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Described is the development of an approach to the algebra of real numbers which includes three areas of mathematics not commonly found in grade 9--the theory of limits of infinite sequences, a frequent use of Cartesian co-ordinates, and algebra of matrices. Seventy per cent of the course is abstract axiomatic algebra and the remaining portion includes intuitive mathematics. The segment of the course, which is based on intuition, includes a brief consideration of problems of measurement and scientific model building in an actual laboratory situation. Considered are engineering drawing and descriptive geometry, empirical probability, mathematical logic, and the history of mathematics. The mathematical content includes 35 subject-matter topics. The remaining portions of the report describe a sequence of learning experience. (RP)

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Experimental Course Report/Grade Nine

I. Background

The Madison Project, a curriculum development project of Syracuse University and Webster College, has, for the past seven years, been engaged in developing mathematics curriculum ideas and materials for Kindergarten through grade 10, and for college courses. The most fully-developed portions of this curriculum provide a supplementary program in modern algebra, logic, and geometry for grades 2 through 8; this material is presented in four books, and in a sequence of films (cf. Appendix A).

At two grade levels the Madison Project program is not supplementary, but forms instead the entire mathematics program for that grade. This occurs at the kindergarten level (because there was no pre-existing established program at this level), and in grade 9, where a complete and unified course is a reasonable expectation.

During the academic year 1963-1964, a 9th grade class of 31 students at Nerinx Hall High School, a Catholic high school in Webster Groves, Missouri, was taught jointly by Professor Robert B. Davis of Syracuse University, and by Sister Francine, S.L., of the Nerinx faculty.

The present report is concerned solely with that portion of the course taught by Professor Davis. As discussed above, this was intended to be the entire course, but could not be, due to a schedule of out-of-town commitments that required Professor Davis to be away from the Nerinx campus about 30% of the time. During Professor Davis's absences, the class was taught by Sister Francine, generally according to the contents of the 9th grade algebra book ordinarily used at Nerinx, namely

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Dolciana, Berman, and Freilich, Modern Algebra: Structure and Method. Book I, Houghton Mifflin Co., Boston, 1962.

The school is an all-girl school, containing (in one building) grades 9 - 12. Since the students had attended grades K - 8 elsewhere, their backgrounds were diverse. It was assumed that none of them had had any previous contact with "modern" school mathematics curricula, and this was an appropriate assumption in nearly all cases.

Because of the presence of Sister Francine's portion of the course, it could be assumed that all essential parts of the "traditional" ninth-grade program were included, although they might not appear in the present report. This does not contradict the assertion that the experimental ninth-grade course outlined here is not merely supplementary, but is intended to become the basic ninth-grade course. Indeed, one might say that the "traditional" topics were supplementary to the modern portion of the course. That such a haphazard arrangement produced an adequately articulated course is due to the wisdom and flexibility of Sister Francine, and to the ninth-grade students themselves. To both, the author wishes to re-affirm his deepest thanks.

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II. General Purpose and Orientation of the Course

To discuss the reason for developing a new ninth-grade mathematics course, we might consider first the "traditional" ninth-grade algebra course which we sought to replace.[1]

The four outstanding attributes of the "traditional" ninth-grade "algebra" course were probably these:

1. The students experienced a long sequence of pedestrian intellectual tasks which were hardly capable of inspiring enthusiasm or commitment, nor of calling forth any sustained, original, and creative effort.

2. The course was intended to cause the student to become able to write an apparently correct mathematical statement, without the need for understanding what he had written.

3. The pace of the course was remarkably slow, and greatly underestimated the potential ability of most students.

4. The student was cast in the passive role of listener, or the merely responsive role of a subject being trained or conditioned. Presumably for this reason, the students actually adopted a passive habit of mind, did not avidly grasp out for knowledge and understanding, and did not learn satisfactorily.

Three points may be left to stand without further comment here, but the second point deserves some discussion.

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[1] For a comparison with other "modern" ninth-grade courses, see Section VII of this report.

The "traditional" ninth-grade "algebra" course was so preoccupied with written symbols that it might have been called a course for typesetters, not for mathematicians. Notice, for example, the preoccupation with written symbolism indicated even in the usual vocabulary: "removing parentheses," "changing signs," "inverting," "simplifying," "multiplying out," "canceling," "transposing," "combining like terms," and so on.

On the other hand, a statement was traditionally written with a mystical optimism concerning the efficacy of notation, but with no concern as to whether it was true, false, open, of presently unknown truth value, implied by the preceding statement, contradicted by the preceding statement, capable of implying the following statement, logically equivalent to the following statement, or whatever. It was merely written. Mathematics thereby achieved the appearance of consisting of a sequence of written statements related by no logical structure that anyone cared to talk about, and describing no identifiable mathematical entities whatsoever. The name became substituted for the thing named. The student who could write

$$\sqrt{2}$$

had somehow penetrated the absolute depths of irrational numbers by the simple act of writing a radical, without the need to consider the theory of limits of infinite sequences or any of the other conceptual paraphernalia which seems to be required by those who choose to think as well as to write.

It is interesting to note that observers of our "new" ninth-grade mathematics class have spontaneously remarked upon the fact that the ratio of discussions to writing during class was far higher than usual.

Axioms. We chose to develop an axiomatic approach to algebra primarily for two reasons: first, any game is more intelligible and more fun if one is allowed to know in advance what the rules are, and, second, an axiomatic approach is capable of showing the man-made choices by which the development of our mathematical structures is shaped. Indeed, the multiplicity of mathematical structures is revealed far more clearly by an axiomatic approach.

"Clean" Mathematics. The ninth-grade course developed by S.M.S.G.,[1] or the axiomatic algebra which appears at the beginning of Moise's Elementary Geometry from an Advanced Standpoint,[2] appeal to us as fine examples of "clean" mathematics, honest, intelligible, and free from murky discussions of things which are ill-defined. We shall not attempt to describe this attribute of "clean-ness" with any precision. We do not mean to deny the important role of intuition, nor do we deny the value of intuitively "sensible" efforts whose precise content becomes revealed only later, *after* they have proved fruitful. We *do* mean the avoidance of that murkiness which is not even based upon sound intuition, which has an "out-of-focus" fuzziness that cannot be excised, and which may even be self-contradictory. An actual example, from a "traditional" course, is:

The absolute value of a number is the numerical value of the

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[1] School Mathematics Study Group, First Course in Algebra, Part I and Part II, Teacher's Commentary, and Part I and Part II, Student's Text, Yale University Press, 1960, 1961.

[2] Edwin Moise, Elementary Geometry from an Advanced Standpoint, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1963.

number, without regard to its condition or sign. Absolute value is neither positive nor negative.

The Various Aspects of Mathematics. Although our course was to be primarily a "clean," abstract, axiomatic approach to the algebra of real numbers, we recognize that mathematics has many faces, and is seen differently by pure mathematicians, applied mathematicians, statisticians, theoretical physicists, experimental physicists, engineers, actuaries, behavioral scientists, logicians, philosophers, lawyers, and so on. Each of these visions has some validity, as their historical survival surely indicates, and as the varied futures of our students forces us to acknowledge. While 70%, or so, of our course was abstract axiomatic algebra, the remaining portion included intuitive mathematics for which a careful foundation was not available, it included a brief consideration of problems of measurement and scientific model building in an actual laboratory situation, it made some use of engineering drawing and descriptive geometry, it dealt with empirical probability, it opened the door for further study of mathematical logic, and it included consideration of some relevant (and revealing) portions of the history of mathematics. The hope was to win as many converts as possible, with due regard for variations among our students.

Limits, Cartesian Co-ordinates, and Matrices. We wished to include three areas of mathematics not commonly found in grade nine, namely the theory of limits of infinite sequences, a frequent (one might say ubiquitous) use of Cartesian co-ordinates, and a nearly-ubiquitous appeal to the algebra of matrices. These topics are discussed in more detail below.

III. Mathematical Content: List of Topics

The course is not adequately described by a mere list of topics included, but such a list provides a good starting point. Here it is:

1. True statements, false statements, open sentences
2. Truth sets
3. Variables
4. Functions
5. Graphs of Truth Sets and of Functions
6. Mathematics in the Laboratory: Problems of Measurement
7. Mathematics in the Laboratory: Making Scientific Models
8. Descriptive Statistics: Average, Variance, Range, "Trimmed" Range, Standard Deviation
9. Empirical Probability
10. Implication, Contradiction, Uniqueness, Truth Tables, Inference Schemes, Mappings of Cartesian Products of Truth-Value Spaces
11. Identities
12. Quantifiers
13. "Shortening Lists" of Identities by Using Implication
14. More Careful Formulation of 13
15. Axioms and Theorems
16. Axioms for the Non-Negative Integers
17. Axioms for the Integers
18. Axioms for Rational Numbers
19. The Algebra of Matrices

20. Models of Axiom Systems (Matrices, Finite Fields, Rational Numbers, etc.)

21. Order Axioms

22. The General Quadratic Equation

23. Simultaneous Equations

24. Isomorphism

25. Linearity, Convexity

26. Transformations or "Mappings"

27. Identities Involving the Distance Function d(p,q)

28. Right-Angle Trigonometry

29. Extension of Definitions (in Various Contexts, Including Page's "Lattices," Exponents, Factorials, Trig Functions, etc.)

30. Trigonometric Identities

31. Complex Numbers via Matrices

32. The Complex Plane

33. The Greek $\sqrt{2}$ Paradox

34. Infinite Sequences, Monotonicity, Convergence

35. Axioms for the Real Number System

The way in which this bare list of "topics" was expanded into a sequence of actual learning experiences -- i.e., into a "course" -- is described in the remaining portions of this report.

Incidentally, three topics -- Mathematical Induction, Finite Difference Methods, and Archimedean Sums (for "Definite Integrals") -- which we had originally hoped to include had to be omitted for lack of time.

IV. General Educational Flavor

This was not a lecture course.

One of the key pedagogical ideas underlying this course is the idea of active, creative, original student participation. The students measure things in a laboratory, and discuss their results. The students choose sets of axioms, and the teacher argues with them about limitations of their chosen set. The teacher accepts "wrong" answers and waits for some students to challenge them.

Just how much direction the teacher injects is a subtle question which we shall not discuss here in detail. In general, the teacher seeks to avoid aimless chaos, but does not avoid controversy, nor does he quickly resolve issues. Where possible, he leaves open questions open, for gradual resolution by the students, often over a period of many weeks (provided, as in the case of the Greek $\sqrt{2}$ paradox, that the matter is sufficiently important to deserve such sustained interest).[1]

Perhaps the pedagogical aspects of the course are best revealed by viewing the films which were made during the 1963 - 1964 academic year, and which show actual classroom lessons. For a listing of these films, consult Appendix D of this report.

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[1] Reference here might be made to the concept of "demand quality" of Wolfgang Köhler, or the "tensions" of Kurt Lewin.

V. Direct Use of Films with Students

Prior to the 1963 - 1964 academic year, and prior to the present ninth-grade course experiment, the Madison Project had made a number of 16mm. sound films showing actual classroom mathematics lessons in grades two through eight. Many of these films dealt with topics which appear in the present ninth-grade course. These films were, of course, intended for teacher training, and are normally used in this way. They are not ordinarily shown to students.

For the present ninth-grade course we made an exception, and experimented with having the students view a few of these films, as specific mathematics learning experiences, rather than as films on pedagogy. Such use of the films appears to have one considerable strength: we wish to present mathematics as an on-going human creation. The films help to do this; in the filmed lesson, problems are posed, and students make up methods for attacking the problems. The methods are often named after the student who discovered them, are extended and generalized where appropriate, and are added to the students' future store of weapons for attacking future problems. There is no doubt as to where the methods came from, or why they were developed.

In a sense, this approach brings the history of mathematics right into the classroom, and lets each student live through important pieces of mathematical history. A "historical break-through" becomes something that the student knows from first-hand experience.

This point of view was maintained with the historical development of mathematics by the Nerinx ninth-graders, and it could be clearly observed in the parallel historical development of mathematics by the students in the films. The important thing was that the "live" ninth-grade class and the filmed lessons shared a consistent point of view concerning the development of mathematics.

Films giving an a-historical and authoritarian presentation of mathematics would not be consistent with this approach. Perhaps for this reason, the Project has never made such films. However, films giving a "problem-to-solution" approach, with all of the exploration, wrong turnings, and gradual accumulation of concepts and techniques, are not inconsistent with the actual development of mathematical systems as an on-going human activity.

