELPHIA AND PITTSBURGH TRANSITION PROBABILITIES

State (Value Class)	Rent Value Class	No. of Observations	Study Area	Probability of Moving (n) States +						
				Losing				Gaining		
				3+	2	1	0	1	2	3+
a_1	0-$25	16	Philadelphia	-	-	-	.25**	.25**	.31**	.19**
	0-$15	30	Pittsburgh	-	-	-	.20*	.40*	.30*	.10*
a_2	$25-30	86	Philadelphia	-	-	.05	.28	.40	.14	.14
	$15-20	158	Pittsburgh	-	-	.06	.34	.40	.14	.06
a_3	$30-35	221	Philadelphia	-	.01	.07	.28	.39	.14	.10
	$20-25	261	Pittsburgh	-	.00	.10	.35	.39	.13	.02
a_4	$35-40	213	Philadelphia	.00	.01	.18	.44	.24	.08	.05
	$25-30	225	Pittsburgh	.00	.03	.17	.42	.28	.07	.03
a_5	$40-45	143	Philadelphia	.00	.05	.38	.34	.18	.03	.02
	$30-35	99	Pittsburgh	.02	.09	.31	.38	.13	.01	.01
a_6	$45-50	107	Philadelphia	.06	.21	.38	.24	.06	.02	.03
	$35-40	36	Pittsburgh	.11*	.19*	.28*	.25*	.14*	.00*	.03*
a_7	$50-55	50	Philadelphia	.14	.28	.28	.14	.10	.02	.04
	$40-45	13	Pittsburgh	.15**	.23**	.46**	.15**	.00**	.00**	.00**

* - Based on less than 50 observations, or 95 percent confidence limits as high as ± 15 percentage points.

** - Based on less than 20 observations, or 95 percent confidence limits as high as ± 22 percentage points.

\+ - i.e., changing rent $5 or more

b. The probability of a block in state a_2 or a_3 moving in value to the next higher state is approximately 0.40 in both study areas. (i.e., experiencing an increase in average rents of from $5 to $10).

c. The probability of a block in state a_2 or a_3 increasing in value enough to move up two states is approximately 0.14 in both study areas. (i.e., experiencing an increase in average rents of from $10 to $15).

2. A census block with a starting state in state a_4 is most likely to remain in that state at the end of a decade, and if it moves, it is likely to move only one state either higher or lower. The probability that such a block will do one of these things varies from 0.87 to 0.88.

a. The probability of a block in state a_4 remaining in state a_4 after a 10-year interval is approximately 0.43 in both study areas.

b. The probability that a block starting in state a_4 will move to the next higher state (0.24-0.28) is only slightly above the probability of moving to the next lower state (o.17-0.18).

3. Census blocks which start in states higher than a_4

become increasingly more likely to move to lower states in a decade, the higher the starting state.

a. As the starting state increases from a_5 through a_7, there is a decrease of approximately 10 percentage points per state in the probability of remaining in the same state at the end of the decade.

b. As the starting increases from a_5 through a_7, there is an increa e greater than 10 percentage points per state in the probability that a block will decline in value by one or more states.

The overall picture suggested by Table 1 shows the lower rent blocks increasing in value, the higher rent blocks decreasing in value and the medium rent blocks retaining their same approximate value. The transition probabilities computed from the Philadelphia and Pittsburgh data demonstrate an amazing level of agreement. This seems to suggest that, by and large, the internal fluctuations of rental values behave in much the same way in the two study areas.

A by product of the Philadelphis and Pittsburgh tests of the model was the development of some guidelines in determing the minimim number of observations required to effectively implement a Markov chain model. Since transition probabilities are a function of the number of observed moves from a given starting state to another state and the total number of moves from the given starting state, the greater one number of observations for a starting state, the more reliable the estimation of transition probabilities from that starting state. It was found in working with the data in this study, that what appeared to be reasonably reliable transition probabilities could be expected if at least 50 observations were available for a starting state. Translated into requirements for a Markov chain model of 10 states, a minimum of 500 observations are required. Even at that, given a normal distribution, it is most likely that the number of observations in the lowest and highest starting states will be less than the desired minimum of 50.

