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The materials in this bulletin consist of a series of daily lesson plans for use by teachers in presenting a modern program of eighth year mathematics. There is an emphasis on (1) an understanding of mathematical structure, (2) growth of a number system, (3) relations and operations in a number system, (4) a development of mathematical skills based on an understanding of mathematical principles, and (5) concept of set in number and in geometry. Classroom materials are developed on such mathematical concepts as measurement, triangles and quadrilaterals, square and cubic measure, systems of numeration, and the set of integers. (RP)

MATHEMATICS

8th YEAR

PART 1

Bureau of Curriculum Development
Board of Education • City of New York

NOTE TO THE TEACHER

Mathematics: 8th Year is presented in two parts of which this is the first. It contains Chapters I through V.

The materials in this bulletin are an extension of the program contained in *Mathematics: 7th Year.*

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MATHEMATICS

8 th YEAR

PART 1

BUREAU OF CURRICULUM DEVELOPMENT
BOARD OF EDUCATION OF THE CITY OF NEW YORK

FOREWORD

Because of the increasing dependence of our society upon trained mathematical manpower, it is essential that vital and contemporary mathematics be taught in our schools.

The mathematics program set forth in this publication has developed as a result of experimentation and evaluation in classroom situations. This is Part I of Mathematics: 8th Year. Part II, a separate bulletin, will be published during the school year 1968.

This bulletin represents a cooperative effort of the Bureau of Curriculum Development, the Bureau of Mathematics, and the Office of Junior High Schools.

We wish to thank the staff members who have so generously contributed to this work.

SEELIG L. LESTER
Deputy Superintendent of Schools
Curriculum and Instruction

January, 1968

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INTRODUCTION

This is the first part of a two part bulletin produced through the cooperative efforts of the Bureau of Curriculum Development, the Bureau of Mathematics, and the Office of Junior High Schools.

Mathematics: Grade 8 presents a modern program for pupils in the eighth grade. Emphasis is placed on:

- an understanding of mathematical structure
- growth of a number system
- relations and operations in a number system
- a development of mathematical skills based on an understanding of mathematical principles
- concept of set in number and in geometry

The program is presented in a series of suggested daily lesson plans for use by the teachers. These lesson plans are the culmination of two years of experimentation in various schools in each of the five boroughs of New York City. They reflect the experiences gained in actual classroom situations and the continued evaluation by teachers and supervisors.

ORGANIZATION OF THIS BULLETIN

The material in this bulletin is arranged in the sequence in which it is to be used. It is expected that each lesson will be presented before the next numbered lesson, and that each chapter will be presented before any work in the ensuing chapter is begun. Although this may seem to be a departure from the cyclical approach, concepts and skills are developed on progressively higher levels throughout the year.

Various topics for enrichment have been included. Labeled optional, they have been placed with the topics of which they are a logical outgrowth.

SUGGESTED PROCEDURE FOR USING THIS BULLETIN

It is suggested that the following procedure be considered in using this publication:

> Before making plans to teach any part of it, the teacher should become acquainted with the content and spirit of Mathematics: 8th Year, and with the relationships among the topics in the course.

Study the introductory discussion in each chapter you plan to present. Note the relationship of each lesson to the one preceding it, and the one following it. Each lesson is organized in terms of:

1. Topic	4. Procedure
2. Aim	5. Practice
3. Specific Objectives	6. Summary Questions

Amplify the practice material suggested for each lesson with additional material from suitable textbooks.

Practice in computation and in the solution of verbal problems should not be confined to the sections in which this work appears in the bulletin. Such work should be interspersed among other topics in order to sustain interest and provide for continuous development and reinforcement of computational skills and of problem-solving skills.

EVALUATION

An evaluation program includes not only the checking of completed work at intervals, but also continual appraisal. It is a general principle of evaluation that results are checked against objectives. The objectives of this course include the development of concepts, principles, and understandings, as well as skills.

Written tests are the most frequently used instrument for evaluation and rating. Test items should be designed to test not only recall of factual items, but also the ability of the pupil to make intelligent application of mathematical principles. Some other writing activities which teachers may use for the purpose of evaluation include written homework assignments, special reports, and short quizzes.

To evaluate pupil understanding continually, there are a number of oral activities which teachers may use such as:

pupil explanations of approaches used in new situations
pupil justification of statements
pupil restatement of problems in their own words
pupil explanation of interrelationship of ideas
pupil discovery of patterns
oral quizzes
pupil reports

Evaluation procedures also include teacher observation of pupil's work at chalkboard and of pupil's work at his desk.

Self-evaluation by pupils can be encouraged through short self-marking quizzes.

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The preparation of this bulletin was under the general supervision of Helene M. Lloyd, Assistant Superintendent, Office of the Superintendent of Schools; William H. Bristow, Assistant Superintendent, Bureau of Curriculum Development; Irving Anker, Assistant Superintendent, Office of Junior High Schools; David A. Abramson, Assistant Director, Bureau of Curriculum Development; and George Grossman, Acting Director, Bureau of Mathematics.

As Chairman of the Junior High School Mathematics Curriculum Committee, Paul Gastwirth, Principal of Edward Bleeker Junior High, coordinated the work of various committees engaged in the initial planning for this eighth year mathematics program.

Frank J. Wohlfort, Acting Assistant Director, Bureau of Mathematics, worked with the coordinators, arranged for the experimental tryout of the program in the junior high schools, and continued to assist in the preparation of these materials.

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Helen K. Halliday, project leader, and Charles J. Goode were the principal writers. Bertha O. Weiss, assisted in the planning, preparation, and editing of the materials for publication. Helen K. Halliday, Charles J. Goode, and Bertha O. Weiss, formerly staff coordinators of the Bureau of Mathematics, are presently staff members of the Bureau of Curriculum Development.

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CHAPTER I

LINEAR MEASUREMENT

The materials in this chapter are concerned with extending the pupil's understanding of measurement. Among the topics developed are:

- nature of measurement
- greatest possible error
- measurement and precision
- relative error and accuracy
- significant digits
- scientific notation

Pupils are reminded that in counting there is a one-to-one correspondence between the set of objects counted and the set of counting numbers. In measurement, a comparison is made between that which is to be measured and a suitable standard unit of measure. A distinction is made between the use of the words measurement and measure. The measurement of an object involves both a number and a unit, while the measure involves only a number. For example, if the measurement is 3 feet, the measure is 3.

The concept of the "greatest possible error," as one-half the unit of measure used, is recalled and reinforced. Pupils are then led to realize that the smaller the unit of measure employed, the greater the precision of measurement. Through a series of practical problems, it is shown that the precision to be used in making a measurement is related to the purpose of the measurement.

The meaning of relative error as the ratio between the greatest possible error and the stated measurement is developed. Pupils are then guided to the realization that the most accurate measurement is the one which has the least relative error. The unique meaning of each of the words - error, precision, and accuracy - when applied to measurement in mathematics is stressed throughout.

The concept of significant digits is explored and a method for determining the number of significant digits in a measurement is developed. Pupils are led to discover the relationship which exists between the number of significant digits and the accuracy of a measurement.

Because of the need created by the frequent use of large numbers in science, the pupils are helped to develop skill in expressing numbers in scientific notation. In addition, pupils learn to write the standard numeral for numbers expressed in scientific notation.

Throughout the work of this chapter ample opportunity is provided for review of decimals, per cent, and both English and metric units of measure.

CHAPTER I

LINEAR MEASUREMENT

Lessons 1-8

Lesson 1

Topic: Measurement

Aim: To reinforce the understanding that all measurements are approximate

Specific Objectives:

To reinforce the understanding that in counting the result is exact
To reinforce the understanding that in measuring the result is approximate

Challenge: There are 6 paperbacks displayed on a 6 inch shelf. Which of the numerals 6 represents a result which is exact? Explain.

I. Procedure

A. To review that in counting the result is exact

1. Have several pupils state the number of pupils sitting in the first row; the number of pencils on a given desk; the number of pieces of chalk on the chalk ledge, etc.

2. Elicit that in each case counting was used to determine the number of the set.

3. Have pupils recall that in counting we set up a one-to-one correspondence between the elements of the set of objects counted and a subset of the natural numbers. In counting, the result is said to be exact.

4. Answer the challenge. Elicit that the number of paperbacks was obtained by counting, and therefore the first 6 represents an exact result.

B. Measurement is always approximate

1. In the challenge question, how was the second 6 obtained? (by measurement)

2. Elicit that measurement is different from counting. In measurement, the length of an object is compared with a standard unit of measure. In the challenge question, what standard unit of measurement is used?

3. Have pupils recall the greatest possible difference between the actual length of an object and the measured length of the object is one-half the unit used. This difference of one-half the unit used is called the greatest possible error. (Remind pupils that "error" in this case does not mean a mistake in measuring.) What is the greatest possible error in the challenge example? (one-half of one inch or $\frac{1}{2}$ inch)

   Could the length of the shelf be less than $5\frac{1}{2}$ in.? more than $6\frac{1}{2}$ in.? Explain.

4. Suppose the length of another shelf is given as $5\frac{1}{2}$ inches, using $\frac{1}{2}$ inch as the standard unit of measure. What is the greatest possible error in this case? ($\frac{1}{2}$ of one-half inch or $\frac{1}{4}$ inch) Between which two measures does the exact length of this shelf lie? (The length of the shelf is greater than $5\frac{1}{4}$ inches, but less than $5\frac{3}{4}$ inches.)

5. Have pupils realize that if we used smaller units to measure the shelf, such as $\frac{1}{8}$", $\frac{1}{16}$", $\frac{1}{32}$", etc., it still would not be possible to determine the exact length of the shelf. Since this is true, we say all measurements are approximate.

II. Practice

A. In which of the following are numbers used to show exact results? Explain your answers.

1. The length of the room is 74 feet.
2. The box contains 36 pencils.
3. There are 100 papers in the package.
4. The distance the boys ran was 100 meters.
5. The Browns gained 232 yds. and scored 4 touchdowns.

B. What is the greatest possible error for each of the following measurements?

1. 3 inches   2. $4\frac{1}{4}$ inches   3. 8 centimeters

C. If a measurement is given as 5 ft., it must be greater than _____ and less than _____.

III. Summary

A. Explain what is meant by a "one-to-one correspondence" in counting.
B. Explain why measurement is said to be approximate.
C. Explain the meaning of the phrase "greatest possible error" in measurement.

Lesson 2

Topic: Precision in Measurement

Aim: To reinforce the understanding of precision in measurement

Specific Objectives:

To review the meaning of precision in measurement
To learn that the precision used often depends on the purpose of a measurement

Challenge: If you were stating the distance from London to New York, how precise would you want to be?

I. Procedure

A. Precision in measurement

1. Have pupils recall that as the unit of measure used becomes smaller and smaller, the difference between measured length and the actual length becomes less and less.

2. Have pupils recall that the unit used in measuring determines the precision of the measurement.

3. Consider that the following measurements of a line segment were carefully made:

3 inches to the nearest inch

$2\frac{1}{2}$ inches to the nearest $\frac{1}{2}$ inch

$2\frac{3}{4}$ inches to the nearest $\frac{1}{4}$ inch

$2\frac{5}{8}$ inches to the nearest $\frac{1}{8}$ inch

$2\frac{11}{16}$ inches to the nearest $\frac{1}{16}$ inch

a. To what unit of measure is the first measurement precise? (inch)
b. To what unit of measure is the last measurement precise? (sixteenth of inch)
c. To what unit of measure is the middle measurement precise? (quarter-inch)
d. How does the unit $\frac{1}{16}$ inch compare with the other four units? (smallest)
e. Which measurement is closest to the actual length? ($2\frac{11}{16}$)
f. Have pupils conclude that the smaller the unit, the more precise the measurement.

B. Precision related to the purpose of a measurement

1. Refer to challenge.
   Should your answer be given to the nearest inch? to the nearest foot? to the nearest mile? to the nearest hundred miles? to the nearest thousand miles.

2. Elicit that the unit of measure would depend upon the purpose for which the measurement was made. For example, a tourist might be satisfied to be precise to the nearest thousand miles, but a ship's navigator might have to be precise to the nearest mile. Why?

3. Tell pupils that the suitable unit of measure to use depends upon the greatest possible error that is acceptable. If a greatest possible error of $\frac{1}{2}$ ft. in measuring the length of a room is acceptable, what unit of measure would be suitable? (1 ft.)

   If a greatest possible error of 50 miles is acceptable in measuring the distance from New York to Chicago, what unit of measure would be suitable? (100 miles)

4. Help pupils conclude that the degree of precision usually depends upon the purpose for which a measurement is made.

II. Practice

A. Which of these measurements is most precise? Explain.

   9 inches $\qquad$ $9\frac{1}{4}$ inches $\qquad$ $9\frac{3}{8}$ inches

B. For each of the following suggest a suitable unit of measure and explain your answer:

   the length of a skirt
   the height of a tree
   the distance from the earth to the sun
   the length of a board for a shelf in a closet

III. Summary

A. In a given number of measurements, how would you decide which is the most precise?

B. How would you determine the degree of precision to be used in a measurement?

Lessons 3 and 4

Topic: Relative Error and Accuracy

Aim: To learn to determine the accuracy of a measurement

Specific Objectives:

To learn the meaning of relative error in measurement
To learn that the measure with the least relative error is the most accurate

Challenge: The distance from Baltimore to Washington is approximately 50 miles. The distance from Boston to Washington is approximately 500 miles. If the distance in each case is measured to the nearest 10 miles, which of these two measurements is more accurate?

I. Procedure

A. Meaning of relative error
Refer to challenge.

1. What unit of measure was used in each case?

2. To what precision were both measurements made?

3. What is the greatest possible error in each case? (5 miles) (Remind pupils again that "error" in this case does not mean a mistake in measurement.)

4. In which case is the error of 5 miles of greater significance? Tell pupils that one way of indicating the significance of an error in a measurement is to compare by means of a ratio the greatest possible error with the stated measurement. This ratio is called the relative error.

$$\text{Relative error} = \frac{\text{greatest possible error}}{\text{stated measurement}}$$

5. In the challenge problem, the relative errors may be computed as follows:

Baltimore to Washington - Relative error $= \frac{5}{50}$ or $\frac{1}{10}$

Boston to Washington - Relative error $= \frac{5}{500}$ or $\frac{1}{100}$

6. To make comparison easier, the relative error is often expressed as a per cent. For example, the relative error of $\frac{1}{10}$ may be expressed as a 10% relative error, the relative error of $\frac{1}{100}$ may be expressed as a 1% relative error.

B. Relative error and accuracy

1. Elicit that a 10% error is greater than a 1% error. Therefore, in the challenge problem, the 5 mile possible error in the 50 mile measurement is more significant than the 5 mile possible error in the 500 mile measurement.

2. Tell pupils that the most accurate of several measurements is the one which has the least relative error. The word "accurate" does not refer to the carefulness with which a measurement is made, nor to the precision of the measurement, but to the relative error.

II. Practice

A. Copy and complete this table.

Measurement	Unit Used	Greatest Possible Error
Distance, N.Y. to Miami	50 miles	?
Distance, earth to moon	10,000 miles	?
Length of runway	1 meter	?
Diameter of tubing	.5 cm	?
Length of a building	1 foot	?

B. Find the greatest possible error and the relative error for each of the following. Express the relative error as a per cent.

1. 50,000 miles (precise to nearest 1,000 miles)

2. .05 inch (precise to nearest .01 inch)

3. 50 millimeters (precise to nearest millimeter)

4. 500 meters (precise to nearest 10 meters)

5. 400 miles (precise to nearest 10 miles)

C. Which of these two measurements is the more accurate? Explain.

1. 5,000 feet measured to the nearest 100 feet

2. 5 feet measured to the nearest foot

III. Summary

A. Explain the difference between "the greatest possible error" and "the relative error" of a measurement.

B. Explain the difference between precision and accuracy of a measurement.

Lessons 5 and 6

Topic: Significant Digits

Aim: To introduce the concept of significant digits

Specific Objectives:

To develop the meaning of the term "significant digits"
To learn how to determine whether a digit is significant
To learn the relationship between significant digits and accuracy

Challenge: How many ten thousandths of an inch are there in 2.6875 inches?

I. Procedure

A. Significant digits

1. Have pupils recall that a measuring instrument may be calibrated in tenths of an inch, hundredths of an inch, etc.

2. Have pupils consider the following measurements:

2.5"   2.25"   2.6875"

What unit of measure was used in each case?

2.5" - tenths of an inch
2.25" - hundredths of an inch
2.6875" - ten thousandths of an inch

3. a. How many tenths of an inch are there in 2.5 inches? (25)

b. How many hundredths of an inch are there in 2.25 inches? (225)

4. Refer to challenge. How many ten thousandths of an inch are there in 2.6875 inches? (26875)

5. In example 3-a, the unit of measure is tenths of an inch, and there are 25 such units. Tell the pupils that the digit 2 and the digit 5 are <u>significant digits</u> because they tell how many times the unit of measure (tenths of an inch) is contained in the measurement 2.5 inches.

6. In example 3-b, what is the unit of measure? How many times is the unit of measure contained in the measurement 2.25 inches?

What then are the significant digits? (2, 2 and 5) Why? (Significant digits tell how many times the unit of measure is contained in the measurement.)

7. Name the significant digits in example 4.

B. Method for determining the number of significant digits

1. Non-zero digits are always significant. Elicit that in each example in section A the number of significant digits is the same as the number of digits in the example. Tell pupils that this is true because all non-zero digits are significant.

2. Zero digits may or may not be significant.

a. Consider the following measurements:

30.2 inches 3.02 inches 4.30 inches

What unit of measure was used in arriving at the measurement 30.2 inches? (.1 inch)

How many times was the unit contained in the measurement? (302)

Which digits are significant? (3, 0 and 2)

How many significant digits? (3)

In like manner, lead pupils to state that in the measurement 3.02 inches, the significant digits are 3, 0, and 2. How many significant digits?

Elicit that in the measurement 4.30 inches the zero indicates that the unit of measurement used was .01 inch. Since 430 units of .01 inch are contained in the measurement, the significant digits are 4, 3, <u>and 0</u>.

b. Consider the following measurements:

.05 inches .0012 inches .060 inches

What unit of measure was used in arriving at the measurement .05 inches? (.01 inch)

How many times was this unit used? (5)

What digit or digits are significant? (only 5 <u>not</u> the zero)

How many significant digits are there? (1)

What unit of measure was used in arriving at the measurement .0012 inches? (.0001 inch)

How many times was this unit used? (12)
Why are the zeros not significant?
How many significant digits are there in .0012? (2)

.060 What unit of measure was used?
How many times was the unit used? (60)
How many significant digits are there?
Which digits are significant? (6 and 0)

C. Significant digits and accuracy

1. Consider the measurements:

5000 feet to the nearest foot
5000 feet to the nearest thousand feet
5000 feet to the nearest hundred feet

a. What unit of measure was used in the first example? (1 foot)
How many such units were contained in the measurement?
Which digits are significant? (5 and the three zeros)

b. What unit of measure was used in the second example? (1000 ft.)
How many such units are contained in the measurement? (5)
Which digits are significant? (only 5)

c. What unit of measure was used in the third example? (100 ft.)
How many such units are contained in the measurement? (50)
Which digits are significant? (5 and only the first zero)

2. a. Have pupils find the relative error in each of the above.

b. Which measurement is the most accurate?

c. Which measurement has the most significant digits?

3. Tell pupils that the greater the number of significant digits in measurement, the more accurate it is.

II. Practice

A. How many significant digits in each of the following?

30.4 in. .057 mm .040 ft. 30.10 cm .00062 m

B. 1. For each of the following measurements, tell which digits are significant:

70,000 yds. to the nearest yard
70,000 yds. to the nearest 1000 yards

2. Which of these is the more accurate? Why?

C. For each of the following tell which zeros are significant?

.030 in.   .43010 m   .006 in.

III. Summary

Have pupils give an example of each of the following generalizations:

A. For any measurement each non-zero digit is significant.

B. For any measurement each zero between non-zero digits is significant.

C. For any measurement the zeros at the right of a decimal are significant when they are the final digits in the numeral.

D. Accuracy may be indicated by the number of significant digits in a measurement.

Lessons 7 and 8

Topic: Scientific Notation

Aim: To learn to express numbers in scientific notation

Specific Objectives:

To review expressing numbers as the product of two factors, one of which is a power of ten
To learn to express numbers in scientific notation
To write the standard numeral for a number, expressed in scientific notation

Note to Teacher: In Grade 8, we shall not express numbers less than one in scientific notation.

Challenge: The star, Betelgeuse, has a diameter of 400,000,000 miles. How can you express this number using fewer digits?

I. Procedure

A. Review expressing numbers as the product of two factors, one of which is a power of ten.

1. Have pupils express the following as powers of ten:

$$100 = 10 \times 10 = 10^2$$

$$1{,}000 = 10 \times 10 \times 10 = 10^3$$

$$10{,}000 = 10 \times 10 \times 10 \times 10 = 10^?$$

$$100{,}000 = \quad ? \quad = ?$$

Have pupils recall that in expressing powers of ten the number of zeros in the standard numeral is the same as the exponent.

2. Express the following numbers as the product of two factors. The first factor must be a number greater than or equal to 1, and less than 10. The second factor must be a power of 10.

$$100 = 1 \times 100 \text{ or } 1 \times 10^2$$
$$300 = 3 \times 100 \text{ or } 3 \times 10^2$$
$$7{,}000{,}000 = 7 \times \ ? \ \text{ or } 7 \times 10^?$$
$$9{,}000{,}000{,}000{,}000 = ? \times \ ? \ \text{ or } ? \times 10^?$$

3. Return to challenge. $400,000,000 = 4 \times ?$ or $4 \times 10^?$

4. Tell pupils that when a number is expressed as the product of a factor greater than or equal to 1 and less than 10, and a power of 10, it is expressed in <u>scientific notation</u>.

   Scientists and others who often work with very great numbers find the use of scientific notation a distinct advantage. The standard numerals for these numbers are difficult to read and write.

B. Expressing numbers in scientific notation

1. Since in scientific notation the first factor must be greater than or equal to one and less than 10, it is sometimes necessary to express the first factor as a decimal fraction.

2. Consider a measurement 2,400,000 miles. How would you express this measurement in scientific notation?

   Since $2,000,000 = 2 \times 1,000,000 = 2 \times 10^6$
   $3,000,000 = 3 \times 1,000,000 = 3 \times 10^6$

   Elicit that $2,400,000 = 2.4 \times 1,000,000 = 2.4 \times 10^6$

3. Express in scientific notation the measurement 360,000 feet.

   Since $300,000 = 3 \times 100,000 = 3 \times 10^5$
   $400,000 = 4 \times 100,000 = 4 \times 10^5$

   Therefore, $360,000 = 3.6 \times 100,000 = 3.6 \times 10^5$

4. In a similar way, have pupils express the following in scientific notation.

   500,000 (5×10^5)
   520,000 (5.2×10^5)
   525,000 (5.25×10^5)

C. Naming with standard numerals numbers expressed in scientific notation

1. What does 2.4×10^7 mean?

$$\begin{aligned} 2.4 \times 10^7 &= 2.4 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 2.4 \times 10,000,000 \\ &= 24,000,000 \end{aligned}$$

   Thus, we can write 2.4×10^7 as 24,000,000.

2. Express each of the following as standard numerals:

2.4×10^5 $\quad$ 8.25×10^8

9.32×10^8 $\quad$ 8.312×10^7

II. Practice

A. Which of the following is not expressed in scientific notation? Explain.

6×10^4 $\quad$ 28×10^3 $\quad$ 4.2×10^5 $\quad$ 62.5×10^2

B. Express in scientific notation the following measures which often appear in newspapers and scientific magazines:

1. The farthest distance (apogee) of the moon from the earth is 240,000 miles.

2. 93,000,000 miles is the average distance of the sun from the earth.

3. The average distance of the closest star to earth is 24,000,000,000,000 miles.

III. Summary

A. What is an advantage of expressing numbers in scientific notation?

B. Describe the two factors of the product represented in scientific notation.

CHAPTER II

TRIANGLES AND QUADRILATERALS

The procedures suggested in this chapter will help pupils develop the following basic geometric concepts:

- methods of classifying triangles
- the sum of the measures of the angles of a triangle
- relationships between the angles of a triangle and the sides opposite the angles
- construction of congruent line segments, congruent angles, and congruent triangles
- similar triangles
- construction of a line parallel to a given line
- some properties of parallelograms

The concept of a polygon as a simple closed curve consisting of the union of line segments is extended and reinforced. Triangles are classified according to the measures of their sides as scalene, isosceles, and equilateral. They are also classified according to the measures of their angles as acute, right, and obtuse. Pupils are then given an opportunity to explore intuitively the possibilities that a triangle can be both right and isosceles, both acute and isosceles, both right and scalene, etc.

Through experimentation pupils are led to the generalization that the sum of the measures of the angles of any triangle is 180°. This inductive method is also employed to lead to an appreciation of the relationships which exist between the angles of a triangle and the sides opposite these angles. No formal proofs are presented at this time.

Using a straightedge and compass pupils learn to construct congruent line segments, congruent angles, and congruent triangles.

The concept of similarity between pairs of triangles is explored. Using the inductive method pupils are led to conclude that similar triangles have corresponding angles congruent, and that the ratios of the measures of their corresponding sides are equivalent.

The meaning of parallel lines and skew lines is recalled. Again, through experimentation, pupils learn that if two parallel lines are intersected by a transversal, pairs of corresponding angles are congruent. Using this information as background, pupils learn to construct a line parallel to a given line through a point not on line.

As an extension of their work with parallel lines, pupils learn to construct parallelograms. Then, by careful observation of their constructions, they discover some of the properties of parallelograms.

CHAPTER II

TRIANGLES AND QUADRILATERALS

Lessons 9-23

Lesson 9

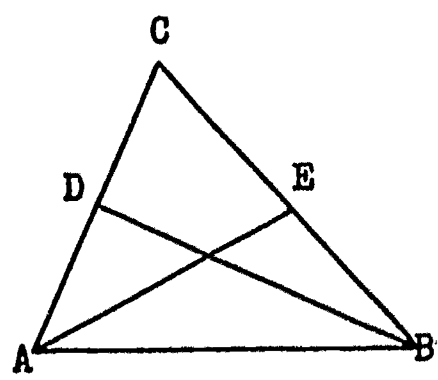
Topic: Classification of Triangles

Aim: To learn to classify triangles according to the measures of their sides

Specific Objectives:

To find the measure of the sides of various triangles
To classify triangles as scalene, isosceles, equilateral - according to the measures of their sides

Challenge: How many triangles can you find in the figure?



I. Procedure

A. To find the measure of the sides of various triangles

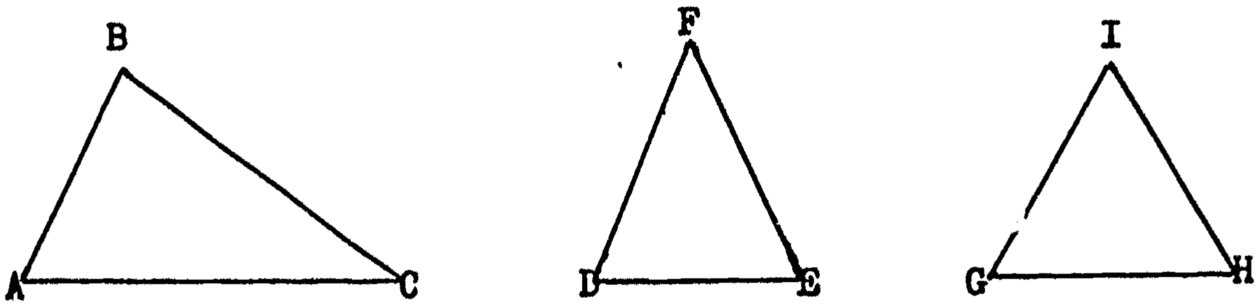
1. How many sides does a triangle have?

2. In the challenge, how many triangles did you find? Name two o: these triangles.

3. In the triangles you have chosen, name the sides.

4. Note: It is suggested that a rexographed sheet with the following three triangles be prepared and distributed to each pupil. A similar transparency on an overhead pro-

jector would be helpful for the teacher.



Have pupils measure the lengths of the sides (to the nearest $\frac{1}{8}$") of each triangle, and tabulate their measurements.

Figure 1		Figure 2		Figure 3	
In triangle ABC		In triangle DEF		In triangle GHI	
Side	Length	Side	Length	Side	Length
$\overline{AB}$		$\overline{FD}$		$\overline{GI}$	
$\overline{AC}$		$\overline{FE}$		$\overline{IH}$	
$\overline{CB}$		$\overline{DE}$		$\overline{GH}$	

a. In Figure 1, compare the measures of the sides. (No two sides have the same measure.)

b. In Figure 2, compare the measures of the sides. (Two sides have the same measure.) Have pupils recall that two line segments which have the same measure are said to be congruent.

c. In Figure 3, compare the measures of the sides. (Three sides are of the same measure, that is, they are congruent.)

5. After examining the tabulations, elicit from the pupils that there are triangles which have:

a. no sides of the same measure or no sides congruent
b. two sides of the same measure or two sides congruent
c. three sides of the same measure or three sides congruent

B. Classifying triangles

1. Have pupils learn the following:

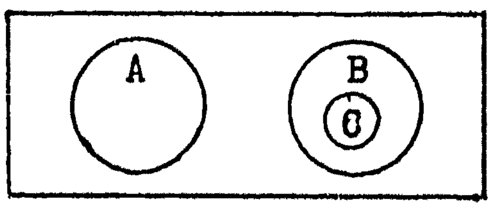
a. A scalene triangle is a triangle in which no two sides have the same measure.

b. An <u>isosceles</u> triangle is a triangle in which <u>at least</u> two sides have the same measure.

c. An <u>equilateral</u> triangle is a triangle in which all three sides have the same measure.