After our experience with the Nerinx ninth-graders, we would recommend continuing exploration of the direct use of such films with students, provided both the "live" course and the films made consistent use of this "developmental" or "accumulation" approach.

In particular, films made for teacher training may have some potential for direct use with students.

VI. The Course

The present section embodies the main part of this report. It is concerned with taking the list of topics given in Section III, and building these topics into complete classroom learning experiences.

1. A "Spiral" Approach. Although we shall discuss the course topic by topic, the actual classroom lessons represented "mixtures" of these topics, in the following ways:

i) Where it seemed desirable, previous topics were reviewed, were fitted into a broader perspective, or were revived for use in a new context.

ii) Sometimes a brief advance notice was used to prepare the way for a future topic, in order to get students thinking about some new problem or some new approach in advance of the time when this problem would make its "official" appearance in class.

iii) Difficult and central topics were spread out over some time, in order for ideas to mature in the students' heads.

iv) Variations were made for the sake of morale and variety.

v) The teacher made some attempt to follow student initiative, which implies some non-sequential organization, since student ideas about generalizations and alternative approaches cannot be predicted in advance. For example, when the teacher was working on a sequence leading to the solution of the general quadratic equation by the method of completing the square, a student (Regina) developed an alternative method, for real roots, based upon analytic geometry. Again, when the teacher was developing an approach to $\sqrt{2}$ via bounded monotonic sequences, a student (Nancy O.)

developed an alternative approach to the equation

$$x^2 = 2$$

by using 2-by-2 matrices.

This "spiral" approach, then, considerably modified a strictly sequential "topic-by-topic" approach. Any given topic would usually make its appearance in many different lessons: as an advance "teaser," as a problem for direct confrontation, as a matter for brief review, as a matter for re-assessment in the light of subsequent developments, as an alternative approach (possibly unexpected by the teacher), or simply as something thrown in for the sake of variety.

2. The First Two Weeks. Because we chose not to assume a previous familiarity with "modern" mathematics courses, we began in September with two weeks devoted to a quick tour through the contents of Discovery in Mathematics[1] -- that is to say, we provided informal preliminary experiences with true statements, false statements, open sentences, truth sets, variables, functions, graphs of functions, and algebraic identities. The tone was informal, honest but not carefully precise, and based upon a tentative use of induction from a variety of instances. In addition to the kind of thing that is contained in Discovery, we included various lessons that were intended to show the diverse faces of mathematics, which we now describe in items 3-8, that follow immediately.

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[1] Robert B. Davis, Discovery in Mathematics, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964.

3. <u>Guessing Functions</u>. This topic is obvious, but is nonetheless gratifying. Some students make up a "rule," such as "whatever number we tell them, they will double it and subtract that from twenty." The class now tell the "rule" team values of x, the "rule" team use their function and tell the class the corresponding numerical value of f(x). It is the task of the class to guess what "rule" f(x) is being used, and to write it in proper algebraic notation.

Many valuable by-products can be derived from this exercise; we mention one: argument will sooner or later arise as to whether [1]

$$(\square + 3) \times 2 = \triangle$$

and

$$(2 \times \square) + 6 = \triangle$$

represent the same "rule" or not. This leads to the distinction (suggested by David Blackwell) between "formula" and "function," and will lead also to a "modern" definition of <u>function</u> as a set of ordered pairs, etc. It is not fair to expect people to guess your <u>formula</u> (cf. for example,

$$\square + 7 - 3 + 2 - 1 = \triangle),$$

but it <u>is</u> fair to expect them to guess your <u>function</u>.

Even the dimensionality of the space of numbers of the form

$$a + b\sqrt{2} \qquad a, b \text{ rational}$$

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[1] In x, y notation, these expressions would read

$$(x + 3) \cdot 2 = y$$

and

$$2x + 6 = y.$$

have made their appearance in this "guessing functions" game, as have properties of primes, of conic sections, of linearity and convexity, of exponential functions, etc.

(Our use of this topic stems from suggestions made by W. Warwick Sawyer.)

4. Mathematics in the Laboratory: Problems of Measurement.[1] We ask four students, independently, to guess the width of the room. We record all four numbers, compute the average, and measure the degree of consistency by computing the range, the inner-quartile range, the average absolute deviation from mean

$$\frac{1}{N} \sum_{i=1}^{n} \left| x_i - \bar{x} \right| ,$$

and the variance

$$\frac{1}{N} \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2 .$$

We then pass out 6-inch plastic rulers, and have four students independently measure the width of the room. With these four numbers we again compute the average, and get measures of the degree of agreement by computing ranges, average absolute deviation, and variance.

We then pass out yardsticks, and repeat the process.

Finally, we repeat the process with four teams using a surveyor's tape-measure.

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[1] This topic, as explored (in slightly modified form) by a 6th grade class, is presented in the film Average and Variance, available from The Madison Project.

The notions of averaging, estimating consistency of independent measurements, and sources of measurement error which are begun in this lesson are continued in other laboratory work (for example, in the work on linear and non-linear elasticity).

This topic is based upon suggestions made by Professor William Walton, of Webster College, and by Professor Frederic Mosteller, of Harvard University.

5. Mathematics in the Laboratory: Making Scientific Models. The students attempt to study the "stress-vs.-strain" relationships for, first, a spring, and, second, a chain of rubber bands. It is easy to record data; it is far harder to decide what the data is telling us.

In working with this data, we consider:

i) graphing the data

ii) whether any seeming linearity is a fact of the physical system or an artifact of our procedure for studying the system

iii) sources of error in measurements

iv) where possible, writing the function algebraically (notice that this builds smoothly on the earlier work in "guessing functions")

v) dependence or non-dependence upon the historical past of the physical system (which is of particular interest in the case of the rubber bands)

vi) range of validity of our study (as in "elastic limit").

This material was originally suggested by Professor Robert Karplus of the University of California (Berkeley).

6. Implication, Contradiction, and Uniqueness. The unit we are about to describe (based upon Professor David Page's Hidden Numbers) is an "experience" unit. We are not concerned (yet) with the elegant formulation of this portion of mathematics; what we are concerned with is providing for our students some experience with implication, contradiction, and uniqueness, on the grounds that many of the students may not have had previous experience with these concepts.

What we do is to play a game, according to the following rules:

i) The teacher writes one or more numerals on a piece of paper. Each numeral refers to a positive integer. (Repetitions are allowed.)

ii) The teacher will begin listing "clues" on the blackboard, identifying clues by letters, as "A," "B," "C," etc.

iii) The teacher's clues are not necessarily true; indeed, some will usually be false, and will be designed to produce contradictions.

iv) The students start with a "credit" of 5 points. What happens to this will be explained next.

v) Whenever a student believes he has found a contradiction in the clues, he must begin by stating precisely which set of clues he is using (e.g., {A, B, D}). He then describes the contradiction. If he is right, the teacher must label all statements used by the student (i.e., A, B, and D) as "True" or "False." If the student is wrong, the "credit" (which was initially 5 points) is reduced by 1.

vi) In citing a contradiction (item "v" above), a student is wrong if the set of

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statements does not contain a contradiction, or if a proper subset contains a contradiction. The student is right if the set he cited does contain a contradiction, and if no proper subset contains a contradiction.

vii) In order to help keep thinking straight, students may write a possible collection of numerals that they think the teacher wrote on the paper. These "possibilities" are accumulated, discussed, and ruled out as additional clues may require. No official scoring is related to this informal list of "possibilities."

viii) From time to time the teacher writes down additional clues (which, again, may be true or may be false).

ix) In order to get the teacher to reveal the numbers which he wrote on the paper, the class must (at an appropriate time) bet the teacher that a certain specific possibility is the only one which is consistent with all the "true" clues that have accumulated.

If the teacher can write down any other collection of numbers that is consistent with all of the "true" clues, then the students' "credit" is reduced to zero.

If the teacher cannot write down some other collection of numbers consistent with all clues marked "T," then he must reveal the paper on which he wrote the original "hidden numbers." This is the normal (and desirable) outcome of the game.

x) Whenever the students' "credit" becomes zero, the teacher takes away the paper on which he wrote the "hidden numbers," and never reveals it to the class. This is the "penalty" outcome of the game.

xi) The teacher adds clues as necessary, until either the students "win" (the outcome where the teacher reveals the hidden paper) or else the students "lose" (the outcome where the student "credit" becomes zero, and the teacher removes the hidden paper without ever revealing it).

xii) Although clues themselves may be "true" or "false," the teacher never cheats in his labeling of clues as "T" or "F," whenever he is required to do so by a student discovery of a contradiction (Rule "v").

Although these rules sound complicated on paper, they have proved simple enough in practice. This game works smoothly in the classroom.

Many -- indeed, potentially any -- concepts of mathematics can be introduced into this game. The following clues give a few suggestions:

"All of the numbers are prime."

"All of the numbers are relatively prime."

"The two numbers are roots of the equation

$$x^2 - 20x + 96 = 0."$$

"The 7 numbers have the form

$$\alpha, \alpha, \beta, \beta, \beta, \beta, \gamma,$$

where

$$\alpha \neq \beta, \quad \beta \neq \gamma, \quad \gamma \neq \alpha."$$

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"All of the numbers are odd."

"The sum of the numbers is less than 37."

"The product of the numbers is 100."

"The sum of the numbers is a minimum, consistent with all other clues."

"The smallest number is 8."

"No two numbers are the same."

"The collection C of numbers that I wrote has the property that, if

$$n \in C,$$

then

$$(n + 1) \in C."$$

The original idea for this topic is due to Professor David Page, of the University of Illinois and Educational Services, Incorporated.

7. Truth Tables, Inference Schemes, and Mappings of Cartesian Products of Truth-Value Spaces. Obviously, at some point we wish to effect a transition from our neo-Egyptian "empirical" mathematics, based upon generalizing from instances, to a modern neo-Greek deductive approach. This will depend upon two things: selecting suitable axioms, and developing a suitable logic.

It should be clear that, in the work described above, we have begun to lay the groundwork for a deductive approach. We now carry this further, by some consideration of simple notions of mathematical logic.

Our approach to logic is divided into three parts.

First, the students are asked to be "sociologists" (or "anthropologists"), and to make up truth tables based upon the way they and their friends use the words "and," "or," "if ... then," "not," etc.

Second, having this before us, we now play a legislative role: we proclaim that, henceforth in this course, the word "or" shall be used as indicated in our truth tables, and so on. This clearly gives a new precision to our use of logical connectives.

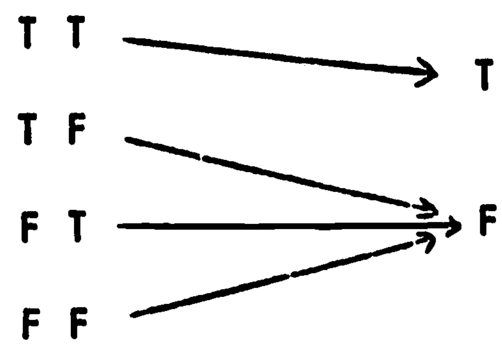
Finally, we behave as mathematicians: we seek abstract representations for what we have done, and we seek generalizations. The truth table entries for "and," for example,

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

can be described as a mapping of the Cartesian product $V \times V$ into V, where V is the "truth value space,"

$$V = \{T, F\}.$$

The "and" mapping can be represented diagrammatically, as follows:



"and"

This formulation suggests many interesting questions, such as:

How many different mappings of V x V into V exist? Does each have some familiar, obvious name? What is the minimum number of mappings in terms of which all mappings of $V \times V \longrightarrow V$ can be expressed (solved originally by H. M. Sheffer in 1913)? What happens to all of this if V contains more than 2 elements? If, say, V contains 3 elements, what would the corresponding truth tables and inference schemes look like?