SECTION IV

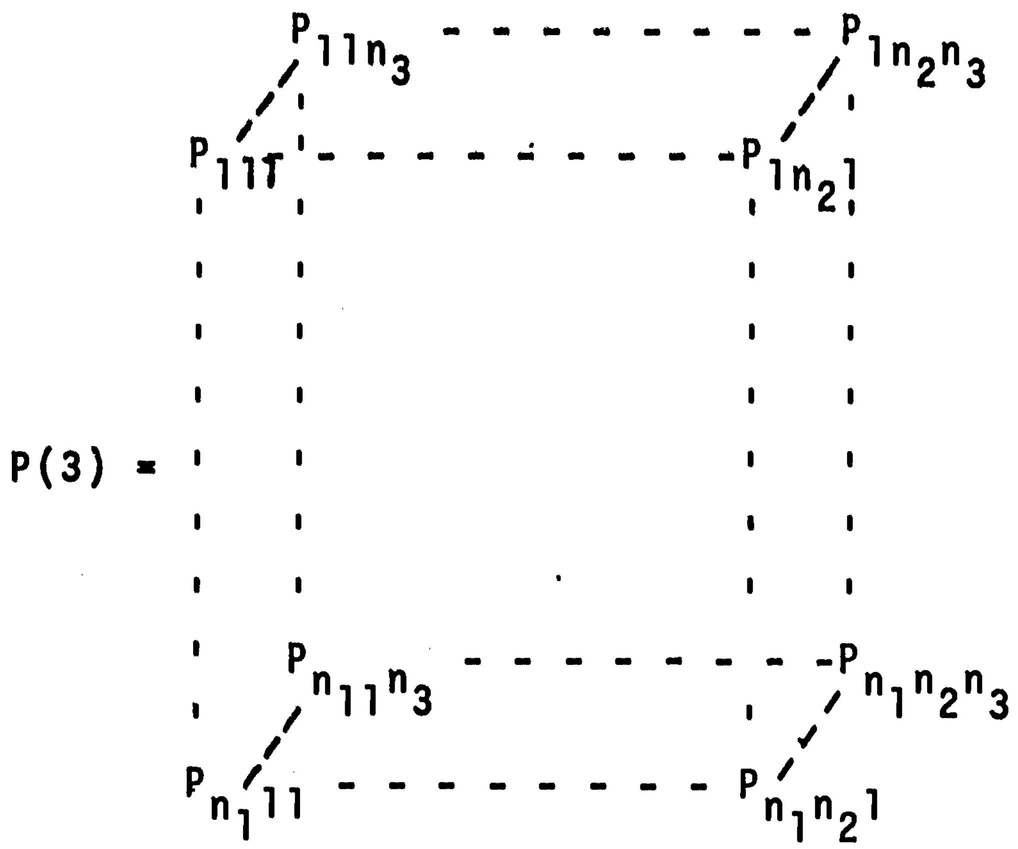
EXPANSION OF THE BASIC MODEL DESIGN AND IMPLEMENTATION

The basic Markov-chain model, as described in the previous section of this report, suffers from the fact that it can deal with only two dimensions, or variables, of which one is typically fixed as time. In this case, rental values were selected as the other variable, and the movement or trace of a census block through rental classes over time became the dimensions of the Markov chain.

Conceptually, it is possible to expand the Markov chain-type model to handle n-dimensions. For example, the transition probability matrix of a two dimensional model takes the familiar form:

$$P(2) = \begin{matrix} P_{11} & \text{-----} & P_{1n_2} \\ \vdots & & \vdots \\ P_{n_1 1} & \text{-----} & P_{n_1 n_2} \end{matrix}$$

In order to encompass a third variable, it is only necessary to conceptualize a transition probability matrix of the form:



The concept of a Markov chain model expanded into more than two-dimensions greatly increases the utility of this class of model. In the case of housing values, a third dimension would permit examination of block movements through rental class in space and time, or, as investigated in this report, block movements through rental and racial classes in time. Conceptually, it is possible to include any number of variables in an expanded Markov-chain model, but as will be seen in the following test, there are significant operational problems in going beyond three dimensions.