<u>Note</u>: Pupils should understand that if a triangle has three congruent sides, it necessarily has two congruent sides. Therefore, the set of equilateral triangles is a subset of the set of isosceles triangles.

This may be shown by means of a Venn diagram.



A = set of scalene triangles

B = set of isosceles triangles

C = set of equilateral triangles

2. Have pupils point out or draw objects which illustrate different types of triangles and classify the triangles as scalene, isosceles, or equilateral.

II. Practice

A. Classify the following triangles according to the measures of their sides. (When we give a side the length 3", we mean the side is exactly 3".)

1. A triangle whose sides are 3", 2", 4"

2. A triangle whose sides are 7", 5", 5"

3. A triangle whose sides are 4", 4", 4"

B. If a side of an equilateral triangle is $4\frac{1}{2}$", what is the perimeter of the triangle?

C. If the perimeter of an isosceles triangle is 7" and one of the congruent sides is 2", how long is each of the other sides? Illustrate with a diagram.

D. Consider the following sets:

A = set of all scalene triangles
B = set of all isosceles triangles
C = set of all equilateral triangles

1. Is set C a subset of set B? Explain.

2. Is set B a subset of set C? Explain.

3. Why are set A and set B disjoint sets?

III. Summary

A. How are triangles classified according to the measures of their sides?

B. Are all equilateral triangles also isosceles triangles? Are all isosceles triangles also equilateral triangles? Explain.

C. What new vocabulary have you learned today?

Lesson 10

Topic: Angle Measurement

Aim: To review the use of the protractor

Specific Objectives:

To reinforce the ability to read the protractor
To measure angles using the protractor

Challenge: Which angle has the greater measure?
How much greater?

Note to Teacher: It is suggested that rexographed sheets be prepared with drawings of the two angles in the challenge and the angles in B-3 and B-4. In the challenge, $\angle A = 50^\circ$ and $\angle B = 40^\circ$. Use of an overhead projector is recommended.

I. Procedure

A. The protractor

1. Have pupils recall that there is an instrument for measuring angles called a protractor.

2. Have pupils study their protractors. Discuss the instrument.

a. The protractor is divided into 180 unit angles. The standard unit of angle measure is one degree, 1°.

b. The common end point of the rays is indicated on the protractor. It is generally pointed to by an arrow. Only part of each ray is shown.

c. The numerals name every tenth ray. From one numeral to the next there are ten divisions.

d. The measure of the angle indicated by any two successive markings is 1°.

e. There are two scales on the protractor, one reading from left to right clockwise, and one reading from right to left counterclockwise.

B. Measuring angles with the protractor

1. Measuring ∠A and ∠B of the challenge

a. Measuring ∠A

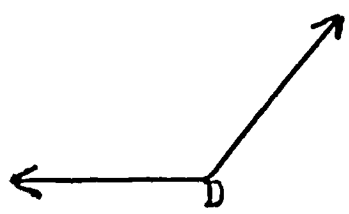
1) What kind of angle is ∠A? (acute) Why?
2) Have pupil place the arrow on the protractor at the vertex of ∠A.
3) Place the protractor so that the ray marked 0 lies along one ray of the angle.
4) Place pencil point at the point on the protractor where the other ray forming the angle crosses the protractor. (Extend ray, if necessary.)
   a) What numerals mark the point? (50, 130)
   b) Recall that ∠A is acute. What numeral on the scale will you select as the measure of the angle? (50) Why?
5) Angle A measures 50° or $\angle A = 50^\circ$.

b. Measuring ∠B

1) Follow procedure used in measuring ∠A.
2) Angle B measures 40° or $\angle B = 40^\circ$.

c. How much greater is ∠A than ∠B?

3. Measuring an angle such as ∠D



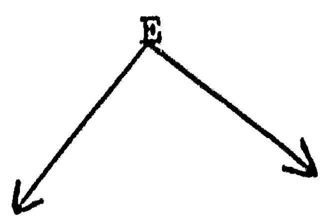
a. What kind of angle is ∠D? Why?

b. Have pupils place arrow of protractor on vertex of angle. Place the edge representing the zero ray on the left along one ray of the angle and note at what point the other ray crosses the protractor.

c. Recall that ∠D is obtuse. What numeral on the scale will you select as the measure of the angle?

d. $\angle D = 135^\circ$.

4. Measuring an angle such as ∠E



a. What kind of angle is ∠E? (Pupils will probably estimate near 90°.)

b. Follow same procedure as in 2 and 3.

c. $\angle E = 90^\circ$.

II. Practice

Measure angles in textbook and in different triangles drawn in students' notebooks.

III. Summary

A. What instrument do we use to measure angles?

B. Describe the method of using a protractor to measure angles.

C. Before using a protractor to measure an angle, we usually classify the angle as acute, right, or obtuse. How is this procedure helpful in selecting the correct scale on the protractor?

Lesson 11

Topic: Angles of a Triangle

Aim: To determine the sum of the measures of the angles of a triangle

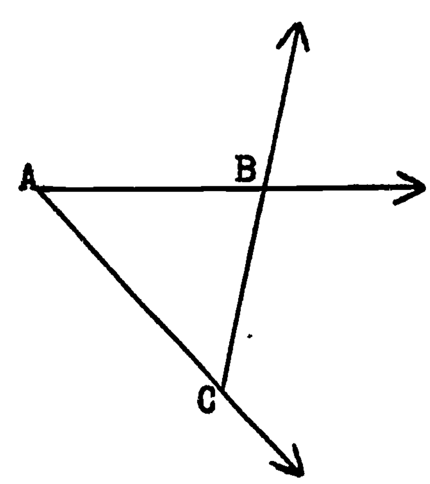
Specific Objectives:

To understand the meaning of an angle of a triangle
To discover that the sum of the measures of the angles of a triangle is 180°

Challenge: In an attic each rafter makes an angle of 40° with the floor. How could we determine the size of the angle at which the rafters meet if it is inconvenient to actually measure the angle?

I. Procedure

A. The meaning of an angle of a triangle



1. In triangle ABC the set of points in $\overline{AB}$ is a subset of the set of points in $\overrightarrow{AB}$, and the set of points in $\overline{AC}$ is a subset of the set of points in $\overrightarrow{AC}$. Since $\angle BAC$ is formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with its vertex at A, we say $\angle BAC$ is determined by $\overline{AB}$ and $\overline{AC}$, sides of triangle ABC.

2. In a similar manner, elicit from pupils that $\angle B$ is determined by $\overline{BA}$ and $\overline{BC}$, and $\angle C$ is determined by $\overline{CA}$ and $\overline{CB}$.

3. Every triangle has associated with it three angles which are called the angles of the triangle. The vertex of each of these angles is a vertex of the triangle.

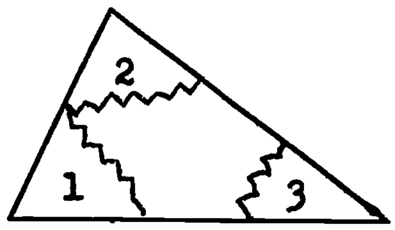
B. The sum of the measures of the angles of a triangle is 180°.

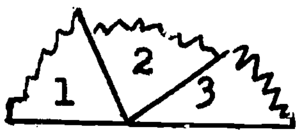
1. Have each pupil draw a triangle. Permit each pupil to draw any triangle he wishes.

2. Have each pupil measure the angles of his triangle and compute the sum of their measures.

3. Elicit that each sum is approximately 180°.

Note to Teacher: At this point, it is suggested that the teacher demonstrate this as follows:





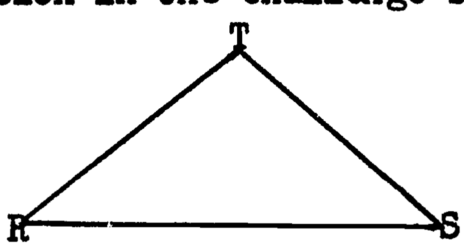
Have each pupil draw a picture of a triangle, cut it out, and tear it apart as shown in Figure A. Place the pieces of paper representing angles 1, 2, and 3 next to each other as shown in Figure B.

What is the sum of the three angles? (180°)

4. Tell pupils that through this experimental (or inductive) method it appears that the sum of the measures of the angles of any triangle is 180°. Since we cannot try all possible triangles, we cannot say with certainty that this is true. In a later course, we will learn a deductive method through which this will be proved. Therefore, we shall agree that the sum of the measures of the angles of any triangle is 180°.

5. Refer to challenge:

   a. Represent the situation in the challenge by triangle RST.



   b. $\angle R = 40^\circ$

      $\angle S = 40^\circ$

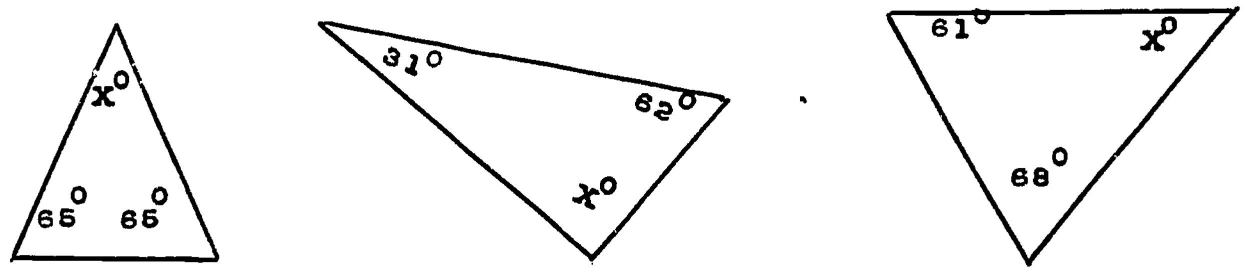
      The sum of these measures is 80°.

   c. Since the sum of the measures of all three angles is 180°, than $\angle T = 180^\circ - 80^\circ$.

      What is the measure of $\angle T$? (100°)

II. Practice

A. Find the size of angle x in each of the following triangles without measuring:



B. In triangle DEF, $\angle E = 40^\circ$ and the other two angles are congruent. What is the measure of each of the other angles?

C. If the three angles of a triangle are congruent, how many degrees are there in each angle?

D. How many right angles can be associated with a triangle? Why?

E. How many obtuse angles can be associated with a triangle? Why?

III. Summary

A. What is the sum of the measures of the angles of a triangle?

B. If a triangle contains a right angle, what is the sum of the measures of the other two angles?

C. If a triangle contains an obtuse angle, what can be said about the sum of the measures of the other two angles?

Lesson 12

Topic: Classification of Triangles

Aim: To learn the classification of triangles according to the measures of their angles

Specific Objectives:

To classify triangles as acute, right, obtuse, or equiangular according to the measures of their angles

To classify triangles according to the measures of their sides and angles

Challenge: How would you classify a triangle which has a right angle and two congruent sides?

I. Procedure

A. Classifying triangles

Note: It is suggested that a rexographed sheet with three triangles, as shown, be prepared and distributed to each pupil. A similar transparency on an overhead projector would be helpful for the teacher.

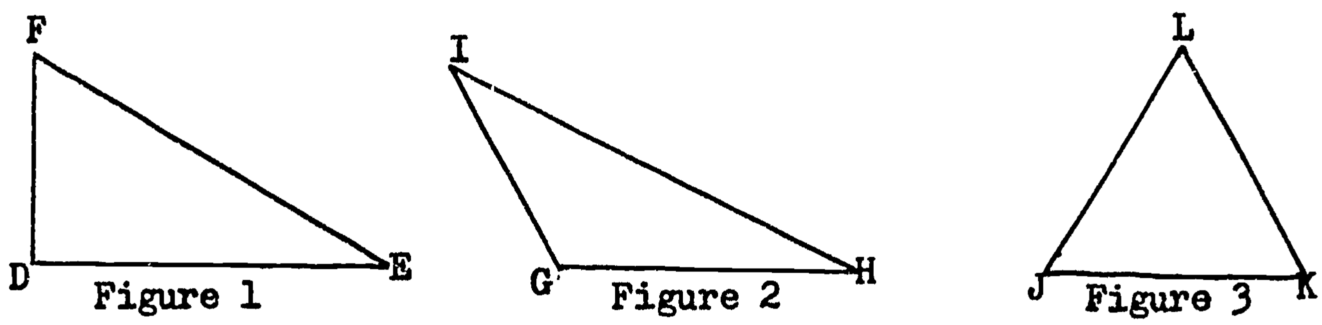


Figure 1 Figure 2 Figure 3

Have pupils measure each of the angles in each triangle and record the results.

1. a. In which of the above figures is a right angle an angle of the triangle?

   b. Tell pupils that when one of the angles of a triangle is a right angle the triangle is called a right triangle.

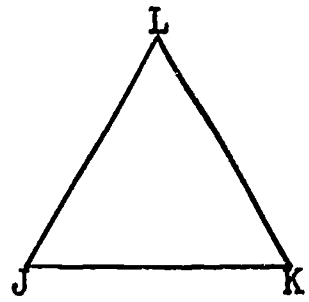
c. How would you classify each of the other two angles of the triangle in Figure 1?

2. a. In which of the above figures is an obtuse angle an angle of the triangle?

   b. Tell pupils that when one of the angles of a triangle is an <u>obtuse angle</u>, the triangle is called an <u>obtuse triangle</u>.

   c. How would you classify each of the other two angles in Figure 2?

3. Have pupils consider triangle JKL.



   a. Why is this not a right triangle?

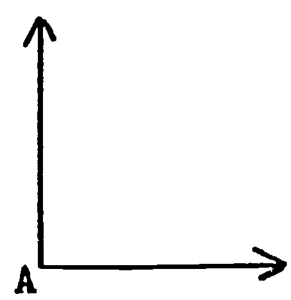
   b. Why is this not an obtuse triangle?

   c. How would you classify each of the three angles of this triangle?

   d. Tell pupils that when a triangle has <u>three acute angles</u>, the triangle is called an <u>acute triangle</u>.

4. Tell pupils that if the three angles of a triangle are equal in size, the triangle is called an <u>equiangular triangle</u>.

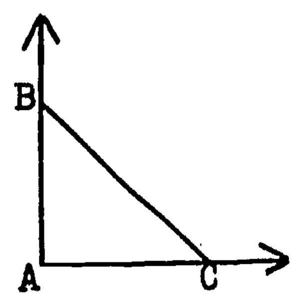
B. Classifying triangles according to the measures of their sides and angles



1. Have pupils draw two rays forming a right angle. Name the vertex A.

2. Have the pupils designate points B and C on the rays, each one inch from the vertex.

3. Draw $\overline{BC}$.



4. Classify this triangle according to the measures of its sides.

5. Classify this triangle according to the measure of its angles.

6. Refer to challenge. Tell pupils that this triangle is classified as an isosceles right triangle.

7. Through similar experimentation, have pupils conclude that an isosceles triangle may also be classified as an isosceles obtuse triangle or an isosceles acute triangle.

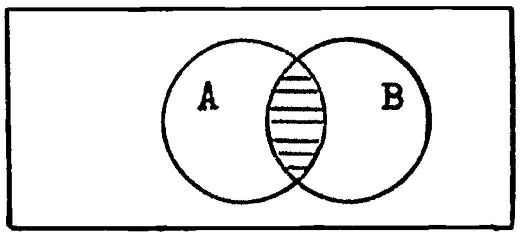
II. Practice

A. Use your protractor to draw a right triangle; an obtuse triangle.

Classify your triangles as scalene or isosceles.

Classify your triangles according to both sides and angles.

B.



A = set of all isosceles triangles

B = set of all right triangles

$A \cap B = ?$

C. In a similar manner, show by means of Venn diagrams the conclusions in B-7.

III. Summary

A. Can an obtuse triangle be a right triangle? Explain.

B. How many acute angles are there in an acute triangle? in a right triangle? in an obtuse triangle? in an equiangular triangle?

C. Can a scalene triangle be a right triangle? an obtuse triangle?

D. Can an isosceles triangle be a right triangle? an acute triangle? an obtuse triangle?

E. What new vocabulary have you learned today?

Lessons 13 and 14

Topic: Triangles: Sides and Angles

Aim: To learn the relationships which exist between the angles of a triangle and the sides opposite these angles

Specific Objectives:

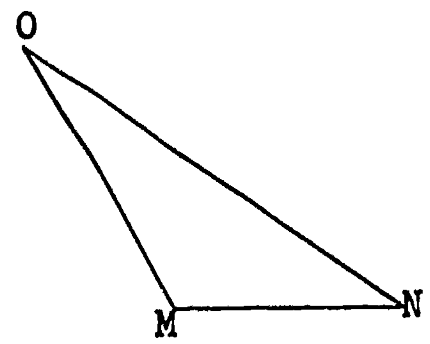
To discover that in any triangle, the longest side is opposite greatest angle
To discover that if two sides of a triangle are congruent, the angles opposite those sides are also congruent
To discover that an equilateral triangle is also equiangular

Challenge: Complete the following statement: If one angle of a triangle is obtuse, the measure of the side opposite that angle is (greater than, less than, or equal to) the measure of either of the other two sides.

I. Procedure

Note: It is suggested that the teacher prepare and distribute rexographed sheets containing Figures 1, 2, and 3. Use of the overhead projector would be helpful.

Figure 1



Side	Length of Side	Angle	Degree Measure
$\overline{OM}$		$\angle O$	
$\overline{MN}$		$\angle M$	
$\overline{ON}$		$\angle N$	

A. In any triangle, the longest side is opposite the greatest angle.

1. In Figure 1, have pupils measure the length of the sides of triangle OMN and record the results in the table.

   Which is the longest side? ($\overline{ON}$)
   What angle lies opposite this side? ($\angle M$)
   Which is the shortest side? ($\overline{MN}$)
   What angle lies opposite this side? ($\angle O$)

2. Have pupils use their protractors to measure the angles of triangle OMN and record the results in the table.

   What kind of triangle is it?
   Which is the greatest angle?
   What side lies opposite this angle?
   Which is the smallest angle?
   What side lies opposite this angle?

   Answer the challenge.

3. Have each pupil draw a right triangle and letter it PQR, with the right angle at Q. Using the same procedure as in 1 and 2, elicit that the longest side is opposite the greatest angle.

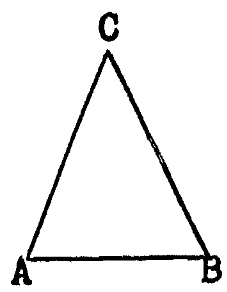
   Tell pupils that in a right triangle, the side opposite the right angle is called the <u>hypotenuse</u>, and the other two sides are called <u>legs</u>.

   Which angle of a right triangle is always the greatest? (the right angle)
   Which side of a right triangle is always the longest? (the hypotenuse)

4. What relationship seems to exist between the longest side of a triangle and the greatest angle of a triangle? between the shortest side of a triangle, and the smallest angle of a triangle? Elicit from pupils that it appears that in a triangle, the longest side is opposite the greatest angle.

B. If two sides of a triangle are congruent, the angles opposite these sides are congruent.

<u>Figure 2</u>



Side	Length of Side	Angle	Degree Measure
$\overline{AC}$		$\angle A$	
$\overline{BC}$		$\angle B$	
$\overline{AB}$		$\angle C$	

1. In Figure 2, have pupils measure the length of the sides of the triangle ABC and record the results in the table.

   What did you discover about the length of the sides of this triangle?
   What kind of triangle is this?
   Which angles lie opposite the congruent sides? ($\angle A$ and $\angle B$)

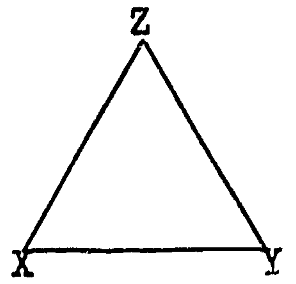
2. Have pupils use their protractors to measure the angles of the triangle and record the results in the table.

   What did you discover about the angles opposite the sides $\overline{AC}$ and $\overline{BC}$?
   When two sides of a triangle are congruent, what seems to be true of the angles opposite these sides?

   Elicit that it appears that if two sides of a triangle are congruent, the angles opposite these sides are also congruent.

C. An equilateral triangle is also equiangular

Figure 3



Side	Length of Side	Angle	Degree Measure
$\overline{XY}$		$\angle X$	
$\overline{YZ}$		$\angle Y$	
$\overline{XZ}$		$\angle Z$	

1. In Figure 3, have pupils measure the length of the sides and size of the angles. Record these measurements in the table.

2. Lead the pupils to discover that if the three sides of a triangle are congruent, the three angles are also congruent.

3. If a triangle with three congruent sides is called equilateral, what might a triangle with three congruent angles be called? (equiangular)

4. Elicit from the pupils that an equilateral triangle seems to be also equiangular.

Note: Have pupils realize that in each of the above cases the conclusions were arrived at through the use of the experimental, or inductive, method.

II. Practice

A. Draw an obtuse triangle. How does the side opposite the obtuse angle compare in length with each of the other two sides?

B. Draw a right triangle. How does the hypotenuse compare in length with each of the legs?

C. Draw an isosceles right triangle. Without using a protractor, what is the sum of the measures of the acute angles? What is the measure of each acute angle?

III. Summary

A. In a scalene triangle, how does the length of the side opposite the greatest angle compare with the lengths of the other two sides?

B. In an obtuse triangle, what can be said of the side opposite the obtuse angle?

C. In a right triangle, what is the name of the side opposite the right angle? How does the measure of the hypotenuse compare with the measure of the legs?

D. If the three sides of a triangle are congruent, what can be said of the measure of the angles?

Lesson 15

Topic: Congruence

Aim: To construct congruent line segments and congruent angles

Specific Objectives:

To construct a line segment which is congruent to a given line segment
To construct an angle which is congruent to a given angle

Challenge: Without using the markings on a ruler, how could you construct a line segment, $\overline{CD}$, which is congruent to $\overline{AB}$?

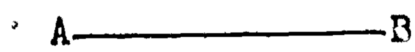
I. Procedure

A. Construction of congruent line segments

1. Have pupils recall that two line segments are said to be congruent if they are equal in measure, i.e., if they have the same length.

2. Refer to challenge. Discuss several methods suggested by the pupils, such as a piece of string, edge of a piece of paper, width between two fingers, etc. Tell pupils that from ancient times mathematicians have made geometric drawings using only a compass and straight edge. Such drawings are called "constructions" (a ruler may be used as a straight edge, if the markings are ignored).

3. Have pupils draw a line segment and name it AB.



4. Have pupils draw a ray and label the end point C. Place the point of a compass on one end point of $\overline{AB}$ and open the compass until the pencil tip is on the other end point of $\overline{AB}$.

Without changing the opening of the compass, place the point of the compass on end point C and draw an arc which intersects the ray. Mark the point of intersection D.

5. Do $\overline{AB}$ and $\overline{CD}$ appear to be congruent?

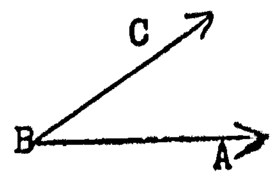
   $\overline{CD}$ has been constructed so that its length is the same as the length of $\overline{AB}$. Therefore, $\overline{CD}$ is congruent to $\overline{AB}$. Tell pupils that mathematicians use the symbol $\cong$ to represent "is congruent to."

   $$\text{If } m(\overline{CD}) = m(\overline{AB}), \text{ then } \overline{CD} \cong \overline{AB}.$$

6. Have pupils repeat this construction with line segments shown in various positions.

B. Construction of congruent angles

1. Have each pupil draw a picture of an acute angle and label the angle ABC.



2. Tell pupils that an angle congruent to angle ABC can be constructed without a protractor by using a compass and straight edge as follows:

   a. Have pupils draw $\overrightarrow{XY}$.

Figure 1

   b. Place the point of the compass on the vertex of $\angle ABC$, and with any opening (radius) draw an arc which intersects $\overrightarrow{BC}$ and $\overrightarrow{BA}$. Mark the intersections E and D respectively.

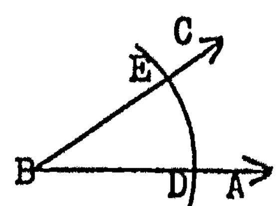


Figure 2

   c. Without changing the opening (radius) of the compass, place the point of the compass on end point of $\overrightarrow{XY}$ and draw a similar arc intersecting $\overrightarrow{XY}$. Mark the point of intersection R.

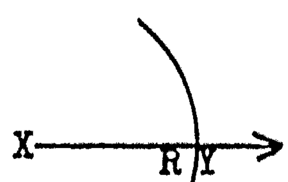


Figure 3

   d. Place the point of the compass on D (Figure 2) and open the compass until the pencil touches point E. Without changing the opening of the compass, place the point of the compass on R (Figure 3) and draw an arc which intersects the arc previously drawn. Name the point of intersection Z. This step is shown in Figure 4.

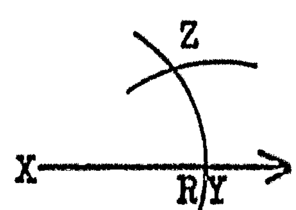


Figure 4

e. Draw $\overrightarrow{XZ}$.

f. Have pupils use their protractors to compare the measures of $\angle ABC$ and $\angle RXZ$.

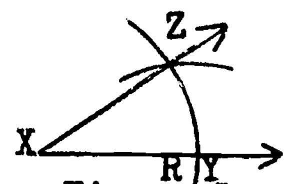


Figure 5

Since $m \angle ABC = m \angle RXZ$, we agree that $\angle ABC \cong \angle RXZ$.

g. Tell pupils that this construction will be proved to be true in a later course.

II. Practice

A. Draw a vertical line segment. Construct a horizontal line segment which is congruent to it.

B. Use a protractor to draw an angle whose measure is 120°. Construct another angle which is congruent to it. Check your construction with a protractor.

III. Summary

A. What are the only instruments which may be used in constructing a geometric figure?

B. If two line segments are congruent and one is vertical, does the other line segment have to be vertical?

If two angles are congruent and one is acute, does the other angle have to be acute?

Lesson 16

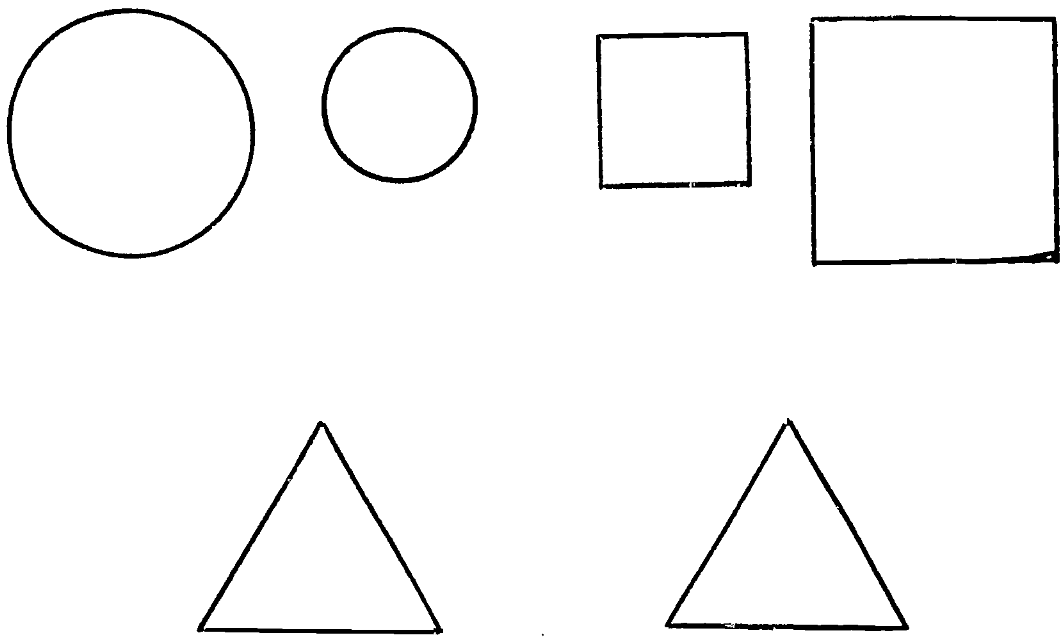
Topic: Congruence

Aim: To learn the meaning of congruence in relation to triangles

Specific Objectives:

To learn the meaning of congruence as applied to geometric figures
To learn the meaning of the corresponding parts of congruent triangles

Challenge: Which of these figures are congruent?



I. Procedure

A. Meaning of congruence as applied to geometric figures

1. Have pupils recall that two line segments are congruent if they have the same measure; that two angles are congruent if they have the same measure.

2. Do the two congruent line segments have the same shape? Do the two congruent angles have the same shape?

3. Tell pupils that, in general, geometric figures that have the same size and the same shape are called congruent.

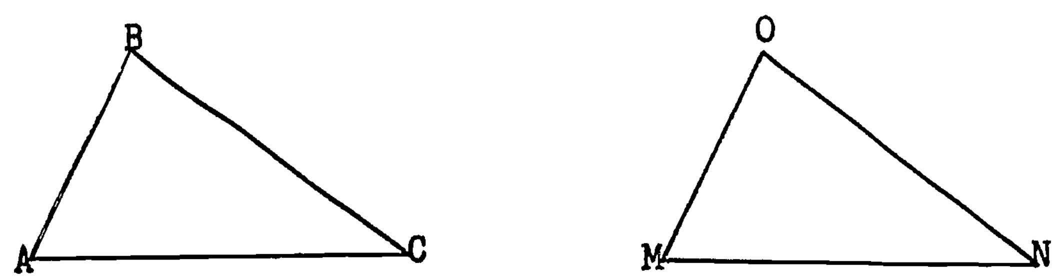
4. Refer to challenge. Elicit that although all circles have the same shape, in the challenge problem the radius of the first circle is greater in length than the radius of the second circle. These circles, therefore, are not the same size and are not congruent.

