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Six tests were constructed, four on a pictorial level and two on a symbolic level, to measure the performance of fourth-, fifth-, and sixth-grade children, in three different ability groups, on problems concerning ratios or fractions. Two variables were of interest in the four tests on a pictorial level: (a) "equal" ratio situations vs. "equal" fraction situations, and (b) missing numerator vs. missing denominator. The two levels of the two variables of interest defined the four tests: (1) "equal" ratios with a missing numerator, (2) "equal" ratios with a missing denominator, (3) "equal" fractions with a missing numerator, and (4) "equal" fractions with a missing denominator. All four tests involved reductions. Variable (b) defined the two tests on a symbolic level. Very low correlations between the pictorial and symbolic tests indicated that a different understanding was required for each; the authors suggest a number of steps that may be taken to develop appropriate abilities in children. (GR)

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THE DEVELOPMENT OF THE CONCEPTS OF RATIO AND FRACTION IN THE FOURTH, FIFTH, AND SIXTH YEARS OF THE ELEMENTARY SCHOOL

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Technical Report No. 49

THE DEVELOPMENT OF THE CONCEPTS OF RATIO AND FRACTION IN THE FOURTH, FIFTH, AND SIXTH YEARS OF THE ELEMENTARY SCHOOL

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Report from the Mathematics Concept Learning Project

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Wisconsin Research and Development
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Madison, Wisconsin

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PREFACE

Contributing to an understanding of cognitive learning by children and youth—and improving related educational practices—is the goal of the Wisconsin R & D Center. Activities of the Center stem from three research and development programs, one of which, Processes and Programs of Instruction, is directed toward the development of instructional programs based on research on teaching and learning and on the evaluation of concepts in subject fields. Since the inception of the Center in 1964, Professor Van Engen and his staff have been concurrently developing and testing "Patterns in Arithmetic," a televised instructional program for Grades 1–6, and conducting related research in children's learning of mathematical concepts.

To ascertain abilities of fourth-, fifth-, and sixth-grade children to solve ratio and fraction problems, the authors of this Technical Report constructed and administere pictorial and symbolic tests of both types of problems. Low correlations between the two tests indicate that a different understanding is required for each; the authors suggest a number of steps that may be taken to develop appropriate abilities in children.

Herbert J. Klausmeier
Director

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ABSTRACT

Six tests were constructed, four on a pictorial level and two on a symbolic level. The tests were designed to measure the performance of fourth-, fifth-, and sixth-grade children, in three different ability groups, on problems which may be classified as ratios or fractions. Two variables were of interest in the four tests on a pictorial level: (I) "equal" ratio situations vs. "equal" fraction situations, and (II) missing numerator vs. missing denominator. The two levels of the two variables of interest defined the four tests: 1) "equal" ratios with a missing numerator, 2) "equal" ratios with a missing denominator, 3) "equal" fractions with a missing numerator and 4) "equal" fractions with a missing denominator. All four tests involved reductions; e.g., $\frac{4}{6} = \frac{\square}{3}$. Variable (II) above defined the two tests on a symbolic level.

Two school systems participated in the study: Madison, Wisconsin, and Janesville, Wisconsin. The data from each system were collected independently by different experimenters and were analyzed independently. The following results were common to both school systems:

1) The ratio-denominator pictorial test was significantly easier than the fraction-denominator pictorial test for the low and middle ability levels in each grade.
2) The fraction-numerator pictorial test was significantly easier than the fraction-denominator pictorial test for each ability group in each grade.
3) The high ability children performed significantly better than the low ability children on each of the four pictorial tests and on the two symbolic tests.
4) The fifth and six graders performed significantly better than the fourth graders on the four pictorial tests and on the two symbolic tests.
5) Very low correlations exist between the scores on the symbolic test and the scores on the pictorial tests.
6) The fraction-denominator pictorial test was the most difficult for each ability group in each grade.

I

INTRODUCTION

It has been long recognized that one of the major goals of mathematical curricula in the elementary school is the acquisition of the concepts of fractions and ratios. However, this goal is not being reached by all children, or even by a majority of children. Continual critical evaluation of all phases of this major goal, and indeed all goals of mathematical curricula in the schools, must be carried on to determine their effectiveness. This section of the report will discuss the mathematical and psychological point of view which inspired the study and which will serve as a basis for interpretation of the results.

MATHEMATICAL BACKGROUND

Fractions

The rational numbers of arithmetic may be thought of as sets of equivalent fractions (17, p. 316), where the fractions are ordered pairs of whole numbers with the second component not zero, and which obey the following conditions (19, pp. 394–95):

a) $\frac{a}{b} \sim \frac{c}{d}$ if and only if $ad = bc$

b) $\frac{a}{b} \oplus \frac{c}{d} \sim \frac{ad + bc}{bd}$

c) $\frac{a}{b} \odot \frac{c}{d} \sim \frac{ac}{bd}$

Other authors reserve the name "fractional numbers" for the above mentioned sets (13, p. 199), where a fraction is the numeral that names the fractional number (12, p. 97). Equivalent fractions are just different names for some fractional number. The only difference between the two approaches is a naming process, for if a rational number of arithmetic is taken to be a number which has a name of the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$ (14, p. 188), then a fractional number is clearly to be considered as a rational number of arithmetic. For the sake of exposition, the first point of view will be adopted.

From the definition of the equivalence ($\sim$) of two fractions given above, it can be seen that $\sim$ is an equivalence relation. It is reflexive since $ab = ab$ implies that $\frac{a}{b} \sim \frac{a}{b}$; it is symmetric since $\frac{a}{b} \sim \frac{c}{d}$ implies $\frac{c}{d} \sim \frac{a}{b}$; and it is transitive since $\frac{a}{b} \sim \frac{c}{d}$ and $\frac{c}{d} \sim \frac{e}{f}$ implies $\frac{a}{b} \sim \frac{e}{f}$. It is well known that such an equivalence relation partitions the set on which it is defined into equivalence sets (11, p. 8). If the original set is taken to be $N \times N^* = a/b$: $a \in N$, $b \in N^*$, where N is the set of natural numbers and N^* is the set of natural numbers without zero, then $\sim$ partitions $N \times N^*$ into equivalence sets. Some of these sets are:

i) $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots, \frac{n}{2n}, \ldots\}$

ii) $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots, \frac{2n}{3n}, \ldots\}$

iii) $\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \ldots, \frac{3n}{8n}, \ldots\}$

iv) $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots, \frac{0}{n}, \ldots\}$

Addition and multiplication of fractions as defined in (b) and (c) above give rise to well-defined operations. That is, if $\frac{a}{b} \oplus \frac{c}{d} \sim \frac{ad + bc}{bd}$, and further if $\frac{a}{b} \sim \frac{e}{f}$ and $\frac{c}{d} \sim \frac{g}{h}$, then $\frac{ad + bc}{bd} \sim \frac{eh + gf}{fh}$. An example will clarify this statement. Take $\frac{1}{2}$ and $\frac{2}{3}$, $\frac{1}{2} \oplus \frac{2}{3} \sim \frac{7}{6}$; $\frac{1}{2} \sim \frac{2}{4}$ and $\frac{2}{3} \sim \frac{6}{9}$. By the definition, $\frac{2}{4} \oplus \frac{6}{9} \sim \frac{42}{36}$. It is easy to see that

$\frac{7}{6} \sim \frac{42}{36}$ Multiplication follows the same pattern. The usual properties of the operations are theorems which can be proved.

The above exposition governs much of what children are expected to know about fractions. This study will be concerned with only the equivalence sets, that is, the definition of the equivalence of two fractions. In most elementary expositions, the equivalence sign is replaced by an equal sign, and the circles are dropped from the two symbols $\oplus$ and $\odot$, a convention which will be followed in further discussions of fractions.

Ratios

Ratios are also taught in some form in the upper elementary school. It is possible to view ratios as forming a mathematical system different from fractions (18). Let Z denote the set of integers, and form $X = \{(a, b): a, b \in Z\}$. The pairs (a, b) will denote ratios. The following definitions will be accepted.

1) $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
2) $(a, b) \oplus (c, d) = (a + c, b + d)$.
3) If $m \in Z$, then $m \odot (a, b) = (ma, mb)$.

Under these definitions, the following theorems are true. The set of integers form the universal set for the variables.

1) $(a, b) \oplus (c, d) = (c, d) \oplus (a, b)$
2) $[(a, b) \oplus (c, d)] \oplus (e, f) = (a, b) \oplus [(c, d) \oplus (e, f)]$
3) $(a, b) \oplus (-a, -b) = (0, 0)$
4) $(a, b) \oplus (0, 0) = (a, b)$
5) $m \odot [(a, b) \oplus (c, d)] = [m \odot (a, b)] \oplus [m \odot (c, d)]$
6) $(m + n) \odot (a, b) = [m \odot (a, b)] \oplus [n \odot (a, b)]$
7) $m \odot [n \odot (a, b)] = mn \odot (a, b)$

The set $z \times z$ along with definitions 1—3 and the theorems 1—7 show that ratios form a two-dimensional vector space (16, p. 42).

In the elementary school, ratios usually arise in problem situations similar to the following. "If John walks 2 miles in 4 hours, how many miles could he walk in 8 hours?" The proportion for this problem is $\frac{2}{4} = \frac{x}{8}$, or $x = 4$. The interpretation given above for ratios would not allow the proportion to be written. The concept of a subspace, however, may be utilized. Consider $L = \{m \odot (2, 4): m \in Z\}$. This set is a subspace of the above vector space since:

1) if $m, n \in Z$; $[m \odot (2, 4)] \oplus [n \odot (2, 4)] = (m + n) \odot (2, 4)$ and
2) if $n \in Z$, $n \odot [m \odot (2, 4)] = nm \odot (2, 4)$,

which shows that L is closed under addition and scalar multiplication, and hence is a subspace.

The problem cited now becomes a matter of finding x so that $(x, 8) \in L$; that is, $(x, 8)$ is a member of the same subspace as $(2, 4)$. To do this, it is sufficient to find some k such that $k \odot (2, 4) = (x, 8)$. From the definition of equality, $2k = x$ and $4k = 8$, which implies $k = 2$, which in turn implies $x = 4$. However, one need not go through all these calculations in order to arrive at a solution, since if (a, b) and (c, d) are in the same subspace, then $ad = bc$. To see this, just consider $k \odot (a, b) = (c, d)$ which implies $ak = c$ and $bk = d$. It is easy to see that $bak = bc$ and $abk = ad$, which means that $ad = bc$. Starting with the statement $ad = bc$, it is true that (a, b) and (c, d) are in the same subspace. It is appropriate, then, to consider two ratios (a, b) and (c, d) as being in the same subspace if and only if $ad = bc$. In symbols, $(a, b) \sim (c, d)$ if and only if $ad = bc$, where "$\sim$" may be interpreted as "in the same linear subspace," which is an equivalence relation.

From the point of view discussed above, there is a similarity between ratios and fractions. Just consider those ratios (x, y) where $x \geq 0$ and $y > 0$. Then, $(a, b) \sim (c, d)$ if and only if $ad = bc$. If (x, y) is interpreted as a fraction, the concept of cross product also may be employed. This study will be concerned with ratios (a, b) which are in the same linear subspace and are such that $a \geq 0$ and $b > 0$, i.e., ratios which are "equivalent." In the study, the usual notation for ratios will be employed.

Quotients

It is possible to approach the rational numbers of arithmetic from a third point of view. If the solution to the equation $ax = b$ is defined as the quotient $\frac{b}{a}$, where a and b are whole numbers and $a \neq 0$, it is easy to show that these quotients are isomorphic to the rational numbers of arithmetic. Using the quotient approach, the concept of equivalent equations is sufficient to define two equivalent fractions. That is, if $ax = b$, then $c(ax) = cb$ is an equivalent equation (a, b, and c whole numbers), which implies $\frac{b}{a} = \frac{cb}{ca}$.

PSYCHOLOGICAL BACKGROUND

Piaget sees formal thinking as evolving about at the age of 11 or 12 years (5, p. 1). He gives quite a detailed exposition of proportionality and its relationship to formal thinking. An attempt will be made here to summarize some of Inhelder and Piaget's work on proportionality and to discuss in part the mental structure by which Piaget characterizes formal thought. The reason for such an exposition will be clarified later when a synthesis of mathematical, psychological, and curricular considerations is attempted.

Inhelder and Piaget reported twelve apparently representative interviews around which the development of the proportionality schema is traced and which they finally interpreted in terms of what they called the INRC Group (5), defined below. The experimental apparatus used was a conventional beam balance along which weights could be hung, with the intervals between the hangers forming congruent segments.

Prior to concrete operational thought, children react to a system of nonhorizontal equilibrium* in a chaotic fashion. An example is given by a recorded interview with a child called Mic (4;6)** (5, p. 166). In this interview, the child was presented with the situation of two equal weights at distances of 14 and 9 units. When asked to make the beam horizontal using weights, he continued to try to balance the two unequal weights by using his own action (e.g., using his hands to make the beam horizontal). The authors interpreted this phenomenon by conjecturing that the child did not differentiate between the instrument and his own actions (5, p. 167). He was clearly at a pre-operational stage.

Children who are at the upper end of the concrete operational stage come quite close to achieving proportionality on the beam balance. An example was given by a child called Fis (10;7). This child concluded that given unequal weights, "...you move up the heaviest [5, p. 171]." The authors were careful to note, however, that Fis did not measure the length even for the ratio 1 to 2 (5, p. 171). Using $A_1, A_2, \ldots$ to represent an increasing sequence of weights and $L_1, L_2, L_3, \ldots$, a decreasing sequence of lengths corresponding to the weights by subscripts, Piaget asserted that Fis and children like him are able to understand that $A_1 \times L_1 = A_2 \times L_2 = A_3 \times L_3 = \ldots$ on a qualitative basis but not on a quantitative basis. It would be a mistake to think of the sign "X" in $A_i \times L_i$ for each i as being a sign indicating multiplication of real numbers, or the equality sign as representing equality in the usual sense. The sign "X" may be interpreted as a logical multiplication (5, p. 172). The equality sign can be interpreted to mean "in horizontal equilibrium." These qualitative operations are inadequate to establish the law of proportionality (5, p. 172). They do, however, allow some correct conclusions, but without a metrical interpretation (5, p. 172). It must be stressed that Piaget felt the above compensations took place without metrical consideration, only on a qualitative plane even though the children were capable of quantifying weights and distance (5, p. 172).

Children who are at the formal level of thinking are capable of the same qualitative logic in the case of the beam balance as are children at the upper end of the concrete operational level. The children capable of formal operations, however, also pay attention to the metrical relationships involved, as is evidenced by the interviews cited. They not only know that a decrease in weight may be compensated by an increase in length, but they are able to reason, using numbers, how much increase in length is needed to compensate for a certain decrease in weight.