While tabulating the average contract rental value per block in the Pittsburgh study area, additional information on the total number of occupied dwelling units per block and the number of units occupied by non-whites per block were collected. These statistics permitted the calculation of the percentage of dwelling units occupied by non-white inhabitants for each block in the study area in 1940, 1950 and 1960. Four racial classes (states) were then postulated as follows: state b_1, 0 to 5% non-white; state b_2, 5% to 25% non-white; state b_3, 25% to 50% non-white, and white b_4, 50% to 100% non-white.

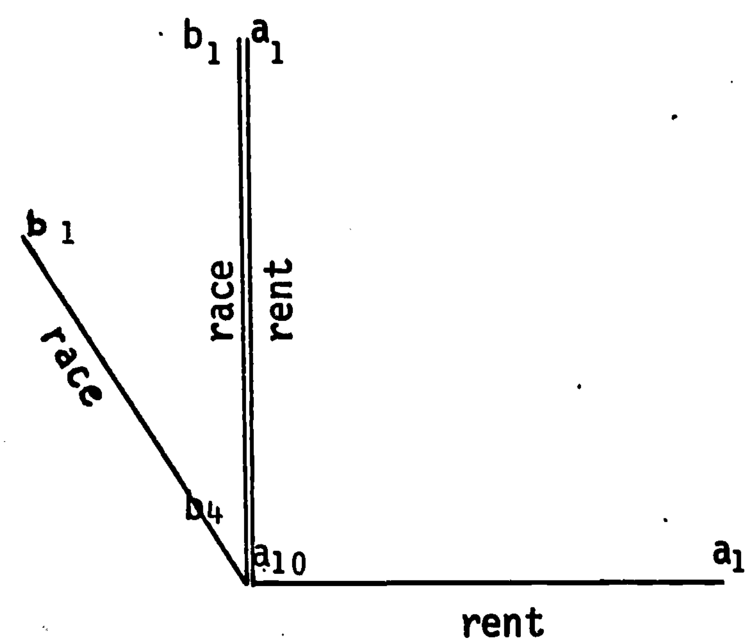


Figure 5

The transition probability matrix for both rental and racial class movements takes the form shown schematically in Figure 5. Operationally, the three dimensional matrix was collapsed into a two dimensional matrix as depicted in Figure 6. In effect, each cell of the 4 by 4 racial matrix became a 10 by 10 racial matrix. Each block then had two starting states, a racial starting state and a rental starting state. The