8. Transformations or "Mappings." Since our work in logic has gotten us well launched on the notion of mappings, we now develop this further, using numerical examples, the concept of isomorphism, logarithms, the projection mapping in E_2, and simple substitution ciphers as mappings of $\mathcal{A}$ onto $\mathcal{A}$, where $\mathcal{A} = \{A, B, C, D, E, \ldots, X, Y, Z\}$,

9. Algebraic Identities. We begin by asking the students if they can write a statement which involves the variable " □," which will become true whenever we make a numerical replacement for the variable, no matter what number we use. This question is easy and interesting, and leads to the accumulation of a big list of identities, such as

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$$\square \times 0 = 0$$

$$0 \times \square = 0$$

$$\square \times 1 = \square$$

$$1 \times \square = \square$$

$$\square + 0 = \square$$

$$0 + \square = \square$$

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

and so on.

Notice that, still lacking any system of axioms, we cannot approach this topic deductively.

10. "Shortening Lists" of Identities by Using Implication. This topic, also, is approached informally for the present. For example, the list of three identities

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

$$A + (B \times C) = (C \times B) + A$$

can evidently be "shortened" to two, namely

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square ,$$

since nothing has been lost thereby; the "missing" identity can be derived from the other two. We work out such derivations with gradually increasing care and attention to detail.

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11. Quantifiers. Over the past several years we have been becoming increasingly aware of the role of quantifiers. We have used them more explicitly than ever before in the present ninth-grade course, and will probably use them even more prominently in future trials of this course.

By "quantifiers" we mean primarily two symbols:

$\exists x$ which means "there exists an x"

and

$\forall_x$ which means "for all x."

We use this last symbol also in a restricted sense, as in

$\forall_{x \neq 0}$ meaning "for all x such that $x \neq 0$."

Virtually every "algebraic" statement may be said to involve quantifiers -- although, of course, they are traditionally omitted. For example, a proper statement of the commutative law of addition might be

$$\forall_x \ \forall_y \quad x + y = y + x.$$

As a further example, we might write

$\forall_{x \neq 0}$ $\exists y$ such that $x \cdot y = 1$; we shall call y the "multiplicative inverse of x." The number y is uniquely determined by the number x.

Somewhat similar to our explicit use of quantifiers is our explicit use of logical inference, as in this example:

P: $x^2 - 5x + 6 = 0$

Q: $(x - 2) \cdot (x - 3) = 0$

$P \longleftrightarrow Q$ (logical equivalence)

R: $x - 2 = 0$ or $x - 3 = 0$

$Q \longleftrightarrow R$

Or, to give a second example,

P: $\sqrt{w - h} = r$

Q: $w - h = r^2$

$P \longrightarrow Q$ ("P implies Q").

We are coming to place similar explicit stress upon the domain of variables, and the truth sets of open sentences. Not that we are always careful; the degree of care is matched to the best of our judgment to the need for care in various situations.

12. A More Careful Approach to "Derivations." We have already seen that the notion that CLM and CLA imply $A + (B \times C) = (C \times B) + A$ has been pursued somewhat informally. We now take more careful look at what is involved.

i) The meaning of " = ." In the first place, we agree to interpret

$$A = B$$

to mean that A names something, and B names something, and (in fact) A and B

<u>name the same thing.</u>

ii) <u>The "Principle of Names."</u> If we examine what we do in making a derivation of

$$A + (B \times C) = (C \times B) + A,$$

using CLA and CLM, we find steps such as the following:

$$\left.\begin{aligned} A + (B \times C) &= (B \times C) + A \\ A + (B \times C) &= (C \times B) + A. \end{aligned}\right\} \text{ using CLM}$$

What we have done, evidently, is to take the known identity

$$A + (B \times C) = (B \times C) + A,$$

to delete a portion of it [namely one occurrence of (B x C)] , getting

$$A + (B \times C) = \underline{\qquad} + A,$$

and thereafter to insert into the "gap" (i.e. "____") another name [i.e. (C X B)] for the same thing.

Attempts to express this in English sentences can be clumsy; we shall make no such serious attempt, but hope that our meaning is clear.

We can formulate this in moderately careful language if we assert, as a <u>rule of our logic</u>, the Principle of Names (abbreviated "P. N."), namely:

> P. N.: If, in any statement, open sentence, or identity a name for a thing is replaced (in one or more occurrences) <u>by another name for the same thing</u>, then the truth value (of the statement), or the truth set (of the open sentence) <u>will not be changed.</u>

We shall henceforth make this a rule of our logic. Note that, for most purposes we permit quantifiers to appear implicitly rather than explicitly when we use P.N. When in doubt, however, we pay careful heed to the quantifiers, or to the respective truth sets and to the replacement sets for the variables.

iii) The "Rule for Substituting." We have, of course, previously established the "rule for substituting," that requires replacement in every occurrence of a variable if the replacement occurs in any. Note that this property sharply distinguishes replacement for a variable as against P.N., where changes in one occurrence need not effect other occurrences of the same original name.

iv) Use of a Variable. Whenever we replace a variable according to the "rule for substituting," we call the process U.V. ("Use of a Variable"), and add this as a permissable operation in our logic.

v) Reflexive Property of " = ". We agree that

$$\forall_x, \qquad x = x.$$

We note that this might be considered a consequence of our meaning for the symbol " = " (i.e., as a restriction on how "names" may be assigned to mathematical entities), or it can be added now as a rule of our logic, or (in fact) it can be inserted as an axiom of our algebra (that is, $\square = \square$ can be added to our list of identities).

With children in grades 3 - 8, we have ordinarily pursued the third course (or, rather, the children have elected the third course and we have gone along with this choice). With the present class of ninth-graders, however, we have preferred the second alternative, and we add

$$\forall_x, \quad x = x$$

as a rule of our logic (known, of course, as "R.P.E." for "reflexive property of equality").

vi) Transitive and Symmetric Properties of Equality. We have similarly added, as further rules of our logic, TPE and SPE, meaning, respectively,

If $A = B$ and $B = C$, then $A = C$.

If $A = B$, then $B = A$.

We would write these as:

$$(A = B \text{ and } B = C) \longrightarrow A = C$$

$$A = B \longleftrightarrow B = A .$$

vii) A "Uniqueness" Axiom. In working with additive inverses and with multiplicative inverses, especially, it is convenient to have a rule of our logic which asserts the following:

Let the set $\mathcal{S}$ contain exactly one element. Let $\alpha \in \mathcal{S}$, and let $\beta \in \mathcal{S}$. Then α names the same thing that β does, i.e.,

$$\alpha = \beta .$$

(In films of the Nerinx class, this rule of our logic is referred to as the "Principle of Maureen," after one of the students in the class.)

We then have a logic which consists of:

an interpretation of " = "

P.N.

U.V.

R.P.E.

S.P.E.

T.P.E.

the "Principle of Maureen"

This gives us a rather systematic tool for reshaping statements, identities, and open sentences. If it is not the elegant and formal tool of the modern logician -- and it is not -- then it can nonetheless quite properly claim to be a very considerable improvement on what has traditionally been done at the pre-college level.

We now have our logic, and it remains to select a suitable set of axioms for our algebra. [1]

13. Selection of Axioms and Theorems. The task of selecting axioms and theorems is -- in principle, at least -- left up to the students. (The teacher does, of course, supply considerable guidance.) In order for a statement or an identity to qualify as a theorem we

[1] Two remarks might be made: first, we choose to separate our logic from our algebra reasonably carefully; second, we have not defined what constitutes a "legal name." Part of this latter task is handled via algebraic closure axioms.

must, of course, make a derivation for it, using the rules of our logic, and using those algebraic axioms which we have previously selected.

Throughout the course, the process of selecting axioms was continuous and cumulative; so was the process of proving theorems. Certain "plateaus" could, however, be identified, as follows:

I. The point at which the class had a set of axioms for which the non-negative integers were a model, and from which it seemed possible to prove many of the algebraic properties of the non-negative integers. These axioms were not, in fact, categorical, nor did they provide for order relations, although the students were not, at this stage, aware of these limitations of the set of axioms which they had chosen.

II. The point at which the class had a set of axioms, generally similar to those described above, dealing with the system of integers (positive, negative, and zero).

III. The point at which the class had a set of axioms dealing with rational numbers.

IV. The point at which order relations were included.

V. The introduction of ("rational") complex numbers, by the use of 2-by-2 matrices, without additional axioms.

VI. The introduction of an axiom dealing with the topological completeness of the real line (stated in terms of sequences), and the consequent ability to deal with real numbers.

These various "plateaus" will be discussed in the next few sections.

14. Axioms for the Non-Negative Integers. As discussed above, this was the first "plateau." The class accumulated a set of axioms that appeared powerful in their ability to generate theorems concerning the multiplicative and additive algebraic structure of the positive integers. These axioms did not, in fact, provide a categorical description of the positive integers, and they omitted order relations, but the students were not, at this point, aware of these shortcomings in the set of axioms which they had chosen.

The axioms were:

i) closure: If A and B are "legal" names, then A + B is a "legal" name.
If A and B are "legal" names, then A x B is a "legal" name.

ii) CLA: $\square + \triangle = \triangle + \square$
(or, alternatively, $\forall_x \forall_y \quad x + y = y + x$)

iii) CLM: $\square \times \triangle = \triangle \times \square$
(or, $\forall_x \forall_y \quad x \cdot y = y \cdot x$)

iv) D.L.: $\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$
$\left(\forall_x \forall_y \forall_z \quad x \cdot (y + z) = (x \cdot y) + (x \cdot z)\right)$

v) ALA: $\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$

vi) ALM: $\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$

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vii) L1: $\square \times 1 = \square$

(i.e., $\exists$ an element 1 $\ni$ $\forall_x$ $x \cdot 1 = x$)

viii) ALZ: $\square + 0 = \square$

(i.e., $\exists$ an element 0 $\ni$ $\forall_x$ $x + 0 = x$)

ix) MLZ:[1] $\square \times 0 = 0$

(i.e., $\forall_x$, $x \cdot 0 = 0$)

[x) (not added until later, when the class was considering "models of axiom systems"):

$$1 \neq 0$$

This axiom moved us one step further toward a categorical description; it was needed for certain later proofs, and was recognized by the class at that time.]

Definition: The usual numerals shall be defined recursively, according to the pattern

$$1 + 1 = 2$$
$$2 + 1 = 3$$
$$3 + 1 = 4$$
$$4 + 1 = 5$$
$$\vdots$$

---

[1] It was later recognized by the students (probably from collateral reading) that "MLZ" is, in fact, a theorem. It is not, however, one which students easily identify as such, and easily prove.

From these axioms it is easy to prove theorems such as these:

Theorem: $\square + \square = 2 \times \square$

Theorem: $(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$

(where we have omitted one set of parentheses on the right hand side, by introducing a suitable convention, namely

$$a + b + c$$

shall mean

$$(a + b) + c \quad)$$

Theorem: $6 + 3 = 9$

Theorem: $A + (B \times C) = (C \times B) + A$

Theorem: $(A + B) \times (C + D) = (D + C) \times (B + A)$

The preceding (with one obvious exception) are to be regarded as identities.

Even at this stage students can begin to get a feeling for algebraic structure. As the selection of axioms proceeds further, they come to get a real feeling for the use of axioms in opening up new algebraical structures.

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15. Axioms for the Integers. To the preceding list of axioms we now add this:

Existence of Additive Inverses (also called "Law of Opposites"; unfortunately the word "opposite" appears to have too many non-mathematical connotations):

$\forall_x$ the open sentence $x + \square = 0$ has exactly one element in its truth set. One name for this element is ${}^{\circ}x$.

We now define "subtraction":

$$\square - \triangle \overset{\text{def}}{=} \square + {}^{\circ}\triangle$$

It is now possible to prove such theorems as:

Theorem: ${}^{\circ}({}^{\circ}A) = A$

(i.e., $\forall_A \quad {}^{\circ}({}^{\circ}A) = A$)

Theorem: $(\square + \triangle) \times (\square - \triangle) = (\square \times \square) - (\triangle \times \triangle)$

(i.e., $\forall_x \forall_y \quad (x + y) \cdot (x - y) = (x \cdot x) - (y \cdot y)$,

or, using exponents, $\forall_x \forall_y \quad (x + y) \cdot (x - y) = x^2 - y^2$)

Theorem: $5 - 3 = 2$

As a result of our new axiom, we (may) now have some new numbers. To help us keep track of them, we introduce some new symbolism:

$^{+}0 = 0$	$^{o}(^{+}0) = {}^{-}0$
$^{+}1 = 1$	$^{o}(^{+}1) = {}^{-}1$
$^{+}2 = 2$	$^{o}(^{+}2) = {}^{-}2$
$^{+}3 = 3$	$^{o}(^{+}3) = {}^{-}3$
⋮	⋮

Notice that we do not yet have the Law of Trichotomy; many different models for this axiom system exist, and $^{+}10$ may name the same number that $^{-}2$ names (to cite one obvious example). We do not yet have "positive" and "negative" in the sense of an order relation. This will appear presently.