The reason children at the formal level of operational thinking are capable of understanding mathematical proportions may be explained by the INRC group (5, pp. 176–181). To explain this group, it is sufficient to let p and q be any statements with the principle of duality applying to p and $\bar{p}$, $\wedge$ (and) and $\vee$ (or).

a) $I(p \wedge q) = p \wedge q$

b) $N(p \wedge q) = \bar{p} \vee \bar{q}$

c) $R(p \wedge q) = \bar{p} \wedge \bar{q}$

d) $C(p \wedge q) = p \vee q$

The transformations I, N, R, and C then form a group with four elements (16, p. 27). Piaget's asserts that the way this group is defined is equivalent to the logical proportion of $\frac{p \wedge q}{\bar{p} \wedge \bar{q}} = \frac{p \vee q}{\bar{p} \vee \bar{q}}$ (5, p. 178). This proportion apparently is to be interpreted as follows. Let α, β, γ and δ be any statements. Then (letting "=" mean logical equivalence of statements) if

1) $\alpha \wedge \delta = \beta \wedge \gamma$

2) $\alpha \vee \delta = \beta \vee \gamma$

3) $\alpha \wedge \bar{\beta} = \gamma \wedge \bar{\delta}$ and $\bar{\alpha} \wedge \beta = \bar{\gamma} \wedge \delta$

*For example, a beam balance with two unequal weights hung at equal distances.

**(4;6) means an age of 4 years and 6 months.

4) $\alpha \wedge \bar{\gamma} = \beta \wedge \bar{\delta}$ and $\bar{\alpha} \wedge \gamma = \bar{\beta} \wedge \delta$

are all true, $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$ (5, p. 315). These four conditions are satisfied by the proportion $\frac{p \wedge q}{\bar{p} \wedge \bar{q}} = \frac{p \vee q}{\bar{p} \vee \bar{q}}$. Making the appropriate substitutions, it is easy to observe that under the definitions, $\frac{I(p \wedge q)}{R(p \wedge q)} = \frac{C(p \wedge q)}{N(p \wedge q)}$.

If p and q are considered, it is apparent that $\frac{p}{q} = \frac{\bar{q}}{\bar{p}}$ is a logical proportion. If the definitions (5, pp. 315, 316, 320)

1) $I(p) = p$
2) $N(p) = \bar{p}$
3) $R(p) = \bar{q}$
4) $C(p) = q$

are made, it is still true that the same relationship holds between I, N, R, and C and the logical proportionality $\frac{p}{q} = \frac{\bar{q}}{\bar{p}}$ as held previously. If p is interpreted as meaning a fixed increase in weight and q as a fixed increase in distance, then $\frac{p}{q} = \frac{\bar{q}}{\bar{p}}$ may be interpreted as follows. An increase in weight is to an increase in distance as a decrease in distance is to a decrease in weight. Since $p \wedge \bar{q} = R(\bar{p} \wedge q)$, the notion of a reciprocal logical proportion may be introduced as $\frac{p}{q} = R\left(\frac{\bar{p}}{\bar{q}}\right)$ (5, p. 179). If w represents a weight and L a length, nw and nL the usual multiplication of numbers, then $\frac{p}{q} = R\left(\frac{\bar{p}}{\bar{q}}\right)$ is isomorphic to $\frac{nw}{nL} = \frac{w \div n}{L \div n}$ (5, p. 180). Piaget stated,

> Before introducing numbers as measurements for weight and distance, the subject usually begins by assuming:
>
> $$p \wedge \bar{q} = R(\bar{p} \wedge q)$$
>
> (increasing the weight and reducing the distance on one of the arms is the same as reducing the weight and increasing the distance on the other arm). However, [the above]* proposition is none other than the proportion $\left[\frac{p}{q} = R\left(\frac{\bar{p}}{\bar{q}}\right)\right]$ * . . . which leads to the metrical proportion (5, p. 180).

*Referred to in the original source by number.

Piaget further elaborated this point.

> Actually, whenever a system of proportions comes into play, before the subject arrives at the calculation of numerical relations, he isolates an anticipatory schema for qualitative proportionality. Second, this schema, simply a logical one at first, leads at a later point to the discovery of metrical proportions [5, pp. 315–316].

For the case of the balance beam discussed above, Piaget stated,

> . . . in the balance problem the subject first discovers that a given increase in weight can be compensated by a proportional increase in the distance from the central axis: in placing a light weight at a great distance and a heavy weight at a small distance, he reaches equilibrium and concludes that the four values have a proportional relationship. But, at first, the compensation as well as the proportion are exclusively qualitative. If we let p stand for the statement of an increase in weight and q an increase in distance, and $\bar{p}$ and $\bar{q}$ the corresponding decreases, the subject begins by conceptualizing simply the following links:
>
> $p = R\bar{q}$ and $\frac{p}{\bar{q}} = \frac{q}{\bar{p}}$, from whence $p \cdot \bar{p} = q \cdot \bar{q}$ and $p \vee \bar{p} = q \vee \bar{q}$; i.e., that the increase in weight is to the corresponding decrease in distance as an increase in distance is to the corresponding decrease in weight (still independently of any measurement).
>
> The qualitative schema which serves as a starting point for the subject in the discovery of proportionality seems to be of this type. Essentially, it is based on the reciprocity of weights and distances which the experiment suggests to him: the increase in weight can be canceled by taking away the added weight (Np = $\bar{p}$, but it is equally possible to compensate this increase by reducing the distance ($p = R\bar{q}$ or $Rp = \bar{q}$). When this is done, the preceding proportion can also be written:
>
> $$\frac{p}{q} = R\left(\frac{\bar{p}}{\bar{q}}\right), \text{ from whence } p \cdot \bar{q} = R(\bar{p} \cdot q).$$
>
> The subject's reasoning usually appears in this latter form: increasing the weight and reducing the distance ($p \cdot \bar{q}$) is equivalent to (R = compensates) decreasing the weight and increasing the distance.
>
> Once these two schemata have been acquired, the subject can at a later point insert the numerical values which are furnished by his measurements [5, pp. 316–317].

One could attempt to criticize Inhelder and Piaget in the beam balance experiment on the grounds that the concept of equilibrium may be impeding the concept of proportionality. However, these authors have observed the same developmental levels in the case of proportionality in such diverse areas as space, speed, probabilities, etc. (5, p. 314). Moreover, "... the general equilibrium schema is not organized before the level of formal operations [5, p. 319]."

Lovell has conducted a followup study of Inhelder and Piaget's work on formal thinking (7). He repeated 10 of the 15 experiments reported by Inhelder and Piaget, of which one is described, in part, above. Lovell concluded that

> The main stages in the development of logical thinking proposed by Inhelder and Piaget have been confirmed. It seems that the authors are correct in suggesting that it is only rarely that average to bright junior school children reach the stage of formal thinking.... the least able of the secondary modern and comprehensive school pupils certainly remain at a low level of logical thought even at 15 years of age, and many of these do not seem to pass beyond the IIA - IIB stage of thinking.
>
> The majority of our protocols show much the same kind of reasoning as those of Inhelder and Piaget, and support many of their statements. For example, the authors maintain that, at the level of formal thought, the child comes to the Projection of Shadows experiment assuming proportionality from the start [7, p. 149].

Lovell discussed the effect of instruction on the results of these experiments.

> Teaching, in the sense of instruction, does not seem to have affected the results as much as had been expected.
>
> We have protocols indicating that a particular experiment had been "taught" in science lessons and yet the subjects were quite unable to reach the stage of formal thinking when they were given the experiment in this study [7, p. 150].

Lunzer performed a study in which he seriously tested Piaget's interpretation of formal reasoning. Not all of this study will be discussed, but only that phase which relates to proportionality. Lunzer stated, "Equally questionable, to my mind, is Piaget's attempt to identify the schema of proportionality with the four-group by accepting quantification as self-evident [8, p. 30]." After Lunzer went through a quotation from Inhelder and Piaget which related the INRC Group to a logical proportion, already done here, he stated further, "There is nothing in this to justify the introduction of numerical multiplication and division. It is true they both apply, but the mathematical statement in terms of proportionality is not a <u>mere</u> equivalent to the logical statement of reciprocity [8, p. 31]." Instead of merely criticizing, however, Lunzer, in the true light of a researcher seeking the truth, tested his own criticisms. He was not only seeking to relate logical proportion to mathematical proportion, but also "... asking whether Piaget is justified in postulating the existence of two successive levels in the development of logical reasoning or, whether, when the child has achieved the level of concrete reasoning and is, therefore, capable of argument in terms of fixed ... terms, his reasoning is <u>ipso facto</u> logical...." (8, p. 31).

In view of the above points, he constructed two types of tests, 1) verbal analogies and 2) numerical items. The test of verbal analogies included four subtests, which are outlined below (a, b, and c represent words, and x and y represent three or four words, one of which was to be selected by the subject).

A. a is to b as c is to x.

B. Same as A, with more difficult content.

C. a is to b as x is to y or x is to y as a is to b.

D. a is to x as y is to b or x is to a as b is to y [8, pp. 32, 33].

The hypotheses were: (8, pp. 33–34)

a) Group A analogies demanded only concrete reasoning, and would be easier than the others.

b) Group D analogies would be most difficult of all.

c) Group C analogies would require some formal qualities of reasoning.

d) Group B analogies would fall between Groups C and D in difficulty.

The difficulties of the four groups of items were as predicted. The subjects ranged from 9 to 17^{+} years of age and were of above average intelligence (8, p. 36). In conclusion, Lunzer stated,

> We are forced to conclude that the elementary analogies of group A represent

something more than mere concrete reasoning. Indeed, the steep rise that occurs after the age of 10 strongly suggests that these problems involve a type of reasoning, that is, formal reasoning, that is not present at the earlier level and is only elaborated at about the age of 11 [8, p. 38].

Thus, an analogy of the form, leather is to shoe as wool is to cardigan, necessarily involves three relations: one between the first two terms, a second between the second pair of terms, and, finally, a third (of identity) between the first two relations. In point of fact, the logical structures of such a system exactly parallels that of a statement of proportionality [8, pp. 40, 41].

This statement leads directly into a consideration of his numerical test, which consisted of two parts, (a) numerical analogies and (b) numerical series. Some of the analogies involved addition and/or subtraction of a constant term(s), items involving addition and/or subtraction involving multiplication and items involving multiplication of a constant term. Both parts (a) and (b) showed a "... general tendency of percentage scores to increase with age, with the sharpest rise occurring between 9 and 11 [8, p. 38]." He then stated, "Analogies, whether verbal or numerical, demand a more complex process of reasoning than is available at the concrete level ... [8, p. 39]." As a concluding remark, he stated, "The main conclusion seemed to be that both verbal and numerical analogies required the application of formal reasoning in the sense that the subject needed to establish second order relations [8, p. 43]."

Lovell and Butterworth (6) have performed a principal component factor analysis on a set of 20 tasks (not all of which involved proportion) to test the following hypotheses:

a) The schema of proportionality depends on some central intellective ability which underpins performance on all tasks involving proportion.

b) In addition to some central intellective ability, specific abilities contribute to the ability to use proportionality in particular tasks.

c) Tasks involving ratio will depend less on the ability indicated under (a) than in the case of the tasks involving proportion (6, p. 2).

Many of Lovell and Butterworth's items seemed to be quite similar to those of Lunzer's. For example, verbal and numerical analogies were included. Average and above average pupils from 9 to 15 years of age for a total of 60 pupils were included in the study. As a conclusion, the authors stated,

As already indicated, there is a central intellective ability which underpins performance on these tasks thus confirming our first hypothesis .... the presence of the bipolar components must be clearly recognized. Their presence confirms our second hypothesis, and they support the now known fact that the schema of proportion and the schema of ratio are not available in all situations and tasks at the same time. As Lunzer has pointed out, content and the nature of the apparatus, as well as structure are important in situations where formal thought is required. Tasks which require the schema of ratio only have, overall, lower loadings on the first general component than the tasks involving proportion.... Our third hypothesis is thus generally confirmed [6, p. 4].

One final remark made by these authors is of interest for this study. "It is not until 14 years of age in these pupils that 18 out of a possible 72 responses are at or near the level of formal operational thought, while even at 15 years of age the number was still a little under 50 percent [6, p. 3]."

Lunzer and Pumfrey have conducted a study on proportionality using 80 children in an age range of 6 to 15 years (9). These children were questioned on each of three situations purportedly designed to measure proportional reasoning. These were (9, p. 10),
(1) Cuisenaire material, (2) A pantograph, and (3) A balance. The results were summarized as follows:

(1) Older children are more successful than younger children. Differences in intelligence also played some part in the success of children.

(2) The cuisenaire apparatus was found to be much easier than the other two situations. The pantograph was considerably easier than the balance.

(3) Even though many children were successful in making predictions for the rods and the pantograph, these very rarely involved them in proportionality reasoning.

(4) Among older subjects, there were many more instances of proportional reasoning in the balance situation than in the other two, and more in the pantograph than the rods.

(5) No children were able to give an adequate explanation of the law of moments in the case of the balance situation. Those who solved the proportionality problems reasoned quite mechanically [9, p. 10–11].

CURRICULAR BACKGROUND

As noted in the section on psychological background, Piaget and Inhelder postulate that formal thinking does not emerge until about 12 years of age. One thing these authors consider as evidence of formal thinking is the ability of children to work successfully in situations involving proportionality. Studies were reviewed which were conducted by Lovell, Lunzer, and Lunzer and Pumfrey which may be taken as corroboration of Piaget and Inhelder's position. These studies justify asking the following basic questions.

1. If the schema of proportionality depends on some central intellective ability generally not present in children less than 12 years of age, how will fourth, fifth, and sixth grade children perform on tests constructed to measure the ability to construct and solve proportionalities ?
2. Since children are not equally successful with respect to all physical embodiments when working with proportionality, are fourth, fifth, and sixth grade children equally successful when working with proportionality situations amenable to an equal ratio interpretation and an equal fraction interpretation ? Are they equally successful when working with proportionalities which are instances of the statements

   a) $\frac{na}{nb} = \frac{x}{b}$ and b) $\frac{na}{nb} = \frac{a}{x}$?
3. Is there a difference in the performance of fourth, fifth, and sixth grade children at differing levels of general ability on tests constructed to measure the ability to construct and solve proportionalities ?

The above three questions cannot be answered independently of the curricular experiences of the children who participate in the study. For this reason, children from two school systems which used different arithmetic series were selected. These two curricula are briefly described below by grade.