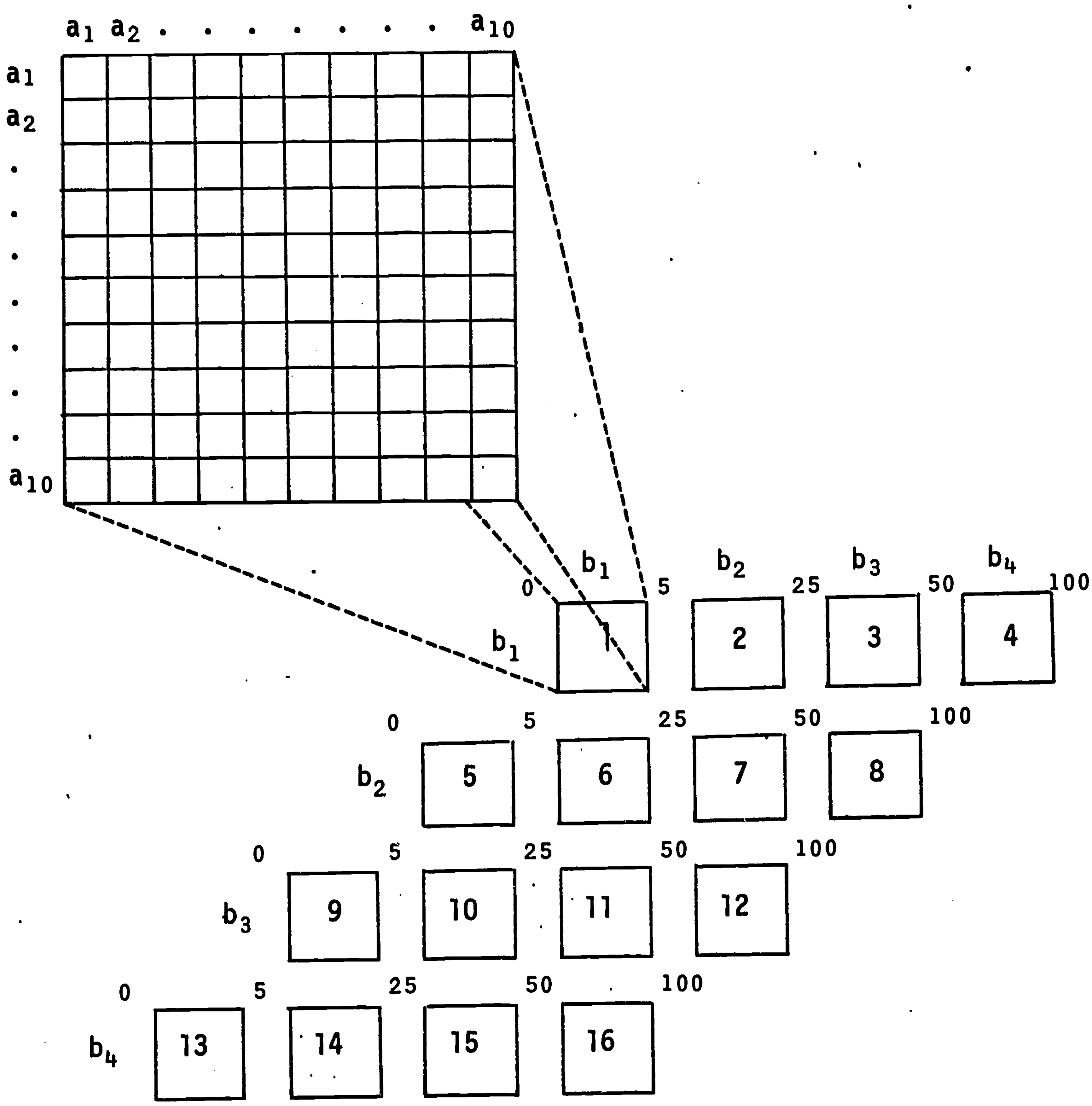


Figure 6

Operational Scheme of Three-Dimensional Markov Chain Matrix movement of a block from its racial starting state through one time period focused on the proper rental matrix, and from that point an entry was made in the rental matrix in the same manner employed in a two dimensional model.

One obvious problem involved in the expansion of the basic Markov-chain model into three or more dimensions is the greatly accelerated data requirements. As an additional dimension is added to the basic model, data requirements increase as an exponential function of the number of states in the new dimension. For example, by adding a racial dimension of four states, the number of observations required to make this model operational was increased 16 times. In terms of the rule of thumb relationship described in Section III, 8000 observations would be needed. If the new dimension had contained 10 states, data requirements would have increased 100 times, or amounted to 50,000 in the cited case.

Results

The 16 cells of the race matrix are presented for reference in Appendix 1 to this report. The study area did not contain sufficient observations to render the three-dimensional model fully operational, but the aggregated summaries in Tables 2 & 3 provide some interesting insights into the potential of such a model.