5. Although all squares have the same shape, in the challenge problem the length of each side of the first square is less than the length of each side of the second square. These squares, therefore, are not the same size and are not congruent.

6. All equilateral triangles have the same shape and in the challenge problem the sides of the two triangles have the same measure. Therefore, these two triangles have the same size and shape and are congruent.

B. Corresponding parts of congruent triangles

Cut out and show pupils models of a pair of congruent triangles (with their interiors).



1. Show pupils that since these triangles are congruent, the model of one of these triangles can be moved onto the model of the other triangle so that the two triangles <u>coincide</u>.

   Have pupils observe that in order to have the two triangles coincide, point M must fall on point A, point N must fall on point C, and point D must fall on point B.

2. Elicit that since the following pairs of sides coincide

   $\overline{AC} \cong \overline{MN}$ $\qquad$ $\overline{AB} \cong \overline{MO}$ $\qquad$ $\overline{BC} \cong \overline{ON}$

and since the angles opposite these sides also coincide,

$\angle ABC \cong \angle MON$ $\angle BCA \cong \angle ONM$ $\angle BAC \cong \angle OMN$

3. Tell pupils that these matching parts in the two triangles are called <u>corresponding parts</u> of the triangles.

4. We may use symbols to show that these triangles are congruent.

   $\Delta\ ABC \cong \Delta\ MON$

   Have pupils note that the order in which the letters appear indicates the corresponding vertices, that is

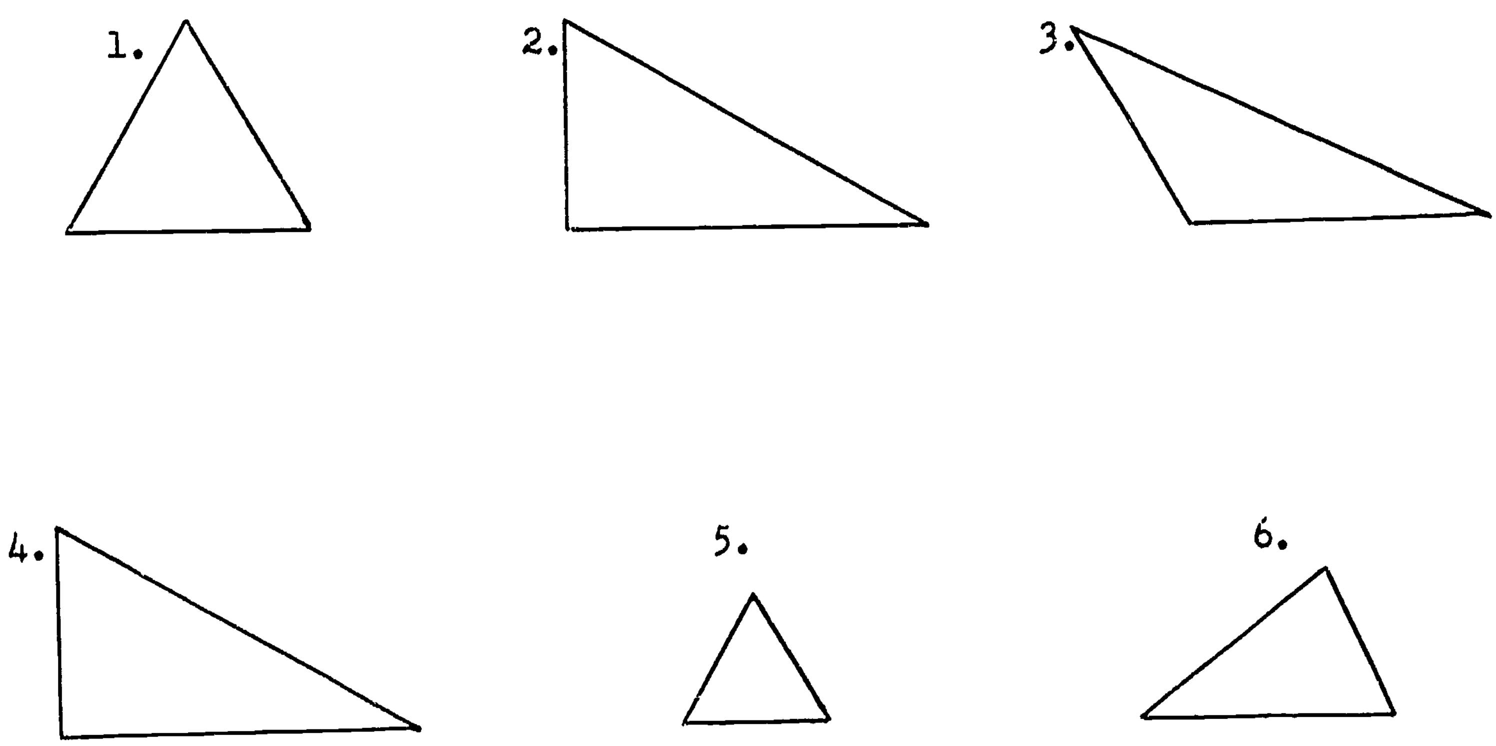
   A corresponds to M

   B corresponds to O

   C corresponds to N

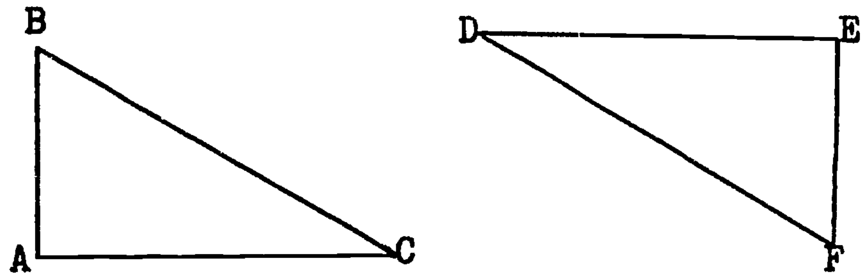
II. Practice

A. Tell which figures appear to be congruent.

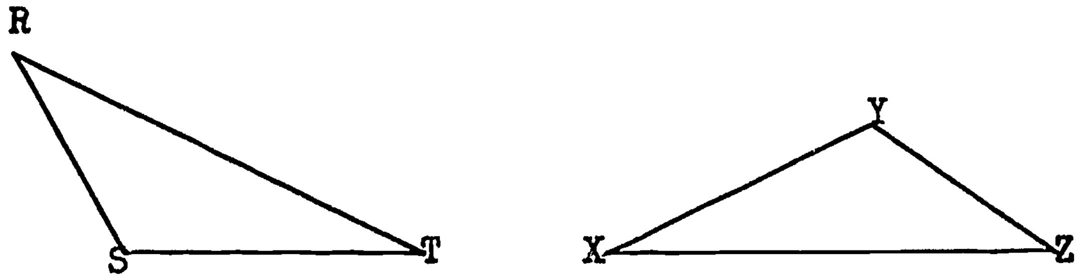


B. For each pair of congruent triangles below, indicate the corresponding sides and the corresponding angles. If necessary, use a ruler or protractor to help you decide.

1.



2.



III. Summary

A. When two geometric figures have the same size and shape, they are said to be _____.

B. Explain what is meant by corresponding parts of two congruent triangles.

C. What is true about the corresponding parts of congruent triangles?

D. What new vocabulary have you learned today?

Lessons 17 and 18

Topic: Congruent Triangles

Aim: To learn to construct a triangle congruent to another triangle

Specific Objectives:

To learn to construct a triangle congruent to another triangle:
- by using the lengths of the three sides
- by using the lengths of two sides and the measure of the included angle
- by using the measures of two angles and the length of the included side

Challenge: Draw a triangle and label it ABC. Is it possible to construct a triangle, DEF, so that $\Delta ABC \cong \Delta DEF$?

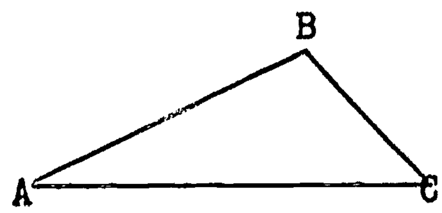
I. Procedure

A. Using the lengths of three sides

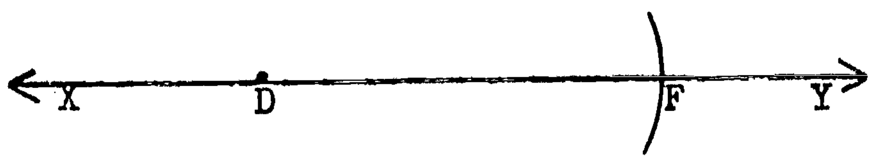
1. Refer to challenge. Tell pupils that we will investigate three methods of constructing congruent triangles.

2. Method I (side, side, side $\cong$ side, side, side)

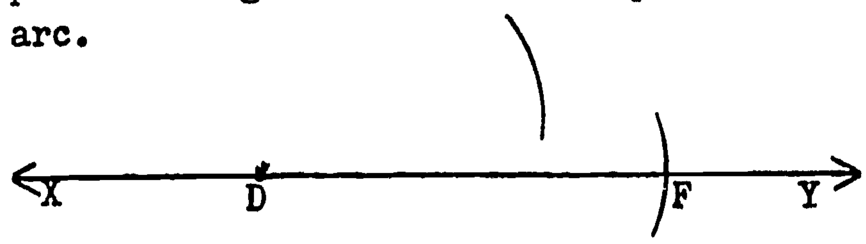
a. Have each pupil draw a triangle and label it ABC.



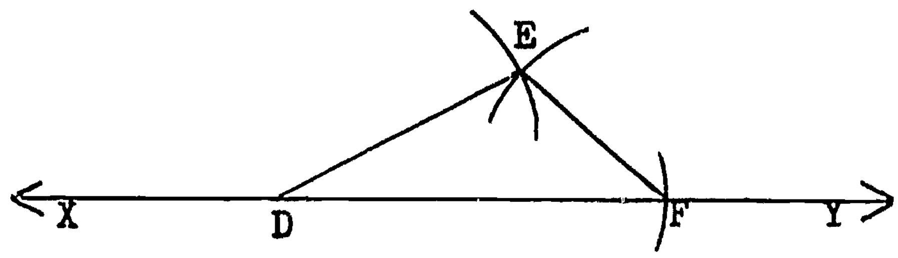
b. Have pupils draw line XY. Using a compass, mark off on $\overleftrightarrow{XY}$ a line segment, $\overline{DF}$, congruent to $\overline{AC}$.



c. Open compass to length of $\overline{AB}$ and with point D as a center draw an arc.



d. Open compass to length of $\overline{BC}$ and with F as a center draw an arc which intersects the arc already drawn. Call the point of intersection E, and draw $\overline{DE}$ and $\overline{FE}$.



3. Elicit that if Δ DEF is to be congruent to Δ ABC, these six conditions must be met.

$\overline{EF} \cong \overline{BC}$	$\angle D \cong \angle A$
$\overline{DF} \cong \overline{AC}$	$\angle E \cong \angle B$
$\overline{DE} \cong \overline{AB}$	$\angle F \cong \angle C$

4. Without measuring, what do we know about the lengths of the corresponding sides?

5. Use your protractor to measure the corresponding angles. Are they congruent?

6. Since the six conditions for congruence have been met, we can state that Δ DEF $\cong$ Δ ABC.

7. Guide pupils to realize that it was <u>not</u> necessary to construct all six parts of the second triangle congruent to the corresponding six parts of the first triangle. If three sides of one triangle are congruent to the corresponding three sides of another triangle, the corresponding angles will also be congruent.

8. For this reason, we say that if three sides of one triangle are congruent to the corresponding three sides of another triangle, the two triangles are congruent. This may be ex-

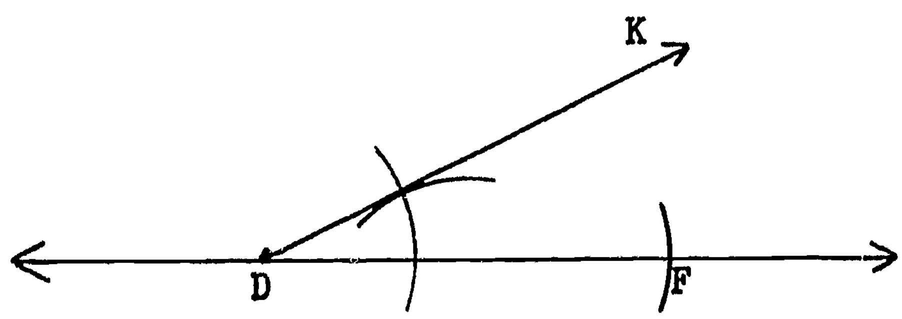
pressed in symbols as: s s s ≅ s s s.

B. Using the lengths of two sides and the measure of the included angle

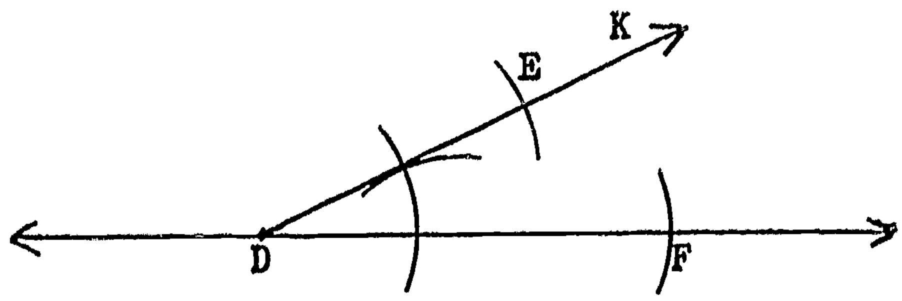
1. Method II (side, angle, side ≅ side, angle, side)

a. Have pupils draw a picture of a line and use a compass to mark off a line segment $\overline{DF}$ congruent to $\overline{AC}$.

b. At point D construct $\angle D \cong \angle A$ and label ray DK.



c. On $\overrightarrow{DK}$ mark off $\overline{DE} \cong \overline{AB}$. Draw EF.



Note to Teacher: Be sure pupils understand the meaning of the term, "included angle" as the angle determined by the two sides.

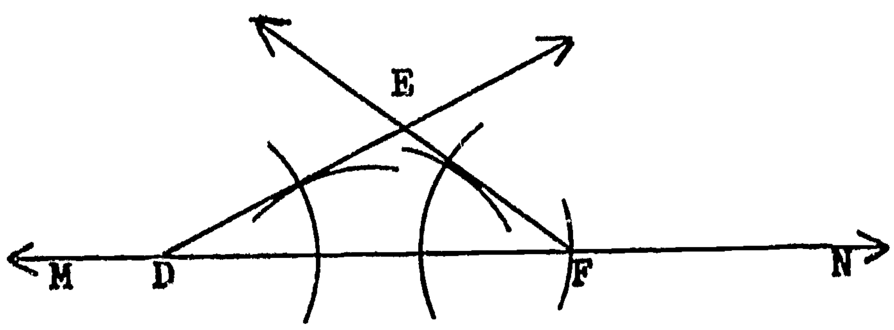
2. Test to see if Δ DEF $\cong$ Δ ABC by measuring the corresponding parts.

3. Refer to A-7 and 8. Using the same procedure, have pupils conclude that if two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the two triangles are congruent. This may be expressed in symbols as: s a s $\cong$ s a s.

C. Using the measures of two angles and the length of the included side

1. Method III (angle, side, angle $\cong$ angle, side, angle)

Have pupils draw $\overleftrightarrow{MN}$ and using a compass, mark off $\overline{DF}$ congruent to $\overline{AC}$. At point D construct $\angle D \cong \angle A$, and at point F construct $\angle F \cong \angle C$. Extend the rays until they intersect, and call the point of intersection E.



<u>Note to Teacher</u>: Be sure pupils understand the meaning of the term, "included side."

2. Test to see that Δ DEF $\cong$ Δ ABC by measuring the corresponding parts.

3. Have pupils conclude that two triangles are congruent if two angles and the included side of one triangle are congruent to two angles and the included side of another triangle. This may be expressed in symbols as: a s a $\cong$ a s a.

II. Practice

A. Draw an acute triangle, a right triangle, and an obtuse triangle. Construct congruent triangles to each using a different method in each case.

B. Show that two triangles are not always congruent if the three angles of one are congruent to the three corresponding angles of the other.

III. Summary

A. What are the three methods used to construct a triangle congruent to another triangle?

B. In constructing one triangle congruent to another, is it necessary to construct all six corresponding parts congruent? Explain.

C. Discuss - Experimental method for verifying or discovering properties of geometric figures.

Lessons 19 and 20

Topic: Similar Triangles

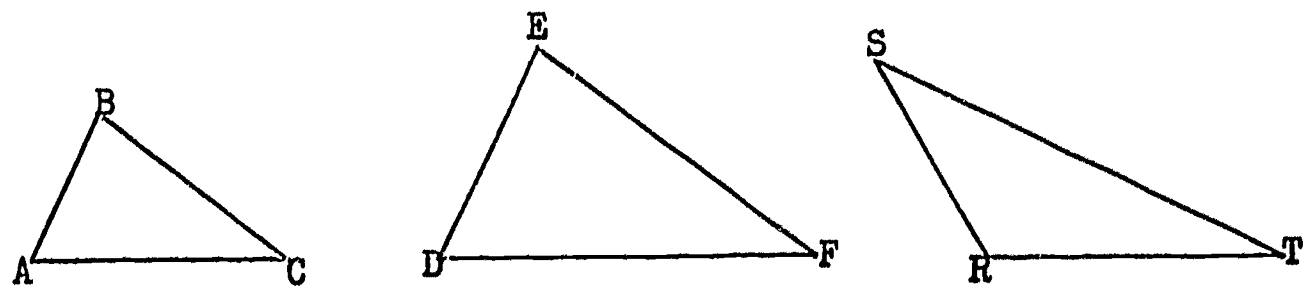
Aim: To learn the meaning of similarity as a relation between pairs of triangles

Specific Objectives:

To understand the meaning of similar triangles
To learn that the corresponding angles of similar triangles are congruent
To learn that ratios of measures of pairs of corresponding sides of similar triangles are equivalent
To learn that if the corresponding angles of two triangles are congruent, the measures of the corresponding sides of these triangles have equivalent ratios and therefore the triangles are similar

Challenge: Which of these triangles appear to have the same shape? Do the triangles with the same shape appear to be congruent?

Note: It is suggested that a rexographed sheet with these three triangles be prepared and distributed to each pupil.



I. Procedure

A. Meaning of similar triangles

1. Discuss with the pupils the general meaning of the word similar, that is, having likeness or resemblance. Have pupils give examples of similarity from everyday life such as a snapshot and its enlargement, two different sized maps of the United States, and two square tiles.

2. Tell pupils that in mathematics the word similar is applied to geometric figures having the same shape, but not necessarily the same size.

3. Refer to challenge.
   Elicit that triangle ABC and triangle DEF seem to have the same shape, but not the same size.

4. Tell pupils that two triangles which have the same shape, but not necessarily the same size, are called <u>similar triangles</u>. The symbol for "is similar to" is $\sim$ and we write: $\Delta\, ABC \sim \Delta\, DEF$.

B. Corresponding angles of similar triangles

1. Consider triangles ABC and DEF in the challenge. Have pupils use their protractors to measure the angles of triangle ABC and triangle DEF.

   Which angle in triangle DEF is congruent to $\angle A$?
   Which angle in triangle DEF is congruent to $\angle B$?
   Which angle is congruent to $\angle C$?

2. Have pupils conclude that it appears that if two triangles are similar, their corresponding angles are congruent.

3. Have pupils consider triangles DEF and RST.

   Do these triangles appear to be similar?
   Have pupils measure the angles of triangle RST.
   Does each angle of triangle RST have a matching congruent angle in triangle DEF?

4. Have pupils conclude that it appears that if two triangles do not have the same shape, their corresponding angles are not congruent.

5. Have pupils realize that these conclusions have been arrived at by the experimental method.

C. Corresponding sides of similar triangles

1. Consider the triangles ABC and DEF in the challenge.

   $\Delta\, ABC \sim \Delta\, DEF$

   Measure $\overline{AC}$ and $\overline{DF}$. What is the ratio of m $(\overline{AC})$ to m $(\overline{DF})$? $(\frac{1}{2})$

   Measure $\overline{BC}$ and $\overline{EF}$. What is the ratio of m $(\overline{BC})$ to m $(\overline{EF})$? $(\frac{1}{2})$

   Without measuring, what do you think is the ratio of m $(\overline{AB})$ to m $(\overline{DE})$?

   Measure the segments to check your answer.

2. After several such examples, have pupils conclude that if two triangles are similar, the ratios of the corresponding sides are equivalent.

3. Lead pupils to realize that similar triangles have

a. corresponding angles congruent and
b. ratios of the measures of corresponding sides equivalent.

4. Have pupils examine another pair of similar triangles and have them notice that the pairs of angles are congruent and the ratios of the measures of the sides are equivalent.

D. If the corresponding angles of two triangles are congruent, the triangles are similar.

1. In order to have a variety of triangles on which to base our conclusion, have pupils draw, with the use of protractors, two triangles of different sizes having their corresponding angles congruent.

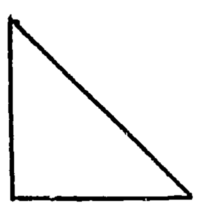
2. Have pupils measure the lengths of the sides of each of their triangles. Help them to see that the ratios of the measures of the corresponding sides are equivalent.

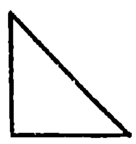
3. Lead pupils to see that in each case when the corresponding angles of two triangles are congruent, the ratios of the measures of the corresponding sides of these triangles are equivalent.

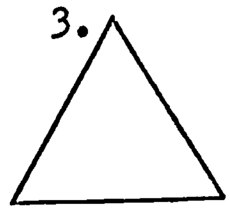
4. Have pupils conclude that if the corresponding angles of two triangles are congruent, the triangles are similar.

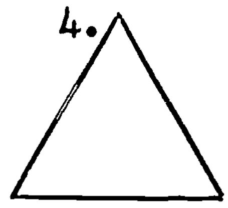
II. Practice

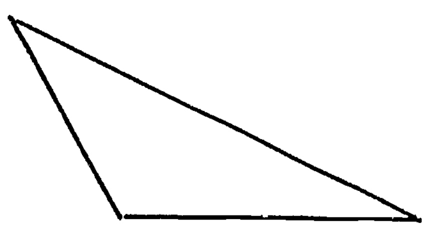
A. Which of the following pairs of figures appear to be similar?

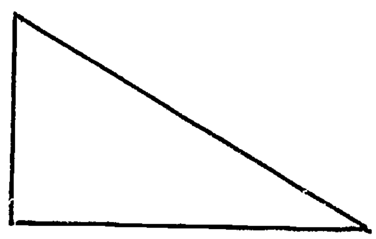
1. 

2. 

3. 

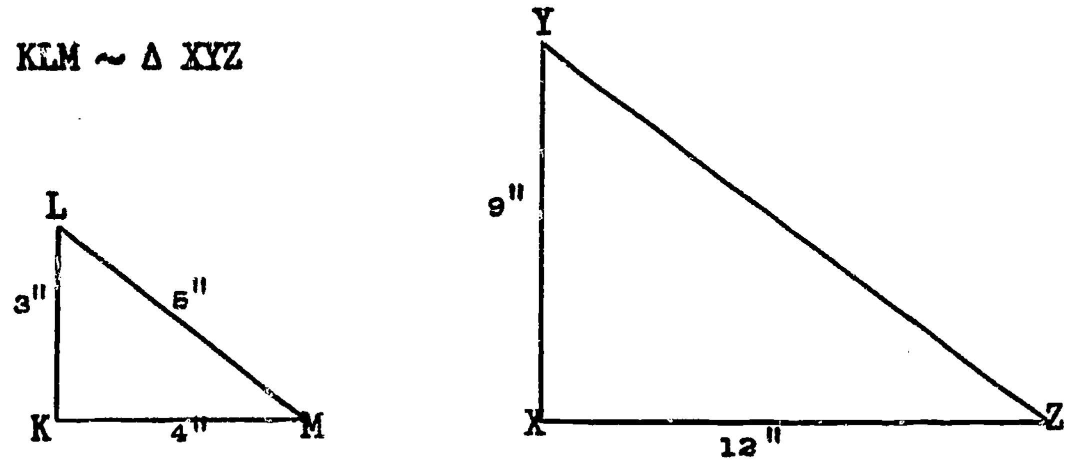
4. 

5. 

6. 

(2)

B. Δ KLM $\sim$ Δ XYZ



1. Use your protractor to find the measures of the pairs of congruent angles.

2. What is the ratio of m ($\overline{KM}$) to m ($\overline{XZ}$)?
   What is the ratio of m ($\overline{LK}$) to m ($\overline{YX}$)?

3. What is the measure of $\overline{YZ}$?

C. In Δ ABC, m $\angle A = 50$, m $\angle B = 100$.
In Δ DEF, m $\angle D = 50$, m $\angle F = 30$.
Are Δ ABC and Δ DEF similar? Why?

D. Explain your answer to each of the following questions.

Are two equilateral triangles always similar?
Are two similar triangles ever congruent?
Are two similar triangles always congruent?
What is the ratio of the corresponding sides of two congruent triangles?

III. Summary

A. What do we mean when we say two triangles are similar?

B. If two triangles are similar, how do the measures of their corresponding angles compare?

C. If two triangles are similar, what is true of the measures of their corresponding sides?

D. If two triangles have their corresponding angles congruent, what is true of the measures of their corresponding sides?

E. What new vocabulary have you learned today?

F. What new symbol did you learn today?

Lesson 21

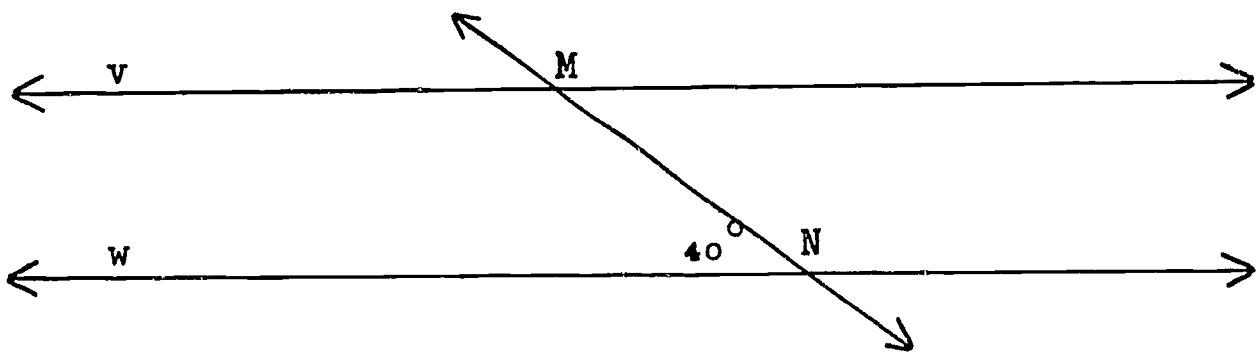
Topic: Parallel Lines

Aim: To discover that if two parallel lines are cut by a transversal, the corresponding angles are congruent

Specific Objectives:

* To review the meaning of parallel lines
  To discover that if two parallel lines are cut by a transversal, the corresponding angles are congruent

Challenge:



If line v is parallel to line w, without using your protractor, find the measures of the angles whose vertex is M.

I. Procedure

A. Review the meaning of parallel lines.

1. Have pupils recall the definition of parallel lines. Emphasize that the lines must be in the same plane and that they do not intersect.

2. Have pupils describe the set of points in the intersection of two parallel lines as the empty (null) set.

3. Elicit illustrations of parallel lines in life situations.

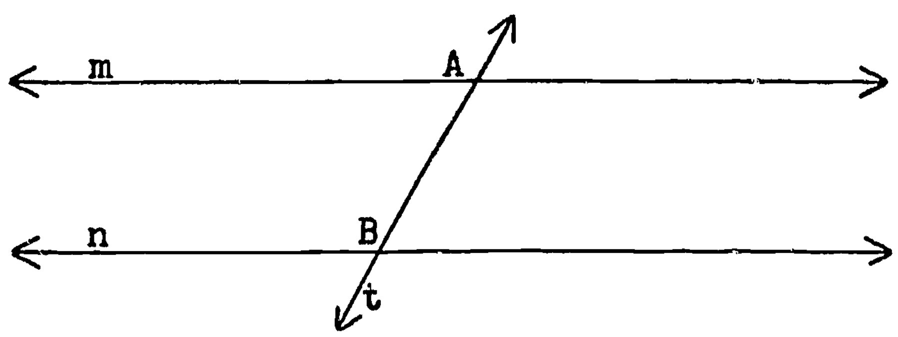
4. Give an example of lines which are not parallel but which never intersect. For example, have pupils consider the two lines suggested by the intersection of the front wall and the ceiling, and the intersection of the side wall and the floor. Will these lines intersect? Are these parallel lines? (No, because they do not lie in the same plane.)

Have pupils recall that two lines that do not intersect, and do not lie in the same plane are called <u>skew</u> lines.

B. To discover that pairs of angles of parallel lines have the same measure

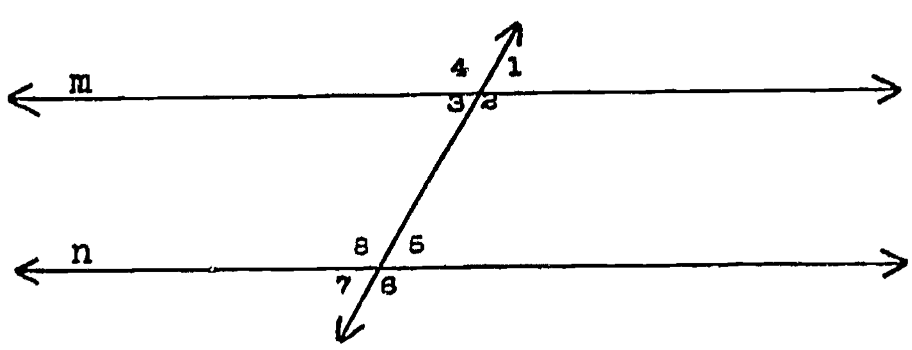
Note to Teacher: It is suggested that drawings representing parallel lines be done in notebooks using the printed lines as guides.