Grade 4

Curriculum I. In this curriculum (4), the mathematical model for ratios and fractions is as given in the section on mathematical background. The quotient approach is not used. Fractions are introduced by means of visual displays. For example, when 1/3 is introduced, the children are told that one of three equal parts of a disc is being considered. The disc is shown in conjunction with the fraction. Six balls partitioned into three sets of two (with one of these three sets being considered) also serves as an interpretation for 1/3. Work is given on various basic pairs which include 1/2, 1/3, 2/5, 1/8, and 1/6. Equal fractions are introduced by virtue of visual displays. For example, in the case of 1/3 = x/6, a disc is presented which is cut into three equal parts, one of which is shaded. Dotted lines are also present which further partition the disc into six equal parts. The task is to determine how many of the six equal parts are shaded. Mixed numbers are taken up and interpretations are given.

Curriculum II. In this curriculum (3), the quotient approach described earlier in the section on mathematical background is used to introduce fractional numbers. The point of view is adopted that any whole number may be assigned to any one, two, or three dimensional region. For example, five units on the number line may be divided into two equal parts, where the midpoint of the segment [0, 5] is assigned the quotient 5/2, which is also viewed as the distance from zero to the midpoint. Along with visual interpretation, the statement is made that "We can divide a whole number into parts. For example, 3/2 is part of three. It is three divided by two. We can read this as '3 divided by 2' or as '(three)* halves [3, p. 187]." Children are also expected to denote shaded or unshaded parts of regions by using quotients. The point of view is adopted that "Fractions that name the same number are <u>equivalent fractions</u> [3, p. 201]." Various interpretations are given for equivalent fractions and the children are to complete statements such as 7/5 = ?/10 = 70/?.

Grade 5

Curriculum I. Approximately one-half of the fifth grade is spent working on ratios and fractions. Visual displays are used as interpretations and learning aids in both cases.

Ratio is interpreted as a comparison between two sets of objects. The ratio 15/10 is used in one example to compare 15 model airplanes to 10 model airplanes. Other numerals are also used in the same comparison, e.g., 3/2 and 9/6, from which equal ratios are taught.

*Note: Parentheses inserted as a mistake appeared in the book.

Ratios are also used to write a sentence for such problems as:

> Sally found 16 shells, and Kathy found 8 shells. Sally found how many times as many shells as Kathy? (4, p. 227)

Fractions are taught as was described under Grade 4. Various geometrical shapes are utilized, the most popular being circles and rectangles.

Curriculum II. Approximately one-third of the fifth-grade textbook is devoted explicitly to quotients and fractional numbers. The nearest to a ratio situation that occurs is in the introduction to fractional numbers. A two to one matching is performed between sets of 18 boys and 9 girls. Other matchings occur. An example follows:

> For his birthday party, George has 6 striped balloons and 18 gray balloons. How many gray balloons are there for each striped balloon? (3, p. 23)

The children are to express this problem by the equation $6 \times \square = 18$ from which they are to find that $\square = 3$, which is to be interpreted as a quotient (3, p. 23). No explicit exercises are given involving equal ratio situations, except as has been noted. Exercises on equivalent fractions, however, are included. Here, the multiplicative property of 1 is utilized. An example given in the text is as follows:

"$6/10 = \frac{2 \times 3}{2 \times 5} = \frac{2}{2} \times \frac{3}{5} = 1 \times \frac{3}{5}$ [3, p. 58]."

After this example, various exercises are given, such as $4/5 = ?/10$. Little time is spent on visual interpretation of fractions.

Grade 6

Curriculum I. A large share of Grade 6 is spent on ratios and fractions and related topics. Instances of the definition $a/b = c/d$ if and only if $ad = bc$ are given for ratios.

Curriculum II. More work is given on fractional numbers where about 22 pages are explicitly devoted to review of the operation of addition. Two pages of examples and exercises are devoted to equivalent fractions.

II

THE DESIGN

THE TESTING INSTRUMENTS

Several tests were constructed with reference to the three questions asked above. The tests and directions appear in Appendix A. Since one of the points of interest was to determine the relative success of children in the fourth, fifth, and sixth grades when working in situations amenable to an equal ratio interpretation and an equal fraction interpretation, these types of situations need to be clarified. They are situations comparable to those which the children had been exposed to in the curriculum.

Ratio Situation

> If there are three squares for every eight triangles, how many squares would there be for twenty-four triangles?

Fraction Situation

> Two of eight equal parts of circle A are shaded. How many of the 16 equal parts of circle B should be shaded so that the same amount will be shaded in both circles (given equal diameters)?

It is not the purpose of this investigation to argue for or against the above interpretation of equal ratios and fractions. The question of interest is to determine whether there is any difference in the success of children when working with the two different situations. No one should argue that the situations are not different.

Two tests were constructed, Test N and Test D. Test N involved statements of the form $\frac{na}{nb} = \frac{x}{b}$ where the subject was asked to find x. Test D involved statements of the form $\frac{na}{nb} = \frac{a}{y}$ where the subject was asked to find y. Test N, the missing numerator test, and Test D, the missing denominator test, each contained two subtests, a pictorial test and a symbolic test. The subtests for Test N were labeled Test NP (missing numerator-pictorial) and Test NS (missing numerator-symbolic). In Test D, the subtests were labeled in a similar fashion (Test DP and Test DS). The pictorial subtests of each test, Test NP and Test DP, were further partitioned into two subtests, an "equal" ratio test and an "equal" fraction test. These tests were labeled Test NPR (missing numerator-pictorial-equal ratios), Test NPF (missing numerator-pictorial-equal fractions), Test DPR (missing denominator-pictorial-equal ratios), and Test DPF (missing denominator-pictorial-equal fractions).

Test NP contained 16 pictorial items partitioned into two sets of 8 items each (8 items constituted Test NPR and 8 items constituted Test NPF). A similar pattern existed in Test DP. For each of the basic pairs $\frac{2}{5}$, $\frac{1}{5}$, $\frac{3}{8}$, and $\frac{1}{3}$, each test has two items involving equal ratios (e.g., two of the items in Test NP which involve equal ratios are instances of the open statements $\frac{4}{10} = \frac{x}{5}$ and $\frac{6}{15} = \frac{x}{5}$); there are also two corresponding items in Test NP which involve fractions. In Test DP, there are four similar items, two of which are instances of the open statement $\frac{4}{10} = \frac{2}{x}$ and two of which are instances of the open statement $\frac{6}{15} = \frac{2}{x}$. All of the items in both tests involve reductions by factors of 2 or 3. In summary, all of the items of Test NP are instances of the open statement $\frac{na}{nb} = \frac{x}{b}$, where $\frac{a}{b}$ is replaced by $\frac{2}{5}$, $\frac{1}{5}$, $\frac{3}{8}$, or $\frac{1}{2}$ and n is replaced by 2 or 3. Similarly, all of the items of Test DP are instances of the open statement $\frac{na}{nb} = \frac{a}{x}$, with the same replacements as Test NP. There is a complete balance of replacements for $\frac{a}{b}$ and n across Tests NP and DP and across the two parts of each test, equal ratios and equal fractions.

As noted, Lunzer has discovered that verbal analogies such as a is to b as c is to x involve a reasoning process not available to children at a concrete level of reasoning. Numerical analogies followed the same pattern of difficulty. Because of this discovery and because of the nature of the items of Tests NP and DP, it was deemed necessary that children be at more than a concrete level of reasoning to be successful when working the tests. However, it was fully recognized that the tests are representative of the types of experiences in the curriculum, so that a large training factor should be present. In the case of the children who participated in Curriculum I, the training factor was present in the case of both ratios and fractions; in the case of those who participated in Curriculum II, the training factor was more heavily weighted in favor of fractions.

The eight items of Tests NS and DS are the symbolic statements which correspond to the items in Tests NP and DP. Some of these items are easy enough to be answered by simple recall of known facts.

One point was given for each correct item so that the total score possible was 16 for Test NP and Test DP; and 8 for Tests NPF, NPR, DPF, DPR, NS, and DS.

THE SAMPLE

As noted earlier, two school systems were involved in the study. Janesville, Wisconsin, was the system which used Curriculum I and Madison, Wisconsin, was the system which used Curriculum II. The sampling procedures were identical in both systems and will be described in general.

The total enrollment for the fourth, fifth and sixth grades was obtained for the buildings which participated in the study. The IQ measure for each child was obtained from the permanent records. On the basis of this measure, three groups of children were identified for each grade, designated as a high group, a middle group, and a low group. An ordered random sample was then selected by grade until each ability group contained 20 children. The first 10 children selected were assigned to Test N and the second 10 children selected were assigned to Test D, which constituted a random assignment to tests. After an ability group contained 20 children, the sampling for that group was terminated.

After the above sample was completed, two alternates for each ability group in each school building were randomly selected, one for Test N and one for Test D. Due to administrative difficulties, it was necessary to discard the last child in each ordered random sample for Janesville, so that 18 instead of 20 children were included in each ability group for each grade. Eight school buildings in Janesville and eleven school buildings in Madison participated. Figure 1 shows the number of children at each grade level at each ability level who participated in the study.

The IQ measures used in each school system were those which were made available by the school system. For Janesville, the total scores from the Lorge-Thorndike Intelligence Test and for Madison, the scores from the California Test of Mental Maturity were used.

In Janesville, the IQ intervals of 80–97, 98–109, and 110–125 were used to define the low, middle, and high groups. These intervals were chosen to reflect approximate thirds of the populations at each grade level.

In Madison, the IQ intervals of 80–100, 105–115, and 120–140 were used to define the low, middle, and high groups. Again, these intervals partitioned the population into thirds as nearly as possible for each grade. The spacing of the intervals occurs due to the fact that a conversion from percentile scores to IQ scores was necessary.

THE PROCEDURE

In each school building, Test N was administered to all of the children assigned to it at one sitting. The same procedure was followed for Test D. No time limit was imposed. Each test took about 45 minutes. Mr. Robert Parr administered all of the tests in Madison and Mr. Leslie Steffe administered all of the tests in Janesville.

When a group of children entered the testing room, the testor told them that the booklet they would receive contained a set of exercises on which they were to do their best. The booklets were then passed out face down. Points 1–5 in the directions were discussed informally. All of the remaining directions were read to the children. Item 0 constituted a warmup item and the children were encouraged to ask questions. If the testor detected any children having trouble with the item, he did his best to insure that the child understood the directions by giving individual consultation. No warmup items were given for the ratio problems. The items were randomized before the booklets were constructed; since it was a group test and the directions were read to the children, the items could not be randomly administered to each child.

Figure 1. Number of Children in the Sample by School, Grade, and Ability

Janesville School	4			5			6		
	L	M	H	L	M	H	L	M	H
Adams	4	2	1	2	3	2	2	3	4
Jackson	0	1	3	1	1	0	3	1	2
Jefferson	0	0	0	6	3	4	4	2	4
Lincoln	5	4	3	7	3	2	3	6	0
Madison	4	4	4	3	1	2	2	0	2
Roosevelt	1	5	5	3	4	6	1	3	3
Washington	1	2	2	2	3	1	3	2	1
Wilson	3	0	0	0	0	1	0	1	2

Madison School	4			5			6		
	L	M	H	L	M	H	L	M	H
Elvehjem	0	3	3	1	1	1	4	1	1
Emerson	8	1	1	4	2	1	1	4	1
Gompers	0	2	2	2	2	4	4	0	4
Crestwood	1	0	2	1	2	1	0	0	0
Van Hise	2	[illegible]	2	0	4	2	0	0	0
Hawthorne	3	2	2	4	0	1	3	5	1
Randall	1	2	2	2	1	3	2	1	5
Mendota	1	2	1	4	1	1	0	4	2
Kennedy	3	6	3	0	2	4	1	1	2
Allis	1	2	2	2	5	2	4	3	3
Falk	0	0	0	0	0	0	1	1	1

THE EXPERIMENTAL DESIGN

As noted above, it was not possible to randomly administer the items independently to each subject. With regard to this point, the following are relevant. 1) The items were randomly arranged in the test booklets. 2) If the items in each test were randomized for each pupil, it would not have been possible to give the test as a group test. Carry-over effects are present in a fixed manner.

Extensive internal-consistency reliability studies were conducted for each grade on each of the ratio and fraction tests. Various sources are available for a full elucidation of the foundations of these studies. *

The experimental design utilized for Tests NP and DP is outlined by Winer (20, Chapter 7) and by Meyers (10, Chapter 8). This design has been successfully used in studies conducted by Boe (1) and Steffe (15). In Figure 2, C denotes grade of which three levels exist, A denotes ability of which three levels exist, T denotes tests of which two levels exist (Test NP and Test DP), and V denotes visuals of which two levels exist (ratio and fraction). All factors are considered as fixed factors. G_{ijk} represents a group of children in grade i, ability level j, and test k.

The above design was utilized for the data in each school system. The study was conceived as being two independent studies, one in Janesville and one in Madison.

The experimental design utilized for Tests NS and DS was a straightforward ANOVA where the factors C, A, and T are included. Again, the data in the two school systems were analyzed separately. Various correlation coefficients were computed between the pictorial and symbolic tests, and subtests thereof.

HYPOTHESES

The above design allows the following hypotheses to be tested. Not all hypotheses tested are stated, only those which are of the most interest to the investigators. The hypotheses were the same for both school systems.

*Frank B. Baker, Empirical Determination of Sampling Distribution of Item Discrimination Indices and a Reliability Coefficient. Contract OE-2-10-071, Nov. 1962.

Figure 2. Diagram of Experimental Design

C	A	T	V: Ratio	V: Fraction
4	H	NP	G_{111}	G_{111}
		DP	G_{112}	G_{112}
	M	NP	G_{121}	G_{121}
		DP	G_{122}	G_{122}
	L	NP	G_{131}	G_{131}
		DP	G_{132}	G_{132}
5	H	NP	G_{211}	G_{211}
		DP	G_{212}	G_{212}
	M	NP	G_{221}	G_{221}
		DP	G_{222}	G_{222}
	L	NP	G_{231}	G_{231}
		DP	G_{232}	G_{232}
6	H	NP	G_{311}	G_{311}
		DP	G_{312}	G_{312}
	M	NP	G_{321}	G_{321}
		DP	G_{322}	G_{322}
	L	NP	G_{331}	G_{331}
		DP	G_{332}	G_{332}

TEST P

1) There are no differences among the mean performances of the children in the fourth, fifth, and sixth grades.
2) There are no differences among the mean performances of the children in the high, medium, and low ability groups.
3) There is no difference between the means of numerator (NP) and denominator (DP) tests.
4) There is no difference between the means of the ratio (NPR and DPR) and fraction subtests (NPF and DPF).
5) The difference between the means of the ratio and fraction subtests is the same across grades.
6) The difference between the means of the ratio and fraction subtests is the same across the ability groups.
7) The difference between the means of the ratio and fraction subtests is the same across the numerator and denominator test.
8) The difference between the means of the ratio and fraction subtests is the same across the ability groups within each grade.