16. The Algebra of Matrices. The present brief outline of the course has not followed the actual time sequence of the Nerinx Hall class. At a much earlier point in the course we had introduced the addition and multiplication of matrices. This presentation followed the Madison Project publication entitled "Matrices, Functions, and Other Topics," and so will not be discussed in detail here. Suffice it to say that the students could add and multiply matrices, and were in the habit of using matrix algebra as a contrast against the algebra of real numbers, as this latter gradually unfolds. Thus, for example, CLA is valid for both systems; CLM holds for real numbers but not for matrices; D.L. holds for both; so does the existence of additive inverses (however, the search for matrix analogues for 0 and 1 is

exciting!). Surprisingly, ALM holds for both.

Moreover, the correspondence

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \longleftrightarrow A$$

introduces the first "operationally valuable" instance of an isomorphism. By exploiting this later, we shall deal with

$$x^2 = {}^-4$$

and even, to a limited extent, with

$$x^2 = 2.$$

As we shall see, subsequent work on simultaneous linear equations will draw on matrix algebra; and some of the work in trigonometry might have done so, but happened not to. The picture of E_2 as a linear vector space began to emerge from the work with matrices, as did the picture of the complex plane.

However, one of the greatest values of the system of 2-by-2 matrices was the important role that it played, alongside modular arithmetic, in providing examples for our discussions of "models for an axiom system."

17. Models of Axiom Systems. As the course progressed, it was the natural point of view of the students to believe that our "careful, legal axiom systems" described some thing or some things.

The question then arises: how many mathematical systems satisfy these axioms?

At the first level, if we retain all of our axioms (as stated above) for the non-negative integers, except that we discard CLM, then we can find many quite different mathematical systems for which the axioms hold. Major examples are:

the system of non-negative integers

the system of (all) integers

the system of rational numbers

the system of 2-by-2 matrices

"clock" arithmetic on a 12-hour clock

"clock" arithmetic on a 5-hour clock.

Moreover, prior to stating the axiom that

$$1 \neq 0,$$

the axioms were satisfied by a system with a single element, where every "legal" name was, in fact, a name for 0.

At the second level (<u>all</u> integers), provided again that we temporarily suspend CLM, all of the structures cited above are still models, except for the first one, which does not admit additive inverses.

Not only did the discussion of models lead to the axiom

$$1 \neq 0,$$

but it also led to the following axiom:

$$\text{"}pq = 0\ \text{Axiom"}: \qquad pq = 0 \longrightarrow (\text{either } p = 0 \text{ or } q = 0).$$

From this point on, then, we shall regard CLM and the "pq = 0 Axiom" as being in force, thereby ruling out the system of 2-by-2 matrices (but notice that certain subsets of the set of 2-by-2 matrices are still valid models!), and ruling out modular arithmetic modulo any non-prime integer.

18. Order Axioms. Before proceeding far with rational numbers, we shall need to be in a position to show that various things are not names for zero. At present our ability to do this is most severely limited; moreover, as our models show, the limitation is not in our logic, but rather in the algebraic axioms themselves. We remedy this forthwith: in effect, we "unroll" our various "clocks," and require them to lie out flat like a properly-behaved number line.

The process of doing this, as we shall see presently, involves us in one of the subtlest or most intricate logical situations that is ever encountered in the entire course.

Our general approach will be to describe an adequate order relation axiomatically. Appealing to our background in models for axiom systems, we shall then ask whether, in fact, our existing mathematical systems can be made to admit of a model for our ordering axioms.

For the axiomatic description of an "adequate" or "reasonable" ordering system, there are several standard approaches. One (cf. E. E. Moise, Elementary Geometry from an

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Advanced Standpoint, Section 1.4, p. 10 ff) states axioms on a relation "$<$," and thereafter defines "positivity" and "negativity" in terms of "$<$." The other common approach (cf. Birkhoff and MacLane, A Survey of Modern Algebra, 1944 edition, p. 7) reverses this procedure, gives an axiomatic description of "positivity," and from this proceeds to define the relation "$<$" (and, of course, "negativity").

For convenience of comparison, we display these two approaches side-by-side:

Axiomatic Description of $<$	Axiomatic Description of "$\mathcal{P}$"
0. If α and β are any elements of our mathematical system, then $\alpha < \beta$ is a statement, and is either true or false.	0′. $\mathcal{P}$ is a set of elements of our mathematical system.
1. (Addition axiom) $(a < b) \longrightarrow [\forall_c, \quad a + c < b + c]$.	1′. $\mathcal{P}$ is algebraically closed under addition: that is, $(\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}) \longrightarrow \alpha + \beta \in \mathcal{P}$.
2. (Multiplication axiom) $(a > 0$ and $b > 0) \longrightarrow a \cdot b > 0$.	2′. $\mathcal{P}$ is algebraically closed under multiplication: i.e., $(\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}) \longrightarrow \alpha \cdot \beta \in \mathcal{P}$.

3. (Law of Trichotomy) If α and β are any elements of our system, then one of the following statements is true, and the other two statements are false:

i) $\alpha < \beta$

ii) $\alpha = \beta$

iii) $\beta < \alpha$

3.' (Law of Trichotomy) If α is any element whatsoever of our mathematical system, then one of the following statements is "True" and the other two statements are "False":

i) $\alpha \in \mathcal{P}$

ii) $\alpha = 0$

iii) $^{\circ}\alpha \in \mathcal{P}$

4. Transitivity:

$$(\alpha < \beta \text{ and } \beta < \gamma) \rightarrow \alpha < \gamma$$

We now define the set $\mathcal{P}$ (of "axiomatically positive" elements) by saying:

$$(\alpha \in \mathcal{P}) \longleftrightarrow (0 < \alpha).$$

The set $\mathcal{N}$ of "axiomatically negative" elements is defined by:

$$(\delta \in \mathcal{N}) \longleftrightarrow (\delta < 0).$$

The elements of $\mathcal{P}$ will be called "axiomatically positive."

The elements $\alpha \ni {}^{\circ}\alpha \in \mathcal{P}$ will be called "axiomatically negative."

The statement $\gamma < \delta$ shall have the same truth value as the statement

$$\delta - \gamma \in \mathcal{P},$$

i.e., $(\gamma < \delta) \leftrightarrow (\delta - \gamma \in \mathcal{P})$.

Notice that these two approaches are in fact equivalent. The correspondence of Trichotomy is obvious. From

$$a < b \longrightarrow [\forall_c, \quad a + c < b + c],$$

together with the transitivity of " $<$," we easily get that $\mathcal{P}$ is algebraically closed under addition:

To prove: $(0 < \alpha \text{ and } 0 < \beta) \longrightarrow 0 < \alpha + \beta$

Proof: $0 < \alpha$

$0 < \beta$

$0 + 0 < \alpha + 0$

$\alpha + 0 < \alpha + \beta$

$\therefore$, by transitivity,

$0 + 0 < \alpha + \beta$.

Q.E.D. (CLA and ALZ being assumed)

It is also obvious that 2 implies $2'$. Consequently, (1, 2, 3, and 4) $\longrightarrow$ ($1'$, $2'$, and $3'$).

Converseley, $3' \longrightarrow 3$ and $2' \longrightarrow 2$ are immediate. Axiom 1 follows from the definition of " $<$," certain algebraic axioms being assumed [so that $(b + c) - (a + c) = b - a$]. That $1'$ implies 4 is also immediate, since $(b - a) + (c - b) = c - a$.

We shall use the axiomatic description of $\mathcal{P}$. Here we encounter an interesting situation (cf. T. M. Apostol, Calculus, Vol. I, p. 16): we have the notion of "positive" in two different senses. We have agreed that the elements of any subset $\mathcal{P}$ of the set of elements in our mathematical system deserves to be called "axiomatically positive" if $\mathcal{P}$ satisfies the axioms above. But, returning to our basic mathematical system, we already have a set of elements called "positive," namely 1 (or $^{+}1$), $2 = 1 + 1$ (or $^{+}2$), $3 = 2 + 1$ (or $^{+}3$), $\cdots$.

Now, the question is, is the set

$$\{1, 2, 3, 4, \cdots\}$$

a legal candidate to be $\mathcal{P}$, the set of "axiomatically positive" elements?

If so, then haven't we had $\mathcal{P}$ all along, so that adding the order axioms on $\mathcal{P}$ has, in fact, really added nothing whatsoever?

A consideration of various models of our axiom system makes it clear that we have, indeed, added something further to our description when we add the order axioms, for they (and they only) rule out the finite fields represented (for example) by the 7-hour clock.

But if, say, we add Axiom $3'$ to our previous list of axioms, the set

$$\{1, 2, 3, 4, \cdots\}$$

is already defined, and the "Axioms" $1'$ and $2'$ should apparently be theorems. Are they?

We did not try to resolve this question with the 9th graders. One approach might have been via the introduction of mathematical induction. We did not pursue this further with the Nerinx class.

If we add Axioms $0'$, $1'$, $2'$, and $3'$ to our list of axioms (even realizing that there lurks here the possibility of either redundancy or contradiction), we can prove all of the theorems that we require.

For example, we can prove that $7 \neq 9$.

Theorem: $7 \neq 9$

Proof:

1) $7 + 1 = 8$

2) $8 + 1 = 9$

3) $(7 + 1) + 1 = 9$ — Note use of P.N.

4) $7 + (1 + 1) = 9$

5) $7 + 2 = 9$

6) Hence (various steps omitted here), $9 - 7 = 2$

7) $2 \in \mathcal{P}$

8) $9 - 7 \in \mathcal{P}$ (Note use of P.N.)

9) $\therefore\ 7 < 9$

10) $\therefore\ 9 \neq 7$ (by Law of Trichotomy)

Q.E.D.

Notice that (as a consideration of various models for our axiom system at various stages of its development shows), it has not always been true (at some earlier stages of our axiom system) that 7 was necessarily different from 9. At various earlier stages it was quite possible that "7" did name the same element that "9" named.

Consequently, we surely have added something additional to our description when we added the order axioms. To look at it another way, we have gained many additional theorems (one of which is: $7 \neq 9$).

19. Axioms for the System of Rational Numbers. Since we are now in a position, thanks to the order axioms, to be able to tell when we do (or do not) have a name for 0, we can proceed to axioms for the rational numbers.

This extension, using "division," parallels precisely our earlier extension to the system of integers, via additive inverses. Evidently, the key axiom here will be:

> $\forall_{x \neq 0}$, the open sentence $x \cdot \square = 1$ has a truth set that contains exactly one element. One name for this element shall be:
>
> $$^{r}x .$$

(As an alternative notation here, we can write:

> $\forall_{x \neq 0}$, let $\mathcal{J}_x$ denote the truth set for the open sentence $x \cdot \square = 1$. Then
>
> $$N(\mathcal{J}_x) = 1,$$
>
> and
>
> $$^{r}x \in \mathcal{J}_x .)$$

We now define "fractions" $\frac{a}{b}$ to mean

$$a \times {}^{r}b ,$$

and we define division in the same way:

$$\alpha \div \beta = \alpha \times {}^{r}\beta .$$

We can now easily prove such theorems as:

Theorem: $4 \div 2 = 2$

Theorem: $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

Theorem: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

and so on.

20. The General Quadratic Equation. Our development here is fairly well documented in three films.[1]

The underlying idea follows Polya's approach to problem-solving,[2] which might be paraphrased roughly as the construction of a suitable sequence of questions which the would-be problem-solver poses to himself, such as:

What do I already know about this topic?

What does this remind me of?

What changes in the problem would make it easier? What changes would make it harder?

What parts of the problem seem to be making it difficult? What parts are unfamiliar to me?

---

[1] The films are: Derivation of the Quadratic Formula -- First Beginnings, Derivation of the Quadratic Formula -- Final Summary, and Quadratic Equations.

[2] George Polya, How to Solve It, Doubleday, 1957.

Now that I have solved such-and-such a problem, where can I go from here? How can I extend my solution? How can I extend my method? Where else can I use my method? What new problems can I reduce to this one that I have now solved?