1. Have pupils represent two parallel lines about two inches apart intersected by a third line.



2. Tell pupils that a line which intersects two or more lines in distinct points is called a transversal of these lines. (Since line t intersects line m at point A and line n at point B, t is a transversal of lines m and n.)

3. If two parallel lines are cut by a transversal, angles are formed. To discuss the relations existing among the angles, we call angles 1, 4, 6, 7 exterior angles, and angles 2, 3, 5, 8 interior angles.



Two angles with different vertices such as $\angle 1$ and $\angle 5$, one exterior and one interior, but on the same side of the transversal, are called corresponding angles. What are the other pairs of corresponding angles? ($\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$, $\angle 3$ and $\angle 7$)

4. Have pupils use their protractors to measure angles 1 and 5. List on the chalkboard the results obtained by several pupils. Similarly, list measures of other pairs of corresponding angles.

5. Elicit that from the results above, it appears that if two parallel lines are intersected by a transversal, pairs of corresponding angles are congruent.

6. Have pupils represent two lines which are not parallel and which are intersected by a transversal. Measure a pair of corresponding angles. List a number of the results obtained by several pupils. Elicit that if the lines are not parallel, it appears that the corresponding angles do not have the same measure. Tell the pupils that this fact will be proven in a future course.

7. Return to challenge:

   In the figure, without using a protractor, find the measures of the other angles with the vertex N. Explain your answers.

   Indicate the measure of one angle with its vertex at M. Explain.

   Indicate the measures of the other three angles with the vertex M.

   If lines v and w were not parallel, could you answer the challenge question? Explain.

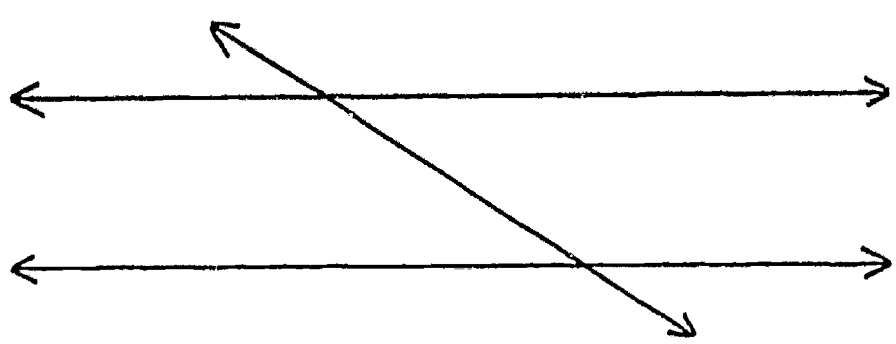
8. OPTIONAL

   Follow the same procedure to develop the concept that when two parallel lines are intersected by a transversal, the alternate interior angles ($\angle 2$ and $\angle 8$; $\angle 3$ and $\angle 5$) are congruent.

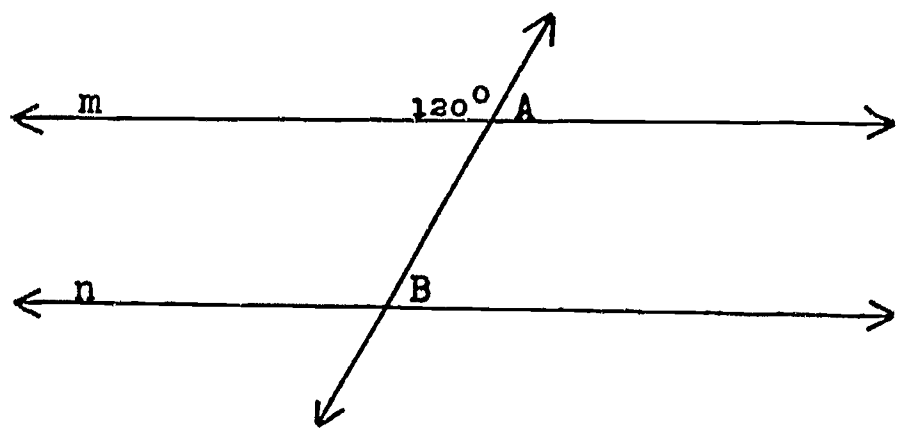
II. Practice

A. Find examples of skew lines in the classroom.

B. Indicate on the following diagram 4 pairs of corresponding angles.



C. In the diagram, if one of the angles with the vertex at A is 120°, find the measures of the other three angles with vertex at A, without using a protractor.



If line m is parallel to line n, indicate the measures of the angles whose vertex is B. Explain.

III. Summary

A. What are two necessary conditions for lines to be parallel?

B. Complete the sentence:

If two _____ lines are intersected by a transversal, the _____ angles are congruent.

C. What new vocabulary have you learned today?

Lesson 22

Topic: Parallel Lines

Aim: To construct a line parallel to a given line through a point not on the line

Specific Objectives:

To learn that if two lines are intersected by a transversal so that a pair of corresponding angles are congruent, the lines are parallel

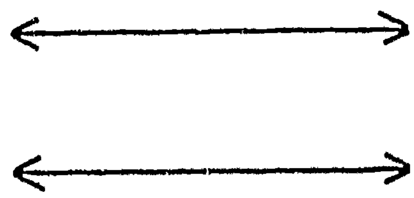
To construct a line parallel to a given line through a point not on the line

Challenge: How can you construct a line through point R which will be parallel to line s?

I. Procedure

A. If two lines are intersected by a transversal so that a pair of corresponding angles are congruent, the lines are parallel.

1. Draw a pair of lines on the chalkboard.



Are these lines parallel?

Discuss the need for a test by which we might determine if the lines are parallel.

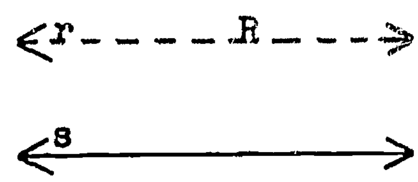
2. Recall that if two parallel lines are intersected by a transversal, the corresponding angles are congruent; if the lines are not parallel, the corresponding angles are not congruent.

3. Suggest drawing a transversal intersecting the lines. Test whether a pair of corresponding angles are congruent. Have pupils conclude that it is reasonable to assume if the corresponding angles are congruent, the lines are parallel. We will use this as a test for parallelism.

B. To construct a line parallel to a given line through a point not on the line

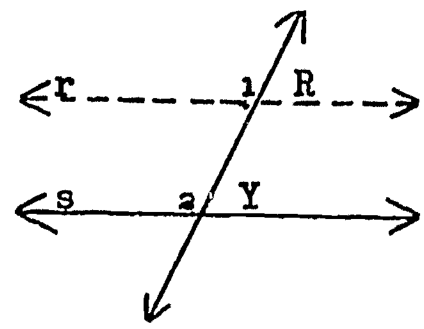
Return to challenge.

1. Consider how the completed drawing would appear.



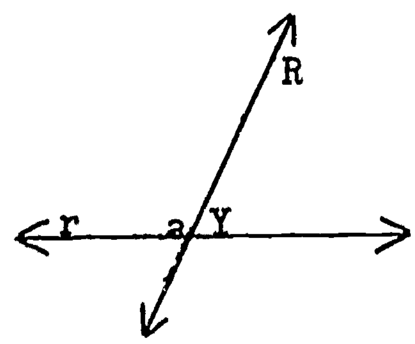
Have pupils draw a dotted line through point R which would appear to be parallel to line s. Name the line r.

2. Discuss the test for parallelism. Have pupils draw a transversal through R intersecting line s at point Y. Indicate a pair of corresponding angles by the numerals 1 and 2 as on the diagram at the right.



If $\angle 1$ and $\angle 2$ are congruent, what is true of lines r and s?

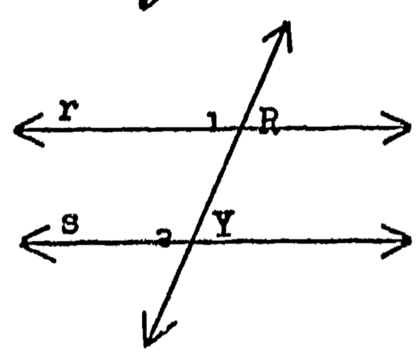
3. Refer to the challenge.



Keeping the trial figure as a guide, use the following steps: (try to elicit these from the class)

a. Draw a line through R cutting line s at Y. Indicate angle 2.

b. With point R as a vertex, use your compass to construct an angle, $\angle 1$, congruent to $\angle 2$. Extend the side of the angle through R and indicate the line r.

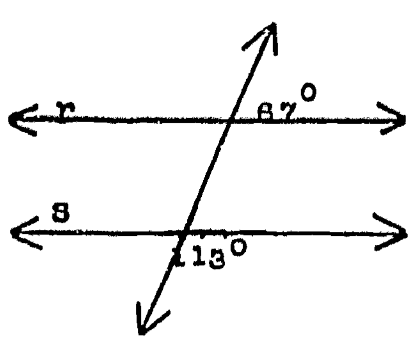


Is line r parallel to line s? Why?

II. Practice

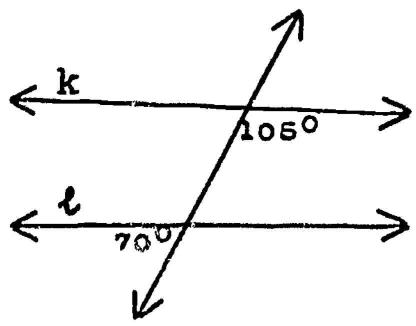
A. Examine the drawings that follow:

1.



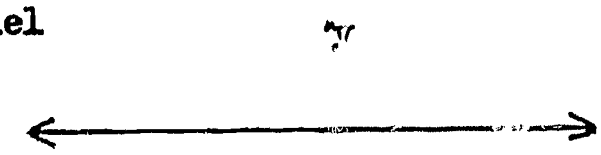
1. Is line r parallel to line s? Explain.

2.



2. Is line k parallel to line ℓ? Explain.

B. Through point y draw a line parallel to line k.



III. Summary

A. What is a test for determining whether two lines are parallel?

B. Describe the steps used in drawing a line parallel to a given line through a point not on the lines.

Lesson 23

Topic: Parallelograms

Aim: To learn some of the properties of a parallelogram

Specific Objectives:

To learn to construct a parallelogram
To discover that the opposite sides of a parallelogram are congruent and the opposite angles of a parallelogram are congruent

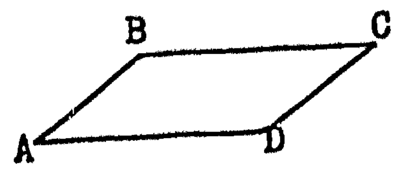
Challenge: Construct a parallelogram so that two of its vertices will be points A and D.

A D

I. Procedure

A. Construction of a parallelogram

1. Have pupils recall that in a quadrilateral, two sides (line segments) which do not intersect are called opposite sides. In the quadrilateral at the right, name the pairs of opposite sides.

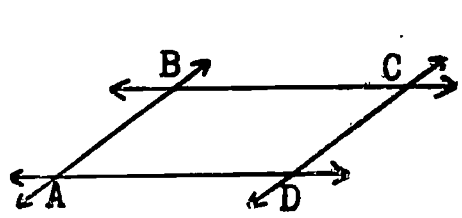


2. A quadrilateral with both pairs of opposite sides parallel is called a <u>parallelogram</u>.

3. In reference to the challenge, have pupils draw a line through A and D and indicate a point B outside the line.

Draw a line through points A and B.

Through B construct a line parallel to $\overleftrightarrow{AD}$ and through D construct a line parallel to $\overleftrightarrow{AB}$. Name the point of intersection of these lines C.



What kind of quadrilateral is formed by the union of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$? Why?

B. The opposite sides of a parallelogram have the same length

1. In their parallelograms, have pupils measure the lengths of sides AD and BC, and the lengths of sides AB and DC. List some of the pupils' results on the chalkboard.

2. Have pupils use protractors to measure $\angle ABC$ and $\angle ADC$. Then measure $\angle BAD$ and $\angle BCD$. List some of the pupils' results on the chalkboard.

3. Lead pupils to see from a study of the results obtained above that although the parallelograms may differ in size and shape, in each case the opposite sides of the parallelogram have the same measure, and the opposite angles have the same measure.

4. Have pupils consider a rectangular sheet of paper. Is this a model of a parallelogram? Lead pupils to realize that a rectangle is a special kind of parallelogram, in which all the angles are right angles.

5. Tell pupils that a parallelogram whose <u>four</u> sides have the same measure is called a <u>rhombus</u>. If a rhombus has four right angles, it is called a <u>square</u>.

II. Practice

A. Which of the following figures are parallelograms, assuming that segments which appear to be parallel are parallel? Why?

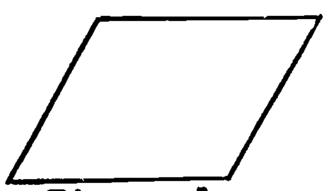


Figure 1

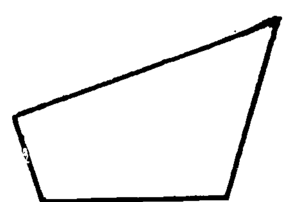


Figure 2

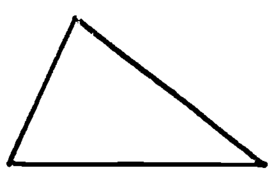


Figure 3

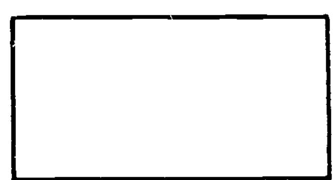


Figure 4

B. Which of these sentences are true? Which are false? Explain your answers.

1. All parallelograms are polygons.

2. All parallelograms are rhombuses.

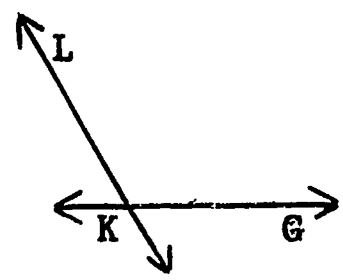
3. All rectangles are parallelograms.

4. All rectangles are squares.

5. All squares are rectangles.

C. If we show by Venn diagram the intersection of the set of rhombuses and the set of rectangles, what set of geometric figures is represented by this intersection? (set of squares)

D. Consider the drawing at the right:



Through point L construct a line parallel to $\overleftrightarrow{KG}$.

Through point G construct a line parallel to $\overleftrightarrow{KL}$.

Call the point of intersection of these lines M.

What kind of geometric figure is LKGM? Why?

III. Summary

A. What is true of the opposite sides of a parallelogram? of the opposite angles?

B. What kind of rhombus is a square?

C. What new vocabulary have you learned today?

(parallelogram, rhombus)

CHAPTER III

SQUARE AND CUBIC MEASUREMENT

In this chapter are suggestions for developing intuitive approaches to the discovery of a formula for each of the following:

area of a parallelogram region
area of a triangular region
area of a circular region
volume of a rectangular solid
volume of a cylindrical solid

The development of understandings and skills associated with the measurement of selected polygonal regions and solids is continued in this chapter.

A simple closed curve such as a rectangle, parallelogram, triangle, circle, etc., separates the plane in which it lies into three sets of points. One set is called the interior and one set is called the exterior. The set of points that constitutes the plane figure itself is called the boundary.

The union of a simple closed curve with its interior is called a region. The concept of region is basic to the understanding of area. Mathematically speaking, a polygon has no area since it is a union of line segments which have length but no thickness. Area, then, is the measure of a region.

Procedures are suggested to help the pupil "transform" a rectangle into a parallelogram without changing the dimension of either the base or the height. They are then led to the realization that the area of this parallelogram region is equal to that of the original rectangular region.

The area formula for a triangular region is developed inductively by relating the area of a triangular region to the area of a parallelogram region. The approach used in developing a formula for the area of a circular region is to compare its area with the area of an inscribed regular polygon region. Pupils are guided toward an intuitive feeling that the formula for the area of a circular region is reasonable.

The understanding that a solid is the union of the set of points on the space figure and the set of points in its interior is developed. The development of the volume formula for the rectangular solid and for the cylindrical solid is inductive and intuitive. The more general volume formula $V = Bh$ has greater applicability than $V = \ell wh$. The general formula can be used in computing the volume of any prism regardless of the polygonal region that constitutes its base. In the form $V = Bh$, the formula serves to unify the concept and computation of volume since it can also be used to find the volume of cylinders.

CHAPTER III

SQUARE AND CUBIC MEASUREMENT

Lessons 24-33

Lessons 24 and 25

Topic: Area of a Parallelogram Region

Aim: To develop a formula for computing the area of any parallelogram region

Specific Objectives:

To review the meaning of area
To find the area of any parallelogram region indirectly by use of a formula
To discover functional relationships in a parallelogram (optional)

Challenge: Which has the greater area, the rectangular region in Figure 1, or the parallelogram region in Figure 2?

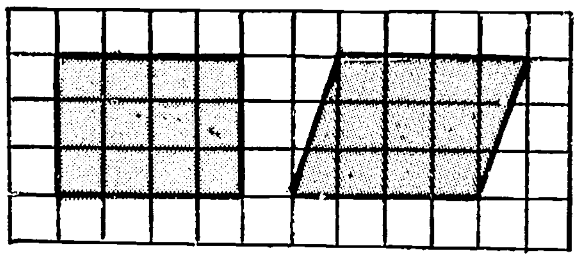


Fig. 1 Fig. 2

I. Procedure

A. Review the meaning of area

1. Recall that the measure of a line segment is called its length. What are some standard units of linear measure? (inch, foot, yard, meter, centimeter)

2. What do we mean by a rectangular region? (the union of the set of points on the rectangle and in its interior) What do we mean by a parallelogram region?

3. What do we call the measure of a closed region? (area)

4. What are some standard units of area measure in the English system? (square inch, square foot, square yard, square mile, acre)

5. What are some standard units of area measure in the metric system? (square meter, square centimeter)

B. Indirect measurement of a parallelogram region

1. Refer to challenge.

   a. Let us consider each of the square regions on the graph paper as the <u>unit</u> of area measure.

      What is a direct method of finding the area of the rectangular region in Figure 1? (Count the number of square regions of unit area that "fit" into the rectangular region. There are 12 such unit areas in the rectangular region.)

   b. Using this same method of counting unit areas (whole units as well as parts), what is the area of the parallelogram region in Figure 2? (It appears to be 12 units.)

   c. Elicit that both regions appear to have the same area.

2. Elicit that direct counting of unit areas is not always necessary and not always possible in determining the measure of a region.

   Have pupils recall the use of the formula $A = \ell \times w$ in computing the area of a rectangular region (indirect measurement of area).

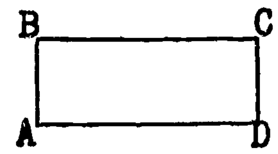
3. Tell pupils that the formula for the area of a rectangular region is often expressed as $A = b \times h$, where b stands for the measure of the base of the rectangle, and h stands for the height or the measure of its altitude.

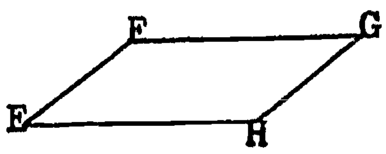
   Have pupils use the formula $A = b \times h$ to compute the area of the rectangular region in the challenge problem, and in several similar problems.

4. Discuss with pupils the desirability of a method of indirect measurement of a parallelogram region.

5. Have pupils consider these figures:

   a. Elicit that in the rectangle ABCD, $\overline{AD}$ is referred to as a base of the rectangle; $\overline{AB}$ is referred to as its altitude.





b. In the parallelogram EFGH, which line segment corresponds to $\overline{AD}$, the base of the rectangle? ($\overline{EH}$)

$\overline{EH}$ is referred to as a base of this parallelogram. Any side of a parallelogram may be considered a base.

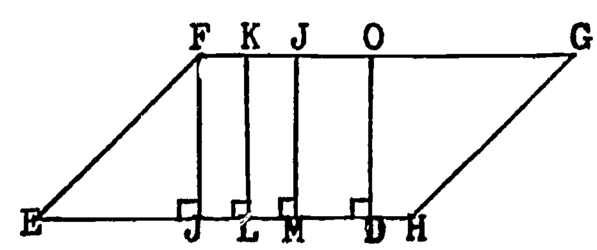
c. Which line segment in the parallelogram corresponds to $\overline{AB}$, the altitude of the rectangle? (No such line segment appears as yet in the diagram.)

Have pupils discuss what is meant by height of a person, room, mountain, etc. Guide them to see that height is the measure of the altitude or is the perpendicular distance an object rises from its base or the level on which it stands.

Have pupils recall that when two lines meet at right angles, they are said to be perpendicular to each other. The right angle may be indicated by a symbol:

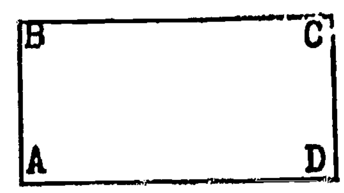
Elicit that the altitude of a parallelogram is a line segment perpendicular to a base, or to the line containing the base, from any point on the side opposite the base.

d. Have pupils see that any of the following perpendicular line segments $\overline{FJ}$, $\overline{KL}$, $\overline{NM}$, $\overline{OP}$, are altitudes if $\overline{EH}$ is the base of the parallelogram EFGH. All these altitudes have the same measure.

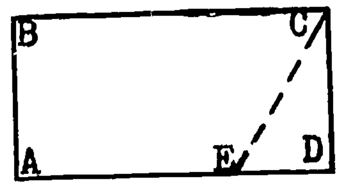


6. Demonstrate that a rectangular region may be "converted" into a parallelogram region without change in area as follows:

a. Using a rectangular sheet of paper or cardboard, label the vertices of the rectangle ABCD.

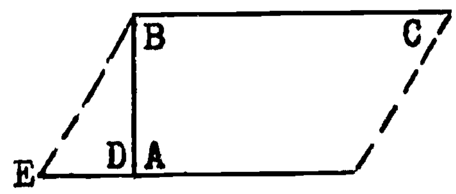


b. Draw dotted line segment CE to form triangular region CED.



Cut out triangular region CED and place it so that $\overline{CD}$ coincides with $\overline{BA}$ as shown in diagram.

c. Which type of polygon seems to have been formed? (parallelogram) Tell pupils that in later courses they will be able to prove that this is a parallelogram.



How does the area of the original rectangular region compare with the area of the parallelogram region?

d. What is the formula for the area of the rectangular region? ($A = b \times h$ or $A = bh$)

e. How does the measure of the base of the original rectangle compare with the measure of the base of the parallelogram formed? (The measures are the same.)

f. How does the measure of an altitude (height) of the rectangle compare with the measure of an altitude of the parallelogram? (Their measures are the same.)

7. Elicit that since the area of the parallelogram region formed is identical with the area of the original rectangular region we may use the formula $A = b \times h$ to compute the area of the parallelogram region, where A stands for the measure of the <u>area</u> of the parallelogram region, b stands for the measure of the <u>base</u> of the parallelogram, and h stands for the measure of its <u>altitude</u> or (<u>height</u>).

8. Guide pupils to discuss the advantage of an indirect method of determining the area of a parallelogram region by using a formula to compute the area.

Lead them to see, however, that in indirect measurement direct measurement is also involved. Thus, we measure the base and the altitude of the parallelogram directly before we use a formula to compute the area of the region. Remind pupils that the measures of the base and of the altitude must be expressed in the same unit.

9. Have pupils use the formula A = bh to compute the area of the parallelogram region in the challenge.

C. OPTIONAL

Functional relationships among the measures of a base and altitude, and the area of a parallelogram

The following procedure may be used to demonstrate how the area of a parallelogram region <u>changes</u> if the size of a base remains the same, but the size of the altitude changes.

1. Use a flexible model of a parallelogram, or make such a model with two pairs of strips of cardboard as shown.

   At the corners, secure the strips with paper fasteners so that the strips may be moved.

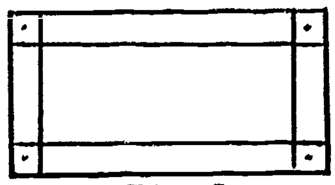


Fig.3

2. Show the model of the parallelogram in position of Figure 4 at the right.

   How do you find the area of the parallelogram region?

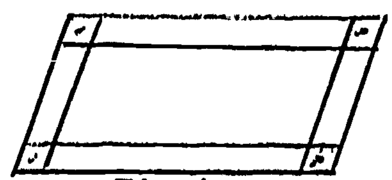


Fig.4

3. Using the same parallelogram, show it in the position of Figure 5.

   Compare the measures of the bases and of the altitudes of the parallelograms in Figures 4 and 5. Compare the areas of the two parallelogram regions.

   Why is the area of the parallelogram region pictured in Figure 5 less than the area of the parallelogram in Figure 4? (The size of the bases is the same in each, but the height of the parallelogram shown in Figure 5 is less.

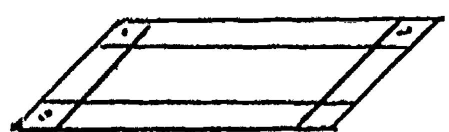
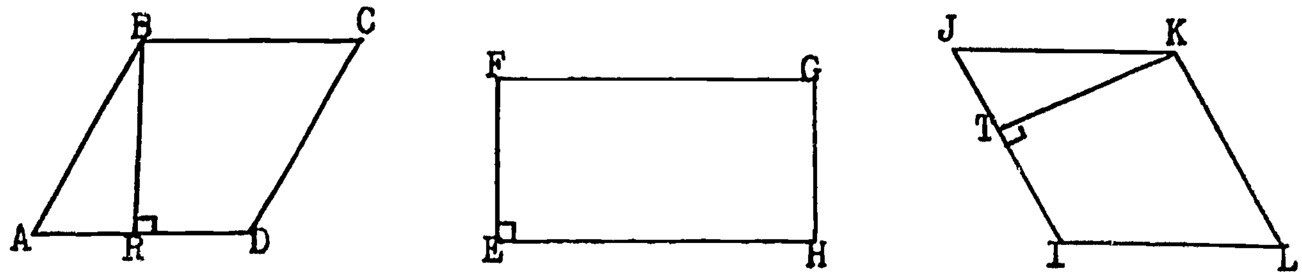


Fig.5

4. What are the only measures that affect the area of a parallelogram region? (the measure of a base and the measure of an altitude to that base)

II. Practice

A. Consider the following parallelograms:



1. In parallelogram ABCD, which line segment is a base to which an altitude has been drawn? Name the altitude.

2. What special kind of parallelogram is EFGH? Name a base. ($\overline{EH}$) Name an altitude to that base. ($\overline{FE}$ or $\overline{GH}$)

3. In parallelogram IJKL, which line segment is a base to which an altitude has been drawn? ($\overline{IJ}$) Name the altitude. ($\overline{KT}$)

B. Sketch a suitable figure and compute the areas of parallelogram regions with the following dimensions. Use the formula: $A = b \times h$.

	Base	Height
1.	12 in.	7 in.
2.	4 ft.	20 in.
3.	9 m	8 m
4.	13.5 cm	9.1 cm
5.	$3\frac{1}{2}$ yd.	$7\frac{3}{4}$ yd.

C. OPTIONAL

The altitudes of two parallelograms are 8 inches and 10 inches respectively. Their bases are each 9 inches. Find the area of each parallelogram region.

What is the ratio of the area of the first region to the second? What is the ratio of the first height to the second?

How does the ratio of the areas compare with the ratio of the heights? What appears to be true concerning the ratio of the areas of two parallelogram regions that have bases of the same size?

The bases of two parallelograms are 12 inches and 9 inches long respectively. Their altitudes are each 7 inches long. Find the area of each parallelogram region.

What is the ratio of the area of the first region to the second? What is the ratio of the measure of the first base to the second?

How does the ratio of the areas compare with the ratio of the measures of the bases?

What appears to be true concerning the ratio of the areas of two parallelogram regions that have altitudes of the same length?

III. Summary

A. What do we call the measure of a line segment? (length)

B. What do we call the number of times a unit square region is contained in a given surface? (the numerical measure of the surface or its <u>area</u>)

C. What is meant by an altitude of a parallelogram?

D. What is the formula used to compute the area of a parallelogram region?
What does each letter of the formula represent?

E. What new vocabulary have we learned today?

(altitude)

Lessons 26 and 27

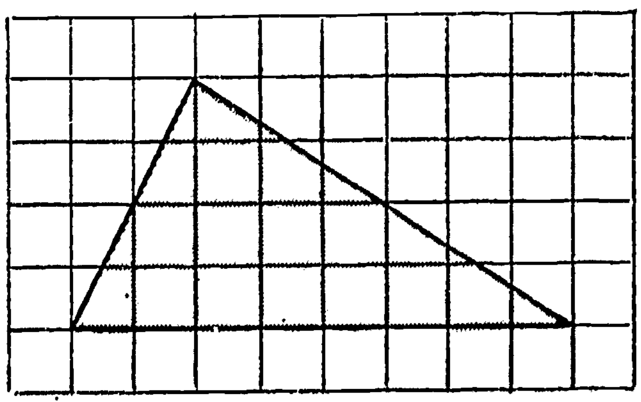
Topic: Area of a Triangular Region

Aim: To develop a formula for computing the area of any triangular region

Specific Objectives:

Meaning of base and altitude of a triangle
Relating the area of a triangular region to the area of a parallelogram region
Formula for computing the area of a triangular region

Challenge: What is the area of the triangular region shown?