TEST S

1) There are no differences among the mean performances of the children in the fourth, fifth, and sixth grades.
2) There are no differences among the mean performances of the children in the different ability groups.
3) There is no difference between the means of the numerator and denominator test.

TESTS P AND S

1) The scores on Test P and Test S and subtests thereof are not correlated.

Figure 3. ANOVA Analysis Test P

Source of Variation	df	MS	F
Between			
C	c-1	MS_C	$MS_{subj\ w.\ groups}$
A	a-1	MS_A	is error for all other
T	t-1	MS_T	mean squares between.
CA	(c-1)(a-1)	MS_{CA}	
CT	(c-1)(t-1)	MS_{CT}	
AT	(a-1)(t-1)	MS_{AT}	
CAT	(c-1)(a-1)(t-1)	MS_{CAT}	
Subj. w. groups	cat(n-1)	$MS_{Subj\ w.\ groups}$	
Within			
V	v-1	MS_V	$MS_{V\ X\ Subj\ w.\ groups}$
VC	(v-1)(c-1)	MS_{VC}	is the error term for
VA	(v-1)(a-1)	MS_{VA}	all other mean squares
VT	(v-1)(t-1)	MS_{VT}	within
VCA	(v-1)(c-1)(a-1)	MS_{VCA}	
VCT	(v-1)(c-1)(t-1)	MS_{VCT}	
VAT	(v-1)(a-1)(t-1)	MS_{VAT}	
VCAT	(v-1)(c-1)(a-1)(t-1)	MS_{VCAT}	
V X Subj w. groups	cat(n-1)(v-1)	$MS_{V\ X\ Subj\ w.\ groups}$	

III

THE RESULTS OF THE STUDY

RELIABILITY STUDIES

An extensive reliability study was conducted by grade for Tests NP (numerator) and DP (denominator). The generally high reliabilities for the total pictorial tests are reported in Table 1.

Table 1

Internal-Consistency Reliability Coefficients: Total Pictorial Tests

System	Grade	Test NP	Test DP
	4	.92	.74
Madison	5	.93	.88
	6	.83	.91
	4	.89	.72
Janesville	5	.85	.87
	6	.92	.83

The modest reliabilities for Test DP for the fourth grades may indicate the difficulty of the tests at that level.

Table 2 contains the results of a reliability study conducted separately on the ratio and fraction subtests of Tests NP and DP. The reliabilities generally are higher for the parts of Test NP than for the corresponding parts of Test DP, a second indication of the relative difficulties of the two pictorial tests (as the difficulty of a test increases, the error variance may increase, and the reliability decrease). The reliabilities seem to be substantial enough, however, to support interpretations from the analysis of the data.

RESULTS OF THE STATISTICAL ANALYSIS: TESTS NP AND DP

The results of the Janesville study will be given first followed by the results of the Madison Study.

Table 2

Internal-Consistency Reliability Coefficients: Ratio and Fraction Tests

System	Grade	NP Ratio	NP Fraction	DP Ratio	DP Fraction
	4	.88	.87	.73	.58
Madison	5	.90	.93	.75	.79
	6	.80	.83	.83	.89
	4	.87	.89	.63	.75
Janesville	5	.84	.85	.86	.79
	6	.90	.91	.76	.83

Janesville Study

The results of the analysis of variance is given in Table 3 for Tests NP and DP. Each source of variation which has a significant *F* ratio is discussed below.

Table 4 contains means for the three grades. Table 5 contains the matrix of differences of these means by pairs, and Table 6 contains the critical values given by the Newman-Keuls test by which the significance of the differences of the means were tested. To obtain these critical values, the standard error of the mean

$S_G = \sqrt{\frac{MS_{S/G}}{n.a.t.v.}}$ is computed and then

$S_G \cdot q_{1-\alpha}(r, d.f.)$, the critical value, is computed where r = the number of steps the two ordered means are apart and $q_{1-\alpha}(r, d.f.) = \frac{\overline{Mi} - \overline{Mj}}{S_G}$, where $\overline{Mi}$ = mean of group i, $\overline{Mj}$ = mean of group j, and q is the studentized range statistic. There were significant differences between Grades 4 and 5 and between 4 and 6 but not between 5 and 6, even though the mean for Grade 6 was greater than the mean for Grade 5.

Table 3

Janesville
ANOVA Table

Source of Variation	df	MS	F
Between			
C (Grades)	2	69.39	9.62**
A (Ability)	2	131.52	18.22**
T (Tests)	1	180.75	25.05**
CA	4	13.47	1.87
CT	2	8.39	1.16
AT	2	7.37	1.02
CAT	4	16.58	2.30
S/G	144	7.22	
Within			
V (Ratio vs. Fraction)	1	91.31	25.84**
CV	2	1.32	<1
AV	2	20.50	5.80**
TV	1	34.68	9.82**
CAV	4	2.08	<1
CTV	2	2.28	<1
ATV	2	5.65	1.60
CATV	4	1.60	<1
V × S/G	144	3.53	

** $p < .01$

Table 4

Janesville
Means by Grade*

Grade	4	5	6
Mean	3.52	4.53	5.10

*These means must be doubled to get the means of the pictorial tests.

Table 5

Janesville
Differences Between All Pairs of Means of the Three Grades

Grade	5	6
4	1.01*	1.58*
5	-	.57

*$p < .05$

Table 6

Janesville
Critical Values for Newman-Keuls Test*

	2	3
$q_{.95}(r, 120)$	2.80	3.36
$S_G\, q_{.95}(r, 120)$	.620	.861

*$q_{.95}(r, 120)$ is used in place of $q_{.95}(r, 144)$

Mean scores contributing to the significant AV and TV interactions and the insignificant ATV interaction are reported in Table 7. In addition, the differences between the means on the ratio and fraction subtests (Factor V) for levels of Factors A and T combined and the results of a t test (20, p. 340) performed on those differences are reported. The differences between the means were significant only for Test DP for the low and medium ability groups, although the difference between the means for Test NP for the medium ability group approached significance. All of these differences are in favor of the ratio subtest (PR).

Table 7

Janesville
Means: Ability by Tests by Visuals

A	T	V: Ratio	V: Fraction	$\bar{R} - \bar{F}$ [a]	t
L	NP	4.52	4.33	.19	<1
	DP	3.15	1.52	1.63	3.19**
M	NP	5.30	4.37	.93	1.82
	DP	5.11	2.04	3.07	6.01**
H	NP	6.19	6.07	.12	<1
	DP	5.22	4.78	.44	<1

[a] $\bar{R}$ and $\bar{F}$ represent means of the ratio test and of the fraction test, respectively.

** $p < .01$

Table 8 represents the TV interaction, which was significant. The difference between the means for the ratio and fraction subtests was significant only for Test DP. The results in Table 7 indicate, however, that due to the ability by visual interaction, this significance is due to only the low and middle ability groups. Test NP was easier than Test DP for both the ratio test and the fraction test. Since ability interacted with factor V, the generality of the preceding statement at each ability level must be checked. Table 9 contains these calculations.

From the results of Tables 8 and 9, it can be safely concluded that for the low ability children Test NP is easier than test DP for each of the ratio and fraction tests; but for the ratio test, at only the .05 level. For the middle and high ability groups, significance is reached only in the case of the fraction test. This interpretation holds for each grade due to the lack of a significant interaction of Grades with any of the other variables.

Table 8

Janesville
Means: Tests by Visuals

T \ V	Ratio	Fraction	$\bar{R} - \bar{F}$	t
NP	5.33	4.93	.40	1.36
DP	4.49	2.78	1.71	5.80**
$\overline{NP} - \overline{DP}$	.84	2.15		
t	2.30*	5.90**		

* $p < .02$

** $p < .01$

Table 9

Janesville
Differences of Means of Test NP and Test DP by AV

A	V	$\overline{NP} - \overline{DP}$	t
L	Ratio	1.37	2.17*
	Fraction	2.81	4.45**
M	Ratio	.19	<1
	Fraction	2.33	3.69**
H	Ratio	.97	1.54
	Fraction	1.29	2.06*

* $p < .05$

** $p < .01$

Table 10

Janesville
Means: Ability by Visuals

A \ V	Ratio	Fraction	$\bar{R} - \bar{F}$	t
L	3.83	2.93	.90	2.49*
M	5.20	3.20	2.00	5.53**
H	5.70	5.43	.27	<1

* $p < .02$

** $p < .01$

Table 10 represents the AV interaction and Table 11 represents the differences of the means of the ability groups at each level of factor V. The matrix of the t values corresponds to the

Table 11

Janesville
Differences of Means of Ability Groups by Visuals

V	$\bar{H} - \bar{M}$	$\bar{M} - \bar{L}$	$\bar{H} - \bar{L}$	t		
Ratio	.50	1.37	1.87	1.12	3.07**	4.19**
Fraction	2.23	.27	2.50	5.00**	<1	5.61**

**p < .01

Table 12

Janesville
Differences of Means of Ability Groups by VT

V	T	$\bar{H} - \bar{M}$	$\bar{M} - \bar{L}$	$\bar{H} - \bar{L}$	t		
Ratio	NP	.89	.78	1.67	1.42	1.25	2.67**
	DP	.11	1.96	2.07	<1	3.13**	3.31**
Fraction	NP	1.70	.04	1.74	2.72**	<1	2.78**
	DP	2.74	.52	3.26	4.38**	<1	5.21**

**p < .01

matrix of the differences. Since T and V interact, in order to better understand the data in Table 11, Table 12 was constructed. The differences between the means of the high and low ability groups were significant for each of Tests NP and DP at each level of factor V. For the fraction test, the differences between the means of the high and middle ability groups were significant for each of Tests NP and DP, but both of these differences were not significant for the ratio test. The differences between the means of the medium and low ability group, however, were significant for Test DP at the ratio level. This information from Table 12 corresponds very nicely with information given in Table 11.

Table 13 represents the interaction of factors G, A, T and V. As shown in Table 3, this interaction is highly insignificant.

The results encompassed in Table 7 show there is no reason to believe that any differences exist in the means of the ratio and fraction test for the high ability groups both for Test NP and DP. Since no interaction with Grades occurs, this result may be generalized to each of the Grades 4, 5, and 6. An inspection of the appropriate means in Table 13 reflects the above result. However, for the middle and low ability groups, the mean of the ratio test is significantly greater than the mean of the fraction test for DP but not NP. An inspection of these means in Table 13 for each of the Grades 4, 5, and 6 when considered in view of the insignificant Grade interactions again reflects the results discussed.

Table 13

Janesville
Means: Grades by Ability by Tests by Visuals

G	A	T	Ratio	Fraction
4	L	NP	3.22	3.78
		DP	3.00	1.67
	M	NP	4.11	3.44
		DP	3.00	.90
	H	NP	6.11	5.44
		DP	4.11	3.44
5	L	NP	6.33	5.67
		DP	2.44	1.00
	M	NP	5.56	3.89
		DP	5.56	2.67
	H	NP	5.67	5.89
		DP	5.00	4.67
6	L	NP	4.00	3.56
		DP	4.00	1.89
	M	NP	6.22	5.78
		DP	6.78	2.56
	H	NP	6.78	6.89
		DP	6.56	6.22

The results encompassed in Table 9 show that the mean for Test NP is greater than the mean for Test DP for each ability group for the fraction test. This result again generalizes to each grade level as is reflected by the appropriate means in Table 13.

The results encompassed in Table 12 show that the mean for the high ability group is significantly greater than the mean for the low ability group for each of Tests NP and DP at each level of factor V. This result again generalizes to each grade level. An inspection of the appropriate means in Table 13 may lead one to believe the result may not be true for Test NP at the fifth grade level. However, this fluctuation is one which can be attributed to chance. The mean for the high ability group is greater than the mean for the middle ability group only for Test NP and DP at the fraction level, which again generalizes to each grade level. The same result did not hold for the ratio test. The mean score for the middle ability group is greater than the mean score for the low ability group only for Test DP at the ratio level. An inspection of the appropriate means in Table 13 would lead one to believe that the result doesn't hold for the fourth grade, but again, this may be a chance fluctuation.

Madison Study

The results of the analysis of variance performed on the data collected in Madison for Tests NP and DP is given in Table 14. Table 15 contains means by grades. Table 16 contains the matrix of differences of these means by pairs and Table 17 contains the critical values given by the Newman-Keuls test. There were significant differences between the means for Grades 4 and 5, 4 and 6, and 5 and 6.

Table 18 contains the means for each ability level and Table 19 contains the matrix of differences of these means. Significant differences occur for each pair of means. Table 20 represents the insignificant GA interaction. Since, as discussed before, both grade and ability were significant, within each grade level, the mean score for the low ability group is significantly less than the mean score for the middle ability group which in turn is significantly less than the mean score for the high ability group.

Table 14 shows a significant interaction of Tests by Visuals. Table 21 represents this interaction and gives the results of a t test for significance of difference of the ratio and fraction subtests at each level of Test P. The fraction subtest (NPF) was easier than the ratio subtest (NPR). For Test DP, however, the fraction subtest (DPF) was much more difficult than the ratio subtest (DPR). There was no difference in the mean scores of the ratio test across Tests NP and DP, but there was a large difference between the mean scores for the fraction test across Tests NP and DP.

Table 14

Madison
ANOVA Table

Source of Variation	df	MS	F
Between			
C	2	96.34	11.64**
A	2	188.21	27.75**
T	1	119.03	14.38**
CA	4	6.77	<1
CT	2	12.66	1.53
AT	2	7.23	<1
CAT	4	1.96	<1
S/G	162	8.28	
Within			
V	1	29.47	10.10**
CV	2	3.10	1.06
AV	2	1.41	<1
TV	1	175.00	59.98**
CAV	4	.44	<1
CTV	2	6.00	2.06
ATV	2	6.07	2.08
CATV	4	2.83	<1
V × S/G	162	2.92	

** $p < .01$

Table 15

Madison
Means by Grade*

Grade	4	5	6
Mean	3.42	4.28	5.21

*These means must be doubled to get the means of the pictorial test.

Table 16

Madison
Differences Between All Pairs of Means of the Three Grades

Grade	5	6
4	.86*	1.79*
5	–	.93*

*$p < .05$

Table 17

Madison
Critical Values for Newman-Keuls Test*

r	2	3
$q_{.95}(r, 120)$	2.80	3.36
$S_G\, q_{.95}(r, 120)$	.74	.88

*$q_{.95}(r, 120)$ is used instead of $q_{.95}(r, 162)$.