TABLE 2

MOVEMENT OF BLOCKS IN RENTAL CLASSES: ALL WHITE BLOCKS
VS. INTEGRATED BLOCKS, PITTSBURGH 1940-1960

State & Rent Class	No. of Observations	Racial Composition of Block	Probabilities of Block Movement: Losing 3+	Losing 2	Losing 1	0	Gaining 1	Gaining 2	Gaining 3+
a_1	20	White	-	-	-	.25*	.40*	.30*	.05*
0-15	10	Integrated	-	-	-	.10**	.40**	.30**	.20**
a_2	95	White	-	-	.08	.39	.42	.10	.01
15-20	63	Integrated	-	-	.03	.25	.38	.21	.13
a_3	113	White	-	.00	.15	.38	.35	.10	.02
20-25	148	Integrated	-	.00	.05	.33	.43	.16	.03
a_4	128	White	.00	.02	.15	.43	.29	.09	.02
25-30	97	Integrated	.00	.03	.21	.40	.28	.04	.04
a_5	66	White	.02	.11	.26	.41	.15	.02	.04
30-35	33	Integrated	.03*	.06*	.42*	.33*	.09*	.00*	.06*

* - Less than 50 observations, or 95 percent confidence limits as high as ± 15 percentage points
** - Less than 20 observations, or 95 percent confidence limits as high as ± 22 percentage points

TABLE 3

MOVEMENT OF BLOCKS IN RENTAL CLASSES BY TRENDS IN RACIAL COMPOSITION, PITTSBURGH, 1940-1960

State & Rent Class	No. of Observations	Racial Composition of Block	Probabilities of Block Movement: Losing 3+	Losing 2	Losing 1	0	Gaining 1	Gaining 2	Gaining 3+
a_1	8	IN	-	-	-	.12**	.38**	.25**	.25*
0-$15	20	CC	-	-	-	.25*	.40*	.30*	.05*
	2	DN	-	-	-	.00**	.50**	.50**	.00*
a_2	29	IN	-	-	.03*	.14*	.24*	.34*	.24*
$15-20	119	CC	-	-	.08	.39	.44	.09	.01
	9	DN	-	-	.00**	.33**	.56**	.11**	.00*
a_3	85	IN	-	.00	.05	.24	.46	.21	.05
$20-25	152	CC	-	.00	.12	.41	.36	.10	.01
	22	DN	-	.00*	.09*	.45*	.36*	.09*	.00*
a_4	61	IN	.00	.03	.16	.38	.30	.07	.06
$25-30	154	CC	.00	.02	.18	.44	.28	.07	.02
	10	DN	.00**	.10**	.20**	.40**	.30**	.00**	.00**
a_5	18	IN	.00**	.06**	.50**	.22**	.11**	.00**	.11*
$30-35	74	CC	.03	.10	.27	.40	.15	.01	.04
	7	DN	.00**	.14**	.29**	.57**	.00**	.00**	.00*

IN - Increasing Non-White Mix
CC - Constant Composition
DN - Decreasing Non-White Mix

* - Based on less than 50 observations
** - Based on less than 20 observations ==(See Table 2 for confidence limits)

Table 2 is based upon two transition matrices derived from the three dimensional model. The all-white matrix, Figure 7, shows movements of blocks which had less than 5% non-white occupied dwellings in all census periods surveyed. The integrated matrix, Figure 8, shows movements of blocks which had more than 5% non-white occupied dwellings in at least one of the census years surveyed.

The results of this comparison of white versus integrated block movements may be summarized by the following points:

1) There is a greater probability of rental class stability, blocks neither gaining nor losing states, for all white blocks than for integrated in all starting states.

2) Integrated blocks of the lower rental classes (states a_1-a_3) tend to have higher probabilities of gaining one or more rental classes in a decade than all white blocks of the same classes. They also tend to have lower probabilities of losing rental classes than comparable all white blocks.

3) Integrated blocks of the higher rental classes (states a_4 and a_5) tend to have higher probabilities of losing one or more rental classes in a decade and lower probabilities of gaining classes than all-white blocks of the same classes.