The triangle is drawn against a background of square units.

Why is counting these units not a very practical method of computing area of the triangular region?

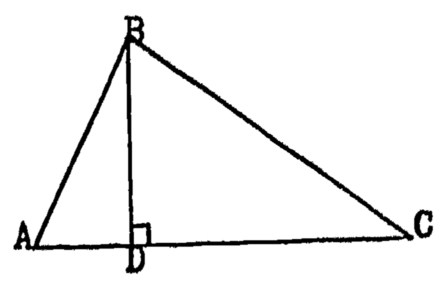
Elicit that there are too many parts of unit squares which are of different sizes to make the counting method practical. Therefore, it is desirable to discover an indirect method for computing the area of a triangular region.

I. Procedure

A. Meaning of base and altitude of a triangle

1. Consider triangle ABC.

a. Have pupils recall that in a parallelogram any side may be considered a base. In the same way, any side of a triangle may be considered a base of the triangle. Thus, $\overline{AC}$ is a base of triangle ABC.



b. Have pupils recall that in a parallelogram an altitude is a line segment which is perpendicular to the base from any point on the side opposite the base.

Elicit from pupils that an <u>altitude</u> of a triangle is a perpendicular line segment from a vertex to the line which <u>contains</u> the opposite side. Which is the altitude to $\overline{AC}$? ($\overline{BD}$)

c. Have pupils practice naming bases and altitudes of triangles such as those indicated below:

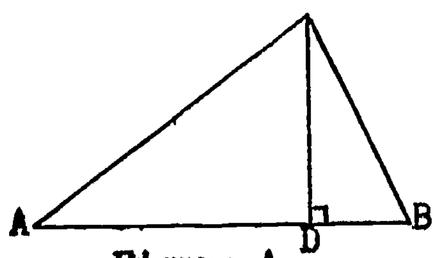


Figure A

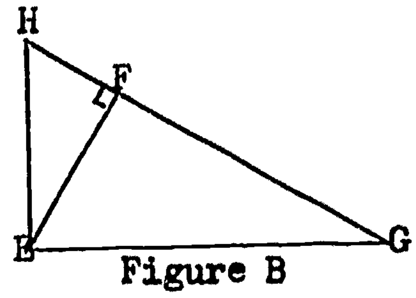


Figure B

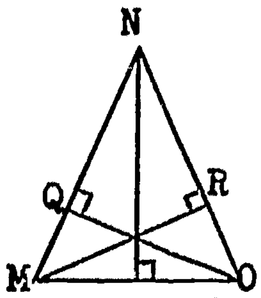


Figure C

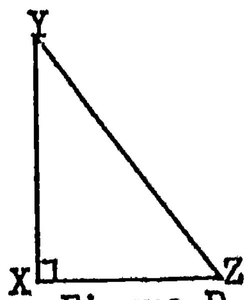


Figure D

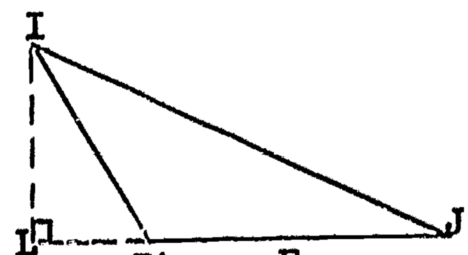


Figure E

1) In Figure A, name the base to which an altitude is drawn. Name the altitude.

2) In Figure B, name the altitude which is drawn. Name the base to which it is drawn.

3) In Figure C, if $\overline{NO}$ is considered the base, the altitude is _____; if $\overline{MO}$ is considered the base, the altitude is _____; if $\overline{MN}$ is considered the base, the altitude is _____?

4) In Figure D, what <u>kind</u> of triangle is triangle XYZ? In triangle <u>XYZ</u>, if $\overline{XZ}$ is considered the base, name the altitude; if $\overline{XY}$ is considered the base, name the altitude.

5) In Figure E, what kind of triangle is IKJ? If $\overline{KJ}$ is considered the base, name the altitude.

Note: Since an altitude is perpendicular to the line which contains the opposite side, have pupils realize that sometimes the altitude lies outside the triangle.

2. What dimensions of a parallelogram are used to find the area of a parallelogram region? (measures of base and altitude)

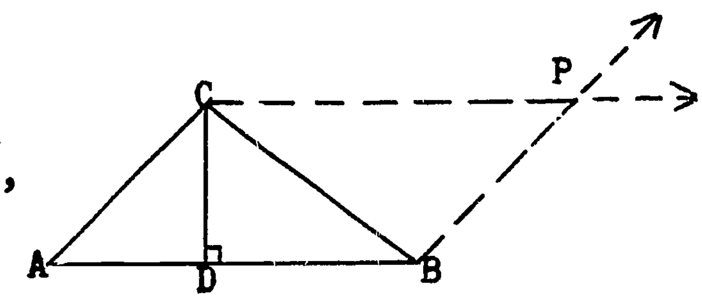
   What dimensions of a rectangle are used to find the area of a rectangular region? (measures of base and altitude)

   What dimensions of a triangle do you think are used to find the area of a triangular region? (measures of base and altitude)

B. Relating area of a triangular region to the area of a parallelogram region

1. Consider any triangle ABC with $\overline{AB}$ as base and $\overline{CD}$, the altitude to that base.

   Through C and B draw lines parallel to $\overline{AB}$ and $\overline{AC}$, meeting in some point, P.



   What geometric figure is ACPB? (parallelogram) Elicit that a base of parallelogram ACPB is $\overline{AB}$, the base of original triangle, and that an altitude of the parallelogram is $\overline{CD}$, the altitude of the original triangle ABC.

2. Consider the two triangular regions, ABC and CBP formed by $\overline{CB}$, the diagonal and the sides of the parallelogram. (A diagonal is a line segment which connects two non-adjacent vertices of a polygon.)

   What seems to be true of the areas of these two triangular regions? (Their areas seem to be the same.)

3. Consider parallelogram ABPC above in B-1. What part of the parallelogram region is triangular region ABC? Elicit that the area of the triangular region ABC seems to be one half of the area of the parallelogram region ABPC.

   Note to Teacher: The truth of this conclusion may be demonstrated as follows:

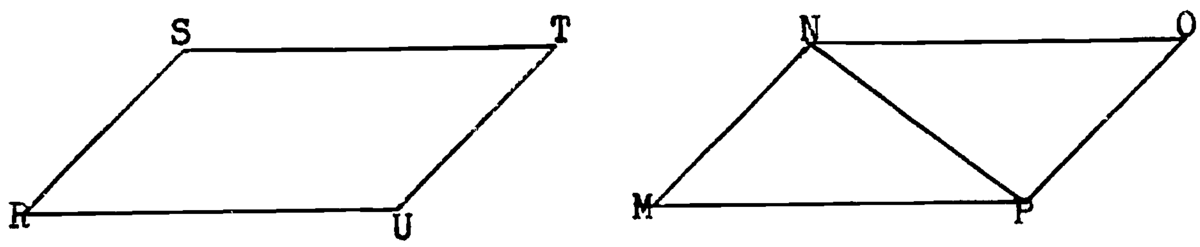
   Cut out two identical cardboard parallelogram regions.

Show the class that one parallelogram region can fit exactly over the other.

Put one aside for reference.

Draw a diagonal of the other parallelogram.

Cut along this diagonal, producing two triangular regions, MNP and PNO.



How can you show that the two triangular regions have the same area? (Show that a model of triangle MNP fits exactly on a model of triangle PNO.)

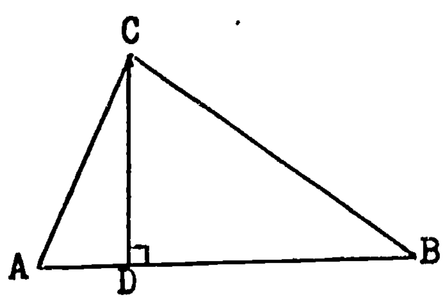
Referring to the reference parallelogram region, show that the two triangular regions cover the whole parallelogram region.

C. Formula for computing the area of a triangular region

1. Refer to parallelogram in B-1. What is the formula for computing the area of parallelogram region ABPC? ($A = bh$)

   What part of this parallelogram region is the triangular region ABC? (one-half) Then, how may we express the formula for computing the area of the triangular region ABC? ($A = \frac{1}{2}bh$)

2. Compute the area of a triangular region where the measure in inches of base AB of triangle ABC is 12, and the measure in inches of the altitude to that base is 10.

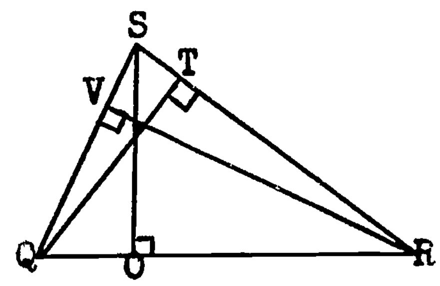


$A = \frac{1}{2}bh$
$A = \frac{1}{2} \times 12 \times 10$
$A = 60$

The area of the triangular region is 60 sq. in.

II. Practice

A. In triangle QRS,

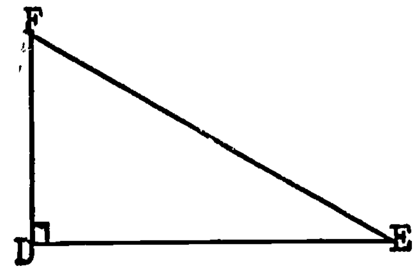


1. if $\overline{SR}$ is the base, name the altitude to that base;

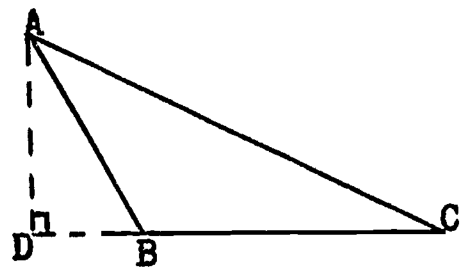
2. if $\overline{RV}$ is the altitude, name the base to which it is drawn;

3. if $\overline{QR}$ is the base, name the altitude to that base.

B. In right triangle DEF, if we consider $\overline{DE}$ the base, name the altitude.



C. In obtuse triangle ABC, name the altitude to base $\overline{BC}$.



D. Sketch a suitable figure and find the area of each triangular region:

Base	Altitude	Area
24 yd.	17 yd.	?
$7\frac{3}{4}$ ft.	10 ft.	?
12.5 cm	15.8 cm	?
6 ft.	28 in.	?

III. Summary

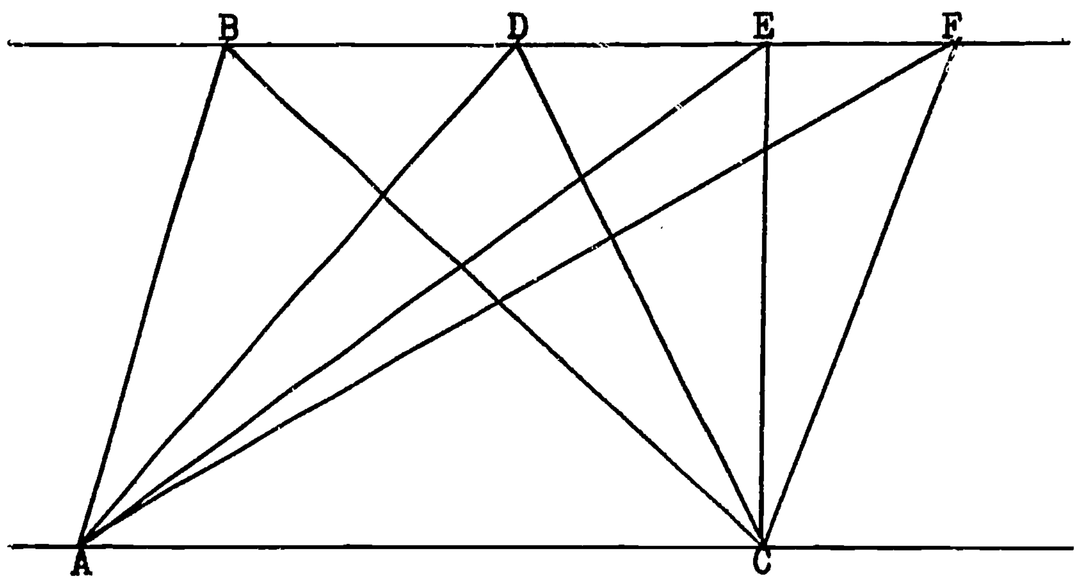
A. What is meant by an altitude of a triangle?

B. How is the formula for the area of a triangular region related to the formula for the area of a parallelogram region?

C. What is the formula used to compute the area of a triangular region?

What does each letter of the formula represent?

D. $\overleftrightarrow{BF}$ is parallel to $\overleftrightarrow{AC}$. Compare the areas of triangular regions ABC, ADC, AEC, AFC. (the same) Explain.



Lessons 28 and 29

Topic: Area of a Circular Region

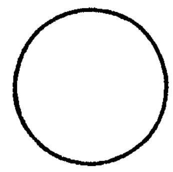
Note to Teacher: It is suggested that Figures 1 through 6 be reproduced on an acetate sheet for use with the overhead projector. Rexographed copies of these figures should be provided for use by the pupils.

Aim: To develop a formula for computing the area of any circular region

Specific Objectives:

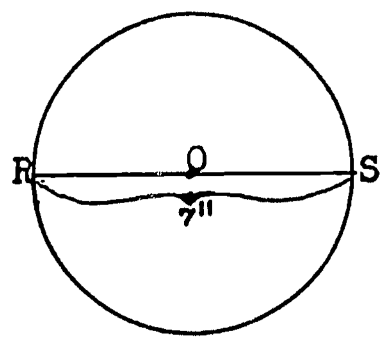
To review terminology connected with a circle and the formula for the circumference of a circle
To explore the direct measurement of a circular region
To develop a method of indirect measurement of a circular region by use of a formula

Challenge: How can we find the area of this circular region?



I. Procedure

A. Review terminology and formula for circumference of a circle



Consider circle, O, at the right.
What does $\overline{RS}$ represent? (diameter)
What does $\overline{OS}$ represent? (radius)
What is the measure in inches of $\overline{RS}$? (7)
What is the measure in inches of $\overline{OS}$? ($3\frac{1}{2}$)
What is the formula for computing the circumference of a circle?
What ratio does π express? (the ratio of the circumference to the diameter of the circle)
Compute the circumference of the circle using the formula:

$$C = 2\,\pi\, r,\ \pi = \frac{22}{7}$$

B. Direct measurement of a circular region

1. What kind of unit is used to measure the circumference of a

circle? (unit of linear measure, i.e., inches, feet, centimeters, meters, etc.)

2. What kind of unit must we use to measure the area of the circular region? (unit of square measure, i.e., square inches, square foot, square centimeters, square meters, etc.)

3. Have pupils try to approximate the area of a circular region by counting the number of square units enclosed by the circle.

   The figure below shows a circle whose radius is 5 units in length, enclosed in a square whose side is 10 units in length. Each small square represents one unit of area.

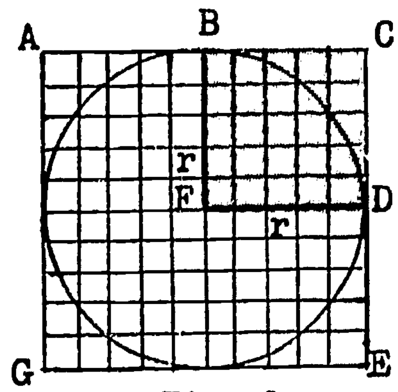


Fig. 1

   Elicit that the area of the square region BCDF may be expressed as $A = r^2$ and $r^2 = 25$.

   The area of region ACEG is how many times the area of region BCDF? (4 times) Then the area of region ACEG may be expressed as $4r^2$ and $4r^2 = 100$.

   Does the area of the circular region F appear to be greater or less than $4r^2$? (less)

   Have the pupils count the approximate number of square units in the circular region F. (about 80)

4. Guide pupils to see that while the number of square units in the circular region F(80) is less than $4r^2$(100), it is greater than $3r^2$(75).

$$75 < 80 < 100$$

and $$3r^2 < 80 < 4r^2$$

   Have them conclude that it seems that the area of a circular region is a little more than 3 times the square of the length of the radius of the circle.

5. Tell pupils that mathematicians have developed a formula by which it is possible to find the area of a circular region by indirect measurement.

C. Indirect measurement of a circular region

1. Consider these figures:

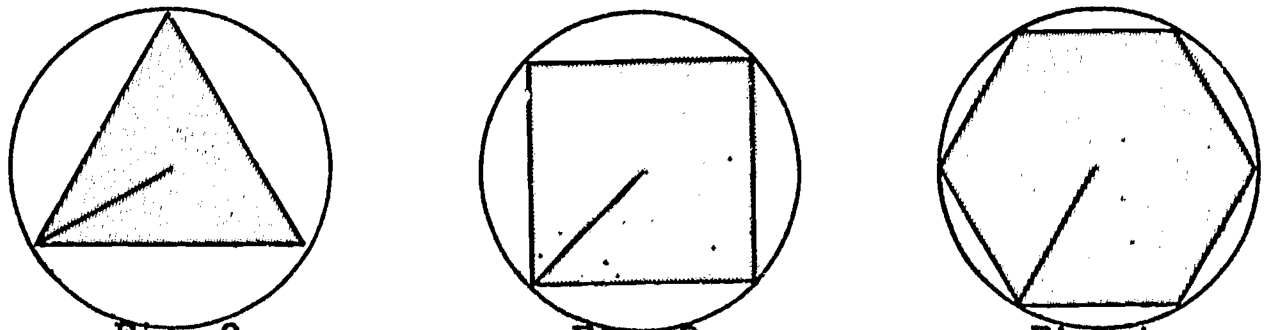


Fig. 2 Fig. 3 Fig. 4

Measure the radius of each circle. Without computing, what can you say about the circumferences of the three circles? Why?

Find the measure of the sides of the polygons in each of these figures.

Find the measure of the angles of the polygons in each of these figures.

Have pupils note that each of these polygons has sides of equal measure, and interior angles of equal measure. Such polygons are called <u>regular</u> polygons. They are said to be inscribed polygons because their vertices lie on the circle.

2. As the number of sides increases, how does the perimeter of the inscribed regular polygon compare with the circumference of the circle? (The length of the perimeter of the polygon gets closer to the circumference of the circle.)

3. As the number of sides increases, how does the area of the inscribed polygon region compare with the area of the circular region?

<u>Note</u>: Where the meaning is clear, we may write "area of a polygon" for "area of a polygon region"; "area of a triangle" for "area of a triangular region," etc. In a similar way, when no confusion will result, we may use the expression "the radius is 2 inches" for the more correct expression "the measure of the radius is 2 inches."

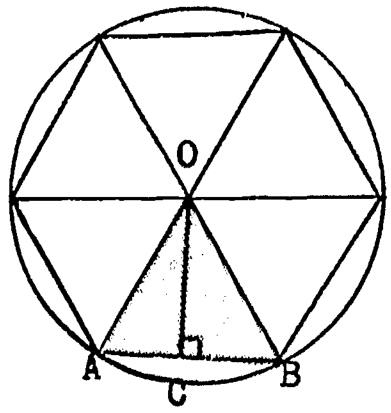


Fig. 5

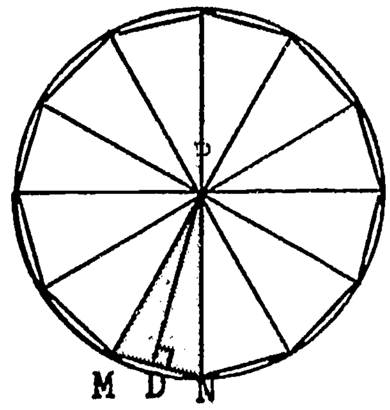


Fig. 6

4. Have pupils consider Figure 5 and Figure 6.

   a. How do the measures of the radii of the two circles compare? How do their circumferences compare?

   b. How does the number of triangles compare with the number of sides of the inscribed regular polygon in Figure 5? in Figure 6?

   c. If we knew the length of $\overline{AB}$, how would we find the perimeter of the polygon in Figure 5? (six times the length of $\overline{AB}$)

   d. If we knew the length of the line segment $\overline{MN}$ in Figure 6, how would we find the perimeter of that polygon? (twelve times the length of $\overline{MN}$)

   e. In general, if we know the length of one side of a regular polygon, how can we find the perimeter? (the measure of one side multiplied by the number of sides)

   f. Measure $\overline{OC}$ and $\overline{PD}$. How does the length of the altitude $\overline{OC}$ compare with the length of the altitude $\overline{PD}$? ($m(\overline{PD}) > m(\overline{OC})$)

   g. Suppose the polygon in Figure 6 had twice as many sides, would the measure of the (height) altitude of the new triangle be greater or less than $m(\overline{PD})$?

   h. Let us suppose we continue to increase the number of sides of the polygon in Figure 6 so that the perimeters of the polygons get closer and closer to the circumference of the circle.

   Suppose we divide each new inscribed polygon into triangles as was done in Figure 6. How will the measure of the altitude of each new triangle compare with the measure of the radius of the circle? (The length of each new altitude gets "closer and closer" to the length of the radius. We indicate this in symbols as: $h \to r$.)

5. Have pupils recall the formula for the area of a triangular region: $A = \frac{1}{2}bh$.

   a. Using this formula, we could find the area of the polygon region in Figure 5 by multiplying the area of one triangular region by the number of triangles, 6.

      How would you find the area of the polygon region in Figure 6?

      In general, how would you find the area of any regular inscribed polygon? (Find the area of one triangle and then multiply by the number of triangles.)

   b. Have pupils recall that the number of triangles is the same as the number of sides. If we indicate the number of sides of a regular inscribed polygon by n, we can express the area of the region as: $A = n \times \frac{1}{2}bh$.

   c. By use of the commutative principle of multiplication, lead pupils to see that this may be written as: $A = \frac{1}{2} \times nbh$.

6. Consider again Figure 5 and Figure 6.

   a. Have pupils recall that as the number of sides of a regular inscribed polygon increases, the perimeter, nb, of the polygon gets closer and closer to the circumference, c, of the circle.

      Since $c = 2\pi r$ $\quad$ $nb \to c$
      $nb \to 2\pi r$

      Also, as the number of sides increases, the altitude, h, of each triangle gets closer and closer to the radius, r, of the circle.

      $h \to r$

   b. Lead pupils to see that as the number of sides increases, the area of the regular inscribed polygon region gets closer and closer to the area of the circular region as a limit.

      Area of polygon region $\to$ Area of circular region

      Since $nb \to 2\pi r$, and $h \to r$, we may say $\frac{1}{2}\, nbh \to \frac{1}{2} \times 2\pi r \times r$, or $\frac{1}{2}\, nbh \to \pi r^2$.

c. Tell the pupils we can approximate the area of a circular region by use of the formula

$$\text{Area of circular region} = \pi r^2$$

II. Practice

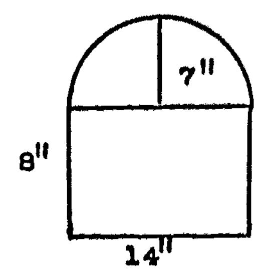
A. Compute the area of each of the circular regions for which the radius of the circle is given below: (Use $3\frac{1}{7}$ as the approximation for π in examples in Column 1, and 3.14 as the approximation for examples in Column 2.

Column 1	Column 2
1. 14 inches	3. 10 cm
2. 7 meters	4. 2.5 feet

B. A pony is tied to a stake 21' long. Over how many square feet of grass can he graze? ($\pi = 3\frac{1}{7}$)

C. A radio station's broadcast can be heard within a radius of 500 miles. Over how many square miles can the station be heard? ($\pi = 3.14$)

D. Compute the area of the region shown in the figure at the right.



E. OPTIONAL

If you double the radius of a circle, what happens to the area of the circular region?

III. Summary

A. What is the formula used to compute the area of a circular region?

B. What does each letter of the formula represent?

Lesson 30

Topic: Volume

Aim: To develop the meaning of volume

Specific Objectives:

To review space figures
To define volume
To show the need for a standard unit of measure of volume

Challenge: Clara made chocolate fudge which she cut into cubes. How many can she place in a box?

I. Procedure

A. Review of space figures

1. Refer to challenge. Why can we not tell how many cubes of fudge Clara can place in the box? (We do not know the size of the cubes or the size of the box.)

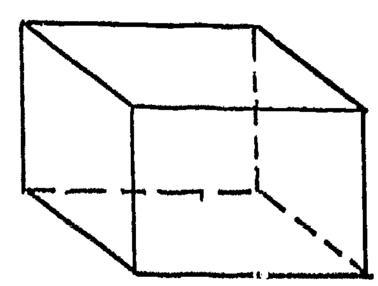
2. Have pupils recall that a plane figure is a geometric figure whose points are all in the same plane, while a space figure is a geometric figure not all of whose points are in the same plane.

3. Elicit that the cubes of fudge and the box are examples of space figures.

4. Have pupils name several objects which are examples of space figures: a book, the classroom, a ball, the teacher's desk, an orange juice can, a coil spring.

5. Recall that we can make drawings on a plane surface to represent a model of a space figure. Have pupils make drawings of: a cube of fudge, a candy box, a ball.

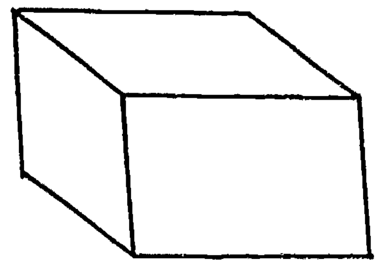
6. Refer to the drawing. Into how many sets of points does a closed space figure divide space? (Three: those outside the space figure, those on the space figure, those inside the space figure.)



Recall that when we speak of a closed space figure, i.e., a rectangular prism, a sphere, etc., we mean the set of points on the figure.

B. Definition of volume

1. Tell pupils that just as a plane region is the union of the set of points on the plane figure and the set of points in its interior, a solid is the union of the set of points on the space figure and the set of points in its interior.



2. Elicit that the measure of a region is called its area. Tell pupils that the measure of a solid is called its volume.

C. Need for a standard unit of measure

1. Elicit that just as we use a unit of length to measure a line segment, and a unit of area to measure a region, we need a unit of space or a unit solid to measure the volume of a solid.

2. Have pupils suggest various unit solids that can be used to measure volume. (rectangular solid, cubical solid, spherical solid)

   Lead pupils to see that a unit cubical solid would be best.

   Have pupils suggest standard units of measure of volume. (cubic inch, cubic foot, cubic centimeter, etc.)

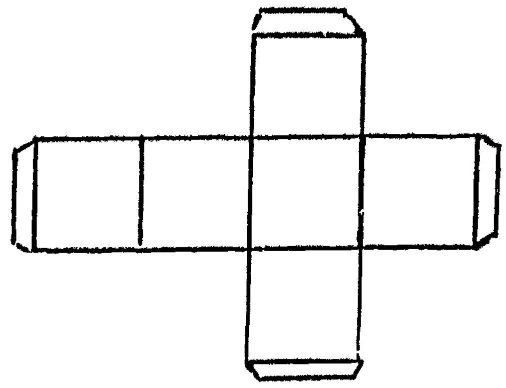
   Note: The abbreviation for cubic centimeter is cc

3. Refer to challenge again. Suppose Clara cuts the fudge into cubic inches. If ten pieces of fudge of this size fit into the box, what is the volume of the box? (10 cubic inches)

II. Practice

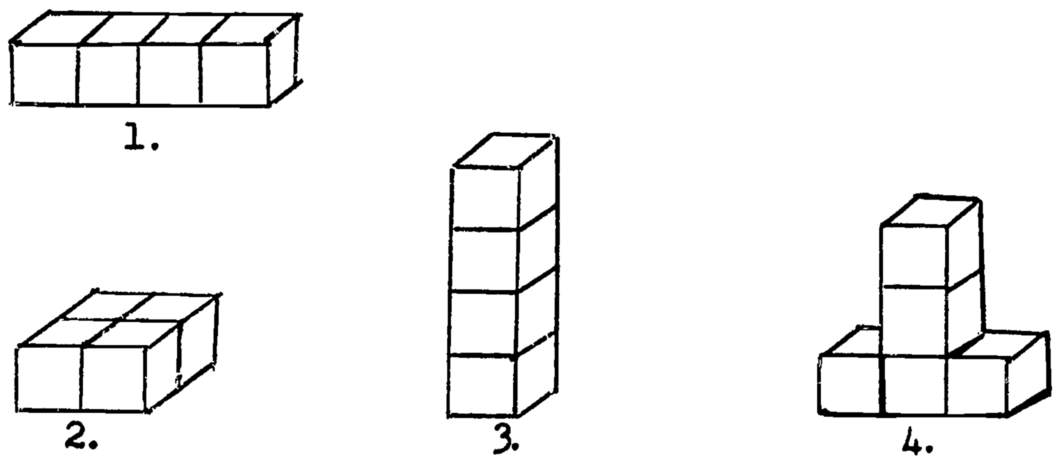
A. Have the pupils make a model of a cubic inch.

B. Have the pupils make models of cubes of various sizes. (This could be done for homework.)



C. Name 5 models of space figures found in the classroom.

D. What is the volume of each of the following figures formed by 1-inch cubes?



III. Summary

A. What do we mean by a solid?

B. What do we mean by volume?

C. What do we call the measure of a line segment? (length)

D. What do we call the measure of a region? (area)

E. What do we call the measure of a solid? (volume)

F. Name a standard unit to measure each.

G. What new vocabulary have we learned today?

(solid, volume, cubic inch, cubic centimeter, etc.)