Table 18

Madison
Means by Ability

Ability	L	M	H
Mean	3.01	4.39	5.51

Table 19

Madison
Differences Between All Pairs of Means of the Three Ability Levels**

Ability	M	H
L	1.38*	2.50*
M	-	1.12*

*$p < .05$

**Critical values obtained from Table 23.

Table 20

Madison
Means: Grade by Ability

G \ A	L	M	H	Total
4	2.38	4.30	4.56	3.42
5	3.10	4.65	5.10	4.28
6	3.55	5.23	6.85	5.21
Total	3.01	4.39	5.51	

RESULTS OF THE STATISTICAL ANALYSIS: TESTS NS AND DS

Janesville Study

The results of the analysis of variance performed on Tests NS and DS are given in Table 22. The effects due to Grades and Ability were significant, at the .01 and .05 levels respectively. Table 23 contains the means by Grade and Ability. The sharp increase between Grades 4 and 5 may be attributed to a training factor.

Table 21

Madison
Means: Tests by Visuals

T \ V	Ratio	Fraction	$\bar{R} - \bar{F}$	t
NP	4.47	5.29	-.82	3.22**
DP	4.71	2.74	1.97	7.73**
$\overline{NP} - \overline{DP}$	-.24	2.55		
t	<1	7.22**		

**$p < .01$

Table 22

Janesville
ANOVA Table

Source of Variation	df	MS	F
C	2	415.64	72.79**
A	2	25.15	4.41*
T	1	4.50	<1
CA	4	10.57	1.85
CT	2	9.24	1.62
AT	2	2.35	<1
CAT	4	4.95	<1
Error	144	5.71	

*$p < .05$

**$p < .01$

Table 23

Janesville
Means: Grade by Ability

G \ A	L	M	H	Total
4	2.00	1.94	1.77	1.89
5	5.11	6.33	7.50	6.32
6	5.78	7.67	7.56	7.[illegible]
Total	4.30	5.32	5.59	

Madison Study

The results of the analysis of variance performed on Tests NS and DS are given in Table 24. The three main effects are significant, where the Test factor is significant at only the 5% level. Table 25 contains the means of the three main effects. Again there is a sharp increase in the mean scores between Grades 4 and 5, which is indicative of a training factor. The three levels of ability were also significant in the direction expected. The means of the two tests favor Test DS. This difference may not be important, as the difference did not exist in the Janesville study.

Table 24

Madison
ANOVA Table

Source of Variation	df	MS	F
C	2	291.24	45.51**
A	2	106.67	16.67**
T	1	30.42	4.75*
CA	4	2.21	<1
CT	2	9.01	1.41
AT	2	.84	<1
CAT	4	4.15	<1
Error	162	6.40	

* $p < .05$

** $p < .01$

THE RESULTS OF THE CORRELATION STUDY

An extensive correlation study was conducted between the scores on the following tests: 1) NPR vs. NS; 2) DPR vs. DS; 3) NPF vs. NS; 4) DPF vs. DS; 5) PR vs. S; 6) PF vs. S; 7) NP vs. NS; 8) DP vs. DS; and 9) P vs. S. Tables 26 and 27 contain the results of this study. In the Madison school system, low to very low correlations exist in most cases. In the Janesville school system, low to very low correlations also exist in most cases.

Table 25

Madison
Means: Grade by Ability by Tests

C	A	NS	DS	Total Grade
4	L	.50	1.80	Four
	M	1.20	1.80	1.98
	H	3.60	3.00	
5	L	1.90	3.70	Five
	M	5.30	5.90	4.98
	H	5.90	7.10	
6	L	5.30	5.30	Six
	M	5.40	6.80	6.28
	H	6.90	8.00	
Total		4.00	4.82	
Total	L	3.08		
	M	4.40		
	H	5.75		

Table 26

Correlation Table: Madison

	NPR vs. NS	NPF vs. NS	DPR vs. DS	DPF vs. DS	PR vs. S	PF vs. S	NP vs. NS	DP vs. DS	P vs. S
All Subjects	.41**	.43**	.51**	.48**	.46**	.36**	.48**	.55**	.47**
6	.30	.38*	.57**	.38*	.42**	.28*	.43*	.51**	.40**
5	.47**	.49**	.39*	.32	.46**	.28*	.54**	.39**	.44**
4	.42*	.27	.30	.27	.35**	.17	.37*	.34	.29*
6H	.06	-.23	-	-	.09	-.10	-.02	-	-
6M	.73**	.79**	.45	.11	.56**	.17	.97**	.35	.45*
6L	-.04	.21	.50	.18	.17	.18	.14	.35	.22
5H	.81**	.57	.33	.28	.61**	.28	.79*	.33	.50*
5M	.13	.50	.24	.02	.17	.24	.38	.16	.26
5L	-.18	.20	.55	.36	.40	.10	.06	.52	.31
4H	.30	-.43	.64*	.74**	.41	.16	-.12	.81**	.30
4M	.15	.19	-.18	-.55	.04	-.08	.18	-.37	-.03
4L	.39	.33	.33	-.26	.36	-.03	.38	.18	.21

Table 27

Correlation Table: Janesville

	NPR vs. NS	NPF vs. NS	DPR vs. DS	DPF vs. DS	PR vs. S	PF vs. S	NP vs. NS	DP vs. DS	P vs. S
All Subjects	.27**	.24*	.48**	.40**	.36**	.29**	.30**	.51**	.37**
6	.54**	.59**	.34	.24	.47**	.42**	.63**	.34	.50**
5	-.14	-.09	.41	.55**	.11	.13	-.15	.53**	.14
4	.21	.12	.36	.25	.31*	.24	.19	.38**	.32*
6H	-	-	.86**	.83**	.46*	.70**	-	.95**	.71**
6M	.33	.50	-.29	.18	.08	.19	.71*	-.06	.26
6L	.54	.54	-.43	-.45	.28	.27	.55	-.51	.29
5H	-.33	-.25	-	-	-.26	-.25	-.42	-	-.31
5M	-.11	-.55	.68*	.62	.13	-.30	-.42	.74*	-.09
5L	.08	.57	.02	.29	.17	.41	.48	.18	.32
4H	.49	.38	.59	.57	.52*	.47*	.55	.70*	.58**
4M	-.01	-.29	-.13	-.63	.03	-.10	-.22	-.32	-.05
4L	.48	.60	.46	-.14	.46*	-.49*	.57	.29	.56*

*p < .05
**p < .01

IV

CONCLUSION AND DISCUSSION

In this chapter, each hypothesis of interest will be stated along with discussion and conclusions relative to that hypothesis.

The results of the reliability studies indicate that substantial reliabilities are associated with each test and subtest for each grade level, which indicates that the results of the statistical analyses may be interpreted with the confidence that the tests are measuring what they were constructed to measure. The results of the study for both school systems will be discussed simultaneously whenever possible.

Test P

1) There are no differences among the mean performances of the children in the fourth, fifth and sixth grades.
5) The difference between the means of the ratio (NPR and DPR) subtest and fraction (NPF and DPF) subtest is the same across grades.

Hypothesis one was rejected ($p < .01$) in both the Janesville and Madison study. In Janesville, the means in terms of percentages were 44, 57 and 64 and in Madison 43, 54, and 65 for Grades 4, 5 and 6 respectively. (See Tables 4 and 15.) In the case of the fourth grade, the means are about what one would expect in view of the limited amount of experience of these children with ratio and fractions. For the fifth grade and especially for the sixth grade, however, the mean scores are modest in view of the extensive emphasis in each curriculum on ratio or fractions. In Janesville, no differences existed between the fifth and sixth grades. No interaction of grades occurred with any other factor.

2) There are no differences among the mean performances of the children in the high, medium and low ability groups.
6) The differences of the means of the ratio subtest (Test PR) and fraction subtest (PF) is the same across the ability groups.

In the Janesville study the high ability group scored significantly better than the low ability group ($p < .01$) for each subtest, i.e., for subtests NPR, NPF, DPR and DPF. The high ability group also scored better ($p < .01$) than the middle ability group for tests both of NPF and DPF. For these two tests, the middle ability group did not score better than the low ability group. Moreover, the high ability group did not score better than the middle ability group for either of tests NPF or DPF. For test DPF, however, the middle ability group did score better ($p < .01$) than the low ability group.

In the Madison study, the effect due to ability was significant ($p < .01$) where the mean scores were 38, 55 and 69 (See Table 26) percent for the low, middle and high groups respectively. The differences of these means taken two at a time are all significant. In Madison, no interaction occurred between Factors A and V. It is impossible to make any comparisons of the above results in the two school systems due to the different testing instruments and intervals used to stratify the two populations. In Janesville, however, the fact that the middle ability group scored as well as the high ability group on both tests NPR and DPR may be attributed to the experiences in ratio situations of both groups.

3) There is no difference in the means of Test NP and Test DP.
4) There is no difference between the means of the ratio (NPR and DPR) and fraction (NPF and DPF) subtests.
7) The difference of the means of the ratio and fraction subtests is the same across the numerator and denominator test.

Factor T and factor V interacted in the analysis of data for both school systems. The manner in which the interactions occurred were not strictly analogous, but similarities do occur. For Test DP (denominator) the ratio subtest was easier than the fraction subtest ($p < .01$) for both school systems. Moreover, for the fraction subtest, the numerator test was easier than the denominator test ($p < .01$) for both school systems. These results indicate that test DPF was extremely difficult for most of the children in the study.

In both school systems, the low subtest score was the fraction-denominator test for every ability group in each grade. In view of these facts, a detailed item difficulty table was constructed and is presented as Table 28. The nomenclature for this table is as follows: in the column headings, 4.1 means grade 4, test NP; 4.2 means grade 4, test DP; etc. In the first column of the table the basic pairs are given. In column two, 4r2 means item 4, ratio and a reduction by a factor of two, etc. The table entries are the percentages scoring a particular item correct. The means are the mean percentages of the four column entries immediately above them. For the numerator test (4.1, 5.1, and 6.1) it is easy to observe a decrease in the means from the two pairs $\frac{1}{3}$ and $\frac{1}{5}$ to the two pairs $\frac{2}{5}$ and $\frac{3}{8}$. For the denominator test (4.2, 5.2, and 6.2) this decrease is very acute, more so for the fraction items than for the ratio items as the above interaction shows.

In the statement of the problem, the assumption was made that a child must be at more than a concrete level of reasoning before he would be successful when working the items. This statement was based on a discovery by Lunzer that verbal and numerical analogies demanded more than concrete reasoning. The empirical evidence given above indicates, however, that the above assumption is not valid for each item of each test. The reason, in the case of the fraction test, may be that in item 14 test NP, for example, a child could solve the problem merely by a visual inspection of the two circles involved. All he had to do was observe that the three shaded parts in the top circle fit into one space in the bottom circle. However, an item such as 10, test DP was not at all conducive to a solution by visual inspection. In this item, it was almost certain that a mathematical proportion had to be established, namely that $\frac{6}{15} = \frac{3}{x}$. The child who was unable to construct this proportion had little chance of success when solving the problem. The high ability sixth graders in both school systems scored quite well on subtest DPF, but for all other grades and ability groups this was not true, which seems to indicate that the high ability sixth graders were the only group who possessed the requisite cognitive structure to be successful when working with a proportionality situation as given in subtest DPF. It must be remembered that the fifth graders had received substantial instruction in ratio or fractions in each school system. These results seem to be very consistent with the facts that (1) Piaget sees formal thinking emerging at about 12 years of age—or at about the sixth grade level, (2) Lunzer observes a steep rise in scores on verbal and numerical analogies after the age of 10 and (3) that Lunzer sees formal reasoning being elaborated only at about the age of 11

No generalizable differences exist between the means of test NP and test DP in the case of ratios. Differences do exist, however, between the means of subtests DPF and DPR except for the high ability groups in Janesville. The differences that do exist are all in favor of subtest DPR. The lack of significance noted immediately above for the high ability groups seems to indicate that the presence in the curriculum of both ratio and fractions may have enhanced the performance of these children on subtest DPF. It must be noted here, however, that with the exception of the 6th graders, the high ability groups performed poorly on subtest DPR in both school systems. The reason subtest DPR was significantly easier than DPF may also be explained by the fact that the visuals involved were more conducive to a solution by inspection rather than a solution by constructing and solving an actual proportion. A second explanation, however, will be offered. It may be true that it is easier to teach children that $\frac{na}{nb}$ expresses the same ratio as $\frac{a}{b}$ than to teach them that $\frac{na}{nb}$ expresses the same measure as $\frac{a}{b}$ (e.g., the same amount of area of two subregions of two congruent regions).

Test S

1) There are no differences among the mean performances of the children in the fourth, fifth and sixth grades.
2) There are no differences among the mean performances of the children in the different ability groups.
3) There are no differences between the means of test NS and test DS.

Hypothesis one was rejected in the case of both school systems ($p < .01$). The means were, in percentages, 25, 62 and 78 for Madison (See Table 25) and 24, 79 and 87 for Janesville (See Table 23) for Grades 4, 5 and 6 respectively. Hypothesis two was rejected for the Janesville study ($p < .05$) and for the Madison study ($p < .01$). The means were, in percentages, 54, 66 and 70 for Janesville and 38, 55 and 72 for Madison for the low, middle and high groups, respectively. Hypothesis

Table 28

Item Difficulty Table

Basic Pair	Item No. Visual Reduction Factor	Janesville 4.1	Janesville 4.2	Janesville 5.1	Janesville 5.2	Janesville 6.1	Janesville 6.2	Madison 4.1	Madison 4.2	Madison 5.1	Madison 5.2	Madison 6.1	Madison 6.2
	4r2	55	59	77	69	77	80	63	63	57	77	80	80
1/5	15r3	59	52	83	66	77	87	63	60	53	73	80	77
	6F2	45	34	67	34	70	53	70	23	70	53	87	57
	14F3	62	45	73	59	70	53	63	37	57	53	80	63
Mean		55.25	47.50	75.00	57.00	73.50	68.25	64.75	45.75	59.25	64.00	81.75	69.25
	11r2	62	55	87	62	80	80	63	60	63	83	73	80
1/3	13r3	62	59	83	72	80	87	60	47	73	70	83	80
	3F2	52	34	67	41	70	60	67	20	73	43	83	67
	14F3	62	45	73	59	70	53	63	37	57	53	80	63
Mean		59.50	48.25	77.50	58.50	75.00	70.00	63.25	41.00	66.50	62.25	79.75	72.50
	9r2	66	49	83	59	77	93	60	57	50	80	67	80
2/5	8r3	48	10	67	38	63	57	33	23	30	33	50	43
	2F2	45	14	60	21	53	40	47	3	63	17	80	40
	16F3	38	7	53	17	57	30	40	3	63	20	63	37
Mean		49.25	20.00	65.75	33.75	62.50	55.00	45.00	21.50	51.50	32.50	65.00	50.00
	7r2	45	28	63	38	63	60	47	33	37	57	53	53
3/8	1r3	34	24	40	31	53	57	30	13	23	43	47	57
	10F2	48	17	70	14	60	40	63	10	57	30	83	30
	5F3	48	10	53	28	63	30	33	3	67	27	60	43
Mean		43.75	19.75	56.50	27.75	59.75	46.75	43.25	14.75	46.00	39.25	60.75	45.75

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three was not rejected for Janesville, but was for Madison ($p < .05$) in favor of the denominator test, where the means were 50 and 62 per cent for the numerator and denominator test, respectively. No interactions occurred in either analysis.