0	15	20	25	30	35	40	45	50	55	∞
15	.25	.40	.30	.05						
20	.08	.39	.42	.10	.01					
25		.15	.38	.35	.10	.02				
30		.02	.15	.43	.29	.09	.02			.01
35		.02	.11	.26	.41	.15	.02	.03		.02
40			.07	.21	.28	.31	.10			.04
45				.09	.18	.55	.18			
50										
55			1.0							
∞										

Figure 7

Pittsburgh Rent Value Transition Probability Matrix: All-white Blocks Only 1940-1950-1960

0	15	20	25	30	35	40	45	50	55	∞
15	.10	.40	.30	.20						
20	.03	.25	.38	.21	.08			.02		.03
25		.05	.33	.43	.16	.01	.01	.01		
30		.03	.21	.40	.28	.04	.03	.01		
35		.03	.06	.42	.33	.09		.06		
40		.28		.14	.28		.28			
45			.50		.50					
50										
55										
∞										

Figure 8

Pittsburgh Rent Value Transition Probability Matrix: Integrated Blocks Only 1940-1950-1960

In summary, integrated census blocks in the Pittsburgh study area undergo greater change of rental values than all white blocks in the same area. The lower value integrated blocks tend to appreciate and the higher value blocks tend to depreciate more rapidly than all white blocks.

The data arrayed in Table 3 are summarized from the three transition matrices presented in Figures 9, 10, and 11. These matrices summarize cells of the three dimensional matrix so that Figure 9 is an aggregated matrix of all cells to the left of the main racial diagonal and describes the movements of those blocks which have decreased in their percentage of non-white occupied housing. Referring back to Figure 6, this matrix is an aggregeation of cells 3, 9, 10, 13, 14 and 15. Figure 10is an aggregation of the main racial diagonal and includes those blocks which did not change racial classes from one decade to the next, or cells 1, 6, 11 and 16 from Figure 6. The matrix in Figure 11is an aggregation of cells 2, 3, 4, 7, 8 and 12 which contain blocks becoming increasingly more non-white with time.

The highlights of Table 3 are summarized in the following points:

0	15	20	25	30	35	40	45	50	55	∞
15		.50	.50							
20		.33	.57	.11						
25		.09	.45	.36	.09					
30		.10	.20	.40	.30					
35			.14	.29	.57					
40										
45			1.0							
50										
55										
∞										

Figure 9

Pittsburgh Rent Value Transition Probability
Matrix: Decreasing Non-white Composition 1940-1950-1960

0	15	20	25	30	35	40	45	50	55	∞
15	.25	.40	.30	.05						
20	.08	.39	.44	.09	.01					
25		.12	.41	.36	.10	.01				
30		.02	.18	.44	.28	.07	.01			.01
35		.03	.10	.27	.40	.15	.01	.03		.01
40		.06	.06	.21	.24	.27	.12			.03
45				.09	.18	.54	.18			
50										
55			1.0							
∞										

Figure 10

Pittsburgh Rent Value Transition Probability
Matrix: Constant Racial Composition 1940-1950-1960

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15	.12	.38	.25	.25						
15–20	.03	.14	.24	.34	.17			.03		.03
20–25		.05	.24	.46	.21	.02	.01	.01		
25–30		.03	.16	.38	.30	.07	.05	.02		
30–35			.06	.50	.22	.11		.11		
35–40					.67		.33			
40–45					1.0					
45–50										
50–55										
55–∞										

Figure 11

Pittsburgh Rent Value Transition Probability
Matrix: Increasing Non-white Composition 1940-1950-1960

1) Census blocks which remain relatively constant in terms of their racial mix tend to have higher probabilities of remaining in the same rental class regardless of their starting state rent value than do blocks experiencing changing racial composition.

2) In the lower rent value classes (states a_1-a_3), blocks which experience an increase in the percentage of non-white residents tend to have higher probabilities of gaining one or more classes than blocks in either of the other two racial mix categories.

3) In the higher rent value classes (states a_4 and a_5) differing racial mix trends appear to have little or no influence on the movement of blocks through rental classes.

In short, racial mix stability tends to correspond with rent value stability, and low average value blocks tend to gain in value more rapidly in cases where the racial composition is changing from white to non-white, than where racial mix is static or changing from non-white to white.