Following such a sequence of questions, the students follow a familiar and "traditional" approach to the quadratic formula, namely, the derivation by "completing the square."[1]

The expected sequence goes like this:

0. We want to solve quadratic equations, by a powerful general method if we can find one.

1. Let's try some easy quadratic equations. Do we know any?

   Yes: $x^2 = 4$ The truth set is $\{^{+}2, ^{-}2\}$.

2. Can we extend this?

[1] Cf. Robert B. Davis, "Solving Problems and Constructing Systems -- Quadratic Equations and Vectors," Report of an Orientation Conference for SMSG Experimental Centers, Chicago, Illinois, September 19, 1959, pp. 97-101; also, Robert B. Davis, Matrices, Functions, and Other Topics, Student Discussion Guide, and Matrices, Functions, and Other Topics, A Text for Teachers, The Madison Project, 1963.

Yes: $x^2 = p$ The truth set is $\{\sqrt{p}, \overset{\circ}{\sqrt{p}}\}$, *provided that we can find a square root of p.*

3. When *can* we find a square root of p?

   i) We can find $\sqrt{p}$ if $p \in \{0, 1, 4, 9, 16, \cdots\}$.

   ii) We can find $\sqrt{p}$ (by using matrices) if $p \in \{^-1, {}^-4, {}^-9, {}^-16, \cdots\}$.

   iii) We can find $\sqrt{p}$ if $p = \frac{a^2}{b^2}$, where a and b are integers (and, obviously, $b \neq 0$).

   iv) Otherwise we have trouble. We'll leave this for the moment, and come back and think about it later.

4. Can we extend our method for $x^2 = p$?

   Yes: $(x - 1)^2 = p$ requires $x - 1 \in \{\sqrt{p}, \overset{\circ}{\sqrt{p}}\}$, hence
   $x \in \{\sqrt{p} + 1, \overset{\circ}{\sqrt{p}} + 1\}$.

5. Can we extend this?

   Yes: The "1" was not crucial. $(x - a)^2 = p$ has the truth set
   $\{\sqrt{p} + a, \overset{\circ}{\sqrt{p}} + a\}$.

6. Are we now able to solve all the quadratic equations in the world? Are there any that we can't solve (immediately, that is)?

There are others: we can't (immediately) solve

$$x^2 - 8x + 16 = 49 ,$$

nor

$$x^2 - 6x + 12 = 4 .$$

7. Can we reduce either of these new problems to our already-solved form

$$(x - a)^2 = p?$$

Yes: For $x^2 - 8x + 16 = 49$, we can use P.N. to delete the left-hand side

$$\underline{\qquad\qquad} = 49$$

and to replace it by $(x - 4)^2$:

$$(x - 4)^2 = 49 .$$

This is justified because of the identity [1]

$$\forall_x , \quad x^2 - 8x + 16 = (x - 4)^2 .$$

---

[1] In the Nerinx Hall class, a girl named Mary Catherine both devised these identities, and used them to solve this particular problem. This was a significant "technological break-through" -- or "historical break-through."

All of the students appreciated the importance of this break-through. We (the teachers) are interested in the fact that this kind of "discovery" course in effect brings mathematical history into the classroom. All of these students really know what a historical break-through means. They have lived through it. They know the pre-dawn doubts: Have we reached the limits? Is it possible to go further? Then they have seen the tentative new suggestion, and gradually grasped its relevance and its utility.

We (the faculty) often found the history of mathematics a meaningless recital of names and dates. For these students, the history of mathematics describes something they have lived through, themselves. Is this the deepest value of this kind of "discovery" course, that it gives us a deeper and personal perspective on our cultural heritage?

Now, the equation

$$(x - 4)^2 = 49$$

is of the form

$$(x - a)^2 = p,$$

and we can write the truth set by a mere <u>replacement</u> <u>of</u> <u>variables</u> (U.V.):

$$4 \longrightarrow a$$

$$49 \longrightarrow p$$

The truth set

$$\left\{\sqrt{p} + a,\ \overset{\circ}{\sqrt{p}} + a\right\}$$

then becomes

$$\left\{\sqrt{49} + 4,\ \overset{\circ}{\sqrt{49}} + 4\right\}$$

which is, evidently,

$$\left\{7 + 4,\ {}^{-}7 + 4\right\}$$

i.e.,

$$\left\{11,\ {}^{-}3\right\}$$

8. Will Mary Catherine's method work for the equation

$$x^2 - 6x + 12 = 4\ ?$$

No, not directly, because we can't find a "Mary Catherine"-type identity

$$x^2 - 6x + 12 = (x - a)^2\ .$$

9. When can we find a suitable "Mary Catherine"-type identity?

Answer (given by Regina): For the expression

$$x^2 - Ax + B,$$

we can find a suitable "Mary Catherine" identity if and only if

$$\frac{A}{2} = \sqrt{B}.$$

10. What can we do about the equation

$$x^2 - 6x + 12 = 4\ ?$$

Answer (given by Kathy):

$$\frac{6}{2} = 3$$

$$3^2 = 9$$

$$12 - 9 = 3$$

Subtract 3 from each side of the equation. (This depends, of course, upon earlier work on "equivalent equations" and "transform operations."[1]

It is easy to complete this chain of reasoning. In the Nerinx Hall class, a major contribution to the final answer was made by Clare.

---

[1] Robert B. Davis, Discovery in Mathematics: A Text for Teachers, Addison-Wesley Publishing Co., Reading, Massachusetts, 1964, pp. 139-153.

What precedes was the line of attack expected by the teacher. As can be seen in the films, it did, in fact, occur with three different classes (grades 5, 7, and 9). However, with the 9th grade class, in addition to this approach, the students devised an alternative approach. If the approach above is described as predominantly algebraic, the alternative approach might be described as an analytic geometry approach. It was invented mainly by Regina.

Regina's geometric method:

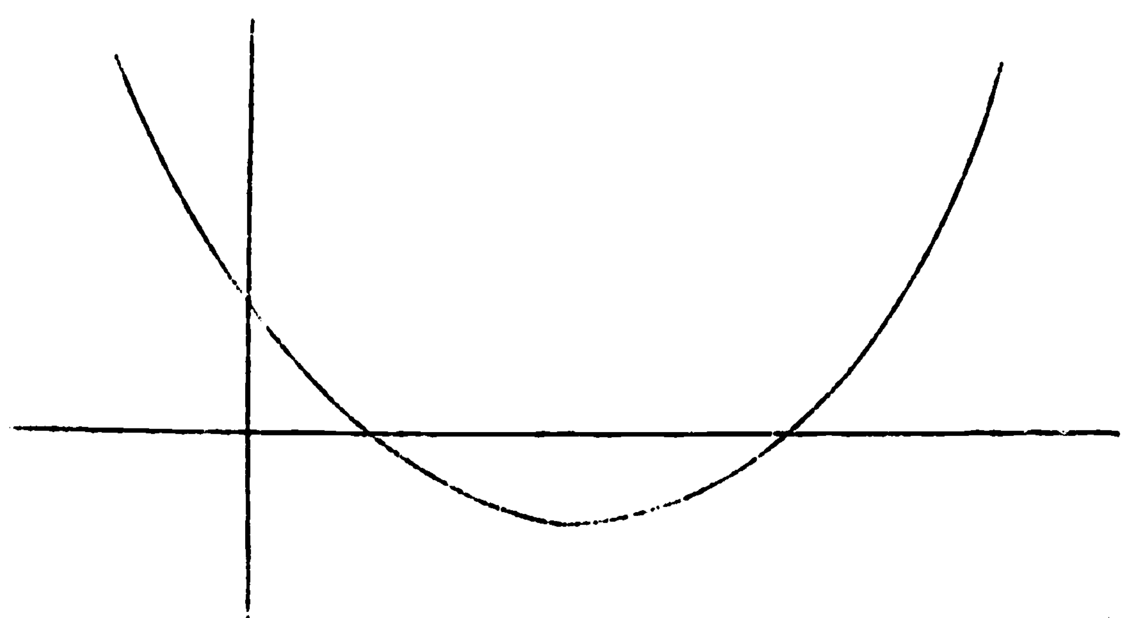
Let $Q(x)$ be a quadratic expression of the form

$$Q(x) = x^2 - Ax + B .$$

Then the graph of

$$y = Q(x)$$

will be a parabola in what we might call the "hanging cable" position:



Now, to solve the quadratic equation

$$Q(x) = p,$$

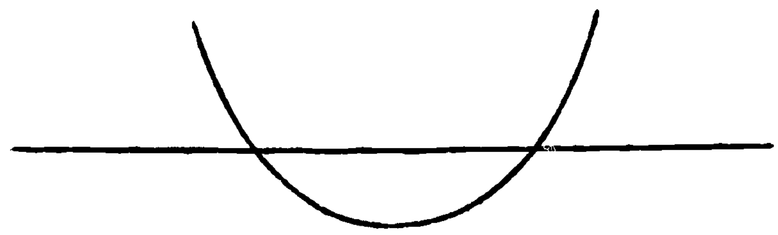
graph $y = Q(x)$, and (on the same axes) graph

$$y = p,$$

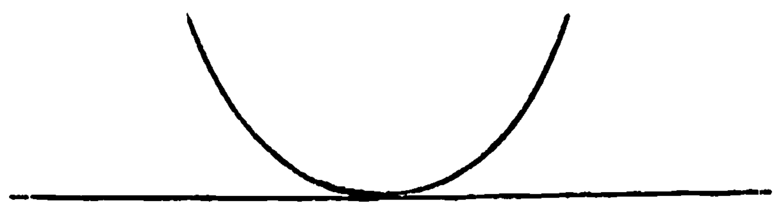
and find the x-coordinates of the points of intersection.

Will Regina's geometric method always work?

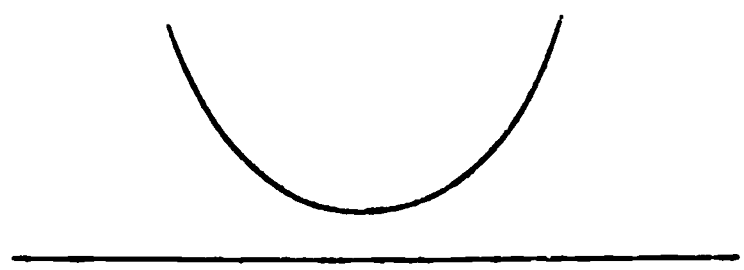
1) It will produce two roots if the parabola and straight line intersect in two points with rational co-ordinates:



2) It will produce one root if the parabola and straight line are tangent:



3) It will produce no roots if the parabola and straight line fail to intersect:



(The teacher, perhaps unwisely, told the class that this case corresponded to the need for matrices -- i.e., complex numbers.)

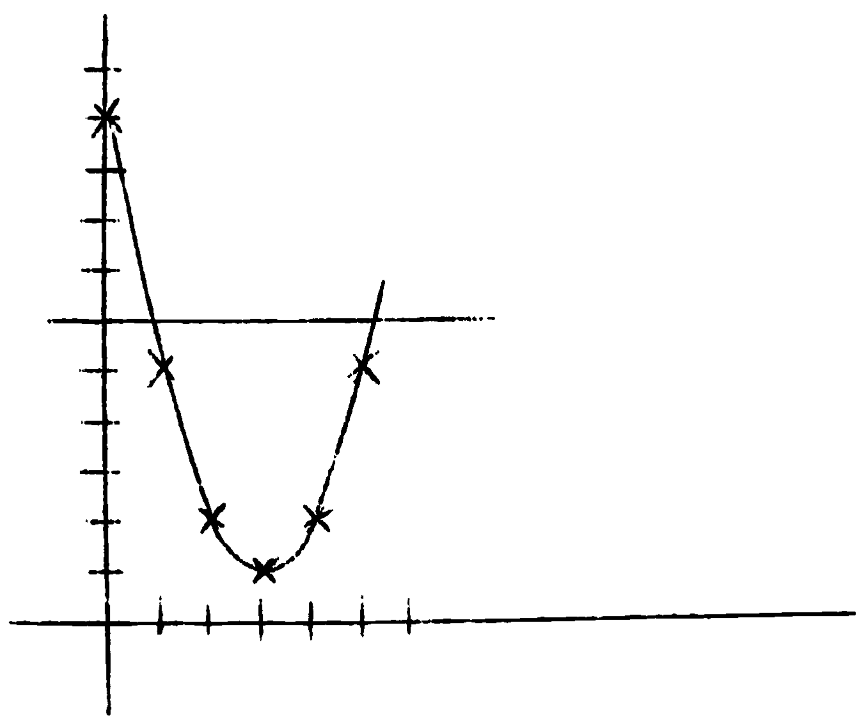
4) The most interesting case occurs when the parabola and straight line intersect in two points whose precise x-coordinates keep eluding our search. For example,

$$x^2 - 6x + 10 = 6.$$

parabola: $y = x^2 - 6x + 10$

x	y
0	10
1	5
2	2
3	1
4	2
5	5

line: $y = 6$



If we seek the x-coordinate for the left-hand point of intersection, we see that

x = 1 is too large

x = 0 is too small.