Lesson 31

Topic: Volume

Aim: To develop the formula for the volume of a rectangular solid

Specific Objectives:

To find the volume of a rectangular solid by direct measurement
To develop the formula for the volume of a rectangular solid:
$V = \ell w h$ or $V = Bh$
To develop the formula for the volume of a cubical solid

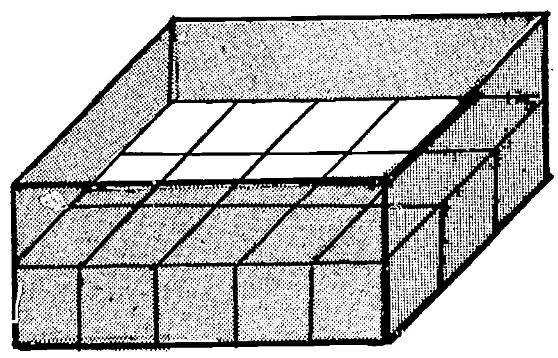
Challenge: A carton measures 5" long, 3" wide, and 2" deep.
How many cubic inches will fill the carton?

Note to Teacher: Bring a small box to class. The inside dimensions of the box may be used in preparing a problem for the challenge.

I. Procedure

A. Finding the volume of a rectangular solid directly

1. Refer to challenge. Elicit that we can find the volume of the carton by filling it with inch cubes. (Use the inch cubes made in the previous lesson.)



2. Have a pupil place one layer of inch cubes in the carton. Elicit that there are 5 rows of 3 cubes each, or 3 rows of 5 cubes each. In either case, one layer contains 15 cubes.

3. What is the height (depth) of the carton? How many layers of 15 cubes each does the box contain? (2) What is the volume of the carton? (two layers of 15 cubic inches or 30 cubic inches)

B. Developing the formulas: $V = \ell w h$ and $V = Bh$

1. Have pupils recall that a plane figure such as a parallelogram has two dimensions. How many dimensions does a space figure such as a rectangular solid have? (3)

2. Have pupils arrange the 30 one-inch cubes to form rectangular solids of different dimensions. Record the measures in table form as shown below:

Length	Width	Height	Volume
15	2	1	30
10	3	1	30
3	2	5	30

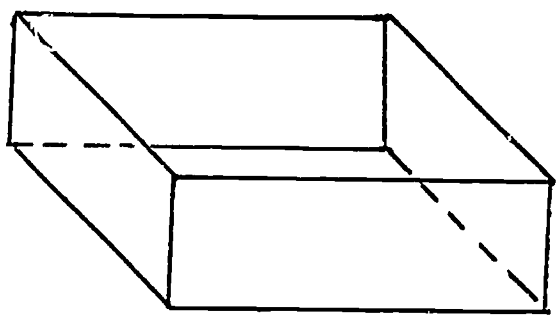
3. Guide pupils to see that the volume of a rectangular solid is equal to the product of the number of units in the length, in the width, and in the height. Tell pupils this can be stated as $V = \ell \times w \times h$ or $V = \ell\, w\, h$.

   Have pupils realize that the same standard unit must be used for each of the three dimensions.

4. Refer to challenge again. Find the volume of the carton using the formula:

   $V = \ell\, w\, h$
   $V = 5 \times 3 \times 2$ or 30 cubic inches

5. Refer to diagram. What does $\ell \times w$ represent? (area of the base)



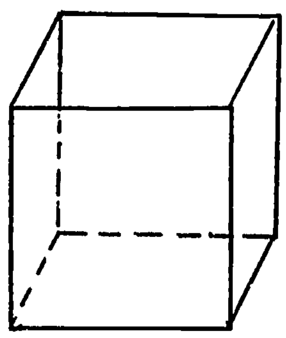
   Let us use B to represent the area of the base. Then we can express the rule for finding the volume of a rectangular solid in another way.

   Volume equals area of base x height. This rule may be expressed in the form $V = Bh$.

C. Formula for the volume of a cubical solid

Note: Tell pupils that in keeping with general practice, we will refer to the volume of a cubical solid as the volume of a cube.

1. Have pupils draw a model of an 8-inch cube.



2. Using the formula: $V = \ell w h$, compute the volume of this solid.

$V = 8 \times 8 \times 8$ or 8^3
$V = 512$

The volume is 512 cubic inches.

3. Elicit that all the edges of a cube have the same measure. If e is the measure of one edge of a cube, in the formula $V = \ell w h$, we may replace $\ell \times w \times h$ by $e \times e \times e$. The formula for the volume of a cube may then be expressed as: $V = e^3$.

II. Practice

A. Using the formula: $V = \ell w h$, compute the volume of the rectangular solids whose dimensions are as follows:

Measure of:

Length	Width	Height	Volume
9 in.	3 in.	2 in.	
1 ft.	4 ft.	6 in.	
3 cm	2 cm	1 cm	

B. The area of the base of a rectangular solid is 25 sq. yd., and its height is 3 yd. Find its volume.

C. An excavation for the basement of a house is to be 35 feet by 8 feet by 24 feet. How many cubic feet of earth must be removed?

D. If each edge of a cube is 3 feet, what is the volume of the cube?

E. Complete each of the following sentences:

1. The volume of a cube whose edge is 5 cm is ____. (125 cc)

2. The volume of a cube whose edge is 10 cm is ____. (1000 cc)

3. The length of the edge of the second cube is ____ times the length of the edge of the first cube.

4. The volume of the second cube is ____ times the volume of the first cube.

F. Show that 1 cubic foot = 1728 cubic inches.

G. If a classroom should have 120 cubic feet of space per pupil, how many pupils would be permitted in a room 20 ft. by 30 ft. by 10 ft.?

H. If the rate for water is 20¢ per 100 cubic feet, how much will it cost to fill a pool 15 ft. by 30 ft. by 10 ft.?

III. Summary

A. State two formulas for computing the volume of a rectangular solid.

($V = \ell\ w\ h$ and $V = Bh$)

B. In the formula: $V = Bh$, what does "B" represent?

C. If the rectangular solid is a cubical solid, what special formula can you use for finding the volume?

D. In the formula for the volume of a cubical solid, what does "e" represent?

Lesson 32

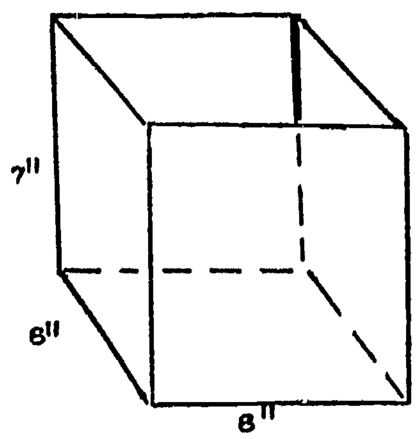
Topic: Volume

Aim: To develop the formula for the volume of a cylindrical solid

Specific Objectives:

To review the formula $V = Bh$ for the volume of a rectangular solid
To develop the formula for the volume of a cylindrical solid

Challenge: Consider the two containers.
Which will hold more sand?



I. Procedure

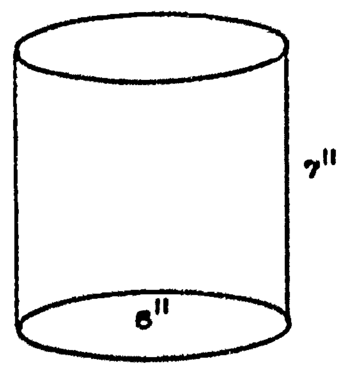
A. Review formula $V = Bh$

1. Have pupils recall the formula $V = Bh$ for finding the volume of a rectangular solid.

2. In the challenge, find the volume of the rectangular container.

$V = Bh$
$B = 6 \times 6$ or 36
$h = 7$
$V = 36 \times 7$ or 252

Volume is 252 cubic inches.



B. Formula for the volume of a cylindrical solid

1. Discuss with the pupils the use of $V = Bh$ to find the volume of the cylindrical container.

2. Elicit that the base of a cylinder has the form of a circle and that the formula for the area of a circular region is πr^2.

3. Replacing B with πr^2, express the formula for the volume of a cylindrical solid. ($V = \pi r^2 h$)

4. Return to the challenge. Have pupils use the formula: $V = \pi r^2 h$ to find the volume of the cylindrical container. (Use $\pi = \frac{22}{7}$)

$V = \frac{22}{7} \times 9 \times 7$

$V = 198$

Volume of this container is 198 cu. in.

5. Answer the challenge.

II. Practice

Using the formula $V = \pi r^2 h$, find the volume of the cylinders whose dimensions are: (Choose the approximation for π which will be most convenient.)

A. radius 7", height 2"

B. radius 5", height 8"

C. radius 9 cm, height 10 cm

D. Find the volume of each of the following:

Cylinder	Measure of Radius	Height
A	7 ft.	14 ft.
B	14 ft.	7 ft.

What is the ratio of the volume of cylinder A to cylinder B?

III. Summary

A. In the formula: $V = Bh$, what does B represent?

B. What is the formula for the area of a circular region?

C. What is the formula for the volume of a cylinder?

Lesson 33

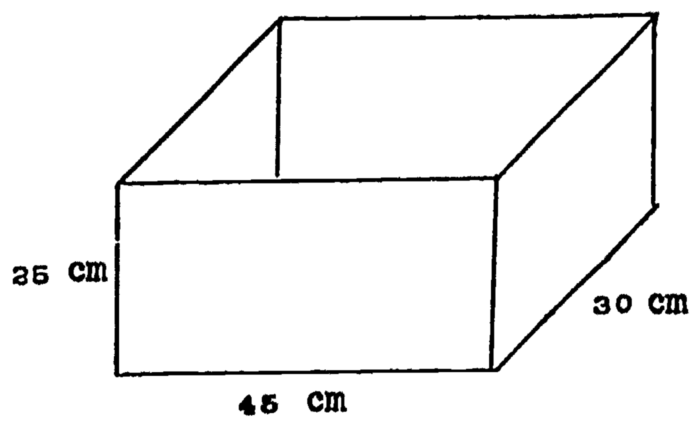
Topic: Volume

Aim: To learn the metric units of liquid measure

Specific Objectives:

To learn the meaning of a liter
To learn other metric units of liquid measure

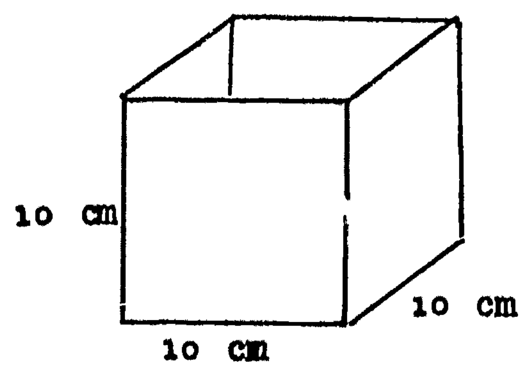
Challenge: How much water will this fish tank hold?



I. Procedure

A. Develop meaning of a liter.

1. Consider a rectangular container each inside edge of which is ten centimeters in length.



2. What is the measure of the space within this container? (10x10x10 = 1000 cc)

3. The amount of liquid, 1000 cc, that this container can hold, i.e., its capacity, is approximately 1 liter (1 ℓ).

4. Tell the pupils that the liter is the basic unit of liquid measure in the metric system, and that a liter is a little larger than a quart.

5. Return to challenge.

a. What is the measure of the space within the tank? ($45 \times 30 \times 25 = 33750$) The measure is 33750 cc.

b. How many cubic centimeters will contain 1 liter? (1000 cc)

c. How many liters of water will the fish tank hold? Since 1000 cc of space will hold approximately 1 liter of liquid, the tank will hold $33750 \div 1000$ or 33.75 ℓ.

B. Other metric units of liquid measure

1. Have pupils recall the meaning of the prefixes used in the metric system. What part of a meter is a decimeter? a centimeter? a millimeter? (.1 m, .01 m, .001 m)

2. Have pupils suggest the meaning of the terms: deciliter (dl), centiliter (cl), milliliter (ml), (.1 ℓ, .01 ℓ, .001 ℓ)

3. Express as liters: 2000 ml, 174 dl, 25 cl.

4. Tell the pupils that the milliliter is the unit of liquid measure they will use most frequently in science.

II. Practice

<u>Note</u>: Equal (=) means is equivalent to.

A. Complete the following:

1000 cc = ____ ℓ

100 cc = ____ ℓ

3500 cc = ____ ℓ

B. Complete the following:

1 ℓ = ____ mℓ

3.5 ℓ = ____ mℓ

2300 mℓ = ____ ℓ

700 mℓ = ____ ℓ

C. If the measure of each inside edge of a cube is 20 cm, how many liters of water will it hold?

III. Summary

A. What is the basic unit of liquid measure in the metric system?
B. Name some other units of liquid measure.
C. Discuss the relationship between the cubic centimeter and the liter.

CHAPTER IV

SYSTEMS OF NUMERATION

In this chapter the binary system of numeration is introduced, and the concepts of the quinary system presented in grade 7 ar' re-inforced and extended.

Among the topics presented are:

meaning of exponents
expressing numbers in expanded form using exponents
converting numbers expressed in base five to base ten; from base ten to base five
introduction to the binary system of numeration
converting numbers expressed in base two to base ten; in base ten to base two
addition and multiplication of numbers expressed as base-five or base-two numerals (optional)

The overall purpose of this chapter is to help the teacher strengthen the pupils' understanding of the decimal system of numeration. Studying numeration systems with a base other than ten enables the pupils to achieve greater insight into place value concepts and into the reasons for the usual procedures and techniques of computation.

A very important step in the development of numeration is the positional arrangement into an ordered sequence of consecutive powers of the base of the particular system of numeration. Pupils develop an understanding of this principle as they express numbers in expanded form as a sequence of powers of 10. This principle is then applied to systems in base five and in base two.

The lessons in addition and multiplication of numbers expressed in base five or base two are marked optional. However, for certain classes these lessons will affort an opportunity for increased appreciation of the techniques used in computing with numbers expressed in the decimal system of numeration.

The practice material in this and succeeding chapters should be supplemented by similar material contained in textbooks. Periodic practice in arithmetic fundamentals is also necessary to maintain computational skills.

CHAPTER IV

SYSTEMS OF NUMERATION

Lessons 34-42

Lesson 34

Topic: Decimal Notation

Aim: To reinforce expressing numbers in expanded form using exponents

Specific Objectives:

To reinforce:

the use of an expanded numeral to represent a number
the meaning of exponent
the writing of expanded numerals using exponents

Challenge: 7546 is the standard numeral for a number.
In what other ways can you express this number?

I. Procedure

A. To reinforce the use of an expanded numeral to represent a number

1. Elicit that 7546 can be expressed as 7 thousands + 5 hundreds + 4 tens + 6 ones.

   7546 could also be expressed as (7x1000) + (5x100) + (4x10) + (6x1).

2. Have pupils recall that (7x1000)+(5x100)+(4x10)+(6x1) is called an expanded numeral for 7546.

3. Have pupils write an expanded numeral for each of the following: 123; 539; 2734.

B. To reinforce the meaning of exponents

1. Elicit that 10x10 can be written as 10^2 and 10x10x10 can be written as 10^3.

2. Have pupils recall that in the expression 10^3, 10 is called the base and 3 the exponent. Elicit that the exponent indicates the number of times the base is a factor.

C. To reinforce the use of exponents to express numbers by expanded numerals

1. Have pupils write 7546 as an expanded numeral using exponents as follows:

$$7546 = (7\times1000) + (5\times100) + (4\times10) + (6\times1)$$

$$= (7\times10^3) + (5\times10^2) + (4\times10) + (6\times1)$$

2. In a similar way:

$$19050 = (1\times10000) + (9\times1000) + (0\times100) + (5\times10) + (0\times1)$$

$$= (1\times10^4) + (9\times10^3) + (0\times10^2) + (5\times10) + (0\times1)$$

3. How can .32 be expressed as an expanded numeral using exponents?

Have pupils recall that $.32 = .3 + .02 = \frac{3}{10} + \frac{2}{100}$

Elicit that a decimal fraction such as .3 may be written as $\frac{3}{10}$ or $3 \times \frac{1}{10}$ and .02 may be written as $\frac{2}{100}$ or $2 \times \frac{1}{10^2}$.

Therefore, $.32 = .3 + .02$

$$= (3\times\frac{1}{10}) + (2\times\frac{1}{10^2})$$

4. Express as expanded numerals using exponents

$$523.42 = 500 + 20 + 3 + .4 + .02$$

$$= (5\times100) + (2\times10) + (3\times1) + (4 \times \frac{1}{10}) + (2 \times\frac{1}{100})$$

$$= (5\times10^2) + (2\times10) + (3\times1) + (4 \times \frac{1}{10}) + (2 \times\frac{1}{10^2})$$

II. Practice

A. Write as expanded nur als: 156, 4312, 75,000

B. In the expression 10^8, what is the base?
What is the exponent?
How many times has ten been used as a factor?

C. Write as expanded numerals using exponents: 520, 895.25, 3.002

III. Summary

A. In the standard numeral 1678, what is the place value of the 6? of the 1? of the 7? of the 8?

B. In the expression 10^4, what does the 4 indicate?

C. How can numbers less than one be expressed with expanded numerals?

D. In the standard numeral .54, what is the place value of the 5? of the 4?

E. In the expression 10^6, what name is given to the 10? to the 6?

Lesson 35

Topic: Other Number Bases

Aim: To reinforce understanding of base ten and base five systems of numeration

Specific Objectives:

Place value in base five
Converting from base five to base ten
Converting from base ten to base five

Challenge: What base-ten numeral will express the same number that the numeral 44_{five} expresses?

I. Procedure

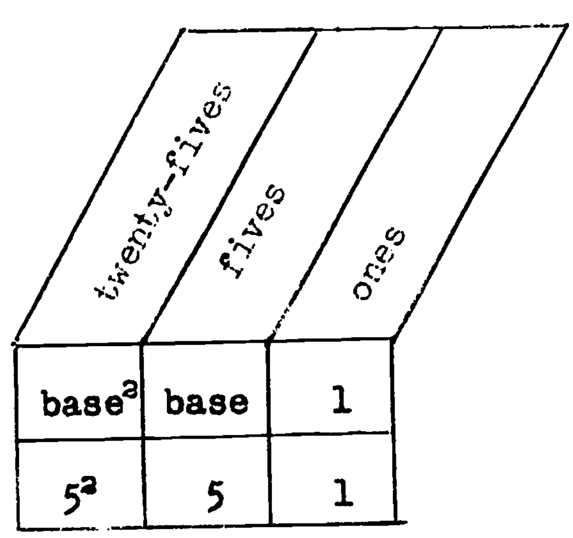
A. Review of place value in base ten and in base five

1. Have pupils recall that

in base ten we use ten digits: 0,1,2,3,4,5,6,7,8,9
in base five we use five digits: 0,1,2,3,4
in base ten we group by tens
in base five we group by fives

2. Have pupils recall that in base ten we use a place value system in which each place has ten times the value of the place immediately to its right. In base five we use a place value system in which each place has five times the value of the place immediately to its right.

Elicit the headings for the place-value chart in base five.



B. Converting from base five to base ten

Refer to challenge.

1. Have a pupil read the challenge aloud. (If necessary, remind the pupils that 44_{five} is not read "forty-four five" but four four, base five.)

2. Have pupils use place value to convert a base-five numeral to an equivalent base-ten numeral.

$$\begin{aligned} 44_{five} &= (4 \times 5) + (4 \times 1) \\ &= \;\; 20 \;\; + \;\; 4 \\ &= \;\; 24 \end{aligned}$$

Thus, $44_{five} = 24$

<u>Note</u>: Remind pupils that a numeral in base ten is written without a base numeral.

3. Have pupils practice converting the following base-five numerals to base ten:

$21_{five} = ?$ $\qquad$ $123_{five} = ?$

C. Converting from base ten to base five

1. Have pupils express 32 as a base-five numeral.

   a. Remind pupils that in base five we group by fives.

   b. Refer to the base five chart:

      Can we find in 32 a group as large as 25? (1 group of 25 and 7 left over)

      Place 1 in column headed "25."

      Can we find in 7 a group as large as 5? (1 group of 5 and 2 ones left over)

      Place 1 in column headed "5" and 2 in column headed "1's."

   c. 32 in base ten is equivalent to 112 base five, because they both name the same number.

      We write $32 = 112_{five}$ and we read this: "32 is equivalent to one one two, base five."

d. Have pupils check:

$$
\begin{aligned}
112_{five} &= (1 \times 25) + (1 \times 5) + (2 \times 1) \\
&= \quad 25 \quad + \quad 5 \quad \div \quad 2 \\
&= \quad 32
\end{aligned}
$$

2. Have pupils practice converting the following base-ten numerals to base five:

$22 = ?$ $\qquad\qquad$ $15 = ?$

II. Practice

A. Convert the following base-five numerals to base-ten numerals:

1. 42_{five} $\qquad$ 3. 32_{five} $\qquad$ 5. 444_{five}

2. 20_{five} $\qquad$ 4. 131_{five} $\qquad$ 6. 240_{five}

B. Convert the following base-ten numerals to base-five numerals:

1. 15 $\qquad$ 3. 38 $\qquad$ 5. 75

2. 27 $\qquad$ 4. 56 $\qquad$ 6. 111

C. Suppose we had a set of hats such as this:

How many hats are in this set?
How would you express your answer using base five?
Is the number of hats still the same?
Are the numerals the same? Explain.

III. Summary

A. In constructing a place-value chart in base five, what multiplier did we use as we moved from right to left?

B. What steps do we take to convert a base-five numeral to a base-ten numeral?

C. What steps do we take to convert a base-ten numeral to a base-five numeral?

Lesson 36

Topic: Other Number Bases

Aim: To introduce the binary system of numeration

Specific Objectives:

To develop place value in base two (the binary system of numeration)
To express base-two numerals in expanded form
To convert base-two numerals to base-ten numerals

Motivation: Most of you have watched TV during the national elections and have seen computers at work. Do you remember seeing lights flashing on and off? What did the flashing lights represent? (Numbers)

These electronic computers depend upon the flow of electric current. When a switch is on, the current flows through the circuit and a light goes on. When the switch is turned off, the light goes out. Is there any other choice?

It is therefore necessary to reduce the symbols for numbers to conform to this "on," "off" principle. Today we will find out how this is done.

I. Procedure

A. To develop place-value chart in the base-two system of numeration

1. Have pupils recall that in base ten we use ten digits: 0, 1, 2, 3, ..., 9. In base five we use five digits: 0, 1, 2, 3, 4.

What relationship have you noticed between the base name and the number of digits used?

How many digits would you expect to use in base two? (2)
What would you expect them to be? (0, 1)

Refer to the motivation: What number would you guess is represented by the light out? (zero)

How is 1 represented?

2. Have pupils construct a place-value chart for base two.

Review with pupils that in base ten each place has a value which is ten times the value of the place to the right. In base five, each place is five times the value of each place to the right.

What would you expect the multiplier to be in setting up a place-value chart for base two?

Complete the headings in the place-value chart.

thirty-twos	sixteens	eights	fours	twos	ones
$base^5$	$base^4$	$base^3$	$base^2$	base	1
2^5	2^4	2^3	2^2	2	1

3. Tell pupils that the base-two system is called the <u>binary</u> system of numeration.

B. To express base-two numerals in expanded form.

1. Place the numerals 11_{two}, 111_{two}, 1011_{two} in the base-two chart and then write each numeral in expanded form.

$$11_{two} = (1\times2) + (1\times1)$$

$$111_{two} = (1\times4) + (1\times2) + (1\times1)$$

or

$$(1\times2^2) + (1\times2) + (1\times1)$$

$$1011_{two} = (1\times8) + (0\times4) + (1\times2) + (1\times1)$$

or

$$(1\times2^3) + (0\times2^2) + (1\times2) + (1\times1)$$

2. Have pupils practice writing the following base-two numerals in expanded form:

10_{two} 1111_{two}

100_{two} 1110_{two}

C. To convert from base two to base ten

Return to the base-two numerals written in expanded form in B above.

What base-ten numeral names the same number as 11_{two}?

1. $11_{two} = (1 \times 2) + (1 \times 1)$
$= 2$
$= 3$

Have pupils read: "one one, base two is equivalent to 3 in base ten."

2. What base-ten numeral is equivalent to 111_{two}?

$$111_{two} = (1 \times 4) + (1 \times 2) + (1 \times 1) = 4 + 2 + 1 = 7$$

Have pupils read: "one one one, base two is equivalent to 7 in base ten."

3. In a similar manner, have pupils convert 1011_{two} to a base-ten numeral.

$$1011_{two} = (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) = 8 + 0 + 2 + 1 = 11$$

Have pupils read: "one zero one one, base two is equivalent to 11 base ten."

II. Practice

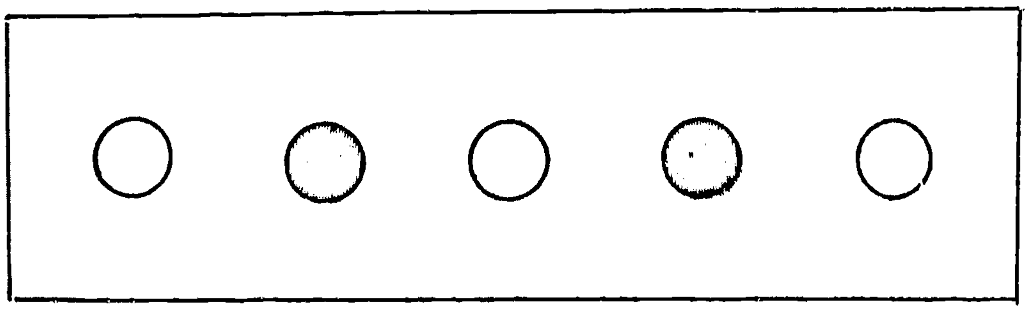
A. Using the base-two place-value chart, have the pupils practice converting the following base-two numerals to base ten numerals.

101_{two} 1010111_{two}

1111_{two} 1111001_{two}

B. Show that 10 in any base is the numeral for the base name.

C. If a computer showed this series of lights, what numeral in base two would be represented?



III. Summary

A. In constructing a place-value chart for base two, what multiplier was used as we moved from right to left?

B. What is the value of 10 in base ten? in base five? in base two?

C. Give the steps you would use to convert a base-two numeral to a base-ten numeral.

D. What new vocabulary have you learned?

(base two, binary)

Lesson 37

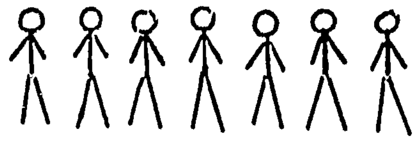
Topic: Other Number Bases

Aim: To convert from base-ten numerals to base-two numerals

Specific Objectives:

To review converting base-ten numerals to base-five numerals
To convert base-ten numerals to base-two numerals

Challenge: How many figures are there in this set?



How would you express the number of the set in base two?

I. Procedure

A. Review converting from base ten to base five

1. Have pupils recall that in base five we group by fives.

2. Have pupils rename 42 as a base-five numeral.

$42 = 1$ twenty-five $+ 3$ fives $+ 2$ ones

$42 = 132_{\text{five}}$

3. In the same manner, have pupils express the following in base five.

$61 = ?$ $\qquad$ $83 = ?$ $\qquad$ $76 = ?$

B. Converting from base ten to base two

1. Have pupils recall that in base ten we grouped by tens.
In base five we grouped by fives.
How shall we group in base two?

2. Return to challenge and help pupils express 7 as a base-two numeral.

a. Refer to place-value chart for base two.
Elicit that 7 is between 4 (or 2^2) and 8 (or 2^3).

b. Can we find in 7 a group as large as eight? (No)
As large as four? (Yes - one group of four and three left over.)
Place 1 in column headed 4.

c. Can we find in three a group as large as two? (Yes - one group of two and one left over.)
Place the numeral 1 in the column headed 2 and place the numeral 1 in ones column.

d. Have pupils write: $7 = 111_{two}$

e. Have pupils check:

$$\begin{aligned} 111_{two} &= (1\times4) + (1\times2) + 1 \\ &= 4 + 2 + 1 \\ &= 7 \end{aligned}$$

f. Have pupils read: "7 is equivalent to one one one, base two."

3. Using similar procedures, have pupils express 6 as a base-two numeral.

4. Help the pupils express 25 as a base-two numeral.

a. Refer to chart and elicit that 25 is between 16 (or 2^4) and 32 (or 2^5).

b. Can we find in 25 a group as large as 16? (Yes - one group of 16 and 9 left over.) Place 1 in column headed 16.

$$16\overline{)25}^{\;1} \quad \begin{array}{r} \underline{16} \\ 9 \end{array} \text{ over}$$

c. Can we find in 9 a group as large as 8? (Yes - one group of 8 and 1 left over.) Place 1 in column headed 8.

$$8\overline{)9}^{\;1} \quad \begin{array}{r} \underline{8} \\ 1 \end{array} \text{ over}$$

d. Can we find in 1 a group as large as 4? (No - place zero in column headed 4)

$$4\overline{)1}^{\;0}$$

e. Can we find in 1 a group as large as 2? (No - place zero in column headed 2.)

$$2\overline{)1}^{\;0}$$

f. Can we find in 1 a group as large as 1? (Yes - one group of ones.) Place 1 in column headed 1.

$$1\overline{)1}^{\;1}$$

g. Have pupils write and read:

$25 = 11001_{two}$ Read: 25 is equivalent to one one zero zero one, base two

h. Have pupils check:

$$11001 = (1\times16) + (1\times8) + (0\times4) + (0\times2) + (1\times1)$$

$$= 16 + 8 + 0 + 0 + 1$$

$$= 25$$

II. Practice

A. Rename each of the following base-ten numerals as base-five numerals:

2 5 10 17 43

B. Rename each of the numerals in A as a base-two numeral.

C. Why is the base-two numeration system used in some computers? (It has only two symbols: 0 and 1, which can be shown by the "on" or "off" position of an electrical switch.)