The above results indicate it may be possible to teach even low ability fifth graders (In Janesville, the mean score for low ability fifth graders was 64 per cent) to be fairly efficient in the case of the reduction of fractions on a symbolic level. No large gain was present, however, for the low ability groups in Janesville between the fifth and sixth grades. (In Janesville, the low ability sixth graders had a mean score of 72 per cent). For the low ability children in the fifth and sixth grades in Madison, a large gain in mean scores occurred. (The mean scores were 35 and 66 per cent for the fifth and sixth grade low ability groups respectively.)

It is not apparent why the numerator test was more difficult than the denominator test in Madison. One possible reason may be that in order to find the numerator, the children had to work with larger numbers (those of the two denominators) than when they found the denominator.

Tests P and S

1) The scores on Tests P and S are not correlated.

In view of the correlations presented in Tables 26 and 27, this hypothesis can not be rejected consistently for the subgroups considered. A low degree of relationship exists between the ratio subtest and symbolic test; the fraction subtest and symbolic test; and the total pictorials test and symbolic test. The fourth-grade children had lower mean scores on the symbolic test than they did on total pictorial test. (In Janesville, 24 vs. 44 per cent and in Madison 25 vs. 45 per cent.) In the fifth and sixth grades, however, just the reverse was true. For Janesville, the comparisons are 79 vs. 47 per cent and 88 vs. 64 per-cent for the fifth and sixth grades respectively and for Madison the comparisons are 62 vs. 54 per cent and 78 vs. 65 per cent for the fifth and sixth grades respectively.

These results indicate that it is questionable whether there is a high degree of relationship between the ability of a majority of fourth, fifth and sixth graders to work with proportionalities at a symbolic level and at a pictorial level.

V

SUMMARY AND IMPLICATIONS

The following summary of the study is warranted based on the statistical analyses and the discussion and conclusions pertaining to these analyses.

1. There is little correlation between the ability of children at the fourth, fifth and sixth grades to perform successfully in proportionality situations at a symbolic level, such as $\frac{6}{15} = \frac{\square}{5}$, and their ability to perform successfully on proportionality situations based on ratio or fractional pictorial data.

2. Children solve many proportionalities presented to them in the form of pictorial data by visual inspection both in the case of ratio and fractional situations.

3. Whenever the pictorial data, which display the proportionalities, are not conducive to solution by visual inspection, the proportionalities become exceedingly difficult for fourth, fifth and sixth grade children to solve, except for the high ability sixth graders. The reason hypothesized for this difficulty is that they are unable to represent the data mathematically. In terms of Piagetian Developmental Psychology, many children below 11 or 12 years of age are not at a formal level of reasoning and thereby do not possess a proportionality schema.

4. For the denominator test, the proportionalities represented pictorially by a ratio situation were easier for the children to solve than the proportionalities represented pictorially by a fractional situation.

5. The children of high intelligence are much more adept at solving proportionalities for both a symbolic and pictorial representation than are children of low intelligence.

6. Subtest NPF was significantly easier than the subtest DPF for each ability group in each grade.

7. The fifth and sixth graders performed significantly better than the fourth graders on all tests and subtests involved.

Various implications are present for curriculum researchers or builders among which are the following:

1. Much more care must be taken in the fifth and sixth grades to develop a sequence of lessons which are designed to enhance children's ability to represent visual data mathematically in the case of ratio or fractions, indeed if that ability can be enhanced.

2. Special attention must be devoted to the lower ability children (40 to 50 per cent of the population) in the active development of a curriculum to insure that sufficient time is being spent on the conceptual development of ratio or fraction for these children.

Implications are also present for further research. Among these implications are the following:

1. Is it possible to construct a "readiness test" for the study of ratio and fractions in the elementary school? Such a test may have its foundation in the psychological theory of Piaget. The ratio or fraction denominator test may be a good starting point.

2. Many experiences are given to children in the elementary school in problem solving which involve ratio and fractions. There is no reason to believe that children are any more successful when solving these problems than they were on subtest DPF. Moreover, if such a test as described in (1) were developed, perhaps its validity could be established in part by assessing its relationship to problem solving abilities. Its validity could also be established in part by establishing its relationship to the ability to add, subtract, multiply or

divide fractions on a symbol level. Such a readiness test may be designed to encompass the ability of children to represent data mathematically.

3. If such a program of research as outlined in (1) and (2) were carried out and was successful, it would then be entirely possible to conduct further experimental research from an instructional point of view.

APPENDIX A: TEST DIRECTIONS AND TESTS

Test NP, Directions

1. Test Books (Face down). Don't turn over your paper until I tell you to do so.
2. This is a set of exercises that we want you to do your very best on. You may find some exercises you cannot complete. Make a cross on those exercises. But remember, do your very best to complete each exercise.
3. Don't turn any pages until I tell you to do so. Make sure to turn only one page at a time.
4. Don't talk when you are doing the exercises.
5. Turn your paper over.

Item 0. Look at the square in the top picture. How many equal parts are there ? (Pause) That's right, four. Two of the four equal parts are shaded. Now look at the bottom picture. This square must have the same amount shaded. Let's read the question.

> How many of the two equal parts should be shaded so that the same amount is shaded in both squares ? The squares are unit squares.

You may do any work you wish on the page. Place your answers in the blank. If you aren't finished when I say turn the page, raise your hand and I will give you more time.

Item 1. Look at the top picture. There are nine squares for twenty-four triangles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

> If there are nine squares for every twenty-four triangles, how many squares would there be for these eight triangles ?

Item 2. Look at the square in the top picture. Four of the ten equal parts are shaded. (Repeat) (Pause) Now look at the bottom picture. This square must have the same amount shaded. (Pause) Now let's read the question.

> How many of the five equal parts should be shaded so that the same amount will be shaded in both squares ? The squares are unit squares.

Item 3. Look at the circle in the top picture. Two of the six equal parts are shaded. (Repeat). (Pause) Now look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> How many of the three equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 4. Look at the top picture. There are two triangles for ten squares. (Pause) Look at the bottom picture. Now let's read the question.

> If there are two triangles for every ten squares, how many triangles would there be for these five squares ?

Item 5. Look at the circle in the top picture. Nine of the twenty-four equal parts are shaded. (Repeat) (Pause) Now look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> How many of the eight equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 6. Look at the circle in the top picture. Two of the ten equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> How many of the five equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 7. Look at the top picture. There are six squares for sixteen triangles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are six squares for every sixteen triangles how many squares would there be for these eight triangles ?

Item 8. Look at the top picture. There are six triangles for fifteen circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are six triangles for every fifteen circles, how many triangles would there be for these five circles ?

Item 9. Look at the top picture. There are four triangles for ten circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are four triangles for every ten circles, how many triangles would there be for these five circles ?

Item 10. Look at the circle in the top picture. Six of the sixteen equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

How many of the eight equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 11. Look at the top picture. There are two squares for six circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are two squares for every six circles, how many squares would there be for these three circles ?

Item 12. Look at the circle in the top picture. Three of the nine equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

How many of the three equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 13. Look at the top picture. There are three squares for nine circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are three squares for every nine circles, how many squares would there be for these three circles ?

Item 14. Look at the circle in the top picture. Three of the fifteen equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

How many of the five equal parts should be shaded so that the same amount will be shaded in both circles ? The circles are unit circles.

Item 15. Look at the top picture. There are three triangles for fifteen squares. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are three triangles for every fifteen squares, how many triangles would there be for these five squares ?

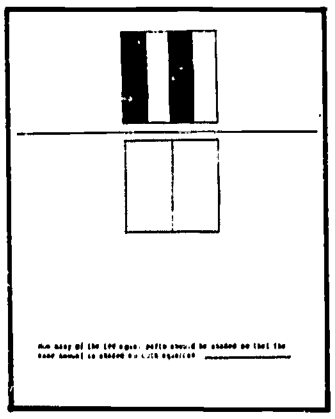
Item 16. Look at the square in the top picture. Six of the fifteen equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This square must have the same amount shaded. (Pause) Now let's read the question.

How many of the five equal parts should be shaded so that the same amount will be shaded in both squares ? The squares are unit squares.

Test NS, Directions

Let's read the directions at the top of the page. In each sentence, find the number that goes in the box. (Repeat) When you find it, place it in the box.

Test NP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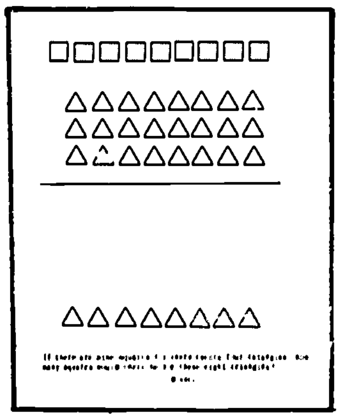
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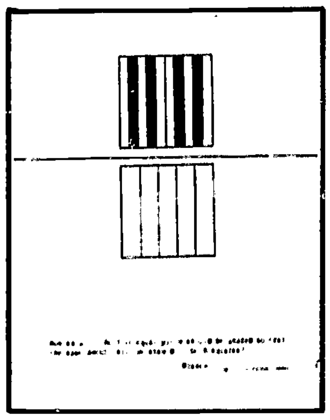
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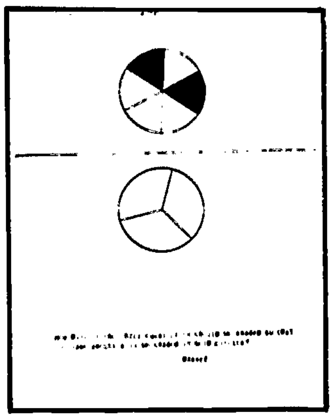
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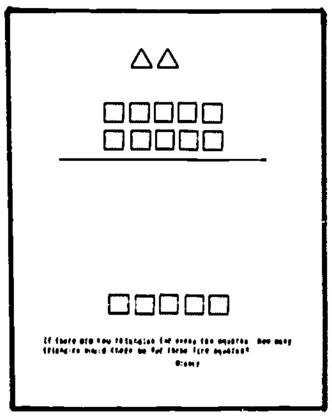
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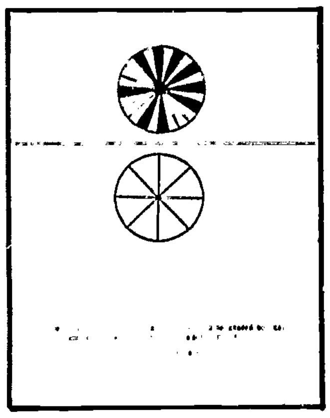
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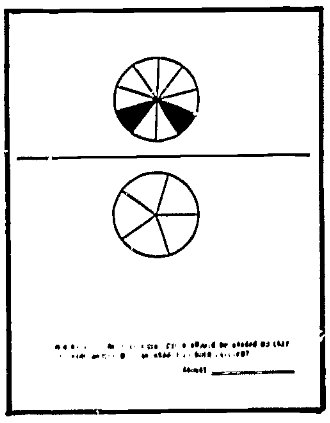
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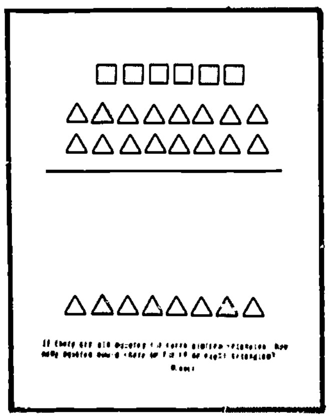
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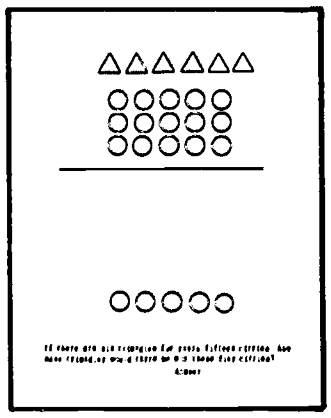
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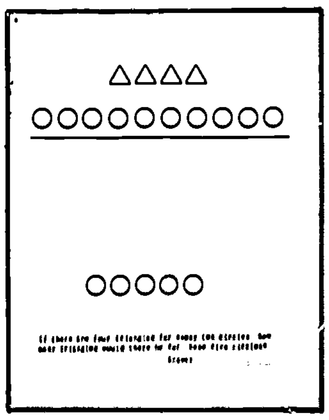
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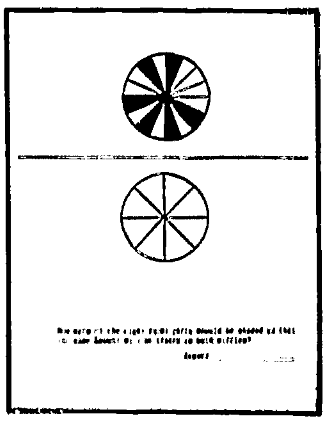
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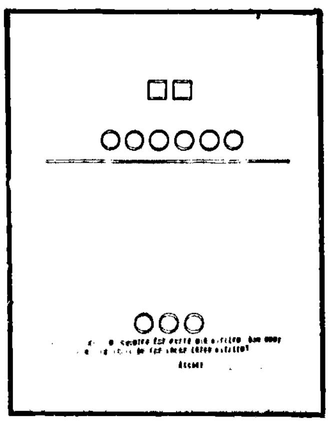
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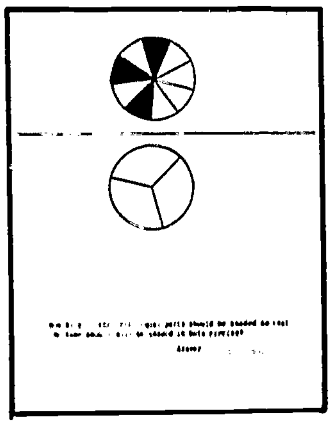

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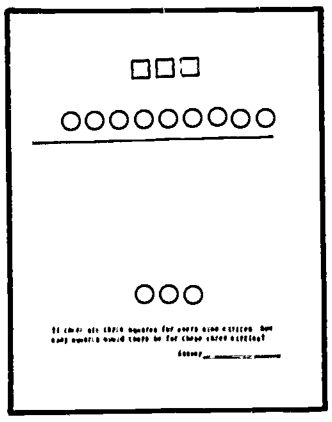

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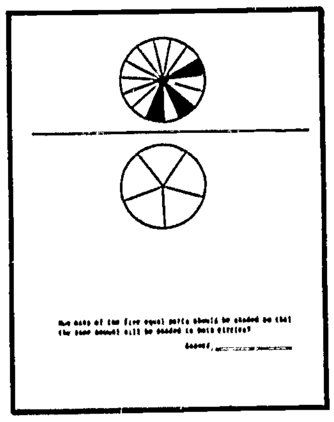

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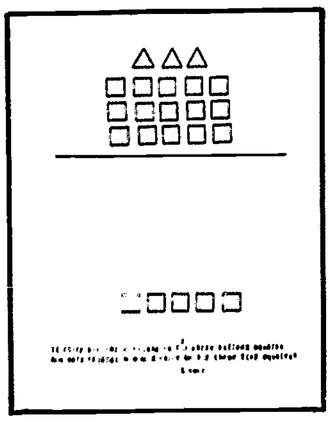

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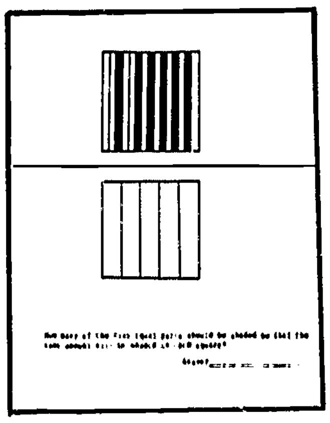

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Test NS

In each sentence, find the number that goes in the box.