We can try $\frac{1}{2}$ as a replacement for the variable x;

$$Q\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{6}{2} + 10 = 10\frac{1}{4} - 3 = 7\frac{1}{4}$$

and so, by looking at the graph, we see that $\frac{1}{2}$ is too small a value for x (we are evidently to the left of the actual point of intersection).

At this point we really brought a significant piece of mathematical history into the classroom, and enacted it as a living reality. We became involved in the most exciting mathematical argument the author has ever witnessed in any of his classes. We re-lived the ancient Greek $\sqrt{2}$ controversy.

21. The Ancient Greek $\sqrt{2}$ Controversy. Pat, looking at the earlier algebraic solution,[1] identified this as the task of finding a number whose square was 5. She pointed out:

$$2 \text{ is too small, since } 2^2 = 4 < 5$$

$$3 \text{ is too large, since } 3^2 = 9 > 5$$

$$2.5 \text{ is too large, since } 2.5^2 = 6.25 > 5$$

$$2.1 \text{ is too small, since } 2.1^2 = 4.41 < 5$$

[1] For the equation $x^2 - 6x + 10 = 6$, subtract 1 from each side, to get $x^2 - 6x + 9 = 5$, or $(x - 3)^2 = 5$.

$$2.2 \text{ is too small, since } 2.2^2 = 4.84 < 5$$

$$2.3 \text{ is too large, since } 2.3^2 = 5.29 > 5$$

and so on.

Pat argued that the "correct" answer would continue to elude us; we could never find a number r such that

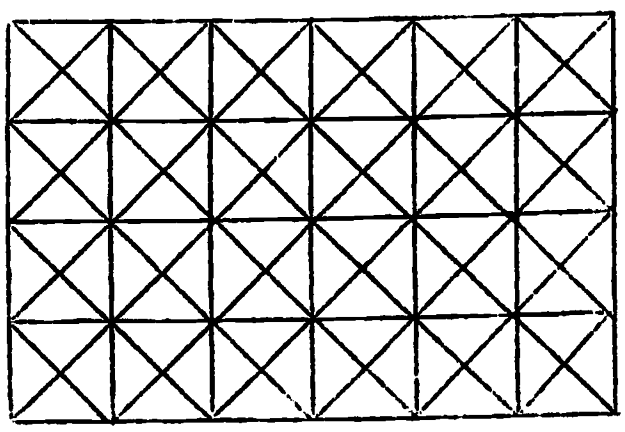
$$r^2 = 5.$$

Nancy F., thinking of the geometric picture, argued that there clearly was a point of intersection of the parabola and the straight line, and that it must have some x-coordinate.

Probably every girl in class chose sides in this argument, though some occasionally shifted on what seemed to be the weight of new evidence.

Where, after all, did this "parabola" come from? First we graphed integers, then fractions -- then we drew a smooth line through them. Were we entitled to? Did we, at this step, fill in lots of little "holes," without any justification for doing so?

The teacher used the pattern



to establish the Theorem of Pythagoras for isosceles right triangles (evidently mainly a geometric matter). Hence the Greeks had a geometric reason for believing that there existed a number s such that

$$s^2 = 2.$$

We then used a standard algebraic argument to show that, p and q being positive integers,

$$\frac{p^2}{q^2} = 2$$

led to a contradiction. Thus the Greek's had an algebraic reason for believing that there did not exist, within Greek arithmetic, any number whatsoever such that

$$s^2 = 2.$$

We had paralleled the Greeks precisely.

They were led -- it is said -- to drink hemlock. What should we do?

Nancy O. contributed a major break-through, but one whose implications are hard to discern, by translating

$$x^2 = 2,$$

via isomorphism, into matrix language

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and solving it (!) with the matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

(Anyone interested in following the contributions of individual students should observe that "Nancy O. $\neq$ Nancy F.")

The further pursuit of this controversy led us to consider the smallest integer that was too large, the largest integer that was too small, the smallest number $N + \frac{A}{10}$ that was too large, etc., thereby getting two sequences,

$$2, \ 1.5, \ 1.42, \ 1.415, \ \cdots$$

$$1, \ 1.4, \ 1.41, \ 1.414, \ \cdots \ .$$

Now, if A and B are any sets, if $A < B$, and if $n(\alpha)$ is any numerical attribute of α, then it must be true that

$$\max_{\alpha \in A} n(\alpha) \leq \max_{\alpha \in B} n(\alpha) \quad ,$$

i.e., the richest person in this building is at least as rich as the richest person in this room.

In this way, one sees that the first sequence

$$2, \ 1.5, \ 1.42, \ \cdots$$

must be monotonically decreasing (that is,

$$\forall_n, \quad u_n \geqslant u_{n+1})$$

because it depends upon minima, and the other sequence must be monotonically increasing (i.e.,

$$\forall_n, \quad u_n \leqslant u_{n+1}).$$

We now turn to the study of monotonic sequences.

22. Monotonic Sequences. By considering various examples, and in particular by trying to fill in every triangular cell in the classification scheme

	bounded above	not bounded above
monotonically increasing	convergent / divergent	convergent / divergent
monotonically decreasing	convergent / divergent	convergent / divergent

the students were lead to conjecture that "every monotonically-increasing sequence that is bounded above converges," and "every monotonically-decreasing sequence that is bounded below converges."

After some thought, a student (very wisely indeed) conjectured that "algebraic" axioms such as CLA, CLM, DL, etc., could never suffice to prove these exciting new statements.

Consequently it was decided, for the time being at least, to add these two new statements as two new axioms.

From this point it was possible to settle the $\sqrt{2}$ controversy, and to discuss the system of real numbers. A completely adequate theory of limits of infinite sequences had to be deferred until next year.[1]

[1] Cf., however, the experimental film Limits.

23. Convergence. It might, however, be worth a remark or two about the approach to convergence that was used. In the first place, the actual word "limit" was never introduced nor used, since in our experience it has many connotations which mislead the intuition, rather than aiding it. Ninth-graders do not have the sophistication of Lewis Carroll, to say "it is a question of who is going to be boss [the word or I]."

Since, however, the task for the class (as it was historically for Newton, Euler, and Cauchy) is to use one's intuition as a foundation on which to build a more formal treatment, we used the following approach, in order to clarify intuitive ideas as much as possible:

For any sequence made up by himself, or by the class, the teacher either assigned an "associated number," or else said that he refused, and labelled the sequence divergent. To give some examples:

sequence	associated number
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$	0
0.9, 0.99, 0.999, ⋯	1
1, 0, 1, 0, 1, 0, ⋯	none (divergent)
1, 1, 1, 1, 1, ⋯	1

and so on.

This brought the activity into a framework often used before by these students: you are to guess "What's My Rule?"

As student guesses revealed part, but not all, of the truth, the teacher proposed new sequences as counter-examples, thereby continuing the process of refining the formal verbal statement.

24. Simultaneous Equations. Actually, the work on infinite sequences closed the year's course, but we shall now go backwards in time and mention various other topics that had been treated earlier.

One such topic was simultaneous equations, especially two equations in two unknowns. Various approaches were used, especially via graphs and intersecting lines, and via matrix inversion. For the actual task of matrix inversion itself several different approaches were again used.

25. Isomorphism. The use of the concept of isomorphism has been discussed above; we list it here for emphasis. This is, above all, a useful concept that aids in the solution of many problems (cf., for example, some of the earlier versions of the UICSM materials, in relation to the "bookstore" problem).

26. Linearity and Convexity. One of the common errors of college freshmen is to replace

$$\frac{\sin 2x}{2x}$$

by

$$\frac{\sin x}{x},$$

or to assume that

$$\sin (A + B) = \sin A + \sin B$$

or

$$\sqrt{A+B} = \sqrt{A} + \sqrt{B}.$$

In an attempt to clarify this issue, we spent some time on linearity and convexity, especially via graphs, and in relation to examples from physics, the "law of diminishing returns" in economics, etc.

27. Transformations or Mappings. The general concept of transformation or mapping was presented in various forms, including:

isomorphisms between number systems

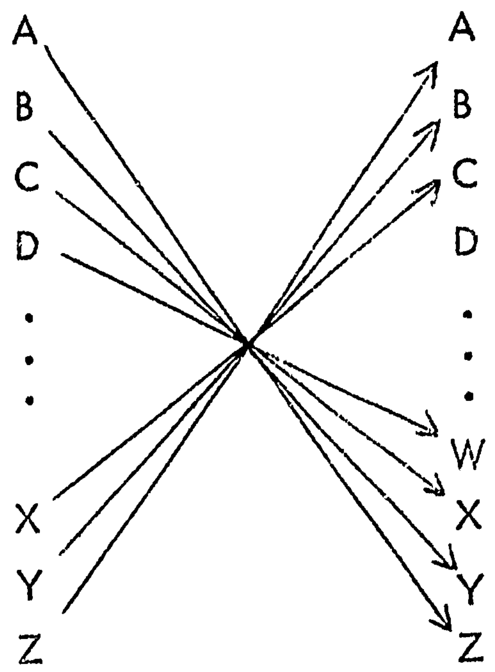
isomorphisms between number systems and matrices

geometric mappings of $E_2 \longrightarrow E_2$

simple substitution ciphers, where the set $\mathcal{A}$ of all letters of the alphabet was mapped onto itself

mappings of $V \times V \longrightarrow V$, where $V = \{T, F\}$, as discussed above in the section on "Logic."

As an example of this "code" technique, which proved popular, we might obtain a cipher from the mapping



28. <u>Identities in the Distance Function d(P,Q).</u> This topic was presented by asking the <u>students</u> to make up identities involving d(P,Q), where P and Q are points on the number line.

Most of the standard results were obtained from student lists, including

$$d(P,Q) = |P - Q|$$

$$d(P,Q) = d(Q,P)$$

$$d(P,Q) \geqslant 0$$

$$d(P,P) = 0$$

$$(P \neq Q) \longrightarrow d(P,Q) \neq 0 \ .$$

29. <u>Complex Numbers and the Complex Plane.</u> As mentioned above, the extension of the rational number system to the "rational" complex number system was achieved by using matrices, and <u>preceded</u> any discussion of irrational numbers. The isomorphism

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \longleftrightarrow A \qquad \underline{A} \text{ rational}$$

permits us to rewrite the equation

$$x^2 = {}^-4$$

as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^-4 & 0 \\ 0 & {}^-4 \end{pmatrix}$$

for which

$$\mathcal{L} = \begin{pmatrix} 0 & {}^-2 \\ {}^+2 & 0 \end{pmatrix}$$

is an element of the truth set. So also is

$$\overset{\circ}{\mathcal{L}} = \begin{pmatrix} 0 & {}^+2 \\ {}^-2 & 0 \end{pmatrix}.$$

We introduce the notation (where α is a rational number)

$$\alpha \begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{\text{def.}}{=} \begin{pmatrix} \alpha A & \alpha B \\ \alpha C & \alpha D \end{pmatrix},$$

and the symbol

$$\lambda = \begin{pmatrix} 0 & {}^-1 \\ {}^+1 & 0 \end{pmatrix}.$$

Involving closure, we are dealing with the system

$$\begin{pmatrix} A & B \\ {}^\circ B & A \end{pmatrix},$$

A and B rational.