SECTION V

CONCLUSIONS AND RECOMMENDED DIRECTIONS FOR FUTURE RESEARCH

The preceeding sections of this study have presented an introduction to the Markov-chain type model, an application of the basic model to housing value data, and an extension of the basic model to encompass a third relevant planning variable. The basic question addressed by the study has been the feasibility of a Markov chain model as a tool for projecting housing values for educational planning. Based entirely upon the concepts and assumptions underlying a Markov chain process model, it must be concluded that such a model does not, by itself, present an adequate vehicle for forecasting future changes in the distribution of housing values. It cannot be assumed with any significant expectation of reliability that the forces which have shaped changes in housing values during some past period, or even an average of two or more periods, will continue to exert influences on values in the same manner through some projected future period.

It should not, however, be assumed that the Markov chain model has no relevance in projecting housing values for educational planning purposes. On the contrary, the implementations made in this study with both the basis and expanded models graphically demonstrated the significance of a Markov chain technique in organizing and describing complex changes in rental values over time. The Markov chain model, therefore, represents a powerful descriptive device which can lead a researcher to the insights necessary to construct a deterministic model for projecting housing values. Map analysis of the movements of individual census blocks through transition states can lead to a better understanding of the spatial aspect of rent value change. Alternatively, the inclusion of a spatial dimension in a three-dimensional model provides an investigator with a comparable spatial viewpoint, which can greatly facilitate the identification of relevant determinants of change, and lead to the efficient postulation of a deterministic model. In short, the value of a Markov chain model in a housing value projection context is not as the projective device, but as an organizing framework which should greatly facilitate the construction of a projection model.

The implementations of Markov-chain models made in this study drew data from relatively small areas. The results of these test runs are highly suggestive, however, and it appears most worth while to pursue the technique in a larger context. For this reason, it is hereby proposed that a large research project be undertaken to apply the Markov chain model to the entire city of Pittsburgh to be used in conjunction with the planning of the Great High Schools. Use of the three dimensional model would permit a simultaneous analysis of both rental values and racial mix, which could provide the analytic framework for developing a deterministic projection model. In this way, planners involved in the Great High Schools project could be provided with projections of both the racial and rent value projections for various sections of the city. Such projections would greatly facilitate the layout school districts to take account of desired levels of socio-economic hetrogeneity.

Technically, the purposed project would be designed to "couple" the Markov chain model with a deterministic projection model and thereby create a flexiable planning tool with a wide range of applicability. Data requirements can be derived directly from published census documents, or, where available, special inter-censal surveys such as the one recently completed for certain sections of Pittsburgh by the Mayor's committee on Human Resources. A city-wide application of a three demensional model of the type introduced in this study would prosper from the additional advantage of having an adequate number of observations to "power" it properly.

It is believed that such a study would produce a valuable planning tool, which could find wide applicability in the layout and design of urban districts throughout the nation.

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APPENDIX

Presented herein are the individual cells of the three-dimensional racial/rental transition probability matrix described in Section IV. It will be recalled that each cell of the racial matrix is in fact a matrix of rental transition probabilities. The summary racial matrix is shown below, and the rental matrices displayed on the following pages are keyed to this summary matrix. The number of observations for each cell of the summary matrix equals the total number of observations for each of the rental matrices.

	b_1 (0–5)	b_2 (5–25)	b_3 (25–50)	b_4 (50–100)
b_1 (0–5)	463 .78	83 .14	32 .05	13 .02
b_2 (5–25)	36 .24	64 .43	29 .20	20 .13
b_3 (25–50)	4 .07	7 .12	19 .33	28 .48
b_4 (50–100)	0 .00	0 .00	4 .16	21 .84

SUMMARY RACIAL MATRIX: OBSERVATIONS AND TRANSITION PRPBABILITIES

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15	.25	.40	.30	.05						
15–20	.08	.40	.42	.10	.01					
20–25		.15	.38	.35	.10	.02				
25–30		.02	.15	.43	.29	.09	.02			.01
30–35		.01	.10	.26	.41	.15	.02	.03		.02
35–40			.07	.21	.28	.31	.10			.04
40–45				.09	.18	.55	.18			
45–50										
50–55			1.0							
55–∞										