1. $\frac{4}{10} = \frac{\square}{5}$

2. $\frac{3}{15} = \frac{\square}{5}$

3. $\frac{9}{24} = \frac{\square}{8}$

4. $\frac{6}{15} = \frac{\square}{5}$

5. $\frac{2}{6} = \frac{\square}{3}$

6. $\frac{6}{16} = \frac{\square}{8}$

7. $\frac{2}{10} = \frac{\square}{5}$

8. $\frac{3}{9} = \frac{\square}{3}$

Test DP, Directions

1. Test Books (Face down). Don't turn over your paper until I tell you to do so.
2. This is a set of exercises that we want you to do your very best on. You may find some exercises you cannot complete. Make a cross on those exercises. But remember, do your very best to complete each exercise.
3. Don't turn any pages until I tell you to do so. Make sure to turn only one page at a time.

Item 0. Look at the square in the top picture. How many equal parts are there? (Pause) That's right, four. Two of the four equal parts are shaded. Now look at the bottom picture. This square must have the same amount shaded. Now let's read the question.

> Into how many equal parts would you have to cut this square so that if you shade one of these equal parts, the same amount will be shaded in both squares? The squares are unit squares.

You may do any work you wish on the page. Place your answer in the blank. If you aren't finished when I say turn the page, raise your hand and I will give you more time.

Item 1. Look at the top picture. There are nine squares for twenty-four triangles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

> If there are nine squares for every twenty-four triangles, for these three squares there would be how many triangles?

Item 2. Look at the square in the top picture. Four of the ten equal parts are shaded. (Repeat) (Pause) Now look at the bottom picture. This square must have the same amount shaded. (Pause) Now let's read the question.

> Into how many equal parts would you have to cut this square so that if you shade two of these equal parts, the same amount will be shaded in both squares? The squares are unit squares.

Item 3. Look at the circle in the top picture. Two of the six equal parts are shaded. (Repeat) (Pause) Now look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> Into how many equal parts would you have to cut this circle so that if you shade one of these equal parts, the same amount will be shaded in both circles? The circles are unit circles.

Item 4. Look at the top picture. There are two triangles for ten squares. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

> If there are two triangles for every ten squares, for this one triangle there would be how many squares?

Item 5. Look at the circle in the top picture. Nine of the twenty-four equal parts are shaded. (Repeat) (Pause) Now look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> Into how many equal parts would you have to cut this circle so that if you shade three of these equal parts, the same amount will be shaded in both circles? The circles are unit circles.

Item 6. Look at the circle in the top picture. Two of the ten equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

> Into how many equal parts would you have to cut this circle so that if you shade one of these equal parts, the same amount will be shaded in both circles? The circles are unit circles.

Item 7. Look at the top picture. There are six squares for sixteen triangles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

> If there are six squares for every sixteen triangles, for these three squares there would be how many triangles?

Item 8. Look at the top picture. There are six triangles for fifteen circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

> If there are six triangles for every fifteen circles, for these two triangles there will be how many circles?

Item 9. Look at the top picture. There are four triangles for ten circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

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If there are four triangles for every ten circles, for these two triangles there would be how many circles ?

Item 10. Look at the circle in the top picture. Six of the sixteen equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

Into how many equal parts would you have to cut this circle so that if you shade three of these equal parts, the same amount will be shaded in both circles ? The circles are unit circles.

Item 11. Look at the top picture. There are two squares for six circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are two squares for every six circles, for this one square there would be how many circles ?

Item 12. Look at the circle in the top picture. Three of the nine equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

Into how many equal parts would you have to cut this circle so that if you shade one of these equal parts, the same amount will be shaded in both circles ? The circles are unit circles.

Item 13. Look at the top picture. There are three squares for nine circles. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are three squares for every nine circles, for this one square there would be how many circles ?

Item 14. Look at the circle in the top picture. Three of the fifteen equal parts are shaded. (Repeat) (Pause) Look at the bottom picture. This circle must have the same amount shaded. (Pause) Now let's read the question.

Into how many equal parts would you have to cut this circle so that if you shade one of these equal parts, the same amount will be shaded in both circles ? The circles are unit circles.

Item 15. Look at the top picture. There are three triangles for fifteen squares. (Repeat) (Pause) Look at the bottom picture. Now let's read the question.

If there are three triangles for every fifteen squares, for this one triangle there would be how many squares ?

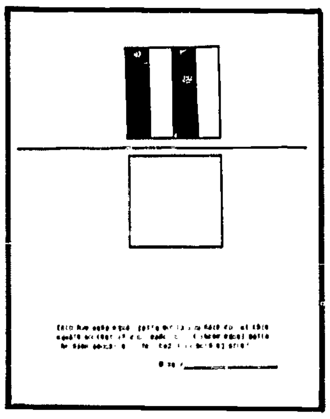
Item 16. Look at the square in the top picture. Six of the fifteen equal parts are shaded. (Repeat) (Pause) Let's read the question.

Into how many equal parts would you have to cut this square so that if you shade two of these equal parts, the same amount will be shaded in both squares ? The squares are unit squares.

Test DS, Directions

Let's read the directions at the top of the page. In each sentence, find the number that goes in the box. (Repeat) When you find it, place it in the box.

Test DP



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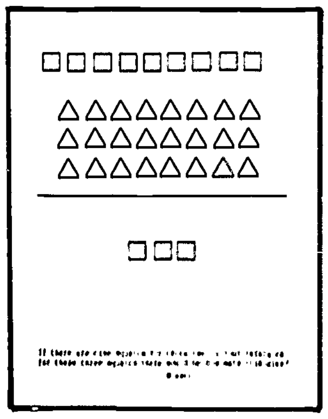

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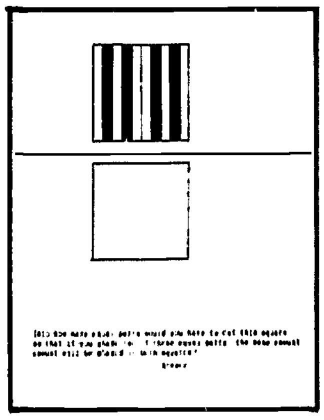

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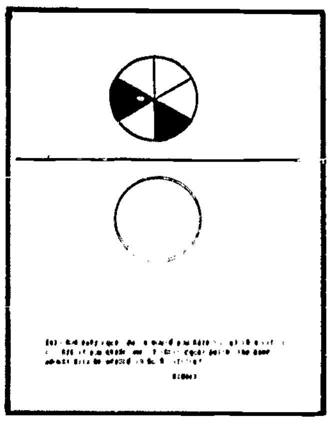

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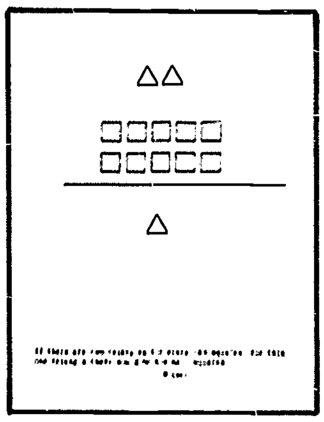

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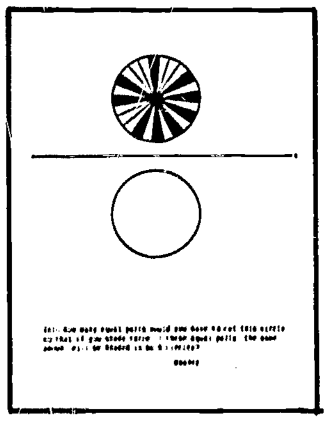

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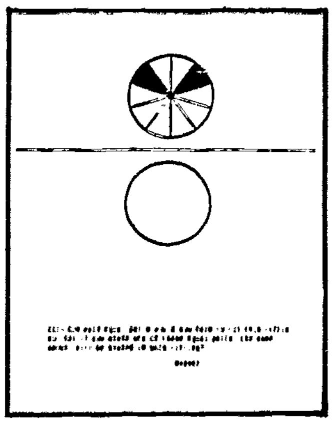

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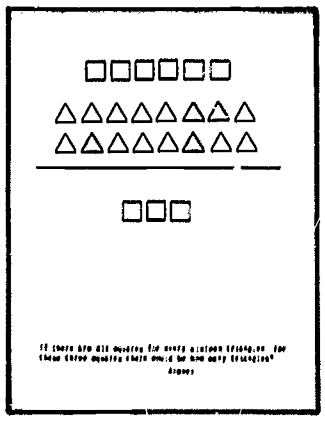

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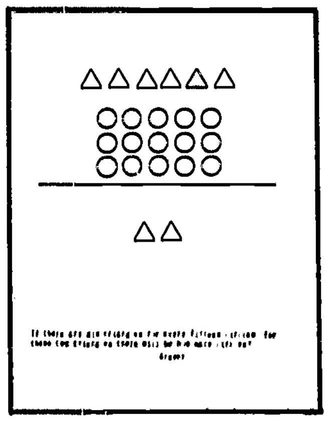

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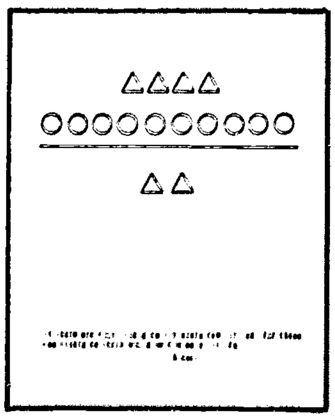

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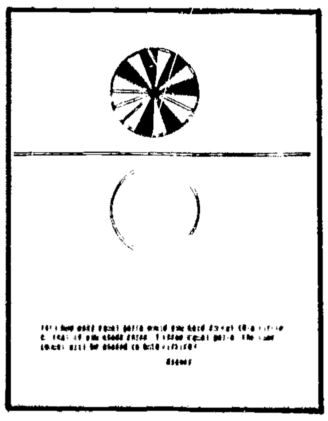

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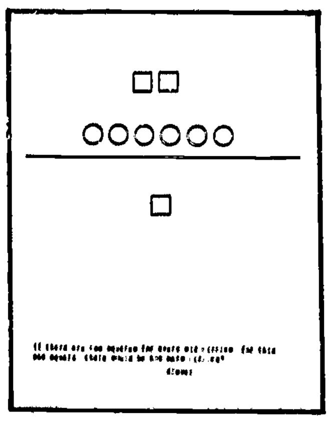

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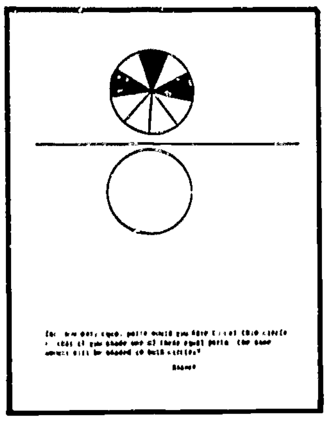

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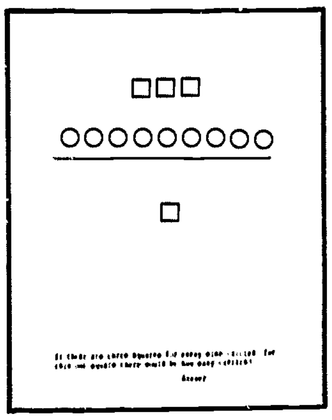

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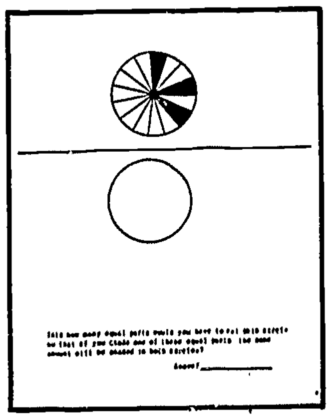

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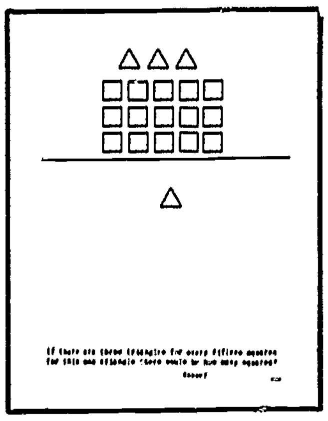

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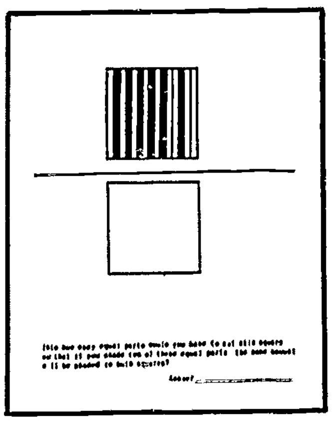

16

Test DS

In each sentence, find the number that goes in the box.