D. What is a disadvantage of a base-two numeration system? (The two symbols must be repeated many times to express even relatively small numbers.)

III. Summary

A. What is the least number which can be expressed by two digits in base ten; base five; base two?

B. What steps do we take to convert from base ten to base two?

C. How can we check our answers?

Lessons 38 and 39 (OPTIONAL)

Topic: Other Bases

Aim: To add numbers expressed in base five

Specific Objectives:

To introduce the base-five number line
To construct a base-five addition table
To perform the operation of addition in base five

Challenge: What is the sum of 2_{five} and 4_{five}?

I. Procedure

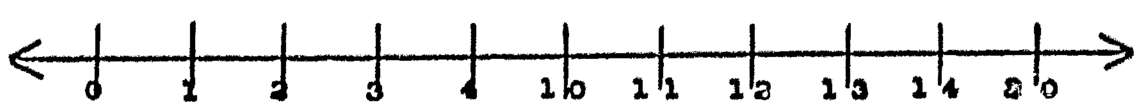
A. Base-five number line

1. Have pupils recall that in base five we use only five digits: 0, 1, 2, 3, 4.

2. Construct a number line using base-five numerals.

a. Have pupils recall that on a number line each whole number is one more than the whole number to its left. Elicit that there is no numeral 5 in base five. Therefore, in base five one more than 4_{five} is 10_{five} (or one group of five and no ones).

b. Draw a number line on the blackboard and have pupils write in the numerals for base five.



3. Using the number line, add:

a. $2_{five} + 2_{five}$ Have pupils place finger on 2 and then count 2 units to right to arrive at 4. Then,

$$2_{five} + 2_{five} = 4_{five}$$

b. $3_{five} + 4_{five}$ Have pupils place finger on 3 and then count 4 units to right to arrive at 12. Then,

$$3_{five} + 4_{five} = 12_{five}$$

Have pupils read answer as "one two, base five."

Elicit that 12_{five} means one group of five and 2 ones.

c. In a similar manner, have pupils add the following numbers:

$1_{five} + 4_{five}$ $\qquad$ $3_{five} + 3_{five}$

$10_{five} + 2_{five}$ $\qquad$ $3_{five} + 2_{five}$

d. Answer the challenge.

B. Addition Table Base Five

1. Elicit that addition on the number line in some cases may be cumbersome. Develop a table of addition facts in base five.

Column

+	0	1	2	3	4
0					
1					
2					
3					
4					

Row

a. In the extreme left column and in the top row, list the digits for base five.

b. Elicit that:
addition is a binary operation
one of the addends is named in the extreme left column and the other addend is named in the top row.

c. Using the number line, help pupils complete the table, as at the right.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

2. Use the addition table to find the following sums:

$1_{five} + 3_{five}$ $\qquad$ $4_{five} + 2_{five}$

$3_{five} + 4_{five}$ $\qquad$ $4_{five} + 4_{five}$

3. Using the table:
Elicit from pupils that zero is the identity element for addition for the numbers expressed in base five, just as it is for the numbers expressed in base ten.

4. Have pupils use the table to find the following sums:

$2_{five} + 1_{five}$ $\qquad$ $1_{five} + 2_{five}$

$3_{five} + 2_{five}$ $\qquad$ $2_{five} + 3_{five}$

$2_{five} + 4_{five}$ $\qquad$ $4_{five} + 2_{five}$

What property of number is illustrated by the above examples? (commutative property of addition)

C. Addition in Base Five Using Expanded Numerals

1. Addition involving no exchange

a. Have pupils add in base ten using the expanded numerals.

$$\begin{array}{rl} 21 = & 2 \text{ tens} + 1 \text{ one} \\ \underline{13} = & \underline{1 \text{ ten} + 3 \text{ ones}} \\ & 3 \text{ tens} + 4 \text{ ones, or } 44 \end{array}$$

b. In a similar manner, have pupils add in base five.

$$\begin{array}{rl} 21_{five} = & 2 \text{ fives} + 1 \text{ one} \\ \underline{13_{five}} = & \underline{1 \text{ five} + 3 \text{ ones}} \\ & 3 \text{ fives} + 4 \text{ ones, or } 34_{five} \end{array}$$

c. Have pupils practice addition using expanded numerals.

$$\begin{array}{cccc} 62 & 11_{five} & 13_{five} & 30_{five} \\ \underline{15} & \underline{10_{five}} & \underline{1_{five}} & \underline{12_{five}} \end{array}$$

2. Addition involving exchange

a. Have pupils recall that in base ten we group by tens, and in base five we group by fives.

b. Have pupils recall how regrouping is used when necessary to add in base ten.

$$\begin{aligned} 38 &= 3 \text{ tens} + 8 \text{ ones} \\ \underline{24} &= \underline{2 \text{ tens} + 4 \text{ ones}} \\ & \quad 5 \text{ tens} + 12 \text{ ones} \\ &= 5 \text{ tens} + (1 \text{ ten} + 2 \text{ ones}) \\ &= (5 \text{ tens} + 1 \text{ ten}) + 2 \text{ ones} \\ &= 6 \text{ tens} + 2 \text{ ones, or } 62 \end{aligned}$$

c. Have pupils add in base five

1) What is the sum of 14_{five} and 23_{five}?

$$\begin{aligned} 14_{five} &= 1 \text{ five} + 4 \text{ ones} \\ \underline{23_{five}} &= \underline{2 \text{ fives} + 3 \text{ ones}} \\ & \quad 3 \text{ fives} + (1 \text{ five} + 2 \text{ ones}) \\ &= (3 \text{ fives} + 1 \text{ five}) + 2 \text{ ones} \\ &= 4 \text{ fives} + 2 \text{ ones, or } 42_{five} \end{aligned}$$

Check by changing to base ten and adding.

$$\begin{aligned} 14_{five} &= (1 \times 5) + (4 \times 1) = 9 \\ \underline{23_{five}} &= (2 \times 5) + (3 \times 1) = \underline{13} \\ 42_{five} &= (4 \times 5) + (2 \times 1) = 22 \end{aligned}$$

2) Find the sum of 32_{five} and 34_{five}.

$$\begin{aligned} 32_{five} &= 3 \text{ fives} + 2 \text{ ones} \\ \underline{34_{five}} &= \underline{3 \text{ fives} + 4 \text{ ones}} \\ & \quad (1 \text{ twenty-five} + 1 \text{ five}) + (1 \text{ five} + 1 \text{ one}) \\ &= 1 \text{ twenty-five} + 2 \text{ fives} + 1 \text{ one, or } 121_{five} \end{aligned}$$

Check by changing to base ten.

II. Practice

A. Using the base five number line, compute the sum of:

$1_{five} + 3_{five}$; $2_{five} + 4_{five}$; $3_{five} + 2_{five}$

Check answers by using table.

B. Compute the sum and then check by changing to base ten.

1. 20_{five} $\underline{11_{five}}$	2. 10_{five} $\underline{24_{five}}$	3. 12_{five} $\underline{11_{five}}$
4. 34_{five} $\underline{11_{five}}$	5. 11_{five} $\underline{24_{five}}$	6. 24_{five} $\underline{34_{five}}$

III. Summary

A. How does the base five number line differ from the base ten number line?

B. How would you compare regrouping in base ten with regrouping in base five?

Lesson 40 (OPTIONAL)

Topic: Other Number Bases

Aim: Multiplication in base five

Specific Objectives:

To construct a base-five multiplication table
To perform the operation of multiplication in base five

Challenge: Find the product: $31_{five} \times 12_{five}$?

I. Procedure

A. Construction of multiplication table

1. Help pupils develop a multiplication table for base five.

Elicit that in the extreme left column and in the top row, we list factors less than five.

x	0	1	2	3	4
0					
1					
2					
3					
4					

Elicit that multiplication is a binary operation.

2. Tell pupils that the multiplication properties of zero and one hold for numbers. Therefore, the properties hold for numbers named in base five. Help pupils fill in the products where one factor is zero; where one factor is one.

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2			
3	0	3			
4	0	4			

3. Have pupils recall that multiplication of whole numbers may be considered in terms of addition. Help pupils complete the table.

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

B. Multiplication in base five

1. Have pupils mu''ply $21_{five} \times 11_{five}$.

$$\begin{array}{r} 21_{five} \\ \times 11_{five} \\ \hline 21 \\ 21 \\ \hline 231_{five} \end{array}$$

2. Check by converting to base ten and multiplying:

$$\begin{array}{rll} 21_{five} & = (2\times 5) + (1\times 1) & = 11 \\ 11_{five} & = (1\times 5) + (1\times 1) & = 6 \\ \hline 231_{five} & = (2\times 25) + (3\times 5) + (1\times 1) & = \overline{66} \end{array}$$

3. Have pupils multiply $11_{five} \times 21_{five}$.

Does the order of the factors affect the product?

4. Answer the challenge.

II. Practice

A. Compute the products:

$30_{five} \times 4_{five}$ $\qquad$ $211_{five} \times 300_{five}$

$21_{five} \times 12_{five}$ $\qquad$ $202_{five} \times 103_{five}$

B. Check each product by converting to base ten.

III. Summary

A. How can we check multiplication in base five?

B. Show by examples that the properties of zero and one hold for multiplication when the numbers are named in base five.

C. How is the operation of multiplication related to the operation of addition?

D. Show by an example in multiplication with base-five numerals that the order of the factors does not affect the product.

Lesson 41 (OPTIONAL)

Topic: Other Number Bases

Aim: To add numbers expressed in base two

Specific Objectives:

To construct a base-two addition table
To perform the operation of addition in base two

Challenge: What is the sum of 11_{two} and 11_{two}?

I. Procedure

A. Construction of addition table
Follow procedure described for construction of addition table in base five.

1. Recall that in base two, we have two digits, namely, 0, 1.

+	0	1
0	0	1
1	1	10

2. Elicit that because we have only two digits in base two, we have only four addition facts to remember.

3. Recall that 10_{two} means 1 group of twos and 0 ones, and is read, "one zero, base$_{two}$."

B. Addition in base two

1. Elicit that in adding in base ten, we group by tens; in base five, we group by fives; in base two, we group by twos.

2. Have pupils add in base two:

$$\begin{array}{rl} 11_{two} & = 1 \text{ two} + 1 \text{ one} \\ \underline{10_{two}} & = \underline{1 \text{ two} + 0 \text{ ones}} \\ & \ 1 \text{ four} + 0 \text{ twos} + 1 \text{ one, or } 101_{two} \end{array}$$

Check by changing to base ten and adding

$$\begin{array}{rll} 11_{two} & = (1 \times 2) + (1 \times 1) & = 3 \\ \underline{10_{two}} & = (1 \times 2) + (0 \times 1) & = \underline{2} \\ 101_{two} & = (1 \times 4) + (0 \times 2) + (1 \times 1) & = 5 \end{array}$$

3. Return to challenge. Have pupils perform the addition and check by changing to base ten.

II. Practice

A. Add and check:

10_{two}	100_{two}	100_{two}
$\underline{11_{two}}$	$\underline{100_{two}}$	$\underline{101_{two}}$

B. Does the order of the addends affect the sum? Explain.

III. Summary

A. In what way is the addition table in base two different from the addition table in base ten?

B. How can we check a base-two addition example?

C. Show by an example in addition in base two that a change in the order of addends does not affect the sum.

CHAPTER IV

SYSTEMS OF NUMERATION

Lessons 34-42

Lesson 34

Topic: Decimal Notation

Aim: To reinforce expressing numbers in expanded form using exponents

Specific Objectives:

To reinforce:

the use of an expanded numeral to represent a number
the meaning of exponent
the writing of expanded numerals using exponents

Challenge: 7546 is the standard numeral for a number.
In what other ways can you express this number?

I. Procedure

A. To reinforce the use of an expanded numeral to represent a number

1. Elicit that 7546 can be expressed as 7 thousands + 5 hundreds + 4 tens + 6 ones.

   7546 could also be expressed as (7x1000) + (5x100) + (4x10) + (6x1).

2. Have pupils recall that (7x1000)+(5x100)+(4x10)+(6x1) is called an expanded numeral for 7546.

3. Have pupils write an expanded numeral for each of the following: 123; 539; 2734.

B. To reinforce the meaning of exponents

1. Elicit that 10x10 can be written as 10^2 and 10x10x10 can be written as 10^3.

2. Have pupils recall that in the expression 10^3, 10 is called the base and 3 the exponent. Elicit that the exponent indicates the number of times the base is a factor.

C. To reinforce the use of exponents to express numbers by expanded numerals

1. Have pupils write 7546 as an expanded numeral using exponents as follows:

$$7546 = (7 \times 1000) + (5 \times 100) + (4 \times 10) + (6 \times 1)$$
$$= (7 \times 10^3) + (5 \times 10^2) + (4 \times 10) + (6 \times 1)$$

2. In a similar way:

$$19050 = (1 \times 10000) + (9 \times 1000) + (0 \times 100) + (5 \times 10) + (0 \times 1)$$
$$= (1 \times 10^4) + (9 \times 10^3) + (0 \times 10^2) + (5 \times 10) + (0 \times 1)$$

3. How can .32 be expressed as an expanded numeral using exponents?

Have pupils recall that $.32 = .3 + .02 = \frac{3}{10} + \frac{2}{100}$

Elicit that a decimal fraction such as .3 may be written as $\frac{3}{10}$ or $3 \times \frac{1}{10}$ and .02 may be written as $\frac{2}{100}$ or $2 \times \frac{1}{10^2}$.

Therefore, $.32 = .3 + .02$

$$= (3 \times \tfrac{1}{10}) + (2 \times \tfrac{1}{10^2})$$

4. Express as expanded numerals using exponents

$$523.42 = 500 + 20 + 3 + .4 + .02$$
$$= (5 \times 100) + (2 \times 10) + (3 \times 1) + (4 \times \tfrac{1}{10}) + (2 \times \tfrac{1}{100})$$
$$= (5 \times 10^2) + (2 \times 10) + (3 \times 1) + (4 \times \tfrac{1}{10}) + (2 \times \tfrac{1}{10^2})$$

II. Practice

A. Write as expanded nur als: 156, 4312, 75,000

B. In the expression 10^8, what is the base?
What is the exponent?
How many times has ten been used as a factor?

C. Write as expanded numerals using exponents: 520, 895.25, 3.002

Lesson 42 (OPTIONAL)

Topic: Other Number Bases

Aim: Multiplication in base two

Specific Objectives:

To construct a base-two multiplication table
To perform the operation of multiplication in base two

Challenge: What is the product of 11_{two} and 11_{two}?

I. Procedure

A. Construction of a base-two multiplication table

1. Elicit from pupils that because we have only the digits 0 and 1 in base two, we have only four multiplication facts to remember.

2. Develop a base-two multiplication table.

x	0	1
0	0	0
1	0	1

B. Multiplication in base two

1. Have pupils multiply:

$$\begin{array}{r} 10_{two} \\ \underline{\times 11_{two}} \\ 10 \\ \underline{10} \\ 110_{two} \end{array}$$

Does the order of the factors affect the product? Explain.

Remind the pupils that the product is read: "one one zero, base two."

pupil's understanding of multiplication as a short form of addition. Later, a multiplication pattern is used to predict the sign of the product of two negative integers.

Division is developed through the related operation, multiplication.

In the final lesson, the pupils are guided to understand that the raised positive sign may be omitted and the raised negative sign may be lowered. This conventional method of writing symbols for directed numbers will be used in all subsequent lessons.

The teacher will find the operations with integers provide an excellent opportunity for practice in the fundamental operations.

CHAPTER V

THE SET OF INTEGERS

Lessons 43-53

Lessons 43 and 44

Topic: Set of Integers

Note to Teacher: For a complete review of the introductory development of this topic, see lessons 123, 124, 125 of Mathematics: 7th Year, Part II (Curriculum Bulletin No. 3b, 1966-1967 Series).

Aim: To review addition of integers using the number line

Specific Objectives

To review and reinforce:
some concepts related to the set of integers
the addition of integers using the number line

Challenge:

On the number line, what numbers are associated with the indicated points to the left of zero?

I. Procedure

A. Concept of the set of integers

1. Have pupils recall that whole numbers may be associated with points on a line.

a. Have pupils draw a number line.

b. Elicit that each whole number associated with a point on the line is one less than the whole number associated with a point on its right.

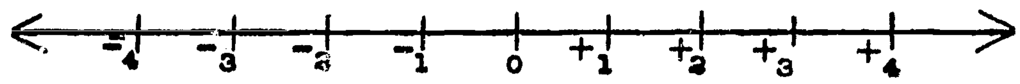
2. Refer to the challenge. Have pupils recall that the number "one less than zero" is named by the symbol $^{-}1$ (read: negative one). How would you indicate the number two less than zero? the number three less than zero?

Answer the challenge and indicate $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$ on the number line. Have pupils note that one less than zero is greater than two less than zero and that $^{-}1$ is associated with a point to the right of $^{-}2$.

3. Integers such as $^{-}1$, $^{-}2$, $^{-}3$ associated with points to the <u>left</u> of zero on the number line are called negative integers. What do you think integers associated with points to the right of zero should be called? (positive integers)

   Help pupils to realize that zero is neither negative nor positive.

4. Tell pupils that a way of designating positive integers is as follows: $^{+}1$, $^{+}2$, $^{+}3$, $^{+}4$, ... That is to say, $^{+}1$ names the same number as 1, $^{+}2$ as 2, and so on.



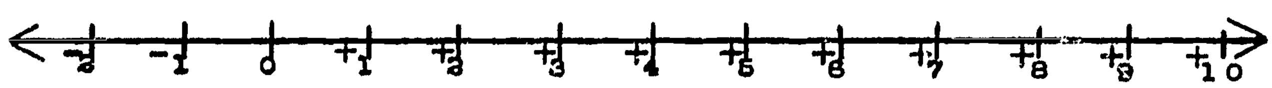
   Have pupils note that the positive integers and zero are associated with the same points on the number line as the whole numbers.

5. Tell pupils that the set of integers is the union of the set of positive integers, zero, and the set of negative integers.

   $I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\}$

B. Addition of integers using the number line

1. Have pupils draw a number line as shown below.



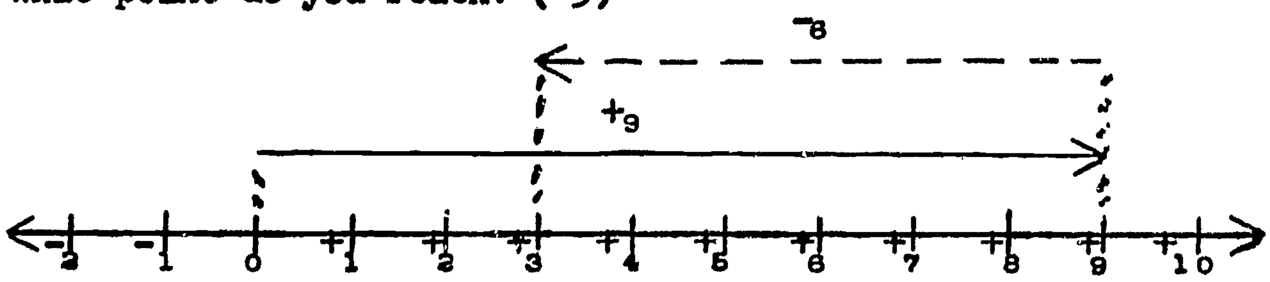
   Remind pupils that we shall consider moving to the right from any point on the number line is moving in a positive direction, and moving to the left from any point on the number line is moving in a negative direction.

2. In the sentence $({}^{+}9) + ({}^{-}6) = \square$, replace the frame to make a true statement.

   Guide the class to use the number line to compute the answer to the problem as follows:

   To add $({}^{+}9)$ to $({}^{-}6)$, start at zero, move 9 units in a positive direction (as shown by the solid arrow), then from $^{+}9$ move 6 units in a negative direction (as shown by the dashed arrow).

What point do you reach? ($^{+}3$)



Thus, $(^{+}9) + (^{-}6) = {}^{+}3$.

Remind pupils that parentheses are used to avoid confusing the sign of the numeral with the sign of the operation to be performed.

5. Using the number line, compute each of the following:

   a. $(^{+}5) + (^{+}4) = \square$ b. $(^{-}3) + (^{-}7) = \square$ c. $(^{-}7) + (^{+}2) = \square$

   d. $(^{+}8) + (^{-}3) = \square$ e. $(^{-}5) + (^{+}9) = \square$

II. Practice

A. Read each of the following:

$^{-}5$ (negative five) $^{+}3$ (positive three) $^{+}12$ $^{-}4$

B. Describe this set, $A = \{0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\}$ (non-negative integers)

Describe this set, $B = \{0, {}^{-}1, {}^{-}2, {}^{-}3, \ldots\}$ (non-positive integers)

C. Where are the following points located on the number line in reference to zero?

$^{+}6$, $^{-}3$, $^{+}15$, $^{-}25$, $^{+}80$, $^{-}120$

D. Refer to the number line to answer each of the following:

1. If John moves from $^{-}6$, 7 units in a positive direction, what point will he reach?

2. If Mary moves from $^{+}3$, 5 units in a negative direction, what point will she reach?

E. Using the number line, compute each of the following:

1. $(^{+}5) + (^{+}7)$	5. $(^{-}6) + (^{+}1)$
2. $(^{-}3) + (^{-}5)$	6. $(^{+}7) + (^{-}7)$
3. $(^{+}12) + (^{-}8)$	7. $(^{+}3) + (0)$
4. $(^{-}6) + (^{+}9)$	8. $(0) + (^{-}2)$

III. Summary

A. What name is given to the set formed by the union of the set of positive integers, zero, and the set of negative integers?

B. Which integer is neither negative nor positive?

C. If two different numbers are paired with two different points on the number line, will the greater number be to the left or right of the lesser number?

D. In order to add integers on the number line, what direction do we assume to be positive? negative?

E. Explain how the sum of two integers is found on the number line.

F. Explain how the following sets differ:

the set of negative integers
the set of non-negative integers
the set of positive integers
the set of non-positive integers
the set of integers

Lesson 45

Topic: The Set of Integers

Aim: To learn the meaning of absolute value

Specific Objectives:

To review the relationship between a number and its opposite
To develop the concept of absolute value

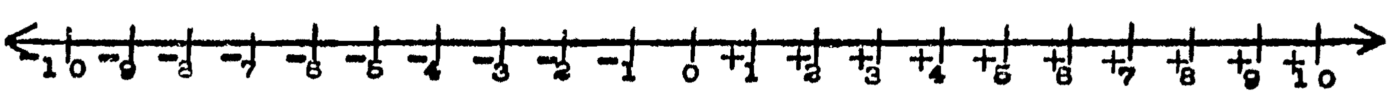
Challenge: What have these number pairs in common?

$(^{+}7, ^{-}7)$ $(^{+}4, ^{-}4)$ $(^{+}10, ^{-}10)$

I. Procedure

A. Relationship between a number and its opposite

1. Draw a number line and with heavy black dots indicate the points associated with the numbers $^{+}7$ and $^{-}7$.



2. How many units from zero is the point associated with $^{+}7$? in which direction?

   How many units from zero is the point associated with $^{-}7$? in which direction?

   Remind pupils that when two numbers represented on the number line by points which are the same distance from zero, but in opposite directions, the numbers are called <u>opposites</u>.

3. Have pupils locate $(^{+}4, ^{-}4)$, $(^{+}10, ^{-}10)$, etc., on the number line and elicit that each of these pairs is an example of a number and its opposite.

4. Have pupils name the opposite of each of the following integers: $^{+}5$, $^{-}6$, $^{+}11$, $^{-}12$

5. Have pupils find the sum of the integer and its opposite, and elicit that in each case the sum is zero.

6. Elicit that of a pair of opposite integers the positive integer is the greater.

B. Concept of absolute value

1. Tell pupils that the greater of any integer and its opposite (except zero, because zero is its own opposite) is called the <u>absolute value</u> of the number.

   a. Find the absolute value of $^{+}7$.
   What is the opposite of $^{+}7$? ($^{-}7$)
   Which is the greater $^{+}7$ or $^{-}7$? ($^{+}7$)

   Therefore, the absolute value of $^{+}7$ is $^{+}7$.

   b. Find the absolute value of $^{-}4$.
   What is the opposite of $^{-}4$? ($^{+}4$)
   Which is the greater $^{-}4$ or $^{+}4$? ($^{+}4$)

   Therefore, the absolute value of $^{-}4$ is $^{+}4$.

2. Tell pupils that the absolute value of a number is denoted by a pair of vertical line segments $|\ \ |$.

   $|^{+}10| = {}^{+}10$ is read: The absolute value of positive ten is positive ten.

   $|^{-}10| = {}^{+}10$ is read: The absolute value of negative ten is positive ten.

   $|0| = 0$ is read: The absolute value of zero is zero.

3. Return to the challenge. Discuss with the pupils several answers such as "their sum is zero," "they are pairs of opposite integers," etc. Have pupils find the absolute value of each pair of opposites. Elicit that each pair has the same absolute value. Have them note that the absolute value of a number is always positive.

II. Practice

A. What is the opposite of each of the following?

$^{-}7$ $\quad ^{+}1$ $\quad ^{-}2$ $\quad ^{-}8$ $\quad 0$ $\quad ^{+}4$ $\quad ^{+}75$ $\quad ^{-}34$

B. State the opposite of each of the following numbers and then state the absolute value of each.

$^{+}9$ (opposite is $^{-}9$; $|^{+}9| = {}^{+}9$ or 9)
$^{-}6$ (opposite is $^{+}6$; $|^{-}6| = {}^{+}6$ or 6)
$^{-}14$
$^{+}27$

C. Replace each frame so that a true statement results.

$|^{+}3| = \square$ $|^{-}12| = \square$

$|^{-}3| = \square$ $|0| = \square$

III. Summary

A. What do we mean when we say that two numbers are "opposites"?

B. Of two opposite numbers, which is the greater?

C. What is the absolute value of zero?

D. Is the absolute value of any number other than zero positive or negative?

E. What symbol do you use to show the absolute value of a number?

Lesson 46

Topic: Operations with Integers

Aim: To learn to add two positive or two negative integers without using the number line

Specific Objectives:

To review addition of two positive or two negative integers using the number line
To discover a method for addition of two positive or two negative integers without using the number line
To formulate a rule for addition of two positive or two negative integers using the concept of absolute value

Challenge: How many units would you need on a number line to add $^+1000$ and $^+810$?

I. Procedure

A. Addition using the number line

1. Use the number line to add the following:

a. $\begin{array}{r} ^+7 \\ \underline{^+4} \end{array}$ b. $\begin{array}{r} ^-7 \\ \underline{^-4} \end{array}$ c. $\begin{array}{r} ^+9 \\ \underline{^+3} \end{array}$ d. $\begin{array}{r} ^-9 \\ \underline{^-3} \end{array}$

2. Answer the challenge. Lead pupils to realize that addition on the number line may be impractical.

3. Suggest to pupils that they try to discover a method for adding integers without the number line.

B. Addition without the number line

1. Have pupils consider examples A-1-a and A-1-c. When both addends are positive, is the sum positive or negative?

2. Have pupils consider examples A-1-b and A-1-d. When both addends are negative, is the sum positive or negative?

3. After doing a number of similar examples by using the number line and examining the answers, lead pupils to generalize that: when two addends are positive, their sum is positive; when two addends are negative, their sum is negative.

C. To formulate a rule for addition of two positive or two negative integers using the concept of absolute value

1. Have pupils compute the following sums using the previous generalization.

$$\begin{array}{r} ^{+}18 \\ \underline{^{+}5} \end{array} \qquad\qquad \begin{array}{r} ^{-}18 \\ \underline{^{-}5} \end{array}$$

How do the two answers differ? (one answer is positive and one is negative)

Compare the absolute values of the answers. ($|^{+}23| = |^{-}23| =$ $^{+}23$ or 23.

2. Guide pupils to see that when two positive integers are added the answer is the sum of the absolute values of the addends.

When two negative integers are added the answer is the negative of the sum of the absolute values of the addends.

Tell pupils that this is generally accepted as the rule for finding the sum of two positive or two negative integers.

II. Practice

A. Replace the frame so that a true statement results.

1. $^{+}10 + ^{+}11 = \square$ 4. $^{-}3 + ^{-}6 = \square$
2. $^{-}8 + ^{-}15 = \square$ 5. $^{+}125 + ^{+}15 = \square$
3. $^{-}6 + 0 = \square$ 6. $^{-}7 + ^{-}7 = \square$

B. Compute the following sums.

1. $^{+}11 + ^{+}8$ $\qquad$ $^{+}8 + ^{+}11$
2. $^{-}3 + ^{-}6$ $\qquad$ $^{-}6 + ^{-}3$
3. $^{-}15 + ^{-}2$ $\qquad$ $^{-}2 + ^{-}15$

Examine each pair of sums. Does the order of the addends affect the sum? What property of addition is illustrated? (Commutative Property of Addition)

C. Find the following sums.

$^{+}171$	$^{-}323$	$^{+}186$	$^{-}451$
$\underline{^{+}\ 99}$	$\underline{^{-}145}$	$\underline{^{+}415}$	$\underline{^{-}451}$

III. Summa

A. When two addends are positive, what is the sign of the sum?

B. When two addends are negative, what is the sign of the sum?

C. In the addition of two positive integers or two negative integers, explain the relationship between the absolute value of the addends and the absolute value of the sum.

D. Illustrate, by means of an example, that the commutative property of addition holds for the addition of two positive integers; for two negative integers.

E. Explain why it is not always practical to use a number line to add two positive or two negative integers.

Lesson 47

Topic: Operations with Integers

Aim: To learn to add a positive integer and a negative integer without using the number line

Specific Objectives:

To review addition of a positive and a negative integer using the number line
To formulate a rule for addition of a positive and a negative integer using the concept of absolute value

Challenge: Consider the following:

$^{+}83 + {}^{-}71 = \square$ $\quad\quad$ $^{-}83 + {}^{+}71 = \square$

What have these sums in common?
In what way are these sums different?