MATRIX 1 (CELL b_{11})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15	[illegible]	.33	.33	.33						
15–20	.08	.25	.25	.17	.25					
20–25		.12	.35	.42	.08	.04				
25–30			.13	.39	.32	.10	.06			
30–35				.50	.20	.20		.10		
35–40					1.0					
40–45										
45–50										
50–55										
55–∞										

MATRIX 2 (CELL b_{12})

0	15	20	25	30	35	40	45	50	55	∞
15	.50		.50							
20			1.0							
25			.30	.50	.10	.10				
30			.30	.40	.30					
35				.60	.40					
40					.50		.50			
45					1.0					
50										
55										
∞										

MATRIX 3 (CELL b_{13})

0	15	20	25	30	35	40	45	50	55	∞
15		1.0								
20				.50						.50
25			.29	.29	.43					
30					1.0					
35								1.0		
40										
45										
50										
55										
∞										

MATRIX 4 (CELL b_{14})

0	15	20	25	30	35	40	45	50	55	∞
15			1.0							
20		.43	.43	.14						
25			.58	.25	.17					
30		.12	.12	.50	.25					
35			.14	.29	.58					
40										
45			1.0							
50										
55										
∞										

MATRIX 5 (CELL b_{21})

0	15	20	25	30	35	40	45	50	55	∞
15										
20	.08	.17	.75							
25		.04	.56	.32	.08					
30			.25	.44	.31					
35		.12		.38	.38	.12				
40		.33		.33			.33			
45										
50										
55										
∞										

MATRIX 6 (CELL b_{22})

0	15	20	25	30	35	40	45	50	55	∞
15										
20		.17	.17	.33	.33					
25			.16	.63	.11		.05	.05		
30		.33			.33	.33				
35				1.0						
40										
45										
50										
55										
∞										

MATRIX 7 (CELL b_{23})

0	15	20	25	30	35	40	45	50	55	∞
15	1.0									
20			1.0							
25		.10	.60	.30						
30			.80			.20				
35		1.0								
40										
45										
50										
55										
∞										

MATRIX 8 (CELL b_{24})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15										
15–20			1.0							
20–25			.50	.50						
25–30					1.0					
30–35										
35–40										
40–45										
45–50										
50–55										
55–∞										

MATRIX 9 (CELL b_{31})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15										
15–20										
20–25		.33	.17	.50						
25–30			1.0							
30–35										
35–40										
40–45										
45–50										
50–55										
55–∞										

MATRIX 10 (CELL b_{32})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15										
15–20		.60	.40							
20–25			.58	.42						
25–30			.33	.50	.17					
30–35										
35–40		1.0								
40–45										
45–50										
50–55										
55–∞										

MATRIX 11 (CELL b_{33})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15			1.0							
15–20			.25	.50			.25			
20–25		.08	.15	.23	.54					
25–30		.10	.30	.30	.20		.10			
30–35										
35–40										
40–45										
45–50										
50–55										
55–∞										

MATRIX 12 (CELL b_{34})

0	15	20	25	30	35	40	45	50	55	∞
15										
20										
25										
30										
35				NO	OBSERVATIONS					
40										
45										
50										
55										
∞										

MATRIX 13 (CELL b_{41})

0	15	20	25	30	35	40	45	50	55	∞
15										
20										
25										
30										
35				NO	OBSERVATIONS					
40										
45										
50										
55										
∞										

MATRIX 14 (CELL b_{42})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15		1.0								
15–20			1.0							
20–25			.50	.50						
25–30										
30–35										
35–40										
40–45										
45–50										
50–55										
55–∞										

MATRIX 15 (CELL b_{43})

	0–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–∞
0–15										
15–20		.50	.25	.25						
20–25		.11	.11	.55	.22					
25–30			.50	.50						
30–35										
35–40										
40–45										
45–50										
50–55										
55–∞										

MATRIX 16 (CELL b_{44})