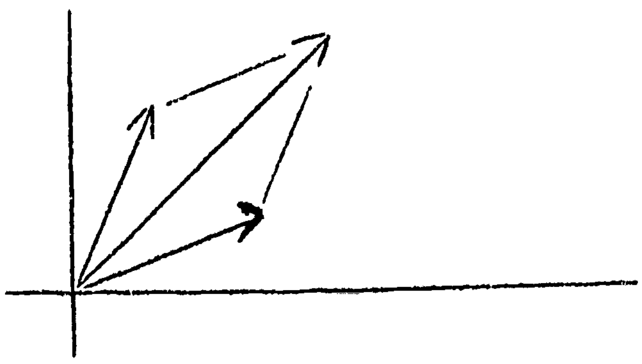
We now try to use the matrices of this last form above to name points in E_2.

The use of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ at (0,0), and of $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ at (A,0) is obvious. Once we elect to use λ as a name for (0,1), the die is cast. All other correspondences between matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

and points of E_2 are now determined.

Addition of such matrices has the obvious geometric meaning of vector addition:



and the question arises, how about multiplication?

This latter question can be regarded as a question in empirical science, and we can formulate "experiments" (i.e., select specific products to work out) from which we hope a general picture will emerge.

If the students knew the "double-angle" identities in trigonometry, they could, of course, settle the matter of multiplication completely. At this point in the course, the Nerinx students did not know these "double-angle" identities.

30. Extensions of Definitions. What we mean here is essentially the celebrated

"correspondence principle" of quantum mechanics, or a primitive analog of "analytic continuation" in classical complex variables.[1]

An effective tool for introducing the subject is David Page's "Maneuvers on Lattices." This goes as follows:

First, we draw part of an infinite array:

$$\begin{array}{cccccccccc} & & & & \vdots & & & & & \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

Then we tell the students that

$$21 \rightarrow$$

is the name of a number; we ask them to find a more familiar name for this number (note that <u>we</u> have not told <u>them</u> what $21 \rightarrow$ means). They respond with

$$21 \rightarrow \; = 22$$

which is what we had hoped for.

The students then work out these names:

$$25 \downarrow \; = 15$$

$$16 \nearrow \; = 27$$

[1] Cf. Eves and Newsom, An Introduction to the Foundation and Fundamental Concepts of Mathematics, Holt, Rinehart, and Winston, 1964, pp. 120-121, concerning the work of George Peacock.

$$27 \searrow = 18$$

$$18 \rightarrow \rightarrow = 20$$

$$8 \leftarrow = 7$$

$$5 \leftarrow \uparrow = 14$$

and so on, working intuitively and with no official meaning for these new symbols.

We note many standard algebraic laws, such as:

$$15 \rightarrow \rightarrow \leftarrow \leftarrow = 15$$

$$16 \nearrow \searrow \leftarrow \leftarrow = 16$$

$$5 \rightarrow \uparrow \swarrow = 5$$

$$3 \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow = 4$$

$$103 \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow = 104$$

and so on.

We have still not said what we mean by these new symbols.

Still working with each individual student's intuitive and unverbalized notion of what $\rightarrow$, $\nearrow$, $\uparrow$, etc., mean, we note certain apparent laws:

i) ("inverses") $\square \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow = \square \rightarrow \rightarrow$, etc.

ii) ("vector addition") $\square \nearrow = \square \uparrow \rightarrow$ or

$\square \nearrow \downarrow \leftarrow = \square$, etc.

iii) ("commutative") $\square \rightarrow \uparrow \nearrow = \square \nearrow \rightarrow \uparrow$

But wait! Do these "laws" really hold? Surely

$$5 \rightarrow \rightarrow \rightarrow \rightarrow = 9,$$

and

$$5 \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow = 7,$$

so

$$5 \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow = 5 \rightarrow \rightarrow .$$

However, suppose we replace the variable $\square$ with 9, rather than 5. Then

$$9 \rightarrow = 10,$$

and what does

$$9 \rightarrow \rightarrow$$

mean?

At this point we ask students to verbalize their various meanings for $\rightarrow$, $\uparrow$, etc. Fortunately, it has always happened in our classes (perhaps because the predominantly geometric orientation of this topic has introduced a psychological "mental set") that most students speak in terms of actual motions on the lattice array of numerals. This definition, which we hoped for, has the pedagogical advantage that it breaks down when we reach the end of the array, and will require suitable extension.

What shall we mean by

$$10 \rightarrow \quad ?$$

Various commonly suggested answers:

$10 \rightarrow = 0$ (when you "fall off the edge," call it zero)

$10 \rightarrow = 10$ (when you can't move, stay put)

$10 \longrightarrow \; = \; 20$ (when you hit a boundary, go up)

$10 \longrightarrow \; = \; 1$ (the array is wrapped around a cylinder)

$10 \longrightarrow \; = \; 11$ (the array is wrapped spirally around a cylinder, like a barber pole)

" $\longrightarrow$ " means "add 1," " $\uparrow$ " means "add 10," " $\downarrow$ " means "subtract 10," " $\nearrow$ " means "add 11," etc. (this always worked in the interior of the array, and it still makes sense at the boundary).

These various suggestions can be checked against what happens to our "laws" such as

$$\square \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \; = \; \square \rightarrow \rightarrow$$

$$\square \nearrow \; = \; \square \uparrow \rightarrow$$

$$\square \nearrow \downarrow \leftarrow \; = \; \square$$

$$\square \rightarrow \uparrow \nearrow \; = \; \square \nearrow \rightarrow \uparrow$$

and so forth. The desirable features of a legitimate extension of a definition can be illustrated clearly.

This notion of the "extension of a definition" can now be brought to bear upon:

i) exponents: $a^m a^n = a^{m+n}$, etc., as we go beyond positive integer exponents.

ii) the definition of 0!

iii) trigonometric functions for $90^\circ \leq \Theta$ or $\Theta \leq 0$.

iv) (if the students are ready) summability of infinite series by Abel summation, Cesaro summation, etc. (not done with the Nerinx Hall class)

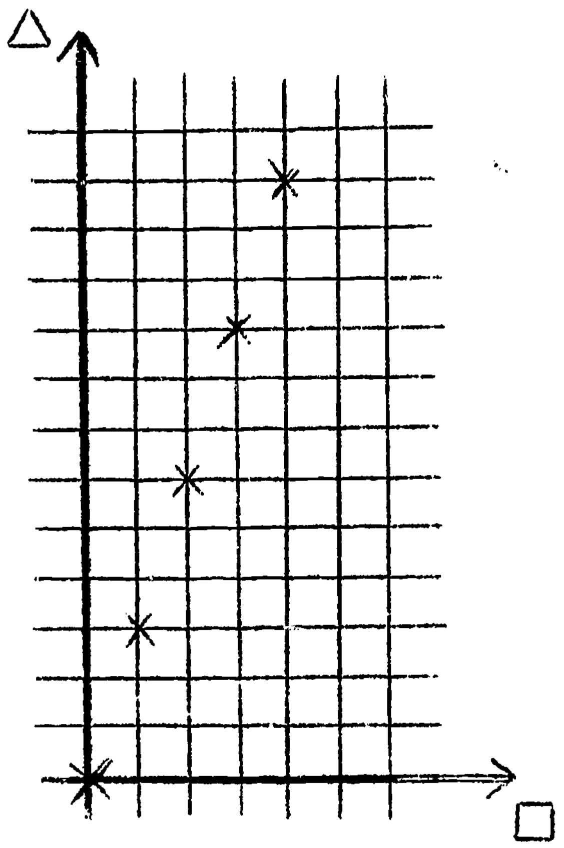
v) the arithmetic of signed numbers.

We illustrate this last instance in more detail:

We assume that the arithmetic of positive integers is already established. We also assume that, using only integral values, the discrete linear graphs are already familiar; that is, for example

$$(3 \times \square) = \triangle$$

with the discrete graph (for integral values)



We shall extend these straight-line patterns to include points in all 4 quadrants. The identification of the proper graph can be carried out independently of the process of multiplication, by extending from a few established cases, and by construing " $^{-}1$ " to mean "down" if " $^{+}1$ " means "up" (as has been done earlier on the number line).

We now define $f_m(x)$ to be the function obtained from the graph, through (0,0), with the pattern "over one to the right and up m." This can be read from suitable graphs, and can be done independently of multiplication.

We now define

$$m \times n$$

to mean

$$f_m(n).$$

It is now simple to verify that, for familiar cases, we get familiar answers

$$f_2(3) = 6$$
$$f_2(2) = 4$$
$$f_1(5) = 5$$
$$f_5(1) = 5$$
$$f_3(2) = 6$$

and so on. It is also simple to use this extension to get "new" results:

$$f_{-1}(^{-}1) = {}^{+}1 .$$

and so forth.

This use of graphs as a foundation for multiplication was suggested by Professor Paul Rosenbloom, of the University of Minnesota. In the present course it is not the only approach to the arithmetic of signed numbers, but is used alongside "models" ("pebbles-in-the-bag" and "postman stories"), and alongside a careful axiomatic approach.[1]

[1] Cf. the Madison Project film entitled Negative One Times Negative One Equals Positive One, the film A Lesson with Second Graders, and the film Postman Stories.

VII Comparison With Other "Modern" Courses

How is the present course different from other "modern" ninth-grade "algebra" courses?

In general, we believe it differs in these respects:

i) the inclusion of the study of infinite sequences, and the use of bounded monotonic sequences as a foundation for irrational numbers

ii) the inclusion of matrix algebra

iii) the emphasis upon an axiomatic approach

iv) the "developmental" approach by which sets of axioms are gradually modified as new needs arise or as new potentialities come into view

v) the inclusion of logic

vi) the inclusion of laboratory experiments, and a statistical approach to measurement problems

vii) the emphasis upon co-ordiante geometry

viii) the unusually high degree of student participation, and the emphasis on student "discovery" (e.g., the students choose the sets of axioms, subject to argument by the teacher)

ix) the considerably faster pace of moving ahead.

Obviously, it is for each teacher to decide on such matters for himself. However, we believe there is enough difference to justify one additional version of ninth-grade algebra.

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Appendix A

The Supplementary Algebra and Geometry Program, for Grades Two Through Eight

The Madison Project's supplementary algebra and geometry program for grades two through eight is presented in four books, and in a sequence of films. The books are:

Robert B. Davis, Discovery in Mathematics: Text for Teachers, Addison-Wesley Publishing Co., Reading, Massachusetts, 1964.

Robert B. Davis, Discovery in Mathematics: Student Discussion Guide, Addison-Wesley Publishing Co., Reading, Massachusetts, 1964.

Robert B. Davis, Matrices, Functions, and Other Topics: Text for Teachers (available from The Madison Project, Webster College, Webster Groves, Missouri, 63119).

Robert B. Davis, Matrices, Functions, and Other Topics: Student Discussion Guide (available from The Madison Project, Webster College, Webster Groves, Missouri, 63119).

The films are:

A Lesson with Second Graders

First Lesson (grades 3 - 7 in an "ungraded" class)

Second Lesson (grades 3 - 7 in an "ungraded" class)

Graphs and Truth Sets (2nd graders)

Experience with Fractions: Number Line and String (2nd graders)

Accumulating a List of Identities

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Average and Variance (6th graders)

Axioms and Theorems (6th graders)

Complex Numbers via Matrices (7th graders)

Circle and Parabola

Dividing Fractions (4th graders)

Education Report: The New Math (grades 2 - 7)

Experience with Area

Experience with Empirical Probability

Experience with Fractions

Experience with identities

Experience with Linear Graphs

Experience Estimating and Measuring Angles

Experience with Angles and Rotations

Graphing an Ellipse (7th graders)

Introduction to Truth Tables and Inference Schemes

Limits (8th graders)

Matrices (5th and 6th graders)

Postman Stories (6th and 7th graders)

Derivation of the Quadratic Formula -- First Beginnings

Derivation of the Quadratic Formula -- Final Summary

Solving Equations with Matrices (6th graders)

Weights and Springs

Appendix B

Partial Listing of Madison Project Personnel

Director: Professor Robert B. Davis, Syracuse University and Webster College

Associate Director: Professor Katharine Kharas, Webster College

Research Associate and Senior Film Editor: Beryl S. Cochran, Weston, Connecticut

Film Producer: Morton Schindel, President, Weston Woods Studios, Weston, Connecticut

Co-ordinators for Syracuse University:

Vice President Frank Piskor
Professor Donald E. Kibbey

Co-ordinator for Webster College: Sister M. Jacqueline, S.L.