I. Procedure

A. Addition using the number line

1. Use the number line to add the following.

a. $\begin{array}{r} ^{-}8 \\ \underline{^{+}2} \end{array}$ $\quad$ b. $\begin{array}{r} ^{+}8 \\ \underline{^{-}2} \end{array}$ $\quad$ c. $\begin{array}{r} ^{-}3 \\ \underline{^{+}4} \end{array}$ $\quad$ d. $\begin{array}{r} ^{+}4 \\ \underline{^{-}3} \end{array}$

2. Suggest to pupils that they attempt to discover a method for adding integers without the use of the number line.

B. Addition without the number line

1. Have pupils consider examples A-1-a and A-1-b.

$\begin{array}{r} ^{-}8 \\ \underline{^{+}2} \\ ^{-}6 \end{array}$ $\quad\quad$ $\begin{array}{r} ^{+}8 \\ \underline{^{-}2} \\ ^{+}6 \end{array}$

What is the absolute value of each answer?
What is the relationship between the absolute value of each answer and the absolute values of the addends? (The absolute value of the answer is the difference between the absolute values of the addends.)

In A-1-a which addend has the greater absolute value? ($^{-}8$)

In A-1-b which addend has the greater absolute value? ($^{+}8$)

What is the relationship between the sign of the answer and the sign of the addend with the greater absolute value? (The sign of the answer and the sign of the addend with the greater absolute value are the same.)

2. Follow the same procedure with A-1-b and A-1-c.

3. After doing several similar examples by using the number line and examining the answers, lead pupils to see that when a positive and a negative integer are added the absolute value of the answer is the <u>difference</u> between the absolute values of the two addends; also, that the sign of the answer is determined by the sign of the addend with the greater absolute value.

   Tell pupils that this is generally accepted as the rule for computing the sum of a positive integer and a negative integer.

4. Have pupils answer the challenge. (The absolute values of the sums are the same. The signs of the sums are different.)

II. Practice

A. Compute the sum for each of the following.

1. $^{+}10 + {}^{-}6$	4. $^{-}3 + {}^{+}5$
2. $^{-}15 + {}^{-}16$	5. $^{-}6 + {}^{+}6$
3. $^{+}4 + {}^{-}12$	6. $^{-}8 + 0$

B. Compute the following sums.

1. $\begin{array}{r} ^{-}266 \\ \underline{^{+}283} \end{array}$ 2. $\begin{array}{r} ^{+}171 \\ \underline{^{-}84} \end{array}$ 3. $\begin{array}{r} ^{-}15 \\ \underline{^{+}120} \end{array}$ 4. $\begin{array}{r} ^{-}543 \\ \underline{^{+}35} \end{array}$

C. Does $^{+}18 + {}^{-}19 = {}^{-}19 + {}^{+}18$? What property of addition is illustrated by this example?

D. To the sum of $^{+}5$ and $^{-}6$ add $^{-}8$.
Add $^{+}13$ to the sum of $^{-}9$ and $^{+}5$.

E. Find the following sums.

1. $(^{+}4 + {}^{-}8) + {}^{+}5$ $\quad$ $^{+}4 + (^{-}8 + {}^{+}5)$

2. $(^{-}3 + {}^{+}6) + {}^{-}7$ $\quad$ $^{-}3 + (^{+}6 + {}^{-}7)$

Examine each pair of sums. Does the regrouping of the addends affect the sum? What property of addition is illustrated by these examples? (Associative Property of Addition)

F. Show by examples that the set of integers is closed under the operation of addition.

III. Summary

A. When one addend is positive and the other addend is negative, what determines the sign of the sum?

B. In the addition of a positive integer and a negative integer, explain the relationship between the absolute values of the addends and the absolute value of the sum.

C. Give illustrations of the commutative, associative, and closure properties of addition in the set of integers.

D. What is the sum of an integer and its opposite?

Lesson 48

Topic: Operations with Integers

Aim: To learn to perform the operation of subtraction by using a related addition problem

Specific Objectives:

To reinforce that, for the set of whole numbers, subtraction and addition are inverse operations
To learn that, in the set of integers, each subtraction problem can be solved by a related addition problem

Challenge: What is the value of $^{+}5 - {}^{-}4$?

I. Procedure

A. In the set of whole numbers, subtraction and addition are inverse operations.

1. Have pupils describe their thinking when performing a subtraction problem such as, "subtract 6 from 10."

$$\begin{array}{r} 10 \\ -\underline{\ 6} \end{array}$$

Elicit that the answer may be found by thinking, "What number added to 6 will give the sum of 10"? That is,

$$\begin{array}{r} 10 \\ -\underline{\ 6} \\ 4 \end{array} \text{ because } 6 + 4 = 10$$

2. Ask pupils how they would check a subtraction example, such as

$$\begin{array}{r} 573 \\ -\underline{362} \\ 211 \end{array}$$

Elicit that to check, we perform the addition $362 + 211 = 573$.

3. Have pupils write an addition example which is related to each of the following subtraction examples:

$8 - 5 = 3$ because $5 + 3 = 8$
$14 - 7 = 7$
$12 - 8 = 4$

4. Have pupils realize that to perform subtraction, addition may be used. Addition and subtraction are said to be <u>inverse operations</u>.

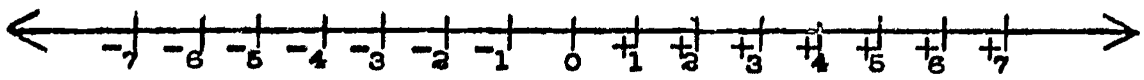
B. In the set of integers, a subtraction problem can be solved by a related addition problem.

1. Have pupils consider the problem: ${}^{+}5 - {}^{+}2 = ?$

<u>Note to Teacher</u>: Have pupils recall that ${}^{+}5 - {}^{+}2$ is read, "positive 5 minus positive 2." The word minus (or plus) is used to indicate an operation, while the word positive (or negative) is used to indicate a directed number.

Can we solve this problem by thinking, "What number added to ${}^{+}2$ will give a sum of ${}^{+}5$"? ${}^{+}2 + ? = {}^{+}5$

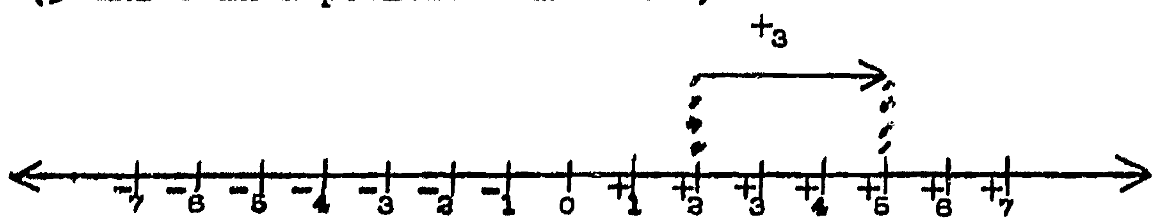
2. Have pupils find the answer by means of the number line.



Have pupils recall that we consider movement to the right to be moving in a positive direction, and movement to the left to be moving in a negative direction.

What number added to ${}^{+}2$ will give a sum of ${}^{+}5$? Think, "I am at a point two units to the right of zero." (${}^{+}2$)

How many units and in what direction must I move to arrive at ${}^{+}5$? (3 units in a positive direction)

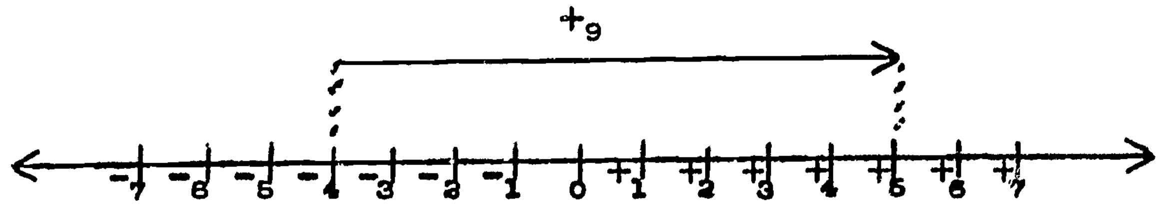


Therefore, ${}^{+}5 - {}^{+}2 = {}^{+}3$ because ${}^{+}2 + {}^{+}3 = {}^{+}5$.

3. Return to challenge: ${}^{+}5 - {}^{-}4 = ?$

This means, "What number must I <u>add</u> to ${}^{-}4$ to arrive at a sum of ${}^{+}5$"?

Think: "I am at a point 4 units to the left of zero (${}^{-}4$). How many units and in what direction must I move to arrive at $^+5$"? (9 units in a positive direction)



Therefore, $^+5 - {}^-4 = {}^+9$, because $^-4 + {}^+9 = {}^+5$.

4. Consider $^-2 - {}^+3 = ?$.

This means, "What number must I <u>add</u> to $^+3$ to arrive at $^-2$"?

Think: "I am at a point 3 units to the right of zero ($^+3$).

How many units and in what direction must I move to arrive at $^-2$"?

Using the number line, have pupils arrive at the answer: $^-2 - {}^+3 = {}^-5$ because $^+3 + {}^-5 = {}^-2$.

5. In a similar manner using the number line, have pupils find the answers to the following. Write the related addition examples.

$^-8 - {}^+4 = ?$	$^+6 - {}^-5 = ?$
$^-5 - {}^-3 = ?$	$^+4 - {}^+6 = ?$

II. Practice

A. Using the set of whole numbers, find the answers to the following and check:

7563 - 642	578 - 89

B. Write a related addition problem to each of the following:

$7 - 3 = 4$	$75 - 75 = 0$
$15 - 9 = 6$	

C. Using the number line, perform the following subtractions:

$^{-}4 - {}^{-}4$	$^{-}3 - {}^{+}6$
$^{+}1 - {}^{-}3$	$^{+}2 - {}^{+}3$

III. Summary

A. A subtraction example may be solved by means of a _____ addition example. (related)

B. Addition and subtraction are said to be _____ operations. (inverse)

C. Read $^{-}4 - {}^{-}2$. Which is the sign of operation? Which signs indicate directed numbers?

D. What new vocabulary have you learned today?

Lesson 49

Topic: Operations with Integers

Aim: To learn to perform the operation of subtraction by using the additive inverse

Specific Objectives:

To learn the meaning of additive inverse
To learn to use the additive inverse to perform the operation of subtraction

Challenge: Subtract $^{+}310$ from $^{-}260$.

I. Procedure

A. The meaning of additive inverse

1. Have pupils recall that $^{-}7$ is the opposite of $^{+}7$. Why? (on the number line, $^{-}7$ is the same number of units from zero as $^{+}7$, but in an opposite direction)

2. What is the sum of $^{+}7$ and $^{-}7$? What other pairs of numbers can you name whose sum is zero?

3. Tell pupils that $^{-}7$ may also be called the additive inverse of $^{+}7$ because $^{-}7 + {}^{+}7 = 0$. What is the additive inverse of $^{-}7$. Why?

4. What is the additive inverse of each of the following: $^{+}3$, $^{-}16$, $^{+}27$, $^{-}10$.

B. Using the additive inverse

1. Have pupils use the number line to find the answers to the following subtraction examples:

SET I

$^{+}4 - {}^{+}9 = ?$	$^{-}4 - {}^{+}6 = ?$
$^{+}5 - {}^{-}3 = ?$	$^{-}2 - {}^{-}1 = ?$

2. Have pupils replace the frames to make the following addition examples true.

SET II

$^{+}4 + \square = {}^{-}5$ $\qquad$ $^{-}4 + \square = {}^{-}10$

$^{+}5 + \square = {}^{+}8$ $\qquad$ $^{-}2 + \square = {}^{-}1$

3. Guide pupils' thinking as follows:

   Compare the operations in Set I and in Set II.
   Compare the answers of the related examples in Set I and in set II.
   Compare the second numbers of the corresponding examples in Set I and in Set II.

4. On the basis of these and similar examples, elicit that the subtraction of a number is equivalent to the addition of its additive inverse.

5. Return to the challenge.
   Write the example in the form $^{-}260 - {}^{+}310 = ?$
   Discuss with pupils the impracticability of using the number line for solving the challenge problem.

   What is the additive inverse of $^{+}310$?

   Using the rules of addition, find sum: $^{-}260 + {}^{-}310 = ?$

II. Practice

A. What is the additive inverse of: $^{-}52$, $^{+}76$, $^{-}15$, $^{+}105$, 0?

B. Using the additive inverse, rewrite these subtraction exercises as additions, and then find their sum:

$^{+}8 - {}^{+}12$ $\qquad$ $^{-}4 - {}^{-}8$

$^{-}5 - {}^{+}3$ $\qquad$ $0 - {}^{-}6$

C. Subtract

$^{+}15 - {}^{-}8$ $\qquad$ $^{-}365 - {}^{-}365$

$^{-}24 - {}^{-}35$ $\qquad$ $^{-}365 - {}^{+}365$

$^{+}687 - {}^{+}786$

D. Show by illustration that the operation of subtraction

1. <u>does not</u> have the property of closure in the set of whole numbers
2. <u>does</u> have the property of closure in the set of integers.

E. Show by illustration that the operation of subtraction is <u>not</u> commutative

1. in the set of whole numbers

2. in the set of integers.

F. Find the value of n in each of the following:

$n = {}^{+}42 - {}^{+}38$	$n = {}^{-}25 - {}^{-}38$
$n = {}^{+}15 - {}^{+}9$	$n = 0 - {}^{-}2$
$n = {}^{+}45 - {}^{-}3$	$n = 0 - {}^{+}3$
$n = {}^{+}19 - {}^{+}5$	$n = {}^{-}6 - {}^{-}6$
$n = {}^{-}3 - {}^{-}2$	$n = {}^{-}6 - {}^{+}6$

III. Summary

A. The opposite of an integer is its _____ inverse.

B. Subtracting a number is the same as _____ its opposite.

C. What new vocabulary have you learned today?

Lesson 50

Topic: Operations with Integers

Aim: To learn to perform the operation of multiplication

Specific Objectives:

To find the product of two positive integers
To find the product of a positive integer and a negative integer - in any order

Challenge: How could you write $^{+}4 \times {}^{-}2$ as an addition example?

I. Procedure

A. Multiplying a positive integer by a positive integer

1. Have pupils recall that multiplication is a short form of addition. For example, in the set of whole numbers 2×5 means the same as $5 + 5$ and is another name for 10.

$$2 \times 5 = 5 + 5 = 10$$

2. Tell pupils that positive integers behave like the numbers of arithmetic. Therefore, in the set of integers $^{+}2 \times {}^{+}5$ is the same as $^{+}5 + {}^{+}5$ and is another name for $^{+}10$.

$$2 \times {}^{+}5 = {}^{+}5 + {}^{+}5 = {}^{+}10$$

3. Have pupils perform the following multiplications:

Numbers of Arithmetic	Integers
$2 \times 5 = 10$	$^{+}2 \times {}^{+}5 = {}^{+}10$
$15 \times 4 = \square$	$^{+}15 \times {}^{+}4 = \square$
$2 \times 0 = \square$	$^{+}2 \times 0 = \square$
$24 \times 6 = \square$	$^{+}24 \times {}^{+}6 = \square$

4. Have pupils recall the meaning of the commutative property of multiplication in the set of whole numbers, i.e.,

$$2 \times 5 = 5 \times 2 = 10$$

Does the commutative property of multiplication appear to hold in the set of integers?

Numbers of Arithmetic	Integers
$5 \times 2 = 10$	$^{+}5 \times {}^{+}2 = \square$
$4 \times 15 = \square$	$^{+}4 \times {}^{+}15 = \square$
$0 \times 2 = \square$	$^{+}0 \times {}^{+}2 = \square$
$6 \times 24 = \square$	$^{+}6 \times {}^{+}24 = \square$

After a number of similar examples, tell pupils that the commutative property of multiplication does hold for all integers.

5. Elicit the generalization that the product of two positive integers is positive.

B. Multiplying a negative integer by a positive integer

1. What is the meaning of 4×2 in terms of addition? 4×2 may be written as $2 + 2 + 2 + 2$ - that is, multiplication is repeated addition. Thus, $4 \times 2 = 2 + 2 + 2 + 2 = 8$.

2. Return to challenge.
What would be another way of writing $^{+}4 \times {}^{-}2$?

   $4 \times {}^{-}2 = {}^{-}2 + {}^{-}2 + {}^{-}2 + {}^{-}2 = {}^{-}8$

3. Using the rules of addition of integers, find the values of:

   $^{+}4 \times {}^{-}3 = {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = {}^{-}12$

   $^{+}2 \times {}^{-}5 = ?$

   $^{+}8 \times {}^{-}1 = ?$

   $^{+}10 \times {}^{-}6 = ?$

4. Elicit that the product of a negative integer by a positive integer is negative.

5. Since multiplication of integers is commutative, $^{+}4 \times {}^{-}3$ should name the same numbers as ${}^{-}3 \times 4$. Similarly, $^{+}2 \times {}^{-}5 = {}^{-}5 \times {}^{+}2 = {}^{-}10$.

6. Elicit the generalization that the product of a positive integer and a negative integer in any order is a negative integer.

II. Practice

A. $^{+}10 \times {}^{+}35$ $\quad$ $^{-}8 \times 0$

$^{+}5 \times {}^{-}15$ $\quad$ $0 \times {}^{-}34$

$^{-}15 \times {}^{+}5$

B. Find the product of $^{+}5 \times {}^{-}6 \times {}^{+}8$.

$^{+}5 \times {}^{-}6 \times {}^{+}8$	$^{+}5 \times {}^{-}6 \times {}^{+}8$
$(^{+}5 \times {}^{-}6) \times {}^{+}8$	$^{+}5 \times (^{-}6 \times {}^{+}8)$
$^{-}30 \times {}^{+}8$	$^{+}5 \times {}^{-}48$
$^{-}240$	$^{-}240$

Does the grouping of the factors affect the product?

C. Find the product:

$^{-}2 \times {}^{+}14 \times {}^{+}5$

$^{-}7 \times {}^{+}37 \times 0$

$^{+}18 \times {}^{-}1 \times {}^{+}10$

III. Summary

A. The product of two positive integers is a _____ integer.

B. The product of a negative and a positive integer is a _____ integer.

C. The product of a positive integer and a negative integer is a _____ integer.

D. In multiplication in the set of integers, the order of the factors does not affect the _____.

E. Give an example to show that the associative property of multiplication holds in the set of integers.

Lesson 51

Topic: Operations with Integers

Aim: To learn to perform the operation of multiplication

Specific Objectives:

To reinforce the multiplication of a positive integer and a negative integer

To find the product of two negative integers

Challenge: What is the product of $^{-}3 \times {}^{-}2$?

I. Procedure

A. Product of a positive integer and a negative integer

1. Have pupils find the following products.

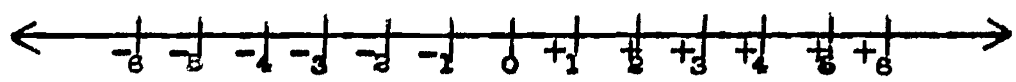
a. $^{+}3 \times {}^{-}2 = ?$

b. $^{+}2 \times {}^{-}2 = ?$

c. $^{+}1 \times {}^{-}2 = ?$

d. $0 \times {}^{-}2 = ?$

2. Have pupils locate these products on the number line.



3. Elicit that each product is two units to the right of the preceding product; therefore, each product is two units more than the preceding product.

4. What do you notice about the factors in the sequence? (The multiplier decreases by one while the multiplicand remains constant.)

B. Multiplying a negative integer by a negative integer

1. What would be the next example in the above sequence? ($^{-}1 \times {}^{-}2$)

2. To preserve the pattern of the product increasing by two, what must be the product of $^{-}1 \times {}^{-}2$? ($^{+}2$) of $^{-}2 \times {}^{-}2$? ($^{+}4$)

3. Answer the challenge: $^-3 \times {}^-2 = ?$ ($^+6$)

4. Have pupils observe that in the above sequence the product of two negative integers is a positive integer.

5. Have pupils observe another pattern:

$^-3 \times {}^+3 = {}^-9$

$^-3 \times {}^+2 = {}^-6$

$^-3 \times {}^+1 = {}^-3$

$^-3 \times 0 = 0$

$^-3 \times {}^-1 = ?$

$^-3 \times {}^-2 = ?$

6. Have pupils generalize: It seems that the product of a negative integer by a negative integer is a positive integer. Tell pupils that this is true.

7. Have the pupils compute the following:

$^+3 \times {}^+6 = ?$ $\qquad$ $^-3 \times {}^-6 = ?$

$^+3 \times {}^-6 = ?$ $\qquad$ $^-3 \times {}^+6 = ?$

Lead the pupils to realize that in each of the above examples, the absolute value of each of the products is the same.

($|{}^+18| = {}^+18$, $|{}^-18| = {}^+18$)

Elicit that the product is positive or negative depending upon the signs of the two factors being multiplied.

II. Practice

A. Find the products.

$^-4 \times {}^-7$ $\qquad$ $^+12 \times {}^+45$

$^+8 \times {}^-6$ $\qquad$ $^-10 \times {}^+20$

B. Find the products.

$^-2 \times {}^+3$ $\qquad$ $^+2 \times {}^+3$

$^+2 \times {}^-3$ $\qquad$ $^-2 \times {}^-3$

Compare the absolute values of these products.

C. Find the products.

$^{+}8 \times {}^{-}3 \times {}^{-}2$ $\qquad$ $^{-}6 \times {}^{-}2 \times {}^{-}5$

$^{-}9 \times {}^{+}5 \times {}^{-}1$ $\qquad$ $^{-}9 \times 0 \times {}^{-}3$

D. Tell pupils that the distributive property of multiplication over addition holds in the set of integers.

$$
\begin{aligned}
{}^{+}4 \times ({}^{+}6 + {}^{-}7) &= ({}^{+}4 \times {}^{+}6) + ({}^{+}4 \times {}^{-}7) \\
&= ({}^{+}24) + ({}^{-}28) \\
&= {}^{-}4
\end{aligned}
$$

also, $^{+}4 \times ({}^{+}6 + {}^{-}7) = {}^{+}4 \times ({}^{-}1)$

$$= {}^{-}4$$

Rewrite the following using the distributive property of multiplication over addition and compute the result.

$^{-}2 \times ({}^{+}5 + {}^{-}3) = ?$

$^{+}5 \times ({}^{-}8 + {}^{-}2) = ?$

III. Summary

A. Have the pupils summarize the ideas presented in Lessons 50 and 51 by answering the following questions:

1. What are the rules for the multiplication of:

   two positive integers
   one positive integer and one negative integer
   two negative integers

2. Give an illustration for each of the following properties of multiplication in the set of integers; closure, commutative, associative, distributive.

Lesson 52

Topic: Operations with Integers

Aim: To learn to divide integers

Specific Objectives:

To review the relationship between multiplication and division in the set of whole numbers
To find the quotient of two integers by using related multiplication
To formulate rules for the division of integers

Challenge: What is the value of $\frac{^{-}36}{^{-}12}$?

I. Procedure

A. Review the relationship between multiplication and division in the set of whole numbers.

1. Have pupils recall that the multiplication and division of the numbers of arthmetic are related. For example,

$8 \div 2 = 4$	means	$4 \times 2 = 8$
$15 \div 3 = ?$	means	$? \times 3 = 15$
$22 \div 11 = ?$	means	$? \times 11 = 22$
$20 \div 5 = ?$	means	$? \times 5 = 20$

2. Elicit that the problem $10 \div 2$ may be written as $\frac{10}{2}$.

$\frac{10}{2} = ?$	means	$? \times 2 = 10$
$\frac{72}{12} = ?$	means	$? \times 12 = 72$
$\frac{324}{18} = ?$	means	$? \times 18 = 324$

3. Have pupils recall that zero as a divisor has no meaning.

$\frac{6}{0} = ?$ means $? \times 0 = 6$

When zero is one factor, there is no number that can be used as the other factor to obtain a product of six. Therefore, we say division by zero is meaningless.

B. Finding the quotient of two integers by using related multiplication

1. Consider $\frac{^+8}{^+2} = ?$

   The related multiplication is $? \times {}^+2 = {}^+8$. The answer is 4. Therefore: $\frac{^+8}{^+2} = {}^+4$.

2. Have pupils write the related multiplications and find the answers.

Division	Multiplication	Answer
$\frac{^+15}{^+5} = ?$	$(?) \times {}^+5 = {}^+15$	$^+3$
$\frac{^-12}{^+4} = ?$		
$\frac{^-10}{^-2} = ?$		
$\frac{^+6}{^-2} = ?$		

3. Using the method on the previous page, have pupils answer the challenge: $\frac{^-36}{^+12} = ?$.

C. Formulating rules for the division of integers

1. Consider the results of the examples in B-2.

   $\frac{^+15}{^+5} = {}^+3$ $\qquad$ $\frac{^-10}{^-2} = {}^+5$

   $\frac{^-12}{^+4} = {}^-3$ $\qquad$ $\frac{^+6}{^-2} = {}^-3$

   Under what conditions were the quotients positive?
   Under what conditions were the quotients negative?

2. Lead the pupils to formulate rules for the division of integers.

   When a positive integer is divided by a positive integer or a negative integer is divided by a negative integer the quotient is positive.

   When a negative integer is divided by a positive integer or a positive integer is divided by a negative integer the quotient is negative.

3. Using the rule just formulated verify the answer to the challenge.

4. Have pupils compute the following:

$\frac{^{+}25}{^{+}5}$ $\frac{^{-}25}{^{-}5}$

$\frac{^{+}25}{^{-}5}$ $\frac{^{-}25}{^{+}5}$

Have pupils realize that in each of the above examples the absolute value of each of the quotients is the same.

$(|^{+}5| = {}^{+}5, |^{-}5| = {}^{+}5)$

Elicit that the quotient is positive or negative depending upon the signs of the dividend and the divisor.

II. Practice

A. Compute the quotients

$\frac{^{+}24}{^{+}12}$ $\frac{^{-}2}{^{+}1}$

$\frac{^{-}36}{^{-}4}$ $\frac{^{+}250}{^{-}50}$

B. Compute and compare the quotients

$\frac{^{+}18}{^{+}9}$ $\frac{^{+}18}{^{-}9}$

$\frac{^{-}18}{^{+}9}$ $\frac{^{-}18}{^{-}9}$

Compare the absolute values of the above quotients.

III. Summary

A. Show by an illustration that multiplication and division are related operations.

B. In the division of integers, when is the quotient positive? When is the quotient negative?

C. Compare the rules governing the sign of the quotient in division of integers with those governing the sign of the product in multiplication of integers.

Lesson 53

Topic: The Set of Integers

Aim: To develop a more convenient method of indicating positive and negative integers

Specific Objectives:

To understand that 9 and $^{+}9$ name the same number
To learn that $^{-}9$ may be expressed as -9

Challenge: Does $^{-}8 + {}^{-}5$ and $-8 + (-5)$ name the same number?

I. Procedure

A. To understand that $^{+}9$ and 9 name the same number

1. Through examples such as the following, have pupils recall that the positive integers and zero behave in the same way as the numbers of arithmetic.

$^{+}7 + {}^{+}2 = {}^{+}9$ $\quad$ $^{+}7 \times {}^{+}2 = {}^{+}14$ $\quad$ $^{+}7 + 0 = {}^{+}7$ $\quad$ $^{+}7 \times 0 = 0$

$7 + 2 = 9$ $\quad$ $7 \times 2 = 14$ $\quad$ $7 + 0 = 7$ $\quad$ $7 \times 0 = 0$

2. Elicit that numbers of arithmetic can be used to obtain the solution to any problem in which you use non-negative integers.

Therefore, the symbol 9 will be used for the symbol $^{+}9$ since they both name the same number.

3. In the same way

$^{+}9 - {}^{+}2$ may be written as $9 - 2$ and

$\frac{^{+}9}{^{+}17}$ may be written as $\frac{9}{17}$

$\frac{^{+}9}{^{+}22}$ may be written as $\frac{9}{22}$

B. Expressing negative numbers

1. Tell pupils that although we can omit the positive signs when using positive numbers, this does not apply to negative numbers.

2. Have pupils recall that the statement, "the opposite or additive inverse of positive 9 is negative 9," may be written in symbols

as: $-^{+}9 = {}^{-}9$.

Elicit that $^{+}9$ may be replaced by 9. Therefore, the above statement may be symbolized: $-9 = {}^{-}9$.

Tell pupils that in subsequent examples $^{-}9$ will be symbolized as -9.

3. Have pupils recall that subtracting a negative number is the same as adding its opposite (additive inverse). For example, $16 - {}^{-}9$ may be written as $16 - (-9)$. Applying the rule for subtraction, the answer will be 25.

4. In a similar way, $16 + {}^{-}9$ may be written as $16 + (-9)$. Using the rule of addition, the answer will be 7.

5. Answer the challenge.

II. Practice

A. Rewrite the following expressions without the use of a raised + or a raised ⁻ symbol.

$^{+}8$ $^{-}5$ $^{-}12$ $^{+}6$

B. Rewrite the following expressions using the new method of indicating positive and negative integers.

$7 - {}^{+}5$	$^{-}5 + 0$	$^{-}8 + {}^{-}5$
$^{-}5 - {}^{-}5$	$0 + {}^{+}5$	$^{-}6 + {}^{+}3$

C. Using the rules for addition or subtraction, compute the following:

$18 - (-6)$ $24 + (-10)$ $7 - (-8)$ $10 + (-15)$

III. Summary

A. Show by examples that positive integers behave in the same way as numbers of arithmetic in the operations of addition, subtraction, multiplication, and division.

B. What is another name for the opposite of a number? (additive inverse)