1. $\frac{4}{10} = \frac{2}{\square}$

2. $\frac{3}{15} = \frac{1}{\square}$

3. $\frac{6}{15} = \frac{2}{\square}$

4. $\frac{6}{16} = \frac{3}{\square}$

5. $\frac{2}{6} = \frac{1}{\square}$

6. $\frac{2}{10} = \frac{1}{\square}$

7. $\frac{9}{24} = \frac{3}{\square}$

8. $\frac{3}{9} = \frac{1}{\square}$

APPENDIX B: RAW DATA

JANESVILLE: TEST N, GRADE 6

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	0	1	1	0	1	1	0	1	0	1	0	1	0	1	0	0	1	1	1	1	1	1	1	1
H	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	6	0	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	8	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	9	0	0	1	1	1	1	1	1	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	10	1	0	1	1	0	1	1	0	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	11	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	12	1	1	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
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	17	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	18	1	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	1	1	1
	19	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	20	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	21	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1
L	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	24	1	0	0	1	1	0	0	0	0	0	1	0	1	1	0	1	0	0	0	0	0	0	0	0
	25	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	26	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1
	27	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1	0	1	0	0

JANESVILLE: TEST N, GRADE 5

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	28	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	29	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	30	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	31	1	0	0	1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	1	1	0	1	1	1
H	32	0	0	0	1	0	0	1	1	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1
	33	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	34	0	1	1	0	1	1	0	0	0	1	0	1	0	1	0	0	1	1	1	1	1	1	1	1
	35	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	36	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

	37	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1
	38	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
	39	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	40	0	0	0	1	0	0	0	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1
M	41	1	0	0	1	0	0	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	42	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	43	1	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	0
	44	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	1	0	1	1	1	1
	45	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	46	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1
	47	0	0	1	0	0	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	0	1	0	0
	48	0	0	0	1	0	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	49	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
L	50	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	51	0	1	0	0	0	0	0	0	1	0	1	0	1	1	1	1	0	0	1	0	0	0	0	0
	52	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
	53	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0
	54	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1

JANESVILLE: TEST N, GRADE 4

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	55	0	1	1	1	1	0	1	1	1	1	1	0	1	0	1	0	1	0	1	0	1	0	0	0
	56	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	57	1	0	1	0	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	58	0	1	0	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0
H	59	0	0	0	1	0	0	1	0	1	0	1	1	1	1	1	0	0	1	0	0	0	0	0	0
	60	0	0	1	1	1	1	0	0	1	1	1	1	1	1	1	1	0	0	1	0	0	0	0	0
	61	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	62	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	0	0	0
	63	0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	64	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	65	1	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	66	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0	0
	67	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
M	68	1	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0
	69	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	70	0	1	1	1	1	1	0	0	1	1	0	1	0	1	0	1	0	0	1	0	0	0	0	0
	71	0	1	1	1	0	0	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	72	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0
	73	0	0	1	1	0	1	1	0	1	0	1	1	1	1	1	1	1	0	1	0	0	0	0	0
	74	1	1	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
	75	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1
	76	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0
L	77	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	1	1	1	0	1	1	1	1
	78	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0
	79	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0
	80	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
	81	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

JANESVILLE: TEST D, GRADE 6

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	1	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
	4	1	1	1	1	0	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0
H	5	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
	6	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

	7	0	1	1	1	1	1	0	0	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1
	8	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
	9	0	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1
	10	0	0	1	1	0	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	11	1	0	0	1	1	0	1	1	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
	12	0	1	1	1	0	1	0	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	13	1	0	0	1	0	0	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1
M	14	0	0	0	1	0	0	0	0	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	15	1	1	1	1	0	1	1	0	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	16	1	0	0	1	0	0	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	17	1	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	1	1	1
	18	0	0	0	1	0	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	19	0	0	0	1	0	1	0	0	1	0	0	0	1	1	1	0	1	1	0	1	1	1	1	1
	20	1	0	1	1	0	1	0	0	1	0	0	1	0	0	1	0	1	1	0	1	1	1	1	1
	21	0	1	1	1	0	1	1	0	1	0	0	1	1	0	1	1	1	0	0	1	0	0	0	0
	22	1	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	1	1	0	1	1	1	1
L	23	0	0	0	1	0	0	0	0	1	0	1	1	0	1	0	0	1	1	1	1	1	1	1	1
	24	1	0	0	1	0	0	1	0	1	1	1	0	1	0	0	1	1	0	0	1	0	0	1	0
	25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1
	26	0	0	0	0	0	0	0	1	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	1
	27	1	0	1	0	0	0	1	1	1	0	0	1	0	1	1	0	1	1	1	1	1	1	1	1

JANESVILLE: TEST D, GRADE 5

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	28	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	29	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1
	30	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	31	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
H	32	0	0	1	1	0	0	0	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	33	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	34	1	0	1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	1	1	1	1	1	1	1
	35	0	1	1	1	1	0	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	36	0	0	1	1	0	1	0	0	1	0	1	0	0	1	0	1	1	1	1	1	1	1	1	1
	37	1	0	0	1	1	0	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	38	0	0	0	1	0	0	0	0	0	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
	39	0	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	0	1	1	0
	40	0	1	0	1	1	0	0	0	0	0	0	0	1	0	1	0	1	1	0	0	1	1	1	0
M	41	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	42	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	43	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	44	0	0	1	0	0	1	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1
	46	0	0	0	1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	C	1	0	0	1
	47	0	0	0	1	0	0	0	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
	49	0	0	0	1	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
L	50	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	51	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	1	1	1	0	0	1
	52	0	0	0	0	0	0	0	0	1	0	1	0	1	1	0	0	1	1	1	1	1	1	1	1
	53	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1	0	1	1	1
	54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1

JANESVILLE: TEST D, GRADE 4

Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
55	1	0	1	1	0	1	0	0	0	0	0	1	1	1	1	0	0	1	0	0	1	1	0	1
56	0	0	1	1	1	0	1	1	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
57	1	0	0	1	0	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
58	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

H	59	0	0	1	1	0	1	0	0	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0
	60	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	61	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0
	62	0	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0
	63	1	1	1	0	0	1	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0
	64	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	65	0	0	0	1	0	0	1	1	1	0	1	0	1	1	1	0	0	1	0	0	0	1	0	0
	66	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1
	67	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
M	68	0	0	0	1	0	1	0	0	0	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0
	69	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
	70	1	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
	71	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	1	0	0
	72	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
	73	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
	74	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	75	0	1	1	1	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
	76	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
L	77	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1
	78	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
	79	0	0	0	1	0	0	1	0	1	1	1	1	1	0	1	0	1	1	0	0	1	1	0	1
	80	0	1	1	0	0	1	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	81	0	0	0	1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0

MADISON: TEST N, GRADE 6

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H	5	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	6	0	1	1	1	0	1	0	0	1	1	0	0	0	0	0	0	1	0	1	0	1	1	1	0
	7	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	8	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	9	0	0	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	10	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	11	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	12	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
	13	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1
	14	0	0	1	1	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
M	15	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	0	1	1
	16	0	0	1	0	0	1	0	0	0	1	0	1	1	1	1	1	1	0	1	0	0	0	0	0
	17	0	1	1	1	0	1	0	0	0	1	0	1	1	1	0	1	1	0	0	0	1	1	1	0
	18	1	1	1	1	1	1	0	0	0	1	0	1	1	1	0	1	1	1	0	0	1	0	1	1
	19	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	20	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	21	0	1	1	1	0	1	0	0	0	1	1	0	1	0	1	0	1	0	0	1	0	0	0	0
	22	0	1	1	1	0	1	0	0	0	1	1	0	0	1	1	0	1	1	1	1	1	1	1	1
	23	1	0	0	0	1	1	0	0	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1
	24	0	1	1	1	0	0	0	0	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	0
L	25	0	1	0	1	0	1	0	1	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1
	26	0	0	1	0	0	1	0	0	0	0	1	1	1	1	1	1	1	0	1	0	0	0	0	0
	27	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	28	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1	1	1	0	1	1	1	1
	29	1	1	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	0	0
	30	1	1	0	1	0	0	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0

MADISON: TEST N, GRADE 5

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	31	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
	32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
	33	0	0	1	1	0	0	0	0	0	0	1	1	1	0	1	1	1	0	1	0	1	0	1	1
	34	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	0	1	0	1	0
H	35	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	36	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1
	37	0	1	1	0	1	1	0	0	0	1	0	0	1	1	0	1	1	0	0	0	1	1	1	0
	38	0	1	1	1	1	1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	1	1	1	0
	39	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1
	40	0	0	1	1	0	1	1	0	1	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1
	41	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0
	42	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	43	0	1	1	1	1	1	0	1	0	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0
	44	0	1	1	0	0	1	0	0	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1
M	45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
	46	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	1
	47	0	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0
	48	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	49	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0
	50	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	0
	51	0	1	1	0	1	1	0	0	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0
	52	0	1	1	0	1	1	0	0	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1
	53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
	54	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	0	0	0	0
L	55	1	0	1	1	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
	56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	57	0	1	1	0	1	1	0	0	0	0	1	1	1	1	0	1	0	0	0	0	0	0	0	0
	58	0	0	1	0	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	59	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
	60	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	0	1	0	0	0	0

MADISON: TEST N, GRADE 4

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	61	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	62	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	63	0	0	1	1	0	1	0	1	1	1	1	1	1	1	1	0	1	1	0	0	1	0	1	1
	64	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
H	65	0	1	1	1	0	1	0	0	0	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0
	66	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	0	1	0	1	1
	67	0	0	0	1	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	68	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	69	0	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	0	1	0	0	0
	70	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	71	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	72	1	1	1	1	0	1	0	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
	73	1	1	1	1	0	0	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0
	74	1	1	1	1	0	1	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
M	75	0	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	1	0	1	0	0	0	0	0
	76	0	1	1	0	0	1	0	0	0	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0
	77	0	0	1	0	0	0	0	0	1	0	1	1	1	1	0	1	0	0	0	0	0	0	0	0
	78	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	79	0	0	1	0	0	1	0	0	0	0	0	1	1	1	0	1	1	0	0	1	0	0	0	0
	80	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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	81	0	1	1	1	0	1	1	1	0	1	1	0	1	1	1	0	1	1	1	0	1	1	0	0
	82	0	0	0	1	0	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	83	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	84	1	1	1	1	1	1	1	1	1	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0
L	85	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	86	1	0	0	0	1	1	0	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
	87	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	88	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	89	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	90	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

MADISON: TEST D, GRADE 6

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H	5	0	0	1	1	0	0	0	0	0	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	6	1	0	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	8	1	1	1	1	1	1	0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	9	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	11	0	0	1	1	0	0	0	0	0	0	1	0	1	1	0	0	1	1	0	1	1	1	1	1
	12	0	0	1	1	0	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	13	1	1	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	14	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1	1	0	1	1	1	1	1
M	15	0	0	0	1	0	1	1	1	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1
	16	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	1	1	1	1	1	1	1
	17	0	0	1	1	0	0	0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	18	1	0	0	1	0	0	1	1	1	0	1	0	1	0	1	0	1	1	0	1	1	1	1	1
	19	1	0	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	20	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	21	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	22	1	0	0	1	0	1	0	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
	23	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	1	1	0	1	1	1	0	1	1
	24	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L	25	1	0	0	1	0	0	0	0	0	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	26	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
	27	0	1	1	1	0	0	0	0	1	0	1	1	1	0	0	0	0	1	1	1	0	1	1	0
	28	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	1	1	1	0	1
	29	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1
	30	0	1	1	1	1	1	0	0	1	0	1	0	1	1	1	0	1	0	0	0	1	1	0	0

MADISON: TEST D, GRADE 5

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	31	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	33	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	34	1	0	0	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	0	1	1	1	1	1
H	35	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	36	0	0	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	37	1	0	0	1	0	0	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1
	38	0	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	1	1	1	0	0
	39	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	40	0	0	1	1	0	1	0	0	1	0	1	1	1	0	0	0	1	0	0	1	1	1	0	0

	41	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	0	1	1	0	1	1	1	0	1
	42	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	1	1	1	1
	43	1	0	0	1	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	44	0	0	1	1	0	1	1	1	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0
M	45	0	0	1	1	0	1	1	0	1	0	1	0	1	1	1	0	1	1	0	0	1	1	1	1
	46	0	0	0	1	0	0	0	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	0	1
	47	0	0	1	1	0	1	0	0	1	0	1	1	1	1	1	0	1	0	0	1	1	1	1	1
	48	1	0	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
	49	0	0	0	1	0	0	0	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	50	1	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	1	0	0	1	1	0	1
	51	0	0	0	1	0	1	0	0	1	0	1	0	1	0	1	0	0	0	0	1	0	0	0	0
	52	0	0	1	0	0	0	0	0	1	1	1	0	0	1	1	0	1	0	0	1	1	1	0	0
	53	0	0	0	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	54	1	1	1	1	0	1	1	1	1	1	1	0	1	0	1	0	1	0	0	1	1	1	1	0
L	55	1	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	0	1	0	0	1	1
	56	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	57	1	0	0	1	0	0	1	0	0	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1
	58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	59	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	60	0	0	0	1	0	1	1	0	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0

MADISON: TEST D, GRADE 4

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	61	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	62	0	0	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
	63	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0
	64	0	0	0	1	0	0	0	0	1	0	1	0	1	0	1	0	1	0	0	1	1	1	0	0
H	65	0	0	0	1	0	0	0	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0
	66	0	0	1	1	0	1	1	0	0	0	1	1	1	1	1	0	1	1	1	0	1	1	0	1
	67	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1
	68	0	0	0	0	0	0	0	1	1	0	1	0	1	1	0	0	0	0	1	0	0	0	0	0
	69	0	0	0	1	0	1	0	0	1	0	0	1	0	0	1	1	0	0	1	0	1	1	0	1
	70	0	0	0	1	0	0	1	1	0	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0
	71	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	1	0	1
	72	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	1
	73	1	0	0	0	0	0	0	0	1	0	1	1	1	1	1	0	0	1	0	0	1	1	0	1
	74	0	0	0	1	0	1	0	1	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	1
M	75	1	0	0	0	0	0	0	0	1	0	0	1	0	1	1	0	1	0	0	0	0	0	0	0
	76	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
	77	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1
	78	0	0	0	1	0	0	1	1	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0
	79	0	0	1	1	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0
	80	0	0	0	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0
	81	0	0	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
	82	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	83	0	0	0	1	0	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	84	0	0	0	1	0	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	1	1	1	1
L	85	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
	86	0	0	0	1	0	0	1	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1
	87	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
	88	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	89	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1	1	0	0	0	1	0	1
	90	0	0	0	1	0	0	1	0	1	0	1	1	1	0	1	0	1	1	0	0	1	1	0	1

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