Director for Early Childhood Education: Doris M. Diamant

Co-ordinator for Physical Sciences: Professor William Walton, Webster College

Specialist Teachers:

Donald Cohen
Gordon Bennett
Lila Page
Frank van Atta
Knowles Dougherty
Sophie Ciola
Marilyn de Santa

Project Fiscal Officer: Roy Hajek

Administrative Assistant: Martha Bowen

Manuscript Production: Bernice Talamante

Design: Peter Geist

Mathematics Advisory Panel:

Professor George Springer, University of Kansas
Professor E. E. Moise, Harvard University
Professor Gail Young, Tulane University
Professor Robert Rosenbaum, Wesleyan University
Professor Alvin N. Feldzamen, University of Wisconsin

Special Advisor on Mathematical Logic: Professor Robert Exner, Syracuse University

Project Psychologists:

Professor Carl Pitts, Webster College
Herbert Barrett, Weston, Connecticut Public Schools

Special Psychological Advisor: Professor Richard de Charms, Washington University

Special Advisor on Education and Measurement: J. Robert Cleary, Educational Testing Service, Princeton, New Jersey

Significant influence on Project work has come from the following persons, not officially associated with the Project:

Professor Andrew Gleason, Harvard University
Professor Stewart Moredock, Sacramento State College
Professor Jerrold Zacharias, Massachusetts Institute of Technology
Professor David Page, University of Illinois
Professor Max Beberman, University of Illinois
Professor Robert Karplus, University of California (Berkeley)
Professor David Hawkins, Educational Services, Incorporated
Paul Merrick, Educational Services, Incorporated
Professor Marshall Stone, University of Chicago
Professor R. C. Buck, University of Wisconsin
Professor Paul Rosenbloom, University of Minnesota
Professor Gerald Thompson, Carnegie Institute of Technology
Professor Erik Hemmingsen, Syracuse University
Professor Patrick Suppes, Stanford University
Professor George Polya, Stanford University
Professor Warwick Sawyer, Wesleyan University

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Professor Paul Johnson, University of California (Los Angeles)
Dr. C. Brooks Fry, M.D., Los Angeles Public Schools
Dr. Carol Fry, M.D., University of California (Los Angeles)
Professor Jerome Bruner, Harvard University
Professor Jerome Kagan, Fels Research Institute
Professor Francis Friedman, Massachusetts Institute of Technology
Dean Lawrence Schmeckebier, Syracuse University

Appendix C

Identification of Classes That Appear on Films

The Nerinx Hall ninth-graders who took the course described in this report had no previous history of Madison Project work, nor of any other "new" mathematics curriculum. This, however, is somewhat unusual in Project work. In general, the Madison Project attempts to follow the same students for as many years as possible. Any serious differences among curricula must surely relate to long-term differences in student growth and attitudes, rather than to short-term effects. In order to help identify the various classes, they have been designated by letters. A portion of this listing is included here:

Class A. Began study of Madison Project materials in 5th grade, during the 1959-1960 academic year. In June, 1964, they are 9th graders.

Name tags read: "Lex," "Bruce," "Geoff," "Jeff," "Ann," "Sarah," "Debby H.," "Ellen," etc.

In grade 8 this class studied the limit of a sequence, and made the experimental film entitled "Limits." Other films: Graphing an Ellipse, and Complex Numbers via Matrices.

Class B. Began the study of Madison Project materials as 4th graders, during the academic year 1959-1960. In June, 1964, they are finishing the 8th grade.

Name tags read: "Beth," "Jean-Anne," "Toby," "Mark," "Flint," etc.

They appear, with Class C, in the film entitled Matrices. When this film was made, Class B were 6th graders.

Class C. Began the study of Madison Project materials when they were 3rd graders, during the academic year 1959-1960.

Name tags read: "Jeff," "Ricky," "Mary," "Pam," "Lilli," "Windy," "Jono," "Geoff," "Greg," "Kris," "Val," "Miklos," "Jill," "Jennifer," etc.

Films include: Matrices (with Class B), Axioms and Theorems, Average and Variance, Weights and Springs, Solving Equations with Matrices.

For information on Classes D - M, inclusive, please refer to:

> Robert B. Davis, Report on Madison Project Activities, September 1962-November 1963. Report submitted to the National Science Foundation, December 16, 1963. Copies available from the Madison Project.

Class N. This is the Nerinx Hall 9th grade class discussed in the present report. They began the study of Madison Project materials in September, 1963, when they were in grade nine.

Name tags read: "Bev," "Marybeth," "Pat C.," "Carol," "Michele," "Pam," "Pat D.," "Susie," "Nancy F." "Suzi," "Maureen," "Eileen," "Donna," "Kathy V.," "Kathy W.," "Cathy," "Regina," "Karen," "Chris Hebert," "Sandy," "Pat H.," "Chris Hohl," "Kathy H.," "Kathy K.," "Marian," "Clare," "Mary Catherine," "Patty," "Mary Ann," "Janice," "Nancy O."

Class N made films on the following dates: December 21, 1963, May 9, 1964, and May 23, 1964.

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Appendix D

Films of the Nerinx 9th Grade Class

This class, designated as "Class N" (see Appendix C), has made the following films during the academic year 1963 – 1964, while in grade nine:

I. Recording session December 21, 1963

1. (no official title assigned as yet) Video Tape Number 35. Taping Session at KETC – TV, St. Louis, Missouri, Saturday, Dec. 21, 1963. There are two topics in this lesson, which runs for 59 minutes. The first topic is concerned with selection of algebraic axioms and selection of rules of logic, with the subsequent proofs of these theorems:

a) $(A + B) \times (C + D) = (D + C) \times (B + A)$

b) $2 + 2 = 4$

c) $A + (B \times C) = (C \times B) + A$

The second topic is some work with implication, contradiction, and uniqueness, based upon a sophisticated version of David Page's Hidden Numbers.

2. (no official title assigned as yet) Video Tape Number 36. Taping Session at KETC – TV, St. Louis, Missouri, Saturday, Dec. 21, 1963.

There are two topics on this tape, which runs 54 minutes. The first topic continues the axiomatic algebra from Video Tape Number 35, with emphasis upon theorems involving additive inverses, including the theorem

$$^{\circ}(^{\circ}A) = A \ .$$

The second topic deals with truth tables, and with the mapping of the Cartesian product V x V into V, where V is the "truth value space,"

$$V = \{T, F\}.$$

3. Negative One Times Negative One Equals Positive One. Video Tape Number 37, recorded at KETC, St. Louis, Dec. 21, 1963. This lesson includes three approaches to the statement

$$^{-}1 \times {}^{-}1 = {}^{+}1 \ .$$

The three approaches are:

i) use of a "model" ("postman stories")

ii) an axiomatic proof of the theorem

iii) an approach via "extension of a definition," using linear graphs, following a suggestion of Paul Rosenbloom.

II. Recording Session May 9, 1964 (KETC, St. Louis)

4. Quadratic Equations. A lesson on the derivation of the quadratic formula,(Video Tape Number 43)

5. Introduction to Infinite Sequences. This is the first time these students encounter this topic. It arises out of two monotonic sequences related to $\sqrt{2}$, which in turn arises out of the attempt to achieve a general quadratic formula, (Video Tape Number 44)

III. Recording Session May 23, 1964 (KETC, St. Louis)

6. What Is Convergence? This continues the work on infinite sequences which was begun on V.T. Number 44. The approach is somewhat similar to the "What's My Rule?" approach to functions: the teacher associates numbers with various sequences, or else declines to do so (saying "that sequence is divergent"), and the students are asked to describe, as precisely they can, the procedure that the teacher is using. (Video Tape Number 46)

7. Bounded Monotonic Sequences. This continues the work on sequences from Video Tapes 44 and 46. Two of the students decide that every bounded monotonic sequence converges, and another student decides that, since it seems unlikely that this could ever be proved from CLA, DL, etc., it needs to be added as an additional axiom. (Video Tape Number 47)

8. Introduction to the Complex Plane. Matrix names are given to points of E_2 . (Video Tape Number 48)

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Appendix E

Some Behavioral Objectives of Madison Project Teaching

It is becoming apparent that some of the differences in mathematics teaching are largely differences in objectives. The linguistic resources available for a discussion of objectives seem inadequate to the task, but we (perhaps unwisely) include a few remarks on the matter. [1]

Within the more narrowly "mathematical" or "technical" abilities that we seek to develop, we would include these:

i) The ability to discover pattern in abstract situations, and (where possible) enough relevant experience to have good judgment in selecting the most significant patterns;

ii) The ability to use independent creative explorations to extend "open-ended" mathematical situations. Cf., for example, the works of Polya, and the booklet Supplementary Problems for 18.01, by A. Mattuck (1963). [2] Professor E. J. McShane of the University of Virginia has cited, as an excellent textbook problem *in mathematics*, the following, which is quoted in full:

A pile of coal catches on fire.

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[1] Cf. the article: Robert B. Davis, "Report on the Madison Project," Science Education News (1962), December, pp. 15-16 (available from the American Association for the Advancement of Science). Cf. also Bert Y. Kersh, "Learning by discovery: what is learned?" The Arithmetic Teacher, Vol. 11, No. 4 (1964), April, pp. 226-232.

[2] Available from the Mathematics Department, Massachusetts Institute of Technology.

iii) The possession of a suitable set of mental symbols that serve to picture mathematical situations in a pseudo-geometrical pseudo-isomorphic fashion, somewhat as described by Polya, Leibnitz, and the psychologist Tolman.[1] (This is the kind of mental imagery that permits one to "visualize" Hilbert space, to "see" orthogonal functions, etc.);[2]

iv) A good understanding of fundamental mathematical concepts -- i.e., those that are operationally and organizationally fundamental, not those that are "logically fundamental." We would include such concepts as: variable, function, Cartesian co-ordinates, open sentence, truth set, matrices, implication, contradiction, axioms and theorems, uniqueness, mapping or transformation, linearity, etc.;

v) Reasonable mastery of important techniques;

vi) Knowledge of mathematical facts;

vii) Ability to read mathematics.

In addition to the objectives listed above, there are some broader or more general behavioral attributes which Madison Project teaching seeks to foster. These include:

i) A belief that mathematics is discoverable. (Indeed, our "ideal" student would probably be a sceptic who believed very little on authority, but who KNEW

[1] Cf. Edward Chace Tolman, "Cognitive Maps in Rats and Men," Chapter 19 of the volume Behavior and Psychological Man, University of California Press, 1958.

[2] Cf. Walter J. Sanders, "The use of models in mathematics instruction," The Arithmetic Teacher, Vol. 11, No. 3 (1964), March, pp. 157-165.

that mathematics is discoverable because he was in the habit of seeing it discovered every day in his classroom.)

ii) A realistic assessment of one's own personal proficiency in discovering mathematics, and a generally positive feeling toward the prospect of further personal growth in this direction.

iii) A personal recognition of the "open-endedness" of mathematics.

iv) Honest and wise personal self-critical ability. (That student is most hopelessly lost who "knows not, and knows not that he knows not.")

v) A personal commitment to the value -- in its proper places -- of abstract rational analysis. (We would not wish to turn out former students who would say "Oh, that's all a lot of theory," or who, in personal, political, or business matters cast rationality to the winds.)

vi) Recognition of the valuable role of "educated intuition."

vii) A feeling that mathematics is "exciting" or "challenging" or "fun" or "rewarding" or "worthwhile." This includes a feeling that the study of mathematics for its own sake is worthwhile and understandable as a human activity, and that the relevance of mathematics to the rest of life is often considerable.

Actually, there is another objective which is both mathematical and cultural: we would wish the student to come to know mathematics as a part of his cultural heritage. This involves a skillful and rare combining of mathematical concepts, dilemmas, and historical breakthroughs, with a cultural history of mathematics. We would seriously claim that this kind of view of one's cultural heritage goes toward answering "who am I" and "who are we who live in the United States in 1964?" Unfortunately, we do not claim great achievements in this direction for our own teaching, but this over-view of mathematics _is_ one of our objectives. The student should come to know, as quickly as possible, "what is mathematics in 1964" and "how did it get this way?"

The highest achievement toward this instructional goal of which we are presently aware is:

> Eves and Newsom, _An Introduction to the Foundations and Fundamental Concepts of Mathematics._ Holt, Rinehart, and Winston, 1964.